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Singular structures for automorphisms of hyperbolic surfaces

PhD dissertation

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Aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation has been obtained by legal means.

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The dissertation is ready to be reviewed.

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Osobliwe struktury automorfizmów powierzchni hiperbolicznych

Słowa kluczowe:
powierzchnia Riemanna, powierzchnia Kleina, automorfizm powierzchni, grupa Fuchsa, grupa NEC, dziedzina fundamentalna, genus algebraiczny, podwójne nakrycie, orbita, punkt okresowy, owal, łańcuch, zbiór okresów, charakter okresów

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20H10, 30F10, 30F35, 30F50, 37F20, 37B99, 37C25, 57S17

Streszczenie

Jednym z podstawowych problemów teorii układów dynamicznych jest badanie istnienia orbit okresowych. Rozprawa poświęcona jest automorfizmom zwartych powierzchni hiperbolicznych, a więc takich których genus algebraiczny jest większy od 1. Uogólniając pytanie dotyczące istnienia orbit okresowych rozpatrywane są struktury osobliwe automorfizmów, rozumiane jako zbiory tych punktów, których orbity są krótsze od rzędu przekształcenia.

W pracy wprowadzono pojęcie charakteru okresów działania grupy cyklicznej $\mathbb{Z}_N$ na zwartej powierzchni hiperbolicznej (a w przypadku powierzchni Riemanna odpowiadającego mu pojęcia zbioru okresów) opisującego struktury osobliwe automorfizmu. Znaleziono i sklasyfikowano wszystkie charaktery okresów w zależności od orientowalności powierzchni wyjściowej i powierzchni ilorazowej. Udowodniono, że przedstawione konstrukcje powierzchni dopuszczających działanie $\mathbb{Z}_N$ zadane danym charakterem okresów prowadzą do powierzchni uniformizowanych przez NEC grupy, dla których miara ich dziedziny fundamentalenj jest najmniejsza. W przypadku powierzchni Riemanna uzyskiwane powierzchnie uniformizowane są oczywiście przez grupy Fuchsa.

Dla konforemnych automorfizmów powierzchni Riemanna rozważono ponadto następujące zagadnienie. Niech $\mathcal{A}$ będzie zbiorem okresów działania $\mathbb{Z}_N$. Każdemu zbiorowi $\mathcal{A}$ możemy przypisać $g_{\mathcal{A}}$ – najmniejszy genus powierzchni, która dopuszcza działanie $\mathbb{Z}_N$ zadane przez $\mathcal{A}$ (jest to równoważne z przypisaniem najmniejszej miary dziedziny fundamentalnej uniformizującej grupy Fuchsa). Dla każdego N znaleziono taki zbiór $\mathcal{A}_{\max}$, że odpowiadający mu genus $g_{\mathcal{A}_{\max}}$ jest największy spośród liczb $g_{\mathcal{A}}$.

W rozprawie odpowiedziano na dwa otwarte pytania dotyczące istnienia homeomorfizmów skończonego rzędu postawione w pracy J. Guaschi, J. Llibre, Orders and periods of algebraically–finite surface maps, Houston J. Math. 23 (1997) 86-97.

Singular structures for automorphisms of hyperbolic surfaces

Keywords and Phrases:
Riemann surface, Klein surface, automorphism of surface, Fuchsian group, NEC group, fundamental region, algebraic genus, complex double, orbit, periodic point, oval, chain, set of periods, character of periods

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Summary

One of the fundamental problems the theory of dynamical systems deals with is the investigation of the existence of periodic orbits. In the dissertation there were studied the automorphisms of compact hyperbolic surfaces i.e. the surfaces with algebraic genus greater than 1. The generalization of the question regarding the existence of periodic orbits brings one to the investigation of singular structures for automorphisms considered here as sets of those points whose orbits are shorter than the order of the map.

In this thesis there was introduced a notion of character of periods of action of a cyclic group $\mathbb{Z}_N$ on a compact hyperbolic surface which describes singular structures for automorphism (in case of a Riemann surface its counterpart is a term set of periods). Based on the orientability character of an initial surface and quotient surface there were found and classified all characters of periods. It was proved that the constructions of surfaces on which a particular character of periods is attained, lead to surfaces which are uniformized by NEC groups whose fundamental region has a minimal measure. In the case of Riemann surfaces we clearly obtain surfaces which are uniformized by Fuchsian groups.

For conformal automorphisms of Riemann surfaces, the following problem was also considered. Let $\mathcal{A}$ be a set of periods of a $\mathbb{Z}_N$–action. For each set $\mathcal{A}$ there is always a number $g_{\mathcal{A}}$ – the minimal genus of a surface on which there exists the action of $\mathbb{Z}_N$ given by $\mathcal{A}$ (which is equivalent to finding an uniformizing Fuchsian group whose fundamental region has minimal measure). For each N there was found a set $\mathcal{A}_{\max}$ for which the corresponding genus $g_{\mathcal{A}_{\max}}$ is maximal among the numbers $g_{\mathcal{A}}$.

In the dissertation, the two open questions left in the paper J. Guaschi, J. Llibre, Orders and periods of algebraically–finite surface maps, Houston J. Math. 23 (1997) 86-97., were solved.

Dedicated to My Parents, Aunts,

Aneta, Agnieszka, Maciek, Mateusz

and to all My Family

Michał Sierakowski

Osobliwe struktury automorfizmów powierzchni hiperbolicznych

Autoreferat pracy doktorskiej

Przedmiotem mojej rozprawy doktorskiej są automorfizmy zwartych powierzchni topologicznych, których genus algebraiczny jest większy od 1. Powierzchnie spełniające powyższy warunek będziemy nazywać hiperbolicznymi. Stosowany w pracy termin *powierzchnie Kleina* odnosi się do powierzchni topologicznych ze strukturą dianalityczną, a więc taką która w charakterze funkcji przejść dopuszcza również odbicia zespolone. W niniejszej rozprawie zajmuję się zatem badaniem *struktur osobliwych* dianalitycznych automorfizmów zwartych hiperbolicznych powierzchni Kleina, rozumianych jako zbiory tych punktów, których orbity są krótsze od rzędu przekształcenia. Zagadnieniem, które rozpatruję jest pytanie o realizację zadanych struktur okresowych wyznaczanych przez działania cyklicznych grup automorfizmów. Genezą moich badań jest pytanie postawione przez prof. Jaume Llibre w poniższej postaci:

Pytanie 1. *Dla rozmaitości zespolonej M znaleźć zbiory okresów orbit okresowych odwzorowań holomorficznych M w siebie.*

Podstawowymi rozmaitościami zespolonymi są powierzchnie Riemanna i to badanie własności automorfizmów krzywych algebraicznych stanowi główny przedmiot obecnej pracy, w której nie rozpatruje się rozmaitości wymiaru (zespolonego) większego od 1.

Jak się okazuje warunek holomorficzności w przypadku powierzchni hiperbolicznych jest założeniem na tyle sztywnym, że determinuje stopień przekształcenia ograniczając jednocześnie jego (skończony) rząd. Co więcej w przypadku powierzchni Kleina i słabszego założenia dianalityczności odwzorowania, otrzymuje się analogiczny wniosek. Na mocy twierdzenia Kerckhoffa [25] każdy okresowy homeomorfizm zwartej hiperbolicznej powierzchni Kleina jest topologicznie sprzężony z dianalitycznym automorfizmem powierzchni Kleina o tym samym typie topologicznym rozumianym jako sygnatura NEC grupy Λ uniformizującej X (tzn. takiej, dla której X jest przestrzenią orbit $\mathbb{H}^2/\Lambda$). Z powyższego zatem można wywnioskować, że pomijając zespoloną strukturę rozmaitości nie traci się ogólności w badaniu dynamicznych własności przekształceń. Jednak badając automorfizmy dianalityczne traktowane jako reprezentanty klas sprzężoności topologicznej homeomorfizmów okresowych można wykorzystać bardzo silne narzędzia analizy zespolonej i geometrii algebraicznej. Dzięki takiemu podejściu udaje się znaleźć odpowiedź na pytanie sformułowane przez Alsedà, Llibre i Misiurewicza:

Pytanie 2 (Alsedà, Llibre and Misiurewicz [1], Open Problem 3.3). *Dla dowolnej powierzchni zwartej wyznaczyć zbiory okresów orbit okresowych dla homeomorfizmów skończonego rzędu, redukowalnych oraz pseudo–Anosowa.*

w części dotyczącej homeomorfizmów skończonego rzędu. Przypomnijmy, że zgodnie z klasyfikacją Nielsena–Thurstona [39] elementy grupy klas odwzorowań $\mathcal{M}(M)$ dowolnej powierzchni M dzielimy właśnie na wymienione w Pytaniu 2 trzy typy.

W pierwszych rozdziałach pracy zajmujemy się analitycznymi przekształceniami powierzchni Riemanna. Rozwiązanie Pytania 1 dla sfery $\hat{\mathbb{C}}$ oraz torusów $\mathbb{T}$ jest znacząco różne od odpowiedzi dla przypadku powierzchni o genusie wynoszącym co najmniej 2. Przypadek sfery opisuje twierdzenie Bakera [4, 15], natomiast zadanie dla torusów zespolonych jest ćwiczeniem bazującym na ogólnej postaci przekształceń holomorficznych $f\colon \mathbb{T} \to \mathbb{T}$ (patrz [31]).

Przyczynami wspomnianych różnic, między przypadkami hiperbolicznym i niehiperbolicznym jest po pierwsze brak górnego ograniczenia na stopień przekształcenia dla $\hat{\mathbb{C}}$ i $\mathbb{T}$. Po drugie zaś własność, że holomorficzne odwzorowania powierzchni hiperbolicznych w siebie są odwracalne już przy słabym założeniu, że ich obrazy nie są jednopunktowe. Wynika to z przytoczonego poniżej twierdzenia Riemanna–Hurwitza:

Twierdzenie 1 (Farkas and Kra [16]). *Niech $f\colon S \to S'$ będzie przekształceniem holomorficznym zwartych powierzchni Riemanna stopnia K (przez co rozumiemy, że zbiór $f^{-1}(Q)$ ma moc K dla prawie wszystkich $Q \in S'$), którego obraz jest różny od punktu. Niech g i γ oznaczają odpowiednio genusy powierzchni S i S'. Wtedy mamy*

$$g = K(\gamma - 1) + 1 + \frac{1}{2}\sum_{P\in S} b_f(P), \tag{1}$$

gdzie $b_f(P) + 1$ jest indeksem rozgałęzienia przekształcenia f w punkcie P.

Zatem holomorficzne odwzorowania $t\colon S \to S$ powierzchni hiperbolicznych nie mają rozgałęzień, a ich stopień jest zawsze równy 1. Tym samym jako przekształcenia "na" i "1–1" są konforemne (przekształcenia odwrotne $t^{-1}\colon S' \to S$ są również konforemne). Co więcej ich rząd jest skończony co wynika z rezultatu Schwarza, który pokazał że grupa automorfizmów analitycznych powierzchni hiperbolicznych jest skończona (patrz [16]).

Dodajmy, że stosowany wielokrotnie w niniejszej pracy wzór (1) jest przede wszystkim wykorzystywany w szczególnym przypadku nakryć rozgałęzionych. Jeśli bowiem $t\colon S \to S$ jest analitycznym automorfizmem powierzchni Riemanna o genusie topologicznym $g \geq 2$, to relacja Riemanna–Hurwitza pozwala na wnioskowanie o indeksach rozgałęzień nakrycia $S \to S/\langle t\rangle$. Przy oznaczeniu przez N rzędu przekształcenia t oraz przez m_i, $i = 1, \ldots, n$ wspomnianych indeksów rozgałęzień mamy na mocy (1):

$$g = N(\gamma - 1) + 1 + \frac{1}{2}\sum_{P\in S} b_f(P) = N(\gamma - 1) + 1 + \frac{1}{2}\sum_{i=1}^{n} \frac{N}{m_i}(m_i - 1),$$

co daje

$$\frac{2(g-1)}{N} = 2(\gamma - 1) + \sum_{i=1}^{n}(1 - \frac{1}{m_i}).$$

Dynamika homeomorfizmów skończonego rzędu, które działają na powierzchniach Riemanna i zachowują orientację jest bardzo prosta, ponieważ posiadają one jedynie skończenie wiele izolowanych orbit okresowych, których okresy są dzielnikami właściwymi rzędu przekształcenia. Znany powszechnie wynik mówi, że dowolny zbiór takich dzielników może być zrealizowany jako zbiór okresów dla pewnego t ([17], patrz również Stwierdzenie 2.4). Zamyka to problem wyznaczenia zbiorów okresów przekształceń holomorficznych zespolonych rozmaitości wymiaru 1. Można jednak pytać o to, czy realizacja zadanego zbioru okresów nakłada wymagania na typ topologiczny powierzchni Riemanna formułując kolejne zagadnienie:

Pytanie 3. *Dla dowolnego N oraz $\mathcal{A}$ – podzbioru zbioru właściwych dzielników N, znaleźć najmniejszy genus hiperbolicznej powierzchni Riemanna, na której można określić odwzorowanie konforemne rzędu N, którego zbiór okresów pokrywa się z $\mathcal{A}$.*

Liczbę spełniającą powyższy warunek nazywamy *genusem $\mathcal{A}$–minimalnym* i oznaczamy $g_{\mathcal{A}}$. Powyższe zadanie zostało rozwiązane metodami kombinatorycznymi w oparciu o teorię grup Fuchsa (Twierdzenie 2.8) przy wykorzystaniu wyników prac Harvey'a [20] i Macbeath'a [28]. Z uwagi jednak na zależność od rozkładu na czynniki pierwsze okresów przekształcenia, nie podajemy zamkniętej formuły na minimalny genus ograniczając się jedynie do wskazania najlepszych oszacowań (Stwierdzenie 2.10). W rozdziale 2.2 rozważamy natomiast *problem maksymalnego genusa*, czyli znalezienia takiego podzbioru dzielników N, którego relizacja jako zbioru okresów automorfizmu analitycznego wymaga modelowania na powierzchni o największym genusie spośród liczb $g_{\mathcal{A}}$ odpowiadających różnym podzbiorom zbioru dzielników własciwych N. Powyższe możemy sformalizować w następującej postaci:

Pytanie 4. *Dla każdego N znaleźć taki zbiór okresów $\mathcal{A}_{\max}$, aby odpowiadający mu genus $\mathcal{A}_{\max}$–minimalny dla każdego $\mathcal{A}$ podzbioru zbioru dzielników właściwych N spełniał warunek $g_{\mathcal{A}} \leq g_{\mathcal{A}_{\max}}$.*

Narzędzia, które zostały wykorzystane do rozwiązania Pytania 4 są standardowymi metodami analizy, teorii grup i teorii mnogości. Uzyskane wyniki wymagały przeprowadzenia serii elementarnych obliczeń, których szczegóły mogłyby się jednak okazać dla Czytelnika nużące i jako takie zostały w pracy pominięte. Ta część rozprawy została opublikowana w artykule [35].

W drugiej części pracy rozpatrujemy wersje wymienionych powyżej Pytań 1 i 3, uogólnione dla homeomorfizmów skończonego rzędu działających na powierzchniach Kleina. Rozważamy następujące zagadnienie:

Pytanie 5. *Dla odwzorowania skończonego rzędu działającego na zwartej powierzchni Kleina znaleźć zbiór punktów, których orbity są krótsze od rzędu przekształcenia oraz wyznaczyć jego okresy.*

Podobnie jak w przypadku homeomorfizmów działających na powierzchniach Riemanna i zachowujących orientację, klasyfikację struktur okresowych uzyskuje się rozważając jedynie podrodzinę homeomorfizmów złożoną z odwzorowań dianalitycznych. Z uwagi na jakościową różnicę w strukturze *zbioru osobliwego* w porównaniu z poprzednim przypadkiem, jaką jest występowanie składowych jednowymiarowych (wymiaru rzeczywistego 1) definiujemy w rozdziale 3 syntetyczną wielkość za pomocą, której opisujemy go w kolejnych częściach pracy. Do tego celu wykorzystujemy *charakter okresów* oznaczany jako $\mathfrak{C}_0$. Zawiera on informacje nie tylko o długościach orbit izolowanych, lecz również informacje o okresach *składników brzegowych*, *jedno–* i *dwustronnych owali* oraz *łańcuchów*. Zauważmy, że wyodrębnienie tak określonych składowych zbioru osobliwego nie jest nowym narzędziem, gdyż pojawiło się już w pracach [42]–[44] oraz w przypadku inwolucji w artykule [9]. Uogólnieniem Pytania 3 jest następujące

Pytanie 6. *Dla dowolnego N oraz charakteru okresów $\mathfrak{C}_0$, znaleźć minimum miary obszaru fundamentalnego NEC grupy Λ, takiej że na powierzchni $\mathbb{H}^2/\Lambda$ można określić dianalityczny automorfizm rzędu N, który realizuje $\mathfrak{C}_0$ jako swój charakter okresów.*

Ponieważ tym razem nie zakłada się, że brzeg jest zbiorem pustym, inaczej niż w przypadku powierzchni Riemanna minimalizacja obszaru fundamentalnego grupy Λ nie jest tożsama z minimalizacją genusa powierzchni X. Wyniki dotyczące analizy poszczególnych przypadków ze względu na orientowalność badanej powierzchni X, powierzchni ilorazowej $X/\langle t\rangle$ oraz parzystość N zostały sformułowane w sześciu twierdzeniach: 5.5, 5.10, 5.17, 5.25, 5.36 i 5.42. Dodajmy przy tym, że stosując modyfikacje metod przedstawionych w rozdziale 4 można również uzyskać formuły minimalizujące genus przy założeniach dotyczących liczby składników brzegowych (lub odwrotnie: liczbę składników brzegowych przy założeniach dotyczących genusa). Podobne wyniki, choć bez rozróżniania zbiorów osobliwych automorfizmów zostały uzyskane w monografii [8].

Prostota implementacji podanych w pracy procedur sprowadza je, w każdym z rozpatrywanych przypadków, do wykonania serii obliczeń bazujacych na zdefiniowanych w pracy własnościach kombinatorycznych zbiorów liczb naturalnych. Zauważmy przy tym, że niektóre zagadnienia związane działaniem cyklicznych grup izometrii na powierzchniach są przedmiotem artykułów popularnych, czego przykładem jest [26].

Dzięki przedstawionym tu konstrukcjom, w Przykładach 2.11 oraz 5.29 udało się odpowiedzieć na dwa otwarte pytania, które postawiono w pracy [18].

Zagadnienia związane z własnościami odwzorowań powierzchni są częstym tematem dysertacji doktorskich. Z niektórymi z nich miałem przyjemność zapoznać się podczas przygotowywania własnej rozprawy: [12, 13, 38] – za co serdecznie dziękuję ich Autorom.

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Part I

Dynamics on Riemann Surfaces

Chapter 1

Dynamics of Analytic Transformations

1.1 Introduction

One of the main problems in the theory of dynamical systems is the determination of the existence of periodic orbits and more generally, the structure of the set of periods, which is considered here as the set of periods of periodic orbits together with its respective multiplicities.

In the present paper we deal with periodic orientation–preserving homeomorphisms of closed orientable surfaces Σ_g of genus $g \geq 2$ and in connection with the above set of periods we define the minimum genus of a surface on which there exists a map realizing a given set of periods. Although the behaviour of the iterations of those homeomorphisms for all initial points is very simple: there is only a finite number of orbits of periods being proper divisors of the order of the homeomorphism. The minimum genus problem in that setting has not been investigated.

For each fixed g, there are only finitely many finite groups G that act on Σ_g by orientation–preserving self–homeomorphisms of Σ_g. By a result of Hurwitz [21] the order of G is bounded by $84(g-1)$ and in particular Wiman [41] improved on this bound for a cyclic group obtaining $2(2g+1)$ as the maximum possible order for a periodic homeomorphism. On the other hand, for each G there is an infinite sequence of values of g such that G acts on Σ_g. This sequence is called *genus spectrum* of G (see [27],[29]).

Let $G = \langle f \rangle$ be a finite cyclic group of order N that acts by conformal automorphisms on a compact Riemann surface S of genus $g \geq 2$. Associated to this is a set $\mathcal{A}$ of periods defined to be the subset of proper divisors d of N such that, for some $x \in S$, x is fixed by f^d but not by any smaller power of f. For an arbitrary subset $\mathcal{A}$ of proper divisors of N, there is always an associated action and, if $g_{\mathcal{A}}$ denotes the minimal genus for such an action, an algorithm is obtained here to determine $g_{\mathcal{A}}$ (Theorem 2.8). It is natural to relate a structure of the set of periods of a $\mathbb{Z}_N$–action to a sequence of genera g of Σ_g on which $\mathbb{Z}_N$ realizes that given structure. We thus get a partition of the genus spectrum of $\mathbb{Z}_N$ into subsequences, which are not necessarily disjoint, that correspond to the possible sets of periods. The smallest member of each of the above subsequences that depend on $\mathcal{A}$ is just the minimal genus $g_{\mathcal{A}}$. It is worth pointing out that the smallest element among all subsequences was obtained by Harvey [20]. Furthermore, in section 2.2 a set $\mathcal{A}_{\max}$ is determined for which $g_{\mathcal{A}}$ is maximal (Theorem 2.16).

Another classification of orientation–preserving periodic maps on compact orientable surfaces up to topological conjugacy was obtained by Nielsen [34], Smith [37] and Yokoyama [42]. Conjugated maps have the same structure of the set of periods but the opposite implication clearly does not hold. In [42] (Theorem 5, p.92), the number of non–conjugated maps is given when the genus of a surface and the structure of periods are fixed.

It is well–known that any N–periodic self–map of hyperbolic surface is an isometry respect to some hyperbolic metric. Therefore our approach involves combinatorial techniques based on Fuchsian groups. We use to a great extent results of Harvey [20] and Macbeath [28]. Harvey's theorem provides necessary and sufficient conditions for the abstract Fuchsian group to be a universal covering transformation group of the cyclic group, while Macbeath gives a formula for the number of fixed points for each non–identity element of a cyclic group of automorphisms of compact Riemann surface. The generalization of that formula to closed non–orientable surfaces was obtained by Izquierdo and Singerman [22].

To fix terminology, let $f : M \to M$ be a self–map of a set M, and n be a positive integer. Let $\mathrm{Fix}(f)$ be the fixed point set of f, and $P_n(f)$ the set of periodic points with least period n

$$\begin{aligned}
\mathrm{Fix}(f) &:= \{x \in M \mid x = f(x)\}, \\
P_n(f) &:= \{x \in M \mid x = f^n(x) \text{ and } x \neq f^k(x) \text{ for any k < n}\} \\
&= \mathrm{Fix}(f^n) \setminus \bigcup_{k<n} \mathrm{Fix}(f^k).
\end{aligned}$$

Denote by $\mathrm{Per}(f)$ the set of positive integers corresponding to least periods of periodic orbits, $\mathrm{Per}(f) := \{n \in \mathbb{N} \mid P_n(f) \neq \emptyset\}$.

In order to assure the existence of periodic orbits we need a certain type of growth for the number of fixed points. Let G be a finite non–trivial group of self–maps of a given set M. By $\langle t \rangle$ we will denote the subgroup generated by t. Note that $\mathrm{ord}t = N$ implies that any period of t divides N. Conversely suppose that d is the least period of a point $x \in M$ and there is $r < d$ such that $N = dm + r$. But now $x = t^N(x) = t^r(t^{dm}(x)) = t^r(x)$, that contradicts our assumption.

The above conclusion leads us to the definition of the set $\mathrm{PPer}(\mathbb{Z}_N)$ of potential periods of a $\mathbb{Z}_N$–action

$$\mathrm{PPer}(\mathbb{Z}_N) = \{\mathcal{A} \mid \mathcal{A} \subseteq \mathcal{D}_0(N)\},$$

where $\mathcal{D}_0(N) = \{d \mid d < N, d \mid N\}$. We restrict our attention to the cases where M is a hyperbolic compact Riemann surface.

1.2 Preliminaries

Recall that the hyperbolic plane is the set $\mathbb{H}^2 = \{x+iy \in \mathbb{C} \mid y > 0\}$ with the metric induced by

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y}.$$

For $U \subset \mathbb{H}^2$, the hyperbolic area is given by

$$\mu(U) = \int_U \frac{dxdy}{y^2}$$

always if this integral exists. It is well known that the set of orientation preserving isometries of $\mathbb{H}^2$ is given by the projective special linear group

$$\begin{aligned} \mathrm{PSL}(2,\mathbb{R}) &= \{A \in \mathrm{GL}(2,\mathbb{R}) \mid \det(\mathrm{A}) = 1\}/\{\pm I\} \\ &= \{z \mapsto \frac{az+b}{cz+d} \mid a,b,c,d \in \mathbb{R}, ab - cd = 1\}. \end{aligned}$$

Let us observe that orientation preserving isometries are the bijective biholomorphic maps from $\mathbb{H}^2$ to itself and form a group under superposition. Unless otherwise stated this group will be denoted as $\mathrm{Aut}(\mathbb{H}^2)$. Throughout the first part of the dissertation S stands for a compact Riemann surface and similarly to the notation above we let $\mathrm{Aut}(S)$ stand for a group of bijective biholomorphic maps from S to itself. Those maps are called conformal automorphisms of S. The following uniformization theorem is the starting point in a combinatorial study of compact Riemann surfaces.

Theorem 1.1 (Farkas and Kra [16]). *Every compact Riemann surface S of genus $g \geq 2$ is conformally equivalent to $\mathbb{H}^2/\Lambda$, where Λ is a freely acting discontinuous group of $\mathrm{Aut}(\mathbb{H}^2)$. Furthermore, $\pi_1(S) \simeq \Lambda$.*

For a Riemann surface S we thus have an unramified holomorphic map $\pi\colon \mathbb{H}^2 \to S$. Moreover the homeomorphism between the *orbit space* $\mathbb{H}^2/\Lambda$ and S induced by the map π gives rise to the unique complex structure on $\mathbb{H}^2/\Lambda$ under which the canonical projection $\mathbb{H}^2 \to \mathbb{H}^2/\Lambda$ is a holomorphic map. We then say that Λ uniformizes S. A *Fuchsian group* is a discrete subgroup of the topological group $\mathrm{PSL}(2,\mathbb{R})$. If a Fuchsian group Γ has compact orbit space it is known that it has a presentation of the form

$$\begin{array}{lll} \text{generators:} & a_1, b_1, ..., a_\gamma, b_\gamma & \text{(hyperbolic)} \\ & x_1, x_2, ..., x_n & \text{(elliptic)} \\ \text{relations:} & x_1^{m_1} = x_2^{m_2} = ... = x_n^{m_n} = 1, \quad m_i \geq 2 & \\ & x_1 x_2 ... x_n \prod_{i=1}^{\gamma} [a_i b_i] = 1, \quad [a_i b_i] = a_i^{-1} b_i^{-1} a_i b_i. & \end{array} \tag{1.1}$$

The integers $m_1, m_2, ..., m_n$ will be called *periods* of the Fuchsian group Γ and γ the *orbit genus.* The symbol $(\gamma; m_1, m_2, ..., m_n)$ will be called the *signature* of Γ. From now on we require each Fuchsian group to be cocompact. Every Fuchsian group has an associated fundamental region, whose hyperbolic area depends only on the signature of the group. For a group with presentation (1.1) is given by $\mu(\Gamma) = 2\pi(2\gamma - 2 + \sum_{i=1}^{n}(1 - m_i^{-1}))$. Recall that an abstract group Γ defined by (1.1) can be realized as a Fuchsian group if and only if $\mu(\Gamma) > 0$. Using this formula we can give explicitly all the exceptional signatures that cannot be attained by Fuchsian groups. These are the following:

$$\begin{gathered}(0; m_1, m_2), (0; 2, 3, 3), (0; 2, 3, 4), (0; 2, 2, m_1), (0; 2, 3, 5) \\ (0; 2, 3, 6), (0; 2, 2, 2, 2), (0; 2, 4, 4), (0; 3, 3, 3).\end{gathered} \tag{1.2}$$

We shall use the above list in section 3 while constructing Fuchsian groups in terms of their signatures.

It is known that if $G \leqslant \mathrm{Aut}(S)$ then orbit space S/G is also a compact Riemann surface. We emphasize that G is not assumed to be the full group $\mathrm{Aut}(S)$. Furthermore the following theorem yields information about the form of G. A Fuchsian group having no elliptic generators will be called a *surface group.*

Theorem 1.2 (Harvey [20]). *A finite group G acts as a group of automorphisms of some compact Riemann surface of genus $g \geq 2$, if and only if G is isomorphic to the factor group Γ/Λ, where Γ is a Fuchsian group with compact orbit space and Λ a Fuchsian surface group with orbit genus g.*

Applying the Riemann–Hurwitz formula to the projection $S \to S/G$ we obtain

$$2(g-1)|G|^{-1} = 2(\gamma - 1) + \sum_{i=1}^{n}\left(1 - m_i^{-1}\right). \tag{1.3}$$

Observe that the group Γ of automorphisms of $\mathbb{H}^2$ is formed by lifting all elements of G. Moreover there is a homomorphism ψ^* from the Fuchsian group Γ onto the group G whose kernel is a surface group, that makes the following diagram commutative

$$\begin{array}{ccc} \Gamma \times \mathbb{H}^2 & \longrightarrow & \mathbb{H}^2 \\ {\scriptstyle\psi^*}\big\downarrow \; {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ G \times S & \longrightarrow & S \end{array} \tag{1.4}$$

and $\ker(\psi^*) = \Lambda \simeq \pi_1(S)$. We then say that the G–action on S is uniformized by *natural* epimorphism $\psi^* \colon \Gamma \to G$. If $t \in G$ then following Macbeath we call a pair (t, S) a *surface transformation* and (G, S) a *surface transformation group.* The transformation group $(\Gamma, \mathbb{H}^2)$ will be called the *universal covering transformation group* of (G, S). A homomorphism $\psi \colon \Gamma \to G$ having a kernel that is a surface group is called *smooth.* Recall that an epimorphism is smooth if and only if it preserves the periods of the elliptic generators (see [20]).

Assume now that $\mathbb{Z}_N$ is acting on a compact Riemann surface S. We shall consider the set of periods of $\mathbb{Z}_N$–actions as the set of $\mathrm{Per}(t)$ taken over all surfaces S and conformal automorphisms $t \in \mathrm{Aut}(S)$ of order N, that is

$$\mathrm{Per}(\mathbb{Z}_N) = \{\mathrm{Per}(t) \mid S \ - \ \text{surface}, \ t \in \mathrm{Aut}(S), \ \mathrm{ord} t = N\}.$$

The definition above involves the sets $\mathrm{Per}(t)$ that clearly depends on the surface S on which t is acting. However by a simple combinatorial argument it does not depend on the choice of generator within the group $\langle t \rangle$. In this sense we may consider the set of periods of a $\mathbb{Z}_N$–action on a particular surface S, which is stated in the following proposition. First we need a lemma ([32], Lemma 1).

Lemma 1.3. *Let $f : M \to M$ be a self map of a set M. If* $\mathrm{ord} f = N$ *and* $(N, m) = m'$, *then*

$$\mathrm{Fix}(f^m) = \mathrm{Fix}(f^{m'})$$

Proposition 1.4. *Suppose that $\langle f \rangle$ is acting as a group of self–maps of a set M. Let* $\mathrm{ord} f = N$ *and $\langle f \rangle \simeq \langle f^m \rangle$. Then, $P_n(f) = P_n(f^m)$ for each n.*

Proof. Suppose $w = f^m$, where $(N, m) = 1$. Since $(N, km) = (N, k)$ then by Lemma 1.3 we obtain equivalence of the following sets

$$\mathrm{Fix}(w^k) = \mathrm{Fix}(f^{(N,k)}) = \mathrm{Fix}(f^k).$$

In consequence $P_n(f) = P_n(f^m)$ as required. □

Note that we have actually proved that it is not only immaterial which generator we choose within the group $\langle t \rangle$ to define $\mathrm{Per}(t)$. Moreover the same (let us say i-th) iteration of any generator has identical fixed point set and thus also the set of periodic points of any least period.

It is well–known that both sets $\mathrm{PPer}(\mathbb{Z}_N)$ and $\mathrm{Per}(\mathbb{Z}_N)$ are equal. However for the convenience of the reader, in the next section we give the proof based on Macbeath's result.

1.2.1 Definitions and Notation I

In this section we establish notation that we shall use.

(0) Let $\mathcal{A} \in \mathrm{Per}(\mathbb{Z}_N)$. It is of interest to know the minimum genus of a surface on which $\mathcal{A}$ is attained as the set of periods. Denote by $g_{\mathcal{A}}$ the minimal genus for such an action of $\mathbb{Z}_N$. We will call the number $g_{\mathcal{A}}$ the $\mathcal{A}$–minimum genus.

(1) $\mathcal{D}_0(N) = \{d \mid d < N, d \mid N\}$, $\mathcal{D}_1(N) = \{d \mid d \neq 1, d \mid N\}$.

(2) Let $\delta_x \colon \mathbb{N} \to \mathbb{N}$ denote the Dirac delta function: $\delta_x(y)$ equals 1 if $x = y$ and 0 otherwise.

Let $\mathcal{B}$ be a subset of divisors m_i of N that all of them are greater than 1 i.e. $\mathcal{B} = \{m_1, ..., m_k\} \subseteq \mathcal{D}_1(N)$. All subsequent definitions in this section apply to $\mathcal{B}$ as defined above.

(3) Let $\mathrm{lcm}\mathcal{B} = \mathrm{lcm}(m_1, ..., m_k)$.

(4) Let $N = p_1^{r_1} ... p_n^{r_n}$ and let $\alpha_{p_i}(N) = r_i$.

Observe that $\alpha_{p_i}(\mathrm{lcm}\mathcal{B}) = \max\limits_{j=1,\dots,n} \alpha_{p_i}(m_j)$. Define also the following sets:

(5) Let $\mathcal{A}_{p_i}(\mathcal{B})$ be the set of elements in $\mathcal{B}$ divisible by the maximum power of the prime factor p_i i.e. $\mathcal{A}_{p_i}(\mathcal{B}) = \{m \in \mathcal{B} \mid \alpha_{p_i}(m) = \alpha_{p_i}(\mathrm{lcm}\mathcal{B})\}$,

(6) If there is only one element in the set $\mathcal{A}_{p_i}(\mathcal{B})$ we call it an *isolated element* and define $F(\mathcal{B})$ to be the set of all isolated elements of $\mathcal{B}$: $F(\mathcal{B}) = \{m \in \mathcal{B} \mid \exists i \ \mathcal{A}_{p_i}(\mathcal{B}) = \{m\}\}$.

(7) Let $C(\mathcal{B})$ be the set of elements of $\mathcal{B}$ which are divisible by the maximum power of 2 but are not isolated: $C(\mathcal{B}) = \mathcal{A}_2(\mathcal{B}) \setminus F(\mathcal{B})$,

(8) If $2 \nmid \sharp C(\mathcal{B})$, let $t(\mathcal{B}) = \min\{m \in \mathcal{B} \mid m \in \mathcal{A}_2(\mathcal{B})\}$ and define

$$G(\mathcal{B}) = \begin{cases} \{t(\mathcal{B})\}, & \text{if } 2 \nmid \sharp C(\mathcal{B}) \\ \emptyset, & \text{if } 2 \mid \sharp C(\mathcal{B}). \end{cases}$$

In this way we obtain that $G(\mathcal{B})$ is either empty or a singleton.

Furthermore, we introduce two auxiliary maps Δ, Δ_2.

(9) Let $\mathcal{B} \subseteq \mathcal{D}_1(N)$ and let Δ, Δ_2 be given by the formulas

$$\begin{aligned} \Delta(\mathcal{B}) &= \sum_{m \in \mathcal{B}} (1 - m^{-1}) + \sum_{m \in F(\mathcal{B})} (1 - m^{-1}) + \sum_{m \in G(\mathcal{B})} (1 - m^{-1}), \\ \Delta_2(\mathcal{B}) &= 1 - N\delta_N(\mathrm{lcm}\mathcal{B}) + \frac{N}{2}\Delta(\mathcal{B}). \end{aligned}$$

Remark 1.5. *Assume that $\emptyset \neq \mathcal{B} \subseteq \mathcal{D}_1(N)$. Then $\mathcal{A}_2(\mathcal{B}) \subseteq F(\mathcal{B})$ implies $G(\mathcal{B}) = \emptyset$.*

Remark 1.6. *Observe that $\mathcal{A} \in \mathrm{Per}(\mathbb{Z}_N)$ and $\mathcal{B} \subseteq \mathcal{D}_1(N)$ are sets. Thus do not contain any repetitions.*

Chapter 2

Orientation Preserving $\mathbb{Z}_N$–actions on Riemann Surfaces

2.1 The Set of Periods of Cyclic Groups Actions

We recall the following theorem of Harvey, that gives necessary and sufficient conditions on a group Γ to have a surface group Λ as a normal subgroup of finite index, such that the factor group Γ/Λ is cyclic.

Theorem 2.1 (Harvey [20]). *Let Γ be a Fuchsian group of the form (1.1) with orbit genus γ, and let $M = \mathrm{lcm}(m_1, ..., m_n)$. There is a smooth epimorphism $\psi : \Gamma \to \mathbb{Z}_N$ if and only if the following conditions are satisfied:*

(i) $\mathrm{lcm}(m_1, ..., \overline{m}_i, ..., m_n) = M$ *for all i, where $\overline{m}_i$ denotes the omission of m_i;*

(ii) M divides N, and if $\gamma = 0$, then $M = N$;

(iii) $n \neq 1$, and, if $\gamma = 0$, then $n \geq 3$;

(iv) if $2|M$, the number of periods of the group Γ divisible by the maximum power of 2 dividing M is even.

For the sake of completeness we recall also a theorem of Macbeath, that we shall need in order to compute the set of periods of a $\mathbb{Z}_N$–action.

Theorem 2.2 (Macbeath [28]). *Let (G, S) be a Riemann surface transformation group and $(\Gamma, \mathbb{H}^2)$ the universal covering group. Let $x_1, ..., x_n$ be generators of the maximal finite cyclic subgroups of Γ of orders $m_1, ..., m_n$ respectively, including exactly one for each conjugacy class. Let ψ^* denote the natural homomorphism of Γ on G. For $t \in G \setminus \{Id\}$ let $\epsilon_i(t)$ be 1 or 0 according as t is or is not conjugate to a power of $\psi^*(x_i)$ in G. Then the number of points of S fixed by t is given by the formula*

$$\sharp\mathrm{Fix}(t) = |N_G(\langle t \rangle)| \sum_{i=1}^{n} \epsilon_i(t) m_i^{-1}, \tag{2.1}$$

where $N_G(\langle t \rangle)$ denotes the normalizer of subgroup $\langle t \rangle$ in G.

If G is cyclic the formula (2.1) is particularly easy to handle although we first need a preliminary result.

Proposition 2.3 (Harvey [20]). *A homomorphism ψ^* from a Fuchsian group Γ onto a finite group G is smooth if and only if it preserves the periods of Γ, i. e. for every elliptic generator x_i, of order m_i, $\psi^*(x_i)$ has order m_i.*

By the above we get $\epsilon_i(t) = 1$ if and only if $\mathrm{ord}t$ divides m_i and consequently

$$\sharp\mathrm{Fix}(t) = N \sum_{m_i, \mathrm{ord}t | m_i} m_i^{-1}.$$

Observe that $\sharp\mathrm{Fix}(t)$ is completely determined by Γ and does not depend on the choice of natural homomorphism. The following proposition yields information about the structure of the set of periods.

Proposition 2.4. *Suppose that* $(\Gamma, \mathbb{H}^2)$, *where* Γ *is given by (1.1), is the universal covering transformation group of a transformation group* $(\langle t\rangle, S)$, $\mathrm{ord}t = N$. *Then, if* $d \mid N$,

$$\sharp P_d(t) = d\ \sharp\{m_i \mid m_i = N/d\}. \tag{2.2}$$

Proof. The proof is by induction on the number of prime factors of d. Observe that

$$\sharp\mathrm{Fix}(t^d) = N \sum_{k|d} \sharp\{m_i \mid m_i = \mathrm{ord}t^k\} m_i^{-1}, \tag{2.3}$$

by Theorem 2.2. Suppose firstly d to be prime. Since any period of t divides N we may assume $d \mid N$ and conclude that $\sharp P_d(t) = N\sharp\{m_i \mid m_i = \mathrm{ord}t^d\}m_i^{-1} = d\sharp\{m_i \mid m_i = N/d\}$.

If now (2.2) holds for the divisors of N having no more than r prime factors counted with multiplicities, we can easily show that it also holds for d that has $r+1$ prime factors. Indeed, since $k < d$, $k \mid d$, k has no more than r prime factors and we obtain

$$\begin{aligned}\sharp\mathrm{Fix}(t^d) - \sharp P_d(t) &= \sum_{k<d, k|d} \sharp P_k(t) = \sum_{k<d,k|d} k\sharp\{m_i \mid m_i = N/k\} \\ &= \sum_{k<d,k|d} N m_i^{-1}\sharp\{m_i \mid m_i = \mathrm{ord}t^k\},\end{aligned}$$

which together with (2.3) gives our assertion. □

We are now in a position to show that $\mathrm{PPer}(\mathbb{Z}_N) = \mathrm{Per}(\mathbb{Z}_N)$.

Corollary 2.5. *The set of periods* $\mathrm{Per}(\mathbb{Z}_N)$ *of the cyclic group* $\mathbb{Z}_N$ *is equal to the set of potential periods* $\mathrm{PPer}(\mathbb{Z}_N)$.

Proof. Suppose that $\mathcal{A} = \{d_1, ..., d_k\} \in \mathrm{PPer}(\mathbb{Z}_N)$ and $\mathcal{A} \neq \emptyset$. We shall construct a Fuchsian group Γ with compact orbit space, such that $(\Gamma, \mathbb{H}^2)$ covers the transformation group $(\langle t\rangle, S)$ and $\mathrm{Per}(t) = \mathcal{A}$, where $\mathrm{ord}t = N$. In order to achieve this, we define

$$\Gamma = (1; N/d_k, N/d_k, ..., N/d_1, N/d_1). \tag{2.4}$$

The group Γ is Fuchsian with orbit genus 1 and two elliptic generators of each period equal to N/d_i. It is easy to check that the group (2.4) satisfies the conditions of Theorem 2.1. By Proposition 2.4 it follows that the transformation group $(\langle t\rangle, S)$ satisfies $\mathrm{Per}(t) = \{d_1, ..., d_k\} \in \mathrm{Per}(\mathbb{Z}_N)$ with S being a surface of genus $g = Nk - \sum_{i=1}^k d_i + 1$ which is a consequence of (1.3).

The only point remaining concerns the case $\mathcal{A} = \emptyset$. Again, by Proposition 2.4 it is clear that Γ has to be a surface group $(\gamma; -)$. Consider a smooth homomorphism of Γ onto $\mathbb{Z}_N$ that maps each hyperbolic generator a_i, b_i onto any element of order N. Note that any such epimorphism is smooth since Γ has no elliptic elements. By the Riemann–Hurwitz formula (1.3), we now obtain

$$\frac{2(g-1)}{N} = 2(\gamma - 1),$$

because the terms depending on the periods of Γ disappear. Since we are investigating only hyperbolic surfaces ($g \geq 2$) it follows that $\gamma > 1$. □

Remark 2.6. *By the above we conclude that the minimum genus of a hyperbolic surface on which $\mathbb{Z}_N$ acts periodic points freely equals $N+1$.*

Example 2.7. *The following example gives surface homeomorphisms of order $N \geq 2$ that acts on Riemann surfaces without periodic points. In order to construct the required maps we consider the closed surface of genus $N+1$ embedded in $\mathbb{R}^3$, modelled as a sum of a torus symmetric with respect to the z–axis with N holes centered in points laying on the external equator corresponding to the multiplicities of the angle $2\pi/N$ and N tori attached to it (see Figure 2.1 below, for $N=3$). Then the rotation t by $2\pi/N$ about the z–axis is a conformal automorphism of order N satisfying $\mathrm{Per}(t) = \emptyset$. The quotient surface is a 2–torus.*

The next theorem shows how to find the signature of a covering group Γ, such that the genus of the underlying surface S is minimal among all surfaces on which $\mathbb{Z}_N$ attains $\mathcal{A}$ as the set of periods.

Theorem 2.8. *Let $N = p_1^{r_1} p_2^{r_2} ... p_n^{r_n}$, $N \neq 2,3,4,6$. Suppose that $\mathcal{A} = \{d_1, d_2, ..., d_k\} \in \mathrm{PPer}(\mathbb{Z}_N)$, $k \geq 2$. Let $\mathcal{B} = \{N/d_1, N/d_2, ..., N/d_k\}$. Then $(\Gamma, \mathbb{H}^2)$, where*

$$\Gamma = (1 - \delta_N(\mathrm{lcm}\mathcal{B}); \mathcal{B}, F(\mathcal{B}), G(\mathcal{B})) \tag{2.5}$$

is a universal covering transformation group of $(\langle t \rangle, S)$, such that $\mathrm{ord}t = N$. Furthermore $\mathrm{Per}(t) = \mathcal{A}$, and genus of S equals $g_{\mathcal{A}}$.

Proof. As we are interested in those Fuchsian groups Γ such that there exists a smooth homomorphisms of Γ onto $\mathbb{Z}_N$, we again apply Harvey's theorem. The repeated periods by means of the set of isolated periods $F(\mathcal{B})$ and the set $G(\mathcal{B})$ correspond to conditions *(i)* and *(iv)* of that theorem. Recall also that due to point *(ii)*, $\gamma = 0$ implies $\mathrm{lcm}\mathcal{B} = N$. Note that genus of the group given by (2.5) equals 0 if and only if $\mathrm{lcm}\mathcal{B} = N$. On the other hand if $\mathrm{lcm}\mathcal{B} \neq N$ then its genus is equal to 1.

We show firstly that Fuchsian group given by (2.5) satisfies Harvey's conditions. Obviously $\sharp\mathcal{A}_{p_i}(\mathcal{B}) + \sharp\mathcal{A}_{p_i}(F(\mathcal{B})) \geq 2$ for every prime factor of $\mathrm{lcm}\mathcal{B}$. Hence condition *(i)* of Theorem 2.1 is satisfied. Observe that in case $k=2$ and $\mathrm{lcm}(N/d_1, N/d_2) = N$ we get that d_1 and d_2 are coprime, hence $\sharp\mathcal{B} + \sharp F(\mathcal{B}) \geq 4$ and condition *(iii)* follows. The set $\mathcal{A}_2(\mathcal{B})$ splits naturally into two subsets, namely $\mathcal{A}_2(\mathcal{B}) \cap F(\mathcal{B})$ and $\mathcal{A}_2(\mathcal{B}) \setminus F(\mathcal{B})$. Since $\mathcal{A}_2(\mathcal{A}_2(\mathcal{B}) \cap F(\mathcal{B})) = \mathcal{A}_2(F(\mathcal{B}))$ we obtain that $\sharp\mathcal{A}_2(\mathcal{B}) + \sharp\mathcal{A}_2(F(\mathcal{B}))$ is even if and only if $\sharp(\mathcal{A}_2(\mathcal{B}) \setminus F(\mathcal{B}))$ is even. Thus in case $2 \nmid \sharp(\mathcal{A}_2(\mathcal{B}) \setminus F(\mathcal{B}))$ we change the parity of $\sharp\mathcal{A}_2(\mathcal{B}) + \sharp\mathcal{A}_2(F(\mathcal{B}))$, by adding the element $\min \mathcal{A}_2(\mathcal{B})$. Thus point *(iv)* of Theorem 2.1 follows. Moreover $\mathrm{Per}(t) = \{d_1, d_2, ..., d_k\} = \mathcal{A}$ by Proposition 2.4.

We now proceed to the proof of minimality of genus of the surface S. Suppose that there is a universal covering transformation group $(\Gamma_1, \mathbb{H}^2)$ and its underlying surface transformation group $(\langle t_1 \rangle, S_1)$ such that $\mathrm{ord}t_1 = N$, $\mathrm{Per}(t_1) = \mathcal{A}$ and genus of surface S_1 equals g_1. Assume $\Gamma_1 = (\gamma_1; m_1', m_2', ..., m_{n_1}')$. It follows that all the periods of Γ_1 are contained in $\mathcal{B}$, *i.e.* $m_i' \in \mathcal{B}$. Otherwise $\mathrm{Per}(t_1) \neq \mathcal{A}$, by Proposition 2.4. Define $m_\Gamma(N/d_i)$ and $m_{\Gamma_1}(N/d_i)$ to be multiplicities of N/d_i appearing as periods of the groups Γ and Γ_1 respectively. Observe

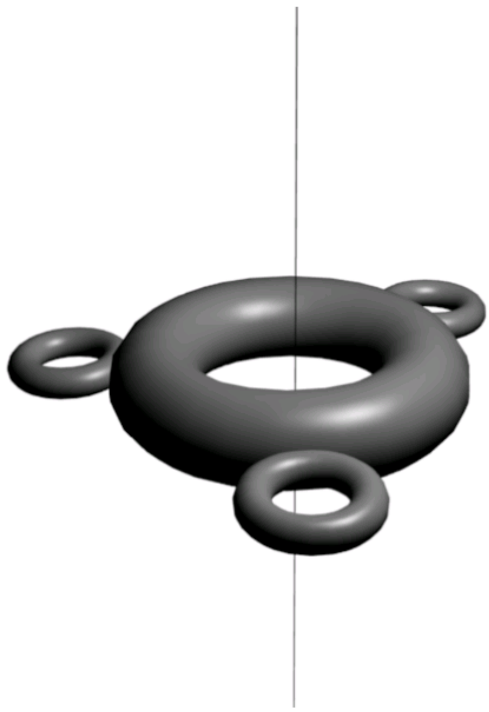

Figure 2.1: Automorphism of order 3 acting freely on the surface of genus 4.

that if $N/d_j \in F(\mathcal{B})$, then there is $1 \le i \le n$ such that $\mathcal{A}_{p_i}(\mathcal{B}) = \{N/d_j\}$. Thus in order to satisfy condition *(i)* of Theorem 2.1 we necessarily must have $m_{\Gamma_1}(N/d_j) \ge 2$. If $N/d_j \neq \min \mathcal{A}_2(\mathcal{B})$ or $G(\mathcal{B}) = \emptyset$, then $m_\Gamma(N/d_j) = \sharp\mathcal{A}_{p_i}(\mathcal{B}) + \sharp\mathcal{A}_{p_i}(F(\mathcal{B})) = 2$. Hence $m_{\Gamma_1}(N/d_j) \ge m_\Gamma(N/d_j)$. In case $\min \mathcal{A}_2(\mathcal{B}) \in F(\mathcal{B})$ and $G(\mathcal{B}) = \{\min \mathcal{A}_2(\mathcal{B})\}$ we obtain $m_\Gamma(\min \mathcal{A}_2(\mathcal{B})) = 3$. Obviously if $m_{\Gamma_1}(\min \mathcal{A}_2(\mathcal{B})) \ge 3$ we certainly have $m_{\Gamma_1}(N/d_j) \ge m_\Gamma(N/d_j)$ for all $1 \le j \le k$. Otherwise $m_{\Gamma_1}(\min \mathcal{A}_2(\mathcal{B})) = 2$ implies that there is $N/d_j \in \mathcal{A}_2(\mathcal{B})$, $N/d_j > \min \mathcal{A}_2(\mathcal{B})$ satisfying $m_{\Gamma_1}(N/d_j) > m_\Gamma(N/d_j)$. Furthermore, observe that

$$1 - (N/d_j)^{-1} > 1 - (\min \mathcal{A}_2(\mathcal{B}))^{-1}. \tag{2.6}$$

Note that in both cases we have $n_1 \ge \sharp\mathcal{B} + \sharp F(\mathcal{B}) + \sharp G(\mathcal{B})$. Moreover $\gamma_1 = 0$ implies $\mathrm{lcm}\mathcal{B} = N$ that finally gives $\gamma = 0$. Since $\gamma \le 1$ the above and (2.6) show that $g \le g_1$ and this is precisely the assertion of the theorem. □

We have just skipped in the last theorem the case $k = 1$. We complete the study by the following remark.

Remark 2.9. *If $k = 1$ and the remaining assumptions of Theorem 2.8 hold then the universal covering group satisfying conditions required there equals*

$$\Gamma = \begin{cases} (0; N, N, N), & \text{if } \mathcal{A} = \{1\},\ 2 \nmid N \\ (0; N, N, N, N), & \text{if } \mathcal{A} = \{1\},\ 2 \mid N, \\ (1; N/d_i, N/d_i), & \text{if } \mathcal{A} = \{d_i\},\ d_i \neq 1. \end{cases}$$

It is worth noting that applying directly the remark above and Theorem 2.8 for $N = 2, 3, 4, 6$ we may obtain exceptional signatures from the list (1.2). As an example take $N = 6$ and $\mathcal{A} = \{1, 2, 3\}$ that would led us to the Euclidean group $(0; 2, 3, 6)$. Analogously for $N = 2$ and $\mathcal{A} = \{1\}$ we would obtain by Remark 2.9 the group $(0; 2, 2, 2, 2)$. In Table 2.1 we consider the excluded cases.

Although in general an exact formula for the $\mathcal{A}$–minimum genus $g_\mathcal{A}$ seems to be complicated, in the next proposition we provide its upper and lower bounds. Note that Harvey in fact found the smallest $\mathcal{A}$–minimum genus.

Proposition 2.10. *Under the assumptions of Theorem 2.8, we have*

$$\frac{1}{2}\left(N(k-2) - \sum_{i=1}^{k} d_i + 2\right) \le g_\mathcal{A} \le Nk - \sum_{i=1}^{k} d_i + 1. \tag{2.7}$$

Proof. By Theorem 2.8 we have $g_\mathcal{A} = \Delta_2(\mathcal{B})$. Clearly $1 - N + N/2\sum_{m\in\mathcal{B}}(1 - m^{-1}) \le \Delta_2(\mathcal{B})$. To show also the upper bound, observe that $\sharp F(\mathcal{B}) + \sharp G(\mathcal{B}) \le \sharp\mathcal{B}$ by Remark 1.5 and in the case where this inequality is sharp our assertion follows. Furthermore, if there is equality and $G(\mathcal{B}) \neq \emptyset$ then $\mathcal{B} \setminus F(\mathcal{B}) \subseteq \mathcal{A}_2(\mathcal{B})$. Hence

$$\Delta(\mathcal{B}) \le \sum_{m\in\mathcal{B}}(1 - m^{-1}) + \sum_{m\in F(\mathcal{B})}(1 - m^{-1}) + \sum_{m\in\mathcal{B}\setminus F(\mathcal{B})}(1 - m^{-1}),$$

since $G(\mathcal{B}) = \{\min \mathcal{A}_2(\mathcal{B})\}$. Finally, $F(\mathcal{B}) = \mathcal{B}$ yields the right–hand bound, which completes the proof. □

In general, for $k \geq 3$, the bounds at (2.7) cannot be sharpened. That is to say, for each N there exists a set of periods $\mathcal{A}$ such that $g_{\mathcal{A}}$ equals the left–hand or the right–term term of inequality (2.7).

As a final remark in this section we give an example of application of the combinatorial argument used in Theorem 2.8.

Example 2.11. *Let Σ_g be a hyperbolic surface of genus g and suppose f to be a finite order orientation–preserving self–homeomorphisms of order N of Σ_g. It was left as an open question in [[18], p.478] as to whether there was a $\mathbb{Z}_N$–action on a surface of genus 3 so that $4 \in \mathrm{Per}(f)$ but $1, 2 \notin \mathrm{Per}(f)$. By (1.4) we are looking for a smooth epimorphism $\psi^* : \Gamma \to \mathbb{Z}_N$. By Proposition 2.4, we must have $N/m_i > 2$ for all i and at least one $N/m_j = 4$ unless $N = 4$ in which case the action must be fixed point free. This latter case cannot arise for $g = 3$ by Remark 2.6. By Theorem 6 of [20] $N = 4k > 4$ implies $g \geq \max\{2, k\}$. Thus $N = 8$ or $N = 12$. For $N = 8$, we have $\mathcal{A} = \{4\}$ and for $N = 12$, $\mathcal{A} = \{4\}, \{3, 4\}, \{4, 6\}, \{3, 4, 6\}$. By Theorem 2.8, the corresponding universal covering groups for the minimal genus $g_{\mathcal{A}}$ are, respectively, $(1; 2, 2)$ for $N = 8$, $(1; 3, 3)$, $(0; 3, 3, 4, 4)$, $(1; 2, 2, 3, 3)$, $(0; 2, 3, 3, 4, 4)$ for $N = 12$. But the minimum genera are then $5, 9, 6, 15, 9$ respectively so that there is no action of the required type on a surface of genus 3.*

2.2 The Maximum Genus Problem

Abstract of the section
Since the content of the actual section comprises mostly some delicate and detailed applications of standard methods based on Harvey's and Macbeath's results we give here a short abstract anticipating the forthcoming investigations. First, recall where we are. Let $G = \langle f \rangle$ be a finite cyclic group of order N that acts by conformal automorphisms on a compact Riemann surface S of genus $g \geq 2$. Associated to this is a set $\mathcal{A}$ of periods defined to be the subset of proper divisors d of N such that, for some $x \in S$, x is fixed by f^d but not by any smaller power of f. For an arbitrary subset $\mathcal{A}$ of proper divisors of N, there is always an associated action and, if $g_{\mathcal{A}}$ denotes the minimal genus for such an action. In the actual section we focus on a set $\mathcal{A}_{\max}$ of proper divisors of N for which $g_{\mathcal{A}}$ is maximal. Furthermore in the general case we observe that $\mathcal{A}_{\max}$ corresponds to the full set $\mathcal{D}_1(N)$ or $\mathcal{D}_1(N)\setminus\{c_1\}$, where the terms $\mathcal{D}_1(N)$ and c_1 were introduced in Subsection 1.2.1. Thus roughly speaking the more periods we require to appear while $\mathbb{Z}_N$ acts on a hyperbolic Riemann surface the higher value of its genus shall be expected. However the above general concept does not cover all the cases forcing us to find all exceptions.

We already know that any set $\mathcal{A}$ of proper divisors of N can be realized as the set of periods of some $\mathbb{Z}_N$–action on a compact Riemann surface. Associating to each of the sets $\mathcal{A}$ the $\mathcal{A}$–minimum genus $g_{\mathcal{A}}$ we may introduce the following relation: $\mathcal{A}_1$ precedes $\mathcal{A}_2$ ($\mathcal{A}_1 \leq_{\mathbb{Z}_N} \mathcal{A}_2$) if $g_{\mathcal{A}_1} \leq g_{\mathcal{A}_2}$. This relation is reflexive and transitive, thus the set $\mathrm{Per}(\mathbb{Z}_N)$ with this relation is a quasi–ordered set. Fix N. We then may ask for the maximal value of $g_{\mathcal{A}} = g_{\mathcal{A}}(N)$ and a corresponding maximal element of $(\mathrm{Per}(\mathbb{Z}_N), \leq_{\mathbb{Z}_N})$. A set for which $g_{\mathcal{A}}$ is maximal will be denoted by $\mathcal{A}_{\max}$. Obviously there may exist more than one maximal element. Nevertheless, since we are interested in maximum value of $g_{\mathcal{A}}$ it is not our purpose to study all maximal elements in $(\mathrm{Per}(\mathbb{Z}_N), \leq_{\mathbb{Z}_N})$. Therefore just the determination of any of them will be regarded as a satisfactory result. We continue to use notation and symbols introduced previously in Subsection 1.2.1. Let $N = p_1^{r_1}...p_n^{r_n}$ and $\mathcal{B} \subseteq \mathcal{D}_1(N)$. Unless otherwise stated we assume that $\mathcal{B} \neq \emptyset, \{N\}$. We define

$$\mathcal{D}_*(N) = \begin{cases} \mathcal{D}_1(N) \setminus \{c_1\}, & \text{if } G(\mathcal{D}_1(N)) = \emptyset,\ 2 \mid N,\ N \neq 2^r \\ \mathcal{D}_1(N), & \text{otherwise,} \end{cases}$$

where $c_1 = \min \mathcal{A}_2(\mathcal{D}_1(N))$. Note that if $N = 2^r M$ for M odd, then $c_1 = 2^r$. Recall that by Theorem 2.8 $\Delta_2(\mathcal{B})$ equals $g_{\mathcal{A}}$, where $\mathcal{B} = \{N/d \mid d \in \mathcal{A}\}$. During the following calculations we will see that in a "general" case we have

$$\Delta_2(\mathcal{B}') \leq \Delta_2(\mathcal{D}_*(N/p_i)) \leq \Delta_2(\mathcal{D}_*(N)), \tag{2.8}$$

where $\mathcal{B}' \subseteq \mathcal{D}_1(N/p_i)$ and nearly all exceptional cases given at the end of this section arise when one of the inequalities above does not hold. Observe that the both maps $\Delta = \Delta(\mathcal{B}, N)$ and $\Delta_2 = \Delta_2(\mathcal{B}, N)$ in fact depend on two arguments: $\mathcal{B} \subseteq \mathcal{D}_1(N)$ a set of divisors of N being the order of a cyclic group and the number N itself. As we fix N, for convenience in notation we ignore the dependence of Δ and Δ_2 on N. Therefore although $\mathcal{B}' \subseteq \mathcal{D}_1(N/p_i)$ we regard $\mathcal{B}'$ as a set of divisors of N.

Observe also that

$$\Delta_2(\mathcal{B}') - \Delta_2(\mathcal{D}_*(N/p_i)) = 2^{-1}N(\Delta(\mathcal{B}') - \Delta(\mathcal{D}_*(N/p_i))). \tag{2.9}$$

We thus begin with a study of the left–hand inequality by considering the map Δ.

Lemma 2.12. *Let $N = p_1^{r_1}p_2^{r_2}...p_n^{r_n}$, where $n \neq 2$ or $n = 2$ and* $\min\{r_1, r_2\}$ ≥ 3. *Then*

$$\max_{\mathcal{B}\subseteq\mathcal{D}_1(N)} \Delta(\mathcal{B}) = \Delta(\mathcal{D}_*(N)).$$

Proof. We have divided the proof into four parts depending on the cardinality of the set of isolated elements $F(\mathcal{B}) = \{m_1, m_2, ..., m_s\}$. The main idea of the proof is to investigate the number of elements of $\mathcal{D}_1(N)$ that do not belong to $\mathcal{B}$ when s varies. It is obvious that for any $m \in \mathcal{D}_1(N)$ we have $1/2 \leq (1-m^{-1}) < 1$. Note that if we consider a sum of elements of the form $(1-m^{-1})$, then by removing $2k$ and adding k summands its value always decreases. We use the above remark in cases $s = 0, 1, 2$ although case $s \geq 3$ differs from this line of argument, which will be explained after the remaining results have been obtained.

Observe firstly that if $G(\mathcal{D}_1(N)) = \emptyset$ but $2 \mid N$, $N \neq 2^r$ we have

$$\Delta(\mathcal{D}_1(N) \setminus \{c_1\}) > \Delta(\mathcal{D}_1(N)).$$

We prove the lemma for $s = 0$. Define $c_2 = \min \mathcal{A}_2(\mathcal{D}_1(N) \setminus \{c_1\})$. If $\sharp(\mathcal{D}_1(N) \setminus \mathcal{B}) \geq 2$, then since $\sum_{m\in\mathcal{D}_1(N)\setminus\mathcal{B}}(1-m^{-1}) \geq 1$, we have $\Delta(\mathcal{D}_1(N)) \geq \Delta(\mathcal{B})$. We thus may assume $\sharp(\mathcal{D}_1(N) \setminus \mathcal{B}) = 1$. We need only consider the cases where $G(\mathcal{B}) \neq \emptyset$, since otherwise we clearly have

$$\sum_{m\in\mathcal{B}}(1-m^{-1}) = \Delta(\mathcal{B}) \leq \Delta(\mathcal{D}_1(N)).$$

By means of the definition in Subsection 1.2.1 it gives $G(\mathcal{B}) = \{t(\mathcal{B})\}$. If $G(\mathcal{D}_1(N)) \neq \emptyset$ then $\Delta(\mathcal{D}_1(N)) - \Delta(\mathcal{B}) = 1 - m^{-1} \geq 2^{-1}$. Otherwise, since $t(\mathcal{B}) \leq c_2$, then $G(\mathcal{D}_1(N)) = \emptyset$ implies $\Delta(\mathcal{D}_1(N) \setminus \{c_1\}) - \Delta(\mathcal{B}) = 1/t(\mathcal{B}) - c_2^{-1} + c_1^{-1} - m^{-1} \geq 0$.

We now turn to the case $F(\mathcal{B}) \neq \emptyset$, that is $s \neq 0$ Suppose $s = 1$. Moreover assume that $n > 1$ and $\mathcal{A}_{p_j}(\mathcal{B}) = \{m_1\}$. Observe that

$$\sharp(\mathcal{D}_1(N) \setminus \mathcal{B}) \geq \prod_{i\neq j}(\alpha_{p_i}(N) + 1) - 1 \geq 3. \tag{2.10}$$

Hence $G(\mathcal{B}) = \emptyset$ gives $\Delta(\mathcal{D}_1(N)) > \Delta(\mathcal{B})$. Suppose then $2 \mid N$. If now $G(\mathcal{D}_1(N)) = \emptyset$ and $G(\mathcal{B}) \neq \emptyset$, then $\mathcal{A}_2(\mathcal{B}) \neq \{m_1\}$, by Remark 1.5. Since $t(\mathcal{B}) > c_1$ gives $\sharp(\mathcal{D}_1 \setminus \mathcal{B}) \geq 4$ we thus get $\Delta(\mathcal{D}_1(N)) - \Delta(\mathcal{B}) > 0$. Hence we may assume $G(\mathcal{B}) = \{c_1\}$. Observe also that there is $m_* \in \mathcal{D}_1(N) \setminus \mathcal{B}$, such that $m_* > c_1$. It follows that $\Delta(\mathcal{D}_1(N)) - \Delta(\mathcal{B}) = \sum_{m \in \mathcal{D}_1(N)\setminus\mathcal{B}}(1 - m^{-1}) - F(\mathcal{B}) - G(\mathcal{B}) \geq c_1^{-1} - m_*^{-1} > 0$. Finally, if $G(\mathcal{D}_1) \neq \emptyset$ then clearly $\Delta(\mathcal{D}_1(N)) > \Delta(\mathcal{B})$. In case $n = 1$ we get $G(\mathcal{B}) = \emptyset$, hence $\mathcal{B} \neq \mathcal{D}_1(N)$ implies $\Delta(\mathcal{D}_1(N)) > \Delta(\mathcal{B})$.

Suppose that $s = 2$. We certainly have $n > 1$. Furthermore, assume $n > 3$ or $n = 3$ and $\max\{r_1, r_2, r_3\} > 1$. We thus obtain

$$\begin{aligned} &\sharp(\mathcal{D}_1(N) \setminus \mathcal{B}) \geq \\ &\prod_{i \neq j}(\alpha_{p_i}(N) + 1) + \prod_{i \neq k}(\alpha_{p_i}(N) + 1) - \prod_{i \neq j,k}(\alpha_{p_i}(N) + 1) - 2 \\ &= (1 + \alpha_{p_j}(N) + \alpha_{p_k}(N)) \prod_{i \neq j,k}(\alpha_{p_i}(N) + 1) - 2 \geq 8 - 2 = 6, \end{aligned} \tag{2.11}$$

and since $\sharp F(\mathcal{B}) + \sharp G(\mathcal{B}) \leq 3$ we are done. If $n = 3$ and $r_1 = r_2 = r_3 = 1$ we conclude that in the case $G(\mathcal{B}) \neq \emptyset$ we have $N = 2p_1p_2$. Suppose $p_1 < p_2$. We then obtain $\Delta(\mathcal{D}_1(N) \setminus \{2\}) - \Delta(\mathcal{B}) \geq (p_2 - 1)/2p_2 > 0$. If now $G(\mathcal{B}) = \emptyset$, then by (2.11) we get $\sharp(\mathcal{D}_1(N) \setminus \mathcal{B}) \geq 4$ and our assertion again follows. For $n = 2$ we have $G(\mathcal{B}) = \emptyset$ and since (2.11) gives $\sharp(\mathcal{D}_1(N) \setminus \mathcal{B}) \geq 5$ we again get the result.

Assume that $s \geq 3$. The proof consists now in the construction of a set $\mathcal{B}'$ such that $\Delta(\mathcal{D}_1(N)) \geq \Delta(\mathcal{B}') \geq \Delta(\mathcal{B})$. We achieve this by enlarging the set of periods of the universal covering group Γ. It results in substitution in the sum that defines the map Δ of all the isolated elements of $\mathcal{B}$ by greater ones. It clearly gives rise to bigger summands of the referred sum. We begin by identifying the structure of the set $F(\mathcal{B})$ that is extremely important for our construction. Recall that according to the definition $m_i \in F(\mathcal{B})$ if and only if there is j such that $\mathcal{A}_{p_j}(\mathcal{B}) = \{m_i\}$. Without loss of generality we may assume that

$$\begin{aligned} \mathcal{A}_{p_j}(\mathcal{B}) &= \{m_1\} \text{ if } k_0 = 1 \leq j < k_1, \\ \mathcal{A}_{p_j}(\mathcal{B}) &= \{m_2\} \text{ if } k_1 \leq j < k_2, \\ &\dots \\ \mathcal{A}_{p_j}(\mathcal{B}) &= \{m_s\} \text{ if } k_{s-1} \leq j < k_s \leq n+1. \end{aligned}$$

Furthermore, assume that

$$m_i = p_{k_{i-1}}^{a_{k_{i-1}}} p_{k_{i-1}+1}^{a_{k_{i-1}+1}} \dots p_{k_i-1}^{a_{k_i-1}} Q_i,$$

where $p_j \nmid Q_i$ for $k_{i-1} \leq j < k_i$. Consider the following terms

$$m'_j = m_j p_{k_j}^{a_{k_j} - \alpha_{p_{k_j}}(Q_j)} \text{ if } 1 \leq j < s, \qquad m'_s = m_s p_1^{a_1 - \alpha_{p_1}(Q_s)}.$$

Note that $m'_j \notin \mathcal{B}$ and $m'_j \neq m'_i$ for $j \neq i$. Set $\mathcal{B}' = \mathcal{B} \cup \{m'_1, m'_2, ..., m'_s\}$. Since for every prime factor of lcm$\mathcal{B}$ there are at least two elements in $\mathcal{B}'$ divisible by its maximum power it follows that $F(\mathcal{B}') = \emptyset$. Observe that $m'_j > m_j$ and $\mathrm{lcm}\mathcal{B} > \min \mathcal{A}_2(\mathcal{B})$. We conclude that $\mathrm{lcm}\mathcal{B} = \mathrm{lcm}\mathcal{B}' > m'_j$ yields $\mathrm{lcm}\mathcal{B}' \notin \mathcal{B}'$. Therefore we get

$$\begin{aligned}\Delta(\mathcal{D}_1(N)) &\geq \sum_{m \in \mathcal{D}_1(N) \setminus \{\mathrm{lcm}\mathcal{B}'\}} (1 - m^{-1}) + 1 - (\mathrm{lcm}\mathcal{B}')^{-1} \\ &\geq \sum_{m \in \mathcal{B}} (1 - m^{-1}) + \sum_{m \in \mathcal{B}' \setminus \mathcal{B}} (1 - m^{-1}) + 1 - 1/t(\mathcal{B}) \geq \Delta(\mathcal{B}),\end{aligned}$$

which completes the proof. □

Our next goal is to investigate in detail the cases that have been omitted in Lemma 2.12. The following proposition provides information about all exceptions of the first inequality of (2.8).

Proposition 2.13. *Let $N = p_1^{r_1} \ldots p_n^{r_n}$. Then*

$$\max_{\mathcal{B} \subseteq \mathcal{D}_1(N)} \Delta(\mathcal{B}) = \Delta(\mathcal{D}_*(N))$$

except the following cases when $\max_{\mathcal{B} \subseteq \mathcal{D}_1(N)} \Delta(\mathcal{B}) = \Delta(\tilde{\mathcal{B}})$

(i) $N = p_1^{r_1} p_2$, $\tilde{\mathcal{B}} = \mathcal{D}_1(N) \setminus \{p_1^{r_1}\}$, $2 \nmid N$, $r_1 > 2$

(ii) $N = p_1^2 p_2$, $\tilde{\mathcal{B}} = \mathcal{D}_1(N) \setminus \{p_1^2\}$, $2 \nmid N$, $2p_2 \geq p_1^2 - p_1 + 2$

(iii) $N = p_1^2 p_2$, $\tilde{\mathcal{B}} = \mathcal{D}_1(N) \setminus \{p_2, p_1^2 p_2\}$, $2 \nmid N$, $2p_2 \leq p_1^2 - p_1 + 2$

(iv) $N = p_1 p_2$, $\tilde{\mathcal{B}} = \{p_1, p_2\}$, $2 \nmid N$

(v) $N = 2p_1^{r_1}$, $\tilde{\mathcal{B}} = \mathcal{D}_1(N) \setminus \{p_1^{r_1}\}$, $2 \nmid r_1$.

Proof. We may assume $n = 2$ and $\min\{r_1, r_2\} \leq 2$. Let $F(\mathcal{B}) = \{m_1, ..., m_s\}$. It is worth noting that the proof of cases $s = 0$ and $s = 3$ of the preceding lemma follows independently on the prime factorization of N. Therefore $\Delta(\mathcal{B}) > \Delta(\mathcal{D}_*(N))$ implies $1 \leq s \leq 2$. Define

$$H(N, \mathcal{B}) = \Delta(\mathcal{D}_*(N)) - \Delta(\mathcal{B}).$$

We will investigate the reductions of the set $\mathcal{D}_1(N)$ that may cause that the value of the map Δ increases. In fact we are looking for those sets $\mathcal{B}$ such that $H(N, \mathcal{B}) < 0$. We will denote the considered cases by *a–b*, which will mean that $\sharp(\mathcal{D}_1(N) \setminus \mathcal{B}) = b$ and $\sharp F(\mathcal{B}) + \sharp G(\mathcal{B}) = a$. Thus in order to augment the value of Δ we shall have $b < 2a$. Since $n = 2$ we also have $a \leq 3$. Moreover if a were equal to 3 there would be $s = 2$. But Remark 1.5 yields $\sharp G(\mathcal{B}) = 0$, which is a contradiction. Observe also that the case *2-3* is empty. Assume firstly $N = 2^{r_2} p_1^{r_1}$. We may dismiss quickly the case $N = 2p_1$ since $\sharp \mathcal{D}_1(2p_1) = 3$. Furthermore $a = 2$ implies

$\mathcal{A}_{p_1}(\mathcal{B}) \cap F(\mathcal{B}) \neq \emptyset$. Note that if $m \in F(\mathcal{B}) \cup G(\mathcal{B})$, then $(1-m^{-1}) \leq (1-N^{-1})$. Observe also that $r_1 \geq 2$ clearly gives $\sharp\mathcal{A}_{p_1}(\mathcal{D}_1(N) \setminus \mathcal{B}) \geq 2$ and thus we obtain

$$\begin{aligned} H(N,\mathcal{B}) &\geq (1-2^{-1}) + \sum_{m \in \mathcal{A}_{p_1}(\mathcal{D}_1(N)\setminus\mathcal{B})} (1-m^{-1}) - 2(1-N^{-1}) \\ &\geq 2^{-1} - 3(2p_1)^{-1} + 2N^{-1} > 0. \end{aligned}$$

Otherwise, if $r_1 = 2$, then 2, p_1 and $2p_1$ are the three smallest elements of $\mathcal{D}_1(N)$ and by the above we again get $H(N,\mathcal{B}) > 0$. If now $2 \nmid N$, then $H(N,\mathcal{B}) = \Delta(\mathcal{D}_1(N)) - \Delta(\mathcal{B})$ and the three smallest elements of $\mathcal{D}_1(N)$ are not smaller than 3, 5 and 9 respectively. Thus

$$H(N,\mathcal{B}) \geq (1-3^{-1}) + (1-5^{-1}) + (1-9^{-1}) - 2(1-N^{-1}) > 2N^{-1} > 0.$$

It follows that we need only consider three cases: *1-1*, *2-1* and *2-2*.

Case 1-1 Since $s = 1$ we have $G(\mathcal{B}) = \emptyset$. By (2.10) we get $N = p_1^{r_1}p_2$ and $\mathcal{B}_1 = \mathcal{D}_1(N) \setminus \{p_1^{r_1}\}$ or $\mathcal{B}_2 = \mathcal{D}_1(N) \setminus \{p_1^{r_1}p_2\}$. Observe that $\Delta(\mathcal{B}_1) > \Delta(\mathcal{B}_2)$. If $2 \nmid N$ then $H(p_1^{r_1}p_2, \mathcal{B}_1) = (1-p_2)/p_1^{r_1}p_2 < 0$. Furthermore $H(2^{r_1}p_2, \mathcal{B}_1) = 0$ and finally $N = 2p_1^{r_1}$, $2 \mid r_1$ gives $H(2p_1^{r_1}, \mathcal{B}_1) = (p_1^{r_1}-1)/2p_1^{r_1} > 0$.

Case 2-1 Assume firstly $s = \sharp G(\mathcal{B}) = 1$. Analogously to the previous case we get $N = p_1^{r_1}p_2$, but now $p_1 \neq 2$. Therefore $N = 2p_1^{r_1}$. Observe that in case $2 \nmid r_1$ we obtain $H(2p_1^{r_1}, \mathcal{B}_1) = -(1+p_1^{r_1-1})/2p_1^{r_1} < 0$. If now $s = 2$ and $G(\mathcal{B}) = \emptyset$, then by (2.11) we have $N = p_1p_2$. If $2 \nmid N$ and $\mathcal{B}_3 = \{p_1, p_2\}$ we get $H(p_1p_2, \mathcal{B}_3) = -1 + (p_1+p_2-1)/p_1p_2 < 0$. Otherwise, for $N = 2p_1$ we get $H(2p_1, \mathcal{B}_3) = 0$.

Case 2-2 Analogously to case *2-1* we shall consider two subcases $\sharp G(\mathcal{B}) = 1$ and $\sharp G(\mathcal{B}) = 0$. Suppose $\sharp G(\mathcal{B}) = 1$. By (2.10) it follows that $N = p_1^{r_1}p_2^{r_2}$ and $F(\mathcal{B}) = \{p_1^{r_1}p_2^{k}\}$, $0 \leq k \leq r_2$. We thus again get $p_2 = 2$. Assume $N = 2p_1^{r_1}$. In order to maximize value of the map Δ we shall have $F(\mathcal{B}) = \{2p_1^{r_1}\}$. But $b = 2$ now yields $\sharp C(\mathcal{B}) = r_1 - 1$, $2 \mid r_1$ or $\sharp C(\mathcal{B}) = r_1$, $2 \nmid r_1$. But the latter case was considered in point *2-1*. It is easy to check that the already mentioned set $\mathcal{B}_1$ gives bigger value of Δ. Hence we shall consider only $\mathcal{B}_4 = \mathcal{D}_1(2p_1^{r_1}) \setminus \{2, p_1^{r_1}\}$, which leads us to $H(2p_1^{r_1}, \mathcal{B}_4) = (p_1^{r_1-1}-1)/2p_1^{r_1} > 0$. Thus $\mathcal{B}_4$ it is not an exceptional set. Assume now $N = 4p_1^{r_1}$. We shall have $F(\mathcal{B}_5) = \{4p_1^{r_1}\}$ and consequently $\sharp C(\mathcal{B}_5) = r_1$, $2 \nmid r_1$. Then let $\mathcal{B}_5 = \mathcal{D}_1(4p_1^{r_1}) \setminus \{p_1^{r_1}, 2p_1^{r_1}\}$. But we again get $H(4p_1^{r_1}, \mathcal{B}_5) = (p_1^{r_1-1}(2p_1-1)-5)/4p^{r_1} \geq 0$. We can now proceed to the case $\sharp G(\mathcal{B}) = 0$. By (2.11) we have $r_1 + r_2 \leq 3$. Observe that $N \neq p_1p_2$ because in that case $b = 2$ implies $s = 1$. Thus $N = p_1^2p_2$. Since $s = b = 2$ we have $p_1^2p_2 \notin \mathcal{B}$. In order to maximize the value of the map Δ we put $\mathcal{B}_6 = \{p_1, p_1p_2, p_1^2\}$. If $2 \nmid N$ we have $H(p_1^2p_2, \mathcal{B}_6) = (-p_1^2 + p_1 - 1 + p_2)/p_1^2p_2$. Furthermore, if $p_2 = 2$ then $H(2p_1^2, \mathcal{B}_6) = (p_1+1)/2p_1^2 > 0$. The same conclusion can be drawn for $p_1 = 2$, namely $H(4p_2, \mathcal{B}_6) = (p_2-2)/2p_2 > 0$.

We are now in a position to enumerate the exceptional sets $\mathcal{B}$ that satisfy $\Delta(\mathcal{B}) > \Delta(\mathcal{D}_*(N))$. Firstly, we observe that $\mathcal{B}_1$, $\mathcal{B}_3$ and $\mathcal{B}_6$ are the only sets on the candidate list. If $N = p_1^{r_1}p_2$, $r_1 > 2$, $2 \nmid N$, then point *(i)* follows from case *1-1*. Similarly, if $N = 2p_1^{r_1}$, $2 \nmid r_1$ then by case *2-1* the map Δ attains its maximum value also in $\mathcal{B}_1$ which is stated in point *(v)*. When $N = p_1p_2$, $2 \nmid N$ we shall compare the values of Δ for $\mathcal{B}_1 = \{p_2, p_1p_2\}$ and $\mathcal{B}_3 = \{p_1, p_2\}$.

But $\Delta(\mathcal{B}_1) - \Delta(\mathcal{B}_3) = (2p_2 + p_1 - p_1p_2 - 2)/p_1p_2 < 0$, which is point *(iv)*. If now $N = p_1^2p_2$, $2 \nmid N$ we compare $\Delta(\mathcal{B}_1)$ and $\Delta(\mathcal{B}_6)$. Observe that $\Delta(\mathcal{B}_1) - \Delta(\mathcal{B}_6) = (2p_2 - p_1^2 + p_1 - 2)/p_1^2p_2$ which falls into cases *(ii)* and *(iii)*. Thus the proposition follows. $\square$

Having disposed of the preliminary results on the first inequality of (2.8) we proceed to investigate the second one, namely $\Delta_2(\mathcal{D}_*(N/p_i)) \leq \Delta_2(\mathcal{D}_*(N))$. In order to get this inequality we need slightly stronger assumptions on the prime factorization of N, than we used in Lemma 2.12. The point of the following corollary is that it allows one to compare values of the map Δ_2 despite of the negative term that appears for $\mathcal{D}_*(N)$.

Lemma 2.14. *Let $N = p_1^{r_1}p_2^{r_2}...p_n^{r_n}$. Suppose that $n > 2$ or $n = 2$, $\min\{r_1, r_2\} \geq 2$. Then we have $\Delta_2(\mathcal{D}_*(N/p_i)) \leq \Delta_2(\mathcal{D}_*(N))$ for any $i \in \{1, 2, ..., n\}$.*

Proof. Let $c = \max_{i=1,...,n} \alpha_{p_i}(N)$. Since construction of $\mathcal{D}_*(N/p_i)$ consists in substituting if necessary the element $c_1 = \min \mathcal{A}_2(\mathcal{D}_1(N/p_i))$ by the one equal to $\min \mathcal{A}_2(\mathcal{D}_1(N/p_i) \setminus \{c_1\})$, then it follows that

$$\Delta_2(\mathcal{D}_*(N/p_i)) \leq \frac{N}{2} \sum_{m \in \mathcal{D}_1(N/p_i)} (1 - m^{-1}) + \frac{N}{2}(1 - 2^{-c+1}) + 1. \tag{2.12}$$

Denote $\mathcal{R} = \mathcal{D}_1(N) \setminus \mathcal{D}_1(N/p_i)$. We have

$$\begin{aligned} \Delta_2(\mathcal{D}_*(N)) &\geq -N + 1 + \\ &\frac{N}{2}\sum_{m\in\mathcal{R}}(1 - m^{-1}) + \frac{N}{2} \sum_{m \in \mathcal{D}_1(N/p_i)} (1 - m^{-1}) \geq \\ &\Delta_2(\mathcal{D}_*(N/p_i)) + \frac{N}{2}\left(\sum_{m\in\mathcal{R}}(1 - m^{-1}) - 3\right). \end{aligned} \tag{2.13}$$

Since $\sharp\mathcal{R} = \prod_{p_j \neq p_i}(\alpha_{p_j}(N)+1)$, thus in case $N = p_1^{r_1}...p_n^{r_n}$, $n \geq 4$ we obtain $\sum_{m\in\mathcal{R}}(1-m^{-1}) \geq 8 \cdot 2^{-1} = 4$ which establishes the inequality. If $n = 3$ then consider

$$L_{j,k}(r_j, r_k) = \sum_{t=0}^{r_j}\sum_{s=0}^{r_k} p_j^{-t}p_k^{-s} \leq \frac{p_j}{p_j - 1}\frac{p_k}{p_k - 1}.$$

Since the function $x \mapsto x(x-1)^{-1}$ defined on $(1, \infty)$ is decreasing we conclude that for $N = p_i^{r_i}p_j^{r_j}p_k^{r_k}$ and $p_i = 2$ we obtain

$$\sum_{m\in\mathcal{R}}(1 - m^{-1}) = \sharp\mathcal{R} - 2^{-r_i}L_{j,k}(r_j, r_k) \geq 4 - 1/2 \cdot 3/2 \cdot 5/4 > 3.$$

Analogously, $p_i \geq 3$ yields $\sum_{m\in\mathcal{R}}(1 - m^{-1}) \geq 4 - 5/6 > 3$. Finally, we consider the case $n = 2$, *i.e.* $N = p_i^{r_i}p_j^{r_j}$. Observe that due to our assumptions we have

$$-2 + \sum_{m\in\mathcal{R}}(1 - m^{-1}) - (1 - 2^{-c+1}) \geq -2 + r_j - 2p_i^{-r_i} + 2^{-c+1}.$$

Furthermore in both cases $r_j > r_i$ and $r_j \leq r_i$ it holds $(-2 + r_j) + (2^{-c+1} - 2p_i^{-r_i}) \geq 0$. Therefore we obtain

$$\Delta_2(\mathcal{D}_*(N)) \geq \frac{N}{2}(-2 + \sum_{m\in\mathcal{R}}(1 - m^{-1}) \quad - \quad (1 - 2^{-c+1}))$$
$$+\frac{N}{2} \sum_{m\in\mathcal{D}_1(N/p_i)} (1 - m^{-1}) + \frac{N}{2}(1 - 2^{-c+1}) \quad + \quad 1 \geq \Delta_2(\mathcal{D}_*(N/p_i)).$$

and by (2.12) the proof is complete. □

We continue in this fashion to obtain the exceptions of the second inequality of (2.8) by considering the cases that have been omitted in the last lemma.

Proposition 2.15. *Let $N = p_1^{r_1}...p_n^{r_n}$, N not prime. Then $\Delta_2(\mathcal{D}_*(N/p_i)) > \Delta_2(\mathcal{D}_*(N))$ if and only if at least one of the following statements holds*

(i) $N = p_1^{r_1}$, $p_i = p_1$

(ii) $N = p_1^2 p_2$, $p_i = p_2$, $2 \nmid N$, $p_2 < p_1^2 + p_1 + 1$

(iii) $N = 2p_1^2$, $p_i = 2$

(iv) $N = 12$, $p_i = 3$

(v) $N = p_1^{r_1} p_2$, $p_i = p_1$, $p_2 \neq 2$

(vi) $N = 2p_1^{r_1}$, $p_i = p_1$, $2 \nmid r_1$.

Proof. ($\Leftarrow$) The proof follows from straightforward calculations.
($\Rightarrow$) By Lemma 2.14 we only consider the cases $n = 1$ or $n = 2$, $\min\{r_1, r_2\} = 1$. Denote

$$H(N, p_i) = 2N^{-1}(\Delta_2(\mathcal{D}_*(N)) - \Delta_2(\mathcal{D}_*(N/p_i))).$$

We are thus looking for N and p_i such that $H(N, p_i) < 0$. Suppose firstly $n = 2$, *i.e.* $N = p_1^{r_1} p_2$. If $2 \nmid N$ then $H(p_1^{r_1}p_2, p_1) = -(p_2 + 1)/p_1^{r_1}p_2 < 0$. But since $H(2^{r_1}p_2, 2) = -2^{r_1-1} < 0$ the assertion follows also for $p_1 = 2$, which is stated in point *(v)*. If $p_2 = 2$ then we shall consider two subcases. Note that $2 \mid r_1$ gives now $H(2p_1^{r_1}, p_1) = (p_1^{r_1-1} - 3)/2p_1^{r_1} \geq 0$ and $2 \nmid r_1$ leads us to $H(2p_1^{r_1}, p_1) = -(p_1^{r_1} + 3)/2p_1^{r_1} < 0$, which is *(vi)*.

We can now proceed analogously to consider the terms of the form $H(p_1^{r_1}p_2, p_2)$. If $2 \nmid N$, then

$$H(p_1^{r_1}p_2, p_2) = -2 + r_1 - \frac{1}{p_2}\frac{p_1^{r_1+1} - 1}{p_1^{r_1}(p_1 - 1)} + \frac{1}{p_1^{r_1}}.$$

Since $r_1 - 3 < H(p_1^{r_1}p_2, p_2) < r_1 - 1$ it remains only to check directly the case $r_1 = 2$. But $H(p_1^2p_2, p_2) = (p_2 - 1 - p_1 - p_1^2)/p_1^2p_2$ and we obtain *(ii)*. If $p_1 = 2$ then we again get

$$r_1 - 3 < H(2^{r_1}p_2, p_2) = -2 + r_1 + 2^{-r_1+1} - 2p_2^{-1} < r_1 - 1.$$

Observe that case $r_1 = 1$ has been already covered by *(vi)*. Furthermore $r_1 = 2$ yields $H(4p_2, p_2) = (p_2 - 4)/2p_2$ which is lower than 0 if and only if $p_2 = 3$ that gives *(iv)*. Let now $p_2 = 2$ and $2 \mid r_1$. We have

$$H(2p_1^{r_1}, 2) = -\frac{3}{2} + r_1 - \frac{1}{2}\frac{p_1^{r_1+1} - 1}{(p_1 - 1)p_1^{r_1}} + \frac{1}{p_1^{r_1}} > -\frac{5}{2} + r_1$$

and $H(2p_1^2, 2) = (1 - p_1)/2p_1^2 < 0$, which shows *(iii)*. Similarly, taking $2 \nmid r_1$ we obtain

$$H(2p_1^{r_1}, 2) = -2 + r_1 - \frac{1}{2p_1}\frac{p_1^{r_1} - 1}{p_1^{r_1-1}(p_1 - 1)} - \frac{p_1^{r_1-1} - 2}{2p_1^{r_1}} > -\frac{5}{2} + r_1$$

and case $r_1 = 1$ has been already covered by *(v)*.

Finally, if $N = p_1^{r_1}$, $r_1 \geq 2$ then we have $H(p_1^{r_1}, p_1) = (-p_1^{r_1} + p_1 - 2)/p_1^{r_1} < 0$, that is stated in point *(i)* and the proof is complete. $\square$

We can now formulate the main result of this section. Recall that for each N we are looking for a set $\mathcal{B}$ that satisfies

$$\max_{\mathcal{B}' \subseteq \mathcal{D}_1(N)} \Delta_2(\mathcal{B}') = \Delta_2(\mathcal{B}). \tag{2.14}$$

Theorem 2.16. *Let $N = p_1^{r_1}...p_n^{r_n}$. The maximal set of periods $\mathcal{A}_{\max}$ in* $(\mathrm{Per}(\mathbb{Z}_N),$ $\leq_{\mathbb{Z}_N})$ *equals $\mathcal{D}_0(N)$, except the following cases*

(i) *If $N = p_1^{r_1}$, $r_1 \leq 2$, then $\mathcal{A}_{\max} = \emptyset$*

(ii) *If $N = p_1^{r_1}$, $r_1 > 2$, then $\mathcal{A}_{\max} = \mathcal{D}_0(N) \setminus \{1\}$*

(iii) *If $N = p_1 p_2$, then $\mathcal{A} = \emptyset$*

(iv) *If $N = p_1^2 p_2$, $2 \nmid N$, then $\mathcal{A}_{\max} = \{p_1 p_2, p_1^2\}$*

(v) *If $N = 2p_1^{r_1}$, $2 \mid r_1$ then $\mathcal{A}_{\max} = \mathcal{D}_0(N) \setminus \{1, 2, 2p_1\}$*

(vi) *If $N = 4p_1$, then $\mathcal{A}_{\max} = \{2, 2p_1\}$*

(vii) *If $N = p_1^3 p_2$, $2 \nmid N$ and $2p_2 \geq p_1^2 - p_1 + 2$ then $\mathcal{A}_{\max} = \{p_1, p_1^2, p_1^3, p_1^2 p_2\}$*

(viii) *If $N = p_1^3 p_2$, $2 \nmid N$ and $2p_1 \leq p_1^2 - p_1 + 2$ then $\mathcal{A}_{\max} = \{p_1^2, p_1 p_2, p_1^2 p_2\}$*

(ix) *If $N = p_1^{r_1} p_2$, $r_1 > 3$, $2 \nmid N$ then $\mathcal{A}_{\max} = \mathcal{D}_0(N) \setminus \{1, p_2, p_1 p_2\}$*

(x) *If $N = 2p_1^{r_1}$, $r_1 \geq 3$, $2 \nmid r_1$ then $\mathcal{A}_{\max} = \mathcal{D}_0(N) \setminus \{1, 2\}$*

(xi) *If $N = 2^{r_1} p_2$, $r_1 > 2$, then $\mathcal{A}_{\max} = \mathcal{D}_0(N) \setminus \{1, p_2, 2p_2\}$*

Proof. Since the proof involves many simple calculations we give only its main ideas. It consists in the construction of a candidate list of exceptional pairs $(N,\mathcal{B})$ that satisfy

$$\Delta_2(\mathcal{B}) > \Delta_2(\mathcal{D}_*(N)). \tag{2.15}$$

Moreover this list will contain all pairs that also satisfy (2.14). The redundant elements will be removed by comparing values of the map Δ_2 for all pairs that share the same N. We still assume $\mathcal{B} \neq \emptyset, \{N\}$.

Suppose firstly that (2.15) holds and $\mathrm{lcm}\mathcal{B} = N$. Since

$$\Delta_2(\mathcal{B}) - \Delta_2(\mathcal{D}_*(N)) = 2^{-1}N(\Delta(\mathcal{B}) - \Delta(\mathcal{D}_*(N))),$$

Proposition 2.13 enumerates all pairs falling into this category.

On the other hand if $\mathrm{lcm}\mathcal{B} \leq N/p_i$ then (2.15) implies

$$\Delta_2(\mathcal{B}) > \Delta_2(\mathcal{D}_*(N/p_i)) \quad \text{or} \quad \Delta_2(\mathcal{D}_*(N/p_i)) > \Delta_2(\mathcal{D}_*(N)).$$

Since $\Delta(\mathcal{B}) = \Delta_2(\mathcal{B})$ for $\mathrm{lcm}\mathcal{B} < N$, it follows that the last two propositions provide necessary conditions on N and $\mathcal{B}$ to satisfy (2.15). Indeed, Proposition 2.15 gives all cases $\Delta_2(\mathcal{D}_*(N/p_i)) > \Delta_2(\mathcal{D}_*(N))$. These then can also be directly attached to the candidate list. By Proposition 2.13 we obtain the pairs $(N,\tilde{\mathcal{B}})$, such that $\max_{\mathcal{B}\subseteq\mathcal{D}_1(N)} \Delta(\mathcal{B}) = \Delta(\tilde{\mathcal{B}})$. Hence for each pair $(N,\tilde{\mathcal{B}})$ we can construct 3 numbers of the form Np_j, where $j = 1, 2, 3$. We then clearly put on the candidate list only those pairs that obey (2.15).

As a final step we choose from the list only those elements $(N,\mathcal{B})$ that correspond to the maximum value of the map Δ_2. Observe that we always have $g_\emptyset > g_{\{1\}}$, where $g_\emptyset$ and $g_{\{1\}}$ are given by Remark 2.6 and 2.9 respectively. It follows that if N does not appear on the candidate list and $\Delta_2(\mathcal{D}_*(N)) \geq g_\emptyset$, then clearly $\max_{\mathcal{B}\subseteq\mathcal{D}_1(N)} \Delta_2(\mathcal{B}) = \Delta_2(\mathcal{D}_*(N))$. We present the results in terms of maximal set of periods of a $\mathbb{Z}_\mathrm{N}$–action on a compact Riemann surface.

In order to show how the above algorithm works we give explicitly all pairs from the candidate list of the form $(2p_1^{r_1},\mathcal{B})$, where $2 \mid r_1$. Suppose that (2.15) holds. By Proposition 2.13 there is no pair, such that $\mathrm{lcm}\mathcal{B} = 2p_1^{r_1}$. By Proposition 2.15 the only pair satisfying $\Delta_2(\mathcal{D}_*(2p_1^{r_1}/p_i) > \Delta_2(\mathcal{D}_*(2p_1^{r_1}))$ is $(2p_1^2, \mathcal{D}_*(p_1^2))$ and we may put it on the candidate list. Finally, assuming that $\Delta_2(\mathcal{B}) > \Delta_2(\mathcal{D}_*(2p_1^{r_1}/p_i))$ holds, by Proposition 2.13 we again obtain $B = \mathcal{D}_1(2p_1^{r_1-1}) \setminus \{p_1^{r_1-1}\}$. Since

$$\Delta_2(\mathcal{D}_*(2p_1^{r_1})) - \Delta_2(\mathcal{D}_1(2p_1^{r_1-1}) \setminus \{p_1^{r_1-1}\}) = -4^{-1}Np_1^{-r_1}(p_1+3) < 0,$$

$(2p_1^{r_1}, \mathcal{D}_1(2p_1^{r_1-1}) \setminus \{p_1^{r_1-1}\})$ is also on the candidate list. We are thus reduced to compare the obtained results with $g_\emptyset$. Note that for $r_1 = 2$ we have $g_\emptyset < \Delta_2(\mathcal{D}_*(p_1^2)) < \Delta_2(\{2, 2p_1\})$, while in case $r_1 \geq 4$ we obtain $g_\emptyset < \Delta_2(\mathcal{D}_1(2p_1^{r_1-1})\setminus\{p_1^{r_1-1}\})$. Thus an equivalent formulation of the above is: if $N = 2p_1^{r_1}$ $2 \mid r_1$, then the maximal set of periods $\mathcal{A}_{\max}$ is equal to $\mathcal{D}_0(N)\setminus\{1, 2, 2p_1\}$, which is point *(v)* of the theorem. The rest of the proof runs as before. □

In order to complete our investigation we give in Table 2.2 the formulas for the signatures of Fuchsian groups that cover cyclic groups in the cases listed in preceding theorem.

N	$\mathcal{A}$	Γ	$g_{\mathcal{A}}$
2	$\emptyset$	$(2;-)$	3
	$\{1\}$	$(0;2,2,2,2,2,2)$	2
3	$\emptyset$	$(2;-)$	4
	$\{1\}$	$(0;3,3,3,3)$	2
4	$\emptyset$	$(2;-)$	5
	$\{1\}$	$(0;4,4,4,4)$	3
	$\{2\}$	$(1;2,2)$	3
	$\{1,2\}$	$(0;2,2,4,4)$	2
6	$\emptyset$	$(2;-)$	7
	$\{1\}$	$(0;6,6,6,6)$	5
	$\{2\}$	$(1;3,3)$	5
	$\{3\}$	$(1;2,2)$	4
	$\{1,2\}$	$(0;3,6,6)$	2
	$\{1,3\}$	$(0;2,2,6,6)$	3
	$\{2,3\}$	$(0;2,2,3,3)$	2
	$\{1,2,3\}$	$(0;2;2,2,3,6)$	4

Table 2.1: Information on the $\mathcal{A}$–minimum genus of the $\mathbb{Z}_N$–actions for $N = 2,3,4,6$.

case	$\Gamma_{\max}$
(i)	$(2;-)$
(ii)	$(1;\mathcal{D}_*(p_1^{r_1-1}),p_1^{r_1-1})$
(iii)	$(2;-)$
(iv)	$(1;p_1,p_1,p_2,p_2)$
(v)	$(1;\mathcal{D}_1(2p_1^{r_1-1})\setminus\{p_1^{r_1-1}\},2p_1^{r_1-1},2)$
(vi)	$(1;2,2,p_1,p_1)$
(vii)	$(1;p_1,p_2,p_1p_2,p_1^2p_2,p_1^2p_2)$
(viii)	$(1;p_1,p_1^2,p_1^2,p_1p_2,p_1p_2)$
(ix)	$(1;\mathcal{D}_*(p_1^{r_1-1}p_2)\setminus\{p_1^{r_1}\},p_1^{r_1-1}p_2)$
(x)	$(1;\mathcal{D}_*(2p_1^{r_1-1}),2)$
(xi)	$(1;\mathcal{D}_*(2^{r_1-1}p_2),2^{r_1-1}p_2)$

Table 2.2: Information on the signature of the universal covering group $\Gamma_{\max}$ corresponding to the maximum value of the $\mathcal{A}$–minimum genus of the cyclic group action.

Part II

Dynamics on Klein Surfaces

Chapter 3

Geometry and Dynamics on the Hyperbolic Plane

3.1 Introduction

In this chapter, we extend our study to embrace also the actions of the automorphisms of surfaces that are not Riemann surfaces. By this we mean that they are non–orientable, have a non–empty boundary, or both. These are Klein surfaces, introduced by Alling and Greenleaf [3], following up ideas by Klein. In order to investigate the dynamics of self–homeomorphisms of such surfaces, we consider counterparts of results from the first part of the thesis, where *NEC groups* play the role of Fuchsian groups. However, the actual general setting requires a more sophisticated treatment, involving a variety of terms and auxiliary results. Since non–orientable surfaces do not admit any analytic structure, we need first a more general notion of automorphism than we have applied for Riemann surfaces. The definition we use is based on the term *dianalyticity* which in turn involves both: analyticity and antianalyticity. Below we recall the required definitions that can be found in [3] and [8].

(1) A *surface* is a Hausdorff, connected, topological space S together with a family $\Sigma = \{(U_i, \phi_i) \mid i \in I\}$ such that $\{U_i \mid i \in I\}$ is an open covering of S and each map $\phi_i \colon U_i \to A_i$ is a homeomorphism onto an open subset A_i of $\mathbb{C}$ or $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \mathrm{Im} z \geq 0\}$. The family Σ is said to be a *topological atlas* on S. The *boundary of* S is the set

$$\partial S = \{x \in S \mid \exists i \in I, x \in U_i, \phi_i(x) \in \mathbb{R} \text{ and } \phi_i(U_i) \subseteq \mathbb{C}^+\}.$$

Each (U_i, ϕ_i) is said to be a *chart*. The *transition functions* of Σ are the homeomorphisms

$$\phi_{i,j} = \phi_i \phi_j^{-1} \colon \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j).$$

(2) Let A be a non–empty open subset of $\mathbb{C}$ and $f \colon \mathbb{C} \to \mathbb{C}$ a map. The map f is analytic on A (resp. antianalytic on A) if $\frac{\partial f}{\partial \bar{z}} = 0$ (resp. $\frac{\partial f}{\partial z} = 0$). The map f is said to be dianalytic on A if its restriction to every connected component of A is either analytic or antianalytic. We also need an extension of the notion of dianalyticity to functions having as a domain an open subset of $\mathbb{C}^+$.

(3) Let A be a non–empty open set in $\mathbb{C}^+$ and $f \colon A \to \mathbb{C}^+$ a map. This map f is said to be analytic (resp. antianalytic) on A if it extends to an analytic (resp. antianalytic) function on some neighbourhood of A in $\mathbb{C}$ into $\mathbb{C}$. If f is analytic or antianalytic on each component of A, then we say that it is dianalytic on A.

(4) Let S be a surface with atlas Σ. We say that Σ is a dianalytic atlas (resp. analytic atlas) on S if all of its transition functions are dianalytic (resp. analytic). Each pair (U_i, ϕ_i) is called a chart of Σ. Clearly, if Σ is analytic, then it is also dianalytic.

(5) Let $\Sigma_U = \{(U_i, \phi_i) \mid i \in I\}$ and $\Sigma_V = \{(V_j, \varphi_j) \mid j \in J\}$ be dianalytic atlases on S. We say that Σ_U and Σ_V are dianalytically equivalent if $\Sigma_U \cup \Sigma_V$ is a dianalytic atlas on S. An equivalence class $\mathcal{S}$ of dianalytic atlases on S will be called a *dianalytic structure* on S.

On account of the above we introduce the category of Klein surfaces:

(6) The surface S equipped with the dianalytic structure induced by a dianalytic atlas Σ is said to be a *Klein surface*. A *morphism* between Klein surfaces S and S' is a continuous map $f\colon S \to S'$, such that

(i) $f(\partial S) \subseteq \partial S'$

(ii) Given $P \in S$, there exist charts (U, ϕ) and (V, φ) at P and $f(P)$ respectively, and an analytic function $F\colon \phi(U) \to \mathbb{C}$ such that the following diagram

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[d, "\phi"'] & & V \arrow[d, "\psi"] \\ \phi(U) \arrow[r, "F"] & \mathbb{C} \arrow[r, "\Psi"] & \mathbb{C}^+ \end{tikzcd}$$

commutes. Here $\Psi\colon \mathbb{C} \to \mathbb{C}^+$ is the *folding map* defined by the formula

$$x + iy \longmapsto x + i|y| \qquad x, y \in \mathbb{R}.$$

Furthermore, the chart (V, φ) must be *positive* which means that $\varphi(V) \subset \mathbb{C}^+$.

Now we are ready to define an extended notion of the automorphisms of Klein surfaces. It differs from the corresponding concept for Riemann surfaces principally in that it lets S "fold" along the boundary components of the quotient surface.

(7) An *automorphism of a Klein surface* S is an isomorphism $t\colon S \to S$ in the category of Klein surfaces.

In order to avoid any ambiguities and separate the actual case from the study of Riemann surfaces, we will denote a Klein surface by the letter X. We denote the full group of automorphisms of X under the composition of maps by $\mathrm{Aut}(X)$. Furthermore if X is orientable we shall denote by $\mathrm{Aut}^+(X)$ the subgroup of orientation preserving elements in $\mathrm{Aut}(X)$. We also write $\mathrm{Aut}^-(X)$ for the set of the orientation reversing elements in $\mathrm{Aut}(X)$. As before, we focus on cyclic subgroups of the group $\mathrm{Aut}(X)$.

The initial idea of this work was to classify all subsets of points on a compact Klein surface whose periodic behaviour under an action of a cyclic group of order N differs from the behaviour of a typical point whose orbit has length N, and to relate their appearance to the topological type of the surface. To be more precise even at this very early stage in the chapter, we give here an initial definition of the *singular set* (for complete definition and conventions see Section 3.3). If $t\colon X \to X$ is an automorphism of order N of a compact surface X, then we introduce $\mathcal{S}(t)$ – the singular set of t as a union of the subset of points of X which are fixed by t^d for at least one $d < N$. By definition the boundary components of X belong to $\mathcal{S}(t)$. By this meaning the points of the singular set are somehow strange, since we may easily distinguish them observing how the iterations of the automorphism t transform the surface under study. Associated to this is a collection of divisors of N that we call here a *character of periods*. For an arbitrary character of periods constrained by the respective conditions according to the orientability character of the surface and its quotient, there is an associated $\mathbb{Z}_N$–action. On the other hand assuming certain $\mathbb{Z}_N$–action, the orientability character of the surface X and the quotient surface $X/\mathbb{Z}_N$ we obtain an effective algorithm to compute the minimal area of a NEC group Λ verifying $X \simeq \mathbb{H}^2/\Lambda$.

The study of the actions of cyclic groups on compact topological surfaces has previously been carried out by several authors. In order to position the actual work in this long–established area, we make below some remarks that refer to articles that deal with similar subjects, emphasizing briefly some differences in the respective approaches.

The singular set in the sense of our definition has been investigated already by Bujalance et al. in [9] in the case of involutions. It is worth noting that the material of Subsection 3.3 is merely an extension of definitions and propositions of the first three sections of the above paper to automorphisms of an order greater than 2. The analysis of the singular set has been also provided by Yokoyama in [42]–[44], although NEC groups have not been exploited there.

The relations between periods of isolated periodic orbits, boundary components and the properties of self–homeomorphisms of surfaces in terms of NEC groups and their homomorphisms have been deeply investigated by Bujalance et al. in [8]. Nevertheless, that study did not include all types of periodic structures that become apparent using the definition of the singular set we have given above.

An already mentioned very technical paper [44] of Yokoyama deals with the complete classification of periodic maps on compact surfaces, up to topological conjugacy. It was preceded by two papers [42] and [43] in which only the orbits of isolated points and boundary components had been considered. Yokoyama's classification is much more precise than ours because of the fact that all conjugated maps share the same character of periods. Indeed, the singular sets of conjugated maps comprise not only the same types of periodic structures, but also the cardinalities of their respective types are equal. Nevertheless, no algorithm to determine the minimal genus of a surface on which there exist given periodic structures is outside the scope of the above papers.

Finally, in two articles [10] and [11] the authors give an algorithm to find all genera of surfaces on which there is a $\mathbb{Z}_N$–action prescribed in terms of so called *topological data*, which includes also the information on the orientability character of the surface and its quotient. However, analogously to [8], no attention is paid to periodic structures other than isolated periodic orbits and boundaries.

Summing up the granularity of the classification of $\mathbb{Z}_N$–actions on compact surfaces we obtain here is finner then the one of [8], [10] and [11]. On the other hand, the complete classification of periodic homeomorphisms of compact surfaces (including orientable and non–orientable cases) up to topological conjugacy was obtained only by Yokoyama, although partial results were also obtained by [14], [34] and [37].

From the perspective of dynamical systems, there is also a variety of papers that deal with the properties of periodic self–homeomorphisms of compact surfaces. The main tool based on the combinatorial approach saturated by the theory of NEC groups used there is the Riemann–Hurwitz formula. A non–complete list of such articles must certainly include the following positions: [1],[17],[18],[24],[40].

We restrict our attention only to Klein surfaces resulting as quotients of the upper half–plane by surface NEC groups. Although we describe some definitions and results from the general theory of NEC groups, the paper is not intended to be a review of the field. The most comprehensive reference is [8]. Our choice is motivated by a result, being a counterpart of the uniformization theorem for compact Riemann surfaces stating that each compact, orientable surface without boundary of genus bigger or equal to 2 is conformally equivalent to $\mathbb{H}^2/\Lambda$, where Λ is a surface Fuchsian group (see Theorem 1.1).

3.2 Preliminaries

Not surprisingly the upper–half plane $\mathbb{H}^2$ is an example of Klein surface. We begin by recalling that its group of automorphisms $\mathrm{Aut}(\mathbb{H}^2)$ can be represented as

$$\begin{aligned} \mathrm{PGL}(2,\mathbb{R}) &= \{A \in \mathrm{GL}(2,\mathbb{R}) \mid \det(\mathrm{A}) = \pm 1\} \\ &= \{z \mapsto \Psi(\frac{az+b}{cz+d}) \mid \Psi \text{ – is the folding map}, a,b,c,d \in \mathbb{R}, ab-cd = \pm 1\}. \end{aligned}$$

Observe that in the context of dianalytic structure the notation $\mathrm{Aut}(\mathbb{H}^2)$ differs from the one used in the first part of the dissertation. Using the actual notation the group of orientation–preserving isometries considered before shall be denoted as $\mathrm{Aut}^+(\mathbb{H}^2)$. We will also write $\mathrm{Aut}^\pm(\mathbb{H}^2)$ for $\mathrm{Aut}(\mathbb{H}^2)$.

Let Γ be a discrete subgroup of $\mathrm{Aut}^\pm(\mathbb{H}^2)$. We say that Γ is a *non–euclidean crystallographic group* (shortly NEC group) if the orbit space $\mathbb{H}^2/\Gamma$ is compact. Likewise in case of Fuchsian groups, the algebraic structure of a NEC group Γ is determined by its signature, which is the symbol of the form

$$\sigma = (\gamma; \pm; [m_1, \ldots, m_n]; C_1, \ldots, C_k) \tag{3.1}$$

The numbers $m_i \geq 2$ are called the *proper periods*, $C_i = (n_{i,1}, \ldots, n_{i,s_i})$ are s_i–uples called *period cycles*, the numbers $n_{i,j} \geq 2$ are the *link periods* and $\gamma \geq 0$ is said to be the *orbit genus* of Γ. If the sign of signature (3.1) equals "$+$" we say that it is *orientable* and *non–orientable* otherwise. We denote the sign of signature of a group Γ by the symbol $\mathrm{sign}(\Gamma)$.

Below we give a presentation of a group Γ with signature (3.1) in canonical generators

generators :	
$x_1, \ldots, x_n$	(elliptic)
$e_1, \ldots, e_k$	(hyperbolic or in some cases elliptic)
$c_{i,j}, \ldots, c_{i,s_i},\ 0 \leq j \leq s_i$	(reflections)
$a_1, b_1, \ldots, a_\gamma, b_\gamma,$ if $\mathrm{sign}\Gamma = "+"$	(hyperbolic)
$g_1, \ldots, g_\gamma,$ if $\mathrm{sign}\Gamma = "-"$	(hyperbolic)

relations:

$$x_1^{m_1} = \ldots = x_n^{m_n} = 1$$

$$c_{i,j}^2 = (c_{i,j-1}c_{i,j})^{n_{i,j}} = c_{i,0}e_i^{-1}c_{i,s_i}e_i = 1, \quad 0 \leq i \leq k,\ 0 \leq j \leq s_i$$

$$\prod_{i=1}^n x_i \prod_{i=1}^k e_i \prod_{i=1}^\gamma [a_i b_i] = 1,\ \text{if } \mathrm{sign}\Gamma = "+" \tag{3.2}$$

$$\prod_{i=1}^n x_i \prod_{i=1}^k e_i \prod_{i=1}^\gamma g_i^2 = 1,\ \text{if } \mathrm{sign}\Gamma = "-" \tag{3.3}$$

Note that the presence of proper periods, period cycles or even link periods in the signature (3.1) is not mandatory. Based on this remark we may distinguish some special types of signatures. For instance the signatures of the form

$$\sigma' = (\gamma; \pm; [\,]; \{(\,)^k\}) \tag{3.4}$$

play a very important role Those groups possess only empty period cycles (there are nor proper periods, nor link periods) and the unique relations involving reflections are the following

$$c_{i,0}^2 = 1 \text{ and } e_i c_{i,0} = c_{i,0} e_i.$$

If a NEC group has a signature (3.4) it is called *surface NEC group*. Note also that any Fuchsian group can be regarded as a NEC group of signature $(\gamma; +; [m_1, \ldots, m_n]; \{\,\})$. Likewise a Fuchsian surface group is an surface NEC group with signature $(\gamma; +; [\,], \{\,\})$.

The *area of* σ is defined to be

$$\mu(\sigma) = 2\pi(\alpha\gamma + k - 2 + \sum_{i=1}^{n}(1 - m_i^{-1}) + 2^{-1}\sum_{i=1}^{k}\sum_{j=1}^{s_i}(1 - n_{i,j}^{-1})), \tag{3.5}$$

where $\alpha = 2$ if $\operatorname{sign}(\sigma) = " + "$ and $\alpha = 1$ otherwise. Associated to Γ there is $F_\Gamma \in \mathbb{H}^2$, a fundamental region of Γ and we define *area of* Γ to be hyperbolic measure of F_Γ. We write $\mu(\Gamma) = \mu(F_\Gamma)$. Recall that $\mu(\Gamma)$ does not depend on the choice of a fundamental region F_Γ. Moreover we have $\mu(\Gamma) = \mu(\sigma)$ (see for instance [8], Theorem 0.2.8). Finally we recall that an abstract signature σ is the signature of some NEC group if and only if $\mu(\sigma) > 0$ and $\alpha + \gamma \geq 2$. Since we will not use in any essential way the equivalence classes of isomorphisms from a NEC group Γ with an abstract signature (3.1) to $\mathrm{PGL}(2, \mathbb{R})$ we do not distinguish between NEC groups and their signatures. This handy convention will be freely used until further notice and we adopt the notation

$$\Gamma = (\gamma; \pm; [m_1, \ldots, m_n]; C_1, \ldots, C_k)$$

instead of (3.1).

Before we formulate the uniformization theorem for compact Klein surfaces we need two notions: of *complex double* and of *algebraic genus of a Klein surface.*

(1) Assume X is not a Riemann surface. By the complex double of a Klein surface X we mean the triple $(X_C, \mathcal{F}, \tau)$, where X_C is a Riemann surface admitting an antianalytic involution τ and morphism $\mathcal{F}\colon X_C \to X$ which verifies $\mathcal{F}\tau = \mathcal{F}$.

(2) Assume X is not a Riemann surface. Let X possess $k(X)$ boundary components and topological genus equal to $g(X)$. The topological genus $g(X_C)$ of X_C is called the algebraic genus of X and we denote it by $p(X)$. If X is a Riemann surface then we

define $p(X) = g(X)$. By the formula involving Euler characteristic, the number of boundary components and the topological genus of an orientable surface we have

$$p(X) = g(X_C) = \begin{cases} 2g(X) + k(X) - 1, & \text{if } X \text{ is orientable and } \partial X \neq \emptyset \\ g(X) + k(X) - 1, & \text{if } X \text{ is non} - \text{orientable.} \end{cases} \tag{3.6}$$

Regarding the above definition of the complex double of a Klein surface X we shall observe that $(X_C, \mathcal{F}, \tau)$ is unique and $X \simeq X_C/\langle\tau\rangle$. See [3] for more details.

Theorem 3.1 (Bujalance et al. [8], Theorem 1.2.3)**.** *Let X be a compact Klein surface with algebraic genus $p \geq 2$. Then there exists a surface NEC group Λ such that X and $\mathbb{H}^2/\Lambda$ are isomorphic as Klein surfaces. Moreover if $\pi\colon \mathbb{H}^2 \to \mathbb{H}^2/\Lambda$ is the canonical projection, then $\Lambda = \{f \in \mathrm{Aut}(\mathbb{H}^2) \mid \pi f = \pi\}$.*

If Γ' is a subgroup of finite index in a NEC group Γ, then it is a NEC group itself (see [8], Proposition 2.1.1) and we have the Riemann–Hurwitz formula

$$[\Gamma : \Gamma'] = \frac{\mu(\Gamma')}{\mu(\Gamma)}.$$

Let t be a generator of a finite cyclic group of order N which acts by automorphisms on a Klein surface X then t lifts to a dianalytic transformation $\tilde{t}$ of $\mathbb{H}^2$ such that $\tilde{t}$ normalizes Λ i.e. $\tilde{t}\Lambda(\tilde{t})^{-1} = \Lambda$. Obviously $\tilde{t}^N \in \Lambda$. Thus NEC group $\Gamma = \langle\tilde{t}, \Lambda\rangle$ contains Λ as a normal subgroup with index N. By Theorems 2.4.2 and 2.4.4 of [8] group Γ has a signature of the form

$$\Gamma = (\gamma; \pm; [m_1, ..., m_n]; \{(\ )^{\lambda}(2^{\mu_1})...(2^{\mu_p})\}) \tag{3.7}$$

for some non–negative integers γ and λ. Moreover the following relations must also be satisfied: $\mu_1, \ldots, \mu_p$ are even and $m_i \geq 2$ for $i = 1, \ldots, n$. Here we use an abbreviate notation standing for

$$(\ )^{\lambda} = \underbrace{(\ )\ldots(\ )}_{\lambda} \qquad (2^{\mu_i}) = \underbrace{(2\ldots2)}_{\mu_i}$$

that is: λ empty period cycles and non–empty period cycle with μ_i link periods equal to 2.

The following theorem is counterpart of Theorem 1.2 in case we consider Klein surfaces.

Theorem 3.2 (Bujalance et al. [8], Remark 1.3.6)**.** *A finite group G is a group of automorphisms of a Klein surface $X = \mathbb{H}^2/\Lambda$ of algebraic genus $p \geq 2$ if and only if G is isomorphic to the factor group Γ/Λ for some NEC group Γ containing Λ as a normal subgroup.*

If G acts by automorphisms on a Klein surface $X \simeq \mathbb{H}^2/\Lambda$, then $\Gamma = N_{\mathrm{Aut}(\mathbb{H}^2)}(\Lambda) = \{\zeta \in \mathrm{Aut}(\mathbb{H}^2) \mid \zeta\Lambda\zeta^{-1} = \Lambda\}$ and there is a smooth epimorphism $\theta\colon \Gamma \to G$ with kernel Λ, such that the following diagram commutes

$$\begin{array}{ccc} \Gamma \times \mathbb{H}^2 & \longrightarrow & \mathbb{H}^2 \\ {\scriptstyle\theta}\big\downarrow \ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ G \times X & \longrightarrow & X. \end{array} \tag{3.8}$$

By smooth, likewise in case of Riemann surfaces and Fuchsian groups, we will understand that $\ker\theta$ is a surface NEC group. Note that Λ is not assumed to be a non–bordered surface group, which means that k appearing in (3.4) may be positive. Recall also that $\mathrm{Aut}(X)$ is finite when algebraic genus $p(X) \geq 2$ (see [8], Corollary 1.3.5).

In case $G \simeq \mathbb{Z}_N \simeq \langle t \rangle$ we will say that epimorphism θ uniformizes or covers a $\mathbb{Z}_N$–action of t on X. The transformation group $(\Gamma, \mathbb{H}^2)$ is called *universal covering transformation group* of $(\mathbb{Z}_N, X)$. Since we restrict ourselves only to study the cases when the factor group Γ/Λ is cyclic we finish this section with an observation that concerns the rigidity of smooth epimorphisms from NEC groups onto cyclic groups.

Proposition 3.3 (Bujalance et al. [8], Proposition 2.4.3)**.** *Let N be an even integer. Let Γ and Λ be NEC groups and $\theta\colon \Gamma \to \mathbb{Z}_N$ be a group epimorphism with* $\ker\theta = \Lambda$*. Let us suppose that Λ is a bordered surface NEC group. Then, if $(n_{i,1}, \dots, n_{i,s_i})$ is a non–empty period cycle in the signature of Γ with associated reflections $\{c_{i,0}, \dots, c_{i,s_i}\}$ it holds*

$$\theta(c_{i,0}) = \theta(c_{i,2l}),\ \ \theta(c_{i,2l-1}) = \theta(c_{i,0})t^{N/2} \text{ for } 1 \leq l \leq s_i/2.$$

Moreover we have $\theta(c_{i,0}) = 1$ or $\theta(c_{i,0}) = t^{N/2}$.

3.3 The Singular Set

From now on, unless otherwise stated, we assume that X is a compact Klein surface of algebraic genus $p \geq 2$. Let $t\colon X \to X$ be an automorphisms of order N. We introduce the *singular set of least period d*

$$\mathcal{S}_d^0(t) = \{x \in X \mid x = t^d(x) \text{ and } x \neq t^k(x) \text{ for any } k < d\}. \tag{3.9}$$

Moreover the *singular set* of t, denoted by $\mathcal{S}(t)$, is defined to be the set

$$\mathcal{S}(t) = \partial X \cup \bigcup_{d<N} \mathcal{S}_d^0(t). \tag{3.10}$$

According to the above, the singular set of t is a subset of X that comprises points belonging to the boundary of X and points with orbit whose length is strictly lower than N. As we shall see, at the end of the section, $\mathcal{S}(t)$ consist of

(1) a finite number of isolated points in int$X$

(2) a finite number of disjoint simple closed curves in int$X$

(3) a finite number of disjoint arcs embedded in X.

Each of the above subsets of $\mathcal{S}(t)$ we will call a *component of* $\mathcal{S}(t)$. Observe that applying definition (3.10) to conformal automorphisms of Riemann surfaces we get isolated points as the only components of the singular set.

Following [9] the last type of components of $\mathcal{S}(t)$ listed above will be called *chains*. A chain of length $2r$ is a set C of r disjoint arcs properly embedded in X which means that the ends of each component of C lay on the boundary of X. For each boundary component B of ∂X, either $C \cap B = \emptyset$ or $C \cap B$ consist of two distinct points a_i, a_{i+1}. Note that a chain of length $2r$ meets the boundary of X in exactly $2r$ points. We distinguish two types of chains subject to their bicollar neighbourhood. To differentiate them we proceed as follows: filling the holes of X with discs we obtain compact surface $\hat{X}$. If C intersects a boundary component B of X we add one of the arcs of B joining a_i and a_{i+1}, thus obtaining a simple closed curve $\hat{C}$ on $\hat{X}$. Then we say that C is one–sided or two–sided if a bicollar neighbourhood of $\hat{C}$ on $\hat{X}$ is a Möbius strip or an annulus.

Simple closed curves in int$X$ will be called *ovals*. Furthermore we also distinguish one–sided and two–sided ovals based on their bicollar neighbourhood.

In Corollary 3.5 we shall observe that the 1–dimensional components of the singular set belong either to ∂X or $\mathcal{S}_{N/2}^0(t)$. Furthermore, the only mapping fixing a boundary pointwise is the identity map. Hence in sense of definition (3.9) the set of possible periods of the 1–dimensional components is very limited. In order to avoid this inconvenience we will call *components of period d* those 1–dimensional components of $\mathcal{S}(t)$ which are setwise fixed by t^d and d is the lowest number with this property. In such a way we obtain *boundary components* *of period d*, *one–sided* and *two–sided ovals of period* d and finally *one–sided* and *two–sided chains of period d.* Unless otherwise stated when referring to 1–dimensional components we use only the above setwise context of periodicity. We denote the singular set of last period d in the setwise context by $\mathcal{S}_d(t)$ and define it formally as follows

$$\mathcal{S}_d(t) = \{L \subseteq \mathcal{S}(t) \mid L \text{ is a component of } \mathcal{S}(t),\ L = t^d(L) \text{ and } L \neq t^k(L) \text{ for all } k < d\}.$$

Obviously for isolated periodic orbits the pointwise and setwise contexts of periodicity are identical.

In order to investigate the relation between pointwise and setwise periodicity in more detail we use once more the term of complex double of a Klein surface X. We recall also a result of [3] by which an automorphism of X can be lifted to an analytic automorphism of symmetric Riemann surface X_C.

Theorem 3.4 (Alling and Greenleaf [3], Theorem 1.11.1)**.** *Let* $(X_C, \mathcal{F}, \tau)$ *be the complex double of the Klein surface* X*. Then*

$$\operatorname{Aut}(X) \simeq (\operatorname{Aut}^+(X_C))^\tau = \{f \in \operatorname{Aut}^+(X_C) \mid \tau f \tau = f\}. \tag{3.11}$$

By the above theorem one may show two results formulated as Corollary 3.5 and Remark 3.6 which are interesting while considering the singular set of an automorphism of a Klein surface.

Corollary 3.5. *Let* X *be a compact Klein surface. Let* $t\colon X \to X$ *be an automorphism of order* N *of* X*. Denote by* L *a 1–dimensional component of* $\mathcal{S}^0_d(t)$*. Then* $L \cap \partial X = \emptyset$ *forces* $2d = N$*, while for* $L \subseteq \partial X$ *we have* $d = N$*.*

Proof. First we take the complex double of X and a point $Q \in L$. If $L \cap \partial X = \emptyset$, then by the construction of the complex double there is a neighbourhood V of Q in X such that $\mathcal{F}^{-1}(V)$ has two components, say $\tilde{V}_1$ and $\tilde{V}_2$. Denote by $\tilde{Q}_1 \in \tilde{V}_1$ and $\tilde{Q}_2 \in \tilde{V}_2$ the two preimages of Q laying on X_C. Let $\mathcal{T}\colon X_C \to X_C$ be a lift of t^d. Take $\tilde{P} \in \mathcal{F}^{-1}(L \cap V) \cap \tilde{V}_1$. Note that $\mathcal{T}(\tilde{Q}_1) = \tilde{Q}_1$ would force

$$\mathcal{F}\mathcal{T}(\tilde{P}) = t^d \mathcal{F}(\tilde{P}) = \mathcal{F}(\tilde{P}),$$

which is impossible since $\mathcal{T}$ is conformal. Hence

$$\begin{aligned} \mathcal{T}(\tilde{Q}_i) &= \tilde{Q}_j, \quad i,j = 1,2 \quad i \neq j \\ \mathcal{T}^2(\tilde{Q}_i) &= \tilde{Q}_i. \end{aligned}$$

Thus we must have $\mathcal{T}^2 = \mathrm{Id}$ which shows that $t^{2d} = \mathrm{Id}$.

On the other hand if $L \subseteq \partial X$, then we also start with a point $Q \in L$ although now the fiber $\mathcal{F}^{-1}(Q)$ comprises only one point. Hence we may find a neighbourhood V of Q on X_C such that $\mathcal{T}_{|L\cap V} = \mathrm{Id}_{|L\cap V}$. Using the same argument as before we now get $\mathcal{T} = \mathrm{Id}$ which leads us to $t^d = \mathrm{Id}$. □

Remark 3.6. *Under the assumptions of Corollary 3.5 denote by B a boundary component intersecting a chain C. Then we have $t^i(B) \cap B = \emptyset$ for $i < N/2$ and $t^{N/2}(B) = B$. Furthermore $t^{N/2}$ fixes exactly two points on B i.e. the intersection of B and C.*

As we see by Corollary 3.5 all ovals and chains are pointwise fixed under $t^{N/2}$. Hence their periods in the setwise context must divide $N/2$ which follows by a simple arithmetic argument (compare the discussion in the last but one paragraph of Section 1.1). Moreover by means of Remark 3.6 the periodic behaviour of the boundaries intersecting chains is special and very simple since their setwise period equals $N/2$. By this reason we introduce another convention concerning the singular structures we are about to study: the boundary components which intersect chains of t are excluded from the set $\mathcal{S}_{N/2}(t)$. We will not take into consideration the period of those boundary components, although we calculate their number (see Remark 3.15).

By Theorem 3.4 it also follows that every group of automorphisms of a Klein surface may be viewed as a group of orientation–preserving automorphisms of symmetric Riemann surface. First we need a definition and lemma (compare with [36], Theorem 1). Let Γ be a *proper NEC* group i.e. not a Fuchsian group and denote by Γ^+ its *canonical Fuchsian subgroup* defined as $\Gamma^+ = \Gamma \cap \mathrm{Aut}^+(\mathbb{H}^2)$.

Lemma 3.7. *Let G be a finite group of automorphisms of a Klein surface $X \simeq \mathbb{H}^2/\Lambda$, which is not a Riemann surface. Suppose that $\theta\colon \Gamma \to G$ is a smooth epimorphism. Then $\theta(\Gamma^+) = G$.*

Proof. Denote $\Lambda = \ker\theta$. We consider two cases when the surface X is non–orientable and orientable.

If X is non–orientable, we clearly have sign$\Lambda =$ " $-$ ". Thus there exists

$$w \in \Lambda \cap (\Gamma \setminus \Gamma^+).$$

We have $\Gamma = \Gamma^+ \cup \Gamma^+ w$. Denote $\theta(\Gamma^+) = G^+$. Then

$$G = \theta(\Gamma) = \theta(\Gamma^+ \cup \Gamma^+ w) = \theta(\Gamma^+) \cup \theta(\Gamma^+)\theta(w) = G^+ \cup G^+ = G^+, \tag{3.12}$$

which yields $\theta(\Gamma^+) = G$.

On the other hand if X is orientable, then $\partial X \neq \emptyset$, since X is not a Riemann surface. Hence Γ takes the form

$$\Gamma = (\gamma; \pm; [m_1, \ldots, m_n]; \{(n_{1,1}, \ldots, n_{1,s_1}), \ldots, (n_{k,1}, \ldots, n_{k,s_k})\}),$$

where $k \geq 1$. However observe that period cycles of Γ may be empty i.e. $s_j = 0$ for some $1 \leq j \leq k$. By Theorems 2.3.1, 2.3.2 and 2.3.3 of [8] there exists canonical reflection $c \in \Gamma$ such that $c \in \ker\theta$. We may write $\Gamma = \Gamma^+ \cup \Gamma^+ c$. Analogously to (3.12) we conclude that

$$G = \theta(\Gamma) = \theta(\Gamma^+ \cup \Gamma^+ c) = \theta(\Gamma^+) \cup \theta(\Gamma^+)\theta(c) = G^+ \cup G^+ = G^+,$$

Thus $\theta(\Gamma^+) = G$. □

Proposition 3.8. *Let the assumptions of Lemma 3.7 hold. Consider $(X_C, \mathcal{F}, \tau)$ the complex double of X and denote by $\kappa\colon (\mathrm{Aut}^+ X_C)^\tau \to \mathrm{Aut} X$ isomorphism given by (3.11). Then the following diagram commutes*

$$\begin{tikzcd} \Gamma^+ \times \mathbb{H}^2 \arrow[r] \arrow[d, "\theta_{|\Gamma^+}"', "\pi_C"] & \mathbb{H}^2 \arrow[d, "\pi_C"] \\ G \times X_C \arrow[r] \arrow[d, "\kappa"', "\mathcal{F}"] & X_C \arrow[d, "\mathcal{F}"] \\ G \times X \arrow[r] & X. \end{tikzcd}$$

Here π_C stands for canonical projection onto Riemann surface X_C.

Proof. Let us denote $\theta^+ = \theta_{|\Gamma^+}$ and $\Lambda = \ker\theta$. We first show that the upper diagram

$$\begin{tikzcd} \Gamma^+ \times \mathbb{H}^2 \arrow[r] \arrow[d, "\theta^+"', "\pi_C"] & \mathbb{H}^2 \arrow[d, "\pi_C"] \\ G \times X_C \arrow[r] & X_C \end{tikzcd} \tag{3.13}$$

is commutative by proving that $\theta^+\colon \Gamma^+ \to G$ is a smooth epimorphism. Due to the preceding lemma θ^+ is certainly an epimorphism. Note that $\ker\theta^+ = \Lambda^+$. It follows by the following two relations:

$$h \in \ker\theta^+ \triangleleft \Gamma^+ \Rightarrow h \in \mathrm{Aut}^+(\mathbb{H}^2) \cap \ker\theta \Rightarrow h \in \Lambda^+ \Rightarrow \ker\theta^+ \leq \Lambda^+$$
$$h' \in \Lambda^+ \Rightarrow \theta(h') = 1 \Rightarrow h' \in \ker\theta^+ \Rightarrow \Lambda^+ \leq \ker\theta^+.$$

The above relations yields also $\Lambda^+ \triangleleft \Gamma^+$. The smoothness of θ^+ can be derived now from the fact that Λ^+ is a Fuchsian surface group. Furthermore we have $\Lambda^+ \simeq \pi_1(X_C)$.

On the other hand by Theorem 3.4 we have $\mathcal{F}f = \kappa(f)\mathcal{F}$, where $f \in G_1 \leq (\mathrm{Aut}^+(X_C))^\tau$, $G_1 \simeq G$. It follows that

$$\begin{tikzcd} G_1 \times X_C \arrow[r] \arrow[d, "\kappa"', "\mathcal{F}"] & X_C \arrow[d, "\mathcal{F}"] \\ G_2 \times X \arrow[r] & X, \end{tikzcd} \tag{3.14}$$

where $G \simeq G_2 \leq \mathrm{Aut}(X)$. Consider $g \in \Gamma^+$ and $z \in \mathbb{H}^2$. Gluing together (3.13) and (3.14) we obtain

$$\kappa(\theta^+(g))\mathcal{F}\pi_C(z) = \mathcal{F}\theta^+(g)\pi_C(z) = \mathcal{F}\pi_C(gz)$$

which yields the commutative diagram

$$\begin{tikzcd} \Gamma^+ \times \mathbb{H}^2 \arrow[r] \arrow[d, "\kappa\theta^+"', "\mathcal{F}\pi_C"] & \mathbb{H}^2 \arrow[d, "\mathcal{F}\pi_C"] \\ G_2 \times X \arrow[r] & X \end{tikzcd} \tag{3.15}$$

as required. □

Remark 3.9. *Assume that $X \simeq \mathbb{H}^2/\Lambda$ is Klein but not Riemann surface and G is a group of its automorphisms. Observe that by Theorem 3.2 we have $G \simeq \Gamma/\Lambda$ for Γ, Λ being proper NEC groups. Obviously (3.15) does not yield $G \simeq \Gamma^+/\Lambda$. Note that Λ contains order reversing elements, which gives $\Lambda \not\leq \Gamma^+$. However by (3.13) we have $G \simeq \Gamma^+/\Lambda^+$.*

In the forthcoming Subsections 3.3.1 – 3.3.3 we describe the structure of the singular set of an automorphism $t\colon X \to X$ of a Klein surface by considering properties of epimorphism $\theta\colon \Gamma \to \mathbb{Z}_N$ uniformizing the action of t. We particularly see the whole spectra of periods of various components of the singular set.

3.3.1 Isolated Orbits

In the easiest way we obtain the number of isolated periodic orbits since their number can be calculated using Macbeath's formula concerning automorphisms of Riemann surfaces. Denote by $\mathcal{P}_d(t)$ the set of isolated periodic points of t with least period d.

Proposition 3.10. *Let X be a compact Klein surface of algebraic genus $p \geq 2$. Let $t\colon X \to X$ be an automorphism of order N of X. If Γ is given by (3.7) and $\theta\colon \Gamma \to \mathbb{Z}_N$ is an epimorphism that uniformizes a $\mathbb{Z}_N$–action given by t, then*

$$\sharp\mathcal{P}_d(t) = d\ \sharp\{m_i \mid m_i = N/d\},$$

where $d \mid N$.

Proof. Denote $\Lambda = \ker\theta$. Assume that map t fixes a point on X. Then we may lift t to $\mathcal{T} \in \mathrm{Aut}(\mathbb{H}^2)$ which has a fixed point in $\mathbb{H}^2$. Thus, using the notation of (3.7), $\mathcal{T}$ is conjugate either to a power of canonical elliptic generator x_i or to a power of the product of two canonical reflections whose fixed points sets i.e. a circle or a line perpendicular to $\mathbb{R}$, do intersect. It follows that these are consecutive canonical reflections $c_{i,j-1}c_{i,j}$. Obviously it follows that t is conjugate to a power of $\theta(x_i)$ or to a power of $\theta(c_{i,j-1}c_{i,j})$. Since $\mathbb{Z}_N$ is abelian we have $\mathrm{ord}\theta(c_{i,j-1}c_{i,j}) = 2$ and the latter case may occur only for involutions. However we show that if we consider cyclic group actions the second scenario does not produce an isolated fixed point on X.

Suppose t is an involution of X and $\mathcal{T} \sim c_{i,j-1}c_{i,j}$. Let us form a fundamental region F_Γ for Γ starting from the common vertex to the sides fixed by the reflections $c_{i,j-1}$ and $c_{i,j}$. Then, in the counter clockwise order it is labelled as follows $\gamma_{i,j-1}\Delta\gamma_{i,j}$, where

(1) Δ represents the other sides of the perimeter of F_Γ

(2) the reflections $c_{i,j-1}$ and $c_{i,j}$ fix the sides $\gamma_{i,j-1}$ and $\gamma_{i,j}$ respectively.

Let us denote the vertex which is common to the sides $\gamma_{i,j}$ and $\gamma_{i,j-1}$ as Q. By Proposition 3.3 we have $\theta(c_{i,j}) = t^{N/2}\theta(c_{i,j-1})$, $\theta(c_{i,j-1}) \in \{1, t^{N/2}\}$. With no loss of generality we may assume $\theta(c_{i,j-1}) = t^{N/2}$. Then, fundamental region for Λ may be generated as follows

$$F_\Lambda = F_\Gamma \cup c_{i,j-1}F_\Gamma.$$

Observe that $c_{i,j-1}(Q) = c_{i,j}(Q) = Q$. Thus Q is a fixed point of the map $c_{i,j-1}c_{i,j}$. On the other hand $Q \in \gamma_{i,j} \cup c_{i,j-1}\gamma_{i,j}$. But $\gamma_{i,j} \cup c_{i,j-1}\gamma_{i,j}$ projects to a boundary component of X, which shows that Q projects to fixed point on X which is not isolated since $\pi(Q) \notin \mathrm{int}X$.

By the above isolated fixed points of t correspond only to powers of θ–images of canonical elliptic generators of Γ which are conjugate to t. By the argument used in the proof of Macbeath's theorem (see for instance [28]) the number of those points equals

$$\sharp\mathrm{Fix}(t) = N \sum_{\mathrm{ord}t \mid m_i} m_i^{-1}.$$

Now, as in the proof of Proposition 2.4, the numbers $\sharp\mathrm{Fix}(t^l)$ for $l \mid d$, enable us to calculate $\sharp\mathcal{P}_d(t)$ which establishes the formula. $\square$

Remark 3.11. *Consider G acting on a Klein surface X by dianalytic automorphisms. Suppose $\theta\colon \Gamma \to G$. It is worth noting that if we consider actions of non–cyclic group G, then there may become apparent fixed points of $g \in G$ which are induced by products of two consecutive canonical reflections of Γ. It happens if and only if the both consecutive reflections do not belong to* $\ker\theta$. *For more details see Theorem 2.2.4 of [8]. See also [19] for examples.*

3.3.2 Boundaries and Ovals

We continue with a theorem that deals with periodic ovals of t and boundaries of X. However, now we restrict our attention only to the period cycles of the surface group Λ being images of empty period cycles of Γ. The remaining boundaries of X that are induced by non–empty period cycles of Γ are considered in the next subsection (see Theorem 3.13). Recall that we may consider periodic ovals only if N is even.

Theorem 3.12. *Let X be a compact Klein surface of algebraic genus $p \geq 2$. Let $t\colon X \to X$ be an automorphism of order N of X and $\theta\colon \Gamma \to \mathbb{Z}_N$ be an epimorphism (3.8) of Γ given by (3.7), that uniformizes a $\mathbb{Z}_N$–action of t. For each generator $c_{i,0}$ we have $\theta(c_{i,0}) = 1$ or $\theta(c_{i,0}) = t^{N/2}$. Let us reorder the reflections $c_{i,0}$ in such a way they hold*

$$\theta(c_{i,0}) = 1 \text{ for } 1 \leq i \leq r \leq \lambda \quad \text{and} \quad \theta(c_{i,0}) = t^{N/2} \text{ for } r+1 \leq i \leq \lambda.$$

Furthermore denote $\theta(e_i) = t^{v_i}$, $i = 1, \ldots, \lambda$. Then we have

(i) if $i \leq r$, then i–th empty period cycle of Γ induces on X a boundary component of period (N, v_i)

(ii) if $r+1 \leq i$ and $\alpha_2(v_i) = \alpha_2(N)$, then i–th empty period cycle of Γ induces on X a two–sided oval of period $(N/2, v_i)$

(iii) if $r+1 \leq i$ and $\alpha_2(v_i) < \alpha_2(N)$, then i–th empty period cycle of Γ induces on X an one–sided oval of period $(N/2, v_i)$.

Proof. Our proof has been motivated by proofs of Theorems 2.3.1 and 2.3.2 of [8] and Proposition 3.2 of [9]. Observe that if we fix i, then from the classical construction of fundamental regions for NEC groups we can find a fundamental region F_Γ for Γ with the perimeter labelled in the counter clockwise order as follows $\varepsilon_i\gamma_{i,0}\varepsilon_i^{'}\Delta$ (see for instance [8]). Here

(1) Δ represents the other sides of the perimeter

(2) the reflection $c_{i,0}$ fixes the side $\gamma_{i,0}$.

(3) for each $i = 1, \dots, \lambda$ we have $\mathrm{e_i(\varepsilon_i^{'}) = \varepsilon_i}$.

We first investigate boundaries of X i.e. we restrict our attention to $i \leq r$. Since Λ is a normal subgroup of Γ with the cyclic factor $\mathbb{Z}_N$ we have

$$\Gamma = \bigcup_{j=1}^{N/\exp_\Lambda e_i} \bigcup_{k=0}^{\exp_\Lambda e_i - 1} \Lambda(\beta_j e_i^k), \tag{3.16}$$

for some $\beta_1, \dots, \beta_{N/exp_\Lambda e_i}$ in Γ, where $\exp_\Lambda e_i$ denotes the least positive power of e_i that belongs to Λ. A fundamental region for Λ may be obtained as follows

$$F_\Lambda = \bigcup_{j=1}^{N/\exp_\Lambda e_i} \bigcup_{k=0}^{\exp_\Lambda e_i - 1} (\beta_j e_i^k) F_\Gamma.$$

It is worth noting that in order to obtain a fundamental region for Λ it suffices to know only that $[\Gamma : \Lambda] = N$. By (3.16) we have more, since we also get the cosets representatives of Γ. By (3.16) we also have

$$\Gamma/\Lambda = \mathbb{Z}_N = \bigcup_{j=1}^{(\mathbb{Z}_N:\langle\theta(e_i)\rangle)} g_j H = \bigcup_{j=1}^{N/\exp_\Lambda e_i} g_j\langle\theta(e_i)\rangle = \bigcup_{j=1}^{N/\exp_\Lambda e_i} \bigcup_{k=0}^{\exp_\Lambda e_i - 1} g_j\theta(e_i^k).$$

Here $g_j \in \mathbb{Z}_N$ are elements satisfying $\theta(\beta_j) = g_j$.

Having disposed of this preliminary step in which we get the structure of the factor group $\mathbb{Z}_N$ we proceed now to find for each divisor d of N the boundaries of X that belong to $\mathcal{S}_d(t)$ i.e. which are setwise fixed under the action of t^d. By Theorem 2.3.1 and Theorem 2.3.2 of [8] the following segment

$$C_{i,j} = \bigcup_{k=0}^{\exp_\Lambda e_i - 1} (\beta_j e_i^k)\gamma_{i,0}$$

of the perimeter of F_Λ generates a hole on X. It means that after gluing the sides of the perimeter of F_Λ according to the identifications given by Λ the segment $C_{i,j}$ will project to a boundary component of X. It is enough to show the two facts:

(i) We must show that there is an element in Λ that pairs the edges $\beta_j\varepsilon_i'$ and $\beta_j e_i^{\exp_\Lambda e_i - 1}\varepsilon_i$ belonging to F_Λ (see Figure **??** which for simplicity of notation is made for $\beta_j = \mathrm{Id}$).

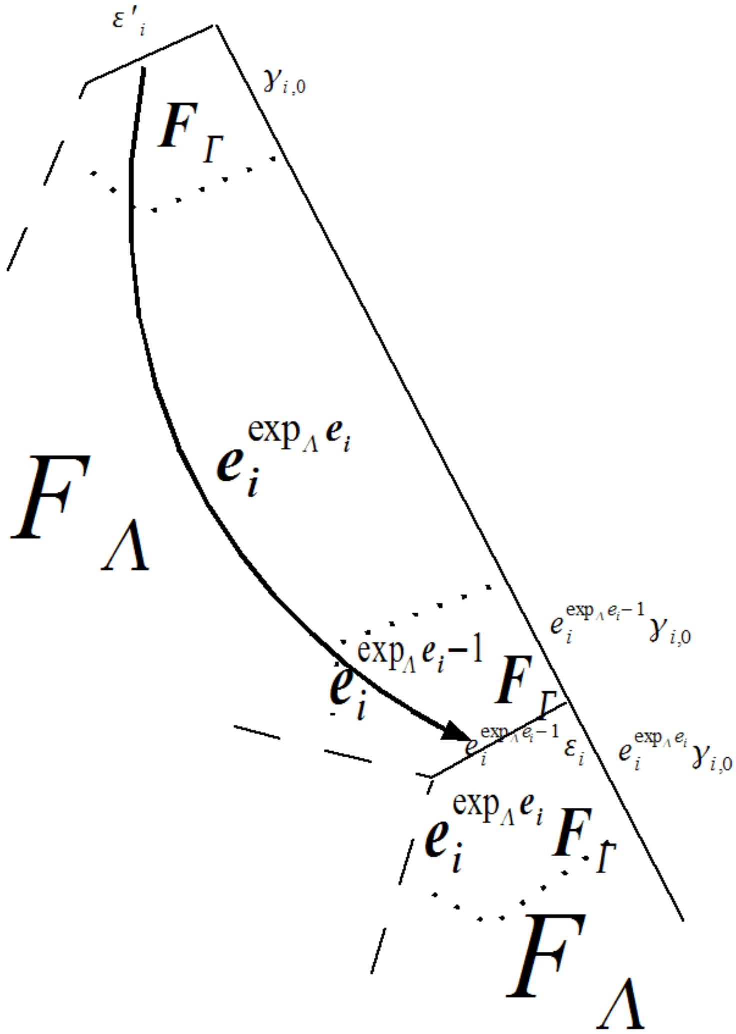

Figure 3.1: A segment generating a boundary component of X, $\beta_j = \mathrm{Id}$.

(ii) We must prove that there are no elements of Λ (other than identity) that would identify a point P belonging to $C_{i,1}$ with another point Q on the perimeter of F_Λ.

For the convenience of the reader we repeat the relevant material from the proofs of the above theorems which helps to make our exposition as self–contained as possible.

Case (i) This point can be easily derived from the relation

$$\beta_j e_i^{\exp_\Lambda e_i} \beta_j^{-1}(\beta_j \varepsilon_i') = \beta_j e_i^{\exp_\Lambda e_i - 1} \varepsilon_i,$$

where $\beta_j e_i^{\exp_\Lambda e_i} \beta_j^{-1} \in \Lambda \lhd \Gamma$.

Case (ii) We will prove that there are no elements of Λ (other than identity) that would identify a point P belonging to $C_{i,1}$ with another point Q on the perimeter of F_Λ. To obtain a contradiction suppose that there is $h \in \Lambda$ such that $h(P) = Q$. We distinguish three different scenarios with a slightly different way of arguing

($ii.1$) $Q \in C_{i,1}$

($ii.2$) $Q \in C_{i,j},\ 2 \le j \le N/\exp_\Lambda e_i$

($ii.3$) Q belongs to the other sides of the perimeter of Λ.

Suppose that $P \in e_i^l(\gamma_{i,0})$ where $0 \le l \le \exp_\Lambda e_i - 1$.

Case ($ii.1$) There is $0 \le l_1 \le \exp_\Lambda e_i - 1$ such that $Q \in e_i^{l_1}(\gamma_{i,0})$. Define $h' \in \Gamma$ by the following formula $h' = e_i^{-l_1} h e_i^l$ and take two points

$$P' = e_i^{-l}(P) \quad \text{and} \quad Q' = e_i^{-l_1}(Q)$$

that belong to $\gamma_{i,0}$. We clearly have $h'(P') = Q'$. Since both points lay on the perimeter of a fundamental region of Γ it must either hold $P' = Q'$ or P' and Q' are common vertices to the sides $\varepsilon_i^{'}$,$\gamma_{i,0}$ and $\gamma_{i,0}$,ε_i respectively. The first possibility leads us to $e^{l_1 - l}(P) = Q$ and this requirement forces $l_1 = l$. Consequently $P = Q$ which is false. On the other hand the second scenario would imply that $e_i \in \Lambda$ which contradicts the assumption $N > 1$.

Case ($ii.2$) There exists $0 \le l_1 \le \exp_\Lambda e_i - 1$ satisfying $Q \in \beta_j e_i^{l_1}(\gamma_{i,0}) \subset C_{i,j}$ where $2 \le j \le N/\exp_\Lambda e_i$. We take

$$P' = e_i^{-l}(P) \quad \text{and} \quad Q' = e_i^{-l_1}\beta_j^{-1}(Q) \tag{3.17}$$

and obtain that $h'(P') = Q'$ for $h' = e_i^{-l_1}\beta_j^{-1} h e_i^l \in \Gamma$. Note that P' and Q' belong to the same side of the perimeter of F_Γ. Thus $P' = Q'$ is a fixed point of h' or, as in *Case* ($i.1$), they are common vertices to the sides $\varepsilon_i^{'}$,$\gamma_{i,0}$ and $\gamma_{i,0}$,ε_i. Since $c_{i,0}$ is the only element of Γ fixing a point on the side $\gamma_{i,0}$ not being a common vertex with $\varepsilon_i^{'}$ or ε_i, we get in case $P' = Q'$ that $c_{i,0} = h' \in \Lambda$. Consequently

$$h' = e_i^{-l_1}\beta_j^{-1} h e_i^l = (\beta_j e_i^{l_1})^{-1} h (\beta_j e_i^{l_1})(\beta_j e_i^{l_1})^{-1} e_i^l = h''(\beta_j e_i^{l_1})^{-1} e_i^l$$

with $h'' \in \Lambda$. But this clearly forces the following relation on cosets $\Lambda = \Lambda(\beta_j e_i^{l_1})^{-1} e_i^l$ which contradicts (3.16) since we have assumed $j \ge 2$.

On the other hand if $\{P', Q'\} = (\varepsilon_i^{'} \cup \varepsilon_i) \cap \gamma_{i,0}$ then $h' = e_i$. It gives us in turn that $h = \beta_j e_i^{1+l_1-l} \in \Lambda$ which also contradicts (3.16).

Case (*ii.3*) Under the actual assumptions we have

$$Q \notin \bigcup_{j=1}^{N/\exp_\Lambda e_i} C_{i,j}. \tag{3.18}$$

Suppose $Q \in \beta_j e_i^{l_1}(F_\Gamma)$, where $1 \leq j \leq N/\exp_\Lambda e_i$. The first step we take in this setting is to transfer P and Q by (3.17) to points P' and Q' laying on the perimeter of F_Γ. Observe that now $P' \in \gamma_{i,0}$ but $Q' \notin \gamma_{i,0}$ by (3.18). Consequently P' can not be paired with Q' by any element of Γ. Hence $h' = e_i^{-l_1}\beta_j^{-1} h e_i^l \notin \Gamma$ a contradiction.

Hence the above argument gives rise to

$$\frac{N}{\exp_\Lambda e_i} = N\frac{(N, v_i)}{N} = (N, v_i) \tag{3.19}$$

different boundary components of X which means that the signature of Λ has (N, v_i) empty period cycles generated by an empty period cycle $\{e_i, c_i\}$ of Γ. Assume now that not only i, but also j is fixed and denote by C a hole on X on which the segment $C_{i,j}$ is projected. Let C' be another hole on X satisfying $C' = \pi(C_{i,j_1})$, $j \neq j_1$. By (3.8) we have

$$\theta(\beta_{j_1}\beta_j^{-1})(C) = \theta(\beta_{j_1}\beta_j^{-1})\pi(C_{i,j}) = \pi(\beta_{j_1}\beta_j^{-1}(C_{i,j})) = \pi(C_{i,j_1}) = C'.$$

Thus C' belongs to the orbit of C under the automorphism t. Hence this orbit counts exactly (N, v_i) boundary components. It yields a period of the boundary component C and consequently we have

$$\bigcup_{i=0}^{(N,v_i)-1} t^i(C) \subseteq \mathcal{S}_{(N,v_i)}(t).$$

Next we proceed to the two remaining cases when $i \geq r+1$ which means that an empty period cycle $\{e_i, c_{i,0}\}$ of Γ will contribute now to the number of ovals on X. We start with the assumption that e_i is mapped to t^{v_i}, where $\alpha_2(v_i) = \alpha_2(N)$. In such a case we have $|\langle\theta(e_i), \theta(c_{i,0})\rangle| = 2\exp_\Lambda e_i$ and we may write

$$\Gamma = \bigcup_{j=1}^{N/(2\exp_\Lambda e_i)} \bigcup_{k=0}^{\exp_\Lambda e_i - 1} \bigcup_{l=0}^{1} \Lambda(\beta_j c_{i,0}^l e_i^k). \tag{3.20}$$

Hence we have

$$F_\Lambda = \bigcup_{j=1}^{N/(2\exp_\Lambda e_i)} \bigcup_{k=0}^{\exp_\Lambda e_i - 1} \bigcup_{l=0}^{1} (\beta_j c_i^l e_i^k) F_\Gamma,$$

which finally gives

$$\begin{aligned}\Gamma/\Lambda &= \mathbb{Z}_N = \bigcup_{j=1}^{(\mathbb{Z}_N:\langle\theta(e_i),\theta(c_{i,0})\rangle)} g_j\langle\theta(e_i),\theta(c_{i,0})\rangle = \bigcup_{j=1}^{N/(2\exp_\Lambda e_i)} g_j\langle\theta(e_i),\theta(c_{i,0})\rangle \\ &= \bigcup_{j=1}^{N/(2\exp_\Lambda e_i)} \bigcup_{k=0}^{\exp_\Lambda e_i-1} \bigcup_{l=0}^{1} g_j\theta(c_i^l)\theta(e_i^k), \qquad (3.21)\end{aligned}$$

$g_j \in \mathbb{Z}_N$. Denote

$$C_{i,j} = \bigcup_{k=0}^{\exp_\Lambda e_i-1} \bigcup_{l=0}^{1} (\beta_j c_i^l e_i^k)\gamma_{i,0} = \bigcup_{k=0}^{\exp_\Lambda e_i-1} (\beta_j e_i^k)\gamma_{i,0}.$$

Unlike the previous case c_i now goes to the element of order 2 in $\mathbb{Z}_N$ and for this reason it has just been used as a representative of a non–identity coset $\Lambda\gamma$ of Γ. It allows us to observe that in a neighbourhood of $\gamma_{i,0}$ we have the situation given on Figure 3.2 (for simplicity of notation the drawing is made for $\beta_j = \mathrm{Id}$). The only images of sides ε_i and ε_i' that lie on the perimeter of F_Λ are the following $\beta_j(\varepsilon_i')$, $\beta_j c_{i,0}(\varepsilon_i')$, $\beta_j e_i^{\exp_\Lambda e_i-1}(\varepsilon_i)$ and $\beta_j c_{i,0} e_i^{\exp_\Lambda e_i-1}(\varepsilon_i)$.

We have

$$\begin{aligned}&\beta_j e_i^{\exp_\Lambda e_i}\beta_j^{-1}\beta_j\varepsilon_i' = \beta_j e_i^{\exp_\Lambda e_i}\varepsilon_i' = \beta_j e_i^{\exp_\Lambda e_i-1}\varepsilon_i \\ &\beta_j e_i^{\exp_\Lambda e_i}\beta_j^{-1}\beta_j c_{i,0}\varepsilon_i' = \beta_j e_i^{\exp_\Lambda e_i} c_{i,0}\varepsilon_i' = \beta_j c_{i,0} e_i^{\exp_\Lambda e_i}\varepsilon_i' = \beta_j c_{i,0} e_i^{\exp_\Lambda e_i-1}\varepsilon_i.\end{aligned}$$

Thus $\beta_j e_i^{\exp_\Lambda e_i}\beta_j^{-1} \in \Lambda$ is a generator pairing the edges $\beta_j(\varepsilon_i' \cup c_{i,0}\varepsilon_i')$and $\beta_j(e_i^{\exp_\Lambda e_i-1}\varepsilon_i \cup c_{i,0}e_i^{\exp_\Lambda e_i-1}\varepsilon_i)$. Hence we obtain

$$\frac{N}{2\exp_\Lambda e_i} = \frac{(N,v_i)}{2} = (\frac{N}{2}, v_i)$$

different two–sided ovals in int$X$. In order to determine their period we follow the same method as before, that is we show that all ovals generated by $\{e_i, c_{i,0}\}$ lie in the same orbit. In consequence their period equals $(N/2, v_i)$.

In the last step of the proof we deal with one–sided ovals. Observe that for $i \geq r+1$ each generator e_i goes to t^{v_i} with $\alpha_2(v_i) < \alpha_2(N)$. Therefore we have $t^{N/2} \in \langle\theta(e_i)\rangle$. Furthermore $|\langle\theta(e_i)\rangle| = |\langle\theta(e_i),\theta(c_{i,0})\rangle| = \exp_\Lambda e_i$. Hence we may represent the group Γ as follows

$$\Gamma = \bigcup_{j=1}^{N/\exp_\Lambda e_i} \bigcup_{k=0}^{(\exp_\Lambda e_i/2)-1} \bigcup_{l=0}^{1} \Lambda(\beta_j c_{i,0}^l e_i^k).$$

It gives a fundamental region for Λ

$$F_\Lambda = \bigcup_{j=1}^{N/\exp_\Lambda e_i} \bigcup_{k=0}^{(\exp_\Lambda e_i/2)-1} \bigcup_{l=0}^{1} (\beta_j c_{i,0}^l e_i^k)F_\Gamma.$$

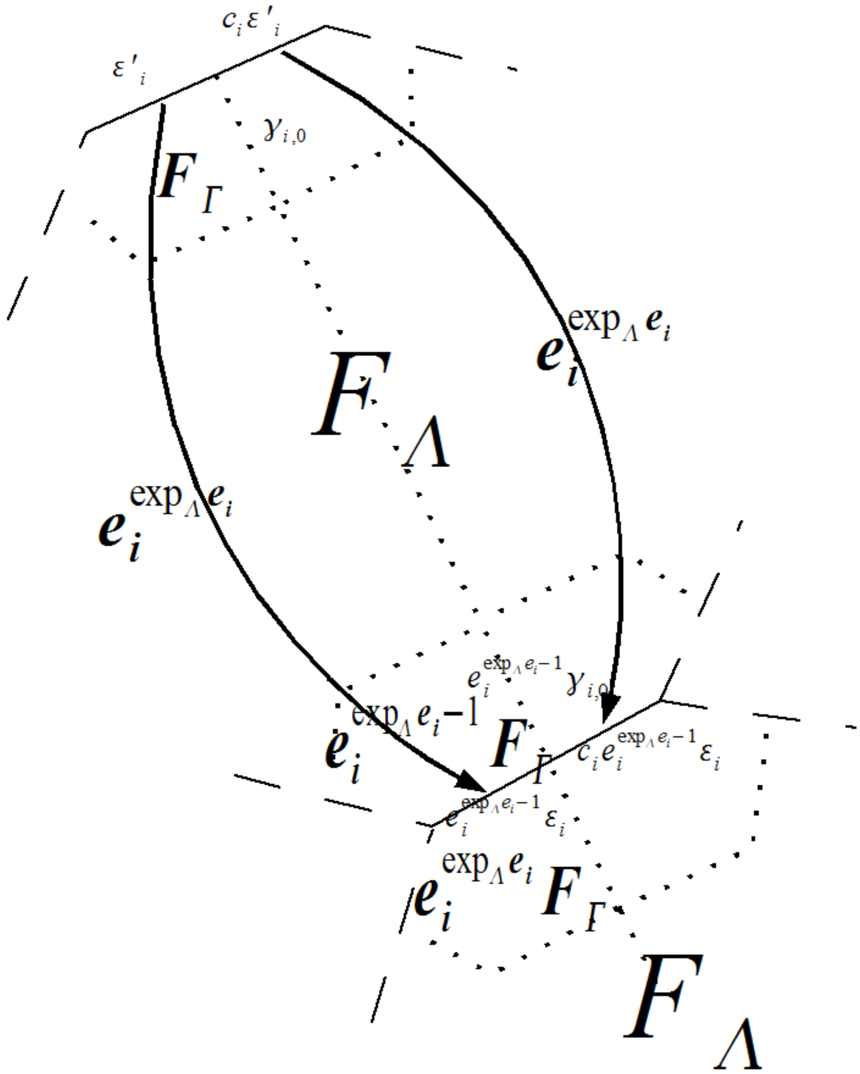

Figure 3.2: A segment generating a two–sided oval on X, $\beta_j = \mathrm{Id}$.

Consequently

$$\Gamma/\Lambda = \mathbb{Z}_N = \bigcup_{j=1}^{(\mathbb{Z}_N:\langle\theta(e_i)\rangle)} g_j\langle\theta(e_i)\rangle = \bigcup_{j=1}^{N/\exp_\Lambda e_i} g_j\langle\theta(e_i)\rangle = \bigcup_{j=1}^{N/\exp_\Lambda e_i}\ \bigcup_{k=0}^{(\exp_\Lambda e_i/2)-1}\ \bigcup_{l=0}^{1} g_j\theta(c_{i,0}^l)\theta(e_i^k).$$

Denote

$$C_{i,j} = \bigcup_{k=0}^{(\exp_\Lambda e_i/2)-1}\ \bigcup_{l=0}^{1}(\beta_j c_{i,0}^l e_i^k)\gamma_{i,0} = \bigcup_{k=0}^{(\exp_\Lambda e_i/2)-1}(\beta_j e_i^k)\gamma_{i,0}.$$

In order to show that each $C_{i,j}$ projects to an oval in int$X$ we apply the technique which has been used before twice. As it is seen on on Figure 3.3 (again on the drawing it is assumed $\beta_j = \mathrm{Id}$) we may find an element of Λ pairing the appropriate edges of F_Λ. Indeed we have

$$\beta_j c_{i,0} e_i^{\exp_\Lambda e_i/2}\beta_j^{-1}\beta_j\varepsilon_i' = \beta_j c_{i,0} e_i^{\exp_\Lambda e_i/2}\varepsilon_i' = \beta_j c_{i,0} e_i^{(\exp_\Lambda e_i/2)-1}\varepsilon_i$$
$$\beta_j c_{i,0} e_i^{\exp_\Lambda e_i/2}\beta_j^{-1}\beta_j c_{i,0}\varepsilon_i' = \beta_j c_{i,0} e_i^{\exp_\Lambda e_i/2} c_{i,0}\varepsilon_i' = \beta_j e_i^{\exp_\Lambda e_i/2} c_{i,0}^2\varepsilon_i' = \beta_j e_i^{(\exp_\Lambda e_i/2)-1}\varepsilon_i$$

where $\beta_j c_{i,0} e_i^{\exp_\Lambda e_i/2}\beta_j^{-1} \in \Lambda$. However it is worth pointing out that the above generator pairing the edges $\beta_j(\varepsilon_i' \cup c_i\varepsilon_i')$ and $\beta_j(e_i^{(\exp_\Lambda e_i/2)-1}\varepsilon_i \cup c_i e_i^{(\exp_\Lambda e_i/2)-1}\varepsilon_i)$ is orientation–reversing. The above leads us to the conclusion that the empty period cycle $\{e_i, c_{i,0}\}$ of Γ induces

$$\frac{N}{\exp_\Lambda e_i} = (N, v_i) = (\frac{N}{2}, v_i)$$

one–sided ovals. Their period equals $(N/2, v_i)$.

□

3.3.3 Chains

As it has been announced before we now proceed to discuss the third type of components of the singular set $\mathcal{S}(t)$ that are called chains. Let us note that boundaries of X as well as ovals contained in int$X$ are always mapped under the projection $X \to X/\langle t\rangle$ onto boundaries of the quotient surface $X/\langle t\rangle$. However those two types of periodic structures on X are not the only ones that "come from" the period cycles appearing in the signature of the covering group Γ. The components of $\mathcal{S}(t)$ of the third type arise from non–empty period cycles of Γ. According to (3.7) these period cycles are of the form (2^μ). From the geometrical point of view they correspond to boundaries of $X/\langle t\rangle$ that contain some cone points i.e. points with ramification indices equal to 2. Since the existence of chains requires N to be even, in the following theorem we assume that order of automorphism under study is even.

Theorem 3.13. *Let X be a compact Klein surface of algebraic genus $p \geq 2$. Let $t\colon X \to X$ be an automorphism of an even order N and $\theta\colon \Gamma \to \mathbb{Z}_N$ be an epimorphism (3.8) of Γ given by (3.7), that uniformizes a $\mathbb{Z}_N$–action of t. Assume that*

$$\theta(e_i) = t^{v_i} \quad \text{for} \quad \lambda + 1 \leq i \leq \lambda + p$$

Then the following conditions hold

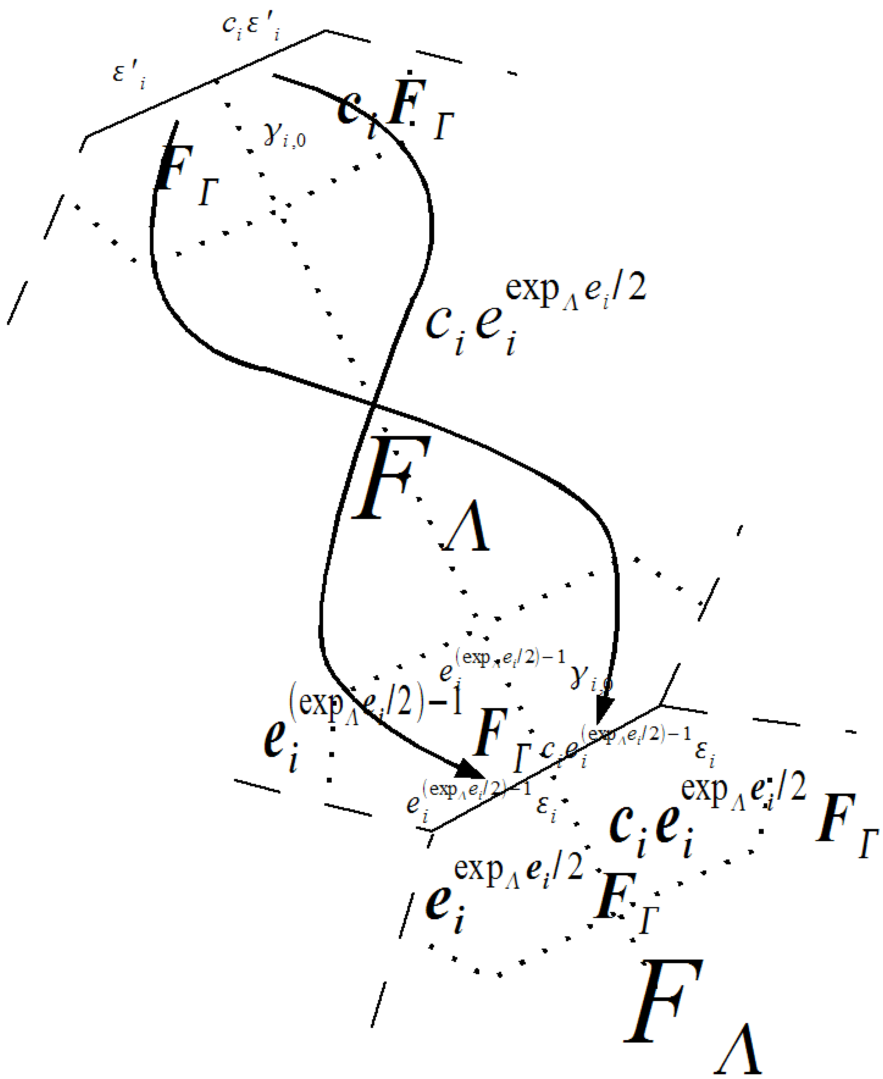

Figure 3.3: A segment generating an one–sided oval on X, $\beta_j = \mathrm{Id}$.

(i) if $\alpha_2(v_i) = \alpha_2(N)$, then i–th non–empty period cycle of Γ induces on X a two–sided chain of period $(N/2, v_i)$

(ii) if $\alpha_2(v_i) < \alpha_2(N)$, then i–th non–empty period cycle of Γ induces on X an one–sided chain of period $(N/2, v_i)$

Proof. The main idea of the proof is similar to the one of the previous theorem. It has been motivated by proofs of Theorem 2.3.3 of [8] and Proposition 3.3 of [9]. For the convenience of the reader we hopefully provide sufficient, but not too much details.

We begin by choosing a fundamental region F_Γ for Γ whose perimeter is labelled as follows $\varepsilon_i\gamma_{i,0}...\gamma_{i,\mu_i}\varepsilon_i'\Delta$. Here

(1) Δ represents the remaining sides of the perimeter

(2) the reflections $c_{i,j}$ fix the respective sides $\gamma_{i,j}$

(3) $e_i(\varepsilon_i^{'}) = \varepsilon_i$

(4) $e_i c_{i,0} = c_{i,s_i} e_i$.

Assume that $\alpha_2(v_i) = \alpha_2(N)$. We proceed to construct a suitable fundamental region for Λ. Without loss of generality we may assume that $c_{i,0} \notin \Lambda$ (see for instance Proposition 3.3). Moreover we see at once that either $c_{i,0}e_i \notin \Lambda$. It follows from the fact that $\theta(c_{i,0}e_i) = t^{N/2+v_i}$ and by the above assumption we have $\alpha_2(N/2+v_i)\alpha_2(N/2) < \alpha_2(N)$. Analogously to (3.20) we observe that

$$\Gamma = \bigcup_{j=1}^{N/(2\exp_\Lambda e_i)} \bigcup_{k=0}^{\exp_\Lambda e_i - 1} \bigcup_{l=0}^{1} \Lambda(\beta_j c_{i,0}^l e_i^k)$$

which enables us to write

$$F_\Lambda = \bigcup_{j=1}^{N/(2\exp_\Lambda e_i)} \bigcup_{k=0}^{\exp_\Lambda e_i - 1} \bigcup_{l=0}^{1} (\beta_j c_{i,0}^l e_i^k) F_\Gamma.$$

Consequently we have

$$\begin{aligned}\Gamma/\Lambda &= \mathbb{Z}_N = \bigcup_{j=1}^{(\mathbb{Z}_N:\langle\theta(e_i),\theta(c_{i,0})\rangle)} g_j\langle\theta(e_i),\theta(c_{i,0})\rangle = \bigcup_{j=1}^{N/(2\exp_\Lambda e_i)} g_j\langle\theta(e_i),\theta(c_{i,0})\rangle \\ &= \bigcup_{j=1}^{N/(2\exp_\Lambda e_i)} \bigcup_{k=0}^{\exp_\Lambda e_i - 1} \bigcup_{l=0}^{1} g_j\theta(c_i^l)\theta(e_i^k),\end{aligned}$$

where $\theta(\beta_j) = g_j$. This is exactly the relation (3.21). Consider the following segment of the perimeter of F_Λ

$$C_{i,j} = \bigcup_{k=0}^{\exp_\Lambda e_i - 1} \bigcup_{l=0}^{1} \bigcup_{m=0}^{\mu_i} (\beta_j c_{i,0}^l e_i^k)\gamma_{i,m}.$$

We show that each $C_{i,j}$ projects to a union of a chain and some boundary components of X. Contrary to both cases of ovals we must be aware that now there are segments of $C_{i,j}$ that are identified by an element of Λ. Indeed we have

$$\lambda\beta_j e_i^k \gamma_{i,2p} = \beta_j c_{i,0} e_i^k \gamma_{i,2p} \quad \text{where} \quad \lambda = \beta_j c_{i,0} c_{i,2p} [c_{i,2p} e_i^k] \beta_j^{-1} \tag{3.22}$$

since $c_{i,2p}(\gamma_{i,2p}) = \gamma_{i,2p}$. Thus the sides $\beta_j e_i^k \gamma_{i,2p}$ and $\beta_j c_{i,0} e_i^k \gamma_{i,2p}$ (see fine dashed arrows on Figure 3.4, for simplicity of notation we assume there $\beta_j = \mathrm{Id}$) are paired by $\lambda \in \Lambda$. Since $\beta_j F_\Gamma \cup \beta_j c_{i,0} F_\Gamma \subseteq F_\Lambda$ we conclude that $\beta_j \gamma_{i,0}$ projects to an arc on X. From (3.22) it also follows that all remaining sides $\beta_j \gamma_{i,2p}$, $p \geq 1$ project to arcs on X.

On the other hand the edges $\beta_j \gamma_{i,2p+1}$ and $\beta_j c_{i,0} \gamma_{i,2p+1}$ project to curves that together form a boundary component of X. Note that the involution $\theta(c_{i,0}) = t^{N/2}$ fixes pointwise all the above arcs and interchanges the curves generated by $\beta_j \gamma_{i,2p+1}$ with those generated by $\beta_j c_{i,0} \gamma_{i,2p+1}$.

In order to show that

$$B_{i,j} = \bigcup_{k=0}^{\exp_\Lambda e_i - 1} \bigcup_{m=0}^{\mu_i} (\beta_j e_i^k) \gamma_{i,m} \subset C_{i,j}$$

projects to a two–sided chain on X we first need to prove that there exists $\lambda \in \Lambda$ with

$$\lambda\beta_j(\varepsilon_i' \cup c_{i,0}\varepsilon_i') = \beta_j(e_i^{\exp_\Lambda e_i - 1}\varepsilon_i \cup c_{i,0} e_i^{\exp_\Lambda e_i - 1}\varepsilon_i).$$

But this is clear since

$$\begin{aligned}
&\beta_j e_i^{\exp_\Lambda e_i} \beta_j^{-1} \beta_j \varepsilon_i' = \beta_j e_i^{\exp_\Lambda e_i} \varepsilon_i' = \beta_j e_i^{\exp_\Lambda e_i - 1} \varepsilon_i \\
&\beta_j c_{i,0} e_i^{\exp_\Lambda e_i} c_{i,0} \beta_j^{-1} \beta_j c_{i,0} \varepsilon_i' = \beta_j c_{i,0} e_i^{\exp_\Lambda e_i} \varepsilon_i' = \beta_j c_{i,0} e_i^{\exp_\Lambda e_i - 1} \varepsilon_i,
\end{aligned}$$

where both elements $\beta_j e_i^{\exp_\Lambda e_i} \beta_j^{-1}$ and $\beta_j c_{i,0} e_i^{\exp_\Lambda e_i} c_{i,0} \beta_j^{-1}$ belong to Λ.

The remaining steps required to show that $\pi(B_{i,j})$ is a chain can be handled in much the same way as *(i.1)*–*(i.3)* in the proof of Theorem 3.12 and as such are superfluous. Hence a non–empty period cycle (2^{μ_i}) of Γ generates

$$\frac{N}{2\exp_\Lambda e_i} = \frac{(N, v_i)}{2} = (\frac{N}{2}, v_i)$$

two–sided chains of period $(N/2, v_i)$. The length of each chain equals $\mu_i \exp_\Lambda e_i$.

Suppose now that $\theta(e_i) = t^{\nu_i}$, where $\alpha_2(v_i) < \alpha_2(N)$. Since we have assumed $c_{i,0} \notin \Lambda$ we may write equivalently $c_{i,0} e_i^{\exp_\Lambda e_i/2} \in \Lambda$ or $\langle \theta(e_i), \theta(c_{i,0}) \rangle = \langle \theta(e_i) \rangle$. The actual case can also be solved using the approach based on a choice of the suitable fundamental region for Λ.

According to our assumptions we now have

$$\Gamma = \bigcup_{j=1}^{N/\exp_\Lambda e_i} \bigcup_{k=0}^{(\exp_\Lambda e_i/2)-1} \bigcup_{l=0}^{1} \Lambda(\beta_j c_{i,0}^l e_i^k),$$

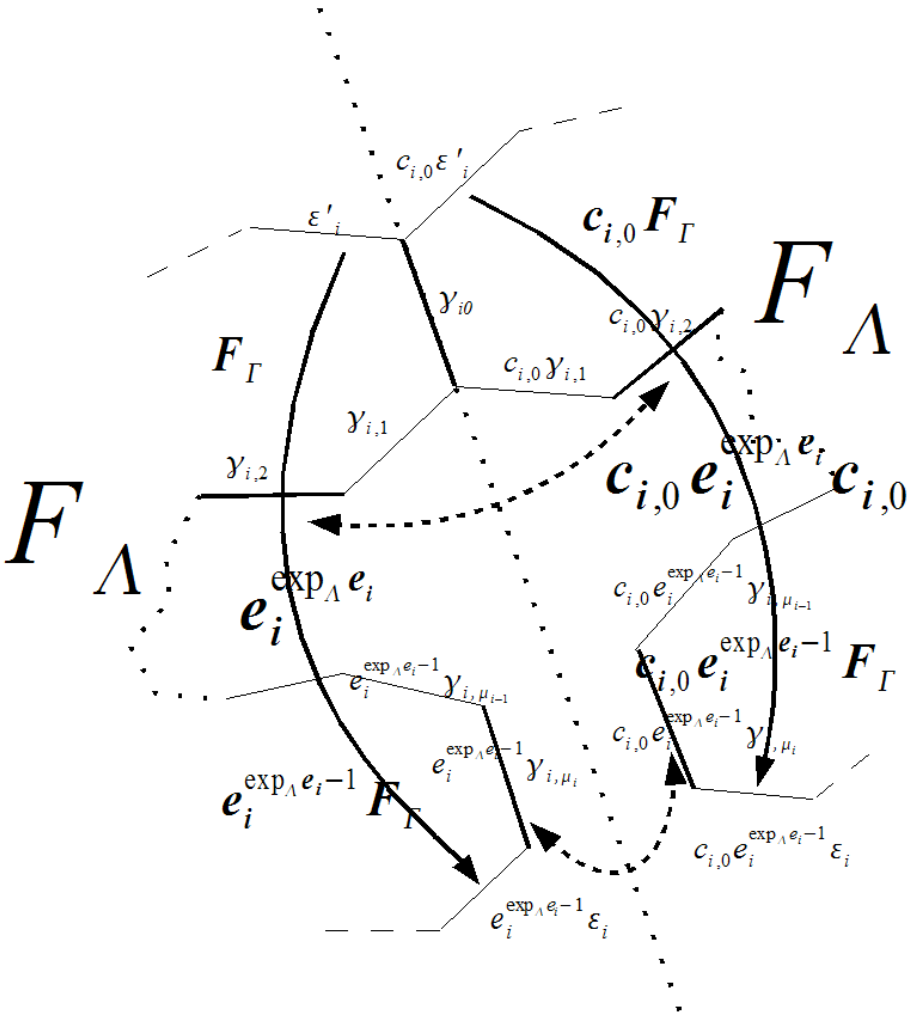

Figure 3.4: A segment generating a two–sided chain on X, $\beta_j = \mathrm{Id}$.

which leads us to

$$F_\Lambda = \bigcup_{j=1}^{N/\exp_\Lambda e_i} \bigcup_{k=0}^{(\exp_\Lambda e_i/2)-1} \bigcup_{l=0}^{1} (\beta_j c_{i,0}^l e_i^k) F_\Gamma.$$

We continue in this fashion obtaining the representation

$$\Gamma/\Lambda = \mathbb{Z}_N = \bigcup_{j=1}^{(\mathbb{Z}_N:\langle\theta(e_i)\rangle)} g_j\langle\theta(e_i)\rangle = \bigcup_{j=1}^{N/\exp_\Lambda e_i} g_j\langle\theta(e_i)\rangle = \bigcup_{j=1}^{N/\exp_\Lambda e_i} \bigcup_{k=0}^{(\exp_\Lambda e_i/2)-1} \bigcup_{l=0}^{1} g_j\theta(c_{i,0}^l)\theta(e_i^k).$$

Let us put

$$C_{i,j} = \bigcup_{k=0}^{(\exp_\Lambda e_i/2)-1} \bigcup_{l=0}^{1} \bigcup_{m=0}^{\mu_i} (\beta_j c_{i,0}^l e_i^k)\gamma_{i,m}$$

We investigate the projection of $C_{i,j}$ on X. Since (3.22) remains true the identifications of segments of $C_{i,j}$ discussed previously are still valid (see fine dashed arrows on Figure 3.5, for simplicity of notation we assume there $\beta_j = \mathrm{Id}$).

The difference between cases $\alpha_2(\nu_i) = \alpha_2(N)$ and $\alpha_2(\nu_i) < \alpha_2(N)$ consists in pairing the images of sides ε_i and ε_i'. Observe that

$$\begin{aligned}
\beta_j c_{i,0} e_i^{\exp_\Lambda e_i/2}\beta_j^{-1}\beta_j\varepsilon_i' &= \beta_j c_{i,0} e_i^{\exp_\Lambda e_i/2}\varepsilon_i' = \beta_j c_{i,0} e_i^{(\exp_\Lambda e_i/2)-1}\varepsilon_i \\
\beta_j e_i^{\exp_\Lambda e_i/2} c_{i,0}\beta_j^{-1}\beta_j c_{i,0}\varepsilon_i' &= \beta_j e_i^{\exp_\Lambda e_i/2}\varepsilon_i' = \beta_j e_i^{(\exp_\Lambda e_i)/2-1}\varepsilon_i,
\end{aligned}$$

where both elements $\beta_j c_{i,0} e_i^{\exp_\Lambda e_i/2}\beta_j^{-1}$ and $\beta_j e_i^{\exp_\Lambda e_i/2} c_{i,0}\beta_j^{-1}$ belong to Λ. Hence the sides

$$\beta_j(\varepsilon_i' \cup c_{i,0}\varepsilon_i') \text{ and} \beta_j(e_i^{(\exp_\Lambda e_i/2)-1}\varepsilon_i \cup c_{i,0} e_i^{(\exp_\Lambda e_i/2)-1}\varepsilon_i)$$

are paired by the above orientation–reversing elements.

We conclude that

$$B_{i,j} = \bigcup_{k=0}^{(\exp_\Lambda e_i/2)-1} \bigcup_{m=0}^{\mu_i} (\beta_j e_i^k)\gamma_{i,m} \subset C_{i,j}$$

projects to one–sided chain on X. It also follows that a non–empty period cycle (2^{μ_i}) of Γ generates

$$\frac{N}{\exp_\Lambda e_i} = (N, v_i) = (\frac{N}{2}, v_i)$$

one–sided chains of period $(N/2, v_i)$. The length of each chain equals $\mu_i \exp_\Lambda e_i/2$. □

We make a remark that goes back to work [9] and shows how the above situation reduces in case of involution.

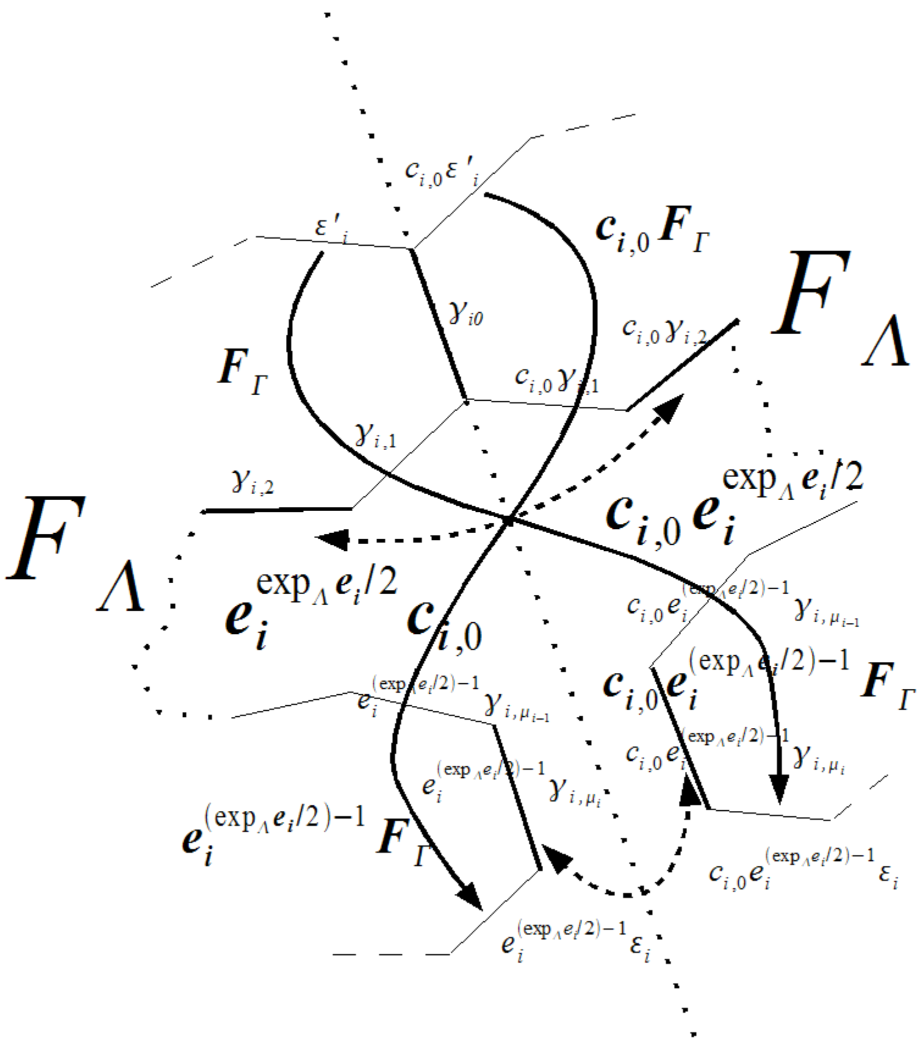

Figure 3.5: A segment generating an one–sided chain on X, $\beta_j = \mathrm{Id}$.

Remark 3.14. *Under the assumptions of Theorem 3.13 with $N = 2$ exactly one of the transformations e_i and $c_{i,0}e_i$ belongs to Λ. Indeed, $v_i = 2$ yields $e_i \in \Lambda$, hence we obtain a two–sided chain of length μ_i. Conversely, if $v_i = 1$ we have $c_{i,0}e_i \in \Lambda$ and consequently we get an one–sided chain of length $\mu_i \exp_\Lambda e_i/2 = \mu_i$. Since $N/2 = 1$, in both cases periods of the above chains equal to 1.*

Remark 3.15. *By results of the last two subsections we may easily calculate the number of boundary components of X in terms of the signature of group Γ and epimorphism $\theta\colon \Gamma \to \mathbb{Z}_N$. Note that the number of boundaries of X that "come from" non–empty period cycles (2^{μ_i}) of Γ can be obtained by the formula*

$$\frac{1}{2}(\text{period of a chain} \times \text{length of a chain})$$

since a boundary component that intersects a chain has exactly 2 common points with it. Hence using the notation of Theorems 3.12 and 3.13 total number of boundary components of X being a sum of those contributed by the empty and non–empty period cycles of Γ equals

$$\sum_{i=1}^{r}(N, v_i) + \frac{N}{2}\sum_{i=1}^{p}\frac{\mu_i}{2}$$

It coincides with the assertion of Theorem 2.4.4 in [8] (see point (2) there). Recall that by the convention we made before the boundary components intersecting chains of t do not form part of the singular set of last period $N/2$ in the setwise context $\mathcal{S}_{N/2}(t)$. These are the boundaries "coming from" non–empty period cycles (2^{μ_i}) of Γ.

Chapter 4

Dynamics of Dianalytic Transformations

4.1 The Character of Periods

We devote the present section to the study of the periodic point behaviour of dianalytic maps of Klein surfaces into itself. Let X be a Klein surface and assume that $t\colon X \to X$ is an automorphism of X. As we know the singular set of such a map may contain isolated periodic points, periodic boundary components and one–sided or two–sided periodic ovals and chains. Recall that one–dimensional components of the singular set of least period d are in fact fixed by t^d only setwise, but do not contain fixed points, except t^d is an involution.

The below notation applies to periods of various components of the singular set of automorphisms of Klein surfaces.

(1) Let $\mathcal{A}_1(t)$ denote the set of periods of isolated periodic points of t.

(2) Let $\mathcal{A}_2(t)$ denote the set of periods of boundary components of X of t that do not intersect chains.

(3) Let $\mathcal{A}_3(t)$ denote the set of periods of two–sided ovals of t.

(4) Let $\mathcal{A}_4(t)$ denote the set of periods of one–sided ovals of t.

(5) Let $\mathcal{A}_5(t)$ denote the set of periods of two–sided chains of t.

(6) Let $\mathcal{A}_6(t)$ denote the set of periods of one–sided chains of t.

The following lemma sums up the results of the last chapter stating precisely properties of the above sets of periods. Recall that due to the notation introduced in Subsection 1.2.1 the symbols $\mathcal{D}_0(N)$ and $\mathcal{D}(N)$ stand for the set of proper divisors and all divisors of N respectively.

Lemma 4.1. *Let $t\colon X \to X$ be an automorphism of a Klein surface X. The sets of periods describing the action of $\langle t\rangle$ on X fulfill the following constraints:*

$$\mathcal{A}_1(t) \subseteq \mathcal{D}_0(N),\ \mathcal{A}_2(t) \subseteq \mathcal{D}(N). \tag{4.1}$$

Furthermore for even N we have

$$\mathcal{A}_3(t), \mathcal{A}_5(t) \subseteq 2^{\alpha_2(N)-1}\mathcal{D}(\frac{N}{2^{\alpha_2(N)}}), \qquad \mathcal{A}_4(t), \mathcal{A}_6(t) \subseteq \mathcal{D}(\frac{N}{2}) \tag{4.2}$$

whereas an odd N forces

$$\mathcal{A}_3(t) = \mathcal{A}_4(t) = \mathcal{A}_5(t) = \mathcal{A}_6(t) = \emptyset. \tag{4.3}$$

Proof. The inclusions (4.1) are immediate provided we remember that by our convention all boundary components belong to the singular set of t. Thus their period may be any number that divides N, including also the N itself. The relations for $\mathcal{A}_3(t)$–$\mathcal{A}_6(t)$ follow from Theorems 3.12 and 3.13 and are determined by the order of the image of the respective e–generator in $\mathbb{Z}_N$. Let $\theta\colon \Gamma \to \mathbb{Z}_N$ be a smooth epimorphism covering the action of t on X. In case of the two–sided structures we have $\theta(e) = t^v$ with $\alpha_2(v) = \alpha_2(N)$, where e is the corresponding e–generator belonging to an empty period cycle in case of an oval or a non–empty period cycle in case of a chain. Hence their periods that equal $(N/2, v)$ hold the relation $\alpha_2((N/2, v)) = \alpha_2(N) - 1$ which justifies the inclusions for $\mathcal{A}_3(t)$ and $\mathcal{A}_5(t)$. On the other hand the sets $\mathcal{A}_4(t)$ and $\mathcal{A}_6(t)$ corresponding to the one–sided structures comprise the numbers of the form $(N/2, v)$ with $\alpha_2(v) < \alpha_2(N)$. Obviously $(N/2, v) \in \mathcal{D}(N/2)$.

If N is odd the reflections c of Γ must be mapped to the identity element in $\mathbb{Z}_N$. Hence the one–dimensional structures of the singular set in the interior of X do not become apparent in that setting. $\square$

In order to extend the notion of set of periods that we used previously in case of automorphisms of Riemann surfaces, we introduce now a term *character of periods* and define it to be a 6–tuple of sets of periods enumerated in points (1)–(6). We use the following notation

$$\mathrm{CPer}(t) = (\mathcal{A}_1(t), \mathcal{A}_2(t), \mathcal{A}_3(t), \mathcal{A}_4(t), \mathcal{A}_5(t), \mathcal{A}_6(t)). \tag{4.4}$$

We shall consider the character of periods of $\mathbb{Z}_N$–actions as the set of $\mathrm{CPer}(t)$ taken over all Klein surfaces X and dianalytic automorphisms $t \in \mathrm{Aut}X$ of order N, i.e.

$$\mathrm{CPer}(\mathbb{Z}_N) = \{\mathrm{CPer}(t) \mid X - \text{Klein surface},\ t \in \mathrm{Aut}X,\ \mathrm{ord}t = N\}.$$

Throughout our exposition we consider various cases that comprise a global study of the dynamics of dianalytic maps of Klein surfaces into itself. At the first stage of our investigation we will focus on the parity of N. The necessity of verification whether N is odd or even, follows from the different restrictions concerning the components of the singular set. Since the former case is not as complex as the latter one we first start with the case of N odd in Section 5.1. Next in Sections 5.2 and 5.3 we will consider the case of even N. Our way of investigating the dynamics of maps of Klein surfaces also takes into account the orientability character of surfaces X and $X/\langle t\rangle$. Let us denote by $\mathrm{CPer}^{+}(\mathbb{Z}_N)$ and $\mathrm{CPer}^{-}(\mathbb{Z}_N)$ the set of characters of periods of $\mathbb{Z}_N$–actions on orientable and non–orientable surfaces respectively. We then may divide those sets with respect to the orientability character of the quotient space $X/\langle t\rangle$. We use the following notation

$$\begin{aligned}
\mathrm{CPer}^{(+,+)}(\mathbb{Z}_N) &= \{\mathrm{CPer}(t) \mid X \text{ orientable},\ X/\langle t\rangle \text{ orientable}\}\\
\mathrm{CPer}^{(-,+)}(\mathbb{Z}_N) &= \{\mathrm{CPer}(t) \mid X \text{ orientable},\ X/\langle t\rangle \text{ non}-\text{orientable}\}\\
\mathrm{CPer}^{(+,-)}(\mathbb{Z}_N) &= \{\mathrm{CPer}(t) \mid X \text{ non}-\text{orientable},\ X/\langle t\rangle \text{ orientable}\}\\
\mathrm{CPer}^{(-,-)}(\mathbb{Z}_N) &= \{\mathrm{CPer}(t) \mid X \text{ non}-\text{orientable},\ X/\langle t\rangle \text{ non}-\text{orientable}\}.
\end{aligned}$$

By theorems of [8] we show in Section 5.1 that the above list for N odd contains only two items since under this assumption we have $\mathrm{CPer}^{(-,+)}(\mathbb{Z}_N) = \mathrm{CPer}^{(+,-)}(\mathbb{Z}_N) = \emptyset$.

In the remainder of the thesis we assume that arbitrary sets denoted here as $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$, $\mathcal{A}_5$, $\mathcal{A}_6$ satisfy the respective conditions (4.1)–(4.3). We write $\mathfrak{C}_0$ for a 6–tuple formed by such sets

$$\mathfrak{C}_0 = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6).$$

In order to consider sets of periods that become apparent on various Klein surfaces we need to separate the notions of covering NEC group and covering epimorphism in the two contexts. First, as defined before, while considering a Klein surface X and an automorphism $t\colon X \to X$ we say that epimorphism θ appearing in the diagram (3.8) covers (or uniformizes) a $\mathbb{Z}_N$–action of t on X and Γ is a NEC group covering a $\mathbb{Z}_N$–action of t on X. On the other hand when we investigate a character of periods given by a 6–tuple of sets of periods $\mathfrak{C}_0$, we say that smooth epimorphism $\theta\colon \Gamma \to \mathbb{Z}_N$ covers a $\mathbb{Z}_N$–action given by $\mathfrak{C}_0$ if there is a Klein surface X and an automorphism $t\colon X \to X$ such that the diagram (3.8) commutes. Observe that in this context we do not assume the orientability character nor of X, nor of $X/\langle t\rangle$. Using the above slightly wider approach we also say that a NEC group Γ covers a $\mathbb{Z}_N$–action given by $\mathfrak{C}_0$.

4.1.1 Definitions and Notation II

Unfortunately we use a quite large number of symbols throughout the thesis. Below we introduce a notation we need in the forthcoming sections. Some of the following terms were defined before in Subsection 1.2.1. However, now they do appear in a much wider context. As in the first part of the exposition, we will work mainly with integers and some very simple structures defined on them. We use a term *set of integers*, assuming that it does not contain any repetitions.

Throughout the remainder of the paper we still denote by $t\colon X \to X$ an automorphism of a Klein surface X. The letter N will stand for the order of a cyclic group $\langle t\rangle$ that acts on X. We introduce a $*$–notation in order to be able to differentiate periods of a $\mathbb{Z}_N$–action on X from orders of elements in that group. All the subsequent constructs (numbers, elements, sets, functions) that refer to orders of elements in $\mathbb{Z}_N$ will be denoted using the $*$–notation. It can also be considered as an advantage for the reader, providing a self–checking general rule stating that

$*$–symbols may be built only on $*$–symbols

On the other hand to shorten notation we shall use some not $*$–symbols built over $*$–symbols. For example we write $F(\mathfrak{C}^*)$ instead of $(F^*(\mathfrak{C}^*))^*$. This convention will be reminded repeatedly as the corresponding constructs come along.

We use the following definitions.

(0) By family of numbers we will understand a sequence of numbers although with no importance on the order of this sequence. By this meaning $\{2, 2, 3, 5\}$ is the same family as $\{2, 3, 2, 5\}$.

Let $\mathcal{D}$ be a family of divisors of N.

(1) If d is a divisor of N, then by d^* we denote the number N/d. Likewise, if $\mathcal{D} = \{d_1, \ldots, d_k\}$ is a family of divisors of N, then $\mathcal{D}^* = \{N/d_1, \ldots, N/d_k\}$. We also write $\mathfrak{C}_0^*$ for a 6–tuple $(\mathcal{A}_1^*, \mathcal{A}_2^*, \mathcal{A}_3^*, \mathcal{A}_4^*, \mathcal{A}_5^*, \mathcal{A}_6^*)$, where $\mathfrak{C}_0 = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6)$ is a character of periods of some $\mathbb{Z}_N$–action on a Klein surface.

(2) The projection of $\mathcal{D}$ is the set of different integers that belong to the family $\mathcal{D}$. It will be denoted by $\pi(\mathcal{D})$.

(3) For an integer d belonging to the family of integers $\mathcal{D}$ we define its multiplicity $m_{\mathcal{D}}(d)$ to be the number of elements of $\mathcal{D}$ that are equal to d.

(4) The cardinality of the family of integers $\mathcal{D}$ is understood in the usual set–theoretic manner and defined as $\sharp\mathcal{D} = \sum_{d\in\pi(\mathcal{D})} m_{\mathcal{D}}(d)$.

(5) Let $\mathcal{D}$ and $\mathcal{D}'$ be two families of integers. We say that $\mathcal{D}'$ includes $\mathcal{D}$ if and only if for every $d \in \pi(\mathcal{D})$ we have $m_{\mathcal{D}}(d) \leq m_{\mathcal{D}'}(d)$. We will write $\mathcal{D} \subset \mathcal{D}'$.

4.1.2 The Induced Action

Suppose that $\theta\colon \Gamma \to \mathbb{Z}_N$ is a smooth epimorphism and recall that by Theorems 2.4.2 and 2.4.4 of [8] group Γ has the representation

$$\Gamma = (\gamma; \pm; [m_1, ..., m_n]; \{(\ )^\lambda(2^{\mu_1})...(2^{\mu_p})\}). \tag{4.5}$$

Observe that in case of epimorphisms uniformizing a $\mathbb{Z}_N$–action on Riemann surfaces by conformal automorphisms, the signature of Γ was a sufficient data to know the set of periods of the underlying action. However, while considering Klein surfaces this is not the case. As we will see in this subsection it is indispensable to know also the orders of images under θ of all canonical generators except the hyperbolic generators corresponding to the orbit genus of Γ. Note that by smoothness of θ the orders of images of elliptic generators of Γ can be easily derived from the signature of Γ. Nevertheless it does not provide us with the information on orders of $\theta(c)$ nor $\theta(e)$, for reflections c and e–generators of Γ.

On the other hand if we know how the canonical generators of Γ are mapped by θ, then by results of the previous chapter we may find all periodic structures of the singular set of the underlying action. In the following remark we state in a precise manner a method of calculating the character of periods of a $\mathbb{Z}_N$–action on a Klein surface.

Remark 4.2. *Let X be a compact Klein surface of algebraic genus $p \geq 2$. Let $t\colon X \to X$ be an automorphism of order N of X and $\theta\colon \Gamma \to \mathbb{Z}_N$ be an epimorphism (3.8) of Γ given by (3.7), that uniformizes a $\mathbb{Z}_N$–action of t. Denote the images of elliptic generators of Γ*

(i) $\theta(x_i) = t^{v_i}$ for $1 \leq i \leq n$

Let us also reorder the reflections $c_{i,0}$ associated to empty period cycles of Γ in such a way they hold

(ii.1) $\theta(c_{i,0}) = 1$ *for* $1 \le i \le r \le \lambda$ *and* $\theta(c_{i,0}) = t^{N/2}$ *for* $r+1 \le i \le \lambda$.

Furthermore we split, as follows, the e–generators associated to empty period cycles into three groups

(ii.2)

$$\begin{aligned}
&\theta(e_i) = t^{v_{n+i}} \ \textit{for}\ 1 \le i \le r \le \lambda\\
&\theta(e_i) = t^{v_{n+i}} \ \textit{for}\ r+1 \le i \le r+v^+,\ \textit{where}\ \alpha_2(v_{n+i}) = \alpha_2(N)\\
&\theta(e_i) = t^{v_{n+i}} \ \textit{for}\ r+v^+ +1 \le i \le \lambda,\ \textit{where}\ \alpha_2(v_{n+i}) < \alpha_2(N).
\end{aligned}$$

Proceeding to non–empty period cycles of Γ we split the corresponding e–generators according to the following

(iii)

$$\begin{aligned}
&\theta(e_i) = t^{v_{n+i}} \ \textit{for}\ \lambda+1 \le i \le r+u^+,\ \textit{where}\ \alpha_2(v_{n+i}) = \alpha_2(N)\\
&\theta(e_i) = t^{v_{n+i}} \ \textit{for}\ \lambda+u^+ +1 \le i \le \lambda+p,\ \textit{where}\ \alpha_2(v_{n+i}) < \alpha_2(N).
\end{aligned}$$

Let us form the following families of numbers

$$\begin{aligned}
\mathcal{G}_1(\Gamma,\theta) &= \{(N, v_j) \mid 1 \le j \le n\}\\
\mathcal{G}_2(\Gamma,\theta) &= \{(N, v_{n+j}) \mid 1 \le j \le r\}\\
\mathcal{G}_3(\Gamma,\theta) &= \{(N/2, v_{n+j}) \mid r+1 \le j \le r+v^+\}\\
\mathcal{G}_4(\Gamma,\theta) &= \{(N/2, v_{n+j}) \mid r+v^+ +1 \le j \le \lambda\}\\
\mathcal{G}_5(\Gamma,\theta) &= \{(N/2, v_{n+j}) \mid \lambda+1 \le j \le \lambda+u^+\}\\
\mathcal{G}_6(\Gamma,\theta) &= \{(N/2, v_{n+j}) \mid \lambda+u^+ +1 \le j \le \lambda+p\}. \qquad (4.6)
\end{aligned}$$

Then the character of periods of t equals

$$\begin{aligned}
\mathrm{CPer}(t) =&(\mathcal{A}_1(t), \mathcal{A}_2(t), \mathcal{A}_3(t), \mathcal{A}_4(t), \mathcal{A}_5(t), \mathcal{A}_6(t))\\
=&(\pi(\mathcal{G}_1(\Gamma,\theta)), \pi(\mathcal{G}_2(\Gamma,\theta)), \pi(\mathcal{G}_3(\Gamma,\theta)), \pi(\mathcal{G}_4(\Gamma,\theta)), \pi(\mathcal{G}_5(\Gamma,\theta)), \pi(\mathcal{G}_6(\Gamma,\theta))), \quad (4.7)
\end{aligned}$$

where $\pi(\mathcal{G}_i(\Gamma,\theta))$ stands for the projection of family $\mathcal{G}_i(\Gamma,\theta)$, $i = 1, \ldots, 6$.

Remark 4.3. *Under the assumptions of Remark 4.2 to obtain families (4.6) it suffices to know the numbers* $\mathrm{ord}t^{v_j}$, $j = 1, \ldots, 6$ *and the type of periodic components they correspond to. We have*

$$\begin{aligned}
\mathcal{G}_i(\Gamma,\theta) &= \{(N, v_j)\} = \{N/\mathrm{ord}t^{v_j}\} = \{(\mathrm{ord}t^{v_j})^*\},\ i = 1, 2\\
\mathcal{G}_i(\Gamma,\theta) &= \{(\frac{N}{2}, v_j)\} = \{\frac{1}{2}(N, v_j)\} = \{\frac{1}{2}(\mathrm{ord}t^{v_j})^*\},\ i = 3, 5\\
\mathcal{G}_i(\Gamma,\theta) &= \{(\frac{N}{2}, v_j)\} = \{(N, v_j)\} = \{(\mathrm{ord}t^{v_j})^*\},\ i = 4, 6, \qquad (4.8)
\end{aligned}$$

where $\mathcal{G}_i(\Gamma,\theta)$ are families given by (4.6).

In order to be able to deal effectively with one of the main problems this thesis is aimed to solve i.e. compute the minimal area of a NEC group uniformizing a $\mathbb{Z}_N$–action we also need an opposite result to the one obtained in Remark 4.3. Namely, based on the character of periods $\mathfrak{C}_0$ of a $\mathbb{Z}_N$–action, we would like to derive the orders of the images of elliptic and e–generators of Γ. The required result is given in the corollary below.

Corollary 4.4. *Let X be a compact Klein surface of algebraic genus $p \geq 2$. Let $t\colon X \to X$ be an automorphism of order N of X fulfilling* $\mathrm{CPer}(t) = \mathfrak{C}_0 = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6)$. *Let $\theta'\colon \Gamma' \to \mathbb{Z}_N$ be an epimorphism uniformizing a $\mathbb{Z}_N$–action of t. Suppose that h is an elliptic or an e–generator corresponding to a periodic component of $\mathcal{S}(t)$. Then*

$$\mathrm{ord}\theta(h) \in \begin{cases} \mathcal{A}_1^*, & \text{if } h \text{ is elliptic} \\ \mathcal{A}_2^*, & \text{if } h \text{ induces a boundary component} \\ (2\mathcal{A}_3)^*, & \text{if } h \text{ induces a two sided oval} \\ \mathcal{A}_4^*, & \text{if } h \text{ induces an one sided oval} \\ (2\mathcal{A}_5)^*, & \text{if } h \text{ induces a two sided chain} \\ \mathcal{A}_6^*, & \text{if } h \text{ induces an one sided chain.} \end{cases}$$

Moreover for each of the families $\mathcal{G}_j(\Gamma', \theta')$, $j = 1, \ldots, 6$ we have

$$\pi(\mathcal{G}_j(\Gamma', \theta')) = \mathcal{A}_j. \tag{4.9}$$

Proof. The properties of $\mathrm{ord}\theta(h)$ can be easily derived from (4.8). The equalities (4.9) are forced by the assumption that θ' uniformizes a $\mathbb{Z}_N$–action given by $\mathfrak{C}_0$ and (4.7). □

By the last corollary the orders of images of all generators of the covering group, except the hyperbolic ones corresponding to the orbit genus are completely determined by the character of periods. This is a kind of rigidity we have mentioned at the beginning of this subsection.

4.2 General Remarks on Epimorphism onto $\mathbb{Z}_N$

In this section we investigate the properties of smooth epimorphisms from NEC groups onto the cyclic ones. We need them to construct epimorphisms uniformizing required characters of periods. The method of proving the existence of such epimorphisms is to construct them explicitly. Furthermore, we will show in the forthcoming sections that the measure of the resulting surface NEC group is the smallest among all surfaces of a given orientability character that admit a required $\mathbb{Z}_N$–action.

Note that also in [8] there were constructed some smooth epimorphisms from NEC groups onto $\mathbb{Z}_N$. However, since we focus on characters of periods of $\mathbb{Z}_N$–actions the choice of the right assignment is much more restricted. It is worth noting that similarly to the proof of Harvey's theorem [20] the key concept of construction of epimorphisms $\theta\colon \Gamma \to \mathbb{Z}_N$ will consist in satisfying the long relation on images of corresponding canonical generators of Γ.

4.2.1 The Order–Preserving Element

Let $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_5, \mathcal{D}_6)$ be a 6–tuple of families of divisors of N i.e.

$$\begin{aligned}(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_5, \mathcal{D}_6) = (\{d_{1,1}, \ldots, d_{1,k_1}\}, \{d_{2,1}, \ldots, d_{2,k_2}\},\\ \{d_{3,1}, \ldots, d_{3,k_3}\}, \{d_{4,1}, \ldots, d_{4,k_4}\}, \{d_{5,1}, \ldots, d_{5,k_5}\}, \{d_{6,1}, \ldots, d_{6,k_6}\}),\end{aligned} \tag{4.10}$$

where $d_{i,j} \in \mathcal{D}(N)$ and $\mathcal{D}_i$ admit elements with multiplicities. We introduce the notion of *order–preserving element* with respect to $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_5, \mathcal{D}_6)$.

(1) We say that an element

$$\begin{aligned}\overline{\eta} = (\{\eta_{1,1}, \ldots, \eta_{1,k_1}\}, \{\eta_{2,1}, \ldots, \eta_{2,k_2}\}, \{\eta_{3,1}, \ldots, \eta_{3,k_3}\},\\ \{\eta_{4,1}, \ldots, \eta_{4,k_4}\}, \{\eta_{5,1}, \ldots, \eta_{5,k_5}\}, \{\eta_{6,1}, \ldots, \eta_{6,k_6}\}),\end{aligned} \tag{4.11}$$

is order–preserving with respect to $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_5, \mathcal{D}_6\}$ if $(d^*_{i,j}, \eta_{i,j}) = 1$ for $d_{i,j} \in \mathcal{D}_i$. Note that in such a setting if $\langle t \rangle = \mathbb{Z}_N$, then $\operatorname{ord} t^{d_{i,j}} = \operatorname{ord} t^{\eta_{i,j} d_{i,j}}$.

(2) If $\mathfrak{C}_0 = (\mathcal{A}_1, \ldots, \mathcal{A}_6)$ is a character of periods of some $\mathbb{Z}_N$–action on a Klein surface, then any set of families (4.10) satisfying $\pi(\mathcal{D}_i) = \mathcal{A}_i$ will be called a *character associated to* $\mathfrak{C}_0$.

(3) For $\mathfrak{D} = (\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_5, \mathcal{D}_6)$ we call $\mathcal{D}_i$ an i–th section of character $\mathfrak{D}$.

For technical purposes we need the following construct

(4) If $\mathfrak{C}_0 = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6)$ is a character of periods, then we define

$$\mathfrak{C} = (\mathcal{A}_1, \mathcal{A}_2, 2\mathcal{A}_3, \mathcal{A}_4, 2\mathcal{A}_5, \mathcal{A}_6)$$

and we call it the *inflated character of periods.*

We also use a notion of character associated to inflated character of periods defined analogously to the above definition (2). Considering characters which may be associated to characters of periods or inflated characters of periods we use symbols that stress this dependency. We write $\mathfrak{D}(\mathfrak{C}_0)$ and $\mathfrak{D}(\mathfrak{C})$ for character associated to character of periods and inflated character of periods respectively.

The reason of using the notion of inflated character of periods is only technical and it is motivated by the following remark.

Remark 4.5. *If $\theta\colon \Gamma \to \mathbb{Z}_N$ uniformizes a $\mathbb{Z}_N$–action given by $\mathfrak{C}_0$, then the orders of images of elliptic and e–generators of Γ are given by d^*, where $d \in \mathfrak{C}$.*

Note that $\mathfrak{C}_0$ does not have the above property. Indeed, we may observe it for sections $\mathcal{A}_3$ and $\mathcal{A}_5$. If $d_0 \in \mathcal{A}_3$ or $\mathcal{A}_5$, then e–generator corresponding to d_0 verifies $\theta(e) = t^v$, where $\alpha_2(v) = \alpha_2(N)$. But $d_0 = (N/2, v) = (N, v)/2$, which gives $d_0^* = 2\mathrm{ord}\theta(e)$. See also (4.8) and Corollary 4.4 where those exceptional cases were mentioned for the first time.

Observe that $\mathfrak{C}_0$ may be also treated as a character associated to itself, since $\pi(\mathfrak{C}_0) = \mathfrak{C}_0$. Therefore, we may formally consider elements $\overline{\eta}$ that are order–preserving with respect to $\mathfrak{C}_0$. Recall, that due to the above definition (1) we must consider elements that are order–preserving with respect to 6–tuples of families. The same remark clearly relates to the inflated character of periods $\mathfrak{C}$. Moreover, it is worth noting that an element $\overline{\eta}$ that is order–preserving with respect to $\mathfrak{C}_0$ is also order–preserving with respect to $\mathfrak{C}$. The opposite relation does not hold.

As the illustration of the deployment of the new terminology we show in the example below how the order–preserving element with respect to a character associated to the inflated character of periods is determined by an epimorphism uniformizing a $\mathbb{Z}_N$–action.

Example 4.6. *Let X be a compact Klein surface of algebraic genus $p \geq 2$. Let $t\colon X \to X$ be an automorphism of order N of X fulfilling*

$$\mathrm{CPer}(t) = \mathfrak{C}_0 = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6) = (\{d_{1,j}\}, \{d_{2,j}\}, \{d_{3,j}\}, \{d_{4,j}\}, \{d_{5,j}\}, \{d_{6,j}\}).$$

Suppose that $\theta\colon \Gamma \to \mathbb{Z}_N$ is an epimorphism uniformizing a $\mathbb{Z}_N$–action of t. Let us form the below families of numbers according to points (i)–(iii) of Remark 4.2

$$\{v_j\}_{j=1}^{n}, \{v_{n+j}\}_{j=1}^{r}, \{v_{n+j}\}_{j=r+1}^{r+v^+}, \{v_{n+j}\}_{j=r+v^++1}^{\lambda}, \{v_{n+j}\}_{j=\lambda+1}^{\lambda+u^+}, \{v_{n+j}\}_{j=\lambda+u^++1}^{\lambda+p}. \tag{4.12}$$

Observe that by (4.6) we defined $(\mathcal{G}_1(\Gamma,\theta), \ldots, \mathcal{G}_6(\Gamma,\theta))$ which is a 6–tuple of families of divisors of N. By (4.7) it is also a character associated to $\mathfrak{C}_0$. Using the above families (4.12) we may get an element which is order–preserving with respect to a character associated to *the inflated character of periods $\mathfrak{C}$. Let us put*

$$\begin{aligned}
\eta_{1j} &= \frac{v_j}{(N,v_j)} && \text{for} \quad 1 \le j \le n \\
\eta_{2j} &= \frac{v_{n+j}}{(N,v_{n+j})} && \text{for} \quad 1 \le j \le r \\
\eta_{3j} = \frac{v_{n+j}}{(N,v_{n+j})} &= \frac{v_{n+j}}{2(\frac{N}{2},v_{n+j})} && \text{for} \quad r+1 \le j \le r+v^+ \\
\eta_{4j} &= \frac{v_{n+j}}{(N,v_{n+j})} && \text{for} \quad r+v^++1 \le j \le \lambda \\
\eta_{5j} = \frac{v_{n+j}}{(N,v_{n+j})} &= \frac{v_{n+j}}{2(\frac{N}{2},v_{n+j})} && \text{for} \quad \lambda+1 \le j \le \lambda+u^+ \\
\eta_{6j} &= \frac{v_{n+j}}{(N,v_{n+j})} && \text{for} \quad \lambda+u^++1 \le j \le \lambda+p
\end{aligned} \tag{4.13}$$

and define

$$\overline{\eta} = (\{\eta_{1,j}\}_{j=1}^{n}, \{\eta_{2,n+j}\}_{j=1}^{r}, \{\eta_{3,n+j}\}_{j=r+1}^{r+v^+}, \{\eta_{4,n+j}\}_{j=r+v^++1}^{\lambda}, \{\eta_{5,n+j}\}_{j=\lambda+1}^{\lambda+t^+}, \{\eta_{6,n+j}\}_{j=\lambda+t^++1}^{\lambda+p}).$$

We then have

$$\begin{aligned}
(d_{ij}^*,\eta_{ij}) &= (\frac{N}{(N,v_{j'})},\eta_{ij}) = (\frac{N}{(N,v_{j'})},\frac{v_{j'}}{(N,v_{j'})}) = 1, \ i=1,2 \\
((2d_{ij})^*,\eta_{ij}) &= (\frac{N}{2(\frac{N}{2},v_{j'})},\eta_{ij}) = (\frac{N}{(N,v_{j'})},\frac{v_{j'}}{(N,v_{j'})}) = 1, \ i=3,5 \\
(d_{ij}^*,\eta_{ij}) &= (\frac{N}{(\frac{N}{2},v_{j'})},\eta_{ij}) = (\frac{N}{(N,v_{j'})},\frac{v_{j'}}{(N,v_{j'})}) = 1,
\end{aligned}$$

where the respective ranges for j' follow from inequalities given by (4.13).

Remark 4.7. *Under the assumptions of Example 4.6*

$$(\mathcal{G}_1(\Gamma,\theta),\mathcal{G}_2(\Gamma,\theta),2\mathcal{G}_3(\Gamma,\theta),\mathcal{G}_4(\Gamma,\theta),2\mathcal{G}_5(\Gamma,\theta),\mathcal{G}_6(\Gamma,\theta))$$

forms a character associated to $\mathfrak{C}$.

4.2.2 Conditions for Epimorphism

We now proceed to show the role that the notion of order–preserving element plays in the construction of epimorphism from NEC groups onto $\mathbb{Z}_N$. Let us assume that $\mathfrak{C}_0$ is a character of periods describing a $\mathbb{Z}_N$–action on a Klein surface. Suppose also that $\theta\colon \Gamma \to \mathbb{Z}_N$ uniformizes the above $\mathbb{Z}_N$–action. As it was shown in Example 4.6 the elliptic generators as well as e–generators of Γ are mapped to elements $t^{\eta d}$, $\langle t\rangle = \mathbb{Z}_N$, where $(d^*,\eta)=1$ and $d \in \mathfrak{C}$. Using this notation we may define for $\theta\colon \Gamma \to \mathbb{Z}_N$ a number

$$L(\overline{\eta}) = \sum_{i,j} \eta_{i,j} d_{i,j}. \tag{4.14}$$

Recall that epimorphism $\theta\colon \Gamma \to \mathbb{Z}_N$ must keep the "long relation", given by (3.2) and (3.3) for Γ orientable and non–orientable respectively, on images of canonical generators of Γ. Thus, we see that

$$L(\overline{\eta}) \equiv 0 \bmod N \qquad \text{if } \operatorname{sign}\Gamma = \text{"} + \text{"} \tag{4.15}$$
$$L(\overline{\eta}) \equiv -2S \bmod N \qquad \text{otherwise,} \tag{4.16}$$

where $\theta(g_1 \dots g_\gamma) = t^S$. Consequently, we are focused on order preserving elements $\overline{\eta}$ with respect to characters associated to $\mathfrak{C}$ fulfilling (4.15) or (4.16) in the respective cases. Note that the number $L(\overline{\eta})$ may be defined provided we know only how θ maps the elliptic and e–generators.

It is worth pointing out that omitting in (4.11) and (4.14) the elements involving $d_{i,j}$ for $i \geq 3$ i.e. excluding from the singular set the one–dimensional components other than boundaries, we obtain a definition of *order–preserving pair* given in [8]. Using the notation of [8] we have $\mathcal{D}_1 = \alpha$, $\mathcal{D}_2 = \beta$ and $L(\mathcal{D}_1, \mathcal{D}_2, \emptyset, \emptyset, \emptyset, \emptyset) = S(\alpha, \beta)$.

Below we give two arithmetic lemmas that establish basic constraints on characters associated to $\mathfrak{C}$ which enable one to find a required $\overline{\eta}$. We will apply the first result mainly in case when a covering group is non–orientable. Roughly speaking, this lemma states that if $\mathfrak{D}(\mathfrak{C})$ is a character associated to $\mathfrak{C}$ and $\overline{\eta}$ is an order–preserving element with respect to the character $\mathfrak{D}(\mathfrak{C})$, then the parity of $L(\overline{\eta})$ does not depend on the choice of $\overline{\eta}$. Thus it is determined only by $\mathfrak{D}(\mathfrak{C})$.

Denote by m the order of image of an x– or e–generator in $\mathbb{Z}_N$ and recall that we consider $\mathbb{Z}_N$–action prescribed by character of periods $\mathfrak{C}_0$. By Remark 4.5, orders m and $d \in \mathfrak{C}$ are related by $m^* = d$, where d belong to the respective sections of $\mathfrak{C}$.

Lemma 4.8. *Let N be even and let $m_1, \dots, m_k$ be positive integers. Then for each sequence of integers $(\eta_i)_{i=1}^k$, satisfying $\operatorname{ord}(t^{\eta_i m_i^*}) = m_i$ we have*

$$\sharp\{m_i \mid \alpha_2(m_i) = \alpha_2(N)\} \equiv \sum_{j=1}^{k} \eta_j m_j^* \bmod 2. \tag{4.17}$$

Proof. We begin by proving that $\eta_i m_i^* \equiv m_i^* \bmod 2$. Indeed, $2 \nmid m_i^*$ implies $\alpha_2(m_i) = \alpha_2(N)$. Since $(\eta_i, m_i) = 1$ we obtain $2 \nmid \eta_i$, hence we conclude that also $2 \nmid \eta_i m_i^*$. Obviously $2 \nmid \eta_i m_i^*$ clearly forces that m_i^* is odd. It follows that the number of odd summands on the right–hand side of sum (4.17) equals the number of m_i which have the property $\alpha_2(m_i) = \alpha_2(N)$. $\square$

Lemma 4.9. *Let $4 \mid N$ and let $m_1, \dots, m_k$ be positive integers satisfying $\alpha_2(m_i) < \alpha_2(N)$. Then for each sequence of integers $(\eta_i)_{i=1}^k$ which verify $\operatorname{ord}(t^{\eta_i m_i^*}) = m_i$ we have*

$$\sum_{j=1}^{k} \eta_j m_j^* \equiv 2 \cdot \sharp\{m_i \mid \alpha_2(m_i) = \alpha_2(N) - 1\} \bmod 4.$$

Proof. It is enough to observe that $\alpha_2(m_i) = \alpha_2(N) - 1$ yields $\alpha_2(m_i^*) = \alpha_2(\eta_i m_i^*) = 1$. The rest of the proof is straightforward. $\square$

Corollary 4.10. *Let $4 \mid N$ and let $m_1, \ldots, m_k$ be positive integers. Suppose that the set $\{m_i \mid \alpha_2(m_i) = \alpha_2(N)\}$ is non–empty and has an even cardinality. Then there exist sequences of integers $(\eta_i)_{i=1}^k$ and $(\eta'_i)_{i=1}^k$ such that*

$$(i)\ \mathrm{ord}(t^{\eta_i m_i^*}) = m_i \text{ and } \sum_{j=1}^{k} \eta_j m_j^* \equiv 0 \mod 4$$

$$(ii)\ \mathrm{ord}(t^{\eta'_i m_i^*}) = m_i \text{ and } \sum_{j=1}^{k} \eta'_j m_j^* \equiv 2 \mod 4.$$

Proof. Take $\eta_1 = \ldots = \eta_k = 1$. By Lemma 4.8 the sum $\sum_{j=1}^k m_j^*$ is even. Assume that $\sum_{j=1}^k m_j^*$ is also divisible by 4. Denote by m_l, $l \le k$ an element satisfying $\alpha_2(m_l) = \alpha_2(N)$. In order to obtain a sum that is not divisible by 4 we switch the respective factor η_l to $N-1$, that is we consider $\eta'_l = N-1$ with the remaining $\eta'_j = 1$, $j \neq l$. We then have

$$\sum_{j=1}^{k} \eta' m_j^* \equiv \sum_{j\neq l} m_j^* + (N-1)m_l^* \equiv \sum_{j=1}^{k} m_j^* - m_l^* + (N-1)m_l^* \equiv \sum_{j=1}^{k} m_j^* + 2 \mod 4.$$

If $\sum_{j=1}^k m_j^*$ is not divisible by 4, then obviously $\sum_{j=1}^k \eta' m_j^*$ defined above is a multiple of 4. In each of the cases we have found both sequences $(\eta_i)_{i=1}^k$ and $(\eta'_i)_{i=1}^k$, as required. □

The third lemma has been already cited in the first part of the thesis, see Theorem 2.1. We recall it once more, just to formulate it within the actual context.

(1) We say that the set of positive integers $\{m_1, \ldots, m_k\}$ verifies the *elimination property* if
$$\mathrm{lcm}(m_1, \ldots, \overline{m}_i, \ldots, m_k) = \mathrm{lcm}(m_1, \ldots, m_k)$$
for each $i = 1, \ldots, k$, where $\overline{m}_i$ denotes the omission of m_i. We also adopt the convention lcm of the empty set is 1. Thus $\{m_1\}$ has the elimination property if and only if $m_1 = 1$.

Lemma 4.11 (Harvey [20], Theorem 4, Bujalance et al. [8], Lemma 3.1.1)**.** *Let $m_1, \ldots, m_k$ be positive integers and denote $M = \mathrm{lcm}(m_1, \ldots, m_k)$. The following statements are equivalent:*

(i) for each multiple N of M, there exist $\eta_1, \ldots, \eta_k$ such that

$$\mathrm{ord}(t^{\eta_i m_i^*}) = m_i \text{ and } \sum_{j=1}^{k} \eta_j m_j^* \equiv 0 \mod N,$$

where $m_i^* = N/m_i$, $i = 1, \ldots, k$

(ii) for each $i = 1, \ldots, k$, $\mathrm{lcm}(m_1, \ldots, \overline{m}_i, \ldots, m_k) = M$, and, if $2 \mid M$, the number of m_i which are multiple of $2^{\alpha_2(M)}$ is even.

Below we recall a theorem which plays an important role while determining the orientability character of normal subgroups of a NEC group. First we need a few definitions. Let Λ' be a normal subgroup of a NEC group Γ.

(2) A canonical generator of Γ is *proper* (with respect to Λ') if it does not belong to Λ'.

(3) The elements of Γ expressible as composition of proper generators of Γ are the *words* of Γ (with respect to Λ').

(4) A given word is *orientable* if it preserves the orientation of $\mathbb{H}^2$. Otherwise is *non–orientable.*

Theorem 4.12 (Bujalance et al. [8], Theorem 2.1.3)**.** *Let Λ' be a normal subgroup of a NEC group Γ with an even index N.*

(i) Let us suppose that Γ is orientable. Then Λ' is orientable if and only if no non–orientable word belongs to Λ'.

(ii) Let us suppose that Γ is non–orientable. Then Λ' is non–orientable if and only if either a glide reflection of the canonical generators of Γ or a non–orientable word belongs to Λ'.

We finish this subsection with a simple lemma concerning a way of identifying $\mathbb{Z}_N$–actions on Klein surfaces by fixing the character of periods $\mathfrak{C}_0$. We will show series of maps that share the same character of periods. Anyway, such a relation between maps is clearly less restrictive than the classification up to topological conjugacy since we do not even assume that maps act on surfaces of the same topological type.

Proposition 4.13. *Let X be a compact Klein surface of algebraic genus $p \geq 2$. Let $t\colon X \to X$ be an automorphism of order N of X. Suppose that $(\Gamma, \mathbb{H}^2)$ is a universal covering transformation group of $(\langle t\rangle, X)$ and $\theta\colon \Gamma \to \mathbb{Z}_N$ is an epimorphism that uniformizes a $\mathbb{Z}_N$–action of t. Denote by J an elliptic canonical generator, an empty period cycle or a non–empty period cycle of Γ. Then for any of the following conditions, (i) and (ii), there is a universal covering transformation group $(\Gamma', \mathbb{H}^2)$ of $(\langle t'\rangle, X')$ and an epimorphism $\theta'\colon \Gamma' \to \mathbb{Z}_N$ such that* $\mathrm{CPer}(t) = \mathrm{CPer}(t')$. *A NEC group Γ' is of the form*

$$\Gamma' = (\gamma'; \pm; [m'_1, ..., m'_{n'}]; \{(\ )^{\lambda'}(2^{\mu'_1})...(2^{\mu'_{p'}})\}),$$

where

(i) $\gamma' > \gamma$

(ii) the number of $J's$ in the signature of Γ' is greater than the number of $J's$ in the signature of Γ.

Moreover the groups Γ and Γ' (respectively $\ker\theta$ and $\ker\theta'$) have the same orientability character.

Proof. Denote $\mathrm{CPer}(t) = \mathfrak{C}_0$. Let us show that the genus of Γ can be increased without affecting the underlying $\mathbb{Z}_N$–action described by means of the character of periods $\mathfrak{C}_0$. Denote

$$\Gamma = (\gamma; \pm; [m_1, ..., m_n]; \{(\ )^\lambda(2^{\mu_1})...(2^{\mu_p})\}).$$

Assume first that the sign$\Gamma = " + "$. In such case we define Γ' as follows

$$\Gamma' = (\gamma + 1; +; [m_1, ..., m_n]; \{(\ )^\lambda(2^{\mu_1})...(2^{\mu_p})\}).$$

We construct epimorphism $\theta' \colon \Gamma' \to \mathbb{Z}_N$ mapping both additional hyperbolic generators $a_{\gamma+1}$ and $b_{\gamma+1}$ to 1. On the remaining canonical generators of Γ' the epimorphism θ' is equal to θ. Observe that we have $\operatorname{sign}\ker\theta' = \operatorname{sign}\ker\theta$ since both $a_{\gamma+1}$ and $b_{\gamma+1}$ are orientable. As we saw in subsection 4.1.2 character of periods of the underlying $\mathbb{Z}_N$ action does not depend on the images of hyperbolic generators corresponding to the orbit genus of a Klein surface. Hence $\mathrm{CPer}(t') = \mathfrak{C}_0$ as required.

On the other hand, if sign$\Gamma = " - "$, then we construct the signature of Γ' by repeating twice each and every canonical generator of Γ. Thus Γ' will posses the following generators

$$\begin{array}{ll} \{x_1, \ldots, x_n, e_1, \ldots, e_{\lambda+p}, & \{g_1, \ldots, g_\gamma, \\ x_{1+A}, \ldots, x_{n+A}, e_{1+A}, \ldots, e_{\lambda+p+A}\}, & g_{1+A}, \ldots, g_{\gamma+A}\}, \end{array}$$

where $A = \gamma + n + \lambda + p$. Clearly, with the generators e_{k+A}, $k = 1, \ldots, \lambda + p$ we add all associated reflections

$$\begin{array}{ll} \{c_{1,0}, \ldots, c_{\lambda,0}, & \{c_{\lambda+l,j}, \\ c_{1+A,0}, \ldots, c_{\lambda+A,0}\} & c_{\lambda+l+A,j}\},\ l = 1, \ldots, p,\ j = 0, \ldots, \mu_l. \end{array}$$

We define θ' by extending θ by the following assignment

$$\begin{array}{ll} \theta'(x_{j+A}) = \theta(x_j),\ j = 1, \ldots, n & \\ \theta'(e_{l+A}) = \theta(e_l), & \theta'(c_{l+A,0}) = \theta(c_{l,0}),\ l = 1, \ldots, \lambda \\ \theta'(e_{\lambda+l+A}) = \theta(e_{\lambda+l}), & \theta'(c_{\lambda+l+A,j}) = \theta(c_{\lambda+l,j}),\ l = 1, \ldots, p,\ j = 0, \ldots, \mu_l \\ \theta'(g_{i+A}) = \theta(g_i),\ i = 1, \ldots, \gamma. & \end{array}$$

For the remaining generators h of Γ', that belong also to Γ, we put $\theta'(h) = \theta(h)$.

Let w' be a word of Γ' with respect to $\ker\theta'$. Note that we may find another word w that is formed only by canonical generators of Γ' with indexes lower than A and verifies $\theta'(w') = \theta'(w)$. We do it simply by subtracting the number A from the indexes of canonical generators that are of the form $i + A$. Hence $w \in \Gamma$ and $\theta'(w) = \theta(w)$. By this argument and Remark 4.2 we have $\mathrm{CPer}(t') = \mathfrak{C}_0$. Furthermore, observe that w is a word with respect to $\ker\theta$ and it has the same orientability character as w'. Thus by point *(ii)* of Theorem 4.12 we eventually obtain $\operatorname{sign}\ker\theta' = \operatorname{sign}\ker\theta$.

We now turn to show that also in case the character of periods is fixed we may still modify a covering group so that the multiplicity of each component of the singular set increases. We achieve this by adding a number of copies of J, i.e. an elliptic generator or an empty, or non–empty period cycle and mapping them on the appropriate elements of $\mathbb{Z}_N$. We give the proof only for the case of J being a non–empty period cycle, i.e. $J = (2^{\mu_i})$ for a fixed i fulfilling $1 \leq i \leq p$. The other cases may be proved in much the same way. We set $\mathrm{sign}\Gamma' = \mathrm{sign}\Gamma$ and put

$$\Gamma' = (\gamma; \pm; [m_1, ..., m_n]; \{(\ )^{\lambda}(2^{\mu_1}) \ldots \underbrace{(2^{\mu_{i,1}}) \ldots (2^{\mu_{i,\mathrm{ord}\theta(e_i)}})}_{\mathrm{ord}\theta(\mathrm{e_i})\ \text{times}} \ldots (2^{\mu_p})\})$$

i.e. we extend Γ by $\mathrm{ord}\theta(e_i)$ period cycles of the form

$$(2^{\mu_{i,k}}) = \underbrace{(2 \ldots 2)}_{\mu_{i,k}}, \ \text{where } \mu_i = \mu_{i,k},\ \ k = 1, \ldots, \mathrm{ord}\theta(e_i).$$

We map the elements of the repeated period cycles $(2^{\mu_{i,k}})$ in the analogous way as θ does with (2^{μ_i}). We define

$$\begin{aligned} &\theta'(c_{i,0}) = \theta'(c_{i,2}) = \ldots = \theta'(c_{i,\mu_{i,k}}) = t^{N/2}, \\ &\theta'(c_{i,1}) = \ldots = \theta'(c_{i,\mu_{i,k}-1}) = 1, \qquad k = 1, \ldots, \mathrm{ord}\theta(e_i) \end{aligned}$$

Moreover, we let all other generators of Γ' be mapped by θ' to their respective images under θ. Since for each word w' of Γ' with respect to $\ker\theta'$ we find a word w of Γ with respect to $\ker\theta$ of the same orientability which also verifies $\theta'(w') = \theta'(w) = \theta(w)$, then we again obtain both equalities $\mathrm{CPer}(t') = \mathfrak{C}_0$, by Remark 4.2, and $\mathrm{sign}\ker\theta = \mathrm{sign}\ker\theta'$ by Theorem 4.12. □

4.3 Prototypes of Covering Groups

Since the global study of the dynamics of dianalytic self–maps of Klein surfaces we deal with in this thesis, involves a quite large number of subcases, in the present section we isolate methods which will be reused many times in Chapter 5. In order to ease application of this approach we introduce a new notation for the inflated character of periods writing

$$\mathfrak{C} = (\mathcal{A}_1, \mathcal{A}_2, 2\mathcal{A}_3, \mathcal{A}_4, 2\mathcal{A}_5, \mathcal{A}_6) = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6).$$

All subsequent definitions apply to $\mathfrak{C}$ as denoted above.

(1) Let $N = p_1^{r_1} \dots p_n^{r_n}$, thus $\alpha_{p_i}(N) = r_i$.

(2) Let $\mathrm{lcm}\mathfrak{C}^* = \mathrm{lcm}\{\mathcal{B}_1^*, \mathcal{B}_2^*, \mathcal{B}_3^*, \mathcal{B}_4^*, \mathcal{B}_5^*, \mathcal{B}_6^*\} = \mathrm{lcm}\bigcup_{i=1}^{6} \mathcal{B}_i^*$.

(3) Let $\mathcal{E}_{p_i}^*(\mathfrak{C}^*)$ be the set of elements of $\mathfrak{C}^*$ divisible by the maximum power of the prime factor p_i i.e. $\mathcal{E}_{p_i}^*(\mathfrak{C}^*) = \{m \in \mathfrak{C}^* \mid \alpha_{p_i}(m) = \alpha_{p_i}(\mathrm{lcm}\mathfrak{C}^*)\}$.

(4) If there is only one element in the set $\mathcal{E}_{p_i}^*(\mathfrak{C}^*)$ we call it an *isolated element* and define $F^*(\mathfrak{C}^*)$ to be the set of all isolated elements of $\mathfrak{C}^*$ i.e. $F^*(\mathfrak{C}^*) = \{m \in \mathfrak{C}^* \mid \exists i\ \mathcal{E}_{p_i}^*(\mathfrak{C}^*) = \{m\}\}$.

(5) Let $C^*(\mathfrak{C}^*)$ be the set of elements of $\mathfrak{C}^*$ that are divisible by the maximum power of 2 but are not isolated i.e. $C^*(\mathfrak{C}^*) = \mathcal{E}_2^*(\mathfrak{C}^*) \setminus F^*(\mathfrak{C}^*)$.

In the forthcoming sections we show how to construct signatures of NEC groups and appropriate epimorphisms onto $\mathbb{Z}_N$, which cover a $\mathbb{Z}_N$–action described by the character of periods $\mathfrak{C}_0$. It is worth noting that the construction of the required NEC signatures involves characters associated to $\mathfrak{C}$ on which we do the $*$–operation (see also Remark 4.5). One of the key points of our construction consists in verifying whether the restrictions imposed by Lemmas 4.8 and 4.11 are obeyed. In order to obtain the required epimorphism we find an element which is order–preserving with respect to a character associated to $\mathfrak{C}$. We always start with the inflated character of periods since, as it was observed before, $\mathfrak{C}$ may be treated as a character associated to $\mathfrak{C}$ itself. If $\mathfrak{C}$ does not give rise to the appropriate epimorphism we extend it by repeating some of the elements of $\mathcal{B}_i$, $i = 1, \dots, 6$. Although there are many ways to extend the inflated character of periods $\mathfrak{C}$ we use only a few definite algorithms.

We use 4 different procedures that are split into two categories. We call them $\mathfrak{O}$, $\mathfrak{N}$, $\mathfrak{N}_0$ and $\mathfrak{N}_{-1}$. We use the first one in case a covering NEC group is orientable and the remaining three if a covering group is non–orientable. We postpone their exact formulation for the next two subsections although here we give basic ideas. The main problem while modifying the inflated character of periods $\mathfrak{C}$ is the choice of elements to repeat, since this choice will be reflected in the measure of fundamental regions of covering groups. By the Riemann-Hurwitz formula we have $\mu(\Lambda) = N\mu(\Gamma)$, thus in order to minimize the area of surface group Λ we consider equivalently the area of Γ. We present a series of results concerning the analysis on how this measure varies.

operation to be done on the signature of Γ	section of $\mathfrak{C}$ which the added element belongs to	$\frac{\mu(\Gamma')-\mu(\Gamma)}{2\pi}$
to increase γ by 1	-	α^1
to add an elliptic generator of order m	$\mathcal{B}_1$	$1-\frac{1}{m}$
to add an empty period cycle	$\mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$	1
to add a non–empty period cycle $(2)^\mu$	$\mathcal{B}_5, \mathcal{B}_6$	$1+\frac{\mu}{4}$
to extend a non–empty period cycle by adding $\mu'-\mu$ reflections	-	$\frac{\mu'-\mu}{4}$

Table 4.1: Information on the increase of measure of the NEC group Γ.

Assume that $\theta\colon \Gamma \to \mathbb{Z}_N$ covers a $\mathbb{Z}_N$–action given by $\mathfrak{C}_0$. By (3.5) the area of NEC group Γ of the form (3.7) is equal to

$$\begin{aligned}\mu(\Gamma) &= 2\pi(\alpha\gamma+\lambda+p-2+\sum_{i=1}^{n}(1-m_i^{-1})+2^{-1}\sum_{i=1}^{p}\sum_{j=1}^{\mu_i}(1-2^{-1}))\\ &= 2\pi(\alpha\gamma+\lambda+p-2+\sum_{i=1}^{n}(1-m_i^{-1})+4^{-1}\sum_{i=1}^{p}\mu_i), \qquad (4.18)\end{aligned}$$

where $\alpha=2$ if $\operatorname{sign}(\sigma)=" + "$ and $\alpha=1$ otherwise. Denote by $\theta'\colon \Gamma'\to\mathbb{Z}_N$ another epimorphism covering the same $\mathbb{Z}_N$–action as above. Below we evaluate the difference between $\mu(\Gamma')$ and $\mu(\Gamma)$ provided we may obtain signature of Γ' by operations on signature of Γ which are defined in the first column of the following table.

Denote $\mathcal{B}_{234}=\mathcal{B}_2\cup\mathcal{B}_3\cup\mathcal{B}_4$ and $\mathcal{B}_{56}=\mathcal{B}_5\cup\mathcal{B}_6$. As we conclude from Table 4.1 in order to minimize the value of $\mu(\Gamma')-\mu(\Gamma)$ we shall require that elements we repeat belong to $\mathcal{B}_1$ rather than to $\mathcal{B}_{234}$ or $\mathcal{B}_{56}$. Analogously, we prefer to extend Γ by adding an empty period cycle (elements belonging to $\mathcal{B}_{234}$), than by adding a non–empty period cycle (elements belonging to $\mathcal{B}_{56}$). However we require that our extensions will satisfy also other conditions which follow from results of [8].

All procedures presented in this section consist in repeating some elements of $\mathfrak{C}$. Note that the selection of elements to repeat is always limited to certain families of divisors of N. Below we present which families become important while considering the respective procedures and which elements will play special roles.

The first procedure $\mathfrak{O}$ is devoted to construct a covering group assuming it is orientable. A special element for this procedure is the one belonging to $\mathcal{E}_2^*(\mathfrak{C}^*)$ which corresponds to an operation listed in Table 4.1 related to the smallest value $\mu(\Gamma') - \mu(\Gamma)$. We denote it by

$$o^*(\mathfrak{C}^*) \;=\; \begin{cases} \min(\mathcal{B}_1^* \cap \mathcal{E}_2^*(\mathfrak{C}^*)) & \text{if } \mathcal{B}_1^* \cap \mathcal{E}_2^*(\mathfrak{C}^*) \neq \emptyset \\ \min(\mathcal{B}_{234}^* \cap \mathcal{E}_2^*(\mathfrak{C}^*)) & \text{if } \mathcal{B}_{234}^* \cap \mathcal{E}_2^*(\mathfrak{C}^*) \neq \emptyset \text{ and} \\ & \mathcal{B}_1^* \cap \mathcal{E}_2^*(\mathfrak{C}^*) = \emptyset \\ \min \mathcal{E}_2^*(\mathfrak{C}^*) & \text{otherwise.} \end{cases} \tag{4.19}$$

Furthermore, on account of the above construct we define

$$G^*(\mathfrak{C}^*) = \begin{cases} \{o^*(\mathfrak{C}^*)\} & \text{if } 2 \nmid \sharp C^*(\mathfrak{C}^*) \\ \emptyset & \text{if } 2 \mid \sharp C^*(\mathfrak{C}^*). \end{cases}$$

Thus $G^*(\mathfrak{C}^*)$ is either empty or singleton.

The remaining procedures $\mathfrak{N}$, $\mathfrak{N}_0$ and $\mathfrak{N}_{-1}$ take into account the case when Γ is non–orientable. We shall need two following families derived from $\mathfrak{C}$

$$\begin{aligned} \mathcal{N}_0^*(\mathfrak{C}^*) &= \{m \in \mathfrak{C}^* \mid \alpha_2(m) = \alpha_2(N)\} \\ \mathcal{N}_{-1}^*(\mathfrak{C}^*) &= \{m \in \mathfrak{C}^* \mid \alpha_2(m) = \alpha_2(N) - 1\}. \end{aligned} \tag{4.20}$$

The former one, $\mathcal{N}_0^*(\mathfrak{C}^*)$, will be considered when applying procedure $\mathfrak{N}_0$ and the latter one in case of procedure $\mathfrak{N}_{-1}$. In both cases we also define special elements. Likewise for $\mathfrak{O}$, these will be the ones belonging to the respective families $\mathfrak{N}_0$, $\mathfrak{N}_{-1}$ which corresponds to operations listed in Table 4.1 related to the smallest value $\mu(\Gamma') - \mu(\Gamma)$. For $i = -1$ and $i = 0$ we put

$$n_i^*(\mathfrak{C}^*) \;=\; \begin{cases} \min(\mathcal{B}_1^* \cap \mathcal{N}_i^*(\mathfrak{C}^*)) & \text{if } \mathcal{B}_1^* \cap \mathcal{N}_i^*(\mathfrak{C}^*) \neq \emptyset \\ \min(\mathcal{B}_{234}^* \cap \mathcal{N}_i^*(\mathfrak{C}^*)) & \text{if } \mathcal{B}_{234}^* \cap \mathcal{N}_i^*(\mathfrak{C}^*) \neq \emptyset \text{ and} \\ & \mathcal{B}_1^* \cap \mathcal{N}_i^*(\mathfrak{C}^*) = \emptyset \\ \min \mathcal{N}_i^*(\mathfrak{C}^*) & \text{otherwise.} \end{cases} \tag{4.21}$$

In the next subsections 4.3.1 and 4.3.2 we deal with procedures $\mathfrak{O}$, $\mathfrak{N}$, $\mathfrak{N}_0$ and $\mathfrak{N}_{-1}$ in much more detail.

Before we start with technical arguments based on intrinsic calculations, we make a simple remark related to an operation listed in the last row of Table 4.1. First we need a lemma.

Lemma 4.14. *Let X be a compact Klein surface of algebraic genus $p \geq 2$. Let $t\colon X \to X$ be an automorphism of order N of X. Denote*

$$\Gamma = (\gamma; \pm; [m_1, ..., m_n]; \{(\ )^\lambda (2^{\mu_1}) ... (2^{\mu_p})\}),$$

and suppose that $\theta\colon \Gamma \to \mathbb{Z}_N$ is an epimorphism that uniformizes a $\mathbb{Z}_N$–action of t on X and $\mathrm{CPer}(t) = \mathfrak{C}_0$. *If the following group Γ'*

$$\Gamma' = (\gamma; \pm; [m_1, ..., m_n]; \{(\ )^\lambda \underbrace{(2^2) \ldots (2^2)}_{\mathrm{p}}\}),\ \ \mathrm{sign}\Gamma' = \mathrm{sign}\Gamma \tag{4.22}$$

is a NEC group, then there is an epimorphism covering a $\mathbb{Z}_N$–action on a Klein surface X' given by $\mathfrak{C}_0$. Furthermore both surfaces X and X' have the same orientability character.

Proof. Let h be a canonical generator of Γ'. Since h belongs also to Γ we may define the following assignment

$$\theta'(h) = \theta(h).$$

By the above we see that θ' is smooth and $\theta'(\Gamma') = \theta(\Gamma) = \mathbb{Z}_N$. The equality $\operatorname{sign}\ker\theta = \operatorname{sign}\ker\theta'$ can be shown by arguments used in the proof of Proposition 4.13 while proving an analogous relation. □

Corollary 4.15. *Under the assumptions of the last corollary we have $\mu(\Gamma') \leq \mu(\Gamma)$.*

What makes our study more interesting is the fact that we do solve the problem of minimizing the area of NEC groups uniformizing a $\mathbb{Z}_N$–action by groups of the form (4.22). It is worth pointing out that the length of a non–empty period cycle of a covering NEC group results only in the length of the induced chain and not in its period.

4.3.1 Procedure $\mathfrak{O}$ for Orientable Covering Groups

For simplicity of notation we will write $F(\mathfrak{C}^*)$, $G(\mathfrak{C}^*)$, $o(\mathfrak{C}^*)$ instead of $(F^*(\mathfrak{C}^*))^*$, $(G^*(\mathfrak{C}^*))^*$ and $(o^*(\mathfrak{C}^*))^*$ respectively.

Observe that the family $\{\mathfrak{C}^*, F^*(\mathfrak{C}^*), G^*(\mathfrak{C}^*)\}$ satisfies condition *(ii)* of Lemma 4.11. First of all it has the elimination property since $\sharp\mathcal{E}^*_{p_i}(\mathfrak{C}^*) + \sharp\mathcal{E}^*_{p_i}(F^*(\mathfrak{C}^*)) \geq 2$. Furthermore we have

$$\sharp\mathcal{E}_2^*(\{\mathfrak{C}^*, F^*(\mathfrak{C}^*), G^*(\mathfrak{C}^*)\}) = 2\sharp(\mathcal{E}_2^*(\mathfrak{C}^*) \cap F^*(\mathfrak{C}^*)) + \sharp C^*(\mathfrak{C}^*) + \sharp G^*(\mathfrak{C}^*). \tag{4.23}$$

Thus the number $\sharp\mathcal{E}_2^*(\{\mathfrak{C}^*, F^*(\mathfrak{C}^*), G^*(\mathfrak{C}^*)\})$ is always even. By Lemma 4.11 we may construct a character associated to $\mathfrak{C}$ and an order preserving element with respect to it. This enable us to define an epimorphism into $\mathbb{Z}_N$ which covers a $\mathbb{Z}_N$–action given by $\mathfrak{C}_0$, provided a covering NEC group is orientable.

We shall proceed as follows. First, we extend the sets $\mathcal{B}_i$, $i = 1, \ldots, 6$ by the elements of $F(\mathfrak{C}^*)$. We form a 6–tuple of families

$$\mathcal{O}_i = \{\mathcal{B}_i, F(\mathfrak{C}^*) \cap \mathcal{B}_i\},\ i = 1, \ldots, 6. \tag{4.24}$$

The second transition in the actual procedure is applied if and only if $G^*(\mathfrak{C}^*) \neq \emptyset$ and consists in adding the element $o(\mathfrak{C}^*)$ given by (4.19) to exactly one of the families $\mathcal{O}_i$. We choose the appropriate family $\mathcal{O}_j$ by the requirement that j is the lowest number satisfying $o(\mathfrak{C}^*) \in \mathcal{B}_j$. In this manner we extend at most one of the families $\mathcal{O}_i$. For abbreviation we do not introduce a new notation and let $\mathcal{O}_i$, $i = 1, \ldots, 6$ stand for the families obtained by the above two–stage extension. Hence we obtain $\mathfrak{O}(\mathfrak{C})$ a character associated to $\mathfrak{C}$

$$\mathfrak{O}(\mathfrak{C}) = (\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_5, \mathcal{O}_6). \tag{4.25}$$

The character $\mathfrak{O}(\mathfrak{C})$ yields the numbers of elliptic and e–generators of a covering NEC group we are about to construct. These numbers equal $\sharp\mathcal{O}_1$ and $\sum_{i=2}^6 \sharp\mathcal{O}_i$ respectively. Observe that $\mathfrak{O}(\mathfrak{C})$ comprises $\sum_{i=1}^6 \sharp\mathcal{O}_i = \sharp\{\mathfrak{C}^*, F^*(\mathfrak{C}^*), G^*(\mathfrak{C}^*)\}$ integers and by Lemma 4.11 there is a sequence $(\eta_j)_{j=1}^{\sum_{i=1}^6 \sharp\mathcal{O}_i}$ by which we easily obtain an element

$$\begin{aligned}
\overline{\eta} &= (\{\eta_1, \ldots, \eta_{\sharp\mathcal{O}_1}\}, \ldots, \{\eta_{\sum_{i=1}^5 \sharp\mathcal{O}_i+1}, \ldots, \eta_{\sum_{i=1}^6 \sharp\mathcal{O}_i}\}) \\
&= (\{\eta_{1,1}, \ldots, \eta_{1,\sharp\mathcal{O}_1}\}, \ldots, \{\eta_{6,\sum_{i=1}^5 \sharp\mathcal{O}_i+1}, \ldots, \eta_{6,\sum_{i=1}^6 \sharp\mathcal{O}_i}\})
\end{aligned}$$

which is order–preserving with respect to $\mathfrak{O}(\mathfrak{C})$. It enables us to define an epimorphism onto $\mathbb{Z}_N = \langle t \rangle$ just by putting

$$\begin{aligned}
&\theta(x_j) = t^{\eta_{1,j} d_{1,j}},\ 1 \leq j \leq \sharp\mathcal{O}_1 \\
&\theta(e_j) = t^{\eta_{i,j} d_{i,j}},\ 2 \leq i \leq 6,\ \sum_{k=1}^{i-1} \sharp\mathcal{O}_k + 1 \leq j \leq \sum_{k=1}^{i} \sharp\mathcal{O}_k.
\end{aligned} \tag{4.26}$$

In order to avoid ambiguity of notation we recall once more that $d_{i,j}$ are elements of $\mathcal{B}_i$ which should not be confused with $\mathcal{A}_i$ – the sections of character of periods $\mathfrak{C}_0$.

In the actual procedure, as well as in case of the three remaining ones, we deal with NEC groups of the form (4.22) i.e. with all link periods equal to 2. In order to finish our construction we need two auxiliary numbers

$$w_1 = \sharp\mathcal{O}_2 + \sharp\mathcal{O}_3 + \sharp\mathcal{O}_4, \qquad w_2 = \sharp\mathcal{O}_5 + \sharp\mathcal{O}_6. \tag{4.27}$$

Eventually, we define epimorphism on reflections associated to e–generators as follows

$$\begin{aligned}
&\theta(c_{i0}) = 1,\ 1 \le i \le \sharp\mathcal{O}_2\\
&\theta(c_{i0}) = t^{N/2},\ \sharp\mathcal{O}_2 + 1 \le i \le w_1\\
&\theta(c_{i0}) = \theta(c_{i2}) = t^{N/2},\ \theta(c_{i1}) = 1,\ w_1 + 1 \le i \le w_1 + w_2.
\end{aligned} \tag{4.28}$$

Summing up, we just have defined a group of signature

$$\Gamma_{\mathfrak{D}} = (0; +; [\mathcal{O}_1^*]; \{(\ )^{w_1}(2^2)^{w_2}\}) \tag{4.29}$$

and epimorphism θ from $\Gamma_{\mathfrak{D}}$ onto a subgroup of $\mathbb{Z}_N$. In the next chapter we extend both the above group $\Gamma_{\mathfrak{D}}$ and the homomorphism $\theta\colon \Gamma_{\mathfrak{D}} \to \mathbb{Z}_N$ in order to obtain a smooth epimorphisms from a NEC group onto $\mathbb{Z}_N$.

The next Lemma 4.16 and Proposition 4.17 we devote to show that the above construction of $\Gamma_{\mathfrak{D}}$ is optimal. We mean that fundamental region of any other NEC group that covers a required $\mathbb{Z}_N$–action must not be smaller that fundamental region of $\Gamma_{\mathfrak{D}}$.

Lemma 4.16. *Assume that $\mathfrak{C}_0 = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6)$ is the character of periods of a $\mathbb{Z}_N$–action. Let* signΓ = " + " *and $\theta\colon \Gamma \to \mathbb{Z}_N$ be an epimorphism that uniformizes a $\mathbb{Z}_N$–action given by $\mathfrak{C}_0$. Recall that by (4.6) we defined a character*

$$(\mathcal{G}_1(\Gamma,\theta), \mathcal{G}_2(\Gamma,\theta), \mathcal{G}_3(\Gamma,\theta), \mathcal{G}_4(\Gamma,\theta), \mathcal{G}_5(\Gamma,\theta), \mathcal{G}_6(\Gamma,\theta))$$

associated to $\mathfrak{C}_0$. We then have

(i) For $i = 1, 2, 4, 6$ families of integers $\mathcal{G}_i(\Gamma,\theta)$ include families $\{\mathcal{A}_i, F(\mathfrak{C}^*) \cap \mathcal{A}_i\}$. On the other hand for $i = 3, 5$ families of integers $\mathcal{G}_i(\Gamma,\theta)$ include families of the form $\{\mathcal{A}_i, \frac{1}{2}(F(\mathfrak{C}^*) \cap 2\mathcal{A}_i)\}$.

(ii) Consider the family of integers

$$\mathcal{R} = \{(\mathcal{G}_1(\Gamma,\theta))^*, (\mathcal{G}_2(\Gamma,\theta))^*, (2\mathcal{G}_3(\Gamma,\theta))^*, (\mathcal{G}_4(\Gamma,\theta))^*, (2\mathcal{G}_5(\Gamma,\theta))^*, (\mathcal{G}_6(\Gamma,\theta))^*\}$$

which comprises orders of images of all elliptic and e–generators of Γ (see Remark 4.3 and Corollary 4.4). If $G^*(\mathfrak{C}^*) \neq \emptyset$, then there are elements $r \in \mathcal{R}$ which satisfy $\alpha_2(r) = \alpha_2(\mathfrak{C}^*)$, but are not contained in $\{\mathfrak{C}^*, F^*(\mathfrak{C}^*)\}$.

Denote the above family by $\mathcal{J}$, i.e.

$$\mathcal{J} = \{r \in \mathcal{R} \mid \alpha_2(r) = \alpha_2(\mathfrak{C}^*)\} \setminus \{\mathfrak{C}^*, F^*(\mathfrak{C}^*)\}. \tag{4.30}$$

(iii) Assume $G^*(\mathfrak{C}^*) \neq \emptyset$. Denote by i_0 the section of the character $\mathfrak{O}(\mathfrak{C})$ given by (4.25) where the element $o(\mathfrak{C}^*)$ belongs to. By i_1 let us denote a section of the character

$$(\mathcal{G}_1(\Gamma,\theta), \mathcal{G}_2(\Gamma,\theta), 2\mathcal{G}_3(\Gamma,\theta), \mathcal{G}_4(\Gamma,\theta), 2\mathcal{G}_5(\Gamma,\theta), \mathcal{G}_6(\Gamma,\theta))$$

which contains an element from the family $\mathcal{J}$ (see also Remark 4.7). Then $i_0 \leq i_1$.

Proof. Observe that by (4.7) each $\mathcal{A}_i$ is contained in family $\mathcal{G}_i(\Gamma,\theta)$. However we need to show more, namely that $\{\mathcal{A}_i, F(\mathfrak{C}^*)\cap\mathcal{A}_i\}$ and $\{\mathcal{A}_i, \frac{1}{2}(F(\mathfrak{C}^*)\cap 2\mathcal{A}_i)\}$ are subfamilies of $\mathcal{G}_i(\Gamma,\theta)$ in the respective cases. Take an order of an elliptic or e–generator of Γ and denote it as u^*. Note that $u^* \in F^*(\mathfrak{C}^*)$ yields that there is only one j satisfying $u \in \mathcal{B}_j$. Hence $u \in \mathcal{G}_i(\Gamma,\theta)$, $i = 1,2,4,6$ or $u/2 \in \mathcal{G}_i(\Gamma,\theta)$, $i = 3,5$. On the other hand by Lemma 4.11 we know that the multiplicity of u^* in the family $\mathcal{R}$ shall be at least two. Thus for $i = 1,2,4,6$ we have

$$\{\mathcal{A}_i, F(\mathfrak{C}^*)\cap\mathcal{A}_i\} \subseteq \mathcal{G}_i(\Gamma,\theta),$$

while for $i = 3,5$ we obtain

$$\{\mathcal{A}_i, \frac{1}{2}(F(\mathfrak{C}^*)\cap\mathcal{B}_i)\} = \{\mathcal{A}_i, \frac{1}{2}(F(\mathfrak{C}^*)\cap 2\mathcal{A}_i)\} \subseteq \mathcal{G}_i(\Gamma,\theta).$$

The point *(ii)* is also a consequence of Lemma 4.11. Indeed, if $G^*(\mathfrak{C}^*) \neq \emptyset$, then the number of elements in $\{\mathfrak{C}^*, F^*(\mathfrak{C}^*)\}$ divisible by $2^{\alpha_2(\mathfrak{C}^*)}$ is odd by (4.23). Thus by Lemma 4.11 the family $\mathcal{J}$ must not be empty.

We now proceed to the proof of point *(iii)*. Since $\mathcal{J}$ is not empty, there is an element in $\mathcal{J}$ that generates a period of a $\mathbb{Z}_N$–action given by $\mathfrak{C}_0$, which belongs to $\mathcal{G}_{i_1}(\Gamma,\theta)$. Denote by ρ^* the above element of $\mathcal{J}$ and observe that $\rho^* \in \mathcal{B}^*_{i_1}\cap\mathcal{E}^*_2(\mathfrak{C}^*)$. Recall that we have required that $o(\mathfrak{C}^*)$ belongs to the family $\mathcal{O}_{i_0}$, such that i_0 is the lowest i satisfying $o(\mathfrak{C}^*) \in \mathcal{B}_i$. Since $o^*(\mathfrak{C}^*) \in \mathcal{E}^*_2(\mathfrak{C}^*)$, then by definition of $o^*(\mathfrak{C}^*)$ we get $i_0 \leq i_1$. □

Proposition 4.17. *Assume that $\mathfrak{C}_0$ is the character of periods of a $\mathbb{Z}_N$–action. Let* sign$\Gamma =$ *" + " and $\theta\colon \Gamma \to \mathbb{Z}_N$ be an epimorphism that uniformizes a $\mathbb{Z}_N$–action given by $\mathfrak{C}_0$. Then $\mu(\Gamma) - \mu(\Gamma_{\mathfrak{O}}) \geq 4\pi\gamma$.*

Proof. We will use the notation introduced in the actual subsection. Denote

$$\Gamma = (\gamma; \pm; [m_1, ..., m_n]; \{(\ )^{\lambda}(2^{\mu_1})...(2^{\mu_p})\}) \tag{4.31}$$

and observe that

$$\sharp\mathcal{G}_1(\Gamma,\theta) = n, \qquad \sum_{i=2}^{4}\sharp\mathcal{G}_i(\Gamma,\theta) = \lambda, \qquad \sum_{i=5}^{6}\sharp\mathcal{G}_i(\Gamma,\theta) = p. \tag{4.32}$$

Note that $G^*(\mathfrak{C}^*) = \emptyset$ yields $\sum_{i=1}^{6}\sharp\mathcal{O}_i = \sharp\{\mathfrak{C}^*, F^*(\mathfrak{C}^*)\}$. On the other hand by point *(i)* of Lemma 4.16 we have $\{\mathcal{A}_1, F(\mathfrak{C}^*)\cap\mathcal{A}_1\} \subset \mathcal{G}_1(\Gamma,\theta)$ and

$$\sum_{i=1}^{6}\sharp\mathcal{G}_i(\Gamma,\theta) \geq \sharp\{\mathfrak{C}^*, F^*(\mathfrak{C}^*)\}.$$

Thus by (4.18) we have

$$\begin{aligned}\mu(\Gamma) &= 2\pi\left(2\gamma + \sum_{i=2}^{6}\sharp\mathcal{G}_i(\Gamma,\theta) - 2 + \sum_{i=1}^{\sharp\mathcal{G}_1(\Gamma,\theta)}(1-m_i^{-1}) + 4^{-1}\sum_{i=1}^{\sharp\mathcal{G}_5(\Gamma,\theta)+\sharp\mathcal{G}_6(\Gamma,\theta)}\mu_i\right)\\ &\geq 4\pi\gamma + 2\pi\left(\sum_{i=2}^{6}\sharp\mathcal{O}_i - 2 + \sum_{i=1}^{\sharp\mathcal{O}_1}(1-m_i^{-1}) + 2^{-1}(\sharp\mathcal{O}_5+\sharp\mathcal{O}_6)\right) = 4\pi\gamma + \mu(\Gamma_{\mathfrak{D}})\end{aligned}$$

as required.

Hence we may assume $G^*(\mathfrak{C}^*) \neq \emptyset$, which by *(ii)* of the same lemma implies that the family $\mathcal{J}$ given by (4.30) is not empty. Take $\rho^* \in \mathcal{J}$ and the smallest i such that $\rho^* \in \mathcal{B}_i^*$, say i_1. Recall that by *(iii)* of Lemma 4.16 we have $o^*(\mathfrak{C}^*) \in \mathcal{B}_{i_0}^*$, where $i_0 \leq i_1$.

If $i_1 = 1$ we have $o^*(\mathfrak{C}^*) \leq \rho^*$, which leads us to

$$\mu(\Gamma) - \mu(\Gamma_{\mathfrak{D}}) \geq 4\pi\gamma,$$

since $1 - o^*(\mathfrak{C}^*)^{-1} \leq 1 - (\rho^*)^{-1}$. Moreover if $i_1 = 2, 3, 4$ then

$$\mu(\Gamma) - \mu(\Gamma_{\mathfrak{D}}) \geq 2\pi(2\gamma + 1 - (1 - o^*(\mathfrak{C}^*)^{-1}))$$

in case $\mathcal{B}_1^* \cap \mathcal{E}_2^*(\mathfrak{C}^*) \neq \emptyset$ or $\mu(\Gamma) - \mu(\Gamma_{\mathfrak{D}}) \geq 4\pi\gamma$ otherwise. Finally if $i_1 = 5, 6$ we have three possibilities according to the definition of $o^*(\mathfrak{C}^*)$. Namely, if $\mathcal{B}_1^* \cap \mathcal{E}_2^*(\mathfrak{C}^*) \neq \emptyset$, then

$$\mu(\Gamma) - \mu(\Gamma_{\mathfrak{D}}) \geq 2\pi(2\gamma + 1 + 4^{-1}\mu_i - (1 - o^*(\mathfrak{C}^*)^{-1})).$$

In the remaining cases we obtain

$$\mu(\Gamma) - \mu(\Gamma_{\mathfrak{D}}) \geq 2\pi(2\gamma + 1 + 4^{-1}\mu_i - 1)$$

if $\mathcal{B}_{234}^* \cap \mathcal{E}_2^*(\mathfrak{C}^*) \neq \emptyset$, but $\mathcal{B}_1^* \cap \mathcal{E}_2^*(\mathfrak{C}^*) = \emptyset$. Otherwise we have

$$\mu(\Gamma) - \mu(\Gamma_{\mathfrak{D}}) \geq 2\pi(2\gamma + 1)$$

which completes the proof. $\square$

4.3.2 Procedures $\mathfrak{N}$, $\mathfrak{N}_0$ and $\mathfrak{N}_{-1}$ for Non-orientable Covering Groups

In this subsection we introduce definitions and constructs which are applicable only in case when a covering NEC group is non–orientable. We will consider actions of cyclic groups which are described by characters of periods $\mathfrak{C}_0$ that belong to $\mathrm{CPer}^{(-,+)}(\mathbb{Z}_N) \cup \mathrm{CPer}^{(-,-)}(\mathbb{Z}_N)$. In order to define adequate groups together with the respective homomorphism we use, as before, characters associated to inflated characters of periods $\mathfrak{C}$. Nevertheless we complete here nor the constructions of covering NEC groups nor the constructions of homomorphisms into $\mathbb{Z}_N$. We only determine prototypes of signatures that will be extended in the forthcoming sections to obtain signatures of NEC groups. Unlike the case of the procedure $\mathfrak{O}$ the assignments we propose below do not keep the "long relation" given by (3.3). We must postpone the closing of the "long relation" until we know by how many glide reflections we extend the respective prototypes of NEC groups.[2] However we define and map here all canonical generators of the candidate NEC groups which are not glide reflections.

The first procedure we start with is called $\mathfrak{N}$. We shall need a character associated to the inflated character of periods $\mathfrak{C}$. We use $\mathfrak{D}(\mathfrak{C})$ defined as follows

$$\mathfrak{D}(\mathfrak{C}) = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6) = (\{d_{1,1}, \ldots, d_{1,\sharp\mathcal{B}_1}\}, \ldots, \{d_{6,1}, \ldots, d_{6,\sharp\mathcal{B}_6}\}). \tag{4.33}$$

Furthermore, we define a signature

$$\Gamma_{\mathfrak{N}} = (0; -; [\mathcal{B}_1^*]; \{(\ )^{w_1}(2^2)^{w_2}\}), \tag{4.34}$$

where

$$w_1 = \sharp\mathcal{B}_2 + \sharp\mathcal{B}_3 + \sharp\mathcal{B}_4, \qquad w_2 = \sharp\mathcal{B}_5 + \sharp\mathcal{B}_6. \tag{4.35}$$

Since we separate the analysis of covering groups into orientable and non–orientable case we use the same letters in (4.27) and (4.35) as no confusion can arise. Finally, to obtain a prototype of homomorphism from a group represented by $\Gamma_{\mathfrak{N}}$ to a cyclic group $\mathbb{Z}_N$ we use (4.14) and element $\overline{\eta}_{\mathfrak{D}(\mathfrak{C})}$ verifying $\eta_{i,j} = 1$. Obviously we have $(d^*_{i,j}, \eta_{i,j}) = 1$ which yields that $\overline{\eta}_{\mathfrak{D}(\mathfrak{C})}$ is order–preserving with respect to $\mathfrak{D}(\mathfrak{C})$. Hence we define the following assignment

$$\begin{aligned} &\theta(x_j) = t^{d_{1,j}},\ 1 \le j \le \sharp\mathcal{B}_1 \\ &\theta(e_j) = t^{d_{i,j}},\ 2 \le i \le 6,\ \sum_{k=1}^{i-1} \sharp\mathcal{B}_k + 1 \le j \le \sum_{k=1}^{i} \sharp\mathcal{B}_k. \end{aligned} \tag{4.36}$$

[2]Since we will consider non–orientable NEC groups we shall observe that the image under a group homomorphism into $\mathbb{Z}_N$ of the product of all canonical glide reflections in general does not vanish. Obviously any homomorphism from an orientable NEC group to an abelian group maps the commutator of hyperbolic generators corresponding to the orbit genus which appear in the "long relation" (3.2) to the identity. It enabled us to close this relation even for the prototype group (4.29).

In order to define a prototype of homomorphism θ on the remaining reflections we put

$$\begin{aligned}
&\theta(c_{i0}) = 1,\ 1 \le i \le \sharp\mathcal{B}_2\\
&\theta(c_{i0}) = t^{N/2},\ \sharp\mathcal{B}_2 + 1 \le i \le w_1\\
&\theta(c_{i0}) = \theta(c_{i2}) = t^{N/2},\ \theta(c_{i1}) = 1,\ w_1 + 1 \le i \le w_1 + w_2.
\end{aligned} \tag{4.37}$$

In both remaining procedures denoted as $\mathfrak{N}_0$ and $\mathfrak{N}_{-1}$ we also determine abstract signatures. We define the appropriate prototypes of homomorphisms into cyclic group $\mathbb{Z}_N$ likewise. The above signature $\Gamma_{\mathfrak{N}}$ is used as a starting point in the subsequent constructions.

In the next procedure $\mathfrak{N}_0$ we will add to $\mathfrak{D}(\mathfrak{C})$ the element $n_0(\mathfrak{C}^*)$ defined by (4.21). Let j be the lowest index satisfying $n_0(\mathfrak{C}^*) \in \mathcal{B}_j$. We form character $\mathfrak{N}_0(\mathfrak{C})$ from $\mathfrak{D}(\mathfrak{C})$ by adding the element $n_0(\mathfrak{C}^*)$ to $\mathcal{B}_j$. Then we denote

$$\mathfrak{N}_0(\mathfrak{C}) = (\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4, \mathcal{N}_5, \mathcal{N}_6). \tag{4.38}$$

It follows that $\mathcal{N}_i = \mathcal{B}_i$ for each $1 \le i \le 6$ except exactly one i. Hence we may define a signature

$$\Gamma_{\mathfrak{N}_0} \quad = \quad (0; -; [\mathcal{N}_1^*]; \{(\ )^{y_1}(2^2)^{y_2}\}) \tag{4.39}$$

where

$$y_1 = \sharp\mathcal{N}_2 + \sharp\mathcal{N}_3 + \sharp\mathcal{N}_4, \qquad y_2 = \sharp\mathcal{N}_5 + \sharp\mathcal{N}_6. \tag{4.40}$$

The construction of prototype of homomorphism $\theta\colon \Gamma \to \mathbb{Z}_N$ runs as before upon specific modifications. We put

$$\begin{aligned}
&\theta(x_j) = t^{d_{1,j}},\ 1 \le j \le \sharp\mathcal{N}_1\\
&\theta(e_j) = t^{d_{i,j}},\ 2 \le i \le 6,\ \sum_{k=1}^{i-1} \sharp\mathcal{N}_k + 1 \le j \le \sum_{k=1}^{i} \sharp\mathcal{N}_k\\
&\theta(c_{i0}) = 1,\ 1 \le i \le \sharp\mathcal{N}_2\\
&\theta(c_{i0}) = t^{N/2},\ \sharp\mathcal{N}_2 + 1 \le i \le y_1\\
&\theta(c_{i0}) = \theta(c_{i2}) = t^{N/2},\ \theta(c_{i1}) = 1,\ y_1 + 1 \le i \le y_1 + y_2.
\end{aligned} \tag{4.41}$$

Observe that the above prototype of homomorphism is defined on elliptic and e–generators of $\Gamma_{\mathfrak{N}_0}$ by an element $\overline{\eta}_{\mathfrak{N}_0(\mathfrak{C})}$ verifying $\eta_{i,j} = 1$, which is order–preserving with respect to $\mathfrak{N}_0(\mathfrak{C})$.

Now we proceed to the last procedure $\mathfrak{N}_{-1}$. Instead of $n_0^*(\mathfrak{C}^*)$ we now use the element $n_{-1}^*(\mathfrak{C}^*)$ defined by (4.21). Let j be the lowest index satisfying $n_{-1}(\mathfrak{C}^*) \in \mathcal{B}_j$. A character $\mathfrak{N}_{-1}(\mathfrak{C})$ is formed from $\mathfrak{D}(\mathfrak{C})$ by adding the element $n_{-1}(\mathfrak{C}^*)$ to $\mathcal{B}_j$. We denote

$$\mathfrak{N}_{-1}(\mathfrak{C}) = (\mathcal{N}_{-1}, \mathcal{N}_{-2}, \mathcal{N}_{-3}, \mathcal{N}_{-4}, \mathcal{N}_{-5}, \mathcal{N}_{-6}) \tag{4.42}$$

and

$$\Gamma_{\mathfrak{N}_{-1}} = (0; -; [\mathcal{N}^*_{-1}]; \{(\ )^{z_1}(2^2)^{z_2}\}), \tag{4.43}$$

where

$$z_1 = \sharp\mathcal{N}_{-2} + \sharp\mathcal{N}_{-3} + \sharp\mathcal{N}_{-4} \qquad z_2 = \sharp\mathcal{N}_{-5} + \sharp\mathcal{N}_{-6}. \tag{4.44}$$

A prototype of homomorphism from $\Gamma_{\mathfrak{N}_{-1}}$ to a cyclic group $\mathbb{Z}_N$ is built analogously to (4.36)–(4.37):

$$\begin{aligned}
&\theta(x_j) = t^{d_{1,j}},\ 1 \le j \le \sharp\mathcal{N}_{-1}\\
&\theta(e_j) = t^{d_{i,j}},\ 2 \le i \le 6,\ \sum_{k=1}^{i-1}\sharp\mathcal{N}_{-k} + 1 \le j \le \sum_{k=1}^{i}\sharp\mathcal{N}_{-k}\\
&\theta(c_{i0}) = 1,\ 1 \le i \le \sharp\mathcal{N}_{-2}\\
&\theta(c_{i0}) = t^{N/2},\ \sharp\mathcal{N}_{-2} + 1 \le i \le z_1\\
&\theta(c_{i0}) = \theta(c_{i2}) = t^{N/2},\ \theta(c_{i1}) = 1,\ z_1 + 1 \le i \le z_1 + z_2.
\end{aligned} \tag{4.45}$$

The above prototype of homomorphism is defined on elliptic and e–generators of $\Gamma_{\mathfrak{N}_{-1}}$ by an element $\overline{\eta}_{\mathfrak{N}_{-1}(\mathfrak{C})}$ verifying $\eta_{i,j} = 1$, which is order–preserving with respect to $\mathfrak{N}_{-1}(\mathfrak{C})$.

We sum up elementary properties of the above abstract signatures in the following remark.

Remark 4.18. *Suppose that $\mathfrak{C}_0 \in \mathrm{CPer}^{(-,+)}(\mathbb{Z}_N) \cup \mathrm{CPer}^{(-,-)}(\mathbb{Z}_N)$ is a character of periods of a $\mathbb{Z}_N$–action by dianalytic transformations on a Klein surface. Let $\Gamma_{\mathfrak{N}}$, $\Gamma_{\mathfrak{N}_0}$ and $\Gamma_{\mathfrak{N}_{-1}}$ be signatures built on a basis of the inflated character of periods $\mathfrak{C}$ which are defined by (4.34), (4.39) and (4.42) respectively. Furthermore assume that $\mathcal{N}_{(2\epsilon+1)j} \setminus \mathcal{B}_j \neq \emptyset$, where $\epsilon \in \{-1, 0\}$.*[3] *Then the following statements hold*

(i) If $j = 1$ and $m \in \mathcal{N}_{2\epsilon+1} \setminus \mathcal{B}_1$, then $\mu(\Gamma_{\mathfrak{N}_\epsilon}) - \mu(\Gamma_{\mathfrak{N}}) = 2\pi(1 - m^{-1})$.

(ii) If $j = 2, 3, 4$, then $\mu(\Gamma_{\mathfrak{N}_\epsilon}) - \mu(\Gamma_{\mathfrak{N}}) = 2\pi$.

(iii) If $j = 5, 6$, then $\mu(\Gamma_{\mathfrak{N}_\epsilon}) - \mu(\Gamma_{\mathfrak{N}}) = 3\pi$.

Let $\mathfrak{C}_0 \in \mathrm{CPer}^{(-,+)}(\mathbb{Z}_N) \cup \mathrm{CPer}^{(-,-)}(\mathbb{Z}_N)$. In the next proposition we obtain a lower bound for the area of NEC groups covering $\mathbb{Z}_N$–action given by $\mathfrak{C}_0$.

Proposition 4.19. *Assume that $\mathfrak{C}_0 = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6)$ is the character of periods of a $\mathbb{Z}_N$–action. Let $\mathrm{sign}\Gamma = " - "$ and $\theta\colon \Gamma \to \mathbb{Z}_N$ be an epimorphism that uniformizes a $\mathbb{Z}_N$–action given by $\mathfrak{C}_0$. Then $\mu(\Gamma) - \mu(\Gamma_{\mathfrak{N}}) \ge 2\pi\gamma$.*

[3] The use of this construct was supposed to ease the notation since $\mathcal{N}_{(2\epsilon+1)j} = \mathcal{N}_{-j}$ for $\epsilon = -1$ and $\mathcal{N}_{(2\epsilon+1)j} = \mathcal{N}_j$ for $\epsilon = 0$.

Proof. By (4.7) each $\mathcal{A}_i$ is contained in the family $\mathcal{G}_i(\Gamma,\theta)$ given by (4.6). Since we may assume that Γ is of the form (4.31), we may also use the relations (4.32). Thus we obtain

$$\begin{aligned}\mu(\Gamma) &= 2\pi\left(\gamma+\sum_{i=2}^{6}\sharp\mathcal{G}_i(\Gamma,\theta)-2+\sum_{i=1}^{\sharp\mathcal{G}_1(\Gamma,\theta)}(1-m_i^{-1})+4^{-1}\sum_{i=1}^{\sharp\mathcal{G}_5(\Gamma,\theta)+\sharp\mathcal{G}_6(\Gamma,\theta)}\mu_i\right)\\ &\geq 2\pi\gamma+2\pi\left(\sum_{i=2}^{6}\sharp\mathcal{B}_i-2+\sum_{i=1}^{\sharp\mathcal{B}_1}(1-m_i^{-1})+2^{-1}(\sharp\mathcal{B}_5+\sharp\mathcal{B}_6)\right)=2\pi\gamma+\mu(\Gamma_{\mathfrak{N}}).\end{aligned}$$

□

4.3.3 Induced Cyclic Subgroups

In the last two subsections we have considered procedures that will be used in the forthcoming paragraphs to obtain signatures of NEC groups that cover certain $\mathbb{Z}_N$–actions on Klein surfaces. Associated to those procedures there are prototypes of homomorphisms to a cyclic group $\mathbb{Z}_N$. Denote Γ to be one of the groups $\Gamma_{\mathfrak{O}}$, $\Gamma_{\mathfrak{N}}$, $\Gamma_{\mathfrak{N}_0}$, $\Gamma_{\mathfrak{N}_{-1}}$ and assume that the respective homeomorphism maps Γ onto a subgroup of $\mathbb{Z}_N$, say $\mathbb{Z}_M$. In order to determine the number M, we present below two arithmetic remarks. Then in the corollary that follows, we show that we may use $\mathrm{lcm}\mathfrak{C}_0^*$ as the order of a subgroup $\theta(\Gamma)\simeq\mathbb{Z}_M$. Let us first observe that

$$\mathrm{lcm}\mathfrak{C}^*=|\langle\theta(x_1),\dots,\theta(x_n),\theta(e_1),\dots,\theta(e_W)\rangle|, \tag{4.46}$$

where W is the number of e–generators in signatures of NEC groups. It is given by (4.27), (4.35), (4.40) and (4.44) in the respective cases.

Remark 4.20. *Let $\mathfrak{C}_0$ be the character of periods of a $\mathbb{Z}_N$–action on a Klein surface. Then*

(i) $\mathrm{lcm}\mathfrak{C}_0^*\geq\mathrm{lcm}\mathfrak{C}^*$

(ii) if $\mathrm{lcm}\mathfrak{C}_0^*>\mathrm{lcm}\mathfrak{C}^*$*, then* $\mathrm{lcm}\mathfrak{C}_0^*=2\mathrm{lcm}\mathfrak{C}^*$

(iii) $\mathrm{lcm}\mathfrak{C}_0^*=2\mathrm{lcm}\mathfrak{C}^*$ *if and only if* $\mathrm{lcm}\mathfrak{C}^*$ *is an odd number*

Remark 4.21. *Let $\mathfrak{C}_0$ be the character of periods of a $\mathbb{Z}_N$–action on a Klein surface. Let Γ be one of the groups $\Gamma_{\mathfrak{O}}$, $\Gamma_{\mathfrak{N}}$, $\Gamma_{\mathfrak{N}_0}$, $\Gamma_{\mathfrak{N}_{-1}}$. Consider the prototypes of homomorphisms $\theta\colon\Gamma\to\mathbb{Z}_N$ given by (4.26)–(4.28), (4.36)–(4.37), (4.41) and (4.45) in the respective cases. We have*

$$|\theta(\Gamma)| = \begin{cases}\mathrm{lcm}\mathfrak{C}^* & \text{if } \bigcup_{i=3}^{6}\mathcal{A}_i=\emptyset\\ \mathrm{lcm}\{2,\mathrm{lcm}\mathfrak{C}^*\} & \text{otherwise.}\end{cases}$$

Corollary 4.22. *Under the assumptions of the previous remarks we have*

$$|\theta(\Gamma)|=\mathrm{lcm}\mathfrak{C}_0^*.$$

Proof. We follow the notation of Remarks 4.20 and 4.21. Assume firstly that $\mathrm{lcm}\mathfrak{C}_0^*$ is odd. Observe that due to (4.2) all sets $\mathcal{A}_i^*$, $i \geq 3$ comprise even numbers. By Remark 4.20 we obtain $\mathrm{lcm}\mathfrak{C}_0^* = \mathrm{lcm}\mathfrak{C}^* = |\theta(\Gamma)|$. On the other hand for $\mathrm{lcm}\mathfrak{C}_0^*$ even we have

$$|\theta(\Gamma)| = \mathrm{lcm}\{2, \mathrm{lcm}\mathfrak{C}^*\} = \mathrm{lcm}\{2, \mathrm{lcm}\mathfrak{C}_0^*\} = \mathrm{lcm}\mathfrak{C}_0^*.$$

□

Remark 4.23. *The assertions of Remark 4.21 and Corollary 4.22 are still valid if we assume only that a NEC group Γ which covers a $\mathbb{Z}_N$–action given by $\mathfrak{C}_0$ has the orbit genus equal to 0.*

Chapter 5

$\mathbb{Z}_N$–actions on Klein Surfaces

5.1 Actions of Groups of Odd Order

In the present section we consider only $\mathbb{Z}_N$–actions on compact surfaces for N odd. We continue using the notation established in the preceding chapter. A particular role in this section will be played by two procedures, $\mathfrak{O}$ and $\mathfrak{N}$. We will apply them together with all accompanying symbols, formulae and related homomorphisms defined in subsection 4.3. Henceforth, we assume that a $\mathbb{Z}_N$–action on a Klein surface X is described by a character of periods $\mathfrak{C}_0$ and uniformized by an epimorphism $\theta\colon \Gamma \to \mathbb{Z}_N$. The groups Γ and $\ker\theta \simeq \Lambda$ are given by (3.7) and (3.4) respectively, although under actual assumptions they take special forms. Since N is assumed to be odd, there are no components of the singular set such as ovals and chains. Thus we have $\mathfrak{C} = \mathfrak{C}_0$ and

$$\Gamma = (\gamma; \pm; [m_1, ..., m_n]; \{(\ )^r\}), \qquad \Lambda = (g; \pm; [\ ]; \{(\ )^k\}).$$

Note that we must have $\theta(c_{i,0}) = 1$, $i = 1, \ldots, r$. It agrees with the notation of Theorem 3.12. We recall some results from the theory of NEC groups that we shall need.

Theorem 5.1 (Bujalance et al. [8], Theorem 2.1.2)**.** *Suppose that Γ and Λ are NEC groups. If Λ is a normal subgroup of Γ of an odd index N, then signatures of both groups have the same sign.*

Following [8] we introduce the below definition of N–pair. Given a NEC group Γ we say that (Γ, Λ) is an *N–pair* if there exists an epimorphism $\theta\colon \Gamma \to \mathbb{Z}_N$ whose kernel is a surface NEC group Λ.

Theorem 5.2 (Bujalance et al. [8], Theorem 3.1.2)**.** *Let us suppose that N is odd and* $\mathrm{sign}\Gamma = " + "$*. Then (Γ, Λ) is an N–pair if and only if:*

(i) for each $i = 1, \ldots, n$, m_i divides N, and $\mathrm{sign}\Lambda = " + "$

(ii) $\mu(\Lambda) = N\mu(\Gamma)$

(iii) there exist positive divisors $l_1, \ldots, l_r$ of N such that

(iii.1) $k = \sum_{i=1}^{r} N/l_i$

(iii.2) the set $\{m_1, \ldots, m_n, l_1, \ldots, l_r\}$ has the elimination property

(iv) if $\gamma = 0$, then $N = \mathrm{lcm}(m_1, \ldots, m_n, l_1, \ldots, l_r)$

Remark 5.3. *The above symbols l_i are the orders of images of e–generators of empty period cycles of Γ under a smooth epimorphism $\theta\colon \Gamma \to \mathbb{Z}_N$ verifying* $\ker\theta = \Lambda$.

Proposition 5.4. *Suppose that N is odd. A character of periods $\mathfrak{C}_0$ belongs to the set* $\mathrm{CPer}^{(+,+)}(\mathbb{Z}_N)$ *if and only if $\mathfrak{C}_0 = (\mathcal{A}_1, \mathcal{A}_2, \emptyset, \emptyset, \emptyset, \emptyset)$.*

Proof. ($\Rightarrow$) The result is forced by the assumption that N is odd. According to Lemma 4.1 we have $\mathcal{A}_3 = \mathcal{A}_4 = \mathcal{A}_5 = \mathcal{A}_6 = \emptyset$.

($\Leftarrow$) Consider a NEC group

$$\Gamma = (2; +; [\mathcal{A}_1^*, \mathcal{A}_1^*]; \{(\ )^{2\sharp\mathcal{A}_2}\}) \tag{5.1}$$

and observe that $\mu(\Gamma) > 0$. Since $\{\mathcal{A}_1^*, \mathcal{A}_1^*, \mathcal{A}_2^*, \mathcal{A}_2^*\} = \{m_1, \dots, m_{2\sharp\mathcal{A}_1}, l_1, \dots, l_{2\sharp\mathcal{A}_2}\}$ has the elimination property, then by Lemma 4.11 there exist numbers $\eta_1, \dots, \eta_{2\sharp\mathcal{A}_1+2\sharp\mathcal{A}_2}$ such that

$$\sum_{i=1}^{2\sharp\mathcal{A}_1} \eta_i m_i^* + \sum_{i=2\sharp\mathcal{A}_1+1}^{2\sharp\mathcal{A}_2} \eta_i l_i^* \equiv 0 \bmod N$$

where $(m_i, \eta_i) = 1$, $i = 1, \dots, 2\sharp\mathcal{A}_1$ and $(l_j, \eta_j) = 1$, $j = 1, \dots, 2\sharp\mathcal{A}_2$. We define the following smooth epimorphism onto $\mathbb{Z}_N = \langle t \rangle$

$$\begin{aligned} \theta(a_i) = \theta(b_i) = t,\ i = 1, 2, \qquad & \theta(x_i) = t^{\eta_i m_i^*},\ i = 1, \dots, 2\sharp\mathcal{A}_1 \\ \theta(e_j) = t^{\eta_j l_j^*}, \qquad & \theta(c_{j,0}) = 1,\ j = 1, \dots, 2\sharp\mathcal{A}_2. \end{aligned} \tag{5.2}$$

By (4.7) we have $\mathrm{CPer}(t) = (\mathcal{A}_1, \mathcal{A}_2, \emptyset, \emptyset, \emptyset, \emptyset)$, as required. □

By the above for N odd we abbreviate $(\mathcal{A}_1, \mathcal{A}_2, \emptyset, \emptyset, \emptyset, \emptyset) \in \mathrm{CPer}^{(+,+)}(\mathbb{Z}_N)$ to $(\mathcal{A}_1, \mathcal{A}_2)^+$. In the next theorem we solve the problem of constructing a NEC group with minimal measure, which covers a $\mathbb{Z}_N$–action given by $\mathfrak{C}_0$ provided $\mathfrak{C}_0 \in \mathrm{CPer}^{(+,+)}(\mathbb{Z}_N)$.

Theorem 5.5. *Let N be odd. Suppose that $\mathfrak{C}_0 = (\mathcal{A}_1, \mathcal{A}_2)^+$, where $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2 \geq 2$.*

(i) If $\mathcal{A}_1 = \mathcal{A}_2 = \{1\}$, then we put

$$\Gamma = (0; +; [N, N]; \{(\ )\})$$

(ii) otherwise

$$\Gamma = (1 - \delta_N(\mathrm{lcm}\mathfrak{C}_0^*); +; [\mathcal{O}_1^*]; \{(\ )^{w_1}\}),$$

where $\mathcal{O}_1$ and w_1 were defined in (4.24) and (4.27) respectively.

In the respective cases the above group Γ is a universal covering group of $(\langle t \rangle, X)$, $\langle t \rangle = \mathbb{Z}_N$. It satisfies $\mathrm{CPer}(t) = \mathfrak{C}_0$. Moreover the area $\mu(\Lambda)$, where $X = \mathbb{H}^2/\Lambda$, is minimal among all orientable surfaces on which $\mathfrak{C}_0$ is attained as the character of periods.

Proof. The proof of case *(i)* is straightforward and follows from direct computations. We define

$$\theta(x_1) = t,\ \theta(x_2) = t^{N-2},\ \theta(e_1) = t,\ \theta(c_{1,0}) = 1,$$

which yields $\mathrm{CPer}(t) = (\{1\}, \{1\})^+$ by (4.7). Furthermore observe that $\Gamma_1 = (0; +; [N]; \{(\ )\})$ is not a NEC group since $\mu(\Gamma_1) < 0$. Observe also that $\mu(\Gamma) - \mu(\Gamma_1) = 2\pi(1 - N^{-1})$. Hence

Γ has minimal measure among all NEC groups covering $\mathfrak{C}_0 = (\{1\}, \{1\})^+$. By Riemann–Hurwitz formula it follows that also $\mu(\Lambda)$ is minimal.

Consider the case *(ii)*. By the procedure $\mathfrak{D}$ we obtain a group $\Gamma_{\mathfrak{D}}$ given by (4.29) together with a homomorphism θ. Corollary 4.22 shows that $\Gamma_{\mathfrak{D}}$ is mapped onto $\mathbb{Z}_M$ with $M = \mathrm{lcm}\mathfrak{C}_0^*$. In order to construct a required epimorphism in case $\mathrm{lcm}\mathfrak{C}_0^* \neq N$, we must add at least one pair of hyperbolic generators. We then map a_1 and b_1 to any element of order N in $\mathbb{Z}_N$.

In order to verify that group Γ obtained in that way is a NEC group we must check whether $\mu(\Gamma)$ is positive. We consider several cases which exhaust all possibilities given by the assumption $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2 \geq 2$. In the following simple calculations we also exploit the assumption that N is odd. Note that since we have $d \leq N/3$ for $d \in \mathfrak{C}$, it follows that $1 - d/N \geq 2/3$. However $\sharp\mathcal{A}_1 \geq 2$ implies more, namely that $\sum_{m\in\mathcal{O}_1^*}(1 - m^{-1}) > 2\sharp\mathcal{O}_1^*/3$. Recall that $m \in \mathcal{O}_1^*$ are of the form N/d, $d \in \mathfrak{C}$.

Assume $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2 = 2$. Observe that $\sharp\mathcal{A}_1 = 2$, $\mathcal{A}_2 = \emptyset$ and $\mathcal{A}_1 = \emptyset$, $\sharp\mathcal{A}_2 = 2$ imply that $\sharp\mathcal{O}_1^* \geq 3$ and $w_1 \geq 3$ respectively, which follows from the procedure $\mathfrak{D}$. In that setting at least one of the periods is repeated. By (4.18) it yields

$$\mu(\Gamma) = 2\pi(2(1 - \delta_N(\mathrm{lcm}\mathfrak{C}_0^*)) + w_1 - 2 + \sum_{m\in\mathcal{O}_1^*}(1 - m^{-1})) > 2\pi(-2 + \frac{2}{3}\cdot 3) \geq 0 \qquad (5.3)$$

for $\sharp\mathcal{A}_1 = 2$, $\mathcal{A}_2 = \emptyset$ and

$$\mu(\Gamma) = 2\pi(2(1 - \delta_N(\mathrm{lcm}\mathfrak{C}_0^*)) + w_1 - 2 + \sum_{m\in\mathcal{O}_1^*}(1 - m^{-1})) > 2\pi(3 - 2) > 0$$

in case $\mathcal{A}_1 = \emptyset$, $\sharp\mathcal{A}_2 = 2$.

Next we consider $\sharp\mathcal{A}_1 = \sharp\mathcal{A}_2 = 1$ and observe that now $\sharp\mathcal{O}_1^* + w_1 \geq 3$ unless $\mathcal{A}_1 = \mathcal{A}_2 = \{d\}$, $d \neq 1$. Consequently $w_1 + \sum_{m\in\mathcal{O}_1^*}(1 - m^{-1}) \geq 7/3 > 2$ which leads us to $\mu(\Gamma) > 0$. On the other hand $\mathcal{A}_1 = \mathcal{A}_2 = \{d\}$ where $d \neq 1$, implies $\delta_N(\mathrm{lcm}\mathfrak{C}_0^*)) \neq 1$. Thus $\mu(\Gamma) \geq 2\pi(1 + 2/3) > 0$.

Finally we consider $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2 > 2$. With this assumption it follows $w_1 + \sum_{m\in\mathcal{O}_1^*}(1 - m^{-1}) > 2$ which immediately gives $\mu(\Gamma) > 0$.

Next we claim that $\mu(\Gamma)$ is minimal. Let Γ be another NEC group that covers a $\mathbb{Z}_N$–action prescribed by $\mathfrak{C}_0$. Denote by γ and γ' the genera of groups Γ and Γ' respectively. By Proposition 4.17 we have

$$\mu(\Gamma') - \mu(\Gamma) = \mu(\Gamma') - (\mu(\Gamma_{\mathfrak{D}}) + 4\pi\gamma) \geq 4\pi(\gamma' - \gamma) \geq 0.$$

We see at once that group Λ is orientable, which follows from Theorem 5.1. $\square$

Since we have skipped the case $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2 < 2$ we complete the study by the following remark that is a natural counterpart of Remark 2.9 from the first part of the thesis.

Remark 5.6. *Under the assumptions of Theorem 5.5 with $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2 < 2$ the universal covering group that covers a $\mathbb{Z}_N$–action prescribed by $\mathfrak{C}_0$ equals*

$$\Gamma = \begin{cases} (2; +; [\,]; \{\,\}), & \text{if } \mathfrak{C}_0 = (\emptyset, \emptyset)^+ \\ (0; +; [N, N, N]; \{\,\}), & \text{if } \mathfrak{C}_0 = (\{1\}, \emptyset)^+ \text{ and } N \neq 3 \\ (0; +; [3, 3, 3, 3]; \{\,\}), & \text{if } \mathfrak{C}_0 = (\{1\}, \emptyset)^+ \text{ and } N = 3 \\ (1; +; [N/d, N/d]; \{\,\}), & \text{if } \mathfrak{C}_0 = (\{d\}, \emptyset)^+ \text{ and } d \neq 1 \\ (0; +; [\,]; \{(\,)^3\}), & \text{if } \mathfrak{C}_0 = (\emptyset, \{1\})^+ \\ (1; +; [\,]; \{(\,)^2\}), & \text{if } \mathfrak{C}_0 = (\emptyset, \{d\})^+ \text{ and } d \neq 1 \end{cases}$$

The area of the respective NEC groups is minimal among all orientable surfaces on which the above 2–tuples of sets are attained as the characters of periods.

Our next concern is to show how to find the signature of a group together with a required epimorphism onto $\mathbb{Z}_N$ in case the underlying surface X is non–orientable. Before we formulate a theorem, we need two preliminary results.

Theorem 5.7 (Bujalance et al. [8], Theorem 3.1.3)**.** *Let us suppose that N is odd and* $\mathrm{sign}\Gamma = " - "$. *Then (Γ, Λ) is an N–pair if and only if:*

(i) for each $i = 1, \dots, n$, m_i divides N, and $\mathrm{sign}\Lambda = " - "$

(ii) $\mu(\Lambda) = N\mu(\Gamma)$

(iii) there exist positive divisors $l_1, \dots, l_r$ of N such that

(iii.1) $k = \sum_{i=1}^{r} N/l_i$

(iv) if $\gamma = 1$, then $N = \mathrm{lcm}(m_1, \dots, m_n, l_1, \dots, l_r)$

Proposition 5.8. *Suppose that N is odd. A character of periods $\mathfrak{C}_0$ belongs to the set* $\mathrm{CPer}^{(-,-)}(\mathbb{Z}_N)$ *if and only if* $\mathfrak{C}_0 = (\mathcal{A}_1, \mathcal{A}_2, \emptyset, \emptyset, \emptyset, \emptyset)$.

Proof. ($\Rightarrow$) The result is forced by the assumption that N is odd and Lemma 4.1.

($\Leftarrow$) Consider a NEC group

$$\Gamma = (3; -; [\mathcal{A}_1^*]; \{(\,)^{\sharp\mathcal{A}_2}\})$$

and observe that $\mu(\Gamma) > 0$. Let $\mathcal{A}_1^* = \{m_1, \dots, m_{\sharp\mathcal{A}_1}\}$ and $\mathcal{A}_2^* = \{l_1, \dots, l_{\sharp\mathcal{A}_2}\}$. Denote

$$\Delta = \sum_{i=1}^{\sharp\mathcal{A}_1} m_i^* + \sum_{j=1}^{\sharp\mathcal{A}_2} l_j^*.$$

We put

$$\begin{aligned} \theta(g_1) = t,\ \theta(g_2) = t^{-1}, \qquad & \theta(x_i) = t^{m_i^*},\ i = 1, \ldots, \sharp\mathcal{A}_1 \\ \theta(e_j) = t^{l_j^*}, \qquad & \theta(c_{j,0}) = 1,\ j = 1, \ldots, \sharp\mathcal{A}_2 \\ & \theta(g_3) = \begin{cases} t^{-\frac{\Delta}{2}}, \text{if } \Delta \text{ is even} \\ t^{-\frac{\Delta}{2}+\frac{N}{2}} \text{ otherwise.} \end{cases} \end{aligned}$$

By (4.7) we have $\mathrm{CPer}(t) = (\mathcal{A}_1, \mathcal{A}_2, \emptyset, \emptyset, \emptyset, \emptyset)$, which finishes the proof.. □

As before, for N odd, we abbreviate $(\mathcal{A}_1, \mathcal{A}_2, \emptyset, \emptyset, \emptyset, \emptyset) \in \mathrm{CPer}^{(-,-)}(\mathbb{Z}_N)$ to $(\mathcal{A}_1, \mathcal{A}_2)^-$. Now we improve on the construction of a NEC group which covers $\mathfrak{C}_0 \in \mathrm{CPer}^{(-,-)}(\mathbb{Z}_N)$ in sense of minimizing measure of its fundamental region.

Lemma 5.9. *Let N be odd and suppose that $\theta\colon \Gamma \to \mathbb{Z}_N$ covers a $\mathbb{Z}_N$–action with the character of periods $\mathfrak{C}_0 = (\mathcal{A}_1, \mathcal{A}_2)^-$. Then $\gamma = 1$ implies $\mathrm{lcm}\mathfrak{C}_0^* = N$.*

Proof. Contrary, assume that $\mathrm{lcm}\mathfrak{C}_0^* < N$. By the "long relation" we have

$$\theta(g_1) \prod_{i=1}^{n} \theta(x_i) \prod_{i=1}^{r} \theta(e_i) = 1,$$

which yields $\theta(g_1) \in \mathbb{Z}_{\mathrm{lcm}\mathfrak{C}_0^*}$. Thus we get $|\theta(\Gamma)| = \mathrm{lcm}\mathfrak{C}_0^*$, which contradicts the assumption that θ is an epimorphism. □

Theorem 5.10. *Let N be odd and suppose that $\mathfrak{C}_0 = (\mathcal{A}_1, \mathcal{A}_2)^-$, where $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2 \geq 2$. Consider*

$$\Gamma = (2 - \delta_N(\mathrm{lcm}\mathfrak{C}_0^*); -; [\mathcal{B}_1^*]; \{(\ )^{\sharp\mathcal{B}_2}\}), \tag{5.4}$$

where the terms $\mathcal{B}_i$ were given in (4.33). The above group Γ is a universal covering group of $(\langle t\rangle, X)$, $\langle t\rangle = \mathbb{Z}_N$. It satisfies $\mathrm{CPer}(t) = \mathfrak{C}_0$. Moreover the area $\mu(\Lambda)$, where $X = \mathbb{H}^2/\Lambda$, is minimal among all non–orientable surfaces on which $\mathfrak{C}_0$ is attained as the character of periods.

Proof. Throughout the proof we use the signature $\Gamma_{\mathfrak{N}}$ given by (4.34). Furthermore a part of our proof is based on procedure $\mathfrak{N}$.

Assume that $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2$ is even. Consider the following character associated to $\mathfrak{C}$: $\mathfrak{D}(\mathfrak{C}) = \mathfrak{C}$. We obtain that $L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$ is even, where $\overline{\eta}_{\mathfrak{D}(\mathfrak{C})}$ stands for order–preserving element with respect to $\mathfrak{D}(\mathfrak{C})$ which verifies $\eta_{i,j} = 1$. Hence we may apply the procedure $\mathfrak{N}$ and define the following assignments

$$\theta(g_1) = t^{-\frac{L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})}{2}} \text{ if } \mathrm{lcm}\mathfrak{C}_0^* = N$$

$$\theta(g_1) = t,\ \theta(g_2) = t^{-\frac{L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})}{2}-1} \text{ if } \mathrm{lcm}\mathfrak{C}_0^* \neq N.$$

On the other hand if $\overline{\eta}$ is an orientation–preserving element with respect to $\mathfrak{D}(\mathfrak{C})$ and $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2$ is odd, then $L(\overline{\eta})$ is odd. Thus we are able to map the required glide reflections as follows

$$\theta(g_1) = t^{-\frac{L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})}{2}+\frac{N}{2}} \text{ if } \mathrm{lcm}\mathfrak{C}_0^* = N$$
$$\theta(g_1) = t,\ \theta(g_2) = t^{-\frac{L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})}{2}-1+\frac{N}{2}} \text{ if } \mathrm{lcm}\mathfrak{C}_0^* \neq N.$$

We proceed to show that the groups defined above have positive measure. Since N is odd we have $\sum_{m\in\mathcal{A}_1^*}(1-m^{-1}) \geq 2/3$. By (4.18) we have

$$\mu(\Gamma) = 2\pi(2 - \delta_N(\mathrm{lcm}\mathfrak{C}_0^*) + \sharp\mathcal{A}_2 - 2 + \sum_{m\in\mathcal{A}_1^*}(1-m^{-1}))$$

Furthermore by the assumption $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2 \geq 2$ we obtain

$$\sharp\mathcal{A}_2 + \sum_{m\in\mathcal{A}_1^*}(1-m^{-1}) \geq \frac{4}{3}$$

which yields $\mu(\Gamma) > 0$.

We are now in a position to show that the measure of Γ constructed above is minimal. Assume that $\theta\colon \Gamma' \to \mathbb{Z}_N$ is another epimorphism that covers $\mathfrak{C}_0$, where

$$\Gamma' = (\gamma'; \pm; [m_1', ..., m_{n'}']; \{(\ )^{r'}\})$$

Observe that $\gamma' = 1$ gives by Lemma 5.9 the equality $\mathrm{lcm}\mathfrak{C}_0^* = N$, which in turn yields $\gamma = 1$. On the other hand $\gamma' \geq 2$ results in $\gamma' \geq \gamma$. Hence by Proposition 4.19 we have

$$\mu(\Gamma') - \mu(\Gamma) = \mu(\Gamma') - (\mu(\Gamma_{\mathfrak{N}}) + 2\pi\gamma) \geq 2\pi(\gamma' - \gamma) \geq 0.$$

Since Theorem 5.1 yields that the group Λ is non–orientable the proof is finished. □

In order to complete the analysis of actions of cyclic groups of odd order on non–orientable surfaces we state a remark that deals with the case $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2 < 2$.

Remark 5.11. *Under the assumptions of Theorem 5.10 with $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2 < 2$ the universal covering group that covers a $\mathbb{Z}_N$–action prescribed by $\mathfrak{C}_0$ equals*

$$\Gamma = \begin{cases} (3; -; [\ ]; \{\ \}), & \text{if } \mathfrak{C}_0 = (\emptyset, \emptyset)^- \\ (1; -; [N, N]; \{\ \}), & \text{if } \mathfrak{C}_0 = (\{1\}, \emptyset)^- \\ (2; -; [N/d]; \{\ \}), & \text{if } \mathfrak{C}_0 = (\{d\}, \emptyset)^- \text{ and } d \neq 1 \\ (1; -; [\ ]; \{(\ )^2\}), & \text{if } \mathfrak{C}_0 = (\emptyset, \{1\})^- \\ (2; -; [\ ]; \{(\ )\}), & \text{if } \mathfrak{C}_0 = (\emptyset, \{d\})^- \text{ and } d \neq 1 \end{cases}$$

The area of the respective NEC groups is minimal among all non–orientable surfaces on which the above 2–tuples of sets are attained as the characters of periods.

We finish this section with an easy remark being an immediate application of the above results.

Corollary 5.12. *Let N be odd. By Remark 5.6 and 5.11 we conclude that the minimum algebraic genus of a surface without boundary on which $\mathbb{Z}_N$ acts without fixed points equals $N+1$.*

Proof. Obviously we have $\mathfrak{C}_0 = (\emptyset, \emptyset)^+$ or $\mathfrak{C}_0 = (\emptyset, \emptyset)^-$. First consider the case of an orientable surface without boundary i.e. a Riemann surface. By Remark 5.6 a NEC group covering fixed point free $\mathbb{Z}_N$–action with minimal measure equals $\Gamma = (2; +; [\,]; \{\,\})$. We define $\theta\colon \Gamma \to \mathbb{Z}_N$ as follows

$$\theta(a_i) = \theta(b_i) = t, \qquad i = 1, 2$$

and obtain $\mu(\ker\theta) = N\mu(\Gamma)$ by Riemann–Hurwitz formula. It yields $\ker\theta = (N{+}1; +; [\,]; \{\,\})$. By definition the algebraic genus of a Riemann surface equals its topological genus.

On the other hand if $\mathbb{Z}_N$ acts on a non–orientable surface X, we put $\Gamma = (3; -; [\,]; \{\,\})$ according to Remark 5.11 and define

$$\theta(g_1) = t, \quad \theta(g_2) = t^{-1}, \quad \theta(g_3) = 1.$$

We now conclude that $\mu(\ker\theta) = 2\pi N$, which gives $\ker\theta = (N+2; -; [\,]; \{\,\})$. By (3.6) we obtain $p(X) = g(X_C) = g(X) - 1 = N + 1$ as required. □

Observe that the above corollary may be considered as a counterpart of Remark 2.6.

In sections 5.2 and 5.3, we proceed to study the actions of cyclic groups of even order on compact surfaces. As before, we reduce the analysis to the study of NEC groups forming N–pairs with prescribed characters of periods. Since the case of N even is more involved than the analogous investigation for N odd we will now apply more procedures and prototypes of covering NEC groups which we introduced in Subsections 4.3.1 and 4.3.2. The following theorems of [8]: Theorem 3.1.5, Theorem 3.1.6, Theorem 3.1.8 and Theorem 3.1.9 were a great motivation for this part of the thesis. For the sake of completeness we recall them on the forthcoming pages. However, we will avoid references to their respective intrinsic proofs by introducing some auxiliary results. For convenience, we separate the assumptions that are shared by all theorems mentioned above. We name these conditions *Common conditions for N–pairs*. Let us suppose that N is even and Γ is a NEC group covering a $\mathbb{Z}_N$–action. Recall that it takes the form

$$\Gamma = (\gamma; \pm; [m_1, ..., m_n]; \{(\ )^{\lambda}(2^{\mu_1})...(2^{\mu_p})\}).$$

Let us denote a surface NEC group Λ by

$$\Lambda = (g; \pm; [\ ]; \{(\ )^k\}) \tag{5.5}$$

Common conditions for N–pairs
(*necessary for* (Γ, Λ) *to form an N–pair*)

(i) for each $i = 1, \ldots, n$ we have $m_i \mid N$

(ii) $\mu(\Lambda) = N\mu(\Gamma)$

(iii) there exist $0 \leq r \leq \lambda$ and some positive divisors $l_1, \ldots, l_r$ of N such that:

(iii.1) $k = \sum_{i=1}^{r} N/l_i + N/2 \sum_{i=1}^{p} \mu_i/2$

In short, we will call the above conditions *the CCN conditions*, and henceforth we refer to them in this way.

5.2 Actions of Groups of Even Order on Orientable Surfaces

Having come up with the auxiliary results and definitions in Chapter 4, we are now in a position to investigate the possible characters of periods of the actions of cyclic groups of even order on orientable surfaces. It is worth noting that in the actual setting either two–sided ovals or two–sided chains may become apparent. In the general case we have $\mathfrak{C} \neq \mathfrak{C}_0$, since we must not assume that $\mathcal{A}_3 \cup \mathcal{A}_5 \neq \emptyset$. Nevertheless, the variety of types of periodic structures that appear on an orientable surface does not embrace either one–sided ovals or one–sided chains. We continue using the notation introduced in the preceding sections.

Theorem 5.13 (Bujalance et al. [8], Theorem 3.1.5). *Let us suppose N is even and* $\mathrm{sign}\Gamma = \mathrm{sign}\Lambda = {}$" $+$ ". *Then* (Γ, Λ) *is an N–pair if and only if the CCN conditions are fulfilled and*

(iii.2) if $4 \mid N$, then $r = \lambda + p$, $\{m_1, \ldots, m_n, l_1, \ldots, l_r\}$ has the elimination property, and for $\gamma = 0$ we have $\mathrm{lcm}(m_1, \ldots, m_n, l_1, \ldots, l_r) = N$

(iii.3) if $4 \nmid N$ and $r = \lambda + p$, condition (iii.2) holds true

(iii.4) if $4 \nmid N$ and $r < \lambda + p$, then each m_i and l_i divides $N/2$. Moreover, if $\gamma = 0$ and $\lambda + p = r + 1$, then $\mathrm{lcm}(m_1, \ldots, m_n, l_1, \ldots, l_r) = N/2$.

We begin by looking for 6–tuples of divisors of N which become apparent as the characters of periods belonging to $\mathrm{CPer}^{(+,+)}(\mathbb{Z}_N)$ for N even.

Lemma 5.14. *Let N be even and suppose that $\theta\colon \Gamma \to \mathbb{Z}_N$ covers a $\mathbb{Z}_N$–action with the character of periods $\mathfrak{C}_0 \in \mathrm{CPer}^{(+,+)}(\mathbb{Z}_N)$ which verifies $\mathcal{A}_3 \cup \mathcal{A}_5 \neq \emptyset$. Then*

(i) $4 \nmid N$

(ii) $\mathcal{A}_1$ and $\mathcal{A}_2$ comprise even periods exclusively.

Proof. Consider point *(i)*. Since $\theta\colon \Gamma \to \mathbb{Z}_N$ is onto $\mathbb{Z}_N$, there is $w \in \Gamma$ satisfying $\theta(w) = t$. Furthermore $\mathcal{A}_3 \cup \mathcal{A}_5 \neq \emptyset$ gives a reflection c which is mapped to $t^{N/2}$. Obviously we have $w^{N/2}c \in \ker\theta$. If now $4 \mid N$, then $w^{N/2}c$ is non–orientable which yields $\mathrm{sign}\ker\theta = {}$" $-$ " by Theorem 4.12, a contradiction.

Now we prove point *(ii)* also by contradiction. Take an odd $d \in \mathcal{A}_1 \cup \mathcal{A}_2$ and let $h_d \in \Gamma$ stand for an x– or e–generator that corresponds to period d. Note that in either case h_d is an orientable word of Γ. Denote $\theta(h_d) = t^v$ which must verify $(N, v) = d$. Thus $\mathrm{ord}\theta(h_d) = N/d$ is an even number, which yields $t^{N/2} \in \langle\theta(h_d)\rangle$. Consequently there is a natural number A satisfying $t^{Av} = t^{N/2}$. Now it is enough to observe that $h_d^A c$ is a non–orientable word and $h_d^A c \mapsto 1$. As before we obtain $\mathrm{sign}\ker\theta = {}$" $-$ ", which finishes the proof. □

Proposition 5.15. *Suppose that N is even. An element $\mathfrak{C}_0$ belongs to the set of characters of periods $\mathrm{CPer}^{(+,+)}(\mathbb{Z}_N)$ if and only if it takes one of the following forms*

(i) $(\mathcal{A}_1, \mathcal{A}_2, \emptyset, \emptyset, \emptyset, \emptyset)$

(ii) $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \emptyset, \mathcal{A}_5, \emptyset)$, where $4 \nmid N$ and $\mathcal{A}_1, \mathcal{A}_2$ consist only of even periods.

Proof. ($\Rightarrow$) Assume that $\theta\colon \Gamma \to \mathbb{Z}_N$ covers a $\mathbb{Z}_N$–action with the character of periods prescribed by $\mathfrak{C}_0$. Denote $\Lambda = \ker\theta$ and let $t\colon \mathbb{H}^2/\Lambda \to \mathbb{H}^2/\Lambda$ be the underlying dianalytic map. Observe that $\mathcal{A}_4 \cup \mathcal{A}_6 = \emptyset$ since $\mathbb{Z}_N$ acts on an orientable surface which does not admit one–sided components of the singular set. Furthermore if $\mathcal{A}_3 \cup \mathcal{A}_5 \neq \emptyset$, then point *(i)* of the last lemma yields $4 \nmid N$. Moreover by point *(ii)* of the same lemma we obtain $\mathcal{A}_1$ and $\mathcal{A}_2$ comprise even periods as required.

($\Leftarrow$) The proof in case *(i)* i.e. $\mathcal{A}_3 \cup \mathcal{A}_5 = \emptyset$ is identical to the proof of the "if" part of Proposition 5.4 based on a NEC group (5.1) and epimorphism given by (5.2).

If we are in case *(ii)*, then instead of (5.1) we shall consider another NEC group

$$\Gamma = (2; +; [\mathcal{O}_1^*]; \{(\ )^{w_1}(2^2)^{w_2}\}),$$

where w_1 and w_2 were defined by (4.27). In order to map Γ onto a cyclic group of order N we apply the homomorphism θ defined in (4.26)–(4.28). Furthermore, we require that θ maps each of the hyperbolic generators a_j, b_j, $j = 1, 2$ to an element of order N in $\mathbb{Z}_N$. Let $t\colon \mathbb{H}^2/\ker\theta \to \mathbb{H}^2/\ker\theta$ be the underlying dianalytic map. By (4.7) we have $\mathrm{CPer}(t) = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \emptyset, \mathcal{A}_5, \emptyset)$. We determine the orientability character of the kernel of the above epimorphism by observing that the only generators of Γ reversing the orientation are the reflections. Since $\mathcal{A}_1$, $\mathcal{A}_2$ comprise even periods and $4 \nmid N$ each number belonging to $\mathcal{A}_1^*$, $\mathcal{A}_2^*$ is odd. Observe that it yields that $\mathrm{lcm}\mathfrak{C}^*$ is an odd number. Thus $\theta(w) \neq t^{N/2}$ for each orientable word w in Γ. Hence there are not non–orientable words belonging to Λ and by Theorem 4.12 a surface NEC group $\ker\theta$ is orientable. It follows that $\mathfrak{C}_0 \in \mathrm{CPer}^{(+,+)}(\mathbb{Z}_N)$. □

Observe that in the proof of the "if" part of the above corollary we have not taken advantage of Theorem 5.13. It becomes natural when we recall that in [8] the authors did not pay attention to periodic structures other than isolated orbits and boundaries i.e. the structures described by $\mathcal{A}_1$ and $\mathcal{A}_2$. Thus the diversity of ways of mapping the e–generators corresponding to ovals and chains are outside of scope of Theorem 5.13. For further reference we point the reader to the proof of Theorem 3.1.5 in [8].

We abbreviate $(\mathcal{A}_1, \mathcal{A}_2, \emptyset, \emptyset, \emptyset, \emptyset)$, $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \emptyset, \mathcal{A}_5, \emptyset) \in \mathrm{CPer}^{(+,+)}(\mathbb{Z}_N)$ to $(\mathcal{A}_1, \mathcal{A}_2)^{(+,+)}$ and $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_5)^{(+,+)}$ respectively.

We will use the term $\mathrm{lcm}\mathfrak{C}_0^*$ to determine genera of NEC groups covering $\mathfrak{C}_0$ which we are about to construct. In the following remark we show why in point *(iii.4)* of Theorem 5.13 it is required that $\mathrm{lcm}(m_1, \ldots, m_n, l_1, \ldots, l_r) = N/2$ and how it relates to $\mathrm{lcm}\mathfrak{C}_0^*$. Recall that if (Γ, Λ) is an N–pair, then according to the proof of Theorem 3.1.5 in [8] the symbols l_i, $i = 1, \ldots, r$ are the orders of images of e–generators of empty period cycles of Γ under a smooth epimorphism $\theta\colon \Gamma \to \mathbb{Z}_N$ verifying $\ker\theta = \Lambda$.

Remark 5.16. *Consider $\mathfrak{C}_0 = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_5)^{(+,+)}$, where $\mathcal{A}_3 \cup \mathcal{A}_5 \neq \emptyset$. Let (Γ, Λ) be an N–pair and assume that $\theta\colon \Gamma \to \mathbb{Z}_N$ is a smooth epimorphism such that $\ker\theta = \Lambda$, which covers a $\mathbb{Z}_N$–action given by $\mathfrak{C}_0$. Observe that by the first part of point (iii.4) of Theorem 5.13 we have $4 \nmid N$ and $\mathrm{lcm}(m_1, \ldots, m_n, l_1, \ldots, l_r) = \mathrm{lcm}(\mathcal{A}_1^*, \mathcal{A}_2^*) \leq N/2$. If now the underlying dianalytic map has exactly one two–sided oval or chain i.e. $\lambda + p = r + 1$ then by the "long relation" of Γ we have*

$$\theta^{-1}(e_{r+1}) = \prod_{i=1}^{n} \theta(x_i) \prod_{i=1}^{r} \theta(e_i),$$

which yields $\theta(e_{r+1}) \in \langle \theta(x_1), \ldots, \theta(e_r) \rangle$. But $\langle \theta(x_1), \ldots, \theta(e_r) \rangle = \mathbb{Z}_{\mathrm{lcm}(\mathcal{A}_1^*, \mathcal{A}_2^*)}$. Observe that by point (ii) of Remark 4.20 the equality $\gamma = 0$ forces

$$N = \mathrm{lcm}\mathfrak{C}_0^* = 2\mathrm{lcm}\mathfrak{C}^* = 2\mathrm{lcm}(\mathcal{A}_1^*, \mathcal{A}_2^*) = 2\mathrm{lcm}(m_1, \ldots, m_n, l_1, \ldots, l_r),$$

which gives $\mathrm{lcm}(m_1, \ldots, m_n, l_1, \ldots, l_r) = N/2$.

Now we formulate one of main results of this section concerning the construction of NEC groups with minimal measure, provided they cover certain $\mathbb{Z}_N$–action on Klein surfaces.

Theorem 5.17. *Let N be even. Suppose that $\mathfrak{C}_0$ fulfils condition (i) or (ii) of the previous proposition and $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2 + \sharp\mathcal{A}_3 + \sharp\mathcal{A}_5 \geq 2$.*

(i) If $\mathcal{A}_5 = \emptyset$, then we put

$$\begin{aligned}
&\Gamma = (0; +; [N, N, N]; \{(\ )\}) \text{ for } \mathfrak{C}_0 = (\{1\}, \{1\})^{(+,+)} \\
&\Gamma = (0; +; [2, 2, 2, 2, 2, 2]; \{\ \}) \text{ for } \mathfrak{C}_0 = (\{1\}, \emptyset)^{(+,+)} \text{ and } N = 2 \\
&\Gamma = (0; +; [2, 2, 4, 4]; \{\ \}) \text{ for } \mathfrak{C}_0 = (\{1, 2\}, \emptyset)^{(+,+)} \text{ and } N = 4 \\
&\Gamma = (0; +; [2, 2, 2, 3, 6]; \{\ \}) \text{ for } \mathfrak{C}_0 = (\{1, 2, 3\}, \emptyset)^{(+,+)} \text{ and } N = 6
\end{aligned}$$

(ii) otherwise

$$\Gamma = (1 - \delta_N(\mathrm{lcm}\mathfrak{C}_0^*); +; [\mathcal{O}_1^*]; \{(\ )^{w_1}(2^2)^{w_2}\}), \tag{5.6}$$

where $\mathcal{O}_1$ and w_1, w_2 were defined in (4.24) and (4.27) respectively.

In the respective cases the above group Γ is a universal covering group of $(\langle t\rangle, X)$, $\langle t\rangle = \mathbb{Z}_N$. It satisfies $\mathrm{CPer}(t) = \mathfrak{C}_0$. *Moreover the area $\mu(\Lambda)$, where $X = \mathbb{H}^2/\Lambda$, is minimal among all orientable surfaces on which $\mathfrak{C}_0$ is attained as the character of periods.*

Proof. The proof of case *(i)* runs by direct computations and therefore it will be omitted.

Consider the case *(ii)*. By the procedure $\mathfrak{O}$ we obtain a group $\Gamma_{\mathfrak{O}}$ given by (4.29) together with the homomorphism θ defined in (4.26)–(4.28). In case $\mathrm{lcm}\mathfrak{C}_0^* \neq N$, we must add at least one pair of hyperbolic generators corresponding to the topological genus of Γ. We then require that θ maps a_1 and b_1 to any element of order N in $\mathbb{Z}_N$. Likewise in the proof of the "if" part of Proposition 5.15 we determine the orientability character of $\ker\theta$ by Theorem 4.12. Hence we get $\mathrm{sign}\Lambda =$ " + ".

In order to verify that group Γ obtained by the above construction is a NEC group we must check if $\mu(\Gamma)$ is positive. We split the analysis of the general situation $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2 + \sharp\mathcal{A}_3 + \sharp\mathcal{A}_5 \geq 2$ into a number of cases considered separately.

We first find the lower bound for $\mu(\Gamma)$ assuming $\mathcal{A}_5 = \emptyset$ and $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2 + \sharp\mathcal{A}_3 > 2$. Simple computations show

$$\begin{aligned}
\mu(\Gamma) &= 2\pi(2(1 - \delta_N(\mathrm{lcm}\mathfrak{C}_0^*)) + w_1 + w_2 - 2 + \sum_{m \in \mathcal{O}_1^*} (1 - m^{-1}) + \frac{1}{2} w_2) \\
&\geq 2\pi(2(1 - \delta_N(\mathrm{lcm}\mathfrak{C}_0^*)) + w_1 - 2 + \sum_{m \in \mathcal{O}_1^*} (1 - m^{-1})) > 0,
\end{aligned}$$

except the cases pointed out explicitly in point *(i)*.

Consequently, once $\mathcal{A}_5 = \emptyset$ it remains to consider only the cases $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2 + \sharp\mathcal{A}_2 = 2$. Below we make calculations in all required cases.

$$
\begin{aligned}
&(\sharp\mathcal{A}_1, \sharp\mathcal{A}_2, \sharp\mathcal{A}_3) = (2,0,0) \Rightarrow \sharp\mathcal{O}_1^* \geq 3 \Rightarrow \sum_{m\in\mathcal{O}_1^*}(1-m^{-1}) \geq 2 \Rightarrow \mu(\Gamma) > 0 \\
&(\sharp\mathcal{A}_1, \sharp\mathcal{A}_2, \sharp\mathcal{A}_3) = (0,2,0) \Rightarrow w_1 \geq 3 \Rightarrow \mu(\Gamma) > 0 \\
&(\sharp\mathcal{A}_1, \sharp\mathcal{A}_2, \sharp\mathcal{A}_3) = (0,0,2) \Rightarrow w_1 \geq 3 \Rightarrow \mu(\Gamma) > 0 \\
&(\sharp\mathcal{A}_1, \sharp\mathcal{A}_2, \sharp\mathcal{A}_3) = (1,1,0) \text{ and } \mathcal{A}_1 \neq \mathcal{A}_2 \Rightarrow w_1 + \sum_{m\in\mathcal{O}_1^*}(1-m^{-1}) > 2 \Rightarrow \mu(\Gamma) > 0 \\
&(\sharp\mathcal{A}_1, \sharp\mathcal{A}_2, \sharp\mathcal{A}_3) = (1,1,0) \text{ and } \mathcal{A}_1 = \mathcal{A}_2 = \{d\}, d \neq 1 \Rightarrow \mu(\Gamma) > 0 \\
&(\sharp\mathcal{A}_1, \sharp\mathcal{A}_2, \sharp\mathcal{A}_3) = (1,0,1) \text{ and } \mathcal{A}_1 \neq \mathcal{A}_3 \Rightarrow w_1 + \sum_{m\in\mathcal{O}_1^*}(1-m^{-1}) > 2 \Rightarrow \mu(\Gamma) > 0 \\
&(\sharp\mathcal{A}_1, \sharp\mathcal{A}_2, \sharp\mathcal{A}_3) = (1,0,1) \text{ and } \mathcal{A}_1 = \mathcal{A}_3 = \{d\} \Rightarrow d \neq 1 \Rightarrow \mu(\Gamma) > 0 \\
&(\sharp\mathcal{A}_1, \sharp\mathcal{A}_2, \sharp\mathcal{A}_3) = (0,1,1) \text{ and } \mathcal{A}_2 \neq \mathcal{A}_3 \Rightarrow w_1 \geq 3 \Rightarrow \mu(\Gamma) > 0 \\
&(\sharp\mathcal{A}_1, \sharp\mathcal{A}_2, \sharp\mathcal{A}_3) = (0,1,1) \text{ and } \mathcal{A}_2 = \mathcal{A}_3 = \{d\} \Rightarrow d \neq 1 \Rightarrow \mu(\Gamma) > 0
\end{aligned}
\tag{5.7}
$$

On the other hand $\mathcal{A}_5 \neq \emptyset$ forces by point Lemma 5.14 that $4 \nmid N$ and $\mathcal{A}_1$ comprises even periods. Thus all $m_i \in \mathcal{A}_1^*$, $i = 1, \ldots, \sharp\mathcal{O}_1$ are odd. It follows that $m_i \geq 3$ and $\sum_{m\in\mathcal{O}_1^*}(1 - m^{-1}) \geq 2\sharp\mathcal{A}_1/3$. Consequently,

$$\mu(\Gamma) \geq 2\pi(w_1 + \frac{3}{2}w_2 - 2 + \sum_{m\in\mathcal{O}_1^*}(1-m^{-1})) \geq 2\pi(\frac{3}{2} + \frac{2}{3}) > 0.$$

Next we claim that $\mu(\Gamma)$ is minimal. Note that $\gamma' \geq \gamma$, where γ' is genus of another NEC group Γ' that covers a $\mathbb{Z}_N$–action prescribed by $\mathfrak{C}_0$ which follows from Remark 4.23. By Proposition 4.17 we have

$$\mu(\Gamma') - \mu(\Gamma) = \mu(\Gamma') - (\mu(\Gamma_{\mathfrak{D}}) + 4\pi\gamma) \geq 4\pi(\gamma' - \gamma) \geq 0.$$

□

Since we have dropped the case $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2 + \sharp\mathcal{A}_3 < 2$ with $\mathcal{A}_5 = \emptyset$ we complete the study by the following remark.

Remark 5.18. *If $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2 + \sharp\mathcal{A}_3 < 2$, $\mathcal{A}_5 = \emptyset$ and the remaining assumptions of Theorem*

5.17 hold then the universal covering group that covers a $\mathbb{Z}_N$–action prescribed by $\mathfrak{C}_0$ equals

$$\Gamma = \begin{cases} (2;+;[\,];\{\,\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\emptyset)^{(+,+)} \\ (0;+;[N,N,N,N];\{\,\}), & \text{if } \mathfrak{C}_0 = (\{1\},\emptyset)^{(+,+)} \\ (1;+;[N/d,N/d];\{\,\}), & \text{if } \mathfrak{C}_0 = (\{d\},\emptyset)^{(+,+)} \text{ and } d \neq 1 \\ (0;+;[\,];\{(\,)^4\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\{1\})^{(+,+)} \\ (1;+;[\,];\{(\,)^2\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\{d\})^{(+,+)} \text{ and } d \neq 1 \\ (0;+;[\,];\{(\,)^4\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\emptyset,\{1\},\emptyset)^{(+,+)} \\ (1;+;[\,];\{(\,)^2\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\emptyset,\{d\},\emptyset)^{(+,+)} \text{ and } d \neq 1 \end{cases}$$

The area of the respective NEC groups is minimal among all orientable surfaces on which the above tuples of sets are attained as the characters of periods.

In the following example we discuss briefly some differences between the above cases.

Example 5.19. *Consider two epimorphism $\theta\colon \Gamma \to \mathbb{Z}_N$ from the same group*

$$\Gamma = (0;+;[\,];\{(\,)^4\})$$

corresponding to the following characters of periods $(\emptyset,\{1\})^{(+,+)}$ and $(\emptyset,\emptyset,\{1\},\emptyset)^{(+,+)}$.

Let us put for $\mathfrak{C}_0 = (\emptyset,\{1\})^{(+,+)}$

$$e_1, e_2 \mapsto t \text{ and } e_3, e_4 \mapsto t^{N-1},$$

which gives $\mathrm{lcm}\mathfrak{C}^* = N$*. Hence a subgroup of $\mathbb{Z}_N$ generated by the images of canonical e–generators of Γ has order N i.e. there is no need to extend Γ by additional hyperbolic generators.*

On the other hand in the latter case we must have $4 \nmid N$ by Lemma 5.14. We define

$$e_i \mapsto t^2,$$

since we consider fixed two–sided ovals. Unlike the previous case we actually have $\mathrm{lcm}\mathfrak{C}^* = N/2$ *i.e. a subgroup of $\mathbb{Z}_N$ generated by the images of canonical orientable generators of Γ does not equal the whole $\mathbb{Z}_N$. However there is no need for additional hyperbolic generators either, because $t^{N/2} \notin \langle t^2 \rangle$ and there is a reflection $c \in \Gamma$ mapped to $t^{N/2}$.*

We now proceed to study the set of characters of periods $\mathrm{CPer}^{(-,+)}(\mathbb{Z}_N)$. Observe that under this assumption we deal only with orientation–reversing automorphisms. We start with recalling another theorem on N–pairs from [8].

Theorem 5.20 (Bujalance et al. [8], Theorem 3.1.9)**.** *Let N be even,* $\mathrm{sign}\Gamma = $ " $-$ " *and* $\mathrm{sign}\Lambda = $ " $+$ "*. Then (Γ,Λ) is an N–pair if and only if the CCN conditions are fulfilled and*

(iii.2) $M = \mathrm{lcm}(m_1,\ldots,m_n,l_1,\ldots,l_r) \neq N$

(iii.3) $2 \mid \sharp\{m_i, l_j \mid \alpha_2(m_i) = \alpha_2(l_j) = \alpha_2(N)\}$

(iii.4) there exists an order–preserving pair (α, β) *with respect to* $\{N/m_1, \dots, N/m_n, N/l_1, \dots, N/l_r\}$ *such that all* $\alpha_i N/m_i$ *and* $\beta_j N/l_j$ *are even numbers; moreover, if* $r = \lambda + p$ *and* $4 \mid N$*, the number* $S(\alpha, \beta)$ *is a multiple of* 4 *if and only if* γ *is even*

(iii.5) if $r < \lambda + p$*, then the numbers* m_i *and* l_j *are odd, and* $4 \nmid N$

(iii.6) if $\gamma = 1$ *and* $r = \lambda + p$*, then* $M = N/2$*.*

Remark 5.21. *Observe that point (iii.3) of Theorem 5.20 follows from the first part of (iii.4). Indeed* $\alpha_2(m_i) = \alpha_2(N)$ *clearly forces that* N/m_i *is odd. But* α_i *is also an odd number since* $(\alpha_i, m_i) = 1$*. Since the same holds for* l_j *the set mentioned in (iii.3) is empty. Hence that point can be dropped. Similarly (iii.4)–(iii.6) yield (iii.2), which is also superfluous.*

We give below a slightly modified version of point *(iii.6)* of the last theorem, where the assumption $r = \lambda + p$ is no longer required.

Lemma 5.22. *Suppose that* $\mathfrak{C}_0 \in \mathrm{CPer}^{(-,+)}(\mathbb{Z}_N)$*. Let* $t\colon X \to X$ *be a dianalytic map and* $\mathrm{CPer}(t) = \mathfrak{C}_0$*. If the genus of a NEC group covering a* $\mathbb{Z}_N$*–action of* t *equals 1, then* $\mathrm{lcm}\mathfrak{C}^* = N/2$*.*

Proof. Suppose that $\theta\colon \Gamma \to \mathbb{Z}_N$ is a surface–kernel epimorphism that uniformizes the required $\mathbb{Z}_N$ action. Recall that we observed in (4.46) that

$$\mathrm{lcm}\mathfrak{C}^* = |\langle \theta(x_1), \dots, \theta(x_n), \theta(e_1), \dots, \theta(e_{\lambda+p})\rangle|.$$

By Theorem 4.12 we obtain $\theta(g_1) \notin \mathbb{Z}_{\mathrm{lcm}\mathfrak{C}^*}$, but $\theta(g_1^2) \in \mathbb{Z}_{\mathrm{lcm}\mathfrak{C}^*}$ by the "long relation" in Γ. Thus $|\langle \theta(g_1), \mathbb{Z}_{\mathrm{lcm}\mathfrak{C}^*}\rangle| = 2\mathrm{lcm}\mathfrak{C}^*$. If there is no reflection $c \in \Gamma$ with $\theta(c) = t^{N/2}$, i.e. $r = \lambda + p$, then $\langle \theta(g_1), \mathbb{Z}_{\mathrm{lcm}\mathfrak{C}^*}\rangle = \mathbb{Z}_N$ and we are done. On the other hand even if there exists such a reflection c in Γ it must hold $\theta(c) \in \langle \theta(g_1), \mathbb{Z}_{\mathrm{lcm}\mathfrak{C}^*}\rangle$, since $\langle t^{N/2}\rangle = \mathbb{Z}_2 \leq \mathbb{Z}_{2\mathrm{lcm}\mathfrak{C}^*} \leq \mathbb{Z}_N$. It again leads us to $N = |\theta(\Gamma)| = 2\mathrm{lcm}\mathfrak{C}^*$. □

Proposition 5.23. *An element* $\mathfrak{C}_0$ *belongs to the set of characters of periods* $\mathrm{CPer}^{(-,+)}(\mathbb{Z}_N)$ *of* $\mathbb{Z}_N$*–actions by dianalytic transformations on orientable surfaces only if* $\mathcal{A}_1$*,* $\mathcal{A}_2$ *comprise even periods and it is of one of the following forms*

(i) $(\mathcal{A}_1, \mathcal{A}_2, \emptyset, \emptyset, \emptyset, \emptyset)$

(ii) $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \emptyset, \mathcal{A}_5, \emptyset)$*, where* $4 \nmid N$*.*

Proof. Assume that $\theta\colon \Gamma \to \mathbb{Z}_N$ covers a $\mathbb{Z}_N$–action with the character of periods prescribed by $\mathfrak{C}_0$. Denote $\Lambda = \ker\theta$ and let $t\colon \mathbb{H}^2/\Lambda \to \mathbb{H}^2/\Lambda$ be the underlying dianalytic map. Likewise in Proposition 5.15 we observe that $\mathcal{A}_4 \cup \mathcal{A}_6 = \emptyset$ since $\mathbb{Z}_N$ acts on an orientable surface and one–sided components of the singular set do not become apparent in this setting. By Remark 5.21 we have $\{m_i, l_j \mid \alpha_2(m_i) = \alpha_2(l_j) = \alpha_2(N)\} = \emptyset$, which shows that $\mathcal{A}_1$ and $\mathcal{A}_2$ may contain only even periods.

It remains to show that $\mathcal{A}_3 \cup \mathcal{A}_5 \neq \emptyset$ yields $4 \nmid N$. Since $\theta\colon \Gamma \to \mathbb{Z}_N$ is onto $\mathbb{Z}_N$, there is $w \in \Gamma$ satisfying $\theta(w) = t$. Furthermore if $\mathcal{A}_3 \cup \mathcal{A}_5 \neq \emptyset$ then there is a reflection $c \in \Gamma$ mapped to $t^{N/2}$. Obviously we have $w^{N/2}c \in \Lambda$. If now $4 \mid N$, then $w^{N/2}c$ would be a non–orientable word belonging to Λ. By point *(ii)* of Theorem 4.12 we would have $\operatorname{sign}\ker\theta = \text{"}-\text{"}$ which is a contradiction. $\square$

On account of the above we denote elements of $\mathrm{CPer}^{(-,+)}(\mathbb{Z}_N)$ as $(\mathcal{A}_1, \mathcal{A}_2)^{(-,+)}$ and $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_5)^{(-,+)}$ instead of writing $(\mathcal{A}_1, \mathcal{A}_2, \emptyset, \emptyset, \emptyset, \emptyset)$ and $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \emptyset, \mathcal{A}_5, \emptyset)$, respectively.

Before we proceed to the next theorem we recall a lemma which will be helpful in determining the orientability character of resulting surface groups.

Lemma 5.24 (Bujalance et al. [8], Notations and remarks 3.1.7 (2))**.** *Suppose that* $\operatorname{sign}\Gamma = \text{"}-\text{"}$, $r = \lambda + p$ *and* $\theta\colon \Gamma \to \mathbb{Z}_N$ *is an epimorphism with kernel* Λ. *Let us write*

$$\{x_1, \ldots, x_n, e_1, \ldots, e_r\},\ \{g_1, \ldots, g_\gamma\}$$

for the sets of orientable (respectively glide reflections) canonical generators of Γ. *Assume that* $\theta(x_i) = t^{v_i}$ *and* $\theta(e_j) = t^{u_j}$, $i = 1, \ldots, n$ *and* $j = 1, \ldots, r$, *where* v_i, u_j *are even. Then* $\operatorname{sign}\Lambda = \text{"}+\text{"}$ *if and only if each* g_k *is mapped onto* t^{q_k} *with* q_k *odd,* $k = 1, \ldots, \gamma$.

In the following theorem we make use for the first time of the procedure $\mathfrak{N}_0$ which was introduced in Subsection 4.3.2.

Theorem 5.25. *Let* N *be even. Suppose that* $\mathfrak{C}_0$ *fulfils condition (i) or (ii) of the previous corollary and* $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2 + \sharp\mathcal{A}_3 \geq 2$ *or* $\mathcal{A}_5 \neq \emptyset$.

(i) If $4 \nmid N$, *then we put*

$$\Gamma = (2 - \delta_N(\mathrm{lcm}\mathfrak{C}_0^*); -; [\mathcal{B}_1^*]; \{(\ )^{w_1}(2^2)^{w_2}\}).$$

(ii) If $4 \mid N$ *and* $\mathcal{N}_{-1}^*(\mathfrak{C}^*) \neq \emptyset$ *we define*

$$\Gamma = (2 - \delta_{\frac{N}{2}}(\mathrm{lcm}\mathfrak{C}_0^*); -; [\mathcal{B}_1^*]; \{(\ )^{w_1}\}) \text{ if } 2 - \delta_{\frac{N}{2}}(\mathrm{lcm}\mathfrak{C}_0^*) \equiv \sharp\mathcal{N}_{-1}^*(\mathfrak{C}^*) \mod 2$$
$$\Gamma = (2 - \delta_{\frac{N}{2}}(\mathrm{lcm}\mathfrak{C}_0^*); -; [\mathcal{N}_{-1}^*]; \{(\ )^{z_1}\}) \text{ if } 2 - \delta_{\frac{N}{2}}(\mathrm{lcm}\mathfrak{C}_0^*) \not\equiv \sharp\mathcal{N}_{-1}^*(\mathfrak{C}^*) \mod 2.$$

(iii) Otherwise, if $4 \mid N$ *but* $\mathcal{N}_{-1}^*(\mathfrak{C}^*) = \emptyset$ *we define*

$$\Gamma = (2; -; [\mathcal{B}_1^*]; \{(\ )^{w_1}\}).$$

The terms $\mathcal{B}_i$ *were given in (4.33),* $\mathcal{N}_{-1}$ *in (4.42),* w_1 *and* w_2 *were defined in (4.35) and* z_1 *was introduced in (4.44).*

In the respective cases the above group Γ *is a universal covering group of* $(\langle t\rangle, X)$, $\langle t\rangle = \mathbb{Z}_N$. *It satisfies* $\mathrm{CPer}(t) = \mathfrak{C}_0$. *Moreover the area* $\mu(\Lambda)$, *where* $X = \mathbb{H}^2/\Lambda$, *is minimal among all orientable surfaces on which* $\mathfrak{C}_0$ *is attained as the character of periods.*

Proof. We have divided the proof into the sequence of steps similar to the one used in the proof of Theorem 5.17. We start with definition of required epimorphisms, after that we show that covering group Γ is in fact a NEC group which leads us also to demonstrate the minimality of $\mu(\Lambda)$. Eventually, we verify that surface NEC group Λ is orientable.

Case *(i)*. We apply here the procedure $\mathfrak{N}$. Recall that by $\overline{\eta}_{\mathfrak{D}(\mathfrak{C})}$ we denoted the order–preserving character with respect to $\mathfrak{D}(\mathfrak{C})$ given by (4.33), which verifies $\eta_{i,j} = 1$. Observe that $L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$ is even since all $d \in \mathcal{B}_i$, $i = 1, 2, 3, 5$ are even. Note that since we have assumed signΓ = " $-$ " together with signΛ = " $+$ " all glide reflections of Γ must be mapped to odd powers of t. Otherwise suppose that a glide reflection g of a covering NEC group Γ is mapped by a surface–kernel epimorphism to t^{2l}, which in the actual setting yields $\theta(g) \in \langle t^2 \rangle$. Denote by w a word of Γ with respect to $\ker\theta$ satisfying $\theta(w) = t^{-1}$. Hence a non–orientable word $w^2 g$ belongs to $\ker\theta$, which by point *(ii)* of Theorem 4.12 gives sign $\ker\theta$ = " $-$ ", a contradiction.

Our proof falls into four cases according to combinations of the two conditions: lcm$\mathfrak{C}_0^*$ equals N or not and $\alpha_2(L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})}))$ equals 1 or not.

Suppose lcm$\mathfrak{C}_0^* = N$ and $4 \nmid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$. Let us define

$$\theta(g_1) = t^{-\frac{L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})}{2}}.$$

On the other hand if $4 \mid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$ while still lcm$\mathfrak{C}_0^* = N$, we may take advantage of the fact that $N/2$ is odd and put

$$\theta(g_1) = t^{-\frac{L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})}{2} + \frac{N}{2}}.$$

Repeated the above technique for lcm$\mathfrak{C}_0^* < N$ we assign

$$\theta(g_2) = t^{-\frac{L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})}{2} - 1},\ \theta(g_1) = t \quad \text{in case } 4 \mid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$$
$$\theta(g_2) = t^{-\frac{L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})}{2} + \frac{N}{2} - 1},\ \theta(g_1) = t \quad \text{in case } 4 \nmid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})}).$$

We proceed to check whether the measure of the groups defined above is positive. Due to $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2 + \sharp\mathcal{A}_3 \geq 2$ we have

$$\mu(\Gamma) = 2\pi(2 - \delta_N(\mathrm{lcm}\mathfrak{C}_0^*) + w_1 + \frac{3}{2}w_2 - 2 + \sum_{m \in \mathcal{A}_1^*}(1 - \frac{1}{m})) \geq -1 + \frac{1}{2} + \frac{2}{3} > 0. \tag{5.8}$$

Moreover $\mathcal{A}_5 \neq \emptyset$ yields $\mu(\Gamma) \geq -1 + 3/2 > 0$.

In order to prove that area of Γ is minimal let us assume that $\theta' \colon \Gamma' \to \mathbb{Z}_N$ is another epimorphism that covers $\mathfrak{C}_0$. Denote by γ and γ' the genera of groups Γ and Γ' respectively. Obviously $\gamma' \geq 1$, since signΓ' = " $-$ ". By Lemma 5.22 equality $\gamma' = 1$ forces lcm$\mathfrak{C}^* = N/2$. But since we have assumed $4 \nmid N$ the last relation gives lcm$\mathfrak{C}_0^* = N$, which by our construction results in $\gamma = 1$. On the other hand we have $\gamma \leq 2$, which leads us to $\gamma' \geq \gamma$. By Proposition 4.19 we then have

$$\mu(\Gamma') - \mu(\Gamma) = \mu(\Gamma') - (\mu(\Gamma_{\mathfrak{N}}) + 2\pi\gamma) \geq 2\pi(\gamma' - \gamma) \geq 0.$$

Now we are in a position to show that the resulting surface NEC group Λ is orientable. In order to verify the orientability character of the surface group Λ we use Theorem 4.12. Observe that the glide reflections go only to odd powers of t, hence do not belong to $\ker\theta$. Thus the task is now to check whether there exists a non–orientable word w in Γ, that belongs to $\ker\theta$. Since $\mathbb{Z}_N$ is abelian we may assume that it has the following form

$$w = w_1 \prod_{i=1}^{j} g_i^{\varepsilon_i} c^{\varepsilon},$$

where $w_1 \in \langle x_1, \dots, x_n, e_1, \dots, e_{\lambda+p}\rangle$, $\varepsilon = 0, 1$ and $j = 1, 2$. Since w is a non–orientable word, the term $\sum_{i=1}^{j} \varepsilon_i + \varepsilon$ must be odd. Denote $\theta(w_1) = t^D$ and observe that $t^D \in \mathbb{Z}_{\mathrm{lcm}\mathfrak{C}^*} \leq \mathbb{Z}_{N/2}$. It follows that D is even. What is left to show is that

$$D + b\varepsilon_1 + \sum_{i=2}^{j} \varepsilon_i + \varepsilon N/2 \not\equiv 0 \mod N,$$

where $\theta(g_1) = t^b$, b an odd number. But this is obvious, since

$$\sum_{i=1}^{j} \varepsilon_i + \varepsilon \equiv D + b\varepsilon_1 + \sum_{i=2}^{j} \varepsilon_i + \varepsilon N/2 \mod 2$$

both for $j = 1$ and $j = 2$.

Case *(ii)*. Note firstly that by *(iii.5)* of the last theorem we have now $\mathfrak{C}_0 = (\mathcal{A}_1, \mathcal{A}_2)^{(-,+)}$. Moreover, all periods contained in $\mathcal{A}_1$ and $\mathcal{A}_2$ are even. Observe that $L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$ is even, which follows from Remark 5.21 and Lemma 4.8. Since $N/2$ is assumed to be even, then by Lemma 5.22 and point *(iii)* of Remark 4.20 the equality $\gamma = 1$ yields $\mathrm{lcm}\mathfrak{C}_0^* = N/2$. Before we define NEC groups Γ and epimorphisms $\theta\colon \Gamma \to \mathbb{Z}_N$ in all required cases considered subsequently, observe that under actual assumptions we get $\mathrm{sign}\ker\theta =$ " $+$ " if and only if the glide reflections are mapped to t^q with q odd.

Let us assume $\mathrm{lcm}\mathfrak{C}_0^* = N/2$ and distinguish two subcases. If $4 \nmid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$, then we apply procedure $\mathfrak{N}$ and define the following assignment for the glide reflection

$$\theta(g_1) = t^{-\frac{L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})}{2}}. \tag{5.9}$$

On the other hand if $4 \mid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$, then by Lemma 4.8 any order–preserving element $\overline{\eta}$ with respect to $\mathfrak{D}(\mathfrak{C})$ gives an value of $L(\overline{\eta})$ that is also divisible by 4. We switch to procedure $\mathfrak{N}_{-1}$ subject to $\mathcal{N}_{-1}^*(\mathfrak{C}^*)$ is non–empty. If this is the case, then by means of procedure $\mathfrak{N}_{-1}$ we add to $\mathfrak{D}(\mathfrak{C})$ the element $n_{-1}^*(\mathfrak{C}^*)$ defined by (4.21). We obtain a character associated to $\mathfrak{C}$ together with the respective order–preserving element $\overline{\eta}_{\mathfrak{N}_{-1}(\mathfrak{C})}$ which verifies that $L(\overline{\eta}_{\mathfrak{N}_{-1}(\mathfrak{C})})$ is even but not divisible by 4. Now we may put

$$\theta(g_1) = t^{-\frac{L(\overline{\eta}_{\mathfrak{N}_{-1}(\mathfrak{C})})}{2}}. \tag{5.10}$$

The analysis for $\mathrm{lcm}\mathfrak{C}_0^* < N/2$ falls into two subcases. Note that now we must have $\gamma \geq 2$. If $4 \mid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$ we use procedure $\mathfrak{N}$ together with the following assignment

$$\theta(g_2) = t^{-\frac{L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})}{2}-1},\ \theta(g_1) = t. \tag{5.11}$$

On the other hand if $4 \nmid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$ then the set $\mathcal{N}_{-1}^*(\mathfrak{C}^*)$ is certainly non–empty. We apply procedure $\mathfrak{N}_{-1}$ and put

$$\theta(g_2) = t^{-\frac{L(\overline{\eta}_{\mathfrak{N}_{-1}(\mathfrak{C})})}{2}-1},\ \theta(g_1) = t. \tag{5.12}$$

Let us verify if all groups defined above are NEC. It suffices to check whether their measure is positive. By a simple calculation we obtain

$$\mu(\Gamma) \geq 2\pi(1 + w_1 + \frac{3}{2}w_2 - 2 + \sum_{m\in\mathcal{A}_1^*}(1-\frac{1}{m})) \geq -1 + \frac{1}{2} + \frac{2}{3} > 0. \tag{5.13}$$

We proceed to show that the area of $\ker\theta$ is minimal. First, we demonstrate that $\gamma' \geq \gamma$, where γ' is the genus of another NEC group Γ' covering $\mathfrak{C}_0$. Let $\theta'\colon \Gamma' \to \mathbb{Z}_N$ be a required smooth epimorphism. Obviously there is nothing to prove in subcases corresponding to (5.9) and (5.10), since $\gamma = 1$. If we consider the scenario $\mathrm{lcm}\mathfrak{C}_0^* < N/2$, then we certainly must have $\gamma' \geq 2$. But due to our construction we have $\gamma = 2$, which gives the desired conclusion. By Proposition 4.19 the inequality $\gamma' \geq \gamma$ suffices to prove the minimality of $\ker\theta$ for all cases where the covering epimorphism is built based on procedure $\mathfrak{N}$, i.e. the cases which verify the relation[1]

$$2 - \delta_{\frac{N}{2}}(\mathrm{lcm}\mathfrak{C}_0^*) \equiv \sharp\mathcal{N}_{-1}^*(\mathfrak{C}^*) \mod 2$$

On the other hand considering the cases that satisfy[2]

$$2 - \delta_{\frac{N}{2}}(\mathrm{lcm}\mathfrak{C}_0^*) \not\equiv \sharp\mathcal{N}_{-1}^*(\mathfrak{C}^*) \mod 2$$

one gets

$$\begin{aligned}\mu(\Gamma') - \mu(\Gamma) &= \mu(\Gamma') - \mu(\Gamma_{\mathfrak{N}}) - (\mu(\Gamma) - \mu(\Gamma_{\mathfrak{N}}))\\ &= \mu(\Gamma') - \mu(\Gamma_{\mathfrak{N}}) - (2\pi\gamma + \mu(\Gamma_{\mathfrak{N}_{-1}}) - \mu(\Gamma_{\mathfrak{N}}))\\ &\geq 2\pi(\gamma'-\gamma) - (\mu(\Gamma_{\mathfrak{N}_{-1}}) - \mu(\Gamma_{\mathfrak{N}})) = 2\pi(\gamma'-\gamma) - (1-m^{-1}).\end{aligned} \tag{5.14}$$

Without loose of generality we may assume that $\Gamma' \not\simeq \Gamma$. Furthermore if $\gamma' > \gamma$, then by the above inequality we are done. However if $\gamma' = \gamma$, then (5.14) is not sufficient. Recall that by (4.7) each $\mathcal{A}_i$ is contained in the families $\mathcal{G}_i(\Gamma,\theta)$ and $\mathcal{G}_i(\Gamma',\theta')$ given by (4.6). Moreover there is at least one section $\mathcal{A}_{i_1}$ of the character $\mathfrak{C}_0$ fulfilling $\mathcal{A}_{i_1} \subsetneq \mathcal{G}_{i_1}(\Gamma',\theta')$. Thus we get

$$\begin{aligned}\mu(\Gamma') - \mu(\Gamma_{\mathfrak{N}}) - 2\pi\gamma' > 0\\ \mu(\Gamma) - \mu(\Gamma_{\mathfrak{N}}) - 2\pi\gamma > 0.\end{aligned}$$

[1] This relation embraces the subcases (5.9) and (5.11).
[2] This relation embraces the subcases (5.10) and (5.12).

However by definition of the element $n^*_{-1}(\mathfrak{C}^*)$ we finally get

$$0 \leq (\mu(\Gamma') - \mu(\Gamma_{\mathfrak{N}}) - 2\pi\gamma') - (\mu(\Gamma) - \mu(\Gamma_{\mathfrak{N}}) - 2\pi\gamma) = \mu(\Gamma') - \mu(\Gamma)$$

since $\gamma' = \gamma$ due to the actual assumptions.

Case *(iii)*. Let $\overline{\eta}$ be an element which is order–preserving with respect to $\mathfrak{D}(\mathfrak{C})$. If $4 \mid N$ and $\mathcal{N}^*_{-1}(\mathfrak{C}^*) = \emptyset$, then independently on $\mathrm{lcm}\mathfrak{C}^*_0$ we get by Lemma 4.8

$$L(\overline{\eta}) \equiv 2 \cdot \sharp\mathcal{N}^*_{-1}(\mathfrak{C}^*) \equiv 0 \bmod 4.$$

Thus $L(\overline{\eta})$ is divisible by 4 and we are limited to manipulate with the genus of the candidate group. We set $\gamma = 2$ and use procedure $\mathfrak{N}$ together with the assignment (5.11) on the glide reflections. By

$$\mu(\Gamma) \geq 2\pi(2 + w_1 - 2 + \sum_{m \in \mathcal{B}^*_1} (1 - \frac{1}{m})) \geq \frac{1}{2} + \frac{2}{3} > 0.$$

the group defined in this way is a NEC group and by Lemma 5.24 it covers a $\mathbb{Z}_N$–action on an orientable surface.

It remains to show that the area of $\ker\theta$ is minimal. By Lemma 5.22 we have $\gamma' \geq 2$ which gives

$$\mu(\Gamma') - \mu(\Gamma) \geq 2\pi(\gamma' - 2) \geq 0.$$

□

Since we have dropped the case $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2 + \sharp\mathcal{A}_3 < 2$, $\mathcal{A}_5 = \emptyset$ we complete the study by the following remark.

Remark 5.26. *If $\sharp\mathcal{A}_1 + \sharp\mathcal{A}_2 + \sharp\mathcal{A}_3 < 2$, $\mathcal{A}_5 = \emptyset$ and the remaining assumptions of Theorem 5.25 hold then the universal covering group that covers a $\mathbb{Z}_N$–action prescribed by $\mathfrak{C}_0$ equals*

$$\Gamma = \begin{cases} (3; -; [\,]; \{\,\}), & \text{if } \mathfrak{C}_0 = (\emptyset, \emptyset)^{(-,+)} \text{ and } 4 \nmid N \\ (4; -; [\,]; \{\,\}), & \text{if } \mathfrak{C}_0 = (\emptyset, \emptyset)^{(-,+)} \text{ and } 4 \mid N \\ (1; -; [N/2, N/2]; \{\,\}), & \text{if } \mathfrak{C}_0 = (\{2\}, \emptyset)^{(-,+)} \text{ and } 4 \nmid N \\ (1; -; [N/2, N/2, N/2]; \{\,\}), & \text{if } \mathfrak{C}_0 = (\{2\}, \emptyset)^{(-,+)} \text{ and } 4 \mid N \\ (2; -; [N/d]; \{\,\}), & \text{if } \mathfrak{C}_0 = (\{d\}, \emptyset)^{(-,+)}, d \neq 2 \text{ and } 4 \nmid N \\ (2; -; [N/d, N/d]; \{\,\}), & \text{if } \mathfrak{C}_0 = (\{d\}, \emptyset)^{(-,+)}, d \neq 2 \text{ and } 4 \mid N \\ (1; -; [\,]; \{(\,)^2\}), & \text{if } \mathfrak{C}_0 = (\emptyset, \{2\})^{(-,+)} \text{ and } 4 \nmid N \\ (1; -; [\,]; \{(\,)^3\}), & \text{if } \mathfrak{C}_0 = (\emptyset, \{2\})^{(-,+)} \text{ and } 4 \mid N \\ (2; -; [\,]; \{(\,)\}), & \text{if } \mathfrak{C}_0 = (\emptyset, \{d\})^{(-,+)}, d \neq 2 \text{ and } 4 \nmid N \\ (2; -; [\,]; \{(\,)^2\}), & \text{if } \mathfrak{C}_0 = (\emptyset, \{d\})^{(-,+)}, d \neq 2 \text{ and } 4 \mid N \\ (1; -; [\,]; \{(\,)^2\}), & \text{if } \mathfrak{C}_0 = (\emptyset, \emptyset, \{1\}, \emptyset)^{(-,+)} \\ (2; -; [\,]; \{(\,)\}), & \text{if } \mathfrak{C}_0 = (\emptyset, \emptyset, \{d\}, \emptyset)^{(-,+)} \text{ and } d \neq 1 \end{cases}$$

The area of the respective NEC groups is minimal among all orientable surfaces on which the above tuples of sets are attained as the characters of periods.

Example 5.27. *Consider a smooth epimorphism $\theta\colon \Gamma \to \mathbb{Z}_N$, $4 \mid N$ from $\Gamma = (4; -; [\,]; \{\,\})$ which covers $\mathfrak{C}_0 = (\emptyset, \emptyset)^{(-,+)}$. Using the notation of [8] (see also comments on page 70) we have $S(\alpha, \beta) = 0$. Note that since there is no ovals nor chains and $\gamma = 4$, it conforms to the assertion of the second part of point (iii.4) of Theorem 5.20.*

The case of actions of cyclic groups of an even order on orientable surfaces by homeomorphisms gives the opportunity to distinguish another property of such maps i.e. preserving or reversing orientation of a manifold. If $t\colon X \to X$ is a map of an orientable Klein surface, then $X/\langle t\rangle$ being a non–orientable surface forces that t reverses orientation. On the other hand either orientation–preserving or orientation–reversing homeomorphisms may lead to orientable quotient surfaces $X/\langle t\rangle$. Below we recall Corollary 3.2.2 from [8] which allows one to determine whether an automorphism of on orientable surface preserves or reverses orientation based on N–pair (Γ, Λ) such that Γ covers a $\mathbb{Z}_N$–action of $t\colon \mathbb{H}^2/\Lambda \to \mathbb{H}^2/\Lambda$.

Corollary 5.28 (Bujalance et al. [8], Corollary 3.2.3)**.** *Let Λ be a surface NEC group such that $X = \mathbb{H}^2/\Lambda$ is orientable. Let $t \in \mathrm{Aut}(X)$ be an element of an even order $N > 1$, and let Γ be a NEC group realizing t i.e. $\langle t\rangle \simeq \Gamma/\Lambda$. As we know, for even N,*

$$\Gamma = (\gamma; \pm; [m_1, ..., m_n]; \{(\ )^{\lambda}(2^{\mu_1})...(2^{\mu_p})\})$$

and all μ_i are even. Let r be the integer for which $c_{i,0} \in \Lambda$ for $1 \le i \le r$ and $c_{i,0} \notin \Lambda$ for $r + 1 \le i \le \lambda$.

(i) if $\mathrm{sign}\Gamma = "-"$*, then t reverses orientation*

(ii) if $\mathrm{sign}\Gamma = "+"$*, then t preserves orientation if and only if $r = \lambda + p$*

(iii) if $4 \mid N$ and $\mathrm{sign}\Gamma = "+"$*, then t preserves orientation.*

We finish this section with the following example which may be treated as a counterpart of a similar problem arisen for orientation–preserving homeomorphisms of finite order which has been already solved in Example 2.11. Unlike the previous case the answer in the actual setting is positive.

Example 5.29. *Let Σ_g be a hyperbolic orientable surface of genus g. It was left as an open question in [[18], p.470] as to whether there exists on Σ_4 a finite order orientation–reversing homeomorphism of order 6 with no points of period less than 6. We find answer to this question by applying a construction based on the combinatorial approach to the theory of NEC groups. Let $\Gamma = (3; -; [\,]; \{\,\})$ and define the following epimorphism $\theta\colon \Gamma \to \mathbb{Z}_6$*

$$\theta(g_1) = t \quad \theta(g_2) = t^{-1} \quad \theta(g_3) = t^3 \tag{5.15}$$

which covers a $\mathbb{Z}_N$–action of a map t. By the last corollary t is orientation–reversing. Let $\ker\theta \simeq \Lambda$*. By Remark 4.2 we have* $\mathrm{CPer}(t) = (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$*. Furthermore* $\mathrm{sign}\Lambda = "+"$ *by Theorem 4.12, since no glide reflection nor a non–orientable word in Γ belongs to Λ.*

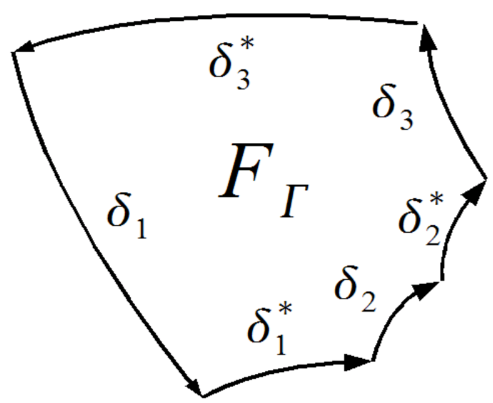

Figure 5.1: Fundamental region for a NEC group $\Gamma = (3; -; [\ ]; \{\ \})$.

Consequently, by Riemann–Hurwitz formula we must have $\mu(\Lambda) = 6\mu(\Gamma) = 12\pi$, *which due to the above partial information on* Λ *yields* $\Lambda = (4; +; [\ ]; \{\ \})$. *Hence* Λ *is an orientable surface group of genus 4.*

Since $\theta(g_1) = t$ *we may use the below representation of* F_Λ

$$F_\Lambda = F_\Gamma \cup g_1^{-1}F_\Gamma \cup g_1^{-2}F_\Gamma \cup g_1^{-3}F_\Gamma \cup g_1^{-4}F_\Gamma \cup g_1^{-5}F_\Gamma.$$

In order to reveal a geometric interpretation of the above $\mathbb{Z}_N$*–action let us first investigate how a fundamental region* F_Λ *is obtained from* F_Γ. *Let the edges of* F_Γ *be labelled as follows* $\delta_1\delta_1^*\delta_2\delta_2^*\delta_3\delta_3^*$ *and use a model of* F_Γ *given on Figure 5.1.*

Thus we obtain a fundamental region F_Λ *represented on Figure 5.1 whose perimeter counts 26 edges. However, observe that the sides of* F_Λ *are paired by elements of* Λ. *Below we list all pairs and for simplicity of notation we relabel the edges as* ζ_j, $j = 1, \ldots, 13$ *(see also fine dashed arrows on Figure 5.2).*

$$\begin{array}{lll}
\delta_2 \sim g_1^{-5}\delta_2^* \sim \zeta_2, & g_1^{-4}\delta_2 \sim g_1^{-3}\delta_2^* \sim \zeta_{10}, & g_1^{-4}\delta_3 \sim g_1^{-1}\delta_3^* \sim \zeta_{12}, \\
g_1^{-1}\delta_2 \sim \delta_2^* \sim \zeta_3, & g_1^{-5}\delta_2 \sim g_1^{-4}\delta_2^* \sim \zeta_{11}, & g_1^{-3}\delta_3 \sim \delta_3^* \sim \zeta_5, \\
g_1^{-2}\delta_2 \sim g_1^{-1}\delta_2^* \sim \zeta_6, & \delta_3 \sim g_1^{-3}\delta_3^* \sim \zeta_4, & g_1^{-2}\delta_3 \sim g_1^{-5}\delta_3^* \sim \zeta_8, \\
g_1^{-3}\delta_2 \sim g_1^{-2}\delta_2^* \sim \zeta_7, & g_1^{-5}\delta_3 \sim g_1^{-2}\delta_3^* \sim \zeta_9, & g_1^{-1}\delta_3 \sim g_1^{-4}\delta_3^* \sim \zeta_{13}, \\
g_1^{-5}\delta_1 \sim \delta_1^* \sim \zeta_1. & &
\end{array}$$

The perimeter Δ *of* F_Λ *is labelled in the counter–clockwise order as follows (see orientations induced on perimeters of* F_Γ, g_1F_Γ,…,$g_1^5F_\Gamma$ *by the respective block circular arrows on Figure 5.2)*

$$\Delta \sim \zeta_1\zeta_2\zeta_3\zeta_4\zeta_5\zeta_6\zeta_7\zeta_8\zeta_9\zeta_{10}\zeta_{11}\zeta_{12}\zeta_{13}\zeta_1^{-1}\zeta_8^{-1}\zeta_9^{-1}\zeta_2^{-1}\zeta_{11}^{-1}\zeta_4^{-1}\zeta_5^{-1}\zeta_{10}^{-1}\zeta_7^{-1}\zeta_{12}^{-1}\zeta_{13}^{-1}\zeta_6^{-1}\zeta_3^{-1}$$

By elementary substitutions we reduce the symbol Δ *in a way which allows us to recognize easily the topological type of a surface on which map* t *is acting. Let us precise two substitutions we shall need (see for instance [[33], p.471])*

(1) $[y_0]aa^{-1}[y_1] \sim [y_0y_1]$ *if* y_0y_1 *comprises at least 4 sides of perimeter of a fundamental region*

(2) $w_0[y_1]a[y_2]b[y_3]a^{-1}[y_4]b^{-1}[y_5] \sim w_0aba^{-1}b^{-1}[y_1y_4y_3y_2y_5]$,

In the below sequence of equivalences we let $\overset{(i)}{\sim}$, $i = 1, 2$ *denote the substitution which yields*

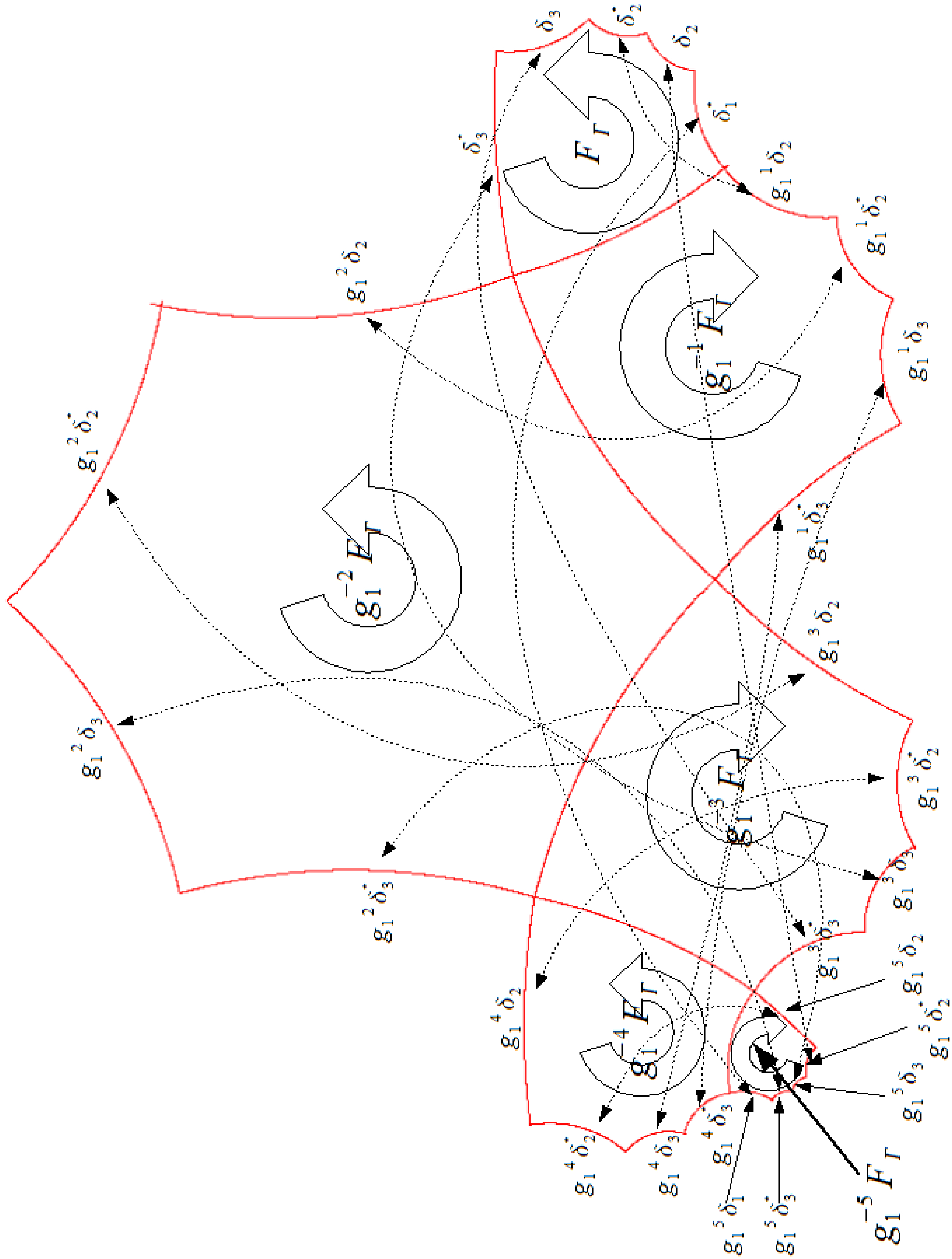

Figure 5.2: Fundamental region $F_\Lambda = \bigcup_{k=0}^{5} g_1^{-k} F_\Gamma$.

the equivalence between the consecutive expressions.

$$
\begin{aligned}
&\Delta \sim \\
&\sim \underbrace{[1]}_{w_0}\underbrace{[1]}_{y_1}\zeta_1\underbrace{[1]}_{y_2}\zeta_2\underbrace{\zeta_3\zeta_4\zeta_5\zeta_6\zeta_7\zeta_8\zeta_9\zeta_{10}\zeta_{11}\zeta_{12}\zeta_{13}}_{y_3}\zeta_1^{-1}\underbrace{\zeta_8^{-1}\zeta_9^{-1}}_{y_4}\zeta_2^{-1}\underbrace{\zeta_{11}^{-1}\zeta_4^{-1}\zeta_5^{-1}\zeta_{10}^{-1}\zeta_7^{-1}\zeta_{12}^{-1}\zeta_{13}^{-1}\zeta_6^{-1}\zeta_3^{-1}}_{y_5} \\
&\overset{(2)}{\sim} \underbrace{\zeta_1\zeta_2\zeta_1^{-1}\zeta_2^{-1}}_{w_0}\underbrace{[1]}_{y_1}\zeta_8^{-1}\underbrace{[1]}_{y_2}\zeta_9^{-1}\underbrace{\zeta_3\zeta_4\zeta_5\zeta_6\zeta_7}_{y_3}\zeta_8\underbrace{[1]}_{y_4}\zeta_9\underbrace{\zeta_{10}\zeta_{11}\zeta_{12}\zeta_{13}\zeta_{11}^{-1}\zeta_4^{-1}\zeta_5^{-1}\zeta_{10}^{-1}\zeta_7^{-1}\zeta_{12}^{-1}\zeta_{13}^{-1}\zeta_6^{-1}\zeta_3^{-1}}_{y_5} \\
&\overset{(2)}{\sim} \underbrace{\zeta_1\zeta_2\zeta_1^{-1}\zeta_2^{-1}\zeta_8^{-1}\zeta_9^{-1}\zeta_8\zeta_9}_{w_0}\underbrace{\zeta_3}_{y_1}\zeta_4\underbrace{[1]}_{y_2}\zeta_5\underbrace{\zeta_6\zeta_7\zeta_{10}\zeta_{11}\zeta_{12}\zeta_{13}\zeta_{11}^{-1}}_{y_3}\zeta_4^{-1}\underbrace{[1]}_{y_4}\zeta_5^{-1}\underbrace{\zeta_{10}^{-1}\zeta_7^{-1}\zeta_{12}^{-1}\zeta_{13}^{-1}\zeta_6^{-1}\zeta_3^{-1}}_{y_5} \\
&\overset{(2)}{\sim} \underbrace{\zeta_1\zeta_2\zeta_1^{-1}\zeta_2^{-1}\zeta_8^{-1}\zeta_9^{-1}\zeta_8\zeta_9\zeta_4\zeta_5\zeta_4^{-1}\zeta_5^{-1}}_{w_0}\underbrace{\zeta_3\zeta_6\zeta_7\zeta_{10}\zeta_{11}}_{y_1}\zeta_{12}\underbrace{[1]}_{y_2}\zeta_{13}\underbrace{\zeta_{11}^{-1}\zeta_{10}^{-1}\zeta_7^{-1}}_{y_3}\zeta_{12}^{-1}\underbrace{[1]}_{y_4}\zeta_{13}^{-1}\underbrace{\zeta_6^{-1}\zeta_3^{-1}}_{y_5} \\
&\overset{(2)}{\sim} \zeta_1\zeta_2\zeta_1^{-1}\zeta_2^{-1}\zeta_8^{-1}\zeta_9^{-1}\zeta_8\zeta_9\zeta_4\zeta_5\zeta_4^{-1}\zeta_5^{-1}\zeta_{12}\zeta_{13}\zeta_{12}^{-1}\zeta_{13}^{-1}\zeta_3\zeta_6\zeta_7\zeta_{10}\zeta_{11}\zeta_{11}^{-1}\zeta_{10}^{-1}\zeta_7^{-1}\zeta_6^{-1}\zeta_3^{-1} \\
&\overset{(1)}{\sim} \zeta_1\zeta_2\zeta_1^{-1}\zeta_2^{-1}\zeta_8^{-1}\zeta_9^{-1}\zeta_8\zeta_9\zeta_4\zeta_5\zeta_4^{-1}\zeta_5^{-1}\zeta_{12}\zeta_{13}\zeta_{12}^{-1}\zeta_{13}^{-1}.
\end{aligned}
$$

Now by relabelling, we obtain a surface symbol of an orientable surface of genus 4

$$[\zeta_1\zeta_2][\zeta_3\zeta_4][\zeta_5\zeta_6][\zeta_7\zeta_8]$$

as required.

Remark 5.30. *The above example is also important in the context of canonical Fuchsian subgroup. In Lemma 3.7 we considered groups of automorphisms of Klein surfaces being not Riemann surfaces. The group $\Gamma = (3; -; [\ ]; \{\ \})$ and epimorphism θ given by (5.15) show that the assumption of not considering Riemann surfaces in Lemma 3.7 is appropriate since here we have $\Gamma^+ = (2; +; [\ ]; \{\ \})$ and $\theta(\Gamma^+) = \mathbb{Z}_3 < \mathbb{Z}_6 = \theta(\Gamma)$.*

5.3 Actions of Groups of Even Order on Non–Orientable Surfaces

This section is devoted to investigating the action of dianalytic homeomorphisms on non–orientable compact surfaces. A brand new type of periodic structures which become apparent in the actual setting embraces the one–sided structures i.e. ovals and chains whose neighbourhood is a Möbius strip.

Let us start with a slightly more constrained case, when the quotient surface is orientable. We begin in the way we did before, by recalling a theorem that specifies the necessary and sufficient conditions on N–pairs.

Theorem 5.31 (Bujalance et al. [8], Theorem 3.1.6)**.** *Let N be even,* $\mathrm{sign}\Gamma = " + "$ *and* $\mathrm{sign}\Lambda = " - "$*. Then (Γ, Λ) is an N–pair if and only if the CCN conditions are fulfilled and*

(iii.2) $r < \lambda + p$

(iii.3) if $\gamma = 0$ and $\lambda + p = r + 1$, then $\mathrm{lcm}(m_1, \ldots, m_n, l_1, \ldots, l_r) = N$.

Corollary 5.32. *An element $\mathfrak{C}_0$ belongs to the set of characters of periods* $\mathrm{CPer}^{(+,-)}(\mathbb{Z}_N)$ *of $\mathbb{Z}_N$–actions by dianalytic transformations on non–orientable surfaces only if it is of the form*

$$(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6),$$

where $\bigcup_{i\geq 3} \mathcal{A}_i \neq \emptyset$.

Proof. Assume that $\theta\colon \Gamma \to \mathbb{Z}_N$ covers a $\mathbb{Z}_N$–action with the character of periods prescribed by $\mathfrak{C}_0$ and denote $\Lambda = \ker\theta$. Since $(\Gamma, \ker\theta)$ is an N–pair and $\ker\theta$ takes the form (5.5), then the result is an immediate consequence of point *(iii.2)* of the last theorem. □

We write $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6)^{(+,-)}$ for $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6) \in \mathrm{CPer}^{(+,-)}(\mathbb{Z}_N)$. Corollary 5.32 gives us the information on condition which is necessary to form a character of periods belonging to $\mathrm{CPer}^{(+,-)}(\mathbb{Z}_N)$. However in Corollary 5.34 we will exclude some of the 6–tuples that do not occur as characters of periods upon the actual assumptions. First we need a lemma.

Lemma 5.33. *Let N be an even number and let $\theta\colon \Gamma \to \mathbb{Z}_N$ be an epimorphism with* $\mathrm{sign}\Gamma = " + "$ *and* $\mathrm{sign}\Lambda = " - "$*. Assume that the character of periods of the underlying $\mathbb{Z}_N$–action equals $\mathfrak{C}_0$. Then $4 \nmid N$ implies* $\mathrm{lcm}\mathfrak{C}^* = \mathrm{lcm}\mathfrak{C}_0^*$.

Proof. By Theorem 4.12 there is a non–orientable word w that belongs to Λ. The group $\mathbb{Z}_N$ is abelian and we may assume that

$$w = w'c,$$

where $w' \in \langle x_1, \ldots, x_n, e_1, \ldots, e_{\lambda+p}\rangle$ and c is a reflection verifying $\theta(c) = t^{N/2}$. Since $\theta(w') \in \mathbb{Z}_{\mathrm{lcm}\mathfrak{C}^*}$ we also have $t^{N/2} \in \mathbb{Z}_{\mathrm{lcm}\mathfrak{C}^*}$. Consequently, $\mathrm{lcm}\mathfrak{C}^*$ must be even which by point *(iii)* of Remark 4.20 gives $\mathrm{lcm}\mathfrak{C}^* = \mathrm{lcm}\mathfrak{C}_0^*$. □

Corollary 5.34. *Let N be an even number satisfying $4 \nmid N$. Suppose that $\mathfrak{C}_0 \in \mathrm{CPer}^{(+,-)}(\mathbb{Z}_N)$. If $\mathcal{A}_4 \cup \mathcal{A}_6 = \emptyset$, then $\mathcal{A}_1 \cup \mathcal{A}_2$ must contain an odd element.*

Proof. Assume that $\theta\colon \Gamma \to \mathbb{Z}_N$ covers a $\mathbb{Z}_N$–action given by $\mathfrak{C}_0$. Suppose that the assertion of the corollary is false. Then $\mathfrak{C}_0 = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \emptyset, \mathcal{A}_5, \emptyset)$ and all subgroups generated by images of canonical elliptic generators and e–generators corresponding to boundary components have odd orders. Recall that also images of the remaining canonical e–generators inducing periods in sections $\mathcal{A}_3$ and $\mathcal{A}_5$ do generate subgroups of odd orders. It follows that subgroup generated by images of all orientable canonical generators of Γ has an odd order, equal to $\mathrm{lcm}\mathfrak{C}^*$. By Corollary 5.32 we get $\mathcal{A}_3 \cup \mathcal{A}_5 \neq \emptyset$, thus $\mathrm{lcm}\mathfrak{C}_0^*$ is an even number. But Lemma 5.33 shows that $\mathrm{lcm}\mathfrak{C}^* = \mathrm{lcm}\mathfrak{C}_0^*$, which is impossible. □

Observe that according to Proposition 5.15 the excluded 6–tuples form characters of periods which belong to $\mathrm{CPer}^{(+,+)}(\mathbb{Z}_N)$.

Lemma 5.35 (Bujalance et al. [8], Remarks 3.1.4 (6))**.** *Let $\theta\colon \Gamma \to \mathbb{Z}_N$ be an epimorphism with kernel Λ and the underlying $\mathbb{Z}_N$–action proscribed by $\mathfrak{C}_0$. If $\bigcup_{i\geq 3} \mathcal{A}_i \neq \emptyset$ and $4 \mid N$, then* $\mathrm{sign}\Lambda =$ " $-$ ".

Theorem 5.36. *Let N be even and suppose that $\mathfrak{C}_0 \in \mathrm{CPer}^{(+,-)}(\mathbb{Z}_N)$ and $\sum_{i=1}^{6} \sharp\mathcal{A}_i \geq 2$.*

(i) We define

$$\Gamma = (0; +; [N, N, N]; \{(\ )\}) \text{ if } \mathfrak{C}_0 = (\{1\}, \emptyset, \emptyset, \{1\}, \emptyset, \emptyset)^{(+,-)}$$
$$\Gamma = (0; +; [2, 2, 2]; \{(2^2)\}) \text{ if } \mathfrak{C}_0 = (\{1\}, \emptyset, \emptyset, \emptyset, \emptyset, \{1\})^{(+,-)} \text{ and } N = 2$$
$$\Gamma = (0; +; [\ ]; \{(\ )^4\}) \text{ if } \mathfrak{C}_0 = (\emptyset, \{1\}, \emptyset, \{1\}, \emptyset, \emptyset)^{(+,-)}$$

(ii) Otherwise

$$\Gamma = (1 - \delta_N(\mathrm{lcm}\mathfrak{C}_0^*); +; [\mathcal{O}_1^*]; \{(\ )^{w_1}(2^2)^{w_2}\}), \tag{5.16}$$

where $\mathcal{O}_1$ and w_1, w_2 were defined in (4.24) and (4.27) respectively.

In the respective cases the above group Γ is a universal covering group of $(\langle t\rangle, X)$, $\langle t\rangle = \mathbb{Z}_N$. It satisfies $\mathrm{CPer}(t) = \mathfrak{C}_0$. Moreover the area $\mu(\Lambda)$, where $X \simeq \mathbb{H}^2/\Lambda$, is minimal among all non–orientable surfaces on which $\mathfrak{C}_0$ is attained as the character of periods.

Proof. Point *(i)* is proved by direct computations. As an example we demonstrate assignment in the first of the exceptional cases i.e. $\mathfrak{C}_0 = (\{1\}, \emptyset, \emptyset, \{1\}, \emptyset, \emptyset)^{(+,-)}$. We use $\Gamma = (0; +; [N, N, N]; \{(\ )\})$ and put

$$x_1, x_2 \mapsto t^{-1} \quad x_3, e_1 \mapsto t \quad \text{and } c_{1,0} \mapsto t^{\frac{N}{2}}.$$

Since $\theta(x_1^{\frac{N}{2}} c_{1,0}) = 1$ and $x_1^{\frac{N}{2}} c_{1,0}$ is a non–orientable word we have $\mathrm{sign}\Lambda =$ " $-$ ". Observe that procedure $\mathfrak{O}$ would give a group $\Gamma_{\mathfrak{O}} = (0; +; [N]; \{(\ )\})$ which is not a NEC group since

$\mu(\Gamma_{\mathfrak{O}}) = -2\pi/N < 0$. The enhancement of $\Gamma_{\mathfrak{O}}$ by adding two elliptic generators results in the smallest increase of its measure.

We now proceed to the proof of case *(ii)*. We first check the orientability character of the quotient surface. By the procedure $\mathfrak{O}$ we define the homomorphism θ on all but hyperbolic canonical generators corresponding to the orbit genus of the group Γ given by (5.16). If $\mathrm{lcm}\mathfrak{C}_0^* \neq N$, then $\Gamma_{\mathfrak{O}}$ given by (4.29) is not mapped by smooth epimorphism θ onto the whole $\mathbb{Z}_N$. We add to $\Gamma_{\mathfrak{O}}$ exactly one pair of hyperbolic generators a_1, b_1 and map them to an element of order N in $\mathbb{Z}_N$. By Theorem 4.12 all reduces to find a non–orientable word $w \in \Gamma$ with $\theta(w) = 1$, which suffices to have $\mathrm{sign}\Lambda = " - "$. If $4 \mid N$, then by Lemma 5.35 we get immediately the desired conclusion. On the other hand if $4 \nmid N$, then we take advantage that $\mathcal{A}_4, \mathcal{A}_6 \subseteq \mathcal{D}(N/2)$. It follows that these sets are empty or comprise odd periods. By Corollary 5.34 there is always an odd period contained in $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_4 \cup \mathcal{A}_6$. Let us denote such a period by d and let $h_d \in \Gamma$ stand for an x– or e–generator that corresponds to d. By (4.26) the element h_d is mapped to $t^{\eta d}$, where $(\eta, N/d) = 1$, i.e. η is odd. Since $\bigcup_{i\geq 3} \mathcal{A}_i \neq \emptyset$ by Corollary 5.32 there is also a reflection c that goes to $t^{N/2}$. Thus one gets

$$\theta(h_d^{\frac{N}{2d}} c) = t^{\eta d \frac{N}{2d}} t^{\frac{N}{2}} = t^{\eta \frac{N}{2}} t^{\frac{N}{2}} = t^{N \frac{\eta+1}{2}} = 1,$$

which is the desired result.

Our next goal is to evaluate the measure of Γ. We have

$$\mu(\Gamma) \quad = \quad 2\pi(2(1 - \delta_N(\mathrm{lcm}\mathfrak{C}_0^*)) + w_1 + w_2 - 2 + \sum_{m\in\mathcal{O}_1^*} (1 - m^{-1}) + \frac{1}{2} w_2).$$

Recall that we have $\sum_{i=3}^{6} \sharp\mathcal{A}_i \geq 1$. Consequently if $\sum_{i=1}^{6} \sharp\mathcal{A}_i > 2$ then it holds

$$\mu(\Gamma) \geq w_1 + w_2 - 2 + \sum_{m\in\mathcal{O}_1^*} (1 - m^{-1}) > 0.$$

It follows that we are reduced to consider the cases when $\sum_{i=1}^{6} \sharp\mathcal{A}_i = 2$. Since the proof goes here by direct computations similar to (5.7) we omit the details. However, all subcases in which this line of arguments fails i.e. the procedure $\mathfrak{O}$ does not lead us to a group Γ with positive measure are considered separately in point (i) of the theorem.

We proceed to show that $\mu(\Lambda)$ is minimal. Let γ' be genus of another NEC group Γ' which covers a $\mathbb{Z}_N$–action given by $\mathfrak{C}_0$. By Remark 4.23 we must have $\gamma' \geq \gamma$. Moreover by Remark 4.17 we have

$$\mu(\Gamma') - \mu(\Gamma) = \mu(\Gamma') - (\mu(\Gamma_{\mathfrak{O}}) + 4\pi\gamma) \geq 4\pi(\gamma' - \gamma) \geq 0,$$

which is our claim. □

Since we have dropped the case $\sum_{i=1}^{6} \sharp\mathcal{A}_i < 2$ we complete our investigation by the following remark.

Remark 5.37. *If $\sum_{i=1}^{6} \sharp\mathcal{A}_i < 2$ and the remaining assumptions of Theorem 5.36 hold then the universal covering group that covers a $\mathbb{Z}_N$–action prescribed by $\mathfrak{C}_0 \in \mathrm{CPer}^{(+,-)}(\mathbb{Z}_N)$ equals*

$$\Gamma = \begin{cases} (0;+;[\ ];\{(\ )^4\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\emptyset,\emptyset,\{1\},\emptyset,\emptyset)^{(+,-)} \\ (1;+;[\ ];\{(\ )^2\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\emptyset,\emptyset,\{d\},\emptyset,\emptyset)^{(+,-)} \text{ and } d \neq 1 \\ (0;+;[\ ];\{(2^2)^2\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\{1\})^{(+,-)} \\ (1;+;[\ ];\{(2^2)^2\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\{d\})^{(+,-)} \text{ and } d \neq 1 \end{cases}$$

The area of the respective NEC groups is minimal among all non–orientable surfaces on which the above tuples of sets are attained as the characters of periods.

Our next concern will be the analysis of the set of characters of periods in case the both surfaces: the one on which a homeomorphism acts and the quotient one are non–orientable.

Theorem 5.38 (Bujalance et al. [8], Theorem 3.1.8)**.** *Let N be even,* $\mathrm{sign}\Gamma = " - "$ *and* $\mathrm{sign}\Lambda = " - "$. *Then (Γ, Λ) is an N–pair if and only if the CCN conditions are fulfilled and*

(iii.2) if $r = \lambda + p$ there exists an order–preserving pair (α, β) with respect to $\{N/m_1, \ldots, N/m_n, N/l_1, \ldots, N/l_r\}$ such that $S(\alpha, \beta)$ is even

(iii.3) if $r = \lambda + p$ and $\gamma = 1$, then $\mathrm{lcm}(m_1, \ldots, m_n, l_1, \ldots, l_r) = N$

(iii.4) assume $r = \lambda + p$, $\gamma = 2$, $4 \mid N$ and every even $S(\alpha, \beta)$ is a multiple of 4, then N/m_i or N/l_j are odd for some i or j.

Likewise in previous cases we introduce here a self–describing notation writing $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6)^{(-,-)}$ for $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6) \in \mathrm{CPer}^{(-,-)}(\mathbb{Z}_N)$. Observe that we do not impose any constraints other than (4.1)–(4.2) regarding 6–tuples of sets as candidates for $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6)^{(-,-)}$.

We need the below auxiliary lemma. We will call it in the proof of Theorem 5.42.

Lemma 5.39. *Let N be even and suppose that $\theta\colon \Gamma \to \mathbb{Z}_N$ covers a $\mathbb{Z}_N$–action with the character of periods $\mathfrak{C}_0 \in \mathrm{CPer}^{(-,-)}(\mathbb{Z}_N)$ which verifies $\bigcup_{i\geq 3} \mathcal{A}_i = \emptyset$. Then $\gamma = 1$ implies* $\mathrm{lcm}\mathfrak{C}^* = N$.

Proof. In order to obtain the assertion we use point *(ii)* of Theorem 4.12 which forces that either there exists a non–orientable word $w \in \ker\theta$ or $g_1 \in \ker\theta$. Since $\mathbb{Z}_N$ is abelian we may assume that $w = w_1 g_1^{\kappa}$ for an odd κ, where w_1 is an orientable word i.e. $w_1 \in \langle x_1, \ldots, x_n, e_1, \ldots, e_{\lambda+p}\rangle$. By the "long relation" it must hold $\theta(g_1^2) \in \mathbb{Z}_{\mathrm{lcm}\mathfrak{C}^*}$, which together with the previous relations results in

$$1 = \theta(w) = \theta(w_1 g_1^{\kappa}) = \theta(w_1 g_1^{\kappa-1})\theta(g_1).$$

Hence $\theta(g_1) \in \mathbb{Z}_{\mathrm{lcm}\mathfrak{C}^*}$ and we get $|\theta(\Gamma)| = \mathrm{lcm}\mathfrak{C}^*$. □

Take $\mathfrak{C}_0 \in \mathrm{CPer}^{(-,-)}(\mathbb{Z}_N)$. In the following lemma we consider properties of $\mathrm{lcm}\mathfrak{C}_0^*$ under some additional conditions involving the number N and the topological genus of a NEC group which covers $\mathfrak{C}_0$. The lemma will be used in the proof of Theorem 5.42.

Lemma 5.40. *Let N be an even number and let $\theta\colon \Gamma \to \mathbb{Z}_N$ be an epimorphism with* $\mathrm{sign}\Gamma = \mathrm{sign}\Lambda = " - "$. *Assume that the character of periods of the underlying $\mathbb{Z}_N$–action equals $\mathfrak{C}_0$, where $\bigcup_{i\geq 3} \mathcal{A}_i \neq \emptyset$. Then*

(i) $\gamma = 1$ *and* $4 \nmid N$ *imply* $\mathrm{lcm}\mathfrak{C}_0^* = N$

(ii) $\gamma = 1$ *and* $4 \mid N$ *imply* $\mathrm{lcm}\mathfrak{C}_0^* \geq N/2$.

Proof. In order to deduce *(i)* we recall again Theorem 4.12. There is a non–orientable word or a glide reflection $w \in \ker\theta$. We may suppose that

$$w = w_1 g_1^{\kappa} c^{\varepsilon},$$

where $w_1 \in \langle x_1, \dots, x_n, e_1, \dots, e_{\lambda+p}\rangle$ and c is a reflection with $\theta(c) = t^{N/2}$, $\varepsilon \in \{0, 1\}$. If $\varepsilon = 0$, then κ is odd and we get $\theta(g_1) \in \mathbb{Z}_{\mathrm{lcm}\mathfrak{C}^*}$ since $\theta(g_1^2) \in \mathbb{Z}_{\mathrm{lcm}\mathfrak{C}^*}$ by the "long relation" in Γ. Thus if $\mathrm{lcm}\mathfrak{C}_0^* < N$, then θ is not onto $\mathbb{Z}_N$. Otherwise, in case $\varepsilon = 1$ we have $w = w_1 g_1^{\kappa} c$, for κ even. Recall our assumption $\bigcup_{i\geq 3} \mathcal{A}_i \neq \emptyset$ and observe that by Remark 4.21 it follows that $\mathrm{lcm}\mathfrak{C}_0^*$ is even. Therefore if $\theta(g_1) \notin \mathbb{Z}_{\mathrm{lcm}\mathfrak{C}_0^*}$, then

$$N = |\theta(\Gamma)| = 2\mathrm{lcm}\mathfrak{C}_0^*.$$

But it contradicts our assumption that $4 \nmid N$. On the other hand $\theta(g_1) \in \mathbb{Z}_{\mathrm{lcm}\mathfrak{C}_0^*}$ yields that θ is not onto a cyclic group of order bigger than $\mathrm{lcm}\mathfrak{C}_0^*$. Hence $N = \mathrm{lcm}\mathfrak{C}_0^*$ as required.

The point *(ii)* can be treated as a completion of the previous case. Indeed $\mathrm{lcm}\mathfrak{C}_0^* < N/2$ would imply that θ is not onto the whole $\mathbb{Z}_N$, which follows from $\theta(g_1^2) \in \mathbb{Z}_{\mathrm{lcm}\mathfrak{C}^*}$. □

In the example below we illustrate point *(ii)* of the above lemma. We show that for $4 \mid N$ and $\mathrm{lcm}\mathfrak{C}_0^* = N/2$ there exists a smooth epimorphism from a NEC group of genus 1 onto $\mathbb{Z}_N$. Note that this situation differs from the case described in point *(iii.3)* of Theorem 5.38 since it takes advantage of the existence of an oval or a chain being a component of the singular set on the surface $\mathbb{H}^2/\Lambda$. It is equivalent to $r > \lambda + p$.

Example 5.41. *Consider a group* $\Gamma = (1; -; [30]; \{(2^2)(2^2)\})$ *with the following epimorphism* $\theta\colon \Gamma \to \mathbb{Z}_{60}$

$$\begin{aligned} \theta(g_1) = t^{-3} \quad & \theta(x_1) = t^2 \\ \theta(e_i) = t^2 \quad & \theta(c_{i,0}) = \theta(c_{i,2}) = t^{N/2}, \;\; \theta(c_{i,1}) = 1. \end{aligned}$$

We have $\mathfrak{C}_0 = (\{2\}, \emptyset, \emptyset, \emptyset, \emptyset, \{2\})$ *and* $\mathrm{lcm}\mathfrak{C}_0^* = 30$. *Moreover* $\mu(\Gamma) = 89\pi/15 > 0$ *and* $\theta((x_1^{-1}g_1^{-1})^{30}c_{i,0}) = 1$. *Since* $(x_1^{-1}g_1^{-1})^{30}$ *is an orientable word in* Γ *we get* $\mathrm{sign}\ker\theta = " - "$.

Theorem 5.42. *Let N be even and suppose that $\mathfrak{C}_0 \in \mathrm{CPer}^{(-,-)}(\mathbb{Z}_N)$ fulfills $\sum_{i=1}^{6} \sharp\mathcal{A}_i \geq 2$. Assume also that X is a non–orientable Klein surface. A NEC group which takes one of the following forms*

(i) $\Gamma = (1; -; [\mathcal{B}_1^*]; \{(\ )^{w_1}(2^2)^{w_2}\})$
(ii) $\Gamma = (2; -; [\mathcal{B}_1^*]; \{(\ )^{w_1}(2^2)^{w_2}\})$
(iii) $\Gamma = (3; -; [\mathcal{B}_1^*]; \{(\ )^{w_1}\})$
(iv) $\Gamma = (1; -; [\mathcal{N}_1^*]; \{(\ )^{y_1}(2^2)^{y_2}\})$
(v) $\Gamma = (2; -; [\mathcal{N}_1^*]; \{(\ )^{y_1}(2^2)^{y_2}\})$
(vi) $\Gamma = (1; -; [\mathcal{N}_{-1}^*]; \{(\ )^{w_1}(2^2)^{w_2}\})$
(vii) $\Gamma = (2; -; [\mathcal{N}_{-1}^*]; \{(\ )^{z_1}\})$,

covers a $\mathbb{Z}_N$–action on X. The above symbols were introduced in (4.33), (4.35), (4.38), (4.40), (4.42) and (4.44). Moreover the area $\mu(\Lambda)$, where $X = \mathbb{H}^2/\Lambda$, is minimal among all non–orientable surfaces on which $\mathfrak{C}_0$ is attained as the character of periods.

Proof. Denote $\mathbb{Z}_N = \langle t \rangle$. Throughout the proof, to each of the cases under consideration, we assign the appropriate covering NEC group listed in points *(i)–(vii)*. We also define the required epimorphisms onto $\mathbb{Z}_N$ and show their properties. In order to make our reasoning more clear we distinguish three auxiliary conditions which help us to divide the proof into cases. These conditions are the following:

(1) the first condition: $\sharp\mathcal{N}_0^*(\mathfrak{C}^*)$ is even

(2) the second condition: $\mathrm{lcm}\mathfrak{C}^* = N$

(3) the third condition: N is divisible by 4.

We will denote the considered subcases by a–b–c, where $a, b, c \in \{0, 1\}$. Using this notation we determine which conditions are fulfilled by switching a, b or c to 1. Observe that our assumption $\sum_{i=1}^{6} \sharp\mathcal{A}_i \geq 2$ together with the condition $\gamma \geq 1$ gives

$$\mu(\Gamma) \geq 2\pi(\gamma + W - 2 + \sum_{i \in \mathcal{B}_1^*}(1 - \frac{1}{m})) \geq 2\pi(-1 + \frac{1}{2} + \frac{2}{3}) > 0, \tag{5.17}$$

where W equals w_1, y_1 or z_1 depending on the particular NEC group listed in the assertion of the theorem. By (5.17) all groups constructed subsequently in the proof are NEC groups.

In each of the cases we deal with $\mathfrak{C}_0$ which obeys certain constraints. In order to ease the way of arguing that our constructs lead to groups of minimal measure we will denote that $\theta' \colon \Gamma' \to \mathbb{Z}_N$ is an epimorphism from NEC group with the underlying $\mathbb{Z}_N$–action given by $\mathfrak{C}_0$. Let γ' be the topological genus of Γ'.

We start with the cases that are explicitly restricted by Theorem 5.38 i.e. $\mathfrak{C}_0 = (\mathcal{A}_1, \mathcal{A}_2, \emptyset, \emptyset, \emptyset, \emptyset)^{(-,-)}$.

Cases 1–1–1 and *1–1–0* Using the notation introduced previously we begin with the following character associated to $\mathfrak{C}$: $\mathfrak{D}(\mathfrak{C}) = \mathfrak{C}$. We need also $\overline{\eta}_{\mathfrak{D}(\mathfrak{C})}$ – the order–preserving element with respect to $\mathfrak{D}(\mathfrak{C})$ which was introduced in Subsection 4.3.2 and verifies $\eta_{i,j} = 1$. We consider both subcases *1–1–1* and *1–1–0* together since upon assumptions (1) and (2), the divisibility of N by 4 does not change the way of arguing. Let us construct Γ based on $\Gamma_{\mathfrak{N}}$ given in (4.34). Furthermore we build epimorphism $\theta\colon \Gamma \to \mathbb{Z}_N$ on account of (4.36)–(4.37) and the following assignment for the only glide reflection

$$\Gamma = (1; -; [\mathcal{B}_1^*]; \{(\ )^{w_1}\}),\ \theta(g_1) = t^{-\frac{L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})}{2}}.$$

Since the second condition is satisfied it shows that there is an orientable word $w \in \Gamma$, such that $\theta(w) = t$. By the first condition $L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$ is even, thus $w_1 = g_1 w^{L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})/2}$ is a non–orientable word that belongs to Λ. It demonstrates that $\operatorname{sign}\Lambda = "-"$. It remains to prove that $\mu(\Gamma)$ is minimal. Observe that $\gamma' \geq 1$. Thus we obtain $\mu(\Gamma') - \mu(\Gamma) \geq 2\pi(\gamma' - 1) \geq 0$ by Proposition 4.19.

The above construct corresponds to point *(i)* of the assertion, where due to the actual assumptions we have $w_2 = 0$.

Case 1–0–1 Observe that in the actual setting, Lemma 5.39 implies $\gamma \geq 2$. However if $\gamma = 2$ and $\overline{\eta}$ is an order–preserving element with respect to $\mathfrak{C}$ which is used to construct a required epimorphism then , as it will be demonstrated below, we shall pay attention to the divisibility of $L(\overline{\eta})$ by 4. For the convenience of the reader we illustrate on the following diagram, the flow of argument and subcases into which the proof of *Case 1–0–1* falls.

$$\begin{tikzcd}
 & \mathit{Case}\ 1-0-1 \arrow[dl] \arrow[dr] & & \\
4 \nmid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})}) \arrow[d] & & 4 \mid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})}) \arrow[dl] \arrow[dr] & \\
(5.18) & \mathcal{N}_0^*(\mathfrak{C}^*) = \emptyset \arrow[d] \arrow[dr] & & \mathcal{N}_0^*(\mathfrak{C}^*) \neq \emptyset \arrow[d] \\
 & \mathcal{N}_{-1}^*(\mathfrak{C}^*) = \emptyset \arrow[d] & \mathcal{N}_{-1}^*(\mathfrak{C}^*) \neq \emptyset \arrow[d] & (5.18) \\
 & (5.21) & (5.20) &
\end{tikzcd}$$

Figure 5.3: Subcases considered in the proof of *Case 1–0–1*.

We start with procedure $\mathfrak{N}$ and an order–preserving element $\overline{\eta}_{\mathfrak{D}(\mathfrak{C})}$. If $4 \nmid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$, then we define

$$\Gamma = (2; -; [\mathcal{B}_1^*]; \{(\ )^{w_1}\}),\ \theta(g_2) = t^{\frac{-L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})+2}{2}},\ \theta(g_1) = t^{-1}. \tag{5.18}$$

We conclude that $(-L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})+2)/2$ is even. Hence θ maps a non–orientable word

$$g_2 g_1^{(-L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})+2)/2} \tag{5.19}$$

to 1, and finally we see that Λ is non–orientable.

On the other hand if $4 \mid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$ we distinguish the cases $\mathcal{N}_0^*(\mathfrak{C}^*) \neq \emptyset$ and $\mathcal{N}_0^*(\mathfrak{C}^*) = \emptyset$.[3] If $\mathcal{N}_0^*(\mathfrak{C}^*) \neq \emptyset$, then we again base our construction on procedure $\mathfrak{N}$. Moreover we define $\gamma = 2$ and use the assignment (5.18) for the glide reflections. Note that the word (5.19) is no longer non–orientable. Let d be an odd period in $\mathcal{B}_i$, $i = 1, 2$ which exists by $\mathcal{N}_0^*(\mathfrak{C}^*) \neq \emptyset$. Denote by h_d a canonical x– or e–generator that corresponds to d. Observe that word $w = g_1^d h_d$ is non–orientable and $\theta(w) = t^{-d} t^d = 1$ due to (4.36). It gives sign$\Lambda =$ " $-$ ".

We now turn to the case $\mathcal{N}_0^*(\mathfrak{C}^*) = \emptyset$ i.e. when all periods of isolated orbits and boundaries are even. If, in addition $\mathcal{N}_{-1}^*(\mathfrak{C}^*) \neq \emptyset$ then we apply the procedure $\mathfrak{N}_{-1}$ and set the topological genus to 2. Moreover, we map the glide reflections according to

$$\Gamma = (2; -; [\mathcal{N}_{-1}^*]; \{(\ )^{z_1}\}),\ \theta(g_2) = t^{\frac{-L(\overline{\eta}_{\mathfrak{N}_{-1}(\mathfrak{C})})+2}{2}},\ \theta(g_1) = t^{-1}. \tag{5.20}$$

Observe that $(-L(\overline{\eta}_{\mathfrak{N}_{-1}(\mathfrak{C})})+2)/2$ is even, which gives that Λ is non–orientable by Lemma 5.24. Nevertheless if $\mathcal{N}_{-1}^*(\mathfrak{C}^*) = \emptyset$, then clearly we are not able to use $\mathfrak{N}_{-1}$. In this case we use once more the procedure $\mathfrak{N}$ and define covering NEC group and the required epimorphism by

$$\Gamma = (3; -; [\mathcal{B}_1^*]; \{(\ )^{w_1}\}),\ \theta(g_3) = t^{\frac{-L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})-2}{2}},\ \theta(g_2) = t,\ \theta(g_1) = 1. \tag{5.21}$$

We get sign$\Lambda =$ " $-$ " by $g_1 \in \ker\theta$.

We proceed to show that the above groups have minimal measure. We are reduced to consider two constructions (5.20) and (5.21) since the minimality of measure of the respective groups in all cases based on (5.18) follow from Proposition 4.19 and Lemma 5.39 which forces $\gamma' \geq 2$.[4]

Consider the case $4 \mid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$ and $\mathcal{N}_0^*(\mathfrak{C}^*) = \emptyset$, but $\mathcal{N}_{-1}^*(\mathfrak{C}^*) \neq \emptyset$. Due to our assumptions we have $n_{-1}(\mathfrak{C}^*) \in \mathcal{B}_1 \cup \mathcal{B}_2$, where $n_{-1}(\mathfrak{C}^*)$ was defined in (4.21). Thus

$$\mu(\Gamma') - \mu(\Gamma) \geq 2\pi(\gamma' - 3) \quad \text{if } \gamma' \geq 3.$$

On the other hand if $\gamma' = 2$, then it must hold

$$\sum_{i=1}^{2} \sharp\mathcal{G}_i(\Gamma', \theta') - \sum_{i=1}^{2} \sharp\mathcal{G}_i(\Gamma, \theta) \geq 0, \tag{5.22}$$

[3]Roughly speaking the first (the second) relation describes the case when not all (all) periods of $\mathfrak{C}_0$ are even.

[4]At the beginning of the proof we have denoted by γ' the topological genus of an arbitrary NEC group covering a $\mathbb{Z}_N$–action given by $\mathfrak{C}_0$.

where the families $\mathcal{G}_i(\Gamma,\theta)$ were given by (4.6). Hence by the choice of the element $n_{-1}(\mathfrak{C}^*)$ we have

$$\mu(\Gamma') - \mu(\Gamma) \geq 0 \quad \text{if } \gamma' = 2.$$

It shows that Γ has minimal measure.

We now proceed to the case $4 \mid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$ and $\mathcal{N}_0^*(\mathfrak{C}^*) = \mathcal{N}_{-1}^*(\mathfrak{C}^*) = \emptyset$. Note that $L(\overline{\eta})$ is now divisible by 4 for every order–preserving element $\overline{\eta}$. Furthermore $\mathcal{N}_0^*(\mathfrak{C}^*) = \emptyset$ gives $\mathrm{lcm}\mathfrak{C}_0^* < N$. Hence if we require θ' to be an epimorphism onto $\mathbb{Z}_N$ it must map a glide reflection to t^v with v odd. We show that upon these conditions we must have $\gamma' \geq 3$. On the contrary suppose that $\gamma' = 2$. Then

$$\theta'(g_1) = t^{-c} \quad \text{and} \quad \theta'(g_2) = t^{-\frac{L(\overline{\eta})}{2}+c}.$$

Since both numbers: c and $c - L(\overline{\eta})/2$ are odd, then by Lemma 5.24 we get $\mathrm{sign}\Lambda = " + "$, a contradiction. Obviously $\gamma' \geq 3$ yields $\mu(\Gamma') - \mu(\Gamma) \geq 0$, as required.

The constructs which have been distinguished in this point correspond in order of appearance to the following items of the assertion: *(ii)*, *(vii)* and *(iii)*.

Cases 0–1–1 and *0–1–0* First observe that $2 \nmid \sharp\mathcal{N}_0^*(\mathfrak{C}^*)$ yields $\mathcal{N}_0^*(\mathfrak{C}^*) \neq \emptyset$. Basing on the procedure $\mathfrak{N}_0$ we may construct a covering NEC group together with the following assignment

$$\Gamma = (1; -; [\mathcal{N}_1^*]; \{(\ )^{y_1}\}),\ \theta(g_1) = t^{-\frac{L(\overline{\eta}_{\mathfrak{N}_0(\mathfrak{C})})}{2}}.$$

Since the whole $\mathbb{Z}_N$ is generated by images of x– and e–generators there exists an orientable word w such that $\theta(w) = t$. Consequently a non–orientable word $g_1 w^{L(\overline{\eta}_{\mathfrak{N}_0(\mathfrak{C})})/2}$ is mapped to 1 which shows that Λ is non–orientable.

We argue that the measure of the above constructed NEC covering group is minimal in a standard way, i.e. by comparing $\mu(\Gamma')$ with $\mu(\Gamma)$. First we show that a construction may not rely on a group $(1; -; [\mathcal{B}_1^*], \{(\ )^{w_1}\})$. Otherwise we would have $g_1 \mapsto t^{-L(\overline{\eta})/2}$, where $\overline{\eta}$ is an order–preserving element with respect to $\mathfrak{C}$. But $L(\overline{\eta})$ is odd regardless of $\overline{\eta}$ which follows from Lemma 4.8. Hence the above group must be enhanced to let us construct a smooth epimorphism onto $\mathbb{Z}_N$. By (5.22) and the choice of $n_0(\mathfrak{C}^*)$ defined in (4.21) we have

$$\mu(\Gamma') - \mu(\Gamma) = (\mu(\Gamma') - 2\pi) - \mu(\Gamma_{\mathfrak{N}_0}) \geq 0.$$

The above construction relates to point *(iv)* of the assertion.

Case 1–0–0 Observe that Lemma 5.39 yields $\gamma \geq 2$. If $4 \nmid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$ then we apply the procedure $\mathfrak{N}$ and set $\gamma = 2$ with the assignment given by (5.18). On the other hand if $4 \mid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$ we put

$$\Gamma = (2; -; [\mathcal{B}_1^*]; \{(\ )^{w_1}\}),\ \theta(g_2) = t^{\frac{-L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})-N+2}{2}},\ \theta(g_1) = t^{-1}.$$

We easily check that $\theta(g_2 g_1^{(-L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})-N+2)/2}) = 1$. Furthermore since $(-L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})}) - N + 2)/2$ is even we conclude that Λ is non–orientable.

The minimality of $\mu(\Gamma)$ in this setting is obvious, since in order to construct required covering group and epimorphism we use character of periods equal to $\mathfrak{C}$. Moreover the group Γ has the minimal possible genus which equals 2.

Both constructions relates to the same point *(ii)* of the assertion.

Cases 0–0–1 and *0–0–0* By $2 \nmid \sharp\mathcal{N}_0^*(\mathfrak{C}^*)$ we have $\mathcal{N}_0^*(\mathfrak{C}^*) \neq \emptyset$. Let d be an odd period contained in $\mathfrak{C}$. By Lemma 5.39 we have $\gamma \geq 2$. Let us denote by h_d a canonical x– or e–generator that corresponds to d. We use the procedure $\mathfrak{N}_0$ and define

$$\Gamma = (2; -; [\mathcal{N}_1^*]; \{(\ )^{y_1}\}),\ \theta(g_2) = t^{\frac{-L(\overline{\eta}_{\mathfrak{N}_0(\mathfrak{C})})+2}{2}},\ \theta(g_1) = t^{-1}.$$

Note that a non–orientable word $g_1^d h_d$ is mapped to 1, proving that sign$\Lambda =$ " $-$ ".

The minimality of $\mu(\Gamma)$ can be shown by Lemma 4.8, the inequality (5.22) and the choice of element $n_0(\mathfrak{C}^*)$. We have

$$\mu(\Gamma') - \mu(\Gamma) = (\mu(\Gamma') - 4\pi) - \mu(\Gamma_{\mathfrak{N}_0}) \geq 0.$$

The above construct corresponds to point *(v)* of the assertion.

Having disposed of the previous steps we can proceed to the situation $\bigcup_{i\geq 3} \mathcal{A}_i \neq \emptyset$. As before, based on conditions (1)–(3) we distinguish 8 subcases. In order to differentiate the cases considered hereafter from the previous ones, we introduce the notation *E.a–b–c*, where $a, b, c \in \{0, 1\}$. The meaning of each symbol is analogous to the previous definition *a–b–c*. Recall that under the assumption $\bigcup_{i\geq 3} \mathcal{A}_i \neq \emptyset$ we now have lcm$\mathfrak{C}_0^* =$ lcm$\{2, \text{lcm}\mathfrak{C}^*\}$ by Corollary 4.22.

Cases E.1–1–1 and *E.1–1–0* Since lcm$\mathfrak{C}^* = N$, there is an orientable word w that goes to t. We use the procedure $\mathfrak{N}$, set $\gamma = 1$ and map g_1 as follows

$$\Gamma = (1; -; [\mathcal{B}_1^*]; \{(\ )^{w_1}(2^{w_2})\}),\ \theta(g_1) = t^{-\frac{L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})}{2}}.$$

Observe that the word $g_1 w^{\frac{L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})}{2}}$ is non–orientable and belongs to Λ, which gives sign$\Lambda =$ " $-$ ".

It is easily seen that $\mu(\Gamma)$ is minimal since

$$\mu(\Gamma') - \mu(\Gamma) \geq 2\pi(\gamma' - 1) \geq 0$$

by Lemma 4.19.

The above group Γ appears in point *(i)* of the assertion.

Case E.1–0–1 Observe that by Remark 5.35 we have sign$\Lambda =$ " $-$ " provided θ is onto $\mathbb{Z}_N$. This fact plays a key role in the actual point and for the simplicity of our reasoning we will not recall it repeatedly once the appropriate epimorphisms are constructed. The proof falls into two parts: lcm$\mathfrak{C}_0^* < N/2$ and lcm$\mathfrak{C}_0^* = N/2$. However it is worth noting that we have assumed now only lcm$\mathfrak{C}^* \neq N$. Let us briefly discuss why we apply here the condition that relates to lcm$\mathfrak{C}_0^*$, rather then the one concerning lcm$\mathfrak{C}^*$. The reasons for such approach consists in arithmetic relations between these two notions. Observe that under the actual assumptions we can have

$$\text{neither} \quad \mathrm{lcm}\mathfrak{C}_0^* = \mathrm{lcm}\mathfrak{C}^* = N, \quad \text{nor} \quad 2\mathrm{lcm}\mathfrak{C}^* = \mathrm{lcm}\mathfrak{C}_0^* = N.$$

The second relation can not hold since $\mathrm{lcm}\mathfrak{C}_0^* = \mathrm{lcm}\mathfrak{C}^*$, according to Remark 4.20. It follows that we are reduced to investigate the cases when $\mathrm{lcm}\mathfrak{C}_0^* \leq N/2$. For the convenience of the reader we illustrate on the following diagram, the flow of argument and subcases into which the proof falls.

$$\begin{tikzcd}
 & & \textit{Case E.1} - \mathit{0} - \mathit{1} \arrow[dll] \arrow[dr] & & \\
\mathrm{lcm}\mathfrak{C}_0^* < \frac{N}{2} \arrow[d] & & & \mathrm{lcm}\mathfrak{C}_0^* = \frac{N}{2} \arrow[dl] \arrow[dr] & \\
(5.23) & & 4 \mid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})}) \arrow[d] \arrow[dr] & & 4 \nmid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})}) \arrow[d] \\
 & & \mathcal{N}_0^*(\mathfrak{C}^*) = \emptyset \arrow[dll, dotted] \arrow[d, dotted] & \mathcal{N}_0^*(\mathfrak{C}^*) \neq \emptyset \arrow[d] & (5.24) \\
\sharp\mathcal{N}_{-1}^*(\mathfrak{C}^*) = 2l > 0 \arrow[d, dotted] & & (5.23) & (5.25) & \\
(5.26) & & & &
\end{tikzcd}$$

Figure 5.4: Subcases considered in the proof of *Case E.1–0–1.*

Let us start with $\mathrm{lcm}\mathfrak{C}_0^* < N/2$. By point *(ii)* of Lemma 5.40 we must have $\gamma \geq 2$. Note that $L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$ is even by Lemma 4.8, thus we may use the following construction built on a basis of the procedure $\mathfrak{N}$

$$\Gamma = (2; -; [\mathcal{B}_1^*]; \{(\ )^{w_1}(2^2)^{w_2}\}),\ \theta(g_2) = t^{-\frac{L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})-2}{2}},\ \theta(g_1) = t^{-1}. \tag{5.23}$$

On the other hand if $\mathrm{lcm}\mathfrak{C}_0^* = N/2$, then we have two branches of subcases depending on the divisibility of $L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$ by 4. Let us consider situation $4 \nmid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$. We use the procedure $\mathfrak{N}$ and put

$$\Gamma = (1; -; [\mathcal{B}_1^*]; \{(\ )^{w_1}(2^2)^{w_2}\}),\ \theta(g_1) = t^{-\frac{L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})}{2}}. \tag{5.24}$$

Observe that $\theta(g_1) \notin \mathbb{Z}_{N/2} \simeq \mathbb{Z}_{\mathrm{lcm}\mathfrak{C}_0^*}$ since it goes to an odd power of t. Consequently, θ is onto the whole $\mathbb{Z}_N$.

Let us now turn to the branch $4 \mid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$. Here we again distinguish two scenarios: $\mathcal{N}_0^*(\mathfrak{C}^*) \neq \emptyset$ and $\mathcal{N}_0^*(\mathfrak{C}^*) = \emptyset$. Suppose $\mathcal{N}_0^*(\mathfrak{C}^*) \neq \emptyset$. By the initial assumption that $\sharp\mathcal{N}_0^*(\mathfrak{C}^*)$ is an even number we may now apply Corollary 4.10. Hence there exists $\overline{\eta}$ – an order preserving element with respect to $\mathfrak{C}$ such that $L(\overline{\eta})/2$ is odd. We take an $\overline{\eta}$ which verifies the above property and based on the procedure $\mathfrak{N}$ we define the following assignment

$$\Gamma = (1; -; [\mathcal{B}_1^*]; \{(\ )^{w_1}(2^2)^{w_2}\}),\ \theta(g_1) = t^{-\frac{L(\overline{\eta})}{2}}. \tag{5.25}$$

Let us now proceed to the scenario $\mathcal{N}_0^*(\mathfrak{C}^*) = \emptyset$. We propose to split it into two parts. The first of the distinguished scenarios embraces the situation when $\sharp\mathcal{N}_{-1}^*(\mathfrak{C}^*) = 2l,\ l \in \mathbb{N} \setminus \{0\}$ and $\mathcal{N}_{-1}^*(\mathfrak{C}^*) \cap \mathcal{B}_1^* \neq \emptyset$. We shall use the procedure $\mathfrak{N}_{-1}$ and set the following assignment

$$\Gamma = (1; -; [\mathcal{N}_{-1}^*]; \{(\ )^{w_1}(2^2)^{w_2}\}),\ \theta(g_1) = t^{-\frac{L(\overline{\eta}_{\mathfrak{N}_{-1}(\mathfrak{C})})}{2}}. \tag{5.26}$$

Note that neither the number of empty, nor the number of non–empty period cycles has changed compared to (5.23)–(5.25), which follows by $\mathcal{N}_{-1}^*(\mathfrak{C}^*) \cap \mathcal{B}_1^* \neq \emptyset$. Moreover by Lemma 4.9 we have $L(\overline{\eta}) = 2 \cdot \sharp\mathcal{N}_{-1}^*(\mathfrak{C}^*)$ for every orientation–preserving element with respect to $\mathfrak{C}$. Thus $L(\overline{\eta})/2$ is even. It follows that $L(\overline{\eta}_{\mathfrak{N}_{-1}(\mathfrak{C})})/2$ is odd, which eventually yields that (5.26) defines an epimorphism onto $\mathbb{Z}_N$, as required.

The second part deals with all remaining cases. We cover them using the procedure $\mathfrak{N}$ enhanced according to (5.23).

The minimality of measure of NEC groups defined above by constructs (5.23) upon the assumption $\mathrm{lcm}\mathfrak{C}_0^* < N/2$, (5.24) and (5.25) is clear by Proposition 4.19 and point *(ii)* of Lemma 5.40.

Consider the case corresponding to construct (5.26). We have

$$\mu(\Gamma') - \mu(\Gamma) \geq 2\pi(\gamma' - 1 + \frac{1}{n_{-1}^*(\mathfrak{C}^*)}) \quad \text{if } \gamma' \geq 2.$$

Furthermore if $\gamma' = 1$, then it holds

$$\sum_{i=1}^{6} \sharp\mathcal{G}_i(\Gamma', \theta') - \sum_{i=1}^{6} \sharp\mathcal{G}_i(\Gamma, \theta) \geq 0. \tag{5.27}$$

By the choice of element $n_{-1}^*(\mathfrak{C}^*)$ we get

$$\mu(\Gamma') - \mu(\Gamma) \geq 2\pi(1 - \frac{1}{n_{-1}^*(\mathfrak{C}^*)}) - 2\pi(1 - \frac{1}{n_{-1}^*(\mathfrak{C}^*)}) = 0.$$

On the other hand in the situation corresponding to (5.23) upon the assumption $\mathrm{lcm}\mathfrak{C}_0^* = N/2$ one has

$$\mu(\Gamma') - \mu(\Gamma) = (\mu(\Gamma') - \mu(\Gamma_{\mathfrak{N}})) + (\mu(\Gamma_{\mathfrak{N}}) - \mu(\Gamma)) \geq 4\pi - 4\pi = 0.$$

The groups which have been built above correspond to points enumerated in the assertion as follows: (5.23) to point *(ii)*, (5.24) and (5.25) to point *(i)* and finally (5.26) to point *(vi)*.

Cases E.0–1–1 and *E.0–1–0* Since $\sharp\mathcal{N}_0^*(\mathfrak{C}^*)$ is odd, then by Lemma 4.8 the expression $L(\overline{\eta})$ takes an odd value for every $\overline{\eta}$ - an order–preserving element with respect to $\mathfrak{C}$. Hence we must repeat an odd period of $\mathfrak{C}$ in order to be able to define a required epimorphism. We apply the procedure $\mathfrak{N}_0$ and use the assignment

$$\Gamma = (1; -; [\mathcal{N}_1^*]; \{(\ )^{y_1}(2^2)^{y_2}\}),\ \theta(g_1) = t^{-\frac{L(\overline{\eta}_{\mathfrak{N}_0(\mathfrak{C})})}{2}},$$

which by the assumption $\mathrm{lcm}\mathfrak{C}^* = |\langle\theta(x_1, \ldots, e_{y_1+y_2})\rangle| = N$ yields an epimorphism onto $\mathbb{Z}_N$. Let us focus on the orientability character of its kernel. Take an orientable word w such that $\theta(w) = t$. It suffices now to observe that a non–orientable word $w^{L(\overline{\eta}_{\mathfrak{N}_0(\mathfrak{C})})/2}g_1$ is mapped to 1. It leads us to $\mathrm{sign}\Lambda = " - "$.

The minimality of $\mu(\Gamma)$ is obvious by Lemma 4.8 and the choice of the element $n_0^*(\mathfrak{C}^*)$.

The group that has just been constructed corresponds to point *(iv)* of the assertion.

Case E.1–0–0 The proof falls into two parts. We distinguish the cases $\mathrm{lcm}\mathfrak{C}_0^* = N$ and $\mathrm{lcm}\mathfrak{C}_0^* < N$.

Let us first consider the situation $\mathrm{lcm}\mathfrak{C}_0^* = N$ and observe that we must have $\mathrm{lcm}\mathfrak{C}^* = N/2$ due to the assumption $\mathrm{lcm}\mathfrak{C}^* < N$ and Remark 4.20. We apply the procedure $\mathfrak{N}$ and define one covering NEC group Γ with two different assignments depending on the divisibility of $L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$ by 4. We put

$$\Gamma = (1; -; [\mathcal{B}_1^*]; \{(\ )^{w_1}(2^2)^{w_2}\}). \tag{5.28}$$

Furthermore, if $4 \mid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$ we put

$$\theta(g_1) = t^{-\frac{L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})}{2}}.$$

On the other hand if $4 \nmid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})$ we change the above assignment by putting

$$\theta(g_1) = t^{-\frac{L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})+N}{2}}.$$

Observe that in both cases g_1 goes to t^u with u even. It enables us to show that $\mathrm{sign}\Lambda = "-"$. Consider a word $w \in \Gamma$ satisfying $\theta(w) = t$, which exists since $\mathrm{lcm}\mathfrak{C}_0^* = N$. It is worth noting that the orientability character of w is not assumed here. We take $w^{-u}g_1$ which is non–orientable word and belongs to $\ker\theta$.

A similar reasoning applies to the case $\mathrm{lcm}\mathfrak{C}_0^* < N$. By point *(i)* of Lemma 5.40 we must have $\gamma \geq 2$. As before we also use the procedure $\mathfrak{N}$, but set $\gamma = 2$

$$\Gamma = (2; -; [\mathcal{B}_1^*]; \{(\ )^{w_1}(2^2)^{w_2}\}). \tag{5.29}$$

Moreover we map $\theta(g_1) = t^{-1}$ and

$$\theta(g_2) = \begin{cases} t^{-\frac{L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})})-2}{2}} & \text{if } 4 \nmid L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})}) \\ t^{-\frac{L(\overline{\eta}_{\mathfrak{D}(\mathfrak{C})}-2-N}{2}} & \text{otherwise.} \end{cases}$$

By the above g_2 is mapped to t^u with u even. Thus $g_1^u g_2$ is a non–orientable word that goes to 1. It follows that sign$\Lambda =$ " $-$ ".

The minimality of measure of covering NEC groups constructed above is obvious by Proposition 4.19 and point *(i)* of Lemma 5.40. We have

$$\mu(\Gamma') - \mu(\Gamma) = \mu(\Gamma') - (\mu(\Gamma_{\mathfrak{N}}) + 2\pi) \geq 2\pi(\gamma' - 1)$$

in case of (5.28) and

$$\mu(\Gamma') - \mu(\Gamma) = \mu(\Gamma') - (\mu(\Gamma_{\mathfrak{N}}) + 4\pi) \geq 2\pi(\gamma' - 2),$$

in case of (5.29), since there holds $\gamma' \geq 2$.

The groups we have defined in this point correspond in order of appearance to groups labeled as *(i)* and *(ii)* in the assertion of the theorem.

Case E.0–0–1 We begin by observing that $L(\overline{\eta})$ is odd for every element $\overline{\eta}$ which is orientation–preserving with respect to $\mathfrak{C}$. Since $\mathcal{N}_0^*(\mathfrak{C}^*) \neq \emptyset$ we apply the procedure $\mathfrak{N}_0$. Recall that by Remark 5.35 we have sign$\Lambda =$ " $-$ " provided we define a smooth epimorphism onto $\mathbb{Z}_N$.

Observe that upon assumptions lcm$\mathfrak{C}^* < N$ and $4 \mid N$ we must have lcm$\mathfrak{C}_0^* \leq N/2$, which follows by Remark 4.20.

We first consider the situation lcm$\mathfrak{C}_0^* < N/2$. By point *(ii)* of Lemma 5.40 we must have $\gamma \geq 2$. Then we built a covering group and a required epimorphism in the following way

$$\Gamma = (2; -; [\mathcal{N}_1^*]; \{(\ )^{y_1}(2^2)^{y_2}\}),\ \theta(g_2) = t^{-\frac{L(\overline{\eta}_{\mathfrak{N}_0(\mathfrak{C})})-2}{2}},\ \theta(g_1) = t^{-1}. \tag{5.30}$$

On the other hand if lcm$\mathfrak{C}_0^* = N/2$ we distinguish the cases $4 \nmid L(\overline{\eta}_{\mathfrak{N}_0(\mathfrak{C})})$ and $4 \mid L(\overline{\eta}_{\mathfrak{N}_0(\mathfrak{C})})$. For $4 \nmid L(\overline{\eta}_{\mathfrak{N}_0(\mathfrak{C})})$ we build a covering NEC group based on the procedure $\mathfrak{N}_0$ and establish a required epimorphism by

$$\Gamma = (1; -; [\mathcal{N}_1^*]; \{(\ )^{y_1}(2^2)^{y_2}\}),\ \theta(g_1) = t^{-\frac{L(\overline{\eta}_{\mathfrak{N}_0(\mathfrak{C})})}{2}}. \tag{5.31}$$

Note that $L(\overline{\eta}_{\mathfrak{N}_0(\mathfrak{C})})/2$ is an odd number which yields $\theta(g_1) \notin \mathbb{Z}_{N/2} \simeq \mathbb{Z}_{\mathrm{lcm}\mathfrak{C}_0^*}$. It shows that $|\theta(\Gamma)| = N$.

Next we proceed to the case $4 \mid L(\overline{\eta}_{\mathfrak{N}_0(\mathfrak{C})})$. We shall consider the following family of positive integers derived from character of periods $\mathfrak{N}_0(\mathfrak{C})$

$$\{\mathcal{N}_1^*, \mathcal{N}_2^*, \mathcal{N}_3^*, \mathcal{N}_4^*, \mathcal{N}_5^*, \mathcal{N}_6^*\} \tag{5.32}$$

i.e. the family of orders of images of the canonical elliptic and e–generators of a potential covering group.[5] Observe that the family (5.32) is non–empty and has an even cardinality. Hence by Corollary 4.10 applied to (5.32) we obtain an element $\overline{\eta}$ which is order–preserving

[5]To be fully compliant to the terminology introduced in Subsection 4.2.1 we shall recall that $\mathfrak{N}_0(\mathfrak{C})$ is character associated to inflated character of periods $\mathfrak{C}$.

with respect to $\mathfrak{N}_0(\mathfrak{C})$ and verifies that $L(\overline{\eta})/2$ is odd. Using such an element we may we define a covering NEC group and the appropriate assignment as follows

$$\Gamma = (1; -; [\mathcal{N}_1^*]; \{(\ )^{y_1}(2^2)^{y_2}\}),\ \theta(g_1) = t^{-\frac{L(\overline{\eta})}{2}}. \tag{5.33}$$

We proceed to demonstrate that measure of the NEC groups constructed above is minimal. In case of (5.30) we get

$$\mu(\Gamma') - \mu(\Gamma) = (\mu(\Gamma') - 4\pi) - \mu(\Gamma_{\mathfrak{N}_0}) \geq 0,$$

by (5.27), the choice of element $n_0(\mathfrak{C}^*)$ and point *(ii)* of Lemma 5.40 which forces $\gamma' \geq 2$. Eventually for (5.31) and (5.33) it holds

$$\mu(\Gamma') - \mu(\Gamma) = (\mu(\Gamma') - 2\pi) - \mu(\Gamma_{\mathfrak{N}_0}) \geq 0$$

by (5.27) and the properties of $n_0(\mathfrak{C}^*)$.

Under the actual assumptions we have distinguished two different types of groups corresponding to points *(v)* and *(iv)* of the assertion.

Case E.0–0–0 Since $\sharp\mathcal{N}_0^*(\mathfrak{C}^*)$ is odd we enhance character of periods $\mathfrak{C}$ by means of the procedure $\mathfrak{N}_0$. Although we assume $\mathrm{lcm}\mathfrak{C}^* < N$ we may still have either $\mathrm{lcm}\mathfrak{C}_0^* = N$ or $\mathrm{lcm}\mathfrak{C}_0^* < N$.

Suppose $\mathrm{lcm}\mathfrak{C}_0^* = N$ which yields $\mathrm{lcm}\mathfrak{C}^* = N/2$. We define the following NEC group

$$\Gamma = (1; -; [\mathcal{N}_1^*]; \{(\ )^{y_1}(2^2)^{y_2}\})$$

and map the glide reflection depending on the divisibility of $L(\overline{\eta}_{\mathfrak{N}_0(\mathfrak{C})})$ by 4 according to

$$\theta(g_1) = \begin{cases} t^{-\frac{L(\overline{\eta}_{\mathfrak{N}_0(\mathfrak{C})})}{2}} & \text{if } 4 \mid L(\overline{\eta}_{\mathfrak{N}_0(\mathfrak{C})}) \\ t^{-\frac{L(\overline{\eta}_{\mathfrak{N}_0(\mathfrak{C})})-N}{2}} & \text{otherwise.} \end{cases}$$

Note that θ is onto $\mathbb{Z}_N$ by $\mathrm{lcm}\mathfrak{C}_0^* = N$. We also observe that the only glide reflection g_1 goes to t^u, with u even. Consider a word verifying $\theta(w) = t$. It is easily seen that a non–orientable word $w^u g_1$ belongs to Λ, which shows $\mathrm{sign}\Lambda = " - "$.

On the other hand $\mathrm{lcm}\mathfrak{C}_0^* < N$ forces by point *(i)* of Lemma 5.40 that $\gamma \geq 2$. Likewise in the above case, we define an unique covering NEC group with the two different assignments. We put

$$\Gamma = (2; -; [\mathcal{N}_1^*]; \{(\ )^{y_1}(2^2)^{y_2}\}).$$

Moreover we always map g_1 to t^{-1} whereas

$$\theta(g_2) = \begin{cases} t^{(-L(\overline{\eta}_{\mathfrak{N}_0(\mathfrak{C})})+2)/2} & \text{if } 4 \nmid L(\overline{\eta}_{\mathfrak{N}_0(\mathfrak{C})}) \\ t^{(-L(\overline{\eta}_{\mathfrak{N}_0(\mathfrak{C})})+2+N)/2} & \text{otherwise.} \end{cases}$$

Since g_2 is mapped to t^u with u even, we conclude that non–orientable word $g_1^u g_2$ fulfils $\theta(g_1^u g_2) = 1$.

The minimality of measure of covering groups which have been considered above follows in case $\mathrm{lcm}\mathfrak{C}_0^* = N$ by (5.27) and the choice of element $n_0^*(\mathfrak{C}^*)$. In case $\mathrm{lcm}\mathfrak{C}_0^* < N$ we additionally use point *(ii)* of Lemma 5.40.

The constructs correspond to items *(iv)* and *(v)* of the theorem. □

Since the case $\sum_{i=1}^{6} \sharp\mathcal{A}_i < 2$ was not considered in Theorem 5.42 we complete our investigation by the following remark.

Remark 5.43. *Let N be even and suppose that $\mathfrak{C}_0 \in \mathrm{CPer}^{(-,-)}(\mathbb{Z}_N)$ fulfils $\sum_{i=1}^{6} \sharp\mathcal{A}_i < 2$. Then the universal covering group that covers a $\mathbb{Z}_N$–action prescribed by $\mathfrak{C}_0 \in \mathrm{CPer}^{(-,-)}(\mathbb{Z}_N)$ equals*

$$\Gamma = \begin{cases}
(3;-;[\,];\{\,\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\emptyset)^{(-,-)} \\
(1;-;[N,N];\{\,\}), & \text{if } \mathfrak{C}_0 = (\{1\},\emptyset,\emptyset,\emptyset,\emptyset,\emptyset)^{(-,-)} \text{ and } N>2 \\
(1;-;[2,2,2,2];\{\,\}), & \text{if } \mathfrak{C}_0 = (\{1\},\emptyset,\emptyset,\emptyset,\emptyset,\emptyset)^{(-,-)} \text{ and } N=2 \\
(2;-;[N/d,N/d];\{\,\}), & \text{if } \mathfrak{C}_0 = (\{d\},\emptyset,\emptyset,\emptyset,\emptyset,\emptyset)^{(-,-)},\ d\neq 1 \text{ and } \alpha_2(d)=0 \\
(2;-;[N/d,N/d];\{\,\}), & \text{if } \mathfrak{C}_0 = (\{d\},\emptyset,\emptyset,\emptyset,\emptyset,\emptyset)^{(-,-)},\ \alpha_2(d)=1 \\
(2;-;[N/d,N/d];\{\,\}), & \text{if } \mathfrak{C}_0 = (\{d\},\emptyset,\emptyset,\emptyset,\emptyset,\emptyset)^{(-,-)},\ \alpha_2(d)>1 \text{ and } \alpha_2(N)=1 \\
(3;-;[N/d,N/d];\{\,\}), & \text{if } \mathfrak{C}_0 = (\{d\},\emptyset,\emptyset,\emptyset,\emptyset,\emptyset)^{(-,-)},\ \alpha_2(d)>1 \text{ and } \alpha_2(N)>1 \\
(1;-;[\,];\{(\,)^2\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\{1\},\emptyset,\emptyset,\emptyset,\emptyset)^{(-,-)} \\
(2;-;[\,];\{(\,)^2\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\{d\},\emptyset,\emptyset,\emptyset,\emptyset)^{(-,-)},\ d\neq 1 \text{ and } \alpha_2(d)=0 \\
(2;-;[\,];\{(\,)\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\{d\},\emptyset,\emptyset,\emptyset,\emptyset)^{(-,-)},\ \alpha_2(d)=1 \\
(2;-;[\,];\{(\,)\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\{d\},\emptyset,\emptyset,\emptyset,\emptyset)^{(-,-)},\ \alpha_2(d)>1 \text{ and } \alpha_2(N)=1 \\
(3;-;[\,];\{(\,)\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\{d\},\emptyset,\emptyset,\emptyset,\emptyset)^{(-,-)},\ \alpha_2(d)>1 \text{ and } \alpha_2(N)>1 \\
(2;-;[\,];\{(\,)\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\emptyset,\{d\},\emptyset,\emptyset,\emptyset)^{(-,-)} \\
(1;-;[\,];\{(\,)^2\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\emptyset,\emptyset,\{1\},\emptyset,\emptyset)^{(-,-)} \\
(2;-;[\,];\{(\,)^2\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\emptyset,\emptyset,\{d\},\emptyset,\emptyset)^{(-,-)},\ d\neq 1 \text{ and } \alpha_2(d)=0 \\
(2;-;[\,];\{(\,)\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\emptyset,\emptyset,\{d\},\emptyset,\emptyset)^{(-,-)},\ \alpha_2(d)>0 \\
(1;-;[\,];\{(2^2)\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\emptyset,\emptyset,\emptyset,\{1\},\emptyset)^{(-,-)},\ \alpha_2(N)=1 \\
(2;-;[\,];\{(2^2)\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\emptyset,\emptyset,\emptyset,\{d\},\emptyset)^{(-,-)},\ d\neq 1 \text{ and } \alpha_2(N)=1 \\
(2;-;[\,];\{(2^2)\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\emptyset,\emptyset,\emptyset,\{d\},\emptyset)^{(-,-)},\ \alpha_2(N)>1 \\
(1;-;[\,];\{(2^2)^2\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\{1\})^{(-,-)} \\
(2;-;[\,];\{(2^2)^2\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\{d\})^{(-,-)},\ d\neq 1 \text{ and } \alpha_2(d)=0 \\
(2;-;[\,];\{(2^2)\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\{d\})^{(-,-)},\ \alpha_2(d)>0,\ \alpha_2(N)=1 \\
(1;-;[\,];\{(2^2)\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\{2\})^{(-,-)},\ \alpha_2(N)>1 \\
(2;-;[\,];\{(2^2)\}), & \text{if } \mathfrak{C}_0 = (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,\{d\})^{(-,-)},\ d>2,\ \alpha_2(d)>0, \\
& \qquad\qquad \text{and } \alpha_2(N)>1
\end{cases}$$

The area of the respective NEC groups is minimal among all non–orientable surfaces on which the above tuples of sets are attained as the characters of periods.

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