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Complexity issues in multimodal logics for multiagent systems

PhD Dissertation

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Aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Abstract

In this thesis we study the complexity of the satisfiability problem for TeamLog, a representative and widely recognized multimodal formalism for multiagent systems. We investigate a fragment of TeamLog, called TeamLog$^{\mathrm{ind}}$, designed for specifying individual agents as well as full TeamLog, that allows for specifying group aspects of multiagent systems, in particular teamwork. We show that the satisfiability problem for TeamLog$^{\mathrm{ind}}$ is PSPACE complete, while in the case of the full TeamLog it is EXPTIME complete.

The complexity results obtained mean, essentially, that from practical point of view tasks involving reasoning in TeamLog are not efficient and certainly not scalable. For this reason it is important to study restrictions of the language that would lead to reduction of the complexity of the problem, or at least would lead to classes of complexity for which effective heuristics based methods could be applied. In the case of TeamLog$^{\mathrm{ind}}$ we consider two restrictions: restricting modal depth of formulas by a constant and restricting the number of propositional symbols by a constant. We show that in the first case the satisfiability problem for TeamLog$^{\mathrm{ind}}$ becomes NPTIME complete, while applying the second restriction leaves the problem PSPACE complete. Combining both restrictions makes the problem solvable in linear time. In this case, however, the constant multiplier depends exponentially on the number of propositional symbols.

In the case of full TeamLog, restricting the modal depth of formulas is not promising, as the satisfiability problem remains EXPTIME complete even if modal depth is bounded by 2. For this reason we introduce an original class of restrictions, called *modal context restrictions*, which generalise modal depth restriction. We study two such restrictions, both resulting in PSPACE completeness of the satisfiability problem. In the case of the less restrictive one of them, called $\mathbf{R}_1$, the problem remains PSPACE hard even if modal depth of formulas is bounded by 2. In the case of the more restrictive $\mathbf{R}_2$, combining it with restricting modal depth by a constant results in NPTIME completeness of the satisfiability problem.

Since in some cases restriction $\mathbf{R}_2$ is too strong, not allowing for expressing important aspects of multiagent systems, we propose a refinement of the restriction $\mathbf{R}_1$. This refinement results in NPTIME completeness of the satisfiability problem, when combined with bounding modal depth of formulas.

Apart from modal context restrictions, we also investigate the effect of bounding the number of propositional symbols by a constant. We show that in this case the problem remains EXPTIME complete. Combining this restriction with restricting modal depth of formulas results in the satisfiability problem for TeamLog being solvable in linear time, but again the constant multiplier depends exponentially on the number of propositional symbols.

The methods we use and the results we obtain can be applied to other multimodal formalisms for multiagent systems.

Streszczenie

W poniższej rozprawie badamy złożoność obliczeniową problemu spełnialności dla reprezentatywnego i uznanego formalizmu wieloagentowego o nazwie TeamLog. Rozważamy dwa fragmenty tego formalizmu: TeamLog$^{\mathrm{ind}}$, przeznaczony do specyfikowania pojedynczych agentów oraz pełen TeamLog, pozwalający na specyfikowanie grupowych aspektów systemów wieloagentowych, w tym pracy zespołowej. Pokazujemy, że problem spełnialności dla TeamLog$^{\mathrm{ind}}$ jest PSPACE zupełny oraz że jest on EXPTIME zupełny w przypadku pełnego formalizmu TeamLog. W rozprawie badamy również w jaki sposób ograniczanie języka powyższych formalizmów wpływa na złożoność problemu spełnialności. W przypadku TeamLog$^{\mathrm{ind}}$ rozważamy dwa ograniczenia: ograniczenie głębokości modalnej formuł przez stałą oraz ograniczenie liczby zmiennych zdaniowych przez stałą. Pokazujemy, że w przypadku pierwszego ograniczenia problem staje się NPTIME zupełny, zaś w drugim przypadku pozostaje PSPACE zupełny. Połączenie obu ograniczeń pozwala na rozwiązanie problemu spełnialności w czasie liniowym. W tym przypadku jednakże stały współczynnik zależy wykładniczo od liczby zmiennych zdaniowych.

Ograniczanie głębokości modalnej pełnego formalizmu TeamLog okazuje się nie być obiecującym kierunkiem, ponieważ problem spełnialności jest EXPTIME zupełny nawet gdy głębokość modalna formuł jest ograniczona przez 2. W związku z tym proponujemy nowy rodzaj ograniczeñ języka, nazwany ograniczaniem kontekstu modalnego. Jest to uogólnienie ograniczenia głębokości modalnej. Wprowadzamy dwa takie ograniczenia, oba prowadzące do PSPACE zupełności problemu spełnialności. W przypadku słabszego z nich, nazwanego $\mathbf{R}_1$, problem spełnialności pozostaje PSPACE trudny nawet wtedy, gdy głębokość modalna formuł jest ograniczona przez 2. W przypadku silniejszego ograniczenia, $\mathbf{R}_2$, połączenie go z ograniczeniem głębokości modalnej przez stałą prowadzi do NPTIME zupełności problemu spełnialności.

W pewnych sytuacjach ograniczenie $\mathbf{R}_2$ okazuje się zbyt silne, nie pozwalając na wyrażenie ważnych własności systemów wieloagentowych, proponujemy wzmocnienie ograniczenia $\mathbf{R}_1$. Prowadzi ono do NPTIME zupełności problemu spełnialności, gdy połączone zostanie z ograniczaniem głębokości modalnej formuł.

Poza ograniczeniami kontekstu modalnego badamy również ograniczenie liczby zmiennych zdaniowych przez stałą. Pokazujemy, że w tym przypadku problem spełnialności dla TeamLog pozostaje EXPTIME zupełny. Połączenie tego ograniczenia z ograniczaniem głębokości modalnej pozwala na rozwiązanie problemu spełnialności w czasie liniowym. Podobnie jak w przypadku TeamLog$^{\mathrm{ind}}$ również tutaj stały współczynnik zależy wykładniczo od liczby zmiennych zdaniowych.

Metody, których używamy oraz wyniki, które otrzymaliśmy mogą zostać zastosowane do innych wielomodalnych formalizmów dla systemów wieloagentowych.

Contents

Chapter 1

Introduction

The research problem addressed in this thesis is the complexity of the satisfiability problem for TeamLog, a representative and widely recognized multimodal formalism for multiagent systems. As we find out in the first part of the thesis, the formalism has EXTPTIME complete satisfiability problem. This means, essentially, that from practical point of view tasks involving reasoning in TeamLog are not efficient and certainly not scalable. For this reason it is important to study restrictions of the language that would lead to reduction of the complexity of the problem, or at least would lead to classes of complexity for which effective heuristics based methods could be applied. We study such restrictions in the second part of the thesis.

1.1 What are multiagent systems about?

Multiagent systems constitute a new approach to analysing, designing and implementing distributed computational applications destined to work in dynamic, unpredictable and open environments. As an area of research, multiagent systems emerged in the early nineties of the last century from distributed artificial intelligence (DAI). Since then it evolved into an independent and interdisciplinary field that incorporates research from various areas such as artificial intelligence, concurrent systems, economics, game theory, philosophy and formal logic, to name a few. This evolution, associated research problems and developments as well as future directions are documented in several 'roadmap' surveys [63, 75, 76].

Application areas most suitable for multiagent approach are those exhibiting dynamism, unpredictability and/or openness. Early applications included manufacturing process management [120, 87, 105, 62], telecommunications [117] and air-traffic control [77]. All these environments are dynamic and unpredictable, however they are usually not open, meaning the number or at least the types of components (including other agents) that may need to interact are not known at the design time. Recent developments of technologies such as service oriented computing and web services, on-line ontologies and universal plug and play provided a base layer infrastructure for open multiagent environments. Therefore increasing use of this approach can be seen in the areas such as e-Commerce and e-Business [82, 118], Grid computing [49, 48] and ambient intelligence [109].

The key abstraction of a multiagent system is an *agent*. According to a classical definition, this notion is intended to capture an entity that is *situated* in some environment and is capable of *flexible* and *autonomous* action involving *social* behaviour. Situatedness means that an agent observes its *environment*, possessing some information about it and can affect it by performing *actions*. Examples of environments could be real world (in the case of robots or agents managing production processes), virtual environments (e.g. Internet or Grid in the case of software agents) or mixed environments (in the case of ambient intelligence). Autonomy means that an agent is capable of working with only partial or lack of any control from humans or other agents. Flexibility means that an agent can adapt to the changing environment, being able to update information about it, selecting tasks and dynamically committing resources to realize these tasks. Social behaviour stands for higher level interactions such as cooperation or competition. Ability to take part in such interactions requires keeping information about other agents, usage of common language and common patterns of low level-interactions (such as negotiations or auctions). It requires also ability to automatically form and sustain bilateral or multilateral dependencies underlying cooperative behaviour.

Main challenges faced by multiagent systems and associated research problems include (but are not restricted to) the following themes:

1. agent architectures,
2. methodologies of agent oriented analysis and design,
3. agent programming languages,
4. knowledge sharing,
5. complex interaction schemes: cooperation, coordination and negotiations,
6. formalization of agents and multiagent systems.

We will briefly introduce these essential subjects below.

1.2 Agent architectures

The first problem tries to answer the question about how an agent capable of flexible, autonomous and social behaviour should be constructed. Since purely deliberative architectures based on symbolic representation of the environment and first principles planning [99, 5, 46] turned out to be too complex and therefore hardly usable in most practical applications [24, 100], and subsequent reactive architectures [19] turned out to be insufficient, hybrid agent architectures were proposed. These architectures are usually based on the layered approach, where lower layers are responsible for reactive behaviour while higher perform more deliberative decision making based on knowledge representation and take into account social aspects. Examples of such architectures are Touring Machines [45] and InteRRaP [81]. A recent methodology and architecture for fusing knowledge represented on different symbolic and non-symbolic levels is CAKE [30].

A family of architectures of agents that gained large popularity in multiagent community are those based on the idea of *practical reasoning*, particularly the *beliefs-desires-intentions (BDI) architecture*. BDI model of agency originates from philosophical study of intentions and their role in practical reasoning and was coined by Bratman [16]. It assumes description of agents in terms of common sense notions referring to their *informational and motivational attitudes*, such as *beliefs*, *desires* or *goals* and *intentions*. While beliefs represent information about the environment possessed by an agent and goals represent the desired states of environment or the tasks to accomplish, intentions stand for those choices or tasks which the agent is currently realizing and to which it commits its resources.

The first high level description of BDI architecture called IRMA (*Intelligent Resource-bounded Machine Architecture*) was given in [17]. Probably the best known of other architectures based on BDI model is PRS (*Procedural Reasoning System*) [51] which was further implemented as dMARS (*distributed Multi-Agent Reasoning System*) [29]. Another implementation of PRS is Dorcas created by Brzeziński and the author of this thesis [21]. Among notable BDI architectures are also COSY [22, 53] and GRATE* [61] (a layered architecture that extends the BDI model by additional notion of *joint-intentions* that are related to cooperation with other agents).

1.3 Agent oriented analysis and design

The research of an agent oriented analysis and design methodologies is strictly connected with research of agent architectures. These methodologies are usually based on existing approaches either from the area of knowledge engineering [60, 18] or object-oriented analysis and design [68, 67, 86].

One of the most influential models, proposed by Kinny, Georgeff and Rao (hence KGR approach) [68, 67, 66], is created for BDI agents. In this approach a multiagent system is modelled from both global and individual perspective. The global perspective identifies types (or roles) of agents in the system, specifies types of interaction they will be involved in and describes protocols of interaction. The individual perspective describes each agent in terms of three models: belief , goal and plan model. The methodology is based on object oriented approach which is extended to allow for expressing agent specific elements, such as goals and actions. KGR approach evolved further into a fully agent oriented and more general methodology called Gaia [123] which does not assume any specific agent model.

1.4 Agent oriented programming

The idea of a new programming paradigm called the agent-oriented programming was first proposed by Shoham [104]. He introduced a prototypical programming language called Agent0. Based on the BDI model, it uses a system of formal logic for defining the mental state of agents. Their behaviour is described using interpreted programming language. Agent0 was further developed in [110], where a programming language called PLACA was presented. PLACA extends Agent0 to allow agents planning and communicating requests for actions.

Probably the most developed agent oriented programming language to date is AgentSpeak(L), proposed in [94] to bridge the gap between theoretical BDI architectures, like IRMA and PRS, and their practical implementations, like dMARS. AgentSpeak(L) was further extended to allow for interaction using speech-act based communication languages [116]. Formal specification of AgentSpeak(L) was given in [28, 15] while formal verification of programs written in it, based on model checking, was studied in [12, 13]. AgentSpeak(L) is also implemented in Jason platform using Java programming language [14].

1.5 Knowledge sharing

The problem of knowledge sharing is fundamental for open multiagent systems, as agents are expected to communicate and 'understand' one another. The first attempt to address this issue is the work by ARPA Knowledge Sharing Effort (KSE) consortium. Three components were identified as necessary for knowledge sharing: a common communication language, knowledge representation language and shared ontologies.

As a communication language KQML (Knowledge Query and Manipulation Language) was suggested. Its idea is based on philosophical speech act theory, where communicating something is viewed as an act resulting in some action being performed by the receiver [7, 102].

Hence such communicates are called performatives. In KQML they range from affecting the knowledge base of the receiver, through requests for performing high level goals, to low level requests related to managing the communication among agents [6, 71]. As a language to specify the content of messages KIF (Knowledge Interchange Format) has been introduced [50]. To ensure proper understanding, agents need to refer to common concepts. Their specification as well as properties and relations between them create an ontology. Ontolingua is a generally agreed mechanism to create and share ontologies used by agents [52].

The approach to knowledge sharing initiated by ARPA KSE was further used by a IEEE Computer Society standards organization called FIPA (Foundation for Physical Intelligent Agents), created for promoting agent-based technology and interoperability of its standards with other technologies. Apart from proposing the whole abstract architecture designed to enable open multi-agent systems, FIPA provided a speech act based FIPA ACL language [1] and several languages for knowledge representation (e.g. FIPA SL [2] and FIPA KIF [3]).

1.6 Cooperation

Possibility to achieve goals that are beyond individual capabilities of agents is arguably one of the most attractive features that multiagent approach has to offer. To ensure cooperation agents must not only be able to communicate but also to act in cooperative and coordinated manner. Two challenges arise in this context. Firstly, understanding what cooperative behaviour actually means and secondly, how to realize it in practice. The first issue is addressed by different formalizations of agents and multiagent systems that are discussed in the next chapter. For now we will focus on the second issue.

Two ways of addressing cooperation can be distinguished in the MAS literature. The first one, rooted in DAI, is based on cooperative multiagent planning. The early techniques, that exploited the idea of complete planning before action, were not suitable for inherently dynamic multiagent environments. Therefore the idea of partial planning was adopted allowing agents to communicate and adapt their local plans to dynamically changing situation. This is called Partial Global Planning (PGP) [37]. The PGP approach was implemented in a generic environment called TÆMS [27].

The second way is based on practical reasoning, rooted in philosophy. It extends the BDI model of agency by introducing the notions of collective (or joint) intentions and collective (or joint) commitments [73, 122, 33, 35, 4]. Two approaches can be distinguished here. In one of them, initiated by Cohen, Levesque and Nunes [73], joint commitments are defined in terms of bilateral commitments between agents in a group and then joint intentions are defined on the basis of individual intentions and joint commitments. In this approach joint intention entails not only a common motivational stand towards some goal in the group, but also it entails existence of a network of bilateral commitments that agents have towards other agents in the group. Since, on the lowest level, these bilateral commitments can be with regard to actions that are to be executed by individual agents, so joint intention determines the individual actions that have to be executed to achieve the common goal. Thus primary notions here are bilateral and then joint commitments, while joint intention is the strongest notion, fully characterizing a cooperating group of agents. This approach was further developed by Jennings and Wooldridge in [122] and was adopted by Wooldridge in a multiagent formalism called $\mathcal{LORA}$ [119]. Another important multiagent formalism that follows this approach is KARO [4], proposed by van der Hoek, van Linder and Meyer [114, 115, 80, 113] and extended to cover group aspects of multiagent systems by Aldewereld [4]. Notable implementations are systems called GRATE* [61] (applied to electric transport management) and STEAM [108] (applied to military training support and RoboCup synthetic soccer).

The second approach, initiated and developed by Dunin-Kęplicz and Verbrugge, is a logical framework called TeamLog [31, 33, 35, 36]. In contrast to the first approach, it follows the view of Castelfranchi [23] and considers collective commitments as the strongest motivational attitude in teamwork. In this context collective intentions are viewed as a component consolidating a cooperating group of agents. On this basis, together with social plan, collective commitments are formed, leading to a certain realization of the collective intention. Thus collective commitments underlie a plan-based team activity. The change of view on the role of motivational attitudes allowed for clarifying the cooperative problem solving process.

Core TeamLog is a static theory concerning collective motivational attitudes. Its pragmatic value has to be verified during teamwork in dynamic and unpredictable environment. The central point here is always the team action. However before the team reaches that point, other forms of teamwork have to be realized: potential recognition, team formation and plan generation, which directly precedes the team action. This model follows from the work of Wooldridge and Jennings [122]. Its realization, called a *reconfiguration algorithm*, was proposed and formally analysed using TeamLog by Dunin-Kęplicz and Verbrugge in [32, 36]. The algorithm addresses, in particular, the problem of possible dynamic changes of the environment that may, on one hand, lead to failure of some of the actions while, on the other hand, pose new opportunities for achieving the common goal.

1.7 Negotiations

Cooperation is one of the fundamental examples of complex interactions between agents. Another important scheme of interaction studied in multiagent systems are *negotiations*. Since agents are assumed to be autonomous, they should be able to negotiate and to decide dynamically on how to share scarce resources. To address these issues, approaches based on game theory and argumentation are used. In classical paper [124] Zlotkin and Rosenschein present and analyse a protocol which allows agents to negotiate a redistribution of tasks that leads to a distribution preferred by agents to the initial one. The authors analyse also possible behaviour of agents following the protocol, showing existence of equilibrium strategies. These ideas were further developed in [98]. An excellent review of automated negotiations based on game theoretic approach is given in [69].

Another approach to reaching agreement is based on argumentation. In this approach agents exchange messages trying to affect beliefs and goals of others until agreement is reached. This technique is used in a system called persuader [106, 107] that was applied to labour negotiations. In [70] a logical model of negotiations through argumentation was given. A recent book [93] gives an overview of argumentation and it applications in artificial intelligence and multiagent systems.

The focus of this thesis is on the last of the research problems listed above: formalization of agents and multiagent systems. We address it in the next chapter, where we concentrate on the TeamLog formalism.

Chapter 2

Formalization of multiagent systems

Formalization of the notions that appear naturally in multiagent systems, such as knowledge, beliefs, goals, intentions, abilities or commitments, is an important and non-trivial task. It allows for creating formal specifications of such systems and for providing reasoning frameworks to formally implement individual agents. Since most of the relevant notions are of intensional character, the formalisms proposed are based on modal logic. The most influential model for individual agents is the BDI model [26, 96, 97]. Formalisms designed to capture group aspects of agency, particularly cooperation, extend this model by specifying collective notions such as common beliefs, mutual intentions or collective commitments [73, 119, 36, 4].

This thesis focuses on the TeamLog logical framework which is a representative and widely recognized formalism for multiagent systems. Below we give a high level introduction to this framework, presenting rationale behind it. Formal presentation of the framework is given in Chapter 3. Starting from that chapter onwards we will focus on TeamLog as a multimodal formalism studying the complexity of the satisfiability problem. In particular, we will be interested in the modalities that are used to characterize group aspects of multiagent systems. Usage of such modalities is a common practice in formalization of MAS (c.f. an overview of three other important formalism given in Appendix B).

2.1 TeamLog: formalizing teamwork in multiagent systems

TeamLog, developed by Dunin-Kęplicz and Verbrugge at the University of Warsaw and the University of Groningen and presented in a series of papers [31, 34, 33, 35] and in a recent book [36], is a logical framework proposed to formalize individual and group aspects of BDI systems. The focus on cooperative aspects and a team of agents inevitably leads to teamwork. The framework was created along with the reconfiguration algorithm [32, 36] – a practical reasoning based approach to team action, starting from potential recognition and group formation, through collective planning up to team action execution.

The full TeamLog is a very reach formalism allowing for expressing and reasoning about various aspects of individual agents and multiagent systems relevant to cooperative problem solving. Moreover, it is intended to be suitable for possible enrichments that could be designed for chosen classes of multiagent systems. The whole framework is organized into four layers, each introducing new elements related to different aspects of multiagent systems [36]:

Individual layer: this layer, called TeamLog$^{\text{ind}}$, forms the basis for the whole TeamLog framework and introduces notions related to individual agents. TeamLog$^{\text{ind}}$ is a BDI model of agency, where each agent is characterised by three components: *beliefs*, representing informational aspects of an agent, *goals*, representing the states of the world

$\mathrm{BEL}(j, \varphi)$	agent j believes that φ
$\mathrm{GOAL}(j, \varphi)$	agent j has the goal to achieve φ
$\mathrm{INT}(j, \varphi)$	agent j has the intention to achieve φ

Table 2.1: Modal operators of TeamLog$^{\mathrm{ind}}$ and their intended meaning

that the agent wants to achieve and *intentions*, representing the subset of goals which the agent is currently trying to attain. Specification of beliefs, goals and intentions of agents describe the multiagent system on the *individual level.*

Group layer: this layer extends the individual layer with notions characterising groups of agents. It introduces *common beliefs*, representing informational aspect of a group of agents and *mutual intentions*, representing motivational and cooperative aspect of a group of agents working together. The notions of team of agents and teamwork are characterised by *collective intentions* adding group awareness to mutual intentions [33]. Mutual intentions and common beliefs describe the multiagent system on the *group level.*

Social layer: this layer introduces actions together with constructs that allow for expressing social plans. On this basis *social commitments* between pairs of agents can be defined. Given a social plan, social commitments describe the distribution of bilateral responsibilities towards the actions from the plan. Collective intentions, together with social commitments and underlying social plan form a basis for *collective commitments*, that reflect how the goal is achieved by the team.

Individual, group and social layers together allow for expressing static aspects of multiagent system and they constitute the core of TeamLog framework. For this reason they are referred to as TeamLog.

Dynamic layer: this layer, called TeamLog$^{\mathrm{dyn}}$, extends TeamLog to allow for expressing and reasoning about actions and plans of agents, along with evolution of static aspects of individual agents and teams, when the actions are executed [35]. The extension adds propositional dynamic logic enriched with parallelism and additional operator $\mathtt{stit}\,\varphi$, representing any course of actions resulting in φ being true [103]. The dynamic layer includes also primitives to express such additional aspects related to teamwork like e.g. *abilities* and *opportunities* of agents [35, 36].

In this thesis we focus on the core TeamLog consisting of the first three layers, presented in detail below.

2.1.1 Individual layer of TeamLog

TeamLog$^{\mathrm{ind}}$, the individual fragment of TeamLog, is a propositional multimodal logic with three groups of modalities used for expressing beliefs, goals and intentions of individual agents. All the modal operators introduced at this layer are defined on the basis of a finite and non-empty set of agents $\mathcal{A}$ and a countable set of propositional symbols $\mathcal{P}$. A summary of modal operators introduced in TeamLog$^{\mathrm{ind}}$ is presented in Table 2.1.

For beliefs, axioms of the standard doxastic modal logic (c.f. [79, 44]) are adopted, forming the $\mathrm{KD45}_n$ system.

Belief distribution

$$\mathrm{BEL}(j, \varphi) \wedge \mathrm{BEL}(j, \varphi \to \psi) \to \mathrm{BEL}(j, \psi).$$

Consistency Beliefs of an agent are consistent:

$$\neg \mathrm{BEL}\,(j, \bot)\,.$$

Positive introspection an agent is aware of what he believes in:

$$\mathrm{BEL}\,(j, \varphi) \to \mathrm{BEL}\,(j, \mathrm{BEL}\,(j, \varphi))\,.$$

Negative introspection an agent is aware of what he does not believe in:

$$\neg \mathrm{BEL}\,(j, \varphi) \to \mathrm{BEL}\,(j, \neg \mathrm{BEL}\,(j, \varphi))\,.$$

For goals the system K_n is used. Hence only goals distribution is assumed,

Goal distribution

$$\mathrm{GOAL}\,(j, \varphi) \wedge \mathrm{GOAL}\,(j, \varphi \to \psi) \to \mathrm{GOAL}\,(j, \psi)\,.$$

Intentions form a consistent subset of goals selected by an agent for realization. Hence the system KD_n is adopted for them.

Intention distribution

$$\mathrm{INT}\,(j, \varphi) \wedge \mathrm{INT}\,(j, \varphi \to \psi) \to \mathrm{INT}\,(j, \psi)\,.$$

Consistency Intentions of an agent are consistent:

$$\neg \mathrm{INT}\,(j, \bot)\,.$$

Different attitudes are interrelated in a particular way. These relations are reflected in additional axioms. The fact that for each agent j intentions are a subset of goals is reflected in the axiom

$$\mathrm{INT}\,(j, \varphi) \to \mathrm{GOAL}\,(j, \varphi)\,.$$

Moreover, each agent j is fully aware about his goals and intentions:

Positive introspection of goals an agent is aware of the goals it has:

$$\mathrm{GOAL}\,(j, \varphi) \to \mathrm{BEL}\,(j, \mathrm{GOAL}\,(j, \varphi))\,,$$

Positive introspection of intentions an agent is aware of the intentions it has:

$$\mathrm{INT}\,(j, \varphi) \to \mathrm{BEL}\,(j, \mathrm{INT}\,(j, \varphi))\,.$$

Negative introspection of goals an agent is aware of what goals it does not have:

$$\neg \mathrm{GOAL}\,(j, \varphi) \to \mathrm{BEL}\,(j, \neg \mathrm{GOAL}\,(j, \varphi))\,,$$

Negative introspection of intentions an agent is aware of what intentions it does not have:

$$\neg \mathrm{INT}\,(j, \varphi) \to \mathrm{BEL}\,(j, \neg \mathrm{INT}\,(j, \varphi))\,.$$

$\text{E-BEL}_G(\varphi)$	every agent in group G believes that φ
$\text{C-BEL}_G(\varphi)$	group G has the common belief that φ
$\text{E-INT}_G(\varphi)$	every agent in group G has the individual intention to achieve φ
$\text{M-INT}_G(\varphi)$	group G has the mutual intention to achieve φ
$\text{C-INT}_G(\varphi)$	group G has the collective intention to achieve φ

Table 2.2: Modal operators of group layer of TeamLog and their intended meaning

2.1.2 Group layer of TeamLog

TeamLog is a propositional multimodal logic extending $\text{TeamLog}^{\text{ind}}$ with modalities used to express properties of groups of agents. The most important notions introduced here are *collective intentions*, characterising the situations when a group of agents achieves something in a fully cooperative and coordinated manner, and *collective commitments* describing a concrete manner of how the goal is achieved. A summary of modal operators introduced at the group layer of TeamLog is presented in Table 2.2.

The introduced operators express the informational and motivational aspects of groups of agents. On their basis social and collective commitments are defined.

2.1.2.1 Common informational attitudes

When logical model of teamwork is considered, agents' *awareness* about the situation is essential. In the approach undertaken in TeamLog, this notion refers to specific informational stance of an agent towards a proposition. It is understood as the state of an agent's beliefs about itself, about other agents and about the environment. Various versions of group notions, based on different levels of awareness, fit different situations, depending on organizational structure, communicative and observational abilities, and so on.

Informational aspects of a group G of agents are expressed by two operators: $\text{E-BEL}_G(\cdot)$ and $\text{C-BEL}_G(\cdot)$. The first one stands for a *general belief* in a group of agents, that is a situation when each agent in the group believes that some state of the world holds. This corresponds to the following axiom:

$$\text{E-BEL}_G(\varphi) \leftrightarrow \bigwedge_{i \in G} \text{BEL}(j, \varphi).$$

The second one, *common belief*, analogous to the operator of common knowledge (c.f. [44, 79]), expresses the situation where there is full awareness in the group about some state of the world. Thus everyone in the group believes that some state of the world holds, everyone in the group believes that there is such a general belief in the group, etc. ad infinitum. With common beliefs one axiom and one rule are included:

$$\text{C-BEL}_G(\varphi) \leftrightarrow \text{E-BEL}_G(\varphi \wedge \text{C-BEL}_G(\varphi)),$$
$$\text{From } \varphi \to \text{E-BEL}_G(\psi \wedge \varphi) \text{ infer } \varphi \to \text{C-BEL}_G(\psi).$$

Common belief of certain facts have been shown to be necessary for coordination in standard examples [44]. Although it is possible to give a very specific procedure that can establish common beliefs, the assumptions needed are very strong [36]. In practical applications the highest level of awareness, represented by common beliefs, is not always necessary. It is possible that beliefs of individual agents within a group, or general beliefs of a subgroup of a group or different degrees of general beliefs of a group about important facts are sufficient (see [36, Chapter 7] for a case study and a discussion). For this reason TeamLog does not assume one fixed representation of awareness, allowing system developer to 'tune' it to the desired level, as required by the application. As discussed in [20, 111], the communication cost of establishing a particular level of awareness is linear with its degree. Hence selecting the optimal level of awareness is vital.

2.1.2.2 Collective motivational attitudes

For motivational aspects of a group of agents two operators: $\text{E-INT}_G(\cdot)$ and $\text{M-INT}_G(\cdot)$, of general intention and mutual intention, are introduced. The notion of *general intention* is analogous to the one of general belief. General intention to achieve φ exists in group G if every agent in the group has intention to achieve φ, which corresponds to the following axiom:

$$\text{E-INT}_G(\varphi) \stackrel{\text{def}}{=} \bigwedge_{i \in G} \text{INT}(j, \varphi)$$

The notion of general intention is insufficient to fully specify the situation when a group of agents intends to achieve something in a fully cooperative fashion. One can imagine that two agents want to reach the same goal but are in a competition, willing to achieve it exclusively. To exclude cases of competition, all agents should intend all members to have the associated individual intention, as well as the intention that all members have the individual intention, and so on (see [36, Chapter 3] for further discussion). The notion that captures that is *mutual intention.* It is defined formally on the basis of general intention, analogously to how common belief is defined on the basis of general belief. This is captured by the following axiom and induction rule:

$$\begin{gathered} \text{M-INT}_G(\varphi) \leftrightarrow \text{E-INT}_G(\varphi \wedge \text{M-INT}_G(\varphi)), \\ \text{From } \varphi \to \text{E-INT}_G(\psi \wedge \varphi) \text{ infer } \varphi \to \text{M-INT}_G(\psi). \end{gathered}$$

In [33, 36] it is argued that for a group of agents to be called a *team*, i.e. a fully cooperating group striving to achieve a common goal in coordinated fashion, it is additionally necessary that there exists a proper awareness about this fact within a group. For this reason a notion of collective intention is introduced which is meant to fully characterise teams of agents. Associated operator is defined as follows, for a given group of agents G:

$$\text{C-INT}_G(\varphi) \stackrel{\text{def}}{=} \text{M-INT}_G(\varphi) \wedge \text{C-BEL}_G(\text{M-INT}_G(\varphi)).$$

2.1.2.3 Collective commitments

While collective intention constitutes a team trying to achieve some goal, collective commitment describes a concrete manner of how the goal is achieved. Having a team of agents established on the basis of collective intention and a plan describing courses of actions the team is to follow, collective commitment specifies how pairwise responsibilities are distributed within a team towards the actions included in the plan. Apart from that, collective commitment specifies information that the agents in the team have about particular building blocks of commitments [31, 35, 36].

To allow for expressing group and bilateral commitments, constructions allowing for expressing actions and social plans are introduced. Individual actions are defined on the basis of a finite set of atomic actions using constructs of regular programs, like in propositional dynamic logic (PDL). Additionally individual actions include a generic 'sees to it that' (or 'brings it about that') action $\texttt{stit}\,\varphi$, that stands for any activity that leads to achieving a state of the world where φ holds (see [103] for extensive discussion of this notion). Using individual actions, social plan expressions that describe activity of a given group of agents are built. Social plans may include sequential or parallel executions of individual actions.

Social and collective commitments (see [36, Chaper 4] for details) are defined using the modalities introduced so far and additional formulas $done(j,\alpha)$ and $constitutes(P,\varphi)$. Formula $done(j,\alpha)$, where j is an agent and α is an individual action, means that agent j has just performed an action α. Formula $constitutes(\varphi, P)$, where P is a social plan expression and φ is a formula, means that successful execution of plan P leads to achievement of the goal φ. Semantics of these formulas is given at the dynamic layer of TeamLog. At this layer each such formula is assumed to be a propositional variable given different values at different worlds of the model.

A social commitment of agent i towards agent j with respect to action α is defined by the following schema:

$$\begin{aligned}\mathrm{COMM}\,(i,j,\alpha) \leftrightarrow\ & \mathrm{INT}\,(j,\alpha) \wedge \mathrm{GOAL}\,(j, done(i,\alpha)) \wedge \\ & awareness_{\{i,j\}}(\mathrm{INT}\,(j,\alpha) \wedge \mathrm{GOAL}\,(j, done(i,\alpha))),\end{aligned}$$

where

$$\mathrm{INT}\,(j,\alpha) \leftrightarrow \mathrm{INT}\,(j, done(j,\alpha))$$

and $awareness_{\{i,j\}}$ can be instantiated by different degrees of general beliefs, up to common belief. Thus agent i is socially committed to agent j to perform action α if i intends to perform α while at the same time j has a goal that the action is performed by i. Additionally both agents have certain degree of information about this fact. This degree depends on the situation and communicational capabilities and in the ideal case the agents have common belief about the fact, in which case social commitment is defined as

$$\begin{aligned}\mathrm{COMM}\,(i,j,\alpha) \leftrightarrow\ & \mathrm{INT}\,(j,\alpha) \wedge \mathrm{GOAL}\,(j, done(i,\alpha)) \wedge \\ & \text{C-BEL}_{\{i,j\}}\left(\mathrm{INT}\,(j,\alpha) \wedge \mathrm{GOAL}\,(j, done(i,\alpha))\right),\end{aligned}$$

Collective commitments are defined for a given group of agents G on the basis of some social plan P by means of the following general schema:

$$\begin{aligned}\text{C-COMM}_{G,P}\,(\varphi) \leftrightarrow\ & \text{M-INT}_G\,(\varphi) \wedge constitutes(P,\varphi) \wedge \bigwedge_{\alpha\in P}\ \bigvee_{i,j\in G} \mathrm{COMM}\,(i,j,\alpha) \wedge \\ & awareness^1_G\,(\text{M-INT}_G\,(\varphi)) \wedge awareness^2_G\,(constitutes(P,\varphi)) \wedge \\ & awareness^{comm}_{G,P}.\end{aligned}$$

This generic description consists of the following ingredients, related to different aspects of teamwork:

1. Mutual intention $\text{M-INT}_G\,(\varphi)$ within a group of agents, allowing them to act as a team. The team exists as long as the mutual intention between team members exists and no teamwork can be considered without a mutual intention among team members.

2. Social plan P for a team on which a collective commitment is based. The social plan provides a concrete manner to achieve a common goal, the object of mutual intention. Furthermore, plan P should correctly lead to achievement of goal φ, as reflected in $constitutes(P,\varphi)$.

3. Pairwise social commitments $\text{COMM}\,(i, j, \alpha)$ for actions occurring in the social plan. Actions from the plan are distributed over team members who accept corresponding social commitments.

Again, different degrees of awareness about these ingredients are represented by different 'dials', $awareness^1_G$, $awareness^2_G$ and $awareness^{comm}_{G,P}$ that can be instantiated separately to reflect different degree of information about these ingredients. In particular, the information about the distribution of bilateral social commitments within the group towards the actions from the plan P, represented by $awareness^{comm}_{G,P}$, can be of two kinds: *global* or *detailed*. This difference is inspired by the *de re / de dicto* distinction stemming from the philosophy of language [92].

In the first case, there is some degree of information within the group that *some* distribution of responsibilities has been established, that is

$$awareness^{comm}_{G,P} \leftrightarrow awareness^3_G \left(\bigwedge_{\alpha \in P} \bigvee_{i,j \in G} \text{COMM}\,(i, j, \alpha) \right).$$

This corresponds to de dicto interpretation, as the information is about *a* distribution. In the second case, there is some degree of information within the group about some particular distribution of responsibilities, that is

$$awareness^{comm}_{G,P} \leftrightarrow \bigwedge_{\alpha \in P} \bigvee_{i,j \in G} awareness^3_G(\text{COMM}\,(i, j, \alpha)).$$

This corresponds to de re interpretation, as the information is about *the* distribution. In both cases the new 'dial' $awareness^3_G$ reflects the degree of information the group has.

The general schema of collective commitment can be instantiated by choosing different degrees of information about the different components as well as by choosing between detailed and general information about the distribution of bilateral social commitments.

Two strongest forms of collective commitment, *robust collective commitment* and *strong collective commitment*, assume the highest degree of awareness. The first one of them assumes detailed information about the distribution of responsibilities, that is:

$$\begin{aligned}\text{R-COMM}_{G,P}\,(\varphi) \leftrightarrow \text{C-INT}_G\,(\varphi) \wedge constitutes(P, \varphi) \wedge \text{C-BEL}_G\,(constitutes(P, \varphi)) \wedge \\ \bigwedge_{\alpha \in P} \bigvee_{i,j \in G} \text{C-BEL}_G\,(\text{COMM}\,(i, j, \alpha))\,.\end{aligned}$$

The second one of them assumes global information about the distribution of responsibilities, that is

$$\begin{aligned}\text{S-COMM}_{G,P}\,(\varphi) \leftrightarrow \text{C-INT}_G\,(\varphi) \wedge constitutes(P, \varphi) \wedge \text{C-BEL}_G\,(constitutes(P, \varphi)) \wedge \\ \text{C-BEL}_G \left(\bigwedge_{\alpha \in P} \bigvee_{i,j \in G} \text{COMM}\,(i, j, \alpha) \right).\end{aligned}$$

As explained in [36], robust collective commitment is most suited for self-leading teams, which are not directly led by a leader. Instead, the team is responsible for achieving some high-level goals, and is entirely free to divide roles, devise a plan, etc. A non-hierarchical team of researchers is a typical example of such a self-leading team establishing a robust collective commitment. Strong collective commitment, on the other hand, may be applicable to teams with one or more leaders and rather separate sub-teams. Even though planning may be done collectively, establishing bilateral commitments is not done publicly, but in subgroups. For example, members might commit to their sub-team leader to do their own part.

The detailed discussion of many other variants collective commitments that can be constructed and their importance in different applications is given in the book [36].

2.2 Complexity results

The formalisms this thesis focuses on are based on modal logics. There are three problems whose decidability and complexity is typically studied in context of such formalisms: satisfiability, validity and model checking.

The satisfiability problem is defined as follows: given a formula φ check whether there exists a model $\mathcal{M}$ (in a given class of models) and a world w in it such that $(\mathcal{M}, w) \models \varphi$.

The validity problem is defined as follows: given a formula φ check whether for every model $\mathcal{M}$ (from a given class of models) and every world w in $\mathcal{M}$, $(\mathcal{M}, w) \models \varphi$. The two problems are strictly related, as φ is valid if and only if $\neg\varphi$ is not satisfiable.

The problem of model checking is defined as follows: given a formula φ and an interpretation $(\mathcal{M}, w)$ check if $(\mathcal{M}, w) \models \varphi$.

The satisfiability problem is important in two tasks that could appear when a multiagent system is developed. The first task is the one faced by the system developer who creates a specification of a multiagent system. It is important to develop automated tools to support the developer with some reasoning tasks that allow for verification of such specifications. Checking whether there exists a system that satisfies the given specification is essentially the satisfiability problem. The second task is related to implementation of individual agents. Often implementations of agents are based on logical formalisms and execution of programs of agents involves reasoning tasks related to the underlying formalism.

The main results on the complexity of the satisfiability problem of basic propositional modal logics with single modality generated by axioms from **K**, **T**, **D**, **4** and **5** have been obtained by Ladner [72]. Using a tableau method, an approach that tries to construct a structure which forms a basis for a model satisfying the investigated formula, he showed that any such system that does not include both axioms **4** and **5** has PSPACE complete satisfiability problem, while the systems including both these axioms (KD45 and S5 in particular) have NPTIME complete satisfiability problems. Halpern and Rêgo [57] completed these results, by showing that it is axiom **5** which is crucial here. Its presence makes the problem NPTIME complete, while in its absence the problem is PSPACE complete. The multimodal versions of these logics were studied by Halpern and Moses in [54], where it was shown that the satisfiability problem for all these systems is PSPACE complete as soon as the number of modalities is at least two. Authors study also the complexity of the satisfiability problem of these logics with group modal operators defined using fixpoint definitions, analogously to common belief or mutual intentions. Relying on the results of Fisher and Ladner [47] and Pratt [91] regarding the complexity of the satisfiability problem of PDL, they showed that the satisfiability problem for these logics is EXTPTIME complete even if the modal depth of formulas is bounded by 3 and $n \geq 2$, in the case of axiom systems including both axioms **4** and **5**. In the case of axiom systems that do not include either axiom **4** or axioms **5** the result holds even if modal depth of formulas is bounded by 2 and $n \geq 1$.

There are not many results on the complexity of the satisfiability problems for formalisms for multiagent systems. As was shown in the previous sections, such formalisms typically combine several basic modal systems with additional axioms interrelating the modalities of different systems. Additionally dynamic or temporal logic may be involved. Although in some cases, where interdependency axioms are simple enough, such a combination does not result in the satisfiability problem being harder than for each of the combined logics, the general results are scarce and involve very simple combinations [11]. In fact there are some very negative results about the transfer of complexity to combined systems. As Blackburn and Spaan show in [11] there are two "very decidable" logics whose combination, even without any interrelation axioms, is undecidable. For B, take a variant of dynamic logic with two atomic programs, both deterministic. Take ; and $\|$ as only operators. Satisfiability of formulas with respect to B, like that for PDL, is in EXPTIME. For C, take the logic of the global operator A (Always), defined as follows: $(\mathcal{M}, w) \models \mathrm{A}\varphi$ iff for all $v \in W$, $(\mathcal{M}, v) \models \varphi$. Satisfiability for C is in NPTIME. The minimal combination of B and C turns out to be undecidable (see also [10][Theorem 6.31]).

In [121] Wooldridge and Fisher proposed a decision procedure based on tableau method for a combination of belief logic and linear time temporal logic. Following this, Rao proposed in [95] a tableau method based decision procedure for the BDI formalism of Rao and Georgeff [96] with the computational tree logic (CTL*) temporal component replaced by linear time logic (LTL). In particular this work involves analysis of different types of axioms interrelating modalities of different types, called realism axioms and their effect on the decision procedure. In [97] Rao and Georgeff proposed a decision procedure for their formalism with temporal component being CTL. Using a translation of this formalism to μ-calculus, Schild showed that the satisfiability problem for it (with CTL temporal component) is EXPTIME complete [101]. Another modal multiagent formalism called KARO (logic for Knowledge, Ability, Results, and Opportunities), developed by van der Hoek, van Linder and Meyer [114, 115, 80, 113] was studied by Hustad et al. in [59]. The proposed decision procedures for a restricted KARO framework. Authors considered a fragment of KARO, called core KARO framework, which is essentially a combination of a multimodal system $\mathrm{S5}_n$ and deterministic propositional dynamic logic without iteration. They showed that the satisfiability problem for the core KARO framework is PSPACE complete.

Due to importance of automated reasoning using modal formalisms, different fragments of their languages were studied to find out how much they have to be restricted so that the complexity of the satisfiability problem is 'lowered',[1] preferably tractable (i.e. solvable in deterministic polynomial time). In [56] Halpern studied how the complexity of the satisfiability problem for multimodal logics generated by axioms from **K**, **T**, **D**, **4** and **5** is affected by bounding the modal depth of formulas and/or the number of propositional symbols. It is shown that for all the systems which either do not include axiom **4** or include both axiom **4** and **5**, the complexity of the satisfiability problem when modal depth of formulas is bounded by a constant is NPTIME complete. Bounding the number of propositional symbols does not affect the completeness results for any of the logics considered, however combining it with bounding the modal depth of formulas results in the satisfiability problem being solvable in linear time (although with a constant that depends exponentially on the number of propositional symbols).

Another restriction that was studied is considering a Horn fragment of modal formulas. The most notable results obtained along this path are by Nguyen [85]. He showed that if the axiom system contains axiom **D** and either contains both axioms **4** and **5** or does not contain axiom **4**, then the satisfiability problem of the Horn fragment of multimodal logics generated by this axiom system is PTIME complete, if the modal depth of formulas is bounded by a

[1] The complexity classes under consideration are PTIME, NPTIME, PSPACE and EXPTIME, and the only facts known so far about relations between them are that PTIME $\neq$ EXPTIME and PTIME $\subseteq$ NPTIME $\subseteq$ PSPACE $\subseteq$ EXPTIME. Hence, formally speaking, the only situation where the complexity is lowered for sure is when it goes from EXPTIME to PTIME. In all other cases the term 'lowered' is used under the assumption that all the subset relations are strict.

constant. In the case of multimodal logics generated by axiom systems not including axiom **D** or including axiom **4** without axiom **5**, the satisfiability problem of the Horn fragment of multimodal logics generated by them is NPTIME complete, if the modal depth of formulas is bounded by a constant. If the modal depth is not bounded, then the satisfiability problem of these logics remains PSPACE complete.

Yet another restriction considered in the literature is limiting the set of propositional operators used in formulas. This approach was taken by Bauland et al. in [8]. It was shown that in the case of basic normal modal system K there is a trichotomy: depending on the boolean operators used, the satisfiability problems are either PSPACE complete, coNPTIME complete or PTIME solvable. In the case of modal system KD a dichotomy was found: the problems are either PSPACE complete or PTIME solvable. Almost complete characterization was also obtained for modal systems T, S4 and S5.

We do not investigate the complexity of model checking in this thesis. For analysis of this problem for multiagent formalisms see [64, 65, 89].

2.3 Contribution of the thesis

In this thesis we study the complexity of the satisfiability problem for TeamLog, aa it is a representative formalism for multiagent systems. We focus, in particular, on the modalities that are used to characterize group aspects in the course of teamwork. Usage of such modalities is a common practice in formalization of MAS, and the methods we applied as well as the results we obtained could be used to study the complexity of the satisfiability problem for similar multiagent formalisms, like KARO or $\mathcal{LORA}$. An overview of multi agent formalisms is given in Appendix B.

In the first part of the thesis we analyse the complexity of the satisfiability problem for unrestricted core TeamLog framework. We show that the problem is PSPACE complete for the individual fragment of TeamLog, TeamLog$^{\mathrm{ind}}$. In the case of the full TeamLog the problem is EXPTIME complete, even if modal depth of formulas is bounded by 2. This result holds also for the complexity of the satisfiability problem of $\mathrm{KD45}_n^+$, and it refines the result obtained by Halpern and Moses in [54]. The algorithm used to show that the satisfiability problem for TeamLog is in EXPTIME is based on the algorithm proposed by Pratt [91] for PDL. It can be naturally extended to the TeamLog framework enriched with dynamic component, TeamLog$^{\mathrm{dyn}}$. These results were partially presented at the Intelligent Agent Technology conference (IAT'05) [40] and discussed in depth in [41].

In the second part of the thesis we investigate how restricting the language of TeamLog affects the complexity of the satisfiability problem. We study two cases separately: the individual fragment of TeamLog, TeamLog$^{\mathrm{ind}}$, and the full TeamLog.

In the case of TeamLog$^{\mathrm{ind}}$ we follow the footsteps of Halpern [56] and check two kinds of restrictions:

- restricting the modal depth of formulas,
- restricting the number of propositional symbols.

The first restriction results in NPTIME completeness of the satisfiability problem, while the second one leaves the problem PSPACE complete. Combining both restrictions makes the problem solvable in linear time, however the constant multiplier depends exponentially on the number of propositional symbols. These results were presented at the Autonomous Agents and Multiagent Systems (AAMAS'07) conference [42] and published in [41].

In the case of the full TeamLog, restricting modal depth of formula turned out to be a non-promising direction due to the fact that the satisfiability problem is EXPTIME complete even if modal depth of formulas is bounded by 2. For this reason we propose an original generalisation of this restriction that restricts modal context of formulas. We study two modal context restrictions and show that both of them result in PSPACE completeness of the satisfiability problem. The first one of them, when combined with bounding modal depth of formulas, results in NPTIME completeness of the satisfiability problem. The second one leaves the problem PSPACE complete even if the modal depth of formulas is bounded by 2. In this case we apply an additional restriction that leads to NPTIME completeness of the satisfiability problem.

Finally, we discuss the restrictions proposed in the context of specifying multiagent systems, focusing on the modal context restrictions. We argue that the first of these restrictions could be applied to the purely informational or purely motivational parts of specifications of agents or groups of agents. However, it is too strong in those parts of the specifications where interrelations between these parts are addressed. These include formulas specifying collective intentions and collective commitments, as they are the key ingredients of the theory of teamwork. In these cases the second, more allowing, restriction is suitable.

The remainder of the thesis is structured as follows. In Chapter 3 we formally present the core of TeamLog framework. In Chapter 4 we study the complexity of the satisfiability problem for TeamLog. In Chapter 5 we study the effect of restricting the modal depth of formulas and the number of propositional symbols on the complexity of the satisfiability problem for the individual fragment of TeamLog. In Chapter 6 we study the effect of restricting the modal context of formulas and the number of propositional symbols on the complexity of the satisfiability problem for the full TeamLog. In Chapter 7 we discuss the language restrictions proposed in the context of multiagent systems specification. We conclude the thesis in Chapter 8, where we discuss, in particular, possible further language restrictions that could be investigated to find tractable fragments of TeamLog and other similar multiagent formalisms.

Chapter 3

The logical framework

Logics considered in this thesis are normal multimodal logics with iterated modal operators. We will focus our attention on one particular such formalism, TeamLog. However, since we will discuss different fragments of this framework and since methods we use can be applied to other multimodal frameworks used for multiagent systems, it will be convenient to define a general multimodal language and related syntactic notions that will be referred to throughout the thesis first, then to define multimodal logics with iterated modalities and their semantics and lastly to introduce the TeamLog framework.

3.1 General multimodal logic

Languages of the logics considered in this thesis are defined on the basis of a countable set of propositional symbols $\mathcal{P}$ and a non-empty set of unary modal operators Ω. We will denote a language based on $\mathcal{P}$ and Ω by $\mathcal{L}[\mathcal{P}, \Omega]$. In further parts of the thesis we will consider restrictions of modal language, which include restricting the set of operators as well as the set of propositional symbols. We will always assume that both these sets are non-empty.

Definition 1 (Language)**.** *Let $\mathcal{P}$ be a countable infinite set of propositional symbols and let Ω be a non-empty set of modal operators. The language $\mathcal{L}[\mathcal{P}, \Omega]$, is a minimal set of formulas satisfying the following properties*

- $\mathcal{P} \subseteq \mathcal{L}[\mathcal{P}, \Omega]$,
- *If* $\varphi \in \mathcal{L}[\mathcal{P}, \Omega]$, *then* $\neg\varphi \in \mathcal{L}[\mathcal{P}, \Omega]$,
- *If* $\varphi_1 \in \mathcal{L}[\mathcal{P}, \Omega]$ *and* $\varphi_2 \in \mathcal{L}[\mathcal{P}, \Omega]$, *then* $\varphi_1 \wedge \varphi_2 \in \mathcal{L}[\mathcal{P}, \Omega]$,
- *If* $\varphi \in \mathcal{L}[\mathcal{P}, \Omega]$ *and* $\Box \in \Omega$, *then* $\Box\varphi \in \mathcal{L}[\mathcal{P}, \Omega]$.

We will also use the following abbreviations:

- $\bot \stackrel{def}{=} p \wedge \neg p$, *where* $p \in \mathcal{P}$,
- $\top \stackrel{def}{=} \neg\bot$,
- $\varphi_1 \vee \varphi_2 \stackrel{def}{=} \neg(\neg\varphi_1 \wedge \neg\varphi_2)$,
- $\varphi_1 \to \varphi_2 \stackrel{def}{=} \neg(\varphi_1 \wedge \neg\varphi_2)$,
- $\varphi_1 \leftrightarrow \varphi_2 \stackrel{def}{=} (\varphi_1 \to \varphi_2) \wedge (\varphi_2 \to \varphi_1)$,

Throughout the thesis we will refer to the notions of length of a formula and modal depth of a formula, which are defined below.

Definition 2 (Length of a formula). *The* length of a formula $\varphi \in \mathcal{L}[\mathcal{P}, \Omega]$*, denoted by* $|\varphi|$*, is defined inductively as follows:*

- $|p| = 1$*, where* $p \in \mathcal{P}$.
- $|\neg\varphi| = |\varphi| + 1$.
- $|\varphi_1 \wedge \varphi_2| = |\varphi_1| + |\varphi_2| + 1$.
- $|\Box\varphi| = |\varphi| + 1$*, where* $\Box \in \Omega$.

Given a set X we will also use $|X|$ to denote the number of elements in X.

Definition 3 (Modal depth). *The* modal depth of a formula $\varphi \in \mathcal{L}[\mathcal{P}, \Omega]$*, denoted by* $\mathrm{dep}(\varphi)$*, is defined inductively as follows:*

- $\mathrm{dep}(p) = 0$*, where* $p \in \mathcal{P}$.
- $\mathrm{dep}(\neg\varphi) = \mathrm{dep}(\varphi)$.
- $\mathrm{dep}(\varphi_1 \wedge \varphi_2) = \max\{\mathrm{dep}(\varphi_1), \mathrm{dep}(\varphi_2)\}$.
- $\mathrm{dep}(\Box\varphi) = \mathrm{dep}(\varphi) + 1$*, where* $\Box \in \Omega$.

Let Φ *be a finite set of formulas, then*

$$\mathrm{dep}(\Phi) = \begin{cases} 0 & \textit{if } \Phi = \varnothing \\ \max\{\mathrm{dep}(\varphi) : \varphi \in \Phi\} & \textit{otherwise} \end{cases}$$

Given a finite set of formulas Φ, we will use $\bigwedge \Phi$ to denote the conjunction of all formulas in the set, and $\bigvee \Phi$ to denote the disjunction of all formulas in the set. We will also use conventions that $\bigwedge \varnothing = \top$ and $\bigvee \varnothing = \bot$.

3.1.1 Semantics

Formulas from $\mathcal{L}[\mathcal{P}, \Omega]$ are interpreted in Kripke models with accessibility relations corresponding to modalities from Ω.

Definition 4 (Kripke frame). *A* Kripke frame *is a tuple* $\mathcal{F} = (W, \{R^\Box : \Box \in \Omega\})$*, where*

- $W \neq \varnothing$ *is the set of possible worlds.*
- *For all* $\Box \in \Omega$, $R^\Box \subseteq W \times W$. *Each relation* $R^\Box$ *stands for the accessibility relation corresponding to the operator* $\Box$.

Definition 5 (Kripke model). *A* Kripke model *is a pair* $\mathcal{M} = (\mathcal{F}, Val)$*, where* $\mathcal{F}$ *is a Kripke frame and*

- $Val : \mathcal{P} \times W \to \{0, 1\}$ *is a valuation function that assigns the truth values to atomic propositions in worlds.*

Given a binary relation $R \subseteq W \times W$ and $w \in W$ we will use $R(w)$ to denote the set of worlds accessible from w, that is $R(w) = \{v \in W : (w, v) \in R\}$.

Definition 6 (Satisfaction)**.** *Let $\mathcal{M}$ be a Kripke model, w be a world in $\mathcal{M}$ and φ be a formula. The notion of φ being* satisfied *(or being* true *or* holding*) in $\mathcal{M}$ at w is defined inductively as follows:*

$$\begin{aligned}
&(\mathcal{M}, w) \models p \textit{ iff } Val(p, w) = 1,\\
&(\mathcal{M}, w) \models \neg\varphi \textit{ iff } (\mathcal{M}, w) \not\models \varphi,\\
&(\mathcal{M}, w) \models \varphi_1 \wedge \varphi_2 \textit{ iff } (\mathcal{M}, w) \models \varphi_1 \textit{ and } (\mathcal{M}, w) \models \varphi_2,\\
&(\mathcal{M}, w) \models \Box\varphi \textit{ iff } (\mathcal{M}, v) \models \varphi, \textit{ for all } v \in R^{\Box}(w).
\end{aligned}$$

Let $\varphi \in \mathcal{L}$ be a formula. We say that φ is *valid* in a Kripke model $\mathcal{M}$ if for every world w in $\mathcal{M}$, $(\mathcal{M}, w) \models \varphi$. We denote this fact by $\mathcal{M} \models \varphi$. We say that φ is *satisfiable in* $\mathcal{M}$ if there exists a world w in $\mathcal{M}$ such that $(\mathcal{M}, w) \models \varphi$. Let $\mathcal{C}$ be a class of Kripke models. We say that φ is valid in $\mathcal{C}$ if $\mathcal{M} \models \varphi$, for every $\mathcal{M} \in \mathcal{C}$. We denote this fact by $\mathcal{C} \models \varphi$. We say that φ is *satisfiable in* $\mathcal{C}$ if there exists $\mathcal{M} \in \mathcal{C}$ such that φ is satisfiable in $\mathcal{M}$.

3.1.2 Deduction system of normal multimodal logic

The logics considered in this thesis are normal multimodal logics, that is logics such that for each $\Box \in \Omega$ the following axioms and inference rules are adopted:

P all instances of propositional tautologies,

K $\Box\varphi \wedge \Box(\varphi \to \psi) \to \Box\psi$, for each $\Box \in \Omega$,

MP from φ and $\varphi \to \psi$ infer ψ (modus ponens),

GEN from φ infer $\Box\varphi$ (generalization), for each $\Box \in \Omega$.

Given a deduction system $\mathcal{D}$, a *proof in* $\mathcal{D}$ is a finite sequence of formulas, each of them either being an axiom of $\mathcal{D}$ or being obtainable from formulas preceding it in the sequence by applying an inference rule of $\mathcal{D}$. A formula φ is *provable in* $\mathcal{D}$ if there is a proof such that φ is an element of it. We denote this fact by $\vdash_{\mathcal{D}} \varphi$.

Let $\mathcal{C}$ be a class of Kripke models and let $\mathcal{D}$ be a deduction system. We say that $\mathcal{D}$ is *sound with respect to* $\mathcal{C}$ is every formula φ such that $\vdash_{\mathcal{D}} \varphi$, φ is valid in $\mathcal{C}$. We say that $\mathcal{D}$ is *complete with respect to* $\mathcal{C}$ if every formula φ such that $\mathcal{C} \models \varphi$ is provable in $\mathcal{D}$.

The deduction system above is sound and complete with respect to the class of all Kripke models (c.f. for example [10])

3.1.3 Multiagent modal logics

Multimodal languages considered in this thesis are based on sets of modal operators that contain subsets of modal operators $\Omega^{\mathcal{A}} = \{\Box_j : j \in \mathcal{A}\}$ indexed by elements of a non-empty and finite set $\mathcal{A}$ called the *set of agents*. Operators from $\Omega^{\mathcal{A}}$ correspond to some aspect of description of agents, like knowledge, beliefs, goals or intentions. $\mathcal{L}^{\mathcal{A}}[\mathcal{P}] = \mathcal{L}[\mathcal{P}, \Omega^{\mathcal{A}}]$ is a multimodal language and its semantics conforms to the definition given in Section 3.1.1. We will use $\mathcal{L}^{\mathcal{A}}$ to denote $\mathcal{L}^{\mathcal{A}}[\mathcal{P}]$ in the cases where we do not put any additional restrictions on the set of propositions $\mathcal{P}$.[1]

Since the modal operators from $\Omega^{\mathcal{A}}$ are uniquely identified by elements from $\mathcal{A}$, so we will use R_j (rather than $R^{\Box_j}$) to denote the relation corresponding to modal operator $\Box_j$.

[1] In Section 5.2 of Chapter 5 and Section 6.4 of Chapter 6 we consider languages where the number of propositional symbols is bounded by a constant.

For all operators from $\Omega^{\mathcal{A}}$ the deduction system described in Section 3.1.2 is adopted. The system is denoted by K_n, where $n = |\mathcal{A}|$. Additionally, the following axioms will be considered (for $j \in \mathcal{A}$):

T $\Box_j\varphi \to \varphi$,

D $\neg\Box_j\bot$,

4 $\Box_j\varphi \to \Box_j\Box_j\varphi$,

5 $\neg\Box_j\varphi \to \Box_j\neg\Box_j\varphi$.

Multimodal logics generated by subsets of axioms **T** – **5**: KD_n, $\mathrm{K4}_n$, $\mathrm{KD45}_n$, etc., are defined in the usual way. In particular, KD_n is obtained by adopting axiom **D** for $\Box j$, for each $j \in \mathcal{A}$ and $\mathrm{KD45}_n$ is obtained by adopting axioms **D**, **4** and **5** for $\Box j$, for each $j \in \mathcal{A}$ (see for example [54, 10] for further details). We will denote the set of all these systems by $\mathcal{S}$ and we will refer to a multimodal logic generated by $S \in \mathcal{S}$ as 'multimodal logic S'.

The axioms above, as far as they do not hold on all frames like **K**, correspond to well-known structural properties on Kripke frames, in the sense that they hold on all frames having certain structural properties (c.f. [112]). Thus, axiom **T** corresponds to *reflexivity* of relations R_j:

- $\forall s(s \in R_j(s))$,

axiom **D** corresponds to *seriality* of relations R_j:

- $\forall s(R_j(s) \neq \varnothing)$,

axiom **4** corresponds to *transitivity* of relations R_j:

- $\forall s,t,u((t \in R_j(s) \wedge u \in R_j(t)) \to u \in R_j(s))$,

and axiom **5** corresponds to *Euclideanity* of relations R_j:

- $\forall s,t,u((t \in R_j(s) \wedge u \in R_j(s)) \to u \in R_j(t))$,

Given a normal multimodal logic from $\mathcal{S}$ and a formula $\varphi \in \mathcal{L}[\mathcal{P}, \Omega^{\mathcal{A}}]$ we say that φ is S *satisfiable* iff it is satisfiable in a class of Kripke models with accessibility relations corresponding to axioms of S. We will call such models S-models. We also say that φ is S provable, denoted by $\vdash_S \varphi$, if there exists a proof of φ that can include axioms from S. A deduction system $S \in \mathcal{S}$ is sound and complete with respect to class of S-models (c.f. for example [10]):

Theorem 3.1. *For any $\varphi \in \mathcal{L}^{\mathcal{A}}$ and $S \in \mathcal{S}$, $S \models \varphi$ iff $\vdash_S \varphi$.*

3.1.4 Multiagent modal logics with iterated modalities

Logical formalisms used to specify multiagent systems, including TeamLog, combine several multiagent modal logics, sometimes with modalities used to specify group aspects of multiagent systems related to individual aspects of agents (e.g. beliefs and common beliefs or intentions and mutual intentions). Typically each such formalisms has a few groups of such operators, like beliefs (and related to them common beliefs), goals and intentions (with related to them mutual intentions). Each group of modalities is associated with a different axioms system (e.g. $\mathrm{KD45}_n$ for beliefs, KD_n for intentions). In this section we introduce a multimodal logics with iterated modalities based on one group of modalities.

The set of operators is based on a finite and non-empty set of agents $\mathcal{A}$ and it is $\Omega^{\mathrm{it}} = \Omega^{\mathcal{A}} \cup \{\Box_G^+ : G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}\}$, where $\{\Box_G^+ : G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}\}$ are modalities corresponding to groups of agents. We will also use the following abbreviation, for $G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}$:

- $\Box_G\varphi \overset{\text{def}}{=} \bigwedge_{j\in G}\Box_j\varphi$.

$\mathcal{L}^{\text{it}} = \mathcal{L}[\mathcal{P},\Omega^{\text{it}}]$ is a multimodal language and its semantics conforms to the definition given in Section 3.1.1. Given a binary relation R, we will use R^+ to denote the transitive closure of R. Moreover, given a family of relations $\{R_j : j \in \mathcal{A}\}$ and a set of agents $G \subseteq \mathcal{A}$, relation $R_G = \bigcup_{j\in G} R_j$. The relation corresponding to modal operator $\Box_G^+$ is $R_G{}^+$. Since the relations corresponding to operators from $\{\Box_G^+ : G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}\}$ are defined on the basis of relations $\{R_j : j \in \mathcal{A}\}$, so we will omit them in the description of Kripke structures and Kripke frames used to interpret formulas from $\mathcal{L}^{\text{it}}$.

Deduction systems for multiagent modal logics with iterated modalities extend systems from $\mathcal{S}$ by introducing a new axiom and a new rule of inference related to modalities associated with groups of agents. The basic normal multimodal logic with iterated modalities, K_n^+, extends the system K_n with the following axioms and rules of inference (for all $G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}$):

C $\quad \Box_G^+\varphi \leftrightarrow \Box_G\big(\varphi \wedge \Box_G^+\varphi\big)$,

RC $\quad$ From $\varphi \to \Box_G(\psi \wedge \varphi)$ infer $\varphi \to \Box_G^+\psi$.

Modalities corresponding to groups of agents share similarity with modal operators corresponding to iteration in propositional dynamic logic (PDL). One could see modalities $\Box_j$ as modalities corresponding to basic programs. Then the operator $\Box_G^+$ could be expressed as $\Big[\Big(\bigcup_{j\in G}\Box_j\Big) \cup \Big(\bigcup_{j\in G}\Box_j\Big)^*\Big]$. Note that inference rule **RC** could be replaced by Segerberg's axiom of induction. Because of similarity between modalities $\Box_G^+$ and iteration in PDL, we will call them *iterated modalities*.

Multimodal logics with modalities extending multimodal logics from $\mathcal{S}$: KD_n, $\mathrm{K4}_n$, $\mathrm{KD45}_n$, etc. with axiom **C** and rule of inference **RC** will be denoted by KD_n^+, $\mathrm{K4}_n^+$, $\mathrm{KD45}_n^+$, etc.. Given a multimodal logic $S \in \mathcal{S}$ we will use S^+ to refer to its extension with iterated modalities.

Given a multimodal logic with iterated modalities S^+ with $S \in \mathcal{S}$, we say that φ is S^+ provable, denoted by $\vdash_{S^+} \varphi$, if there exists a proof of φ that can include axioms from S^+. A deduction system S^+ is sound and complete with respect to class of S-models (c.f. for example [54])

Theorem 3.2. *For any $\varphi \in \mathcal{L}[\mathcal{P},\Omega^{\text{it}}]$ and $S \in \mathcal{S}$, $S \vDash \varphi$ iff $\vdash_{S^+} \varphi$.*

3.2 TEAMLOG: A Logical Framework For Multiagent Systems

TEAMLOG introduces three groups of modal operators based on a finite and non-empty set of agents $\mathcal{A}$, used to represent beliefs, goals and intentions of agents, as well as two groups of modal operators used to represent common beliefs and mutual intentions within a group of agents. As we explained before, throughout this and the following chapters, where we study complexity of the satisfiability problem of TEAMLOG and its different subsets, we will use a more compact notation for operators of TEAMLOG, replacing that used in Chapter 1. The compact notation is more convenient for presentation of proofs and algorithms. A summary of correspondence between the compact notation and the standard TEAMLOG notation is given in Table 3.1.

The set of modal operators of TEAMLOG, based on a non-empty set of agents, $\mathcal{A}$, is $\Omega^{\mathrm{T}} = \Omega^{\mathrm{B}+} \cup \Omega^{\mathrm{G}} \cup \Omega^{\mathrm{I}+}$, where $\Omega^{\mathrm{B}+} = \Omega^{\mathrm{B}} \cup \{[\mathrm{B}]_G^+ : G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}\}$, $\Omega^{\mathrm{I}+} = \Omega^{\mathrm{I}} \cup \{[\mathrm{I}]_G^+ : G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}\}$, $\Omega^{\mathrm{B}} = \{[\mathrm{B}]_j : j \in \mathcal{A}\}$, $\Omega^{\mathrm{G}} = \{[\mathrm{G}]_j : j \in \mathcal{A}\}$ and $\Omega^{\mathrm{I}} = \{[\mathrm{I}]_j : j \in \mathcal{A}\}$. Moreover, we will also refer to the set of individual modalities $\Omega^{\mathrm{Tind}} = \Omega^{\mathrm{B}} \cup \Omega^{\mathrm{G}} \cup \Omega^{\mathrm{I}}$.

$[\mathrm{B}]_j\varphi \equiv \mathrm{BEL}\,(j,\varphi)\,,$	$[\mathrm{B}]_G\varphi \equiv \text{E-BEL}_G\,(\varphi)$
$[\mathrm{G}]_j\varphi \equiv \mathrm{GOAL}\,(j,\varphi)\,,$	$[\mathrm{B}]^+_G\varphi \equiv \text{C-BEL}_G\,(\varphi)$
$[\mathrm{I}]_j\varphi \equiv \mathrm{INT}\,(j,\varphi)\,,$	$[\mathrm{I}]_G\varphi \equiv \text{E-INT}_G\,(\varphi)$
	$[\mathrm{I}]^+_G\varphi \equiv \text{M-INT}_G\,(\varphi)$

Table 3.1: Standard TeamLog notation and its compact substitute.

The language of TeamLog, based on an a given set of propositional symbols $\mathcal{P}$ and the set of operators Ω^{T} will be denoted by $\mathcal{L}^{\mathrm{T}}[\mathcal{P}] = \mathcal{L}[\mathcal{P}, \Omega^{\mathrm{T}}]$. We will also denote it by $\mathcal{L}^{\mathrm{T}}$ if we do not put any additional constraints on $\mathcal{P}$. Additionally, we will consider a subset of TeamLog framework covering individual aspects of multiagent system, TeamLog$^{\mathrm{ind}}$, with language $\mathcal{L}^{\mathrm{Tind}}[\mathcal{P}] = \mathcal{L}[\mathcal{P}, \Omega^{\mathrm{Tind}}]$ (we will denote it by $\mathcal{L}^{\mathrm{Tind}}$ in the case of $\mathcal{P}$ being infinite and countable). We will also use the following abbreviations, for $G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}$:

- $[\mathrm{B}]_G\varphi \stackrel{\text{def}}{=} \bigwedge_{j\in G}[\mathrm{B}]_j\varphi$,
- $[\mathrm{I}]_G\varphi \stackrel{\text{def}}{=} \bigwedge_{j\in G}[\mathrm{I}]_j\varphi$.

Languages $\mathcal{L}^{\mathrm{T}}$ and $\mathcal{L}^{\mathrm{Tind}}$ are multimodal languages and definitions of their semantics conform to the definition given in Section 3.1.1. Similarly to $\mathcal{L}^{\mathrm{it}}$, the modal operators from Ω^{T} are uniquely identified by elements from $\mathcal{A}$ and a label $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$, so we will use R^O_j to denote the accessibility relation corresponding to modal operator $[O]_j$. The accessibility relations corresponding to iterated modalities $[O]^+_G$ with $O \in \{\mathrm{B}, \mathrm{I}\}$ are R^{O+}_G and they are defined on the basis of relations $\{R^O_j : j \in \mathcal{A}\}$. Thus, like in the case of $\mathcal{L}^{\mathrm{it}}$, we will omit them in the description of Kripke structures and Kripke frames used to interpret formulas from $\mathcal{L}^{\mathrm{T}}$.

3.2.1 Deduction system

TeamLog combines axiom systems $\mathrm{KD45}^+_n$, associated with modal operators from Ω^{B}, K_n, associated with modal operators from Ω^{G}, and KD^+_n, associated with modal operators from Ω^{I}. TeamLog$^{\mathrm{ind}}$ is a subset of TeamLog that does not have axioms **C** and rules **RC** associated with iterated modalities $[\mathrm{B}]^+$ and $[\mathrm{I}]^+$. That is, it combines systems $\mathrm{KD45}_n$, K_n and KD_n. Additionally, TeamLog (and TeamLog$^{\mathrm{ind}}$) introduce axioms interrelating operators from different groups, called *mixed axioms* (for $j \in \mathcal{A}$):

BG4 $[\mathrm{G}]_j\varphi \to [\mathrm{B}]_j[\mathrm{G}]_j\varphi$,

BG5 $\neg[\mathrm{G}]_j\varphi \to [\mathrm{B}]_j\neg[\mathrm{G}]_j\varphi$,

BI4 $[\mathrm{I}]_j\varphi \to [\mathrm{B}]_j[\mathrm{I}]_j\varphi$,

BI5 $\neg[\mathrm{I}]_j\varphi \to [\mathrm{B}]_j\neg[\mathrm{I}]_j\varphi$,

IG $[\mathrm{I}]_j\varphi \to [\mathrm{G}]_j\varphi$.

Like in the case of axioms **T** – **5**, the axioms above correspond to certain properties of Kripke frames. Axioms $\mathbf{B}\boldsymbol{O}\mathbf{4}$, where $O \in \{\mathrm{G}, \mathrm{I}\}$, correspond to the following property

- $\forall s,t(t \in R^{\mathrm{B}}_j(s) \to R^O_j(t) \subseteq R^O_j(s))$.

We will call this property a *generalized transitivity.* Axioms $\mathbf{B}\boldsymbol{O}\mathbf{5}$, where $O \in \{\mathrm{G}, \mathrm{I}\}$, correspond to the property

- $\forall s, t(t \in R_j^{\mathrm{B}}(s) \to R_j^O(s) \subseteq R_j^O(t))$.

We will call this property a *generalized Euclideanity.* Finally, axiom **IG** corresponds to the property

- $R_j^{\mathrm{G}} \subseteq R_j^{\mathrm{I}}$.

Proofs of these correspondences are given in [35]. The class of all Kripke frames with accessibility relations $\{R_j^O : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}$ satisfying the properties above will be called the class of TeamLog frames. Analogically the class of TeamLog models is defined. We also say that φ is TeamLog$^{\mathrm{ind}}$ provable, denoted by $\vdash_{\mathrm{Tind}} \varphi$, if there exists a proof of φ that includes axioms from TeamLog$^{\mathrm{ind}}$. Similarly, φ is called TeamLog provable, denoted by $\vdash_{\mathrm{T}} \varphi$, if there exists a proof of φ that includes axioms from TeamLog. The deduction systems TeamLog$^{\mathrm{ind}}$ and TeamLog are sound and complete with respect to class of TeamLog models, as was shown in [33].

Theorem 3.3. *Let* T *be the class of* TeamLog *models. Then for any* $\varphi \in \mathcal{L}^{\mathrm{Tind}}$

$$\mathrm{T} \vDash \varphi \text{ iff } \vdash_{\mathrm{Tind}} \varphi$$

and for any $\varphi \in \mathcal{L}$

$$\mathrm{T} \vDash \varphi \text{ iff } \vdash_{\mathrm{T}} \varphi.$$

Chapter 4

The complexity of the satisfiability problem for TeamLog framework

In this thesis we focus our attention on the complexity of checking satisfiability and validity of formulas belonging to languages of logical frameworks TeamLog and TeamLog$^{\text{ind}}$ as well as some of their fragments. Let us present shortly issues of decidability and complexity of the satisfiability and validity problems. Logical frameworks studied in this thesis are normal multimodal frameworks extended, possibly, with iterated modalities. We are interested in finding out, given a formula φ, how much time or space (in terms of the length of φ) is needed to compute whether φ is satisfiable, i.e. whether there is a suitable Kripke model $\mathcal{M}$ (from the class of structures corresponding to the logic) and a world w in it, such that $(\mathcal{M}, w) \models \varphi$. From this, the complexity of the validity problem (truth in all worlds in all suitable Kripke models) follows immediately, because φ is valid if and only if $\neg\varphi$ is not satisfiable. *Model checking*, i.e. evaluating truth of a given formula in a given world and model $(\mathcal{M}, w) \models \varphi$ is the most important related problem, and is easily seen to be less complex than both the satisfiability and validity problems.

Thus, for example, if the satisfiability problem of some logic is NPTIME-complete, then its validity problem is coNPTIME-complete. We do not investigate the complexity of model checking here, but see [64] for such an analysis of some MAS logics; in any case, various methods have already been developed that can perform model checking in a reasonable time, as long as the considered models are not too large.

Unfortunately, even the satisfiability in propositional logic is NPTIME-complete. Thus, if indeed PTIME $\neq$ NPTIME, then modal logics interesting for MAS, all containing propositional logic as a subsystem, do not have tractably solvable satisfiability problems. Even though a single algorithm performing well on all inputs is not possible, it is still important to discover in which complexity class a given logical theory falls. In our work we take the point of view of the system developer who wants to reason about, specify and verify a multiagent system to be constructed. It turns out that for many of the interesting formulas appearing in such human reasoning, the satisfiability tends to be easier to compute than suggested by the worst-case labels like "PSPACE-complete" and "EXPTIME-complete" [54]. This motivates our study of complexity of restricted fragments of the logical framework in question.

Of many single-agent modal logics with one modality, the complexity has long been known. An overview is given in [54], which extends these results to multi-agent logics, though still containing only a single modality (either knowledge or belief). For us, the following results are relevant. The satisfiability problems for the systems $\mathrm{S5}_1$ and $\mathrm{KD45}_1$, modelling knowledge and belief of one agent, are NPTIME-complete. Thus, perhaps surprisingly, they are no more complex than propositional logic. Extending these systems to more than one agents case makes the satisfiability problem PSPACE-complete. PSPACE-completeness is a property of the satisfiability problems for many other modal logics, for both the single and the multiagent case; examples are the basic system K_n (that is adopted for goals in TEAMLOG framework) and the system KD_n (that is adopted for intentions). As soon as notions of seemingly infinite character modelled by iterated modalities appear, the satisfiability problem becomes EXPTIME-time complete. Intuitively, trying to find a satisfying model for a formula containing a common belief operator by the tableau method, one may need to look exponentially deep in the tableau tree to find it, while for simpler modal logics like K_n, a depth-first search through a polynomially shallow tree suffices for all formulas.

When investigating the complexity of multi-modal logics, one might like to turn to general results on the transfer of the complexity of the satisfiability problems from single logics to their combinations: is not a combination of a few PSPACE-complete logics, with some simple interdependency axioms, automatically PSPACE-complete again? However, it turns out that the positive general results that do exist (such as those in [11]) apply mainly to minimal combinations, without added interdependencies, of two NPTIME-complete systems, each with a single modality. Even more dangerously, there are some very negative results on the transfer of complexity to combined systems. There exist two "very decidable" logics whose combination, even without any interrelation axioms, is undecidable.

For one logic, denoted by B, take a variant of dynamic logic with two atomic programs, both deterministic. Take ; and $\|$ as only operators. The satisfiability of formulas with respect to B, like that for propositional dynamic logic itself, is in EXPTIME. For the other logic, denoted by C, take the logic of the global operator A (*always*), defined as follows:

$$(\mathcal{M}, w) \models A(\varphi) \text{ iff for all } v \in W,\ (\mathcal{M}, v) \models \varphi.$$

The satisfiability problem for C is in NPTIME-complete. In [11] (see also [10, Theorem 4.14.31]), it is shown that the minimal combination of B and C is not only *not* in EXPTIME, but even undecidable in any finite time. This goes to show that one needs to be very careful with any assumptions about generalizations of complexity results to combined systems. Thus we start by investigating the complexity of the satisfiability problems of unrestricted logical frameworks $\text{TEAMLOG}^{\text{ind}}$ and TEAMLOG.

4.1 Preliminary definitions

Before we analyse the complexity of the satisfiability problem of TEAMLOG and $\text{TEAMLOG}^{\text{ind}}$ logical frameworks, we will introduce notions that will be used throughout the thesis.

We will refer to the notion of "single negation". Given a formula φ,

$$\sim\varphi = \begin{cases} \psi, & \text{if } \varphi = \neg\psi \text{ for some formula } \psi, \\ \neg\varphi, & \text{otherwise.} \end{cases}$$

A set of formulas Φ is *closed under single negation* iff for all $\varphi \in \Phi$, it holds that $\sim\varphi \in \Phi$. Given a set of formulas Φ we will use $\neg\Phi$ to denote the smallest set containing Φ and closed under single negation.

Let $\mathcal{L}[\mathcal{P}, \Omega]$ be a multimodal language and let $\varphi \in \mathcal{L}[\mathcal{P}, \Omega]$, then

$$\mathrm{Sub}(\varphi) = \{\psi : \psi \text{ is a subformula of } \varphi\}$$

is the set of all subformulas of φ. Let $\mathrm{PT}(\varphi)$ be defined inductively as follows:

1. $\mathrm{PT}(p) = \{p\}$, where $p \in \mathcal{P}$,
2. $\mathrm{PT}(\neg\psi) = \{\neg\psi\} \cup \mathrm{PT}(\psi)$,
3. $\mathrm{PT}(\psi_1 \wedge \psi_2) = \mathrm{PT}(\psi_1) \cup \mathrm{PT}(\psi_2)$.
4. $\mathrm{PT}(\Box\psi) = \{\Box\psi\}$, where $\Box \in \Omega$.

The set $\mathrm{PT}(\varphi)$ contains subformulas of φ taken with respect to propositional operators.

We will also refer to the notion of blatantly inconsistent set of formulas. A set of formulas Φ is called *blatantly inconsistent* if there exists $\psi \in \Phi$ such that $\neg\psi \in \Phi$. Given a set of formulas Φ we call any set $M \subseteq \Phi$ which is not blatantly inconsistent and maximal among such subsets of Φ a *maximal consistent subset of* Φ.

Our analysis of the complexity of the satisfiability problems and associated algorithms is based on notions of modal tableaux designed for a given multimodal logic. The notion of modal tableau is based on the notion of model graph, which we define below.

Definition 7 (Model graph). *Let $\mathcal{L}[\mathcal{P}, \Omega]$ be a multimodal language. A* model graph *$\mathcal{T}$ associated with Ω is a tuple*

$$\mathcal{T} = (W, \{R^{\Box} : \Box \in \Omega\}, L)$$

where W and $R^{\Box}$ are defined like in a Kripke frame and L is a labelling function *associating with each state $w \in W$ a set $L(w)$ of formulas.*

4.2 The complexity of the satisfiability problem of TEAMLOG$^{\mathrm{ind}}$

In this section we show that the TEAMLOG satisfiability problem for formulas from $\mathcal{L}^{\mathrm{Tind}}$ is decidable and is PSPACE-complete. The results obtained in this section have already been published in [40, 41]. We decided to make the presentation of these results in this thesis different, to facilitate the presentation of the results in Chapters 5 and 6.

For the lower bounds of the complexity of this problem, the following fact will be useful.

Fact 4.1. *Let $\varphi \in \mathcal{L}[\mathcal{P}, \Omega^{O}]$, where $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$, be a formula build with modal operators based on a non-empty set of agents $\mathcal{B} \subseteq \mathcal{A}$, with $|\mathcal{B}| = m$. Then φ is S-satisfiable iff φ is* TEAMLOG *satisfiable, where $S = KD45_m$ if $O = \mathrm{B}$, $S = K_m$ if $O = \mathrm{G}$ and $S = KD_m$ if $O = \mathrm{I}$.*

Proof. For the left to right implications, if φ is S-satisfiable and *model* is an S-model with a world w in it such that $(\mathcal{M}, w) \models \varphi$, then a TEAMLOG model $\mathcal{M}'$ can be constructed by setting $R^{O}_{j}(v)$ to $\{v\}$, with $O \in \{\mathrm{B}, \mathrm{I}\}$ and $j \in \mathcal{A}$, at all worlds of $\mathcal{M}$ where $R^{O}_{j}(v) = \varnothing$. It is easy to see that $(\mathcal{M}', w) \models \varphi$.

For the right to left implication, if φ is TEAMLOG satisfiable then a TEAMLOG model for it is an S-model for it as well. □

By Fact 4.1, each of the problems of checking K_n, KD_n and $\mathrm{KD45}_n$ satisfiability can be reduced to the problem of checking TEAMLOG satisfiability of $\mathcal{L}^{\mathrm{Tind}}$. Each of these problems is known to be PSPACE-hard (in the case of $\mathrm{KD45}_n$, $n \geq 2$ is required for this result to hold) (see for example [54]).

To show that the TEAMLOG satisfiability problem is in PSPACE we use the tableau method, as presented in [54]. The approach can be divided into the following steps:

1. Define the notion of modal tableau for the logic in question. A modal tableau is a model graph with labels of states and accessibility relations satisfying additional properties associated with axioms generating a considered modal logic.

2. Show that any formula of the logic is satisfiable iff there is a tableau for it.
3. Show the algorithm for checking the satisfiability of a formula. The algorithm constructs a tree-like structure called a pre-tableau which forms a basis for a tableau for the formula.
4. Show that the algorithm has a stop property and is valid.
5. Analyse the computational complexity of the algorithm.

4.2.1 TeamLog$^{\mathbf{ind}}$ tableau

We start with introducing the notion of TeamLog$^{\mathrm{ind}}$ tableau. Roughly speaking, a TeamLog$^{\mathrm{ind}}$ tableau is a model graph with labels of states being *propositional tableaux* and accessibility relations satisfying additional properties corresponding to the axioms of TeamLog$^{\mathrm{ind}}$.

Definition 8 (Propositional tableau)**.** *A* propositional tableau *is a set $\mathcal{T}$ of formulas such that $\mathcal{T}$ is not blatantly inconsistent and:*

1. *If $\neg\neg\psi \in \mathcal{T}$ then $\psi \in \mathcal{T}$.*
2. *If $\varphi \wedge \psi \in \mathcal{T}$ then $\varphi \in \mathcal{T}$ and $\psi \in \mathcal{T}$.*
3. *If $\neg(\varphi \wedge \psi) \in \mathcal{T}$ then either $\{\sim\varphi, \psi\} \subseteq \mathcal{T}$ or $\{\varphi, \sim\psi\} \subseteq \mathcal{T}$ or $\{\sim\varphi, \sim\psi\} \subseteq \mathcal{T}$.*

A *propositional tableau for a formula* φ is a minimal propositional tableau $\mathcal{T}$ such that $\varphi \in \mathcal{T}$. It is easy to see that every propositional tableau for φ is a maximal consistent subset of $\neg\mathrm{PT}(\varphi)$. Notice that, by definition, a propositional tableau cannot be blatantly inconsistent.

Below we present a definition of a general modal tableau, together with its extension to a TeamLog$^{\mathrm{ind}}$-tableau.

Definition 9 (Modal tableau)**.** *A* modal tableau *$\mathcal{T}$ associated with multimodal language $\mathcal{L}[\mathcal{P}, \Omega]$ is a model graph $\mathcal{T} = (W, \{R^{\Box} : \Box \in \Omega\}, L)$ such that for all $w \in W$, $L(w)$ is a propositional tableau. Moreover, for any $\Box \in \Omega$ the following conditions have to be satisfied, for any $w \in W$:*

T1 *If $\Box\varphi \in L(w)$ and $v \in R^{\Box}(w)$, then $\varphi \in L(v)$.*

T2 *If $\neg\Box\varphi \in L(w)$, then there exists $v \in R^{\Box}(w)$ such that $\sim\varphi \in L(v)$.*

*The following conditions have to be satisfied if $\Box \in \Omega$ is associated with additional axioms from **D** – **5** (c.f. [54]):*

- *If $\Box$ is associated with axiom **D**, then the following condition has to be satisfied, for any $w \in W$:*

  TD *If $\Box\varphi \in L(w)$, then either $\varphi \in L(w)$ or $R^{\Box}(w) \neq \varnothing$.*

- *If $\Box$ is associated with axiom **4**, then the following condition has to be satisfied, for any $w \in W$:*

  T4 *If $v \in R^{\Box}(w)$ and $\Box\varphi \in L(w)$, then $\Box\varphi \in L(v)$.*

- *If $\Box$ is associated with axiom **5**, then the following condition has to be satisfied, for any $w \in W$*

T5 *If* $v \in R^{\Box}(w)$ *and* $\Box\varphi \in L(v)$*, then* $\Box\varphi \in L(w)$.

In the case of $\mathcal{L}^{\mathrm{Tind}}$ *and mixed axioms of* TEAMLOG^{ind} *the following additional conditions, associated with axioms* ***BO4***, ***BO5*** *(with* $O \in \{\mathrm{G}, \mathrm{I}\}$*) and* ***IG*** *have to be satisfied for all* $j \in \mathcal{A}$ *and* $w \in W$*:*

TB*O*4 *If* $v \in R^{\mathrm{B}}_j(w)$ *and* $[O]_j\varphi \in L(w)$*, then* $[O]_j\varphi \in L(v)$.

TB*O*5 *If* $v \in R^{\mathrm{B}}_j(w)$ *and* $[O]_j\varphi \in L(v)$*, then* $[O]_j\varphi \in L(w)$.

TIG *If* $v \in R^{\mathrm{G}}_j(w)$ *and* $[\mathrm{I}]_j\varphi \in L(w)$*, then* $\varphi \in L(v)$.

A TEAMLOG$^{\mathrm{ind}}$ tableau *is a modal tableau satisfying conditions* ***T1*** *and* ***T2*** *(for all* $[O]_j$ *with* $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ *and* $j \in \mathcal{A}$*), condition* ***TD*** *(for all* $[O]_j$ *with* $O \in \{\mathrm{B}, \mathrm{I}\}$ *and* $j \in \mathcal{A}$*), conditions* ***T4*** *and* ***T5*** *(for all* $[\mathrm{B}]_j$ *with* $j \in \mathcal{A}$*) and conditions* ***TBG4***, ***TBG5***, ***TBI4***, ***TBI5*** *and* ***TIG***.

Given a formula φ*, we say that* $\mathcal{T}$ *is a tableau for* φ *if there exists a state* $w \in W$ *such that* $\varphi \in L(w)$.

The following theorem links the TEAMLOG$^{\mathrm{ind}}$ satisfiability problem and the notion of TEAMLOG$^{\mathrm{ind}}$ tableau.

Theorem 4.2. *A formula* $\varphi \in \mathcal{L}^{\mathrm{Tind}}$ *is satisfiable iff there is a* TEAMLOG^{ind} *tableau for* φ.

Proof. For the left to right implication assume that φ is satisfiable, and that $(\mathcal{M}, w) \models \varphi$ where

$$\mathcal{M} = (W, \{R^O_j : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, Val)$$

and $w \in W$. Consider a model graph

$$\mathcal{T} = (W, \{R^O_j : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, L),$$

where $\psi \in L(v)$ iff $(\mathcal{M}, v) \models \psi$. Since $\varphi \in L(w)$, so it is enough to show that $\mathcal{T}$ is a TEAMLOG$^{\mathrm{ind}}$ tableau.

We start by showing that for any $v \in W$, $L(v)$ is a propositional tableau. First of all it is easy to see that $L(v)$ is not blatantly inconsistent, for any $v \in W$. If $L(v) = \varnothing$ than the condition is satisfied. Assume that $L(v) \neq \varnothing$. Take any $\psi \in L(v)$. Then it holds that $(\mathcal{M}, v) \models \psi$ and so $(\mathcal{M}, v) \not\models \neg\psi$. Thus $\neg\psi \notin L(v)$. For the remaining conditions of TEAMLOG$^{\mathrm{ind}}$ tableau we will show that for any $v \in W$ and $\psi \in \mathcal{L}^{\mathrm{Tind}}$, $\mathcal{T}$ satisfies them. In continuation of this part of the proof we consider only the kinds of formulas which are relevant for remaining conditions of TEAMLOG$^{\mathrm{ind}}$ tableau.

Assume that ψ is of the form $\neg\neg\xi$. The only condition relevant here is condition 1. Assume that $\psi \in L(v)$. Since $\neg\neg\xi \in L(v)$, so $(\mathcal{M}, v) \models \neg\neg\xi$ and so $(\mathcal{M}, v) \not\models \neg\xi$. Suppose that $(\mathcal{M}, v) \not\models \xi$. Then $(\mathcal{M}, v) \models \neg\xi$ and we get a contradiction. Thus it must be that $(\mathcal{M}, v) \models \xi$ and so $\xi \in L(v)$. This shows that condition 1 is satisfied.

Assume that ψ is of the form $\xi \wedge \zeta$. The only condition relevant here is condition 2. Assume that $\psi \in L(v)$. Since $\xi \wedge \zeta \in L(v)$, so $(\mathcal{M}, v) \models \xi$ and $(\mathcal{M}, v) \models \zeta$. Thus $\xi \in L(v)$ and $\zeta \in L(v)$. This shows that condition 2 is satisfied.

Assume that ψ is of the form $\neg(\xi \wedge \zeta)$. The only condition relevant here is condition 3. Assume that $\psi \in L(v)$. Since $\neg(\xi \wedge \zeta) \in L(v)$, so $(\mathcal{M}, v) \models \neg(\xi \wedge \zeta)$ and so $(\mathcal{M}, v) \not\models \xi \wedge \zeta$. Suppose that $(\mathcal{M}, v) \models \xi$ and $(\mathcal{M}, v) \models \zeta$. Then $(\mathcal{M}, v) \models \xi \wedge \zeta$, and we get a contradiction. Hence it must be that either $(\mathcal{M}, v) \not\models \xi$ or $(\mathcal{M}, v) \not\models \zeta$, and so it must be either $\sim\xi \in L(v)$ or $\sim\zeta \in L(v)$. Thus condition 3 is satisfied and we have shown that $L(v)$ is a propositional tableau.

Assume that ψ is of the form $[O]_j\xi$, where $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$. We start with showing that condition **T1** holds. Suppose that $[O]_j\xi \in L(v)$ and that $u \in R_j^O(v)$. Then $(\mathcal{M}, v) \vDash [O]_j\xi$, and for any $u \in R_j^O(v)$, it holds that $(\mathcal{M}, u) \vDash \xi$. Hence $\xi \in L(u)$ and condition **T1** is satisfied.

Arguments for conditions **T4**, **TBG4** and **TBI4** are similar for each $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and we will consider them together. Suppose that $u \in R_j^{\mathrm{B}}(v)$ and that $[O]_j\xi \in L(v)$. Take any $t \in R_j^O(u)$. Then, by properties of accessibility relations (transitivity, in the case of $O = \mathrm{B}$, or generalized transitivity, in the case of $O \in \{\mathrm{G}, \mathrm{I}\}$), it must be that $t \in R_j^O(v)$. Suppose that $(\mathcal{M}, u) \nvDash [O]_j\xi$. Then there must be $t \in R_j^O(u)$ such that $(\mathcal{M}, t) \nvDash \xi$. Since it also holds, that $t \in R_j^O(v)$, so $(\mathcal{M}, v) \nvDash [O]_j\xi$. But this leads to contradiction, as $[O]_j\xi \in L(v)$ implies $(\mathcal{M}, v) \vDash [O]_j\xi$. Hence it must be, that for any $t \in R_j^O(u)$ it holds that $(\mathcal{M}, t) \vDash \xi$. Thus $(\mathcal{M}, u) \vDash [O]_j\xi$ and $[O]_j\xi \in L(u)$. This shows that conditions **T4**, **TBG4** and **TBI4** are satisfied.

In the case of conditions **T5**, **TBG5** and **TBI5** we will proceed similarly to the case above. Suppose that $u \in R_j^{\mathrm{B}}(v)$ and $[O]_j\xi \in L(u)$. Take any $t \in R_j^O(v)$. By properties of accessibility relations (Euclidean property, in the case of $O = \mathrm{B}$ or generalized Euclidean property, in the case of $O \in \{\mathrm{G}, \mathrm{I}\}$), it must be that $t \in R_j^O(u)$. Suppose that $(\mathcal{M}, v) \nvDash [O]_j\xi$. Then there must be $t \in R_j^O(v)$ such that $(\mathcal{M}, t) \nvDash \xi$. Since it also holds that $t \in R_j^O(u)$, so $(\mathcal{M}, u) \nvDash [O]_j\xi$. But this leads to contradiction, as $[O]_j\xi \in L(u)$ implies $(\mathcal{M}, v) \vDash [O]_j\xi$. Hence it must be, that for any $t \in R_j^O(v)$, it holds that $(\mathcal{M}, t) \vDash \xi$. Thus $(\mathcal{M}, v) \vDash [O]_j\xi$ and $[O]_j\xi \in L(v)$. This shows that conditions **T5**, **TBG5** and **TBI5** are satisfied.

Now consider condition **TD**. Assume that $[O]_j\xi \in L(v)$ with $O \in \{\mathrm{B}, \mathrm{I}\}$. Observe that since R_j^O is serial, so there must exist $u \in R_j^O(v)$. Moreover, since $[O]_j\xi \in L(v)$, so $(\mathcal{M}, v) \vDash [O]_j\xi$ and so, in particular, $(\mathcal{M}, u) \vDash \xi$. Hence $\xi \in L(u)$ and so condition **TD** is satisfied.

For the last case, consider condition **TIG**. Suppose that $[O]_j\xi \in L(v)$, $O = \mathrm{I}$ and $u \in R_j^{\mathrm{G}}(v)$. It follows that $(\mathcal{M}, v) \vDash [\mathrm{I}]_j\xi$ and, since $R_j^{\mathrm{G}} \subseteq R_j^{\mathrm{I}}$, so $(\mathcal{M}, u) \vDash \xi$, for any $u \in R_j^{\mathrm{G}}(v)$. Hence $(\mathcal{M}, v) \vDash [\mathrm{G}]_j\xi$ and $[\mathrm{G}]_j\xi \in L(v)$. This shows that condition **TIG** is satisfied.

Assume that ψ is of the form $\neg[O]_j\xi$, where $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$. The only conditions to consider for this case is condition **T2**. Suppose that $\neg[O]_j\xi \in L(v)$. Then $(\mathcal{M}, v) \vDash \neg[O]_j\xi$, and so $(\mathcal{M}, v) \nvDash [O]_j\xi$. Thus there must exist $u \in R_j^O(v)$ such that $(\mathcal{M}, u) \nvDash \xi$. Hence $(\mathcal{M}, u) \vDash {\sim}\xi$ and ${\sim}\xi \in L(u)$. This shows that condition **T2** is satisfied.

For the right to left implication let

$$\mathcal{T} = (W, \{R_j^O : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, L)$$

be a TEAMLOG$^{\mathrm{ind}}$ tableau for φ, so that $\varphi \in L(w)$ for some $w \in W$. We will show how to construct a TEAMLOG model in which φ is satisfied. Consider

$$\mathcal{M} = (W, \{R_j'^O : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, Val),$$

where

$$Val(p, v) = \begin{cases} 1, & \text{if } p \in L(v) \\ 0, & \text{if } p \notin L(v) \end{cases}$$

Before defining accessibility relations $R_j'^{\mathrm{B}}$, $R_j'^{\mathrm{G}}$ and $R_j'^{\mathrm{I}}$, we will introduce a notion that will be useful here. We will say that states v and u are R_j^{B}-connected with a sequence of states $s_0, \ldots, s_m$ if $v = s_0$, $u = s_m$, $m > 0$ and for any $0 < j \leq m$, either $s_j \in R_j^{\mathrm{B}}(s_{j-1})$ or $s_{j-1} \in R_j^{\mathrm{B}}(s_j)$. We will say that states v and u are R_j^{B}-*connected* if there is a sequence of states $s_0, \ldots, s_m$ such that v and u are R_j^{B}-connected with it.

Relation R'^{B}_j is defined as follows. Let

$$\bar{R}^{\mathrm{B}}_j = R^{\mathrm{B}}_j \cup \{(v,v) \in W \times W : R^{\mathrm{B}}_j(v) = \varnothing\}.$$

Then $(v,u) \in R'^{\mathrm{B}}_j$ iff $(v,u) \in \bar{R}^{\mathrm{B}}_j$ or there exists s such that v and s are R^{B}_j-connected and $u \in R^{\mathrm{B}}_j(s)$ (notice that it means, in the case of $(v,u) \in R^{\mathrm{B}}_j$ or in the latter case, that v and u are R^{B}_j-connected).

Relation R'^{G}_j is defined as follows. A pair of states $(v,u) \in R'^{\mathrm{G}}_j$ iff $(v,u) \in R^{\mathrm{G}}_j$ or there exists s, such that v and s are R^{B}_j-connected and $u \in R^{\mathrm{G}}_j(s)$.

Relation R'^{I}_j is defined as follows. Let

$$\bar{R}^{\mathrm{I}}_j = R^{\mathrm{I}}_j \cup \{(v,v) \in W \times W : R^{\mathrm{I}}_j(v) = \varnothing\}.$$

Then $(v,u) \in R'^{\mathrm{I}}_j$ iff $(v,u) \in \bar{R}^{\mathrm{I}}_j \cup R'^{\mathrm{G}}_j$ or there exists s such that v and s are R^{B}_j-connected and $u \in R^{\mathrm{I}}_j(s)$.

To show that $\mathcal{M}$ is a TEAMLOG model, we have to show that all required properties of accessibility relations are satisfied. For the start, take any relation R'^{B}_j. Relation R'^{B}_j is serial, as it contains $\bar{R}^{\mathrm{B}}_j$ as a subset. For transitivity, take any v, u, t such that $u \in R'^{\mathrm{B}}_j(v)$ and $t \in R'^{\mathrm{B}}_j(u)$. If $v = u$ or $u = t$, then $t \in R'^{\mathrm{B}}_j(v)$ is obvious. Assume that $v \neq u$ and $u \neq t$. Then there must be a sequence of states $s_0, \dots, s_m$ such that v and u are R^{B}_j-connected with it and a sequence of states $r_0, \dots, r_k$ such that u and t are R^{B}_j-connected with it. Thus v and t are R^{B}_j-connected with a sequence of states $s_0, \dots, s_m, r_0, \dots, r_k$ and there must be r such that $r_k \in R^{\mathrm{B}}_j(r)$. This shows that $t \in R'^{\mathrm{B}}_j(v)$.

For Euclidean property, take any v, u and t such that $u \in R'^{\mathrm{B}}_j(v)$ and $t \in R'^{\mathrm{B}}_j(v)$. If $v = u$, then $t \in R'^{\mathrm{B}}_j(u)$ is obvious. Assume that $v \neq u$. Then there must be a sequence of states $s_0, \dots, s_m$ such that v and u are R^{B}_j-connected with it. If v and t are not R^{B}_j-connected, then $v = t$ and $R^{\mathrm{B}}_j(v) = \varnothing$. Thus it must be that $s_0 \in R^{\mathrm{B}}_j(s_1)$, so there is s such that $v \in R^{\mathrm{B}}_j(s)$. Since the relation of being R^{B}_j-connected is symmetric, so $t \in R'^{\mathrm{B}}_j(u)$. If there is a sequence of states $r_0, \dots, r_k$ such that v and t are R^{B}_j-connected with it, then u and t are R^{B}_j-connected with a sequence of states $s_m, \dots, s_0, r_0, \dots, r_k$ and there must be r such that $r_k \in R^{\mathrm{B}}_j(r)$. This shows that $t \in R'^{\mathrm{B}}_j(u)$.

Now take any relation R'^{G}_j. For generalized transitivity with relation R'^{B}_j, take any v, u and t such that $u \in R'^{\mathrm{B}}_j(v)$ and $t \in R'^{\mathrm{G}}_j(u)$. If $v = u$, then $t \in R'^{\mathrm{G}}_j(v)$ is obvious. Assume that $v \neq u$. Then there must be a sequence of states $s_0, \dots, s_m$ such that v and u are R^{B}_j-connected with it. If $t \in R^{\mathrm{G}}_j(u)$, then $t \in R'^{\mathrm{G}}_j(v)$, by definition of R'^{G}_j. Otherwise there must be a state r and a sequence of states $r_0, \dots, r_k$ such that u and r are R^{B}_j-connected with it and $t \in R^{\mathrm{G}}_j(r)$. Thus v and r are R^{B}_j-connected with a sequence of states $s_0, \dots, s_m, r_0, \dots, r_k$ and $t \in R^{\mathrm{G}}_j(r)$. This shows that $t \in R'^{\mathrm{G}}_j(v)$.

For generalized Euclidean property with relation R'^{B}_j, take any v, u and t such that $u \in R'^{\mathrm{B}}_j(v)$ and $t \in R'^{\mathrm{G}}_j(v)$. If $v = u$, then $t \in R'^{\mathrm{G}}_j(u)$ is obvious. Assume that $v \neq u$. Then there must be a sequence of states $s_0, \dots, s_m$ such that v and u are R^{B}_j-connected with it. If $t \in R^{\mathrm{G}}_j(v)$, then $t \in R'^{\mathrm{G}}_j(u)$ by definition of R'^{G}_j and the fact that relation of being R^{B}_j-connected is symmetric. Otherwise there must be a state r and a sequence of states $r_0, \dots, r_k$ such that v and r are R^{B}_j-connected with it and $t \in R^{\mathrm{G}}_j(r)$. Thus u and r are R^{B}_j-connected with a sequence of states $s_m, \dots, s_0, r_0, \dots, r_k$ and $t \in R^{\mathrm{G}}_j(r)$. This shows that $t \in R'^{\mathrm{G}}_j(u)$.

Lastly, take any relation R'^{I}_j. Relation R'^{I}_j is serial, as it contains $\bar{R}^{\mathrm{I}}_j$ as a subset. It is also easy to see that $R^{\mathrm{G}}_j \subseteq R'^{\mathrm{I}}_j$. For generalized transitivity with relation R'^{B}_j, take any v, u and t such that $u \in R'^{\mathrm{B}}_j(v)$ and $t \in R'^{\mathrm{I}}_j(u)$. If $v = u$, then $t \in R'^{\mathrm{I}}_j(v)$ is obvious. Assume that $v \neq u$. Then there must be a sequence of states $s_0, \ldots, s_m$ such that v and u are R^{B}_j-connected with it. If $t \in R^{\mathrm{I}}_j(u)$ or $t \in R^{\mathrm{G}}_j(u)$, or $u = t$ and $R^{\mathrm{I}}_j(t) = \varnothing$, then $t \in R'^{\mathrm{I}}_j(v)$ by definition of R'^{I}_j. Otherwise, there must be a state r and a sequence of states $r_0, \ldots, r_k$ such that u and r are R^{B}_j-connected with it and either $t \in R^{\mathrm{G}}_j(r)$ or $t \in R^{\mathrm{I}}_j(r)$, or $r = t$ and $R^{\mathrm{I}}_j(t) = \varnothing$. Since v and r are R^{B}_j-connected with a sequence of states $s_0, \ldots, s_m, r_0, \ldots, r_k$, so $t \in R'^{\mathrm{I}}_j(v)$.

For generalized Euclidean property with relation R'^{B}_j, take any v, u and t such that $u \in R'^{\mathrm{B}}_j(v)$ and $t \in R'^{\mathrm{I}}_j()$. If $v = u$, then $t \in R'^{\mathrm{I}}_j(u)$ is obvious. Assume that $v \neq u$. Then there must be a sequence of states $s_0, \ldots, s_m$ such that v and u are R^{B}_j-connected with it. If $t \in R^{\mathrm{I}}_j(v)$ or $t \in R^{\mathrm{G}}_j(v)$, or $u = t$ and $R^{\mathrm{I}}_j(t) = \varnothing$, then $t \in R'^{\mathrm{I}}_j(u)$, by definition of R'^{I}_j and the fact that relation of being R^{B}_j-connected is symmetric. Otherwise there must be a state r and a sequence of states $r_0, \ldots, r_k$ such that v and r are R^{B}_j-connected with it and either $t \in R^{\mathrm{G}}_j(r)$ or $t \in R^{\mathrm{I}}_j(r)$, or $r = t$ and $R^{\mathrm{I}}_j(t) = \varnothing$. Thus u and r are R^{B}_j-connected with a sequence of states $s_m, \ldots, s_0, r_0, \ldots, r_k$ and so $t \in R'^{\mathrm{I}}_j(u)$.

We have shown that $\mathcal{M}$ is a TeamLog model. Now we will show, using induction on the length of formulas, that for any $\psi \in \mathcal{L}^{\mathrm{Tind}}$ and $v \in W$, $\psi \in L(v)$ implies $(\mathcal{M}, v) \vDash \psi$. Assume that $\psi = p$, where $p \in \mathcal{P}$. Suppose that $p \in L(v)$. Then, by definition of Val, $Val(p, v) = 1$ and $(\mathcal{M}, v) \vDash p$.

For $\psi = \neg\xi$, each of the possible forms of ξ will be considered separately. For now, assume that $\psi = \neg p$, where $p \in \mathcal{P}$, and $\neg p \in L(v)$. Then by definition of Val, $Val(p, v) = 0$ and $(\mathcal{M}, v) \nvDash p$, that is $(\mathcal{M}, v) \vDash \neg p$.

Assume that $\psi = \xi \wedge \zeta$ and $\xi \wedge \zeta \in L(v)$. Then, by condition 2, $\xi \in L(v)$ and $\zeta \in L(v)$, and, by the induction hypothesis, $(\mathcal{M}, v) \vDash \xi$ and $(\mathcal{M}, v) \vDash \zeta$. Thus $(\mathcal{M}, v) \vDash \xi \wedge \zeta$.

Assume that $\psi = \neg(\xi \wedge \zeta)$ and $\neg(\xi \wedge \zeta) \in L(v)$. Then, by condition 3, either $\sim\xi \in L(v)$ or $\sim\zeta \in L(v)$, and, by the induction hypothesis, either $(\mathcal{M}, v) \vDash \sim\xi$ or $(\mathcal{M}, v) \vDash \sim\zeta$. Suppose that $(\mathcal{M}, v) \vDash \xi \wedge \zeta$. This implies that $(\mathcal{M}, v) \vDash \xi$ and $(\mathcal{M}, v) \vDash \zeta$, and we get contradiction. Thus it must hold that $(\mathcal{M}, v) \nvDash \xi \wedge \zeta$, and so $(\mathcal{M}, v) \vDash \neg(\xi \wedge \zeta)$.

Assume that $\psi = [\mathrm{B}]_j\xi$ and $[\mathrm{B}]_j\xi \in L(v)$. Take any $u \in R'^{\mathrm{B}}_j(v)$. If $u \in R^{\mathrm{B}}_j(v)$, then by condition **T1** it holds that $\xi \in L(u)$ and, by the induction hypothesis, $(\mathcal{M}, u) \vDash \xi$. Otherwise, if $u \in \bar{R}^{\mathrm{B}}_j(v) \setminus R^{\mathrm{B}}_j(v)$, then it must be that $u = v$ and $R^{\mathrm{B}}_j(v) = \varnothing$. Thus, by condition **TD** it holds that $\xi \in L(u)$ and by the induction hypothesis $(\mathcal{M}, u) \vDash \xi$. If none of the above holds, then v and u must be R^{B}_j-connected with some sequence of states $s_0, \ldots, s_m$ and there must be s, such that $s_m \in R^{\mathrm{B}}_j(s)$. By conditions **T4** and **T5** and by simple induction on the length of the sequence of states $s_0, \ldots, s_m$, it can be shown that $[\mathrm{B}]_j\xi \in L(s)$. Then, by condition **T1**, it holds that $\xi \in L(u)$ and, by the induction hypothesis, $(\mathcal{M}, u) \vDash \xi$. Hence $(\mathcal{M}, v) \vDash [\mathrm{B}]_j\xi$.

Assume that $\psi = [\mathrm{G}]_j\xi$ and $[\mathrm{G}]_j\xi \in L(v)$. Take any $u \in R'^{\mathrm{G}}_j(v)$. If $u \in R^{\mathrm{G}}_j(v)$, then, by condition **T1** it holds that $\xi \in L(u)$ and, by the induction hypothesis, $(\mathcal{M}, u) \vDash \xi$. Otherwise there must be a state t and a sequence of states $s_0, \ldots, s_m$ such that v and t are R^{B}_j-connected with the sequence and $u \in R^{\mathrm{G}}_j(t)$. By conditions **TBG4** and **TBG5** and simple induction on the length of the sequence of states $s_0, \ldots, s_m$ it can be shown that $[\mathrm{G}]_j\xi \in L(t)$. Then, by condition **T1**, it holds that $\xi \in L(u)$ and, by the induction hypothesis, $(\mathcal{M}, u) \vDash \xi$. Hence $(\mathcal{M}, v) \vDash [\mathrm{G}]_j\xi$.

Assume that $\psi = [\mathrm{I}]_j\xi$ and $[\mathrm{I}]_j\xi \in L(v)$. Take any $u \in R'^{\mathrm{I}}_j(v)$. If $u \in R^{\mathrm{I}}_j(v)$, then, by condition **T1** it holds that $\xi \in L(u)$ and, by the induction hypothesis, $(\mathcal{M}, u) \vDash \xi$. Otherwise, if $u \in R^{\mathrm{G}}_j(v)$, then, by condition **TIG** it holds that $\xi \in L(u)$ and, by the induction hypothesis,

$(\mathcal{M}, u) \models \xi$. Otherwise, if $u \in \bar{R}^{\mathrm{I}}_j(v)$, then it must be that $u = v$ and $R^{\mathrm{I}}_j(v) = \varnothing$. Thus, by condition **TD** it holds that $\xi \in L(u)$ and, by the induction hypothesis, $(\mathcal{M}, u) \models \xi$. If none of the above holds, then there must be a state t and a sequence of states $s_0, \ldots, s_m$ such that v and t are R^{B}_j-connected with the sequence and either $u \in R^{\mathrm{G}}_j(t)$ or $u \in R^{\mathrm{I}}_j(t)$, or $t = u$ and $R^{\mathrm{I}}_j(u) = \varnothing$. By conditions **TBI4** and **TBI5** and by simple induction on the length of the sequence of states $s_0, \ldots, s_m$, it can be shown that $[\mathrm{I}]_j\xi \in L(t)$. If $u \in R^{\mathrm{G}}_j(t)$, then, by condition **TIG**, it holds that $\xi \in L(u)$. If $u \in R^{\mathrm{I}}_j(t)$, then, by condition **T1**, it holds that $\xi \in L(u)$. If $t = u$ and $R^{\mathrm{I}}_j = \varnothing$, then, by condition **TD**, $\xi \in L(u)$. Since $\xi \in L(u)$, so, by the induction hypothesis, $(\mathcal{M}, u) \models \xi$. Hence $(\mathcal{M}, v) \models [\mathrm{I}]_j\xi$.

Assume that $\psi = \neg[O]_j\xi$ (where $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$, and $\neg[O]_j\xi \in L(v)$. Then, by condition **T2**, there must exist $u \in R^{O}_j(v)$ such that $\sim\xi \in L(u)$. Since $R^{O}_j \subseteq R'^{O}_j$, so $u \in R'^{O}_j(v)$ and, by the induction hypothesis, $(\mathcal{M}, u) \models \sim\xi$. Hence $(\mathcal{M}, u) \not\models \xi$ and $(\mathcal{M}, u) \models \neg[O]_j\xi$.

This shows that for any $\psi \in \mathcal{L}^{\mathrm{Tind}}$ and $v \in W$, $\psi \in L(v)$ implies $(\mathcal{M}, v) \models \psi$, and, in particular, $\varphi \in L(w)$ implies $(\mathcal{M}, w) \models \varphi$, that is φ is satisfiable. □

4.2.2 Algorithm for deciding TEAMLOG$^{\mathbf{ind}}$ satisfiability

In this section we present an algorithm for checking TEAMLOG satisfiability of a formula $\varphi \in \mathcal{L}^{\mathrm{Tind}}$. The algorithm extends algorithms presented in [54] designed for checking satisfiability for multimodal logics generated by different combinations of axioms from **K**, **T**, **D**, **4** and **5**. Generally speaking the algorithm below and the algorithms presented in [54] try to construct a *pre-tableau* – a tree-like structure that forms the basis for a modal tableau for φ associated with given multimodal logic.

Before we formally define this notion let us give a brief general introduction first to explain how it is connected with the notion of modal tableau. A pre-tableau constits of nodes connected with a successor relation. Each node can have zero or more successors and each of them has zero or one predecessor. There is at most one node in the pre-tableau that has no predecessors and it is called a *root of the pre-tableau.* Each node is labelled with a set of formulas. The nodes of a pre-tableau can be divided into two groups: *internal nodes* and *states.* Successors of states correspond to accessibility relations and are created for formulas of the form $\neg\Box\psi$ in the label of states and, in the case of modal operators with which axiom **D** is associated, for formulas of the form $\Box\psi$ in the label of states. To construct a modal tableau based on a given pre-tableau, a subset of states of the pre-tableau is selected and the accessibility relations are constructed on the basis of successor relations for states. Labels of states must be propositional tableaux (possibly satisfying some additional requirements). Internal nodes correspond to subsequent steps of constructing labels of states.

Now we turn to defining a pre-tableau for TEAMLOG$^{\mathrm{ind}}$. Because axioms **5**, **BG5** and **BI5** are associated with operators $[\mathrm{B}]_j$, additional requirements need to be put on labels of states.[1]

Definition 10 ([B]-expanded tableau)**.** *A* [B]-expanded tableau *is a propositional tableau $\mathcal{T}$ such that for all $j \in \mathcal{A}$:*

4. If $[\mathrm{B}]_j\varphi \in \neg\mathcal{T}$ and $[O]_j\psi \in \neg\mathrm{PT}(\varphi)$, where $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$, then $[O]_j\psi \in \neg\mathcal{T}$.

Similarly to the case of propositional tableau above, a [B]-*expanded tableau for a formula* φ is a minimal [B]-expanded tableau $\mathcal{T}$ such that $\varphi \in \mathcal{T}$.

[1] In [41] we used a different extension of propositional tableau (inspired by [54]) requiring labels of states to be propositional tableaux closed under subformulas. A more subtle version used here is needed in Chapter 6. We decided to introduce it at this point, to be able to extend the algorithm gradually in further chapters.

We also define the following notion. Let $\varphi \in \mathcal{L}^{\text{Tind}}$ and let $\text{OT}_{[\text{B}]}(\varphi)$ be defined inductively as follows:

1. $\text{OT}_{[\text{B}]}(p) = \{p\}$, where $p \in \mathcal{P}$,
2. $\text{OT}_{[\text{B}]}(\neg\psi) = \{\neg\psi\} \cup \text{OT}_{[\text{B}]}(\psi)$,
3. $\text{OT}_{[\text{B}]}(\psi_1 \wedge \psi_2) = \text{OT}_{[\text{B}]}(\psi_1) \cup \text{OT}_{[\text{B}]}(\psi_2)$,
4. $\text{OT}_{[\text{B}]}([O]_j\psi) = \{[O]_j\psi\}$, where $O \in \{\text{G}, \text{I}\}$,
5. $\text{OT}_{[\text{B}]}([\text{B}]_j\psi) = \{[\text{B}]_j\psi\} \cup \bigcup_{O\in\{\text{B},\text{G},\text{I}\}} \bigcup_{[O]_j\xi\in\neg\text{PT}(\psi)} \text{OT}_{[\text{B}]}([O]_j\xi)$.

The set $\text{OT}_{[\text{B}]}(\varphi)$ extends $\text{PT}(\varphi)$ by adding to it properly selected subformulas of formulas starting with modal operators $[\text{B}]_j$. It is easy to see that every [B]-expanded tableau for φ is a maximal consistent subset of $\neg\text{OT}_{[\text{B}]}(\varphi)$.

Below we give a formal definition of pre-tableau associated with $\text{TEAMLOG}^{\text{ind}}$.

Definition 11 (Pre-tableau)**.** *A* pre-tableau based on formula φ *is a tuple*

$$(N, \text{root}, succ, \{R_j^O\text{-}succ : j \in \mathcal{A},\ O \in \{\text{B}, \text{G}, \text{I}\}\}, L),$$

such that

- N *is the set of* nodes.
- $\text{root} \in N$ *is the* root *of the pre-tableau.*
- L *is the* labelling function *associating with each node* $n \in N$ *a set* $L(n)$ *of formulas such that* $L(n) \subseteq \neg\text{Sub}(\varphi)$*. If* $L(n)$ *is a* [B]*-expanded tableau, then* n *is called a* state*. The set of states is denoted by* S*. Any node which is not a state is called an* internal node.
- $succ \subseteq N \times N$ *is a* successor relation*. If* $n \in succ(m)$*, then* n *is called a* successor *of* m *and* m *is called the* predecessor *of* n*.*
- $R_j^O\text{-}succ \subseteq S \times N$*, where* $O \in \{\text{B}, \text{G}, \text{I}\}$ *and* $j \in \mathcal{A}$*, are* R_j^O successor relations *between states and nodes, associated with modal operators.*

Having defined the notion of a pre-tableau we introduce several additional notions that will be useful in describing it. A sequence of nodes $n_0, \ldots, n_k$ in a pre-tableau such that $k > 0$ and for any $0 < j \leq k$, $(n_{j-1}, n_j) \in succ \cup \bigcup_{O\in\{\text{B},\text{G},\text{I}\}} \bigcup_{j\in\mathcal{A}} R_j^O\text{-succ}$ is called a *path* between n_0 and n_k. The length of the path is the number of elements in the sequence, not counting the first element. Nodes n and m are *connected* if there is a path between them. Node m is called a *descendant* of n in this case. A node n that does not have successors of any kind, i.e. $succ(n) \cup \bigcup_{O\in\{\text{B},\text{G},\text{I}\}} \bigcup_{j\in\mathcal{A}} R_j^O\text{-succ}(n) = \varnothing$, is called a *leaf.*

Given a node $n \neq \text{root}$ such that root and n are connected, *height* of n is the length of the shortest path between root and n. Height of the root is 0. Any node on the path between root and n, excluding n, is called an *ancestor* of n. Given a state $s \neq \text{root}$ such that root and s are connected, *state height* $\text{sheight}(s)$ of s is the number of states on the path from the root to s, excluding s. If the root is a state, then its state height is 0.

The following relations on states can be defined on the basis of the relations *succ* and R_j^O-succ, for $O \in \{\text{B}, \text{G}, \text{I}\}$, $j \in \mathcal{A}$ and $G \in \text{P}(\mathcal{A}) \setminus \{\varnothing\}$:[2]

[2]We will use capital letters in names of the relations between states and small letters in names of the relations between nodes. So for example R_j^O-succ is a relation between a state and a node and R_j^O-Succ is a relation between states.

- R^O_j-Succ $\subseteq S \times S$, the relation of being a R^O_j-*Successor*, $t \in R^O_j$-Succ(s) if there is a path $n_0, \ldots, n_m$ such that $n_0 = s$, $n_m = t$, $n_1 \in R^O_j$-succ(n_0) and s and t are the only states on the path. Symmetrically, the relation R^O_j-Prec $\subseteq S \times S$ of being R^O_j-*Predecessor* is defined.
- R^O_G-Succ $= \bigcup_{j \in G} R^O_j$-Succ, the relation of being a R^O_G-*Successor*. Symmetrically the relation R^O_G-Prec of being a R^O_G-*Predecessor* is defined.
- R^O-Succ $= R^O_{\mathcal{A}}$-Succ, the relation of being a R^O-*Successor*. Symmetrically the relation R^O-Prec of being a R^O-*Predecessor* is defined.

Given a set of formulas Φ, $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $j \in \mathcal{A}$, we define the following sets of formulas

$$\begin{aligned}
\Phi/[O]_j &= \{\psi : [O]_j\psi \in \Phi\},\\
\Phi \sqcap [O]_j &= \{[O]_j\psi : [O]_j\psi \in \Phi\},\\
\Phi \sqcap \neg[O]_j &= \{\neg[O]_j\psi : \neg[O]_j\psi \in \Phi\},\\
\Phi \sqcap j &= (\Phi \sqcap [\mathrm{B}]_j) \cup (\Phi \sqcap [\mathrm{G}]_j) \cup (\Phi \sqcap [\mathrm{I}]_j) \cup\\
&\quad (\Phi \sqcap \neg[\mathrm{B}]_j) \cup (\Phi \sqcap \neg[\mathrm{G}]_j) \cup (\Phi \sqcap \neg[\mathrm{I}]_j).
\end{aligned}$$

The following set of formulas will be used in the algorithm to define labels of the newly created successors of a state ($O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $j \in \mathcal{A}$)

$$\Phi^{\neg[O]_j}(\psi) = \{\sim\psi\} \cup \Phi^{[O]_j}.$$

The definition of set $\Phi^{[O]_j}$ depends on the axioms associated with $[O]$:

$$\begin{aligned}
\Phi^{[\mathrm{I}]_j} &= \Phi/[\mathrm{I}]_j,\\
\Phi^{[\mathrm{G}]_j} &= (\Phi/[\mathrm{G}]_j) \cup \Phi^{[\mathrm{I}]_j}\\
\Phi^{[\mathrm{B}]_j} &= (\Phi/[\mathrm{B}]_j) \cup (\Phi \sqcap j).
\end{aligned}$$

Given a state s and its label $L(s)$, we will write $L^{[O]_j}(s)$ to denote $(L(s))^{[O]_j}$ and $L^{\neg[O]_j}(s, \psi)$ to denote $(L(s))^{\neg[O]_j}(\psi)$. The following lemma will be useful.

Lemma 4.3. *Let Φ be a set of formulas, $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $j \in \mathcal{A}$. Then for any* TeamLog *model $\mathcal{M}$ and any world v in it such that $(\mathcal{M}, v) \vDash \bigwedge \Phi$, for any $u \in R^{[O]_j}(v)$ it holds that $(\mathcal{M}, u) \vDash \bigwedge \Phi^{[O]_j}$.*

Proof. Suppose that $(\mathcal{M}, v) \vDash \bigwedge \Phi$ and take any $u \in R^{[O]_j}(v)$. We will show first that for any $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $j \in \mathcal{A}$ it holds that $(\mathcal{M}, u) \vDash \bigwedge(\Phi/[O]_j)$. By definition of $\Phi/[O]_j$, for any formula $\psi \in \Phi/[O]_j$ there is a formula $[O]_j\psi \in \Phi$ and since $(\mathcal{M}, v) \vDash \bigwedge \Phi$, so $(\mathcal{M}, v) \vDash [O]_j\psi$. Thus $(\mathcal{M}, u) \vDash \psi$ and since this holds for all $\psi \in \Phi/[O]_j$, so $(\mathcal{M}, u) \vDash \bigwedge(\Phi/[O]_j)$.

Since $(\mathcal{M}, u) \vDash \bigwedge(\Phi/[\mathrm{I}]_j)$, for all $u \in R^{[\mathrm{I}]_j}(v)$, so $(\mathcal{M}, u) \vDash \bigwedge \Phi^{[\mathrm{I}]_j}$. Moreover, since for all $u \in R^{[\mathrm{G}]_j}(v)$ it holds that $(\mathcal{M}, u) \vDash \bigwedge(\Phi/[\mathrm{G}]_j)$, for all $u \in R^{[\mathrm{I}]_j}(v)$ it holds that $(\mathcal{M}, u) \vDash \bigwedge \Phi^{[\mathrm{I}]_j}$ and $R^{[\mathrm{G}]_j}(v) \subseteq R^{[\mathrm{I}]_j}(v)$, so $(\mathcal{M}, u) \vDash \bigwedge \Phi^{[\mathrm{G}]_j}$. For $O = \mathrm{B}$ take any $u \in R^{[\mathrm{B}]_j}(v)$. Take any formula of the form $[O]_j\psi \in \Phi$ with $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$. Then $(\mathcal{M}, v) \vDash [O]_j\psi$. Moreover, by transitivity of R^{B}_j and generalized transitivity of R^{G}_j and R^{I}_j with R^{B}_j, it holds that $R^O_j(u) \subseteq R^O_j(v)$. Thus for any $t \in R^O_j(v)$ we have $(\mathcal{M}, t) \vDash \psi$ and, consequently, $(\mathcal{M}, u) \vDash [O]_j\psi$. Now take any formula of the form $\neg[O]_j\psi \in \Phi$ with $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$. Then $(\mathcal{M}, v) \vDash \neg[O]_j\psi$ and there exists $t \in R^O_j(v)$ such that $(\mathcal{M}, t) \vDash \neg\psi$. Moreover, by Euclideanity of R^{B}_j and generalized Euclideanity of R^{G}_j and R^{I}_j with R^{B}_j, it holds that $R^O_j(v) \subseteq R^O_j(u)$. Thus $t \in R^O_j(u)$ and, consequently, $(\mathcal{M}, u) \vDash \neg[O]_j\psi$. This shows that $(\mathcal{M}, u) \vDash \bigwedge(\Phi \sqcap j)$ and since $(\mathcal{M}, u) \vDash \bigwedge(\Phi/[\mathrm{B}]_j)$ so $(\mathcal{M}, u) \vDash \bigwedge \Phi^{[\mathrm{B}]_j}$. $\square$

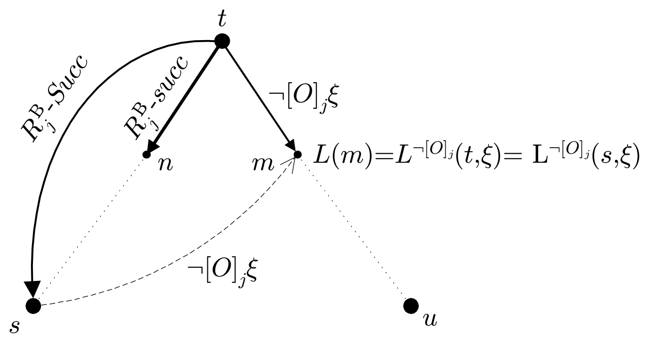

Figure 4.1: Creation of $\neg[O]_j\xi$-successor of s is blocked by its R_j^{B}-Predecessor t. Dotted lines depict sequences of internal nodes (these sequences can be empty, in which case the starting node coincides with the ending state). As we will show below (Lemma 4.4) creation of $\neg[O]_j\xi$-successor m of t cannot be blocked, as t cannot be a R_j^{B}-Predecessor. Situation in the case of $[O]_j\xi$-successors is analogous.

Algorithm 4.1, for checking TeamLog satisfiability of a formula φ, works in two general stages. First it attempts to construct a pre-tableau based on φ and then it marks nodes of the constructed pre-tableau, either as `sat` or `unsat`. The satisfiability of φ is decided on the basis of how the root of the pre-tableau is marked.

The stage of pre-tableau construction consists of two steps. Each of the steps is realized as long as possible, then execution of the algorithm moves to the next step. If anything changes during the current step, execution of the algorithm moves back to the first of the steps. Firstly, leaves of the pre-tableau are selected that are not states. Then a formula in the label of a leaf violating some condition preventing the leaf from being a state is selected. Such a formula is called a *witness* to violation of the condition. Next a new node is added to the pre-tableau, as a successor of the considered leaf. Label of this leaf is modified so that the condition violation is removed. Thus nodes created during this stage of the algorithm can be seen as substeps of states creation. The full procedure is presented in Procedure 4.2.

In the second step, R_j^{B}-, R_j^{G}- and R_j^{I}-successors are created for a selected leaf states and for all formulas in the label of this state, for which a successor creation is possible. This is described in Procedures 3, 4 and 5. An R_j^{O}-successor of state s created for a formula $\psi \in L(s)$ is called a *ψ-successor* of s. Notice that before a successor of a state for a given formula is created, it is checked if there is no R_j^{B}-Predecessor of the state for which a successor with the same label could have been created. Thus creation of a successor of a state can be blocked by its predecessor. This situation is illustrated in Figure 4.1.

Notice also that whenever a successor of a node is added during execution of the algorithm, it is added as a successor of a leaf. Thus any pre-tableau constructed by the algorithm has a tree-like structure.

Now we will show that for any input formula $\varphi \in \mathcal{L}^{\mathrm{Tind}}$, the algorithm for checking the satisfiability stops. We start with the following lemma that will be useful for proving the stop property.

Lemma 4.4. *Let s and $t \in R_j^{\mathrm{B}}$-Succ(s) be states of the pre-tableau constructed by Algorithm 4.1 for some input formula $\varphi \in \mathcal{L}^{\mathrm{Tind}}$. Then the following hold for $O \in \{\mathrm{B},\mathrm{G},\mathrm{I}\}$:*

1. $\neg[O]_j\xi \in L(s)$ *iff* $\neg[O]_j\xi \in L(t)$.
2. $[O]_j\xi \in L(s)$ *iff* $[O]_j\xi \in L(t)$.

Algorithm 4.1: DecideSatisfiability

```
Input: a formula φ
Output: a decision whether φ is satisfiable or not
/* Pre-tableau construction                                                  */
Construct a pre-tableau consisting of single node root, with L(root) = {φ} and all
successor relations being empty;
repeat
    Let Z be the set of all leaves of the pre-tableau with labelling sets that are not
    blatantly inconsistent;
    if there is n ∈ Z such that n is not a state and ψ ∈ L(n) is a witness to that then
        FormState(n,ψ);
    else if there is s ∈ Z then
        foreach ψ ∈ L(s) do
            CreateSuccessorsB(s,ψ);
            CreateSuccessorsG(s,ψ);
            CreateSuccessorsI(s,ψ);
until no change occurred;
/* Marking nodes and deciding satisfiability                                 */
repeat
    MarkNodes;
until no new node marked;
if root is marked sat then
    return sat;
else
    return unsat;
```

Procedure 4.2: FormState

```
Input: a node n and a formula ψ
if ψ is of the form ¬¬ξ then
    Create a successor m of n and set L(m) := L(n) ∪ {ξ};
else if ψ is of the form ξ1 ∧ ξ2 then
    Create a successor m of n and set L(m) := L(n) ∪ {ξ1, ξ2};
else if ψ is of the form ¬(ξ1 ∧ ξ2) then
    Create three successors m1, m2 and m3 of n and set L(m1) := L(n) ∪ {∼ξ1, ξ2},
    L(m2) := L(n) ∪ {ξ1, ∼ξ2} and L(m3) := L(n) ∪ {∼ξ1, ∼ξ2};
else if ψ is of the form [B]_j ξ or of the form ¬[B]_j ξ with a formula [O]_j ζ ∈ ¬PT(ξ) such
that [O]_j ζ ∉ ¬L(n) then
    Create two successors m1 and m2 of n and set L(m1) := L(n) ∪ {[O]_j ζ} and
    L(m2) := L(n) ∪ {¬[O]_j ζ};
```

Procedure 4.3: CreateSuccessorsB

Input: a state s and a formula $\psi \in L(s)$

if ψ *is of the form* $\neg[\mathrm{B}]_j\xi$ **then**
 If there is no R_j^{B}-Predecessor t of s such that $L^{\neg[\mathrm{B}]_j}(t,\xi) = L^{\neg[\mathrm{B}]_j}(s,\xi)$, then create an R_j^{B}-successor u of s with $L(u) = L^{\neg[\mathrm{B}]_j}(s,\xi)$;

else if ψ *is of the form* $[\mathrm{B}]_j\xi$ *and there are no formulas of the form* $\neg[\mathrm{B}]_j\zeta \in L(s)$ **then**
 If there is no R_j^{B}-Predecessor t of s such that $L^{[\mathrm{B}]_j}(t) = L^{[\mathrm{B}]_j}(s)$, then create an R_j^{B}-successor u of s with $L(u) = L^{[\mathrm{B}]_j}(s)$;

Procedure 4.4: CreateSuccessorsG

Input: a state s and a formula $\psi \in L(s)$

if ψ *is of the form* $\neg[\mathrm{G}]_j\xi$ **then**
 If there is no R_j^{B}-predecessor state t of s such that $L^{\neg[\mathrm{G}]_j}(t,\xi) = L^{\neg[\mathrm{G}]_j}(s,\xi)$, then create an R_j^{G}-successor u of s with $L(u) = L^{\neg[\mathrm{G}]_j}(s,\xi)$;

Procedure 4.5: CreateSuccessorsI

Input: a state s and a formula $\psi \in L(s)$

if ψ *is of the form* $\neg[\mathrm{I}]_j\xi$ **then**
 If there is no R_j^{B}-Predecessor t of s such that $L^{\neg[\mathrm{I}]_j}(t,\xi) = L^{\neg[\mathrm{I}]_j}(s,\xi)$, then create an R_j^{I}-successor u of s with $L(u) = L^{\neg[\mathrm{I}]_j}(s,\xi)$;

else if ψ *is of the form* $[\mathrm{I}]_j\xi$ *and there are no formulas of the form* $\neg[\mathrm{I}]_j\zeta \in L(s)$ **then**
 If there is no R_j^{B}-Predecessor t of s such that $L^{[\mathrm{I}]_j}(t) = L^{[\mathrm{I}]_j}(s)$, then create an R_j^{I}-successor u of s with $L(u) = L^{[\mathrm{I}]_j}(s)$;

Procedure 4.6: MarkNodes

```
foreach node n of the pre-tableau do
    if n an unmarked state then
        if n has a successor marked unsat then
            Mark n unsat;
        else if n does not have an unmarked successor then
            Mark n sat;
    else if n is an unmarked internal node then
        if L(n) is blatantly inconsistent or all its successors are marked unsat then
            Mark n unsat;
        else if n has at least one successor marked sat then
            Mark n sat;
```

3. $L^{[O]_j}(s) = L^{[O]_j}(t)$.

4. $L^{\neg[O]_j}(s,\xi) = L^{\neg[O]_j}(t,\xi)$.

Proof. For points 1 and 2 notice first that if $\neg[O]_j\xi \in L(s)$, then $\neg[O]_j\xi \in L^{[\mathrm{B}]_j}(s)$ and $\neg[O]_j\xi \in L(t)$, as t is an R_j^{B}-Successor of s. Similarly, if $[O]_j\xi \in L(s)$, then $[O]_j\xi \in L(t)$, as t is an R_j^{B}-Successor of s. This shows the left to right implications of the two points.

For the right to left implications, we will show first that if $[O]_j\xi \in \neg L(t)$, then it holds that $[O]_j\xi \in \neg L(s)$. We will use induction on modal depth of $[O]_j\xi$. Suppose that modal depth of $[O]_j\xi$ is maximal in $\neg L(t)$ (of all the formulas of the form $[O]_j\zeta \in \neg L(t)$ with O being either B, G or I). Then it must be that $[O]_j\xi \in \neg L(s)$, as otherwise there would have to be a formula $[\mathrm{B}]_j\psi \in \neg L(s)$ such that $[O]_j\xi \in \neg\mathrm{PT}(\psi)$ and, from the left to right implications shown above, it would hold that $[\mathrm{B}]_j\psi \in \neg L(t)$, which would contradict the assumption of maximality of modal depth of $[O]_j\xi$. For the induction step suppose that modal depth of $[O]_j\xi$ is not maximal in $L(t)$. Then either there is a formula $[\mathrm{B}]_j\psi \in \neg L(s)$ such that $[O]_j\xi \in \neg\mathrm{PT}(\psi)$ or there is a formula $[\mathrm{B}]_j\psi \in \neg L(t)$ such that $[O]_j\xi \in \neg\mathrm{PT}(\psi)$. If the second case holds, then, by the induction hypothesis, it must hold that $[\mathrm{B}]_j\psi \in \neg L(s)$. Since $L(s)$ is a [B]-expanded tableau, so it must hold that $[O]_j\xi \in \neg L(s)$.

Now, if $\neg[O]_j\xi \in L(t)$, then $[O]_j\xi \in \neg L(s)$ and it must hold that $\neg[O]_j\xi \in L(s)$, as otherwise it would be $[O]_j\xi \in L(s)$ and $[O]_j\xi \in L(t)$. Since $\neg[O]_j\xi \in L(t)$, so this would contradict the assumption that t is a state and cannot be blatantly inconsistent. If $[O]_j\xi \in L(t)$, then it must be that $[O]_j\xi \in L(s)$ by analogous arguments. Thus we have shown that points 1 and 2 hold.

For point 3, three cases of $O \in \{\mathrm{B},\mathrm{G},\mathrm{I}\}$ will be considered separately. Firstly assume that $O = \mathrm{B}$. Let $\psi \in L^{[\mathrm{B}]_j}(s)$. Then $\psi \in L(t)$, as t is R_j^{B}-Successor of s. Moreover it must be that either ψ is of one of the forms $[O]_j\xi$ or $\neg[O]_j\xi$, or $[\mathrm{B}]_j\psi \in L(s)$. In the first case $\psi \in L^{[\mathrm{B}]_j}(t)$, by definition of $L^{[\mathrm{B}]_j}(\cdot)$. In the second case $[\mathrm{B}]_j\psi \in L(t)$, as $[\mathrm{B}]_j\psi \in L^{[\mathrm{B}]_j}(s)$ and so $\psi \in L^{[\mathrm{B}]_j}(t)$. Now let $\psi \in L^{[\mathrm{B}]_j}(t)$. Then it must be that either ψ is of one of the forms $[O]_j\xi$ or $\neg[O]_j\xi$, and $\psi \in L(t)$, or $[\mathrm{B}]_j\psi \in L(t)$. If the first case holds, then $\psi \in L(s)$ follows immediately from points 1 and 2. If the second case holds, then, by point 2, $[\mathrm{B}]_j\psi \in L(s)$ and so $\psi \in L^{[\mathrm{B}]_j}(s)$.

Secondly assume that $O = \mathrm{I}$. Let $\psi \in L^{[\mathrm{I}]_j}(s)$. Then it must be that $[\mathrm{I}]_j\psi \in L(s)$ and, consequently, $[\mathrm{I}]_j\psi \in L(t)$, as t is an R_j^{B}-Successor of s. Thus $\psi \in L^{[\mathrm{I}]_j}(t)$. Now let $\psi \in L^{[\mathrm{I}]_j}(t)$. Then it must be that $[\mathrm{I}]_j\psi \in L(t)$ and, by point 2, it must be that $[\mathrm{I}]_j\psi \in L(s)$. Hence $\psi \in L^{[\mathrm{I}]_j}(s)$.

Lastly assume that $O = \mathrm{G}$. Let $\psi \in L^{[\mathrm{G}]_j}(s)$. Then it must be that $[\mathrm{G}]_j\psi \in L(s)$ or $[\mathrm{I}]_j\psi \in L(s)$ and, consequently, $[\mathrm{G}]_j\psi \in L(t)$ or $[\mathrm{I}]_j\psi \in L(t)$, as t is an R_j^{B}-Successor of s. Thus $\psi \in L^{[\mathrm{G}]_j}(t)$. Now let $\psi \in L^{[\mathrm{G}]_j}(t)$. Then it must be that $[\mathrm{G}]_j\psi \in L(t)$ or $[\mathrm{I}]_j\psi \in L(t)$ and, by point 2, it must be that $[\mathrm{I}]_j\psi \in L(s)$ or $[\mathrm{G}]_j\psi \in L(s)$. Hence $\psi \in L^{[\mathrm{G}]_j}(s)$. This shows that point 3 holds.

For point 4, recall that $L^{\neg[O]_j}(v,\xi) = \{\sim\xi\} \cup L^{[O]_j}(v)$. By point 3, $L^{[O]_j}(s) = L^{[O]_j}(t)$ and, by point 3, $\neg[O]_j\xi \in L(s)$ iff $\neg[O]_j\xi \in L(t)$. This shows that point 4 holds. $\square$

Lemma 4.5. *The maximal state height of a state of the pre-tableau constructed by Algorithm 4.1 for input* $\varphi \in \mathcal{L}^{\mathrm{Tind}}$ *is* $\leq 2\mathrm{dep}(\varphi)$ *and the maximal height of a node of the pre-tableau is* $\leq 2\mathrm{dep}(\varphi)|\varphi|$.

Proof. For any node n in the pre-tableau constructed by the algorithm $|L(n)| \leq 2|\varphi|$, as $L(n) \subseteq \neg\mathrm{Sub}(\varphi)$ (if $L(s)$ is not blatantly inconsistent then $|L(s)| \leq |\varphi|$). Thus the path between any subsequent states s and t can contain at most $|\varphi| - 1$ internal nodes.

If s and t are states such that t is an R^{G}_j-Successor or an R^{I}_j-Successor of s, then $\mathrm{dep}(L(t)) < \mathrm{dep}(L(s))$. If t is an R^{B}_j-Successor of s and u is an R^{B}_k-Successor of t, where $j \neq k$, then $\mathrm{dep}(L(u)) < \mathrm{dep}(L(s))$. If t is an R^{B}_j-Successor of s then, by Lemma 4.4, t cannot have an R^{B}_j-Successor. Thus, for any successor node u of t, $\mathrm{dep}(L(s)) < \mathrm{dep}(L(u))$.

All the arguments above show that the maximal height of a node of the pre-tableau constructed by the algorithm must be $\leq 2\mathrm{dep}(\varphi)|\varphi|$ and the maximal state height of a state of the pre-tableau must be $\leq 2\mathrm{dep}(\varphi)$. □

The following is an immediate consequence of Lemma 4.5.

Lemma 4.6. *For any input formula $\varphi \in \mathcal{L}^{\mathrm{Tind}}$ Algorithm 4.1 terminates.*

Proof. Since the size of pre-tableau constructed by the algorithm is bounded, so the stage of pre-tableau construction ends. The stage of marking nodes ends as well, since in every loop of the stage at least one node is marked and the loop is executed as long as any new node can be marked. □

Before showing the validity of Algorithm 4.1, we state the following lemma which will be useful.

Lemma 4.7. *Let n be an internal node in the pre-tableau constructed by Algorithm 4.1 for some input formula $\varphi \in \mathcal{L}^{\mathrm{Tind}}$. For any Kripke model $\mathcal{M}$ and a world v in it such that $(\mathcal{M}, v) \vDash \bigwedge L(n)$ there exists a successor m of n such that $(\mathcal{M}, v) \vDash \bigwedge L(m)$.*

Proof. Let $\mathcal{M}$ be a Kripke model with a world v in it such that $(\mathcal{M}, v) \vDash \bigwedge L(n)$. Suppose that a successor m of n was created for a witness of the form $\neg\neg\xi$. Then it holds that $(\mathcal{M}, v) \vDash \neg\neg\xi$ and so $(\mathcal{M}, v) \nvDash \neg\xi$. If $(\mathcal{M}, v) \nvDash \xi$ then we have a contradiction, so it must be $(\mathcal{M}, v) \vDash \xi$. Thus $(\mathcal{M}, v) \vDash \bigwedge(L(n) \cup \{\xi\})$, that is $(\mathcal{M}, v) \vDash \bigwedge L(m)$.

Now suppose that a successor m of n was created for a witness of the form $\xi_1 \wedge \xi_2$. Then it holds that $(\mathcal{M}, v) \vDash \xi_1 \wedge \xi_2$ and so $(\mathcal{M}, v) \vDash \xi_1$ and $(\mathcal{M}, v) \vDash \xi_2$. Thus $(\mathcal{M}, v) \vDash \bigwedge(L(n) \cup \{\xi_1, \xi_2\})$, that is $(\mathcal{M}, v) \vDash \bigwedge L(m)$.

Next, suppose that successors m_1 and m_2 of n were created for a witness of the form $\neg(\xi_1 \wedge \xi_2)$. Thus $(\mathcal{M}, v) \nvDash \xi_1 \wedge \xi_2$. Suppose that $(\mathcal{M}, v) \vDash \xi_1$ and $(\mathcal{M}, v) \vDash \xi_2$. Then $(\mathcal{M}, v) \vDash \xi_1 \wedge \xi_2$ and we get a contradiction. Hence it must be either $(\mathcal{M}, v) \nvDash \xi_1$ or $(\mathcal{M}, v) \nvDash \xi_2$, and so either $(\mathcal{M}, v) \vDash \bigwedge(L(n) \cup \{\sim\xi_1\})$ or $(\mathcal{M}, v) \vDash \bigwedge(L(n) \cup \{\sim\xi_2\})$. Thus either $(\mathcal{M}, v) \vDash \bigwedge L(m_1)$ or $(\mathcal{M}, v) \vDash \bigwedge L(m_2)$.

Lastly, suppose that successors of n were created during [B]-expanded tableau formation. Since for any formula $\xi \in \mathcal{L}$ it holds that either $(\mathcal{M}, v) \vDash \xi$ or $(\mathcal{M}, v) \vDash \neg\xi$. So either $(\mathcal{M}, v) \vDash \bigwedge(L(n) \cup \{\xi\})$ or $(\mathcal{M}, v) \vDash \bigwedge(L(n) \cup \{\neg\xi\})$. Hence one of the successors of n must be satisfied in $(\mathcal{M}, v)$. □

Now we are ready to prove the validity of the algorithm.

Lemma 4.8. *A formula $\varphi \in \mathcal{L}^{\mathrm{Tind}}$ is satisfiable iff Algorithm 4.1 returns* `sat` *on the input φ.*

Proof. For the left to right implication we start by showing, for any node n of the pre-tableau constructed by the algorithm, that if n is marked `unsat`, then $\bigwedge L(n)$ is unsatisfiable. The proof is by induction on the maximal length of paths from a node to one of its successor leaves. Suppose that n is a leaf. If n is marked `unsat`, then it must be blatantly inconsistent. Thus $\bigwedge L(n)$ cannot be satisfiable. For the induction step, consider a node n which is not a leaf. Let n be an internal node. Since n is marked `unsat`, so all its successors must be marked `unsat` and, by the induction hypothesis, for any successor m of n, $\bigwedge L(m)$ must be unsatisfiable.

Suppose that $\bigwedge L(n)$ is satisfiable. By Lemma 4.7 there exists a successor m of n such that $\bigwedge L(m)$ is satisfiable and we get a contradiction. Thus $\bigwedge L(n)$ must be unsatisfiable in this case.

Let n be a state. Since it is marked `unsat`, so it must have an R_j^O-successor m, with $j \in \mathcal{A}$ and $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$, which is marked `unsat` as well and, by the induction hypothesis, $\bigwedge L(m)$ must be unsatisfiable. Suppose that $L(n)$ is satisfiable and that $(\mathcal{M}, v) \models \bigwedge L(n)$. If m is a $\neg[O]_j\xi$-successor of n, then it must be that $\neg[O]_j\xi \in L(n)$ and $(\mathcal{M}, v) \models \neg[O]_j\xi$. Thus there must be $u \in R_j^{\mathrm{O}}(v)$ such that $(\mathcal{M}, u) \not\models \xi$. Since it also holds, by Lemma 4.3, that $(\mathcal{M}, u) \models \bigwedge L^{[O]_j}(n)$, so $(\mathcal{M}, u) \models \bigwedge(L^{[O]_j}(n) \cup \{\sim\xi\})$, that is $(\mathcal{M}, u) \models \bigwedge L(m)$, which contradicts the assumption that $\bigwedge L(m)$ is not satisfiable. If m is a $[O]_j\xi$-successor of n (which is possible for $O \in \{\mathrm{B}, \mathrm{I}\}$), then it follows from Lemma 4.3, that $(\mathcal{M}, u) \models \bigwedge L^{[O]_j}(n)$, which contradicts the assumption that $\bigwedge L(m)$ is not satisfiable. Hence, if n has an R_j^{O}-successor that is marked `unsat`, then $\bigwedge L(n)$ must be unsatisfiable.

This finishes the inductive proof and shows, for any node n of the pre-tableau constructed by the algorithm, that if n is marked `unsat` then $\bigwedge L(n)$ is unsatisfiable. Now, if the algorithm does not return `sat` on some input φ, then it means that root is marked `unsat` and, by what we just have shown, φ is unsatisfiable. Thus the algorithm must return `sat` for any satisfiable input formula φ.

For the right to left implication, let

$$(N, \mathrm{root}, \mathit{succ}, \{R_j^{\mathrm{O}}\text{-succ} : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, L)$$

be the pre-tableau constructed by the algorithm for the input φ. We will show how to construct, on the basis of this pre-tableau, a TeamLog$^{\mathrm{ind}}$ tableau for φ such that the number of states of the tableau is $\leq |\varphi|^h$, where h is the state height of the pre-tableau. In further part of the proof we will refer to the following set, defined for a given node n:

$$SS(n) = \begin{cases} \{n\}, & \text{if } n \text{ is a state,} \\ \bigcup_{m \in \mathit{succ}(n)} SS(m), & \text{otherwise,} \end{cases}$$

which is the set of states in the subtree of the pre-tableau with the root n that are closest to n. Notice that n is marked `sat` if and only if there exists $s \in SS(n)$ such that s is marked `sat`. Moreover, for any $s \in SS(n)$ it holds that $L(n) \subseteq L(s)$, as labels of successors created during the steps of propositional tableau formation and [B]-expanded tableau formation extend the labels of their predecessors.

Consider a model graph

$$\mathcal{T} = (W, \{R_j^{\mathrm{O}} : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, L|_W),$$

where W is constructed as follows. We start with W consisting of a state marked `sat` from $SS(\texttt{root})$. Then, for each state $w \in W$ whose R^O-Successors were not added to the set yet, we take, for each R^O-successor node n of w, a state $v \in SS(n)$ which is marked `sat`. We proceed like that until leaves of the pre-tableau are reached. Since each state of the pre-tableau has at most $|\varphi|$ R^O-successors (because the number of elements in its label is bounded by $|\varphi|$, as the label of a state is not blatantly inconsistent), so W has $\leq |\varphi|^h$ elements. Labelling function L is like in the pre-tableau, restricted to W and the accessibility relations are defined as follows

- $R_j^{\mathrm{B}} = R_j^{\mathrm{B}}\text{-Succ} \cap W \times W) \cup \{(v, u) \in W \times W : \text{there exists } w \in W \text{ such that } \{v, u\} \subseteq R_j^{\mathrm{B}}\text{-Succ}(w)\}$,
- $R_j^{\mathrm{G}} = (R_j^{\mathrm{G}}\text{-Succ} \cap W \times W) \cup \{(v, u) \in W \times W : \text{there exists } w \in W \text{ such that } v \in R_j^{\mathrm{B}}\text{-Succ}(w) \text{ and } u \in R_j^{\mathrm{G}}\text{-Succ}(w)\}$,

- $R^{\mathrm{I}}_j = (R^{\mathrm{I}}_j\text{-Succ} \cap W \times W) \cup \{(v,u) \in W \times W$: there exists $w \in W$ such that $v \in R^{\mathrm{B}}_j\text{-Succ}(w)$ and $u \in R^{\mathrm{I}}_j\text{-Succ}(w)\}$.

First notice that $W \neq \varnothing$, as root is marked `sat` and so there is $s \in SS(\mathrm{root})$ that is marked `sat`. Moreover there is $s \in W$ such that $\varphi \in L(s)$, as $\varphi \in L(\mathrm{root})$ and $L(\mathrm{root}) \subseteq L(s)$. Since during the step of state formation the sets of formulas labelling added nodes are created by extending labels of their predecessors, so it must be that $\varphi \in L(s)$. Thus it is enough to show that $\mathcal{T}$ is a TeamLog$^{\mathrm{ind}}$ tableau and, by Theorem 4.2, it will follow that φ is satisfiable. We have to show that conditions of TeamLog$^{\mathrm{ind}}$ tableau are satisfied for $\mathcal{T}$. Since all elements of W are states, so they must be [B]-expanded tableaux. All other conditions are shown below.

For condition **T1** let $v \in W$ and take any $u \in W$ such that $u \in R^O_j(v)$, where $O \in \{\mathrm{B},\mathrm{G},\mathrm{I}\}$. If u is an R^O_j-Successor of v, then, by construction of Procedures 3, 4 and 5 of successor creation and the fact that during step of state formation labels of added successors extend labels of their predecessors, it is straightforward to see that $[O]_j\xi \in L(v)$ implies $\xi \in L(u)$. If u is not an R^O_j-Successor of v, then there must be $w \in W$ such that $v \in R^{\mathrm{B}}_j\text{-Succ}(w)$ and $u \in R^O_j\text{-Succ}(w)$. Take any formula $[O]_j\xi \in L(v)$. By Lemma 4.4 it holds that $[O]_j\xi \in L(w)$ and, by what was shown above, $\xi \in L(u)$. Thus condition **T1** is satisfied.

For condition **T2** let $v \in W$ and let $\neg[O]_j\xi \in L(v)$, where $O \in \{\mathrm{B},\mathrm{G},\mathrm{I}\}$. Assume that R^O_j-successor n of v was created with $L(n) = L^{\neg[O]_j}(v,\xi)$ and, in particular, $\sim\xi \in L(n)$. Since v is marked `sat`, so n must be marked `sat` as well and there must be a state $u \in SS(n)$ which is marked `sat` (and so $u \in W$) and which is an R^O_j-Successor of v. Since $\sim\xi \in L(n)$ and $L(n) \subseteq L(u)$, so $\sim\xi \in L(u)$. If a successor of v with label $L^{\neg[O]_j}(v,\xi)$ could not be created, then there must be an R^{B}_j-Predecessor w of v with $\neg[O]_j\xi \in L(w)$. Moreover, an R^O_j-successor n of w with label $L(n) = L^{\neg[O]_j}(v,\xi)$ can be created as, by Lemma 4.4, a pre-tableau constructed by the algorithm cannot have two subsequent R^{B}_j-Successors (see Figure 4.1). Since $v \in W$, so w must be marked `sat` and $w \in W$. Hence, by analogous arguments to those used above, there must be a state $u \in SS(n)$ such that $u \in W$ and $\neg\xi \in L(u)$. Moreover, by construction of relation R^{B}_j it holds that $u \in R^O_j(v)$. Thus condition **T2** is satisfied.

For condition **TD** let $v \in W$ and $[O]_j\xi \in L(v)$, where $O \in \{\mathrm{B},\mathrm{I}\}$. If an R^O_j-successor n of v was created, then there must exists a state $u \in SS(n)$ which is marked `sat` and, consequently, $u \in W$ and $u \in R^O_j(v)$. Moreover, as was shown in the case of condition **T1**, $\xi \in L(u)$. If an R^O_j-successor of v could not be created, then there must be an R^{B}_j-predecessor w of v such that $w \in W$ and such that an R^O_j-successor n was created for it (see Figure 4.1). Moreover, n must be marked `sat` and there must exist $u \in SS(n)$ such that u is marked `sat` as well. Thus $u \in R^O_j(v)$ and, as was shown in case of condition **T1**, $\xi \in L(u)$. Thus condition **TD** is satisfied.

For conditions **T4**, **TG4** and **TI4**, let $v \in W$ and $u \in R^{\mathrm{B}}_j(v)$. If $u \in R^{\mathrm{B}}_j\text{-Succ}(v)$, then, by point 2 of Lemma 4.4, the conditions are satisfied. Otherwise there must be a state $w \in W$ such that $\{v,u\} \subseteq R^{\mathrm{B}}_j\text{-Succ}(w)$ and again, by point 2 of Lemma 4.4, the conditions are satisfied. Conditions **TB5**, **TB5** and **TB5** can be shown by similar arguments, using point 2 of Lemma 4.4.

Finally, for condition **TIG** let $v \in W$ and $u \in R^{\mathrm{G}}_j(v)$. Since $L^{[\mathrm{I}]_j}(v) \subseteq L^{[\mathrm{G}]_j}(v)$, so, by arguments similar to those used for condition **T1**, $[\mathrm{I}]_j\xi \in L(v)$ implies $\xi \in L(u)$. This shows that condition **TIG** is satisfied. Thus we have shown that $\mathcal{T}$ is a TeamLog$^{\mathrm{ind}}$ tableau for φ and that φ is satisfiable. □

The following theorem states lower and upper bounds for complexity of the satisfiability problem for TeamLog$^{\mathrm{ind}}$ logical framework.

Theorem 4.9. *The satisfiability problem for* TeamLog^{ind} *is PSPACE-complete.*

Proof. As we observed already, the problem is PSPACE-hard by Fact 4.1 and the fact that the problem of KD_n satisfiability is PSPACE-hard. To show that the problem is in PSPACE, we have to that Algorithm 4.1 can be run by a deterministic Turing machine using polynomial space with respect to $|\varphi|$. To check the satisfiability of φ a pre-tableau is constructed and the decision with regard to the satisfiability is made on the basis of how the root of this pre-tableau is marked. Since the decision on how each node is marked in this pre-tableau depends on how the descendants of this node are marked, so for deciding how the root node should be marked, the pre-tableau could be traversed in depth first search like manner. By Lemma 4.5, the depth of the pre-tableau constructed by Algorithm 4.1 is $\leq 2\mathrm{dep}(\varphi)$. Hence the algorithm can be run by a deterministic Turing machine using $\mathcal{O}(\mathrm{dep}(\varphi))$ space and so the problem is in PSPACE. □

4.3 The complexity of the satisfiability problem of TeamLog

In this section we show that the satisfiability problem for TeamLog is decidable and is EXPTIME-complete. The results presented in this section have already been publishe in [41]. Like in the case of the results presented in Section 4.2, we decided to make the presentation of these results different to that publication to facilitate the presentation of the results in Chapter 6.

We show first that TeamLog logical framework has the small model property, in the sense that for each satisfiable formula $\varphi \in L$, a satisfying model of size $\mathcal{O}(2^{|\varphi|})$ can be found. From this result we conclude that the satisfiability problem for TeamLog is satisfiable and that it can be solved by a non-deterministic algorithm working in exponential time. Next we propose a deterministic algorithm for checking TeamLog satisfiability that works in exponential time. The idea of the algorithm is based on Pratt's algorithm for checking satisfiability of PDL [91], as presented in [58]. However our presentation is based on a notion of modal tableau defined for TeamLog logical framework. Finally, we show that checking TeamLog satisfiability is EXPTIME-complete, by showing that certain problem related to two person tiling games can be reduced to checking TeamLog satisfiability.

4.3.1 Small model property of TeamLog

To show that TeamLog logical framework has the small model property we will use the filtration technique (see e.g. [10]). The idea behind this method is to, given some model $\mathcal{M}$, construct a "smaller" model by identifying states of $\mathcal{M}$ using a properly constructed equivalence relation. This relation is constructed on the basis of a set of formulas closed for subformulas and satisfying some additional properties.

Definition 12 (Closed set of formulas)**.** *A set of formulas $\Phi \subseteq \mathcal{L}$, closed for subformulas, is* closed *if it satisfies the following, for all $G \subseteq \mathcal{A}$ and $O \in \{\mathrm{B}, \mathrm{I}\}$:*

Cl *if $[O]^+_G\varphi \in \Phi$, then $\{[O]_j[O]^+_G\varphi, [O]_j\varphi : j \in G\} \subseteq \Phi$,*

Given a formula $\varphi \in \mathcal{L}$, we will use $\mathrm{Cl}(\varphi)$ to denote the smallest closed set of formulas containing φ. Similarly, given a set of formulas Φ we will use $\mathrm{Cl}(\Phi)$ to denote the smallest closed set of formulas having Φ as a subset. Let

$$\mathcal{M} = (W, \{R^O_j : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, Val)$$

be a TEAMLOG model, Φ a closed set of formulas, and let $\equiv_f^{\Phi} \subseteq W \times W$ be defined as follows

$$w \equiv_f^{\Phi} v \text{ iff for any } \varphi \in \Phi,\ (M, w) \vDash \varphi \text{ iff } (M, v) \vDash \varphi.$$

It is easy to see that $\equiv_f^{\Phi}$ is an equivalence relation. Let

$$\mathcal{M}_{\Phi}^{f} = (W^f, \{R_j^{Of} : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, Val^f)$$

be defined as follows:

F0 $W^f = W/\equiv_f^{\Phi}$, $Val^f(p, [v]) = Val(p, v)$,

F1 $R_j^{\mathrm{B}f} = \{([v], [u]) :$ for any $[\mathrm{B}]_j\varphi \in \Phi$, $(\mathcal{M}, v) \vDash [\mathrm{B}]_j\varphi$ implies $(\mathcal{M}, u) \vDash \varphi$ and for any $[O]_j\varphi \in \Phi$, $(\mathcal{M}, v) \vDash [O]_j\varphi$ iff $(\mathcal{M}, u) \vDash [O]_j\varphi\}$, where $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$,

F2 $R_j^{\mathrm{G}f} = \{([v], [u]) :$ for any $[\mathrm{G}]_j\varphi \in \Phi$, $(\mathcal{M}, v) \vDash [\mathrm{G}]_j\varphi$ implies $(\mathcal{M}, u) \vDash \varphi$ and for any $[\mathrm{I}]_j\varphi \in \Phi$, $(\mathcal{M}, v) \vDash [\mathrm{I}]_j\varphi$ implies $(\mathcal{M}, u) \vDash \varphi\}$,

F3 $R_j^{\mathrm{I}f} = \{([v], [u]) :$ for any $[\mathrm{I}]_j\varphi \in \Phi$, $(\mathcal{M}, v) \vDash [\mathrm{I}]_j\varphi$ implies $(\mathcal{M}, u) \vDash \varphi\}$.

As we show below, $\mathcal{M}_{\Phi}^{f}$ is a filtration of $\mathcal{M}$ through Φ.

Fact 4.10. *If Φ is a closed set, then, for any $p \in \mathcal{P}$, $j \in \mathcal{A}$ and $G \subseteq \mathcal{A}$,*

1. $W^f = W/\equiv_f^{\Phi}$, $Val^f(p, [v]) = Val(p, v)$,
2. *if* $(v, u) \in R_j^O$, *then* $([v], [u]) \in R_j^{Of}$, *where* $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ *and* $j \in \mathcal{A}$,
3. *if* $(v, u) \in R_G^{O+}$, *then* $([v], [u]) \in R_G^{Of^+}$, *where* $O \in \{\mathrm{B}, \mathrm{I}\}$ *and* $G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}$,
4. *if* $([v], [u]) \in R_j^{Of}$, *then, for all* $[O]_j\varphi \in \Phi$, *if* $(\mathcal{M}, v) \vDash [O]_j\varphi$, *then* $(\mathcal{M}, u) \vDash \varphi$, *where* $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$,
5. *if* $([v], [u]) \in R_G^{Of^+}$, *then, for all* $[O]_G^+\varphi \in \Phi$, *if* $(\mathcal{M}, v) \vDash [O]_G^+\varphi$, *then* $(\mathcal{M}, u) \vDash \varphi$, *where* $O \in \{\mathrm{B}, \mathrm{I}\}$,

that is $\mathcal{M}_{\Phi}^{f}$ is a filtration of $\mathcal{M}$ through Φ.

Proof. Point 1 holds by point **F0** of definition of $\mathcal{M}_{\Phi}^{f}$.

For point 2, take any $(v, u) \in R_j^O$, where $j \in \mathcal{A}$ and $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$, and assume that $(\mathcal{M}, v) \vDash [O]_j\varphi$, for some $[O]_j\varphi \in \Phi$. Since $u \in R_j^O(v)$, so $(\mathcal{M}, u) \vDash \varphi$. This shows that point 2 holds for $R_j^{\mathrm{I}f}$. For relations $R_j^{\mathrm{G}f}$ and $R_j^{\mathrm{B}f}$ we need to show additional properties.

For $R_j^{\mathrm{G}f}$, assume that $(\mathcal{M}, v) \vDash [\mathrm{I}]_j\varphi$, for some $[\mathrm{I}]_j\varphi \in \Phi$. Since $\mathcal{M}$ is a TEAMLOG model, so $R_j^{\mathrm{G}} \subseteq R_j^{\mathrm{I}}$ and so $u \in R_j^{\mathrm{I}}(v)$ and $(\mathcal{M}, u) \vDash \varphi$. This and the fact shown above show that point 2 holds for $R_j^{\mathrm{G}f}$.

For $R_j^{\mathrm{B}f}$, assume that $(\mathcal{M}, v) \vDash [O]_j\varphi$, for some $[O]_j\varphi \in \Phi$, where $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$. Take any $t \in R_j^O(u)$. Then, by generalized transitivity, $t \in R_j^O(v)$ and so $(\mathcal{M}, t) \vDash \varphi$. Thus $(\mathcal{M}, u) \vDash [O]_j\varphi$.

On the other hand, assume that $(\mathcal{M}, u) \vDash [O]_j\varphi$, for some $[O]_j\varphi \in \Phi$, where $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$. Take any $t \in R_j^O(v)$. Then, by generalized Euclidean property, $t \in R_j^O(u)$ and so $(\mathcal{M}, t) \vDash \varphi$. Thus $(\mathcal{M}, v) \vDash [O]_j\varphi$. This and facts shown above show that point 2 holds for $R_j^{\mathrm{B}f}$.

For point 3, take any $(v, u) \in R_G^{O+}$, where $G \subseteq \mathcal{A}$ and $O \in \{\mathrm{B}, \mathrm{I}\}$. Then there exists a sequence of worlds $s_0, \ldots, s_m$ such that $s_0 = v$, $s_m = u$ and $s_k \in R_{j_k}^O(s_{k-1})$ with $j_k \in G$, for all $1 \leq k \leq m$. Thus, by point 2, $[s_k] \in R_{j_k}^{O\,f}([s_{k-1}])$ and so $[u] \in R_G^{O\,f^+}([v])$.

For point 4, take any $([v], [u]) \in R_j^{O\,f}$, where $j \in \mathcal{A}$ and $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$, and assume that $(\mathcal{M}, v) \vDash [O]_j\varphi$, for some $[O]_j\varphi \in \Phi$. Then, by definition of $\mathcal{M}_\Phi^f$, $(\mathcal{M}, u) \vDash \varphi$ and so point 4 is satisfied.

For point 5, take any $([v], [u]) \in R_G^{O\,f^+}$, where $G \subseteq \mathcal{A}$ and $O \in \{\mathrm{B}, \mathrm{I}\}$, and assume that $(\mathcal{M}, v) \vDash [O]_G^+\varphi$, for some $[O]_G^+\varphi \in \Phi$. Since $([v], [u]) \in R_G^{O\,f^+}$, so there exists a sequence of worlds $[s_0], \ldots, [s_m]$ such that $[s_0] = [v]$, $[s_m] = [u]$ and $[s_k] \in R_{j_k}^O([s_{k-1}])$ with $j_k \in G$, for all $1 \leq k \leq m$. Moreover, by properties of closed set, if $[O]_G^+\varphi \in \Phi$, then $[O]_j[O]_G^+\varphi \in \Phi$ and $[O]_j\varphi \in \Phi$, for any $j \in G$. Hence, by simple induction over length of sequence $[s_0], \ldots, [s_m]$ and definition of relations $R_j^{\mathrm{B}\,f}$ and $R_j^{\mathrm{I}\,f}$, it holds that $(\mathcal{M}, s_k) \vDash [O]_G^+\varphi$ and $(\mathcal{M}, s_k) \vDash \varphi$, for all $1 \leq k \leq m$. In particular $(\mathcal{M}, u) \vDash \varphi$, so point 5 is satisfied. □

Since $\mathcal{M}_\Phi^f$ is a filtration of $\mathcal{M}$ through Φ, so the following filtration theorem holds (see [10, Theorem 2.39] for general statement and proof).

Theorem 4.11 (Filtration Theorem)**.** *If $\mathcal{M}$ is a* TeamLog *model and Φ is a closed set of formulas then for all $\varphi \in \Phi$ and all $v \in W$, $(\mathcal{M}, v) \vDash \varphi$ iff $(\mathcal{M}_\Phi^f, [v]) \vDash \varphi$.*

Filtration theorem leads immediately to the small model property and to decidability of satisfiability problem of TeamLog formulas, as stated below.

Theorem 4.12 (Small model property)**.** *If a formula $\varphi \in \mathcal{L}$ is satisfiable, then it is satisfiable in a finite model containing at most $2^{(2|\mathcal{A}|+1)|\varphi|}$ worlds.*

Proof. Suppose that a formula $\varphi \in \mathcal{L}$ is satisfiable in $\mathcal{M}$. Then, by Theorem 4.11, it is also satisfiable in $\mathcal{M}_{\mathrm{Cl}(\varphi)}^f$. Since any element $[v]$ of $W/\equiv_f^{\mathrm{Cl}(\varphi)}$ is uniquely identified by a set of formulas from $\mathrm{Cl}(\varphi)$ satisfied in $(\mathcal{M}, v)$, so $\left|W/\equiv_f^{\mathrm{Cl}(\varphi)}\right| \leq |P(\mathrm{Cl}(\varphi))|$. Since $|\mathrm{Cl}(\varphi)| \leq (2|\mathcal{A}|+1)|\mathrm{Sub}(\varphi)|$ (as any subformula of φ may require adding $2|\mathcal{A}|$ new formulas to get $\mathrm{Cl}(\varphi)$), so number of worlds in $\mathcal{M}_{\mathrm{Cl}(\varphi)}^f$ is $\leq 2^{(2|\mathcal{A}|+1)|\varphi|}$. □

Corollary 4.13. *Checking* TeamLog *satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}$ is decidable.*

Proof. The satisfiability of a formula $\varphi \in \mathcal{L}^{\mathrm{T}}$ can be checked by the following non-deterministic Algorithm 4.7.

Algorithm 4.7: DecideSatisfiabilityNonDeterministic

Input: a formula φ
Output: a decision whether φ is satisfiable or not

Guess a TeamLog model $\mathcal{M}$ with world w in it such φ is satisfied in $\mathcal{M}$ at w;
if $(\mathcal{M}, w) \vDash \varphi$ **then**
  return `sat`;

Validity of the algorithm follows from the small model property of TeamLog. □

4.3.2 TeamLog tableau

A TeamLog tableau is a TeamLog$^{\text{ind}}$ tableau with labels of states being *closed propositional tableaux* and accessibility relations satisfying additional conditions corresponding to iterated modalities $[\mathrm{B}]^+_G\cdot$ and $[\mathrm{I}]^+_G\cdot$.

Definition 13 (Closed propositional tableau)**.** Closed propositional tableau *is a propositional tableau satisfying condition* ***Cl****.*

Definition 14 (TeamLog tableau)**.** *A* TeamLog tableau *is a* TeamLog^{ind} *tableau* $\mathcal{T} = (W, \{R^O_j : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, L)$ *such that for all* $w \in W$

- *L(w) is a closed propositional tableau*

and the following property is satisfied, for all $w \in W$*,* $O \in \{\mathrm{B}, \mathrm{I}\}$ *and* $G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}$

TC *If* $\neg[O]^+_G\varphi \in L(w)$*, then there exists* $v \in R^{O+}_G(w)$ *such that* $\sim\varphi \in L(v)$*.*

Given a formula φ *we say that* $\mathcal{T} = (W, \{R^O_j : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, L)$ *is a* TeamLog *tableau for* φ *if* $\mathcal{T}$ *is a* TeamLog *tableau and there is a state* $w \in W$ *such that* $\varphi \in L(w)$*.*

The following theorem links existence of TeamLog tableau for a formula with its satisfiability.

Theorem 4.14. *A formula* $\varphi \in \mathcal{L}$ *is satisfiable iff there is a* TeamLog *tableau for* φ*.*

Proof. Proof is an extension of proof of Theorem 4.2 with kinds of formulas that extend $\mathcal{L}^{\text{Tind}}$ to $\mathcal{L}$. For the left to right implication, a TeamLog tableau for φ is constructed on the basis of a model $\mathcal{M}$ with a world w in it such that $(\mathcal{M}, w) \vDash \varphi$, in the same way as a TeamLog$^{\text{ind}}$ tableau for a satisfiable formula in the proof of Theorem 4.2. Let

$$\mathcal{T} = (W, \{R^O_j : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, L)$$

be a tableau constructed as in the proof of Theorem 4.2. The only new thing we need to show here is that for any $v \in W$, $L(v)$ is a closed propositional tableau and that condition **TC** is satisfied.

Showing that $L(v)$ is a propositional tableau can be done by the same arguments as those used in proof of Theorem 4.2. Thus what remains to be shown is that condition **Cl** is satisfied. Let $v \in W$ and assume that $[O]^+_G\xi \in L(v)$ with $O \in \{\mathrm{B}, \mathrm{I}\}$. Then $(\mathcal{M}, v) \vDash [O]^+_G\xi$. Take any $u \in R^O_j(v)$, for some $j \in G$. Then $u \in R^{O+}_G(v)$, and so $(\mathcal{M}, u) \vDash \xi$. Moreover for any $t \in R^{O+}_G(u)$, it also holds that $t \in R^{O+}_G(v)$, by transitivity of R^{O+}_G. Hence $(\mathcal{M}, u) \vDash [O]^+_G\xi$. Thus $(\mathcal{M}, v) \vDash [O]_j\xi$ and $(\mathcal{M}, v) \vDash [O]_j[O]^+_G\xi$, and so $[O]_j\xi \in L(v)$ and $[O]_j[O]^+_G\xi \in L(v)$. This shows that condition **Cl** is satisfied.

For condition **TC**, let $v \in W$ and assume that $\neg[O]^+_G\xi \in L(v)$ with $O \in \{\mathrm{B}, \mathrm{I}\}$. Then $(\mathcal{M}, v) \nvDash [O]^+_G\xi$ and there must be $u \in R^{O+}_G(v)$ such that $(\mathcal{M}, u) \nvDash \xi$. Thus $(\mathcal{M}, u) \vDash \sim\xi$ and $\sim\xi \in L(u)$. Hence condition **TC** is satisfied.

For the right to left implication let

$$\mathcal{M} = (W, \{R'^O_j : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, Val)$$

be a model constructed on the basis of a TeamLog tableau for φ as in the proof of Theorem 4.2. Part of the proof showing that $\mathcal{M}$ is a TeamLog model is not affected by extending from

$\mathcal{L}^{\text{Tind}}$ to $\mathcal{L}$. We need to extend inductive proof showing, that for any $\psi \in \mathcal{L}$ and $v \in W$, $\psi \in L(v)$ implies $(\mathcal{M}, v) \vDash \psi$.

Assume that $\psi = [O]^+_G \xi$, with $O \in \{\mathrm{B}, \mathrm{I}\}$, and $[O]^+_G \xi \in L(v)$. Take any $u \in R'^{O+}_G(v)$. By condition **Cl**, $\{[O]_j \xi, [O]_j [O]^+_G \xi\} \subseteq L(v)$, for any $j \in G$. Moreover, by simple induction on the length of sequences over G, it can be shown that that for any $u \in R'^{O+}_G(v)$ it holds that $\{\xi, [O]_j \xi, [O]_j [O]^+_G \xi\} \subseteq L(u)$. Thus, by the induction hypothesis, $(\mathcal{M}, u) \vDash \xi$ and so $(\mathcal{M}, v) \vDash [O]^+_G \xi$.

Now let $\psi = \neg[O]^+_G \xi$, with $O \in \{\mathrm{B}, \mathrm{I}\}$, and $\neg[O]^+_G \xi \in L(v)$. By condition **TC** there exists $u \in R^{O+}_G(v)$ such that $\sim\xi \in L(u)$. Thus, by the induction hypothesis and the fact that $R^O_j \subseteq R'^O_j$, for all $j \in \mathcal{A}$, it holds that $u \in R'^{O+}_G(v)$ and $(\mathcal{M}, u) \vDash \sim\xi$. Hence $(\mathcal{M}, u) \nvDash \xi$ and $(\mathcal{M}, v) \vDash \neg[O]^+_G \xi$.

As in proof of Theorem 4.2, we have shown that for any $\psi \in \mathcal{L}$ and $v \in W$, $\psi \in L(v)$ implies $(\mathcal{M}, v) \vDash \psi$, and, in particular, $\varphi \in L(w)$ implies $(\mathcal{M}, w) \vDash \varphi$, that is φ is satisfiable. □

4.3.3 Algorithm for deciding TeamLog satisfiability

In this section we present a deterministic exponential time algorithm for checking TeamLog satisfiability of a formula φ. The idea of the algorithm and associated proof of its validity are based on Pratt's algorithm for checking satisfiability for PDL and associated proof of validity, as presented in [58]. However, to make our presentation consistent with other parts of the thesis, we will base this algorithm on the notion of a TeamLog tableau.

The algorithm tries to construct a TeamLog tableau for a given input formula φ, by starting from a model graph

$$\mathcal{T} = (W, \{R^O_j : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, L)$$

with labels of states being maximal consistent subsets of $\neg\mathrm{Cl}(\varphi)$ and accessibility relations being maximal binary relations on the set of states satisfying conditions **T1** (for all $j \in \mathcal{A}$ and $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$), **T4**, **T5** (for all $j \in \mathcal{A}$ and $O = \mathrm{B}$), **TBG4**, **TBG5**, **TBI4**, **TBI5** and **TIG** (for all $j \in \mathcal{A}$) of TeamLog tableau.

After the first stage of model graph initialization, in the second stage all states with labels that are not propositional tableaux are removed. Then, in the third stage, states violating remaining conditions of TeamLog tableau are deleted until a TeamLog tableau is obtained or all states are removed. In the fourth stage satisfiability of input formula is decided.

Lemma 4.15. *For any input formula $\varphi \in \mathcal{L}$ Algorithm 4.8 terminates after $\mathcal{O}(2^{|\varphi|})$ steps.*

Proof. The algorithm starts with model graph $\mathcal{T}$ having $|W| \leq 2^{|\mathrm{Cl}(\varphi)|}$ states and in each step of the algorithm at least one state is removed from W. Since $|\mathrm{Cl}(\varphi)| \leq (2|\mathcal{A}|+1)|\mathrm{Sub}(\varphi)|$, as for each $\psi \in \mathrm{Sub}(\varphi)$ there are $\leq 2|\mathcal{A}|$ additional formulas in $\mathrm{Cl}(\varphi) \setminus \mathrm{Sub}(\varphi)$, so $|W| \leq 2^{(2|\mathcal{A}|+1)|\varphi|}$.

Each step of the algorithm can be realized in polynomial time with respect to $|W|$ and number of steps is $\leq |W|$. Hence the algorithm must terminate after $\mathcal{O}(2^{|\varphi|})$ steps. □

Lemma 4.16. *A formula $\varphi \in \mathcal{L}$ is satisfiable iff Algorithm 4.8 returns `sat` on the input φ.*

Proof. For the left to right implication assume that φ is satisfiable, and that $(\mathcal{M}, w) \vDash \varphi$ where

$$\mathcal{M} = (W, \{R^O_j : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, Val)$$

and $w \in W$. By Theorem 4.11, $(\mathcal{M}^f_{\mathrm{Cl}(\varphi)}, [w]) \vDash \varphi$, where

$$\mathcal{M}^f_{\mathrm{Cl}(\varphi)} = (W^f, \{R^{Of}_j : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, Val^f)$$

Algorithm 4.8: DecideSatisfiability

Input: a formula φ
Output: a decision whether φ is satisfiable or not

```
InitializeModelGraph(φ);
Remove from W all states with labels that are not propositional tableaux;
UpdateModelGraph;
while there is w ∈ W violating one of the conditions T2, TD or TC do
    W := W \ {w};
    UpdateModelGraph;
if there there is w ∈ W such that φ ∈ L(w) then
    return sat;
else
    return unsat;
```

Procedure 4.9: InitializeModelGraph

Input: a formula φ

$W := \varnothing$;
foreach *maximal consistent* $\Phi \subseteq \neg\mathrm{Cl}(\varphi)$ **do**
  Add a new state s to W and set $L(s) := \Phi$;
forall the $j \in \mathcal{A}$ **do**
  $R^{\mathrm{B}}_j := \{(s,t):$ for any $[\mathrm{B}]_j\varphi \in \mathrm{Cl}(\varphi)$, $[\mathrm{B}]_j\varphi \in L(s)$ implies $\varphi \in L(t)$ and for any $[O]_j\varphi \in \mathrm{Cl}(\varphi)$, $[O]_j\varphi \in L(s)$ iff $[O]_j\varphi \in L(t)\}$, where $O \in \{\mathrm{B},\mathrm{G},\mathrm{I}\}$;
  $R^{\mathrm{G}}_j := \{(s,t):$ for any $[\mathrm{G}]_j\varphi \in \mathrm{Cl}(\varphi)$, $[\mathrm{G}]_j\varphi \in L(s)$ implies $\varphi \in L(t)$ and for any $[\mathrm{I}]_j\varphi \in \mathrm{Cl}(\varphi)$, $[\mathrm{I}]_j\varphi \in L(s)$ implies $\varphi \in L(t)\}$;
  $R^{\mathrm{I}}_j := \{(s,t):$ for any $[\mathrm{I}]_j\varphi \in \mathrm{Cl}(\varphi)$, $[\mathrm{I}]_j\varphi \in L(s)$ implies $\varphi \in L(t)\}$;

is a filtration of $\mathcal{M}$ through $\mathrm{Cl}(\varphi)$. Consider a model graph

$$\mathcal{T}^f = (W^f, \{R_j^{Of} : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, L^f),$$

with

$$L^f([v]) = \{\psi \in \neg\mathrm{Cl}(\varphi) : (\mathcal{M}^f_{\mathrm{Cl}(\varphi)}, [v]) \models \psi\},$$

where $[v] \in W^f$.

Obviously $L([v]) \subseteq \neg\mathrm{Cl}(\varphi)$, for any $[v] \in W$. Moreover, for all $[v] \in W$, $L([v])$ is not blatantly inconsistent and for any $\psi \in \neg\mathrm{Cl}(\varphi)$ either $\psi \in L([v])$ or $\neg\psi \in L(v)$. Hence $L([v])$ is a maximal consistent subset of $\neg\mathrm{Cl}(\varphi)$ which is not blatantly inconsistent. Also, it can be shown, using the same argumentation as the one used in proofs of Theorem 4.2 and Theorem 4.14, that $\mathcal{T}^f$ is a TeamLog tableau.

Let

$$\mathcal{T}^0 = (W^0; \{R^{0O}_j : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, L^0)$$

be a model graph created after first stage of the algorithm. Then $\mathcal{T}^0$ contains $\mathcal{T}^f$ in the sense that (for all $j \in \mathcal{A}$ and $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$)

- $W^f \subseteq W^0$,
- $R_j^{Of} = R^{0O}_j \cap W^f \times W^f \subseteq B^0_j$ and
- $L^f = L^0|_{W^f} \subseteq L^0$.

No element of W^f can be removed throughout the second and the third stage of the algorithm, as all states in W^f satisfy conditions of TeamLog tableau. Hence all model graphs created in these stages will contain $\mathcal{T}^f$, in particular the model graph created after the third stage of the algorithm will do so as well. Thus there will be $[w] \in W^f$ with $\varphi \in L([w])$ before last stage of the algorithm and so `sat` will be returned.

For the right to left implication suppose that Algorithm 4.8 returned `sat` on some input $\varphi \in \mathcal{L}$. Then a model graph

$$\mathcal{T} = (W, \{R_j^O : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, L)$$

constructed by the algorithm is such that there exists $w \in W$ with $\varphi \in L(w)$. We will show that $\mathcal{T}$ is a TeamLog tableau.

First notice that, by construction of the algorithm, conditions **TD**, **T2** and **TC** are satisfied for $\mathcal{T}$. For the remaining conditions, notice that conditions **T1**, **T4**, **T5**, **TBG4**, **TBG5**, **TBI4**, **TBI5** and **TIG** are satisfied for the model graph created in the first step of Algorithm 4.8. Moreover, labels of all states of this model graph are closed sets of formulas. Removing states from W and updating accessibility relations and labelling functions does not affect these properties, as labels of states that remain in W are not affected by such change and accessibility relations are changed by removing pairs of states only. Hence conditions **T1**, **T4**, **T5**, **TBG4**, **TBG5**, **TBI4**, **TBI5** and **TIG** must be satisfied after the second stage of the algorithm. Also, after the second stage, $L(v)$ must be a closed set of formulas and a propositional tableau, for all $v \in W$. This is because during that stage all the states with labels that are not propositional tableaux are removed. Labels of states will remain closed propositional tableaux after the third stage of the algorithm, as they are not affected then. Conditions **T1**, **T4**, **T5**, **TBG4**, **TBG5**, **TBI4**, **TBI5** and **TIG** are also satisfied after the third stage of the algorithm. Hence $\mathcal{T}$ is a TeamLog tableau. Since $\mathcal{T}$ is a TeamLog tableau with $\varphi \in L(w)$ for some $w \in W$, so, by Theorem 4.14, φ is satisfiable. □

Procedure 4.10: UpdateModelGraph

forall the $j \in \mathcal{A}$ **do**
  $R_j^{\mathrm{B}} := R_j^{\mathrm{B}} \cap W \times W;$
  $R_j^{\mathrm{G}} := R_j^{\mathrm{G}} \cap W \times W;$
  $R_j^{\mathrm{I}} := R_j^{\mathrm{I}} \cap W \times W;$
$L := L|_W;$

The following corollary from Lemma 4.15 and Lemma 4.16 states upper bound for complexity of the TEAMLOG satisfiability problem.

Lemma 4.17. *The satisfiability problem for* TEAMLOG *is in EXPTIME.*

4.3.4 Lower bound

We will show that checking TEAMLOG satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}$ is EXPTIME-complete. Since we have shown already that the problem is in EXPTIME, so it is enough to show that the problem is EXPTIME-hard. The following fact analogous to Fact 4.1 can be shown by the same arguments as those used in the proof of Fact 4.1.

Fact 4.18. *Let* $\varphi \in \mathcal{L}[\mathcal{P}, \Omega^{O^+}]$*, where* $O \in \{\mathrm{B}, \mathrm{I}\}$*, be a formula build with modal operators based on a non-empty set of agents* $\mathcal{B} \subseteq \mathcal{A}$*, with* $|\mathcal{B}| = m$*. Then* φ *is S-satisfiable iff* φ *is* TEAMLOG *satisfiable, where* $S = KD45_m^+$ *if* $O = \mathrm{B}$ *and* $S = KD_m^+$ *if* $O = \mathrm{I}$.

By Fact 4.18, each of the problems of checking KD_n^+ and $\mathrm{KD45}_n^+$ satisfiability can be reduced to the problem of checking TEAMLOG satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}$. As was shown in [54], each of these problems is EXTPIME-hard (in the case of $\mathrm{KD45}_n^+$, $n \geq 2$ is required for this result to hold). Hence the problem of checking TEAMLOG satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}$ is EXPTIME-hard as well. EXPTIME-hardness of KD_n^+ and $\mathrm{KD45}_n^+$ satisfiability problems is shown [54] by constructing formulas for an exponential time Turing machine and its input such that these formulas are satisfiable if and only if the machine accepts the input. In the case of KD_n^+ satisfiability the formula has modal depth equal to 2, hence it is shown that checking KD_n^+ satisfiability of formulas with modal depth bounded by 2 is EXPTIME-hard. This result alone implies that checking TEAMLOG satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}$ with modal depth bounded by 2 is EXPTIME-hard. In the case of $\mathrm{KD45}_n^+$ it is required in [54] that $n \geq 2$ and the formula has modal depth equal to 3. Hence it is shown that checking $\mathrm{KD45}_n^+$ satisfiability of formulas with $n \geq 2$ and modal depth bounded by 3 is EXPTIME-hard. We refine this result by showing that even if modal depth is bounded by 2, the $\mathrm{KD45}_n^+$ satisfiability problem is EXPTIME-hard.

The proof we give here is inspired by the proof of EXPTIME hardness of the satisfiability problem for PDL given in [10, Ch. 6.8]. Like in that proof, we will show a reduction of EXPTIME-hard problem related to certain kind of two person tiling game to the problem of TEAMLOG satisfiability.

Theorem 4.19. *The* $KD45_2^+$ *satisfiability problem for formulas with modal depth* ≤ 2 *is EXPTIME-hard.*

Proof. To show EXPTIME-hardness of the $\mathrm{KD45}_2^+$ satisfiability problem, we will use a two-person corridor tiling game. A tile is a 1×1 square, with fixed orientation and a colour assigned to each side. There are two players taking part in the game and a referee who starts the game. The referee gives the players a finite set of $\{T_1, \dots, T_s\}$ of tile types. Players will use tiles of these types to arrange them on the grid in such a way that the colours on the common sides of adjacent tiles match. Additionally there are two special tile types T_0 and T_{s+1}. T_0 is white on all sides and is used merely to mark the boundaries of the corridor inside which the two players will place their tiles. T_{s+1} is a special winning tile that can be placed only in the first column.

At the start of the game the referee fills in the first row (places $\{1, \dots, m\}$) of the corridor with m initial tiles of types $\{T_1, \dots, T_s\}$ and places two columns of T_0 type tiles in columns 0 and $m+1$ marking the boundaries of the corridor. Now the two players A and B place their tiles in alternating moves. Player A is the one to start. The corridor is to be filled row by row from bottom to top and from left to right. Thus the place of the next tile is determined and the only choice the players make is the type of tile to place. The colour of a newly placed tile must fit the colours of its adjacent tiles. We will use $C(T', T, T'')$ to denote that T can be placed to the right of T' and above tile T'', so that $right(T') = left(T)$ and $top(T'') = bottom(T)$, where $right$, $left$, top and $bottom$ give the colours of respective sides of a tile.

If after finitely many rounds a tiling is constructed in which a tile of type T_{s+1} is placed in the first column, then player A wins. Otherwise, that is if no player can make a legal move or if the game goes on infinitely long and no tile of type T_{s+1} is placed in the first column, player B wins. The problem of deciding if for a given setting of the game there is a winning strategy for player A is known to be an EXPTIME-hard problem [25]. Following [10, Ch. 6.8] we will show that this problem can be reduced to the $\mathrm{KD45}_2^+$ satisfiability problem.

In the proof of [10, Ch. 6.8] a formula is constructed for a given tiling game, such that it is satisfied in a model that is the game tree for given settings of the game and at the world being a current state. States of the tree contain information about the actual configuration of the tiles, the player who is to move next, and the position at which the next tile is to be placed. The depth of the tree is bounded by m^{s+2}. Note that after m^{s+2} rounds, repetition of rows must have occurred and if A can win a game with repetitions, A can also win a game without them, thus it is enough to consider m^{s+2} rounds only. The formula from the proof of [10, Ch. 6.8] has modal depth equal to 2 and uses two PDL modal operators $[a]$ and $[a^*]$. These operators could be replaced by $[\mathrm{I}]_1$ and $[\mathrm{I}]'^{+}_{\{1\}}$, respectively, where $[\mathrm{I}]'^{+}_G\varphi \stackrel{\text{def}}{=} [\mathrm{I}]^+_G\varphi \wedge \varphi$, for any $G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}$ (recall that $[a^*]$ is reflexive and $[\mathrm{I}]^+$ is not). The proof would remain the same. Thus an alternative proof of EXPTIME-hardness of the problem of checking KD_1^+ satisfiability could be provided. Below we show a slightly modified version of the [10, Ch. 6.8] proof, adapted for $\mathrm{KD45}_2^+$. In this case $n \geq 2$ is required. This is not surprising, as for $n = 1$, $[\mathrm{B}]^+$ and $[\mathrm{B}]$ are the same modality, because, by axioms **T4** and **T5**, $[\mathrm{B}]_1\varphi$ and $[\mathrm{B}]_1[\mathrm{B}]_1\varphi$ are equivalent.

Let $\mathcal{G} = (m, \mathcal{T}, (I_1, \dots, I_m))$, where $\mathcal{T} = \{T_0, \dots, T_{s+1}\}$ and $R^{\mathrm{I}}_j \in \mathcal{T}$ for $0 \leq j \leq m$, be a setting for a two person corridor tiling game described above. Here, $(I_1, \dots, I_m)$ is the row of types of the initial tiling of the first row of the corridor. We construct a formula $\varphi(G)$ such that it is satisfiable iff player A has a winning strategy. The following propositional symbols are used to construct a formula:

- a to indicate that A has the next move; we will also use p_1 to denote a and p_2 to denote $\neg a$ in order to shorten some formulas,
- $pos_1, \dots, pos_m$ to indicate the column in which a tile is to be placed in the current round,
- $col_i(T)$, for $0 \leq i \leq m+1$ and $T \in \mathcal{T}$, to indicate that a tile previously placed in column i is of type T,[3]

[3]Note that $col_i(T)$ is a parametrized name of a propositional symbol.

- *win* to indicate that the current position is a winning position for A,
- $q_1, \dots, q_N$, where $N = \lceil \lg(m^{s+2}) \rceil$, to enumerate states; boolean values of these variables in a given state can be treated as a representation of a binary number with q_1 being the least significant bit and q_N being the most significant one; we will give the same number to all states belonging to the same round; we will use the notation $round = k$ as a short cut for the formula expressing that the number encoded by $q_N \dots q_1$ is equal to k.

The formula $\varphi(G)$ is composed of the following formulas describing settings of the game and giving necessary and sufficient conditions for the existence of a winning strategy for A. In what follows, $k \in \{1, 2\}$, $0 \le i \ne j \le m+1$, $0 \le x \ne y \le s+1$ and $\{T, T', T''\} \subseteq \mathcal{T}$ (if not stated differently). We will use $[\mathrm{B}]'^{+}_{G}\varphi$ as an abbreviation for $[\mathrm{B}]^{+}_{G}\varphi \wedge \varphi$.

$$a \wedge pos_1 \wedge col_0(T_0) \wedge col_{m+1}(T_0) \wedge col1(I_1) \wedge \dots \wedge col_m(I_m) \tag{4.1}$$

$$[\mathrm{B}]^{+}_{\{1,2\}} \left(pos_1 \vee \dots \vee pos_m\right) \tag{4.2}$$

$$[\mathrm{B}]'^{+}_{\{1,2\}} \left(pos_i \to \neg pos_j\right), 1 \le i \ne j \le m \tag{4.3}$$

$$[\mathrm{B}]^{+}_{\{1,2\}} \left(col_i(T_0) \vee \dots \vee col_i(T_{s+1})\right) \tag{4.4}$$

$$[\mathrm{B}]'^{+}_{\{1,2\}} \left(col_i(T_x) \to \neg col_i(T_y)\right) \tag{4.5}$$

$$[\mathrm{B}]^{+}_{\{1,2\}} \left(col_0(T_0) \wedge col_{m+1}(T_0)\right) \tag{4.6}$$

$$\begin{aligned}[\mathrm{B}]'^{+}_{\{1,2\}}(\neg pos_i \to (&(col_i(T_x) \to [\mathrm{B}]_k col_i(T_x)) \wedge \\ &(\neg col_i(T_x) \to [\mathrm{B}]_k \neg col_i(T_x)))\end{aligned} \tag{4.7}$$

$$\begin{aligned}&[\mathrm{B}]'^{+}_{\{1,2\}}((pos_m \wedge p_k \to [\mathrm{B}]_k pos_1) \wedge \\ &(pos_1 \wedge p_k \to [\mathrm{B}]_k pos_2) \wedge \dots \wedge (pos_{m-1} \wedge p_k \to [\mathrm{B}]_k pos_m))\end{aligned} \tag{4.8}$$

$$[\mathrm{B}]'^{+}_{\{1,2\}} \left((a \to [\mathrm{B}]_1 \neg a) \wedge (\neg a \to [\mathrm{B}]_2 a)\right) \tag{4.9}$$

$$\begin{aligned}[\mathrm{B}]'^{+}_{\{1,2\}}(\; &pos_i \wedge col_{i-1}(T_0) \wedge col_i(T'') \wedge p_k \to \\ &[\mathrm{B}]_k \left(\bigvee \{col_i(T) : C(T', T, T'')\}\right)), 1 \le i \le m\end{aligned} \tag{4.10}$$

$$[\mathrm{B}]'^{+}_{\{1,2\}} \left(pos_n \to [\mathrm{B}]_k \left(\bigvee \{col_n(T) : right(T) = white\}\right)\right) \tag{4.11}$$

$$[\mathrm{B}]'^{+}_{\{1,2\}}(\neg a \wedge pos_i \wedge col_i(T'') \wedge col_{i-1}(t_0) \rightarrow \bigwedge\{\neg[\mathrm{B}]_k\neg col_i(T) : C(T', T, T'')\}), 1 \leq i \leq m \tag{4.12}$$

$$win \wedge [\mathrm{B}]'^{+}_{\{1,2\}}(win \rightarrow (col_1(T_{s+1}) \vee (a \wedge \neg[\mathrm{B}]_1\neg win) \vee (\neg a \wedge [\mathrm{B}]_2 win))) \tag{4.13}$$

$$[\mathrm{B}]'^{+}_{\{1,2\}}\,((round = N) \rightarrow [\mathrm{B}]_k\neg win) \tag{4.14}$$

Formulas (4.1 – 4.7) describe the settings of the game. The initial setting is as described by (4.1). During the game tiles are placed in exactly one of the columns $1..m$ (4.2 – 4.3) and in every column exactly one tile type was previously placed (4.4 – 4.5). The boundary tiles are placed in columns 0 and $m+1$ (4.6) and nothing changes in columns where no tile is placed during the game (4.7).

Formulas (4.8 – 4.11) describe the rules of the game. Tiles are placed from bottom to top, row by row from left to right (4.8); thus, the first conjunct of (4.8) represents the flipping of one row to the next. The players alternate (4.9). Tiles that are placed have to match adjacent tiles (4.10 – 4.11). The formula (4.12) ensures that all possible moves by player B are encoded in the model.

Formula (4.13) gives properties of states that can be marked as winning positions for the player A and formula (4.14) states that all states reached after $\geq N$ rounds can not be winning positions for A.

Similarly to [10, Lemma 6.51], one can force exponentially deep models of TeamLog for satisfying some, properly constructed, formulas. Specifically, to enumerate the states according to rounds of the game we will need the following additional formula.

$$\bigwedge_{j=1}^{N} \neg q_j \wedge [\mathrm{B}]'^{+}_{\{1,2\}}\left(\mathsf{INC}_0 \wedge \bigwedge_{j=1}^{N-1} \mathsf{INC}_1(j)\right), \tag{4.15}$$

where

$$\mathsf{INC}_0 = \neg q_1 \rightarrow \left([\mathrm{B}]_1 q_1 \wedge \bigwedge_{j=2}^{N}((q_j \rightarrow [\mathrm{B}]_1 q_j) \wedge (\neg q_j \rightarrow [\mathrm{B}]_1\neg q_j))\right) \tag{4.16}$$

$$\mathsf{INC}_1(i) = \left(\neg q_{i+1} \wedge \bigwedge_{j=1}^{i} q_j\right) \rightarrow \left([\mathrm{B}]_2\left(q_{i+1} \wedge \bigwedge_{j=1}^{i} \neg q_j\right) \wedge \bigwedge_{j=i+2}^{N}((q_j \rightarrow [\mathrm{B}]_2 q_j) \wedge (\neg q_j \rightarrow [\mathrm{B}]_2\neg q_j))\right) \tag{4.17}$$

Formula (4.15) enforces that the root of the model receives a number $(0\ldots0)_2$ and worlds corresponding to states in subsequent rounds of the game receive subsequent numbers in binary representation. The formula INC_0 is responsible for increasing even numbers, and $\mathsf{INC}_1(i)$ is responsible for increasing odd numbers ending with a sequence of i digits 1 and having digit 0 at the position $i+1$.

The formula $\varphi(G)$ is a conjunction of formulas (4.1 – 4.15) and it is of size polynomial with respect to m. It can be easily seen that if A has a winning strategy in the particular game, then the formula $\varphi(G)$ is satisfiable in a model built on the basis of a game tree for this game. Edges corresponding to turns of player A are the basis for accessibility relation R_1^{B} and those corresponding to turns of player B are the basis for accessibility relation R_2^{B}. To satisfy the properties of the model, R_1^{B} and R_2^{B} are extended by identity in worlds that violate the seriality property. All other relations R_j^{B} and R_j^{I} are set to identity and R_j^{G} are set to $\varnothing$. Valuation of propositional variables in the worlds of the model is automatically determined by the description of the situation in the corresponding states of the game.

On the other hand, if $\varphi(G)$ is satisfiable, A can use a model of $\varphi(G)$ as a guide for his winning strategy. At the beginning, he chooses a transition (represented by accessibility relation R_1^{B}) to a world where win is true, and plays accordingly. Player A does analogously in all subsequent rounds of the game. He can track the worlds corresponding to states of subsequent rounds of the game, by following relations R_1^{B} and R_2^{B} alternately. Notice that for all worlds v corresponding to states where A is to play and where A has a winning strategy (that is win is true), it must be that $v \notin R_1^{\mathrm{B}}(v)$, as guaranteed by (4.9). The same holds for R_2^{B} and states where B is to play. Notice also that formula (4.14) guarantees that A will reach a winning position in a finite number of steps if he plays as described above. $\square$

Chapter 5

The Complexity of the satisfiability problem for restricted TeamLog$^{\text{ind}}$ framework

In this chapter we study the effects of restricting language $\mathcal{L}^{\text{Tind}}$ on the complexity of satisfiability problem of TeamLog$^{\text{ind}}$. Two restrictions are studied: restricting modal depth of formulas and restricting the number of propositional symbols. Combination of the two restrictions is also analysed. The results presented in this chapter have already been published in [42].

5.1 Restricting modal depth of TeamLog$^{\text{ind}}$

As was shown in [56], bounding modal depth of formulas by a constant results in NPTIME-completeness of the satisfiability problem for modal logics K_n, KD_n and $\mathrm{KD45}_n$.[1] Analogous result holds for TeamLog$^{\text{ind}}$, as we show in this section.

Theorem 5.1. *For any fixed k the* TeamLog *satisfiability problem for formulas from $\mathcal{L}^{\text{Tind}}$ with modal depth bounded by k is NPTIME-complete.*

Proof. By Lemma 4.5 and the construction of TeamLog$^{\text{ind}}$ tableau based on the pre-tableau constructed by Algorithm 4.1 presented in Lemma 4.8, the size of the tableau for a satisfiable formula φ is bounded by $\mathcal{O}\left(|\varphi|^{2\mathrm{dep}(\varphi)}\right)$. Hence, if modal depth of φ is bounded by k, then the size of the tableau is bounded by $\mathcal{O}\left(|\varphi|^{2k}\right)$. This means that the satisfiability of φ with bounded modal depth can be checked by the following non-deterministic Algorithm 5.1.

Algorithm 5.1: DecideSatisfiabilityNonDeterministic

Input: a formula φ
Output: a decision whether φ is satisfiable or not

Guess a tableau $\mathcal{T}$ satisfying φ;
if *$\mathcal{T}$ is a tableau for φ* **then**
  return `satisfiable`;

Since tableau $\mathcal{T}$ constructed by Algorithm 5.1 is of polynomial size, so checking if it is a tableau for φ can be realized in polynomial time. This shows that satisfiability of φ can be

[1] In fact in [56] logic T_n (not KD_n) is considered, but all the proofs there that work for T_n work also for KD_n as well.

checked in NPTIME. The problem is also NPTIME-complete, as the satisfiability problem for propositional logic is NPTIME-hard. □

5.2 Restricting number of propositional symbols of TEAMLOG$^{\mathbf{ind}}$

Another constraint on the language is bounding the number of propositional symbols. As was shown in [56], constraining the language of logics K_n, KD_n (for $n \geq 1$) and $\mathrm{KD45}_n$ (for $n \geq 2$) this way does not change the hardness of the satisfiability problem for them, even if $|\mathcal{P}| = 1$. Hence, by Fact 4.1, the problem of checking TEAMLOG satisfiability of formulas from $\mathcal{L}^{\mathrm{Tind}}$ with the number of propositional symbols bounded by 1 is PSPACE-hard as well.

Theorem 5.2. *The* TEAMLOG *satisfiability problem for formulas from* $\mathcal{L}^{\mathrm{Tind}}[\{p\}]$ *is PSPACE-complete.*

Similarly to [56] we can show that if bounding the number of propositional atoms is combined with bounding the modal depth of formulas, then the complexity is the satisfiability problem is reduced to linear time. However, the constant factor depends exponentially on the number of propositional symbols.

Theorem 5.3. *For any fixed* $k, l \geq 1$*, if the number of propositional symbols is bounded by* l *and the modal depth of formulas is bounded by* k*, then the* TEAMLOG *satisfiability problem can be solved in linear time.*

Proof. By the same argument as in [56], if $|\mathcal{P}| \leq l$, then there is a finite number of equivalence classes (based on logical equivalence) of formulas of modal depth bounded by k in language $\mathcal{L}^{\mathrm{Tind}}$. This can be shown by induction on k (see for example [10, Proposition 2.29]). Thus there is a finite set $\varphi_1, \ldots, \varphi_N$ of satisfiable formulas, each representing a different equivalence class all of whose members are satisfiable (the number of these classes is $\mathcal{O}\big(2^{|\mathcal{P}|}\big)$). There is also a corresponding fixed finite set of models $M_1, \ldots, M_K$ with $K \leq N$ satisfying these formulas. To check the satisfiability of a formula, it is enough to check whether it is satisfied in one of the models $M_1, \ldots, M_N$, and this can be done in time linear in the length of the formula (as the set of relevant models is fixed, it only contributes to the constant factor). □

Chapter 6

The complexity of the satisfiability problem for restricted TeamLog framework

In this chapter we study the effects of restricting language $\mathcal{L}^{\mathrm{T}}$ on the complexity of satisfiability problem of TeamLog. Since we know already, from proof of Theorem 4.19, that even if modal depth of formulas is bounded by 2, the satisfiability problem is still EXPTIME hard, so bounding modal depth of formulas would be a very restrictive method for making the complexity of the satisfiability problem solvable in polynomial space or non-deterministic polynomial time. Therefore we will study the effects of restricting modal context of formulas, which can be seen as generalization of restricting modal depth of formulas. Initial investigation of such restriction have been done in [39] where a more forbidding restriction than the one studied in this thesis was proposed. In [38] restrictions of modal context of various basic multimodal logics with iterated modalities generated by various combinations of axioms **K**, **D**, **T**, **4**, **5** were studied. The restrictions presented in this chapter could be seen as extensions and adaptations of those from [38] for TeamLog framework, particularly to the situation where there are iterated modalities and axioms interrelating different groups of modal operators.

We will propose two modal context restrictions. Both of them result in PSPACE completeness of the satisfiability. However, only one of them leads to NPTIME completeness of the satisfiability problem, when modal depth of formulas is bounded by a constant. In the case of another one, the problem remains PSPACE complete if modal depth of formulas is ≥ 2. Since we consider the first of these restrictions too restrictive, as it does not allow to express important properties of multiagent systems such as collective intentions, we study possible refinements of the second restriction. As the result we propose further restriction of the language which, when combined with the second modal context restriction and with bounding modal depth of formulas by a constant, leads to NPTIME completeness of the satisfiability problem. The effect of restricting the number of propositional symbols is also studied in the case of $\mathcal{L}^{\mathrm{T}}$, as well as its combination with restricting modal depth of formulas.

6.1 Modal context restriction

We start by defining the notion of modal context restriction for general language of multimodal logic. First we need a notion of modal context of a formula within a formula. Let $\mathcal{L}[\mathcal{P}, \Omega]$ be a multimodal language.

Definition 15 (Modal context of a formula within a formula)**.** *Let* $\{\varphi, \xi\} \subseteq \mathcal{L}[\mathcal{P}, \Omega]$. *The* modal context of formula ξ within formula φ *is a set of finite sequences over* Ω*,* $\mathrm{cont}\,(\xi, \varphi) \subseteq \Omega^*$*, defined inductively as follows:*

- $\mathrm{cont}\,(\xi, \varphi) = \varnothing$*, if* $\xi \notin \mathrm{Sub}(\varphi)$*,*
- $\mathrm{cont}\,(\varphi, \varphi) = \{\varepsilon\}$*,*
- $\mathrm{cont}\,(\xi, \neg\psi) = \mathrm{cont}\,(\xi, \psi)$*, if* $\xi \neq \neg\psi$*,*
- $\mathrm{cont}\,(\xi, \psi_1 \wedge \psi_2) = \mathrm{cont}\,(\xi, \psi_1) \cup \mathrm{cont}\,(\xi, \psi_2)$*, if* $\xi \neq \psi_1 \wedge \psi_2$*,*
- $\mathrm{cont}\,(\xi, \Box\psi) = \Box \cdot \mathrm{cont}\,(\xi, \psi)$*, if* $\xi \neq \Box\psi$ *and* $\Box \in \Omega$*,*

where $\Box \cdot S = \{\Box \cdot s : s \in S\}$*, for* $\Box \in \Omega$ *and* $S \subseteq \Omega^*$*.*

Definition 16 (Modal context restriction)**.** *A* modal context restriction *is a set of finite sequences over* Ω*,* $R \subseteq \Omega^*$*, constraining possible modal contexts of subformulas within formulas. We say that a formula* $\varphi \in \mathcal{L}[\mathcal{P}, \Omega]$ satisfies a modal context restriction $R \subseteq \Omega^*$ *iff for all* $\xi \in \mathrm{Sub}(\varphi)$ *it holds that* $\mathrm{cont}\,(\xi, \varphi) \subseteq R$*.*

6.2 Restricting modal context of TeamLog

In this chapter we study two modal context restrictions of the language of TeamLog that lead to PSPACE completeness of the satisfiability problem. The restrictions are presented below.

Definition 17 (Restriction $\mathbf{R}_1$)**.** *Let*

$$\mathbf{R_1} = \Omega^* \setminus \left(\Omega^* \cdot \left[\bigcup_{G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}} (S_{\mathrm{I}}(G) \cup S_{\mathrm{IB}}(G)) \cup \bigcup_{G \in \mathrm{P}(\mathcal{A}), |G| \geq 2} S_{\mathrm{B}}(G) \right] \cdot \Omega^* \right),$$

where

$$\begin{aligned} S_{\mathrm{IB}}(G) &= \bigcup_{j \in G} [\mathrm{I}]_G^+ \cdot \left\{ [\mathrm{B}]_j, [\mathrm{B}]_{\{j\}}^+ \right\}^* \cdot T_{\mathrm{B}}(\{j\}) \cdot T_{\mathrm{I}}(\{j\}), \text{ and} \\ S_O(G) &= [O]_G^+ \cdot T_O(G), \\ T_O(G) &= \{[O]_j : j \in G\} \cup \{[O]_H^+ : H \in \mathrm{P}(\mathcal{A}), H \cap G \neq \varnothing\}, \end{aligned}$$

for $O \in \{\mathrm{B}, \mathrm{I}\}$*. The set of formulas in* $\mathcal{L}^{\mathrm{T}}$ *satisfying restriction* $\boldsymbol{R}_1$ *will be denoted by* $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$*.*

Definition 18 (Restriction $\mathbf{R}_2$)**.** *Let*

$$\mathbf{R_2} = \Omega^* \setminus \left(\Omega^* \cdot \left[\bigcup_{G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}} (S_{\mathrm{I}}(G) \cup S_{\mathrm{IB}}(G)) \cup \bigcup_{G \in \mathrm{P}(\mathcal{A}), |G| \geq 2} \tilde{S}_{\mathrm{B}}(G) \right] \cdot \Omega^* \right),$$

where

$$\tilde{S}_{\mathrm{B}}(G) = [\mathrm{B}]_G^+ \cdot \left(\{[\mathrm{G}]_j : j \in G\} \cup \bigcup_{O \in \{\mathrm{B}, \mathrm{I}\}} T_O(G) \right)$$

and S_{IB}*,* S_{I} *and* T_O*, for* $O \in \{\mathrm{B}, \mathrm{I}\}$*, are defined like in the case of restriction* $\boldsymbol{R}_1$*. The set of formulas in* $\mathcal{L}^{\mathrm{T}}$ *satisfying restriction* $\boldsymbol{R}_2$ *will be denoted by* $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$*.*

Restriction $\mathbf{R}_1$ forbids any operator $[O]_j$ or $[O]^+_H$, with $O \in \{\mathrm{B}, \mathrm{I}\}$ in the context of $[O]^+_G$, if $j \in G$ or $H \cap G \neq \varnothing$. Additionally the restriction forbids subsequences contained in S_{IB}. Forbidding subsequences from S_{IB} is related to mixed axioms **BI4** and **BI5**. Notice that it is possible to rewrite the formulas used in proof of Theorem 4.19 by replacing operators $[\mathrm{B}]^+_{\{1,2\}}$ with $[\mathrm{I}]^+_{\{1\}}$ and operators $[\mathrm{B}]_1$ and $[\mathrm{B}]_2$ with $[\mathrm{B}]_1[\mathrm{I}]_1$ and still have a reduction of the winning strategy in two person corridor tiling game problem to the TeamLog satisfiability problem. For this reason forbidding sequences from S_{IB} is needed.

Restriction $\mathbf{R}_2$ is a refinement of restriction $\mathbf{R}_1$ as it forbids any operator $[O]_j$ or $[O]^+_H$, with $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ in the context of $[\mathrm{B}]^+_G$, if $j \in G$ or $H \cap G \neq \varnothing$. Thus any formula $\varphi \in \mathcal{L}^{\mathrm{T}}$ satisfying restriction $\mathbf{R}_2$, satisfies restriction $\mathbf{R}_1$ as well, that is $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}} \subseteq \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$. Notice that if $|\mathcal{A}| = 1$, then $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}} = \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$.

Example 1. *The following formulas satisfy restrictions $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$.*

$$[\mathrm{I}]^+_{\{1,2\}}[\mathrm{B}]_1 p, \qquad [\mathrm{B}]^+_{\{1,2\}}\left(q \vee [\mathrm{B}]_3 p\right).$$

The following formulas satisfy restriction $\boldsymbol{R}_1$ and violate restriction $\boldsymbol{R}_2$

$$[\mathrm{B}]^+_{\{1,2\}}[\mathrm{I}]^+_{\{1,2\}} p, \qquad [\mathrm{B}]^+_{\{1,2\}}[\mathrm{G}]_1 p.$$

The following formulas violate both restrictions $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$

$$[\mathrm{I}]^+_{\{1,2\}}[\mathrm{B}]_1[\mathrm{I}]_1 q, \qquad [\mathrm{B}]^+_{\{1,2\}}[\mathrm{B}]_1 p.$$

6.3 The complexity of the satisfiability problem

In this section we study the complexity of checking TeamLog satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ and $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$. For checking the TeamLog satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$ we will use the method based on pre-tableau construction presented in Section 4.2. However, adopting a similar algorithm for $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ would not work. This is because, as we show below, formulas of $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ may require an exponentially deep model with respect to the size of input formula, while all the algorithms based on the pre-tableau method perform a depth first search constructing sequences of nodes that constitute the tree-like structure of the pre-tableau for a given input.

Theorem 6.1. *Let $|\mathcal{A}| \geq 2$. Then there exists a* TeamLog *satisfiable formula $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ such that any* TeamLog *model $\mathcal{M}$ in which it is satisfied contains a sequence of pairwise different worlds of length exponential with respect to $|\varphi|$.*

(Proof of the theorem is moved to the Appendix.)

6.3.1 The algorithm for $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$

Similarly to the algorithm presented in Section 4.2, Algorithm 6.1 for checking TeamLog satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$ presented in this section constructs a pre-tableau which forms a basis for a TeamLog tableau for φ. The definition of a pre-tableau associated with TeamLog extends the definition given in Section 4.2.2 for TeamLog$^{\mathrm{ind}}$, by putting additional requirements on nodes that are states of the pre-tableau. Before we discuss these extensions, we need to introduce a new notion of $[O]^+$-expanded set of formulas, for $O \in \{\mathrm{B}, \mathrm{I}\}$, related to iterated modalities.

Definition 19 ($[O]^+$-expanded set of formulas)**.** *A set of formulas $\Phi \subseteq \mathcal{L}^{\mathrm{T}}$ is* $[O]^+$-expanded*, with $O \in \{\mathrm{B}, \mathrm{I}\}$ and $G \subseteq \mathcal{A}$, if the following condition is satisfied:*

CE *If $\neg[O]^+_G\psi \in \Phi$, then for all $j \in G$, $\{[O]_j\psi, [O]_j[O]^+_G\psi\} \subseteq \neg\Phi$ and there exists $j \in G$ such that either $\neg[O]_j\psi \in \Phi$ or $\neg[O]_j[O]^+_G\psi \in \Phi$.*

The notion of [B]-expanded tableau has to be extended as follows.

Definition 20 ([B]-expanded tableau)**.** *A* [B]-expanded tableau *is a* $[\mathrm{B}]^+$*-expanded and* $[\mathrm{I}]^+$*-expanded closed propositional tableau* $\mathcal{T}$ *such that for all* $j \in \mathcal{A}$*:*

4. *If $[\mathrm{B}]_j\varphi \in \neg\mathcal{T}$ and $[O]_j\psi \in \neg\mathrm{PT}(\varphi)$, then $[O]_j\psi \in \neg\mathcal{T}$, where $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$.*
5. *If $[\mathrm{B}]_j\varphi \in \neg\mathcal{T}$ and $[O]^+_G\psi \in \neg\mathrm{PT}(\varphi)$ with $j \in G$, then $[O]_j\psi \in \neg\mathcal{T}$ and $[O]_j[O]^+_G\psi \in \neg\mathcal{T}$, where $O \in \{\mathrm{B}, \mathrm{I}\}$.*

A [B]-expanded tableau for a formula φ *is a minimal* [B]*-expanded tableau* $\mathcal{T}$ *such that* $\varphi \in \mathcal{T}$*.*

We extend the definition of $\mathrm{OT}_{[\mathrm{B}]}(\varphi)$ from Section 4.1, associated with the notion of [B]-expanded tableau, to the case of $\mathcal{L}^{\mathrm{T}}$. In this case $\mathrm{OT}_{[\mathrm{B}]}(\varphi)$ is defined inductively with points 1 – 4 remaining unchanged and with point 5 being replaced by points 6 and 7 below:

6. $\mathrm{OT}_{[\mathrm{B}]}([O]^+_G\psi) = \bigcup_{j\in G}(\mathrm{OT}_{[\mathrm{B}]}([O]_j\psi) \cup \{[O]_j[O]^+_G\psi\})$, where $O \in \{\mathrm{B}, \mathrm{I}\}$.
7. $\mathrm{OT}_{[\mathrm{B}]}([\mathrm{B}]_j\psi) = \{[\mathrm{B}]_j\psi\} \cup \bigcup_{O\in\{\mathrm{B},\mathrm{G},\mathrm{I}\}} \bigcup_{[O]_j\xi\in\neg\mathrm{PT}(\psi)} \mathrm{OT}_{[\mathrm{B}]}([O]_j\xi) \cup$
$$\bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \bigcup_{[O]^+_{G\cup\{j\}}\xi\in\neg\mathrm{PT}(\psi)} \mathrm{OT}_{[\mathrm{B}]}\left([O]^+_{G\cup\{j\}}\xi\right).$$

In this case subformulas of φ of the form $[O]^+_{G\cup\{j\}}\xi$ are expanded and their subformulas are properly added to the set $\mathrm{OT}_{[\mathrm{B}]}(\varphi)$ as well. It is easy to see that every [B]-expanded tableau for φ is a maximal consistent subset of $\neg\mathrm{OT}_{[\mathrm{B}]}(\varphi)$.

The notion of pre-tableau associated with TeamLog extends the notion of pre-tableau associated with $\mathrm{TeamLog}^{\mathrm{ind}}$ as follows. Nodes of a pre-tableau constructed for an input formula φ are labelled with subsets of $\neg\mathrm{Cl}(\varphi)$. States are nodes with labels being [B]-expanded tableaux.

We start with presenting Algorithm 6.1 for checking TeamLog satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$. The algorithm is an extension of Algorithm 4.1. Like in the case of that algorithm, it consists of two stages. First it attempts to construct a pre-tableau based on the input formula and then it marks nodes of the constructed pre-tableau. The difference lies in realization of these two stages. Firstly, two steps, of state construction and of successors creation, that constitute the first stage are changed, so that states of the pre-tableau are [B]-expanded tableaux in accordance with the new definition and issues related to existence of iterated modalities in the step of successors creation are addressed. Secondly, the procedures of creating R^{B}_j-successors and R^{I}_j-successors are modified to address the issues related to existence of iterated modalities. When creating $\neg[O]_j[O]^+_G\xi$-successor, with $O \in \{\mathrm{B}, \mathrm{I}\}$ and $j \in G$, $L^{[O]_j}(\cdot)$ of a $[O]^+_G\psi$*-Ancestor* is checked. A state t is called a $[O]^+_G\xi$*-Ancestor* of state s if t is an ancestor of s and for every state u on the path from t to s such that $u \neq s$, there exists $j \in G$ such that u is a $\neg[O]_j[O]^+_G\xi$-Successor. If the label of a potential R^O_j-successor, with $O \in \{\mathrm{B}, \mathrm{I}\}$, of a state s is equal to the label of a successor node n of some $[O]^+_G\psi$*-Ancestor* of s which is on the path from t to s, then construction of the successor of s is blocked by n. This is illustrated in Figure 6.1.

To decide whether a node containing a formula of the form $\neg[O]^+_G\xi$ is satisfiable, it has to be checked whether an appropriate sequence of states can be constructed, that would indicate that this formula is satisfiable. Since creation of successors for formulas of the form $\neg[O]_j[O]^+_G\xi$, with $j \in G$, may be blocked by some ancestor node, the decision whether such an appropriate

Algorithm 6.1: DecideSatisfiability2

```
Input: a formula φ
Output: a decision whether φ is satisfiable or not

/* Pre-tableau construction                                              */
Construct a pre-tableau consisting of single node root, with L(root) = {φ} and all
successor relations being empty;
repeat
    Let Z be the set of all leaves of the pre-tableau with labelling sets that are not
    blatantly inconsistent;
    if there is n ∈ Z such that n is not a state and ψ ∈ L(n) is a witness to that then
        FormState2(n,ψ);
    else if there is s ∈ Z then
        foreach ψ ∈ L(s) do
            CreateSuccessorsB2(s,ψ);
            CreateSuccessorsG(s,ψ);
            CreateSuccessorsI2(s,ψ);
until no change occurred;

/* Marking nodes and deciding satisfiability                             */
repeat
    MarkNodes2;
until no new node marked;

if root is marked sat then
    return sat;
else
    return unsat;
```

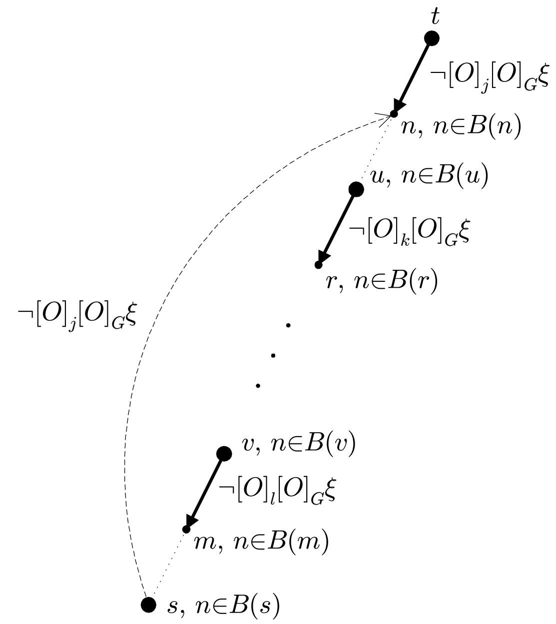

Figure 6.1: Creation of $\neg[O]_{[O]_j^+\xi}$-successor of s is blocked by its ancestor n which is a $\neg[O]_{[O]_j^+\xi}$-successor of a $[O]_G^+\xi$-*Ancestor* t of s. Dotted lines depict sequences of internal nodes (these sequences can be empty, in which case the starting node coincides with the ending state).

sequence of states can be constructed may have to be suspended until the satisfiability of the ancestors is checked. Therefore for each node n there is a set $B(n)$ associated with it and containing weak ancestors[1] that block creation successors of states in the n-subtree of the pre-tableau (c.f. Figure 6.1).[2]. Whenever a new node n is created by the algorithm, the associated set of nodes $B(n)$ is set to $\varnothing$.

During the stage of marking nodes, nodes of the pre-tableau are marked either `sat`, `unsat`, or `undec`. A node n being marked `undec` indicates that satisfiability of $\bigwedge L(n)$ could not be decided due to existence of a formula of the form $\neg[O]_G^+\psi$ in its label for which an appropriate sequence of states was not constructed yet. We call such formulas unresolved in a given node, as defined below.

Definition 21 (Unresolved formula)**.** *Let n be a node in a pre-tableau and let $\neg[O]_G^+\psi \in L(n)$ with $O \in \{\mathrm{B}, \mathrm{I}\}$. A formula $\neg[O]_G^+\psi$ is* unresolved in n *if one of the following holds:*

- *n is an internal node and a $\neg[O]_j[O]_G^+\psi$-descendant with $j \in G$, none of its successors is marked* `sat`*, there exists a successor of n marked* `undec` *and $B(n) \neq \varnothing$,*
- *n is a state and a $\neg[O]_j[O]_G^+\psi$-Successor with $j \in G$, $B(n) \neq \varnothing$, none of $\neg[O]_k[O]_G^+\psi$-successors of n, with $k \in G$, is marked* `sat` *and $[O]_k\psi \in L(n)$, for all $k \in G$.*

Notice that if $B(n) = \varnothing$, then a node cannot be marked `undec`.

We show first that for any input formula $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$ the algorithm for checking satisfiability stops. We start with an auxiliary lemma extending Lemma 4.4 used in the analysis of the algorithm for checking TeamLog satisfiability of formulas from $\mathcal{L}^{\mathrm{Tind}}$.

[1]A *weak ancestor* of n is either an ancestor of n or n.

[2]The *n-subtree* is a subtree of the pre-tableau with n being its root

Procedure 6.2: FormState2

Input: a node n and a formula ψ

if *ψ is of the form* $\neg\neg\xi$ **then**
- Create a successor m of n and set $L(m) := L(n) \cup \{\xi\}$;

else if *ψ is of the form* $\xi_1 \wedge \xi_2$ **then**
- Create a successor m of n and set $L(m) := L(n) \cup \{\xi_1, \xi_2\}$;

else if *ψ is of the form* $\neg(\xi_1 \wedge \xi_2)$ **then**
- Create three successors m_1, m_2 and m_3 of n and set $L(m_1) := L(n) \cup \{\sim\xi_1, \xi_2\}$, $L(m_2) := L(n) \cup \{\xi_1, \sim\xi_2\}$ and $L(m_3) := L(n) \cup \{\sim\xi_1, \sim\xi_2\}$;

else if *ψ is of the form* $[\mathrm{B}]_j\xi$ *or of the form* $\neg[\mathrm{B}]_j\xi$ **then**
- **if** *there is* $[O]_j\zeta \in \neg\mathrm{PT}(\xi)$ *such that* $[O]_j\zeta \notin \neg L(n)$ **then**
  - Create two successors m_1 and m_2 of n and set $L(m_1) := L(n) \cup \{[O]_j\zeta\}$ and $L(m_2) := L(n) \cup \{\neg[O]_j\zeta\}$;
- **else if** *there is* $[O]^+_G\zeta \in \neg\mathrm{PT}(\xi)$ *with* $j \in G$ *such that either* $[O]_j\zeta \notin \neg L(n)$ *or* $[O]_j[O]^+_G\zeta \notin \neg L(n)$ **then**
  - Create four successors m_1, m_2, m_3 and m_4 of n and set $L(m_1) := L(n) \cup \{[O]_j\zeta, [O]_j[O]^+_G\zeta\}$, $L(m_2) := L(n) \cup \{[O]_j\zeta, \neg[O]_j[O]^+_G\zeta\}$, $L(m_3) := L(n) \cup \{\neg[O]_j\zeta, [O]_j[O]^+_G\zeta\}$, $L(m_4) := L(n) \cup \{\neg[O]_j\zeta, \neg[O]_j[O]^+_G\zeta\}$;

else if *ψ is of the form* $\neg[O]^+_G\xi$ **then**
- **foreach** $(H_1, H_2) \subseteq \mathrm{P}(G) \times \mathrm{P}(G)$ *such that* $H_1 \cup H_2 \neq \varnothing$ **do**
  - Create a successor m of n and set $L(m) := L(n) \cup \bigcup_{j\in H_1}\{\neg[O]_j\xi\} \cup \bigcup_{j\in G\setminus H_1}\{[O]_j\xi\} \cup \bigcup_{j\in H_2}\{\neg[O]_j[O]^+_G\xi\} \cup \bigcup_{j\in G\setminus H_2}\{[O]_j[O]^+_G\xi\}$;

else if *ψ is of the form* $[O]^+_G\xi$ **then**
- Create a successor m of n and set $L(m) := L(n) \cup \bigcup_{j\in G}\{[O]_j\xi, [O]_j[O]^+_G\xi\}$;

Procedure 6.3: CreateSuccessorsB2

Input: a state s and a formula $\psi \in L(s)$

if *ψ is of the form* $\neg[\mathrm{B}]_j\xi$ **then**
- **if** *there is an* R^{B}_j*-Predecessor* t *of* s *such that* $\neg[\mathrm{B}]_j\xi \in L(t)$ *and* $L^{\neg[\mathrm{B}]_j}(t,\xi) = L^{\neg[\mathrm{B}]_j}(s,\xi)$ **then**
  - **if** $\xi = [\mathrm{B}]^+_G\zeta$ *with* $j \in G$ *and* s *is a* $\neg[\mathrm{B}]_j\xi$*-Successor of* t **then**
    - For every descendant m of t on the path from t to s set $B(m) := B(m) \cup \{n\}$, where n is an R^{B}_j-successor of t on the path from t to s;
- **else if** $\xi = [\mathrm{B}]^+_G\zeta$ *with* $j \in G$ *and there is a* $\neg[\mathrm{B}]^+_G\zeta$*-Ancestor* t *of* s *such that its* $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\zeta$*-successor* n *is on the path from* t *to* s *and* $L^{[\mathrm{B}]_j}(s) = L^{[\mathrm{B}]_j}(t)$ **then**
  - For every descendant m of t on the path from t to s set $B(m) := B(m) \cup \{n\}$;
- **else** Create an R^{B}_j-Successor v of s with $L(v) = L^{\neg[\mathrm{B}]_j}(s,\xi)$;

else if *ψ is of the form* $[\mathrm{B}]_j\xi$ *and there are no formulas of the form* $\neg[\mathrm{B}]_j\zeta \in L(s)$ **then**
- If there is no R^{B}_j-Predecessor t of s such that $[\mathrm{B}]_j\xi \in L(t)$ and $L^{[\mathrm{B}]_j}(t) = L^{[\mathrm{B}]_j}(s)$ and $L^{[\mathrm{B}]_j}(s) \not\subseteq L(s)$, then create an R^{B}_j-successor u of s with $L(u) = L^{[\mathrm{B}]_j}(s)$;

Procedure 6.4: CreateSuccessorsI2

Input: a state s and a formula $\psi \in L(s)$

if *ψ is of the form $\neg[\mathrm{I}]_j\xi$* **then**
- **if** *there is no R^{B}_j-predecessor state t of s such that $\neg[\mathrm{I}]_j\xi \in L(t)$ and $L^{\neg[\mathrm{I}]_j}(t,\xi) = L^{\neg[\mathrm{I}]_j}(s,\xi)$* **then**
  - **if** *$\xi = [\mathrm{I}]^+_G\zeta$ with $j \in G$ and there is a $\neg[\mathrm{I}]^+_G\zeta$-Ancestor t of s such that its $\neg[\mathrm{I}]_j[\mathrm{I}]^+_G\zeta$-successor n is on the path from t to s and $L^{[\mathrm{I}]_j}(s) = L^{[\mathrm{I}]_j}(t)$* **then**
    - For every descendant m of t on the path from t to s set $B(m) := B(m) \cup \{n\}$;
  - **else** Create an R^{I}_j-successor v of s with $L(v) = L^{\neg[\mathrm{I}]_j}(s,\xi)$;

else if *ψ is of the form $[\mathrm{I}]_j\xi$ and there are no formulas of the form $\neg[\mathrm{I}]_j\zeta \in L(s)$* **then**
- If there is no R^{B}_j-Predecessor t of s such that $[\mathrm{I}]_j\xi \in L(t)$ and $L^{[\mathrm{I}]_j}(t) = L^{[\mathrm{I}]_j}(s)$ and $L^{[\mathrm{I}]_j}(s) \nsubseteq L(s)$, then create an R^{I}_j-successor u of s with $L(u) = L^{[\mathrm{I}]_j}(s)$;

Procedure 6.5: MarkNodes2

repeat
- **if** *n is an unmarked state* **then**
  - **if** *n has a successor that is marked* `unsat` **then**
    - Mark n `unsat`;
  - **else if** *n does not have an unmarked successor* **then**
    - **if** *there is a formula $\neg[O]^+_G\psi \in L(n)$ which is unresolved in n* **then**
      - **if** $B(n) = \{n\}$ **then**
        - Mark n `unsat`;
      - **else**
        - Mark n `undec`;
    - **else**
      - Mark n `sat`;
- **else if** *n is an unmarked internal node* **then**
  - **if** *$L(n)$ is blatantly inconsistent or all successors of n are marked* `unsat` **then**
    - Mark n `unsat`;
  - **else if** *there exists a successor of n marked* `sat` **then**
    - Mark n `sat`;
  - **else if** *n does not have an unmarked successor* **then**
    - **if** *there exists a formula $\neg[O]^+_G\psi \in L(n)$ which is unresolved in n and $B(n) = \{n\}$* **then**
      - Mark n `unsat`;
    - **else**
      - Mark n `undec`;

until *no new node marked*;

Lemma 6.2. *Let s and $t \in R^{\mathrm{B}}_j$-Succ(s) be states of the pre-tableau constructed by Algorithm 6.1 for some input formula $\varphi \in \mathcal{L}^{\mathrm{T}}$. Then the following hold for $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$:*

1. $\neg[O]_j\xi \in L(s)$ *iff* $\neg[O]_j\xi \in L(t)$.
2. $[O]_j\xi \in L(s)$ *iff* $[O]_j\xi \in L(t)$.
3. $L^{[O]_j}(s) = L^{[O]_j}(t)$.
4. $L^{\neg[O]_j}(s,\xi) = L^{\neg[O]_j}(t,\xi)$.

Proof. For points 1 and 2, notice that the left to right implications can be shown by the same arguments as those used in proof of Lemma 4.4. For the right to left implication we will show first that $[O]_j\xi \in \neg L(t)$ implies $[O]_j\xi \in \neg L(s)$. Assume that there is a formula $[O]_j\xi \in \neg L(t)$. Then one of the following cases holds:

(i). $[O]_j\xi \in \neg L(s)$,

(ii). there is a formula $[\mathrm{B}]_j\psi \in \neg L(s)$ such that $[O]_j\xi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\psi)$.

If case (i) holds, then the claim holds. If case (ii) holds, then $[O]_j\psi \in \neg L(s)$, as s is a state and $L(s)$ is a [B]-expanded tableau.

Now, $\neg[O]_j\xi \in L(t)$ implies $[O]_j\xi \in \neg L(s)$ and it must hold that $\neg[O]_j\xi \in L(s)$, as otherwise it would be $[O]_j\xi \in L(s)$ and $[O]_j\xi \in L(t)$, which would contradict the assumption that t is a state and $L(t)$ cannot be blatantly inconsistent. If $[O]_j\xi \in L(t)$, then it must be that $[O]_j\xi \in L(s)$ by similar arguments. Hence points 1 and 2 hold. Points 3 and 4 are straightforward implication of points 1 and 2 and the definitions of $L^{[O]_j}(\cdot)$, for $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$. □

We will show that for any formula $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$, the height of a state of the pre-tableau constructed for φ by Algorithm 6.1 is bounded by a polynomial depending on $|\varphi|$, while its state height is bounded by a polynomial depending on $\mathrm{dep}(\varphi)$. The main difficulty and difference from proof of analogous fact for pre-tableaux constructed by the algorithm presented in Section 4.2.2 lies in showing that the length of sequences of states with unchanged modal depth is bounded by a polynomial depending on the length of input formula.

The main problem here are the formulas of the form $[O]^+_G\psi$ or $\neg[O]^+_G\psi$. This is because if t is a R^O_j-Successor of s in a pre-tableau constructed by Algorithm 6.1, then any formula $[O]^+_G\psi \in L(s)$ with $j \in G$ is in $L(t)$ as well. Similarly with a formula of the form $\neg[O]^+_G\psi \in L(s)$, if t is a $\neg[O]_j[O]^+_G\psi$-Successor of s. Moreover, if additionally u is a R^O_k-Successor of t and $k \in G$, then for any formula $\xi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\psi)$, $\xi \in L(t)$ (as it is added during [B]-expanded tableau formation) and $\xi \in L(u)$, as $[O]^+_G\psi \in L(u)$. Thus formulas of the form $[O]^+_G\psi$ may carry over to the label of the R^O_j-Successor formulas from $\neg\mathrm{OT}_{[\mathrm{B}]}(\psi)$. Similarly, the may carry over the formulas $[O]_j[O]^+_G\psi$ and $[O]_j\psi$, that are added to the label during the closed propositional tableau formation.

To analyse length of sequences of R^O-Successors in a pre-tableau constructed by Algorithm 6.1, we need to separate the formulas in labels of states which are carried by some other formulas from those which are not carried by any other formula. We will say that a formula $[O]^+_G\psi$ *carries* a formula ξ if $\xi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\psi)$ or $\xi \in \widetilde{\mathrm{Cl}}([O]^+_G\psi)$, where $\widetilde{\mathrm{Cl}}([O]^+_G\psi) = \{[O]_j\psi : j \in G\} \cup \{[O]_j[O]^+_G\psi : j \in G\}$. Similarly, a formula $\neg[O]^+_G\psi$ carries a formula ξ if $\xi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\psi)$ or $\xi \in \widetilde{\mathrm{Cl}}(\neg[O]^+_G\psi)$, where $\widetilde{\mathrm{Cl}}(\neg[O]^+_G\psi) = \neg\widetilde{\mathrm{Cl}}([O]^+_G\psi)$. Given a set of formulas Φ and a formula ξ we will say that ξ is *carried by Φ* if there is a formula in Φ which carries it.

First we will consider the carried formulas of the form $[O]^+_H\zeta$ or $\neg[O]^+_H\zeta$. Notice that in this case such a formula is carried by some formula $[O]^+_G\psi$ or $\neg[O]^+_G\psi$ if and only if it is in $\neg\mathrm{PT}(\psi)$. Given a set of formulas Φ, let[3]

$$\mathrm{Gr}(\Phi) = \bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \left(\left(\Phi \sqcap [O]^+\right) \;\cup\; \left(\Phi \sqcap \neg[O]^+\right)\right).$$

Let $\mathcal{F}_\Phi : \mathrm{P}\left(\mathcal{L}^{\mathrm{T}}\right) \to \mathrm{P}\left(\mathcal{L}^{\mathrm{T}}\right)$ be defined as follows, for $\Psi \subseteq \mathcal{L}^{\mathrm{T}}$,

$$\mathcal{F}_\Phi(\Psi) = \mathrm{Gr}(\Phi) \setminus \neg\mathrm{PT}\left(\bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \neg\,\Psi/[O]^+\right),$$

where

$$\begin{aligned}
\Phi/[O]^+ &= \{\psi : [O]^+_G\psi \in \Phi,\ \text{for some } G \in \mathrm{P}(\mathcal{A}) \setminus \varnothing\},\\
\Phi \sqcap [O]^+ &= \{[O]^+_G\psi : [O]^+_G\psi \in \Phi,\ \text{for some } G \in \mathrm{P}(\mathcal{A}) \setminus \varnothing\},\\
\Phi \sqcap \neg[O]^+ &= \{\neg[O]^+_G\psi : \neg[O]^+_G\psi \in \Phi,\ \text{for some } G \in \mathrm{P}(\mathcal{A}) \setminus \varnothing\}.
\end{aligned}$$

The operator $\mathcal{F}_\Phi$, when applied to a set of formulas Ψ, removes from Φ all the formulas from $\mathrm{Gr}(\Phi)$ which are carried by Ψ.

Given a set of formulas Φ and a formula ψ we will say that ψ *is uncarried in* Φ if $\psi \in \Phi$ and ψ is not carried by Φ. We will be interested in sets of formulas which are *carry-free*, that is Φ such that all the formulas in Φ are uncarried in it. Given $i \in \mathbb{N}$, let

$$F^{(i)}_\Phi = \begin{cases} \varnothing, & \text{if } i = 0,\\ \mathcal{F}_\Phi\left(F^{(i-1)}_\Phi\right), & \text{if } i > 0 \end{cases}$$

and let $F^{(\infty)}_\Phi = \lim_{i\to\infty} F^{(i)}_\Phi$. As we show below, for any $\Phi \in \mathcal{L}^{\mathrm{T}}$ the limit $F^{(\infty)}_\Phi$ exists.

Lemma 6.3. *For any $\Phi \subseteq \mathcal{L}^{\mathrm{T}}$, $F^{(\infty)}_\Phi$ exists.*

Proof. Notice that for all $i \in N$, $F^{(i)}_\Phi \subseteq \mathrm{Gr}(\Phi)$ and $\mathrm{dep}\big(F^{(i)}\big) \leq \mathrm{dep}(\mathrm{Gr}(\Phi))$. We will show first that for all $i \in \mathbb{N}$, $F^{(2i)}_\Phi \subseteq F^{(2(i+1))}_\Phi$ and

$$F^{(2i)}_\Phi \cap \neg\mathrm{PT}\left(\bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \neg F^{(2i+1)}_\Phi/[O]^+\right) = \varnothing.$$

We will use induction on i. For $i = 0$ the claim is obvious, as $F^{(0)}_\Phi = \varnothing$. Let $i \geq 1$. Since, by the induction hypothesis, $F^{(2(i-1))}_\Phi \subseteq F^{(2i)}_\Phi$, so

$$\neg\mathrm{PT}\left(\bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \neg F^{(2(i-1))}_\Phi/[O]^+\right) \subseteq \neg\mathrm{PT}\left(\bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \neg F^{(2i)}_\Phi/[O]^+\right)$$

and so $F^{(2i+1)}_\Phi \subseteq F^{(2i-1)}_\Phi$. Since, by the definition of $F^{(2i)}_\Phi$,

$$F^{(2i)}_\Phi \cap \neg\mathrm{PT}\left(\bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \neg F^{(2i-1)}_\Phi/[O]^+\right) = \varnothing$$

[3] Then name Gr is from 'group', as it selects the formulas starting with modalities $[O]^+$ related to properties of groups of agents.

and $F_\Phi^{(2i+1)} \subseteq F_\Phi^{(2i-1)}$, so

$$F_\Phi^{(2i)} \cap \neg\mathrm{PT}\left(\bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \neg F_\Phi^{(2i+1)}/[O]^+\right) = \varnothing.$$

This, together with the definition of $F_\Phi^{(2(i+1))}$, implies that $F_\Phi^{(2(i))} \subseteq F_\Phi^{(2(i+1))}$.

Since $\mathrm{Gr}(\Phi)$ is finite and, for all $i \in \mathbb{N}$, $F_\Phi^{(i)} \subseteq \mathrm{Gr}(\Phi)$ and, as we have shown above, $F_\Phi^{(2(i))} \subseteq F_\Phi^{(2(i+1))}$, so there exists $n \in \mathbb{N}$ such that for all $i > n$, $F_\Phi^{(2i)} = F_\Phi^{2(i+1)}$.

Secondly, for $i \in \mathbb{N}$,

$$F_\Phi^{(2i+3)} \setminus F_\Phi^{(2i+2)} = F_\Phi^{(2i+3)} \setminus \left(\mathrm{Gr}(\Phi) \setminus \neg\mathrm{PT}\left(\bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \neg F_\Phi^{(2i+1)}/[O]^+\right)\right)$$

and since $F_\Phi^{(2i+3)} \subseteq \mathrm{Gr}(\Phi)$, so

$$F_\Phi^{(2i+3)} \setminus F_\Phi^{(2i+2)} = F_\Phi^{(2i+3)} \cap \neg\mathrm{PT}\left(\bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \neg F_\Phi^{(2i+1)}/[O]^+\right).$$

Thus if $F_\Phi^{(2i+1)} \setminus F_\Phi^{(2i)} \neq \varnothing$, then

$$\mathrm{dep}\left(F_\Phi^{(2i+1)} \setminus F_\Phi^{(2i)}\right) > \mathrm{dep}\left(F_\Phi^{(2i+3)} \setminus F_\Phi^{(2i+2)}\right)$$

and since, for all $i \in \mathbb{N}$, $\mathrm{dep}\left(F^{(i)}\right) \leq \mathrm{dep}(\mathrm{Gr}(\Phi))$, so there exists $n \in \mathbb{N}$ such that for all $i > n$, $F_\Phi^{(2i+1)} \setminus F_\Phi^{(2i)} = \varnothing$. Hence there exists $n \in \mathbb{N}$, such that $F^{(i)} = F^{(i+1)}$, for all $i > n$, and so $F^{(\infty)}$ exists. $\square$

Let $\widehat{\mathrm{Gr}}(\Phi) = F_\Phi^{(\infty)}$, that is

$$\widehat{\mathrm{Gr}}(\Phi) = \mathrm{Gr}(\Phi) \setminus \neg\mathrm{PT}\left(\bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \neg\widehat{\mathrm{Gr}}(\Phi)/[O]^+\right).$$

The set $\widehat{\mathrm{Gr}}(\Phi)$ it the maximal carry-free subset of $\mathrm{Gr}(\Phi)$ containing all the formulas which are uncarried in $\mathrm{Gr}(\Phi)$. To see how this set is constructed consider the following example. Let $\Phi = \{\psi_1, \psi_2, \psi_3, \psi_4\}$, where

$$\begin{aligned}
\psi_1 &= [\mathrm{I}]^+_{\{1,2\}}(p \wedge [\mathrm{I}]^+_{\{3,4\}}(p \wedge [\mathrm{I}]^+_{\{1,2\}}(p \wedge [\mathrm{I}]^+_{\{3,4\}}p))),\\
\psi_2 &= [\mathrm{I}]^+_{\{3,4\}}(p \wedge [\mathrm{I}]^+_{\{1,2\}}(p \wedge [\mathrm{I}]^+_{\{3,4\}}p)),\\
\psi_3 &= [\mathrm{I}]^+_{\{1,2\}}(p \wedge [\mathrm{I}]^+_{\{3,4\}}p),\\
\psi_4 &= [\mathrm{I}]^+_{\{3,4\}}p.
\end{aligned}$$

Notice that $\neg\mathrm{PT}(\psi_i/[\mathrm{I}]^+) = \neg\{p, \psi_{i+1}\}$, for $i \in \{1,2,3\}$. Notice also that $\mathrm{Gr}(\Phi) = \Phi$. To construct the set $\widehat{\mathrm{Gr}}(\Phi)$ we start with $F_\Phi^{(1)} = \mathrm{Gr}(\Phi)$. The set $F_\Phi^{(2)} = \{\psi_1\}$ is exactly the set of formulas which are uncarried in $\mathrm{Gr}(\Phi)$. The next application of the operator $\mathcal{F}_\Phi$ removes from $\mathrm{Gr}(\Phi)$ the formulas which are carried by $F_\Phi^{(2)}$. Thus $F_\Phi^{(3)} = \{\psi_1, \psi_3, \psi_4\}$. The next application of $\mathcal{F}_\Phi$ removes from $\mathrm{Gr}(\Phi)$ the formulas which are carried by $F_\Phi^{(3)}$, which leads to $F_\Phi^{(4)} = \{\psi_1, \psi_3\}$. Further applications of $\mathcal{F}_\Phi$ change nothing, so $\widehat{\mathrm{Gr}}(\Phi) = \{\psi_1, \psi_3\}$.

Now we turn to the uncarried formulas which are not in $\mathrm{Gr}(\Phi)$ but can 'carry' other formulas to successor labels. These are those formulas in Φ which are (possibly negated) formulas of the form $[O]_j\xi$. More precisely, we will be interested in those of such formulas which are not carried by $\widehat{\mathrm{Gr}}(\Phi)$ nor are elements of $\widetilde{\mathrm{Cl}}(\Phi)$, where

$$\begin{aligned}\widetilde{\mathrm{Cl}}(\Phi) = {} & \{[O]_j\psi : [O]_G^+\psi \in \neg\Phi \text{ and } j \in G\} \cup \\ & \{\neg[O]_j\psi : \neg[O]_G^+\psi \in \Phi \text{ and } j \in G\} \cup \\ & \{[O]_j[O]_G^+\psi : [O]_G^+\psi \in \neg\Phi \text{ and } j \in G\} \cup \\ & \{\neg[O]_j[O]_G^+\psi : \neg[O]_G^+\psi \in \Phi \text{ and } j \in G\}.\end{aligned}$$

The set of such formulas is[4]

$$\mathrm{Ind}(\Phi) = \left(\bigcup_{j\in\mathcal{A}} \Phi \sqcap j\right) \setminus \left(\widetilde{\mathrm{Cl}}(\Phi) \cup \neg\mathrm{OT}_{[\mathrm{B}]}\left(\bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \neg\widehat{\mathrm{Gr}}(\Phi)/[O]^+\right)\right).$$

To see what formulas are contained in this set, consider the following example. Let $\Phi = \{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8, \psi_9\}$, where

$$\begin{aligned}\psi_1 &= [\mathrm{I}]_{\{1,2\}}^+(p \wedge [\mathrm{B}]_4[\mathrm{G}]_4p \wedge [\mathrm{I}]_{\{3,4\}}^+(p \wedge [\mathrm{I}]_1p)),\\ \psi_2 &= [\mathrm{I}]_{\{3,4\}}^+(p \wedge [\mathrm{I}]_1p),\\ \psi_3 &= [\mathrm{I}]_{\{1,2\}}^+(p \wedge [\mathrm{I}]_3p),\\ \psi_4 &= [\mathrm{I}]_{\{3,4\}}^+q,\\ \psi_5 &= [\mathrm{I}]_1p,\\ \psi_6 &= [\mathrm{I}]_1[\mathrm{I}]_1p,\\ \psi_7 &= [\mathrm{I}]_3p,\\ \psi_8 &= [\mathrm{G}]_5p,\\ \psi_9 &= [\mathrm{I}]_4q.\end{aligned}$$

Notice that $\widehat{\mathrm{Gr}}(\Phi) = \{\psi_1, \psi_3, \psi_4\}$ and $\mathrm{Ind}(\Phi) = \{\psi_5, \psi_6\}$. Formula ψ_7 is not in $\mathrm{Ind}(\Phi)$, because $\psi_7 \in \neg\mathrm{OT}_{[\mathrm{B}]}(\psi_3/[\mathrm{I}]^+)$. Formula ψ_8 is not in $\mathrm{Ind}(\Phi)$, because $\psi_8 \in \neg\mathrm{OT}_{[\mathrm{B}]}(\psi_1/[\mathrm{I}]^+)$. Formula ψ_9 is not in $\mathrm{Ind}(\Phi)$, because $\psi_9 \in \neg\widetilde{\mathrm{Cl}}(\psi_4)$.

Although the modal depth of $\widehat{\mathrm{Gr}}(\cdot)$ of labels may not change between R^O-Successors in the sequence of states in a pre-tableau constructed by Algorithm 6.1, there is one more parameter of these sets that will change. Given a formula ψ and $O \in \{\mathrm{B}, \mathrm{I}\}$, we define a set[5]

$$\mathrm{ag}(\psi, [O]^+) = \begin{cases} G, & \text{if } \psi \text{ is of the form } [O]_G^+\xi \text{ or } \neg[O]_G^+\xi,\\ \mathcal{A} \cup \{\omega\}, & \text{otherwise,}\end{cases}$$

where $\omega \notin \mathcal{A}$. Given a set of formulas $\Phi \neq \varnothing$ and $O \in \{\mathrm{B}, \mathrm{I}\}$ we define

$$\mathrm{ag}(\Phi, [O]^+) = \begin{cases} \bigcap_{\psi\in\Phi} \mathrm{ag}(\psi, [O]^+), & \text{if } \Phi \neq \varnothing,\\ \mathcal{A} \cup \{\omega\}, & \text{otherwise.}\end{cases}$$

[4] The name Ind comes from 'individual', as the formulas it selects start with modal operators associated with individual properties of agents.

[5] The name ag comes from 'agents', because it relates to the sets of agents associated with group modalities.

Notice that $\omega \in \mathrm{ag}(\Phi, [O]^+)$ implies that there are no formulas of the form $[O]^+_G\xi$ nor $\neg[O]^+_G\xi$ in Φ. Also, when formulas are removed from Φ, then $\mathrm{ag}(\Phi, [O]^+)$ either remains unchanged or increases.

When analysing how labels of subsequent states change, we will divide them into subsets (levels) of different modal depth of formulas and then we will look at the sets $\mathrm{ag}(\cdot, [O]^+)$ at different levels. Given a set of formulas Φ let $\Phi_d = \{\psi \in \Phi : \mathrm{dep}(\psi) = d\}$. Also, let

$$\mathrm{ag}(\Phi, [O]^+, d) = \mathrm{ag}(\Phi_d, [O]^+).$$

Notice that $\mathrm{ag}(\Phi, [O]^+, d)$ is well defined even for $d > \mathrm{dep}(\Phi)$. Is it simply $\mathcal{A} \cup \{\omega\}$ then. Similarly for levels $d \le \mathrm{dep}(\Phi)$ at which there are no formulas of the form $[O]^+_G\xi$ nor $\neg[O]^+_G\xi$. Notice also, that $\mathrm{ag}(\Phi, [O]^+, 0) = \mathcal{A} \cup \{\omega\}$.

We start the analysis of lengths of sequences of R^O-Successors with R^{I}-Successors. The following lemma gives properties of $\widehat{\mathrm{Gr}}(\cdot)$ and $\mathrm{Ind}(\cdot)$ that follow from modal context restriction $\mathbf{R}_1$. This restriction guarantees that if a formula of the form $[O]^+_H\xi$ is carried by a formula $[O]^+_G\psi$, then it must be that $G \cap H = \varnothing$. Similarly, if a formula of the form $[O]_j\xi$ is carried by a formula $[O]^+_G\psi$, then it must be that $j \notin G$. Thus if $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\Big)$ and there are no formulas of the form $[\mathrm{B}]^+_G\psi \in \neg\Phi_d$ at levels $d > D$, then any formula of the form $[\mathrm{I}]^+_H\xi \in \neg\Phi$ with modal depth $\ge D$ must be uncarried in Φ. Also, if a formula $[\mathrm{I}]_j\xi$ is in $\neg\widetilde{\mathrm{Cl}}(\Phi)$, then it must be in $\neg\widetilde{\mathrm{Cl}}(\widehat{\mathrm{Gr}}(\Phi))$.

Lemma 6.4. *Let $\Phi \subseteq \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ and let $D \ge 0$ and $j \in \mathcal{A}$ be such that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{B}]^+, d\Big)$, for all $d \ge D+1$. Then the following hold*

(i). if $[\mathrm{I}]^+_G\psi \in \neg\Phi$ with $j \in G$ and $\mathrm{dep}([\mathrm{I}]^+_G\psi) \ge D$, then $[\mathrm{I}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(\Phi)$,

(ii). if $\mathrm{dep}(\mathrm{Ind}(\Phi)) \le D$ and $[\mathrm{I}]_j\psi \in \neg\Phi$ with $\mathrm{dep}([\mathrm{I}]_j\psi) \ge D+1$, then $[\mathrm{I}]_j\psi \in \neg\widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(\Phi)\Big)$.

Proof. For point (i) take any formula of the form $[\mathrm{I}]^+_G\psi \in \neg\Phi$ with $j \in G$ and $\mathrm{dep}([\mathrm{I}]^+_G\psi) \ge D$ and suppose that $[\mathrm{I}]^+_G\psi \notin \neg\widehat{\mathrm{Gr}}(\Phi)$. Then either there is a formula $[\mathrm{I}]^+_H\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ or a formula $[\mathrm{B}]^+_H\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ with $[\mathrm{I}]^+_G\psi \in \neg\mathrm{PT}(\xi)$. The first case is impossible as $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\Big)$ for all $d \ge D+1$ and so it must hold that $j \in H$ which would violate modal context restriction $\mathbf{R}_1$. The second case is impossible as well since $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{B}]^+, d\Big)$, for all $d \ge D+1$. Hence it must be that $[\mathrm{I}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(\Phi)$.

For point (ii) take any formula of the form $[\mathrm{I}]_j\psi \in \neg\Phi$ with $\mathrm{dep}([\mathrm{I}]_j\psi) \ge D+1$. Since $\mathrm{dep}(\mathrm{Ind}(\Phi)) \le D$ so $[\mathrm{I}]_j\psi \notin \neg\mathrm{Ind}(\Phi)$. Thus either there is a formula $[\mathrm{I}]^+_G\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ or a formula $[\mathrm{B}]^+_G\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ with $[\mathrm{I}]_j\psi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$, or $[\mathrm{I}]_j\psi \in \neg\widetilde{\mathrm{Cl}}(\Phi)$. The first case is impossible as $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\Big)$ for all $d \ge D+1$ and so it must hold that $j \in G$ which would violate modal context restriction $\mathbf{R}_1$.[6] The second case is impossible as well, as $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{B}]^+, d\Big)$, for all $d \ge D+1$. Hence it must be that $[\mathrm{I}]_j\psi \in \neg\widetilde{\mathrm{Cl}}(\Phi)$. Thus either there is a formula $[\mathrm{I}]^+_G\psi \in \neg\Phi$ such that $j \in G$ or ψ is of the form $[\mathrm{I}]^+_G\xi$ with $j \in G$ and $\psi \in \neg\Phi$. Hence, by point (i), it holds that $[\mathrm{I}]_j\psi \in \neg\widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(\Phi)\Big)$. □

[6] Notice that for this argument to hold it is necessary to forbid sequences $S_{\mathrm{IB}}(G)$ in modal context of formulas.

Lemma 6.4 allows us to analyse the origins of formulas in labels of R^{I}_j-Successors. The corollary below points out the sources of formulas in the successor state with modal depth not smaller than $\mathrm{dep}(\mathrm{Ind}(\cdot))$ of the predecessor state. Roughly speaking all such formulas are either added when the label of the successor states is being closed or are carried by the uncarried formulas from the label of the predecessor state, or are uncarried formulas inherited from the label of the predecessor state.

Corollary 6.5. *Let t be an R^{I}_j-Successor of state s in the pre-tableau constructed by Algorithm 6.1 for an input $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$, with $D \geq 0$ such that $\mathrm{dep}(\mathrm{Ind}(L(s))) \leq D$, $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, d\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, d\Big)$, for all $d \geq D+1$. Then for all $\psi \in L(t)$ with $\mathrm{dep}(\psi) \geq D$ one of the following holds*

(i). $\psi \in \widetilde{\mathrm{Cl}}(L(t))$ *or*

(ii). there exists a formula $[\mathrm{I}]^+_G\xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ with $j \in G$ such that $\psi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$, or

(iii). ψ is of the form $[\mathrm{I}]^+_G\eta$ with $j \in G$ and $\psi \in \neg\widehat{\mathrm{Gr}}(L(s))$, or

(iv). ψ is of the form $\neg[\mathrm{I}]^+_G\eta$ with $j \in G$, $\psi \in \widehat{\mathrm{Gr}}(L(s))$, $\neg[\mathrm{I}]_j[\mathrm{I}]^+_G\eta \in L(s)$ and t is a $\neg[\mathrm{I}]_j[\mathrm{I}]^+_G\eta$-Successor of s.

Proof. Take any $\psi \in L(t)$ with $\mathrm{dep}(\psi) = d \geq D$. If $\psi \in \widetilde{\mathrm{Cl}}(L(t))$, then the claim holds. Otherwise there exists a formula $[\mathrm{I}]_j\xi \in \neg L(s)$ such that $\psi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$. Since $\mathrm{dep}([\mathrm{I}]_j\xi) \geq D+1$ so, by point (ii) of Lemma 6.4, it holds that $[\mathrm{I}]_j\xi \in \neg\widetilde{\mathrm{Cl}}(\widehat{\mathrm{Gr}}(L(s)))$. Hence either there is a formula $[\mathrm{I}]^+_G\xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ with $j \in G$ or ψ is of the form $[\mathrm{I}]^+_G\eta$ with $j \in G$, $[\mathrm{I}]^+_G\eta \in \neg\widehat{\mathrm{Gr}}(L(s))$ and $[\mathrm{I}]_j[\mathrm{I}]^+_G\eta \in \widetilde{\mathrm{Cl}}(\widehat{\mathrm{Gr}}(L(s)))$, or ψ is of the form $\neg[\mathrm{I}]^+_G\eta$ with $j \in G$, $\neg[\mathrm{I}]^+_G\eta \in \widehat{\mathrm{Gr}}(L(s))$ and $[\mathrm{I}]_j[\mathrm{I}]^+_G\eta \in \neg\widetilde{\mathrm{Cl}}(\widehat{\mathrm{Gr}}(L(s)))$.

In the last case it must be that $\neg[\mathrm{I}]_j[\mathrm{I}]^+_G\eta \in \widetilde{\mathrm{Cl}}(\widehat{\mathrm{Gr}}(L(s)))$, as otherwise it would be $[\mathrm{I}]_j[\mathrm{I}]^+_G\eta \in \widetilde{\mathrm{Cl}}(\widehat{\mathrm{Gr}}(L(s)))$ and $[\mathrm{I}]^+_G\eta \in L(t)$, which would contradict the assumption that t is a state and $L(t)$ cannot be blatantly inconsistent. Moreover, in this case it must be that t is a $\neg[\mathrm{I}]_j[\mathrm{I}]^+_G\eta$-Successor of s. To see why assume the opposite. Then there must be a formula $[\mathrm{I}]_j\xi \in L(s)$ such that $[\mathrm{I}]^+_G\eta \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$. As we already observed, it cannot be that $\xi = [\mathrm{I}]^+_G\eta$. This, together with the assumption that $[\mathrm{I}]^+_G\eta \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$, implies that ξ cannot be of the form $[\mathrm{I}]^+_H\zeta$. Now, since $\mathrm{dep}([\mathrm{I}]_j\xi) \geq D+1$ so, by point (ii) of Lemma 6.4, it must be that $[\mathrm{I}]_j\xi \in \neg\widetilde{\mathrm{Cl}}(\widehat{\mathrm{Gr}}(L(s)))$. Since ξ cannot be of the form $[\mathrm{I}]^+_H\zeta$, so there must be formula $[\mathrm{I}]^+_H\xi \in \widehat{\mathrm{Gr}}(L(s))$ with $j \in H$. But this is impossible, as it violates modal context restriction $\mathbf{R}_1$. □

We are now ready to state and prove the lemma about the bounds of the length of a sequence of R^{I}-Successors with unchanged modal depth of labels in the pre-tableau constructed by Algorithm 6.1 for some input φ. It is enough that φ satisfies modal context restriction $\mathbf{R}_1$ for the lemma to hold. To assess lengths of sequences of R^{I}-Successors with unchanged modal depth of labels, we will show that the sets $\mathrm{ag}(\widehat{\mathrm{Gr}}(\cdot), [\mathrm{I}]^+, d)$ and $\mathrm{ag}(\widehat{\mathrm{Gr}}(\cdot), [\mathrm{B}]^+, d)$ must gradually increase proceeding top down, from $d = \mathrm{dep}(\Phi)$ to $d = 1$. For this reason we will need to assess for how long these sets may remain unchanged at different levels. This is expressed by the following properties of states, for $O \in \{\mathrm{B}, \mathrm{I}\}$:

P*O*1 $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [O]^+, d\Big) = \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [O]^+, d\Big)$, We say that the sequence of states $s_0, \ldots, s_m$ satisfies $\mathbf{P}\boldsymbol{O}\mathbf{1}$ if for all $0 < k \leq m$, states s_{k-1} and s_k satisfy $\mathbf{P}\boldsymbol{O}\mathbf{1}$.

Additional factor that needs to be taken into account is the set of formulas of the form $\neg[O]^+_G\xi$ at different levels of the set $\widehat{\mathrm{Gr}}(\cdot)$. The following property states that this set remains unchanged between two states:

$\mathbf{P}\boldsymbol{O}\mathbf{2}$ $\left(\widehat{\mathrm{Gr}}(L(s)) \sqcap \neg[O]^+\right)_d = \left(\widehat{\mathrm{Gr}}(L(t)) \sqcap \neg[O]^+\right)_d.$

We say that the sequence of states $s_0, \ldots, s_m$ satisfies $\mathbf{P}\boldsymbol{O}\mathbf{2}$, if for all $0 < k \leq m$, states s_{k-1} and s_k satisfy $\mathbf{P}\boldsymbol{O}\mathbf{2}$.

Lemma 6.6. *The maximal length of a sequence of R^{I}-Successors with unchanged modal depth of labels in the pre-tableau constructed by Algorithm 6.1 for an input $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ is $\leq \mathcal{O}\big(\mathrm{dep}(\varphi)^{2|\mathcal{A}|}\big)$.*

Proof. The structure of the proof is as follows. First we prove three claims that are crucial for the result to hold, then we assess how the length of a sequence of R^{I}-Successors with unchanged modal depth of labels can be bounded. The general idea is as follows. We show that the sets $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(\cdot)), [O]^+, d\Big)$ for subsequent states in the sequence must gradually increase at subsequent levels d, starting from the topmost level. This relies on the following observations. If subsequent states s and t, such that t is a R^{I}_j-Successor of state s, satisfy properties **PB1**, **PI1** and **PI2** from above some level $D \geq \mathrm{dep}(\mathrm{Ind}(L(s)))$, then it must be that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d\Big)$ at levels $d \geq D$ (Claim 6.7). Moreover, in this case the sets $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\cdot), [\mathrm{I}]^+, d\Big)$ and $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\cdot), [\mathrm{B}]^+, d\Big)$ can only increase between s and t at the level D and $\mathrm{dep}(\mathrm{Ind}(L(t)))$ must either be below D or be empty. Also, there can be at most one formula of the form $\neg[\mathrm{I}]^+_G\psi$ in $\widehat{\mathrm{Gr}}(L(t))$ at level D or above, and if there is one, then t must be a $\neg[\mathrm{I}]_j[\mathrm{I}]^+_G\psi$-Successor of s (Claim 6.8). Notice that since properties **PB1**, **PI1** and **PI2** are always satisfied at levels above $\mathrm{dep}(L(s_0))$ for all the sets in the sequence (where s_0 is the first state in the sequence), so either they will have to be satisfied at levels above $\mathrm{dep}(L(s_0)) - 1$ starting from the second state in the sequence or the sets $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(\cdot)), [O]^+, d\Big)$ will have to increase at this level. This observation applies to lower levels of this set for subsequent states in the sequence and leads to recurrence Equations (6.1) and (6.4) which are used to assess the maximal length of the sequence.

The basis of these recurrence equations is the length of a sequence where $\mathrm{Ind}(L(\cdot)) = \varnothing$ and properties **PB1**, **PI1** and **PI2** are satisfied at all levels. In such a case the only differences in the labels the successor nodes of states get come from formulas in $\widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(\cdot))\Big)$ (Claim 6.9). By construction of the algorithm, if the labels the successor nodes of states are equal, no new successor can be added to the sequence. Since there can be at most one formula of the form $\neg[\mathrm{I}]^+_G\psi$ in $\widehat{\mathrm{Gr}}(L(\cdot))$ starting from the second state of the sequence, so these differentiating formulas come from a very restricted set and we show that within a constant number of steps repetition of the label of the successor node in the sequence must occur. The detailed analysis of the bounds of the sequence is given after the claims.

Claim 6.7. *Let t be an R^{I}_j-Successor of state s in the pre-tableau constructed by Algorithm 6.1 for an input $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ and let $D \geq \mathrm{dep}(\mathrm{Ind}(L(s)))$. If s and t satisfy properties **PB1**, **PI1** and **PI2**, for all $d > D$, then the following hold, for all $d \geq D$ and $\psi \in \mathcal{L}^{\mathrm{T}}$ with $\mathrm{dep}(\psi) \geq D$*

(i). $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d\Big)$,

(ii). $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d\Big)$,

(iii). if $[\mathrm{I}]^+_G\xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ *and* $\psi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$*, then* $\psi \notin \mathrm{Ind}(L(t))$ *and* $\psi \notin \widehat{\mathrm{Gr}}(L(t))$.

Proof. Take any $d \geq D$. Notice that if $d > \mathrm{dep}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$, then points (i) and (ii) hold for it. Also if $d \geq \mathrm{dep}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$, then, since $\mathrm{dep}\Big(\widehat{\mathrm{Gr}}(L(s))\Big) = \mathrm{dep}(\mathrm{Gr}(L(s)))$ and $\mathrm{dep}(\widehat{\mathrm{Gr}}(L(t))) = \mathrm{dep}(\mathrm{Gr}(L(t)))$, so point (iii) holds for it as well.

For $d \leq \mathrm{dep}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$ we will use induction, starting with maximal value of d. So suppose that $d = \mathrm{dep}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$. As we observed above, point (iii) holds for d and we need to show points (i) and (ii) only.

For point (i), assume that $j \notin \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d\Big)$. Then there must be a formula $[\mathrm{I}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(t))$ with $\mathrm{dep}([\mathrm{I}]^+_G\psi) = d$ and $j \notin G$. Hence there must be a formula $[\mathrm{I}]_j\xi \in \neg L(s)$ such that $[\mathrm{I}]^+_G\psi \in \neg\mathrm{Sub}(\xi)$. Since $\mathrm{dep}([\mathrm{I}]_j\xi) > \mathrm{dep}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$, so either $[\mathrm{I}]_j\xi \in \mathrm{Ind}(L(s))$ or $[\mathrm{I}]_j\xi \in \widetilde{\mathrm{Cl}}(\mathrm{Gr}(L(s)))$. The first case is impossible, as $\mathrm{dep}([\mathrm{I}]_j\xi) > \mathrm{dep}(\mathrm{Ind}(L(s)))$. Suppose that the second case holds. Since $\mathrm{dep}\Big(\widehat{\mathrm{Gr}}(L(s))\Big) \geq \mathrm{dep}(\mathrm{Gr}(L(s)))$, so $\mathrm{dep}([\mathrm{I}]_j\xi) > \mathrm{dep}(\mathrm{Gr}(L(s)))$ and the only possibility for this case to hold is that ξ is of the form $[\mathrm{I}]^+_H\zeta$ with $j \in H$. But then, from the assumptions that $[\mathrm{I}]^+_G\psi \in \neg\mathrm{Sub}(\xi)$ and $\mathrm{dep}([\mathrm{I}]^+_G\psi) = d$, it would mean that $G = H$ and $\zeta = \psi$, which is impossible, as $j \notin G$. Hence this case is impossible as well and it must be that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d\Big)$.

For point (ii), assume that $\omega \notin \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d\Big)$. Then there must be a formula $[\mathrm{B}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(t))$ with $\mathrm{dep}([\mathrm{B}]^+_G\psi) = d$. Hence there must be $[\mathrm{I}]_j\xi \in \neg L(s)$ such that $[\mathrm{B}]^+_G\psi \in \neg\mathrm{Sub}(\xi)$. Again this is impossible, as $\mathrm{dep}([\mathrm{I}]_j\xi) > \mathrm{dep}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$ and $\mathrm{dep}([\mathrm{I}]_j\xi) > \mathrm{dep}(\mathrm{Ind}(L(s)))$. Thus it must be that $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d\Big)$.

For the induction step, suppose that $d < \mathrm{dep}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$. For point (iii) notice that if $[\mathrm{I}]^+_G\xi \in \neg L(s)$ and $\psi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$ and $\mathrm{dep}(\psi) \geq D$, then $\mathrm{dep}([\mathrm{I}]^+_G\xi) \geq d+1$. Moreover since, by point (i), $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d+1\Big)$ so, by property **PB1** it holds that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, d+1\Big)$ and so $j \in G$. Thus if $[\mathrm{I}]^+_G\xi \in \widehat{\mathrm{Gr}}(L(s))$, then $[\mathrm{I}]^+_G\xi \in L(t)$ and if $\neg[\mathrm{I}]^+_G\xi \in \widehat{\mathrm{Gr}}(L(s))$, then $[\mathrm{I}]^+_G\xi \in \neg L(t)$, by condition **PB2**. Since $j \in G$ and $\mathrm{dep}([\mathrm{I}]^+_G\xi) > d$ so, by point (i) of Lemma 6.4, $[\mathrm{I}]^+_G\xi \in \neg\widehat{\mathrm{Gr}}(L(t))$. Hence it must be that $\psi \notin \mathrm{Ind}(L(t))$ and $\psi \notin \widehat{\mathrm{Gr}}(L(t))$.

For point (i) assume that $j \notin \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d\Big)$. Then there must be a formula $[\mathrm{I}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(t))$, with $\mathrm{dep}([\mathrm{I}]^+_G\psi) = d$ and $j \notin G$. By the induction hypothesis it holds that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d'+1\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d'+1\Big)$, for all $d' \geq d$. Moreover, by properties **PI1** and **PB1** it holds that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, d'+1\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, d'+1\Big)$, for all $d' \geq d$. Thus, by Corollary 6.5, there exists a formula $[\mathrm{I}]^+_H\xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ such that $[\mathrm{I}]^+_G\psi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$ (notice that since $j \notin G$, so neither point (iii) nor point (iv) of Corollary 6.5 can apply here). Then, by point (iii) it holds that $[\mathrm{I}]^+_G\psi \notin \neg\widehat{\mathrm{Gr}}(L(t))$ which contradicts our assumptions. Hence it must be that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d\Big)$.

For point (ii) assume that $\omega \notin \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d\Big)$. Then there must be a formula

$[\mathrm{B}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(t))$ with $\mathrm{dep}([\mathrm{B}]^+_G\psi) = d$. As in the case of point (i), by Corollary 6.5 there must exist a formula $[\mathrm{I}]^+_H\xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ such that $[\mathrm{B}]^+_G\psi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$ and, by point (iii), it must hold that $[\mathrm{B}]^+_G\psi \notin \neg\widehat{\mathrm{Gr}}(L(t))$. This contradicts our assumptions and so it must be that $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d\Big)$. □

Claim 6.8. *Let t be an R^{I}_j-Successor of state s in the pre-tableau constructed by Algorithm 6.1 for an input $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ and let $D \geq \mathrm{dep}(\mathrm{Ind}(L(s)))$ such that s and t satisfy properties **PB1**, **PI1** and **PI2**, for all $d > D$. Then the following hold:*

(i). $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, D\Big) \subseteq \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, D\Big)$ *and* $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, D\Big) \subseteq \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, D\Big)$.

(ii). Either $\mathrm{dep}(\mathrm{Ind}(L(t))) < D$ *or* $\mathrm{Ind}(L(t)) = \varnothing$.

(iii). There can be at most one formula of the form $\neg[\mathrm{I}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(t))$ with $\mathrm{dep}([\mathrm{I}]^+_G\psi) \geq D$. Moreover, if there is such a formula, then $\neg[\mathrm{I}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(s))$, $\neg[\mathrm{I}]_j[\mathrm{I}]^+_G\psi \in L(s)$ and t is a $\neg[\mathrm{I}]_j[\mathrm{I}]^+_G\psi$-Successor of s.

Proof. Point (i)

For the fact that $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, D\Big) \subseteq \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, D\Big)$ assume that the opposite holds. Then there must exist a formula $[\mathrm{I}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(t))$ such that $\mathrm{dep}([\mathrm{I}]^+_G\psi) = D$ and $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, D\Big) \nsubseteq G$. Notice that by point (i) of Claim 6.7 it holds that $j \in G$. By points (i) and (ii) of Claim 6.7 and properties **PI1** and **PB1** it holds that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, d\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, d\Big)$, for all $d > D$. Thus, by Corollary 6.5, either there exists a formula $[\mathrm{I}]^+_H\xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ such that $[\mathrm{I}]^+_G\psi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$, or $[\mathrm{I}]^+_G\psi \in \neg L(s)$. The first case is impossible, as it implies that $j \in H$ and so it violates modal context restriction $\mathbf{R}_1$. Thus it must be that the second case holds and, by the fact that $j \in G$ and by point (i) of Lemma 6.4, it must be that $[\mathrm{I}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(s))$. But then it must hold that $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, D\Big) \subseteq G$, which contradicts our assumptions. Hence this case is impossible as well and it must be that $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, D\Big) \subseteq \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, D\Big)$.

For the fact that $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, D\Big) \subseteq \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, D\Big)$ notice that, by point (ii) of Claim 6.7, it holds that $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, D\Big)$ and so $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, D\Big) = \mathcal{A} \cup \{\omega\}$. Hence it holds that $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, D\Big) \subseteq \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, D\Big)$.

Point (ii)

Assume that the opposite holds. Then $\mathrm{dep}(\mathrm{Ind}(L(t))) \geq D$ and there exists a formula $\psi \in \mathrm{Ind}(L(t))$ such that $\mathrm{dep}(\psi) \geq D$. By points (i) and (ii) of Claim 6.7 and properties **PB1** and **PI1**, it holds that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, d\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, d\Big)$, for all $d > D$. Thus, by Corollary 6.5, there exists a formula $[\mathrm{I}]^+_G\xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ such that $\psi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$. By point (iii) of Claim 6.7, this implies that $\psi \notin \mathrm{Ind}(L(t))$ which contradicts our assumptions. Hence it must be that either $\mathrm{dep}(\mathrm{Ind}(L(t))) < D$ or $\mathrm{Ind}(L(t)) = \varnothing$.

Point (iii)

Take any formula $\neg[\mathrm{I}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(t))$ with $\mathrm{dep}([\mathrm{I}]^+_G\psi) \geq D$. By point (i) of Claim 6.7 it must be that $j \in G$. By points (i) and (ii) of Claim 6.7 and by properties **PB1** and **PI1**, it holds that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, d\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, d\Big)$, for all $d > D$. Thus, by Corollary 6.5, either there exists a formula $[\mathrm{I}]^+_H\xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ with $j \in H$ such that $[\mathrm{I}]^+_G\psi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$, or $\neg[\mathrm{I}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(s))$, $\neg[\mathrm{I}]_j[\mathrm{I}]^+_G\psi \in L(s)$ and t is a $\neg[\mathrm{I}]_j[\mathrm{I}]^+_G\psi$-Successor of s. This implies, in particular, that there can be at most one formula of the form $\neg[\mathrm{I}]^+_G\psi$ in $\widehat{\mathrm{Gr}}(L(t)) \sqcap \neg[\mathrm{I}]^+$ with $\mathrm{dep}(\neg[\mathrm{I}]^+_G\psi) \geq D$. □

Claim 6.9. *Let t be an R^{I}_j-Successor of state s in the pre-tableau constructed by Algorithm 6.1 for an input $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$. If s and t satisfy properties **PB1**, **PI1** and **PI2**, for all $d \geq 0$ and* $\mathrm{Ind}(L(s)) = \varnothing$, *then for all $k \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+\Big)$ and $\psi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ it holds that*

(i). $\psi \in L(s)/[\mathrm{I}]_j$ implies $\psi \in L(t)/[\mathrm{I}]_k$ or $\neg[\mathrm{I}]_k\psi \in \widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(t))\Big)$, and

(ii). $\psi \in L(t)/[\mathrm{I}]_k$ implies $\psi \in L(s)/[\mathrm{I}]_j$ or $\neg[\mathrm{I}]_j\psi \in \widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$.

Proof. Notice first that since t is an R^{I}_j-Successor of s so, by point (i) of Claim 6.7, it holds that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+\Big)$. Moreover, by property **PI1**, it holds that $\{j,k\} \subseteq \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+\Big)$.

For point (i), let $\psi \in L(s)/[\mathrm{I}]_j$. Then there exists a formula $[\mathrm{I}]_j\psi \in L(s)$ and, by point (ii) of Lemma 6.4, $[\mathrm{I}]_j\psi \in \neg\widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$. Thus there exists a formula $[\mathrm{I}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(s))$ or ψ is of the form $[\mathrm{I}]^+_G\xi$ and $\psi \in \neg\widehat{\mathrm{Gr}}(L(s))$. Suppose that the first case holds. If $\neg[\mathrm{I}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(s))$, then $\neg[\mathrm{I}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(t))$, by property **PI2**. Otherwise $[\mathrm{I}]_j[\mathrm{I}]^+_G\psi \in L(s)$, as s is a state and $L(s)$ is a closed propositional tableau. Thus $[\mathrm{I}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(t))$, as $j \in G$, $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+\Big)$ and point (i) of Lemma 6.4 applies. Since $[\mathrm{I}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(t))$, $k \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+\Big)$ and $L(t)$ is a $[\mathrm{I}]^+$-expanded closed propositional tableau, so either $\psi \in L(t)/[\mathrm{I}]_k$ or $\neg[\mathrm{I}]_k\psi \in \widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(t))\Big)$. Suppose now that the second case holds. Then, by arguments analogous to those used for the first case, it holds that $[\mathrm{I}]^+_G\xi \in \neg\widehat{\mathrm{Gr}}(L(t))$ and the point holds by the fact that t is a state and $L(t)$ is a $[\mathrm{I}]^+$-expanded tableau.

For point (ii), let $\psi \in L(t)/[\mathrm{I}]_k$. Notice that, by point (ii) of Claim 6.8, $\mathrm{Ind}(L(t)) = \varnothing$. Hence, by arguments analogous to those used above either there is a formula $[\mathrm{I}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(t))$ or ψ is of the form $[\mathrm{I}]^+_G\xi$ and $[\mathrm{I}]^+_G\xi \in \neg\widehat{\mathrm{Gr}}(L(t))$. Suppose that the first case holds. We will show that $[\mathrm{I}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(s))$. If $\neg[\mathrm{I}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(t))$, then $\neg[\mathrm{I}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(s))$, by property **PI2**. Otherwise $[\mathrm{I}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(t))$ and, by Corollary 6.5, either there exists a formula $[\mathrm{I}]^+_H\zeta \in \neg\widehat{\mathrm{Gr}}(L(s))$ such that $[\mathrm{I}]^+_G\psi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\zeta)$, or $[\mathrm{I}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(s))$. The first case is impossible because $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+\Big)$ and so $j \in H$, which violates modal context restriction $\mathbf{R}_1$. Hence it must be that $[\mathrm{I}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(s))$. By arguments analogous to those used for point (i) it can be shown that either $\psi \in L(s)/[\mathrm{I}]_j$ or $\neg[\mathrm{I}]_j\psi \in \widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$. Suppose now that the second case holds. By analogous arguments to those used for the first case, it holds that $[\mathrm{I}]^+_G\xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ and either $[\mathrm{I}]^+_G\xi \in L(s)/[\mathrm{I}]_j$ or $\neg[\mathrm{I}]_j[\mathrm{I}]^+_G\xi \in \widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$. Thus the point holds, as $\psi = [\mathrm{I}]^+_G\xi$. □

Consider a sequence of states $s_0, \ldots, s_m$ such that for any $0 < k \leq m$, s_k is an $R^{\mathrm{I}}_{j_k}$-Successor of s_{k-1}. Suppose that for any $0 < k \leq m$ it holds that $\mathrm{Ind}(L(s_{k-1})) = \varnothing$ and the sequence satisfies properties **PI1**, **PB1** and **PI2**, for all $d \geq 0$. If $\widehat{\mathrm{Gr}}(L(s_0)) \sqcap \neg[\mathrm{I}]^+ = \varnothing$ then, by Claim 6.9, the length of such sequence must be ≤ 2. This is because for any $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(s_1), [\mathrm{I}]^+\Big)$ it holds that $L(s_1)/[\mathrm{I}]_j \subseteq L(s_0)/[\mathrm{I}]_{j_1} \subseteq L(s_1)$ and there are no formulas of the form $\neg[\mathrm{I}]_j\psi \in L(s_1)$. Hence $L^{[\mathrm{I}]_j}(s_1) \subseteq L(s_1)$ and no R^{I}_j-Successor of s_1 can be created. On the other hand, if $j \notin \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(s_1), [\mathrm{I}]^+\Big)$, then for any R^{I}_j-Successor s_2 of s_1, property **PI1** will not be satisfied for s_1 and s_2. If there is more than one formula of the form $\neg[\mathrm{I}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(s_0))$, then the length of such sequence must be ≤ 2 as if it was larger then, by point (iii) of of Claim 6.8, property **PI2** would have to be violated.

Lastly, if $\widehat{\mathrm{Gr}}(L(s_0)) \sqcap \neg[\mathrm{I}]^+ = \{\neg[\mathrm{I}]^+_G\psi\}$, then the length of the sequence must be $\leq 2|G|+1$. To see why, assume the opposite, that is $m > 2|G|+1$. Notice that, by point (iii) of Claim 6.8, for all $0 < k \leq m$ it must be that $\widehat{\mathrm{Gr}}(L(s_k)) \sqcap \neg[\mathrm{I}]^+ = \{\neg[\mathrm{I}]^+_G\psi\}$ and s_k must be a $\neg[\mathrm{I}]_{j_k}[\mathrm{I}]^+_G\psi$-Successor of s_{k-1} with $j_k \in G$. Hence, by Claim 6.9, for any two subsequent states s_{k-1} and s_k in this sequence, with $k < m$, it must hold that $L^{[\mathrm{I}]_{j_k}}(s_{k-1}) \subseteq L^{[\mathrm{I}]_{j_{k+1}}}(s_k) \cup \{\psi\}$ and $L^{[\mathrm{I}]_{j_k}}(s_{k-1}) \cup \{\psi\} \subseteq L^{[\mathrm{I}]_{j_{k+1}}}(s_k)$. To see why, consider the first inclusion and take any formula $\xi \in L^{[\mathrm{I}]_{j_k}}(s_{k-1})$. Then $\xi \in L(s_{k-1})/[\mathrm{I}]_{j_k}$. Suppose that $\xi \notin L^{[\mathrm{I}]_{j_{k+1}}}(s_k)$. Since the sequence satisfies properties **PI1**, **PB1** and **PI2**, for all $d \geq 0$, so, by point (i) of Claim 6.7, it must be that $j_{k+1} \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d\Big)$. Since $L^{[\mathrm{I}]_{j_{k+1}}}(s_k) = L(s_k)/[\mathrm{I}]_{j_{k+1}}$ so, by point (i) of Claim 6.9, $\neg[\mathrm{I}]_{j_{k+1}}\xi \in \widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(s_k))\Big)$. Since $\widehat{\mathrm{Gr}}(L(s_k)) \sqcap \neg[\mathrm{I}]^+ = \{\neg[\mathrm{I}]^+_G\psi\}$, so it must be that either $\xi = \psi$ or $\xi = [\mathrm{I}]^+_G\psi$. The second case is impossible, as it would imply that $[\mathrm{I}]^+_G\psi \in L^{[\mathrm{I}]_{j_k}}(s_{k-1}) \subseteq L(s_k)$, while we already have $\neg[\mathrm{I}]^+_G\psi \in L(s_k)$. Thus it must be that $\xi = \psi$. The second inclusion can be shown by analogous arguments, using point (ii) of Claim 6.9. Since for any two subsequent states s_{k-1} and s_k in the sequence, with $k < m$, it must hold that $L^{[\mathrm{I}]_{j_k}}(s_{k-1}) \subseteq L^{[\mathrm{I}]_{j_{k+1}}}(s_k) \cup \{\psi\}$ and $L^{[\mathrm{I}]_{j_k}}(s_{k-1}) \cup \{\psi\} \subseteq L^{[\mathrm{I}]_{j_{k+1}}}(s_k)$, so the analogous fact holds for any two states in the sequence (possibly excluding the last state).

If $m > 2|G|+1$, then there must exist $0 < k_1 < k_2 < k_3 \leq m$ such that $j_{k_1} = j_{k_2} = j_{k_3}$. By what we have shown above the sets $L^{[\mathrm{I}]_{j_{k_1}}}(s_{k_1-1})$, $L^{[\mathrm{I}]_{j_{k_2}}}(s_{k_2-1})$ and $L^{[\mathrm{I}]_{j_{k_3}}}(s_{k_3-1})$ may differ by at most one formula, ψ, which either appears in them or not. Thus at least two of them must be equal. But then the $\neg[\mathrm{I}]_{j_i}[\mathrm{I}]^+_G\psi$-successor of one of the states s_{k_1-1}, s_{k_2-1} or s_{k_3-1} with $i = 1, 2$ or 3, respectively, cannot be created, which contradicts the assumption that all s_{k_1}, s_{k_2} and s_{k_3} are in the sequence. Hence the length of the sequence must be $\leq 2|G|+1$.

Let $G \subseteq \mathcal{A}$, $D \geq 0$ and let T^G_D denote the maximal length of a sequence of R^{I}-Successors in the pre-tableau constructed by Algorithm 6.1 such that for each state s in the sequence $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, d\Big) \subseteq G$, for all $d \geq D$ and

1. properties **PI1** and **PB1** are satisfied for the sequence for all $d \geq D$,
2. property **PI2** is satisfied for the sequence for all $d > D$,
3. for each state s in the sequence $\mathrm{dep}(\mathrm{Ind}(L(s))) \leq D$ and
4. there is exactly one formula of the form $\neg[\mathrm{I}]^+_H\psi \in \widehat{\mathrm{Gr}}(L(s))$ with $\mathrm{dep}([\mathrm{I}]^+_H\psi) > D$.

Then $T^G_D \leq \overline{T}^{|G|}_D$, where

$$\overline{T}^n_m = \begin{cases} 2n+1, & \text{if } m = 0 \\ 2 + \sum_{i=1}^n \overline{T}^i_{m-1}, & \text{if } m > 0 \end{cases} \tag{6.1}$$

To show that this inequality holds, we will use induction on D. The fact that $T_0^G \leq 2|G|+1$ follows from what we have shown above. The fact that $T_D^G \leq 2 + \sum_{i=1}^{|G|} \overline{T}_{D-1}^i$, for $D > 0$, follows from Claims 6.7 and 6.8. To see why, notice that by point (iii) of Claim 6.8, starting from the second state in the sequence under consideration, property **PI2** is satisfied for the remaining subsequence, for all $d \geq D$. Thus, by point (ii) of Claim 6.8, any subsequence of the sequence under consideration with dep(Ind($L(s)$)) remaining unchanged for its every state s, can have length at most 1. Hence starting from the third state in the sequence dep(Ind($L(s)$)) $\leq D-1$.

To assess the length of the remaining part of the sequence, we divide it into parts marked by the first appearance of a new element $j \in G$ in the set $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(\cdot)), [\mathrm{I}]^+, D-1\Big)$. By doing this we divide the sequence into $|G|$ parts, $P_1, \ldots, P_{|G|}$. A part P_i is a subsequence $s_j, \ldots, s_{k-1}$ such that some i'th element of G appeared for the first time in $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s_j)), [\mathrm{I}]^+, D-1\Big)$ and some $i+1$'th element of G appeared for the first time in $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s_k)), [\mathrm{I}]^+, D-1\Big)$. Notice that it may be that for some state s in the sequence more than one element of G appears for the first time in $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, D-1\Big)$. In such cases we can assume that the length of some of the sequences P_i is 0.

Let $G^{(1)}, \ldots, G^{(|G|)}$ be the sequence of subsets of G such that $G^{(i)}$ is the set of all j that appear in the sets $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, D-1\Big)$ for s in the sequence P_i. By point (ii) of Claim 6.8 and point (i) of Claim 6.8, $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, D-1\Big)$ for every state s in the sequence. Hence the condition **PB1** is satisfied for all $d \geq D-1$ and every state of the sequence. Moreover, by point (i) of Claim 6.8, starting from the first occurrence of some $j \in G$ in $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, D-1\Big)$ we have $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, D-1\Big)$ for all the remaining states t of the sequence. Hence the condition **PI1** is satisfied for all $d \geq D-1$ on every part P_i of the sequence. Thus the length of a each part P_i is $\leq T_{D-1}^{G^{(i)}}$. Hence, by the induction hypothesis, the length of the sequence consisting of the parts $P_1, \ldots, P_{|G|}$ with points $1-4$ being satisfied is $\leq \sum_{i=1}^{|G|} \overline{T}_{D-1}^i$.

To solve 6.1 we use the following fact (proved in the Appendix).

Fact 6.10. *Let X_m^n be defined as follows, for $m \geq 0$ and $n \geq 1$:*

$$X_m^n = \begin{cases} 2n+1, & \text{if } m = 0 \\ B + \sum_{i=1}^n X_{m-1}^i, & \text{if } m > 0. \end{cases} \tag{6.2}$$

Then, for $n, m \geq 1$,

$$X_m^n = B\binom{n+m-1}{m-1} + (n+2)\binom{n+m-1}{m}. \tag{6.3}$$

By Fact 6.10, from (6.1) we get (for $D > 0$)

$$T_D^G \leq 2\frac{(|G|+D-1)!}{|G|!(D-1)!} + (|G|+2)\frac{(|G|+D-1)!}{(|G|-1)!D!} = \mathcal{O}\Big(D^{|G|}\Big).$$

Let now S_D^G denote the maximal length of the sequence of R^{I}-Successors in the pre-tableau constructed by Algorithm 6.1 such that for each state s in the sequence $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, d\Big) \subseteq G$, for all $d \geq D$ and points $1-3$ are satisfied for it. Then $S_D^G \leq \overline{S}_D^{|G|}$, where

$$\overline{S}_m^n = \begin{cases} 2n+1, & \text{if } m = 0 \\ 2 + T_{m-1}^n + \sum_{i=1}^n \overline{S}_{m-1}^i, & \text{if } m > 0. \end{cases} \tag{6.4}$$

Explanation for this equation is similar to that of Equation (6.1). The only new thing is $\overline{T}^{|G|}_{D-1}$ in the case of $D > 0$. It comes from the fact the after two states in the sequence, by point (iii) of Claim 6.8 there can be at most one formula of the form $\neg[\mathrm{I}]^+_H\xi \in \widehat{\mathrm{Gr}}(L(s))_{D-1}$. If there is no such a formula in $\widehat{\mathrm{Gr}}(L(s))_{D-1}$, then, by point (iii) of Claim 6.8 property **PI2** will be satisfied for the remaining part of the sequence for all $d \geq D$, and there can be at most $\sum_{i=1}^{|G|} \overline{S}^i_{D-1}$ states in this remaining part. However, if there is exactly one such formula in $\widehat{\mathrm{Gr}}(L(s))_{D-1}$, then the maximal length of the subsequence in which it remains is bounded by T^G_{D-1}. After that, there can be no formula of the form $\neg[\mathrm{I}]^+_H\xi \in \widehat{\mathrm{Gr}}(L(s))_{D-1}$ and there can be at most $\sum_{i=1}^{|G|} \overline{S}^i_{D-1}$ states in the remaining part of the sequence.

By Fact 6.10, we get (for $D > 0$)

$$S^G_D \leq \left(2 + T^G_{D-1}\right) \frac{(|G| + D - 1)!}{|G|!(D-1)!} + (|G| + 2)\frac{(|G| + D - 1)!}{(|G| - 1)!D!} = \mathcal{O}\Big(D^{2|G|}\Big)\,.$$

Thus the maximal length of a sequence of R^{I}-Successors with the same modal depth of labels is $\leq S^{\mathcal{A}}_{\mathrm{dep}(\varphi)+1} = \mathcal{O}\big(\mathrm{dep}(\varphi)^{2|\mathcal{A}|}\big)$. □

Next we move to the analysis of lengths of sequences of R^{B}-Successors. The general approach is similar to that used for sequence of R^{B}-Successors. There is one important difference, however, when the length of a sequence of R^{B}-Successors with $\mathrm{Ind}(L(\cdot)) = \varnothing$ and properties **PB1**, **PI1** and **PB2** satisfied at all levels is assessed. In this case modal context restriction $\mathbf{R}_2$ is necessary to obtain bounds similar to those obtained in the case of R^{I}-Successors. As shown in Theorem 6.1, restriction $\mathbf{R}_1$ may lead to sequences of states with length which is exponential with respect to the length of the input formula. In Section 6.3.2 we propose a modification of Algorithm 6.1 which solves the satisfiability problem for formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ using polynomial space with respect to the length of input formula (but not with respect to modal depth of the input formula).

There is one more thing that differentiates the analysis of the case of R^{B}-Successors from the case of R^{I}-Successors. If t is a R^{B}_j-Successor of state s, then, by construction of the algorithm, $L(s) \sqcap j \subseteq L(t)$. For this reason we will consider the set $\mathrm{Ind}(\Phi) \sqcap j$ rather than the set $\mathrm{Ind}(\Phi)$. The difference is of technical nature and does not affect the general line of argumentation.

Like in the case of R^{I}-Successors we start with a lemma giving properties of $\widehat{\mathrm{Gr}}(\cdot)$ and $\mathrm{Ind}(\cdot)$ that follow from modal context restriction $\mathbf{R}_1$. The lemma has additional point, that requires modal context restriction $\mathbf{R}_2$.

Lemma 6.11. *Let $\Phi \subseteq \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ and let $D \geq 0$ and $j \in \mathcal{A}$ be such that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{B}]^+, d\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\Big)$, for all $d \geq D+1$. Then the following hold*

(i). if $[\mathrm{B}]^+_G\psi \in \neg\Phi$ with $j \in G$ and $\mathrm{dep}([\mathrm{B}]^+_G\psi) \geq D$, then $[\mathrm{B}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(\Phi)$,

(ii). if $\mathrm{dep}(\mathrm{Ind}(\Phi) \sqcap j) \leq D$ and $[\mathrm{B}]_j\psi \in \neg\Phi$ with $\mathrm{dep}([\mathrm{B}]_j\psi) \geq D+1$, then $[\mathrm{B}]_j\psi \in \neg\widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(\Phi)\Big)$,

(iii). if $\Phi \subseteq \mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$, $\mathrm{dep}(\mathrm{Ind}(\Phi) \sqcap j) \leq D$, $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\Big)$, for all $d \geq D$ and $[O]_j\psi \in \neg\Phi$ with $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $\mathrm{dep}([O]_j\psi) \geq D+1$, then $[O]_j\psi \in \neg\widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(\Phi)\Big)$ and $O = \mathrm{B}$.

Proof. For point (i) take any formula of the form $[\mathrm{B}]^+_G\psi \in \neg\Phi$ with $j \in G$ and $\mathrm{dep}([\mathrm{B}]^+_G\psi) \geq D$ and suppose that $[\mathrm{B}]^+_G\psi \notin \neg\widehat{\mathrm{Gr}}(\Phi)$. Thus either there is a formula $[\mathrm{B}]^+_H\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ or a formula $[\mathrm{I}]^+_H\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ with $[\mathrm{B}]^+_G\psi \in \neg\mathrm{PT}(\xi)$. The first case is impossible as $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{B}]^+, d\Big)$ for all $d \geq D+1$ and so it must hold that $j \in H$ which would violate modal context restriction $\mathbf{R}_1$. The second case is impossible as well since $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\Big)$, for all $d \geq D+1$. Hence it must be that $[\mathrm{B}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(\Phi)$.

For point (ii) take any formula of the form $[\mathrm{B}]_j\psi \in \neg\Phi$ with $\mathrm{dep}([\mathrm{I}]_j\psi) \geq D+1$. Since $\mathrm{dep}(\mathrm{Ind}(\Phi)) \leq D$ so $[\mathrm{B}]_j\psi \notin \neg\mathrm{Ind}(\Phi)$. Thus either there is a formula $[\mathrm{B}]^+_G\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ or a formula $[\mathrm{I}]^+_G\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ with $[\mathrm{B}]_j\psi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$, or $[\mathrm{B}]_j\psi \in \neg\widetilde{\mathrm{Cl}}(\Phi)$. The first case is impossible as $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{B}]^+, d\Big)$ for all $d \geq D+1$ and so it must hold that $j \in G$ which would violate modal context restriction $\mathbf{R}_1$. The second case is impossible as well, as $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\Big)$, for all $d \geq D+1$. Hence it must be that $[\mathrm{B}]_j\psi \in \neg\widetilde{\mathrm{Cl}}(\Phi)$. Thus either there is a formula $[\mathrm{B}]^+_G\psi \in \neg\Phi$ such that $j \in G$ or ψ is of the form $[\mathrm{B}]^+_G\xi$ with $j \in G$ and $\psi \in \neg\Phi$. Hence, by point (i), it holds that $[\mathrm{B}]_j\psi \in \neg\widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(\Phi)\Big)$.

Before showing point (iii), we will show that if $\Phi \subseteq \mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$ and there is a formula of the form $[\mathrm{I}]^+_G\psi \in \neg\Phi$ with $j \in G$ and $\mathrm{dep}([\mathrm{I}]^+_G\psi) \geq D$, then $[\mathrm{I}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ **(*)**.

So take any formula of the form $[\mathrm{I}]^+_G\psi \in \neg\Phi$ with $j \in G$ and $\mathrm{dep}([\mathrm{I}]^+_G\psi) \geq D$ and suppose that $[\mathrm{I}]^+_G\psi \notin \neg\widehat{\mathrm{Gr}}(\Phi)$. Then either there is a formula $[\mathrm{B}]^+_H\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ or a formula $[\mathrm{I}]^+_H\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ with $[\mathrm{I}]^+_G\psi \in \neg\mathrm{PT}(\xi)$. The first case is impossible as $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{B}]^+, d\Big)$ for all $d \geq D+1$ and so it must hold that $j \in H$ which would violate modal context restriction $\mathbf{R}_2$. The second case is impossible as well since $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\Big)$, for all $d \geq D+1$. Hence it must be that $[\mathrm{I}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(\Phi)$.

For point (iii), take any formula of the form $[O]_j\psi \in \neg\Phi$ with $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $\mathrm{dep}([O]_j\psi) \geq D+1$. Since $\mathrm{dep}(\mathrm{Ind}(\Phi) \sqcap j) \leq D$ so $[O]_j\psi \notin \neg\mathrm{Ind}(\Phi)$. Thus either there is a formula $[\mathrm{B}]^+_G\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ or a formula $[\mathrm{I}]^+_G\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ with $[O]_j\psi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$, or $[O]_j\psi \in \neg\widetilde{\mathrm{Cl}}(\Phi)$.

The first case is impossible as $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{B}]^+, d\Big)$ for all $d \geq D+1$ and so it must hold that $j \in G$ which would violate modal context restriction $\mathbf{R}_2$. The second case is impossible as well, since $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\Big)$, for all $d \geq D+1$. Hence it must be that $[O]_j\psi \in \neg\widetilde{\mathrm{Cl}}(\Phi)$. Thus $O \in \{\mathrm{B}, \mathrm{I}\}$ and either there is a formula $[O]^+_G\psi \in \neg\Phi$ such that $j \in G$ or ψ is of the form $[O]^+_G\xi$ with $j \in G$ and $\psi \in \neg\Phi$. Hence, by point (i) and by **(*)**, it holds that $[O]_j\psi \in \neg\widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(\Phi)\Big)$. Since $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\Big)$, for all $d \geq D$, so it must be that $O = \mathrm{B}$. □

The corollary below points out the sources of formulas in the successor state with modal depth not smaller than $\mathrm{dep}(\mathrm{Ind}(\cdot))$ of the predecessor state. The main difference to analogous corollary showed in the case of R^{I}-Successors comes from the fact that in the case of R^{B}_j-Successor formulas from $\psi \in L(s) \sqcap j$ are inherited to $L(t)$.

Corollary 6.12. *Let t be an R^{B}_j-Successor of state s in the pre-tableau constructed by Algorithm 6.1 for an input $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$, with $D \geq 0$ such that* $\mathrm{dep}(\mathrm{Ind}(L(s)) \sqcap j) \leq D$, $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, d\Big)$ *and* $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, d\Big)$*, for all $d \geq D+1$. Then for all $\psi \in L(t)$ with* $\mathrm{dep}(\psi) \geq D$ *one of the following holds* *(i).* $\psi \in L(s) \sqcap j$ *or*

(ii). $\psi \in \widetilde{\mathrm{Cl}}(L(t))$ *or*

(iii). *there exists* $[\mathrm{B}]^+_G\xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ *with* $j \in G$ *such that* $\psi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$ *or*

(iv). ψ *is of the form* $[\mathrm{B}]^+_G\eta$ *with* $j \in G$ *and* $\psi \in \neg\widehat{\mathrm{Gr}}(L(s))$ *or*

(v). ψ *is of the form* $\neg[\mathrm{B}]^+_G\eta$ *with* $j \in G$, $\psi \in \widehat{\mathrm{Gr}}(L(s))$, $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\eta \in L(s)$ *and* t *is a* $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\eta$*-Successor of* s.

Proof. Take any $\psi \in L(t)$ with $\mathrm{dep}(\psi) = d \geq D$. If $\psi \in \widetilde{\mathrm{Cl}}(L(t))$ or $\psi \in L(s) \sqcap j$, then the claim holds. Suppose otherwise. Notice that if ψ was added to $L(t)$ during [B]-expanded tableau formation, then ψ must be of the form $[\mathrm{B}]_k\xi$ or $\neg[\mathrm{B}]_k\xi$. Moreover, it must be that $k \neq j$ as, by Lemma 6.2, it holds that $L(s) \sqcap j = L(t) \sqcap j$. Hence if neither $\psi \in \widetilde{\mathrm{Cl}}(L(t))$ nor $\psi \in L(s) \sqcap j$, then there must be a formula $[\mathrm{B}]_j\xi \in \neg L(s)$ such that $\psi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$. By point (ii) or Lemma 6.11 it holds that $[\mathrm{B}]_j\xi \in \neg\widetilde{\mathrm{Cl}}(\widehat{\mathrm{Gr}}(L(s)))$. Hence either there is a formula $[\mathrm{B}]^+_G\xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ with $j \in G$ or ψ is of the form $[\mathrm{B}]_G\eta$ with $j \in G$, $[\mathrm{B}]^+_G\eta \in \neg\widehat{\mathrm{Gr}}(L(s))$ and $[\mathrm{B}]_j[\mathrm{B}]^+_G\eta \in \widetilde{\mathrm{Cl}}(L(s))$, or ψ is of the form $\neg[\mathrm{B}]^+_G\eta$ with $j \in G$, $\neg[\mathrm{B}]^+_G\eta \in \widehat{\mathrm{Gr}}(L(s))$ and $[\mathrm{B}]_j[\mathrm{B}]^+_G\eta \in \neg\widetilde{\mathrm{Cl}}(L(s))$. In the last case it must hold that $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\eta \in \widetilde{\mathrm{Cl}}(L(s))$ as otherwise it would be $[\mathrm{B}]_j[\mathrm{B}]^+_G\eta \in \widetilde{\mathrm{Cl}}(L(s))$ and $[\mathrm{B}]^+_G\eta \in L(t)$, which would contradict the assumption that t is a state and $L(t)$ cannot be blatantly inconsistent. Moreover, in this case it must be that t is a $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\eta$-Successor of s. To see why assume the opposite. Then there must be a formula $[\mathrm{B}]_j\xi \in L(s)$ such that $[\mathrm{B}]^+_G\eta \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$. As we already observed, it cannot be that $\xi = [\mathrm{B}]^+_G\eta$. This, together with the assumption that $[\mathrm{B}]^+_G\eta \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$, implies that ξ cannot be of the form $[\mathrm{B}]^+_H\zeta$. Now, by point (ii) or Lemma 6.11, it must be that $[\mathrm{B}]_j\xi \in \neg\widetilde{\mathrm{Cl}}(\widehat{\mathrm{Gr}}(L(s)))$. Since ξ cannot be of the form $[\mathrm{B}]^+_H\zeta$, so there must be formula $[\mathrm{B}]^+_H\xi \in \widehat{\mathrm{Gr}}(L(s))$ with $j \in H$. But this is impossible, as it violates modal context restriction $\mathbf{R}_1$. □

Now we are ready to prove the lemma on bounds of the length of a sequence of R^{B}-Successors with unchanged modal depth of labels in the pre-tableau. This time the proof requires that the input formula satisfies modal context restriction $\mathbf{R}_2$.

Lemma 6.13. *The maximal length of sequence of* R^{B}*-Successors with unchanged modal depth of labels in the pre-tableau constructed by Algorithm 6.1 for an input* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$ *is* $\leq \mathcal{O}\big(\mathrm{dep}(\varphi)^{2|\mathcal{A}|}\big)$.

Proof. The structure of the proof is very similar to that of proof of Lemma 6.6. Claims 6.14, 6.15 and 6.16 are analogous to Claims 6.7, 6.8 and 6.9, respectively, used in proof of Lemma 6.6. This time, however, we need an additional Claim 6.17 which is used (together with Claim 6.16) to assess the length of a sequence of R^{B}-Successors where $\mathrm{Ind}(L(\cdot)) = \varnothing$ and properties **PB1**, **PI1** and **PB2** are satisfied at all levels. Proof of Claim 6.17 uses the assumption that the input formula satisfies modal context restriction $\mathbf{R}_2$. For all the remaining claims modal context restriction $\mathbf{R}_1$ is a sufficient assumption. Proofs of the claims are moved to the Appendix.

Claim 6.14. *Let* t *be an* R^{B}_j*-Successor of state* s *in the pre-tableau constructed by Algorithm 6.1 for an input* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ *and let* $D \geq \mathrm{dep}(\mathrm{Ind}(L(s)) \sqcap j)$. *If* s *and* t *satisfy properties* ***PI1***, ***PB1*** *and* ***PB2***, *for all* $d > D$, *then the following hold for all* $d \geq D$ *and* $\psi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ *with* $\mathrm{dep}(\psi) \geq D$

(i). $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d\Big)$,

(ii). $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d\Big)$,

(iii). *if* $[\mathrm{B}]^+_G\xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ *and* $\psi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$, *then* $\psi \notin \mathrm{Ind}(L(t))$ *and* $\psi \notin \widehat{\mathrm{Gr}}(L(t))$.

Claim 6.15. *Let* t *be an* R^{B}_j*-Successor of state* s *in the pre-tableau constructed by Algorithm 6.1 for an input* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ *and let* $D \geq \mathrm{dep}(\mathrm{Ind}(L(s)) \sqcap j)$ *be such that* s *and* t *satisfy properties* ***PI1***, ***PB1*** *and* ***PB2***, *for all* $d > D$. *Then the following hold*

(i). $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, D\Big) \subseteq \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, D\Big)$ *and*
$\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, D\Big) \subseteq \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, D\Big)$.

(ii). *Either* $\mathrm{dep}(\mathrm{Ind}(L(t)) \setminus (L(t) \sqcap j)) < D$ *or* $\mathrm{Ind}(L(t)) \setminus (L(t) \sqcap j) = \varnothing$.

(iii). *There can be at most one formula of the form* $\neg[\mathrm{B}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(t))$ *with* $\mathrm{dep}([\mathrm{B}]^+_G\psi) \geq D$. *Moreover, if there is such a formula, then* $\neg[\mathrm{B}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(s))$, $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(s)$ *and* t *is a* $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$*-Successor of* s.

Claim 6.16. *Let* t *be an* R^{B}_j*-Successor of* s *in the pre-tableau constructed by Algorithm 6.1 for an input* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$. *If* s *and* t *satisfy properties* ***PB1***, ***PI1*** *and* ***PB2***, *for all* $d \geq 0$, *and* $\mathrm{Ind}(L(s)) \sqcap j = \varnothing$, *then for all* $k \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\Big)$ *and* $\psi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ *it holds that*

(i). $\psi \in L(s)/[\mathrm{B}]_j$ *implies* $\psi \in L(t)/[\mathrm{B}]_k$ *or* $\neg[\mathrm{B}]_k\psi \in \widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(t))\Big)$, *and*

(ii). $\psi \in L(t)/[\mathrm{B}]_k$ *implies* $\psi \in L(s)/[\mathrm{B}]_j$ *or* $\neg[\mathrm{B}]_j\psi \in \widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$.

Claim 6.17. *Let* t *be an* R^{B}_j*-Successor of* s *in the pre-tableau constructed by Algorithm 6.1 for an input* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$. *If* s *and* t *satisfy properties* ***PB1***, ***PI1*** *and* ***PB2***, *for all* $d \geq 0$, *and* $\mathrm{Ind}(L(s)) \sqcap j = \varnothing$, *then for all* $k \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\Big)$ *and* $\psi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$ *it holds that*

(i). $[O]_j\psi \in \neg L(s)$ *implies* $O = \mathrm{B}$ *and* $[O]_k\psi \in \neg L(t)$, *and*

(ii). $[O]_k\psi \in \neg L(t)$ *implies* $O = \mathrm{B}$ *and* $[O]_j\psi \in \neg L(s)$.

Consider a sequence of states $s_0, \ldots, s_m$ such that for any $0 < k \leq m$, s_k is an $R^{\mathrm{B}}_{j_k}$-Successor of s_{k-1}. Suppose that for any $0 < k \leq m$ it holds that $\mathrm{Ind}(L(s_{k-1})) \sqcap j_k = \varnothing$ and the sequence satisfies properties **PB1**, **PI1** and **PB2** for all $d \geq 0$. If $\widehat{\mathrm{Gr}}(L(s_0)) \sqcap \neg[\mathrm{B}]^+ = \varnothing$ then, by Claim 6.16, the length of such sequence must be ≤ 2. This is because for any $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(s_1), [\mathrm{B}]^+\Big)$ it holds that $L(s_1)/[\mathrm{B}]_j \subseteq L(s_0)/[\mathrm{B}]_{j_1} \subseteq L(s_1)$, $L(s_1) \sqcap j \subseteq L(s_1)$ and there are no formulas of the form $\neg[\mathrm{B}]_j\psi \in L(s_1)$. Hence $L^{[\mathrm{B}]_j}(s_1) \subseteq L(s_1)$ and no R^{B}_j-Successor of s_1 can be created. On the other hand, if $j \notin \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(s_1), [\mathrm{B}]^+\Big)$, then for any R^{B}_j-Successor s_2 of s_1, property **PB1** will not be satisfied for s_1 and s_2. If there is more than one formula of the form $\neg[\mathrm{B}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(s_0))$, then the length of such sequence must be ≤ 2 as if it was larger then, by point (iii) of Claim 6.15, property **PB2** would have to be violated.

Lastly, if $\widehat{\mathrm{Gr}}(L(s_0)) \sqcap \neg[\mathrm{B}]^+ = \{\neg[\mathrm{B}]^+_G\psi\}$, then the length of the sequence must be $\leq 2|G|+1$. To see why, assume the opposite, that is $m > 2|G|+1$. Notice that, by point (iii) of Claim 6.15, for all $0 < k \leq m$ it must be that $\widehat{\mathrm{Gr}}(L(s_k)) \sqcap \neg[\mathrm{B}]^+ = \{\neg[\mathrm{B}]^+_G\psi\}$ and s_k must be a $\neg[\mathrm{B}]_{j_k}[\mathrm{B}]^+_G\psi$-Successor of s_{k-1} with $j_k \in G$. By Claim 6.16 and arguments analogous to those used in proof of Lemma 6.13, for any two states s_{i-1} and s_{l-1} in the sequence, with $i, l \leq m$, it holds that $L(s_{i-1})/[\mathrm{B}]_{j_i} \subseteq L(s_{l-1})/[\mathrm{B}]_{j_l} \cup \{\psi\}$ and $L(s_{l-1})/[\mathrm{B}]_{j_l} \cup \{\psi\} \subseteq L(s_{i-1})/[\mathrm{B}]_{j_i}$. Moreover, Claim 6.17 together with Claim 6.16 implies that for any two subsequent states s_{k-1} and s_k, with $k < m$, $L(s_{k-1}) \sqcap j_k \subseteq L(s_k) \sqcap j_{k+1} \cup \{[\mathrm{B}]_{j_{k+1}}\psi\}$ and $L(s_k) \sqcap j_{k+1} \subseteq L(s_{k-1}) \sqcap j_k \cup \{[\mathrm{B}]_{j_k}\psi\}$. To see why, consider the first inclusion and take any $\xi \in L(s_{k-1}) \sqcap j_k$. By point (i) of Claim 6.17 it must be that ξ is either of the form $[\mathrm{B}]_{j_k}\zeta$ or $\neg[\mathrm{B}]_{j_k}\zeta$ and $[\mathrm{B}]_{j_{k+1}}\zeta \in \neg L(s_k)$. Suppose that the first case holds. Then it must be that $\zeta \in L(s_{k-1})/[\mathrm{B}]_{j_k}$ and, by the fact that $L(s_{k-1})/[\mathrm{B}]_{j_k} \subseteq L(s_k)/[\mathrm{B}]_{j_{k+1}} \cup \{\psi\}$, either $[\mathrm{B}]_{j_{k+1}}\zeta \in L(s_k) \sqcap j_{k+1}$ or $\zeta = \psi$. Suppose now that the second case holds. If $\neg[\mathrm{B}]_{j_{k+1}}\zeta \notin L(s_k)$, then $[\mathrm{B}]_{j_{k+1}}\zeta \in L(s_k)$ and, consequently, $\zeta \in L(s_k)/[\mathrm{B}]_{j_{k+1}}$. Since $L(s_k)/[\mathrm{B}]_{j_{k+1}} \subseteq L(s_{k-1})/[\mathrm{B}]_{j_k} \cup \{\psi\}$, so either $[\mathrm{B}]_{j_k}\zeta \in L(s_{k-1})$ or $\zeta = \psi$. The first case is impossible, as $\neg[\mathrm{B}]_{j_k}\zeta \in L(s_{k-1})$, and so it must be that $\zeta = \psi$ and $[\mathrm{B}]_{j_{k+1}}\psi \in L(s_k)$. Since for any two subsequent states s_{k-1} and s_k, with $k < m$, $L(s_{k-1}) \sqcap j_k \subseteq L(s_k) \sqcap j_{k+1} \cup \{[\mathrm{B}]_{j_{k+1}}\psi\}$ and $L(s_k) \sqcap j_{k+1} \subseteq L(s_{k-1}) \sqcap j_k \cup \{[\mathrm{B}]_{j_k}\psi\}$, so the analogous fact holds for any two states in the sequence (possibly excluding the last state).

If $m > 2|G|+1$, then there must exist $0 < k_1 < k_2 < k_3 \leq m$ such that $j_{k_1} = j_{k_2} = j_{k_3}$. By what we have shown above the sets $L^{[\mathrm{B}]j_{k_1}}(s_{k_1-1})$, $L^{[\mathrm{B}]j_{k_2}}(s_{k_2-1})$ and $L^{[\mathrm{B}]j_{k_3}}(s_{k_3-1})$ may differ by at most two formulas, ψ and $[\mathrm{B}]_j\psi$. Moreover, each of these sets either contains both these formulas or does not contain ψ and contains $\neg[\mathrm{B}]_j\psi$. Thus at least two of the sets must be equal. But then the $\neg[\mathrm{B}]_{j_i}[\mathrm{B}]^+_G\psi$-successor of one of the states s_{k_1-1}, s_{k_2-1} or s_{k_3-1} with $i = 1, 2$ or 3, respectively, cannot be created, which contradicts the assumption that all s_{k_1}, s_{k_2} and s_{k_3} are in the sequence. Hence the length of the sequence must be $\leq 2|G|+1$.

Let $G \subseteq \mathcal{A}$, $D \geq 0$ and let T^G_D denote the maximal length of a sequence of R^{B}_G-Successors in the pre-tableau constructed by Algorithm 6.1 such that

1. properties **PB1** and **PI1** are satisfied for the sequence for all $d \geq D$,
2. property **PB2** is satisfied for the sequence for all $d > D$,
3. for each state s in the sequence $\mathrm{dep}(\mathrm{Ind}(L(s))) \leq D$ and
4. there is exactly one formula of the form $\neg[\mathrm{B}]^+_H\psi \in \widehat{\mathrm{Gr}}(L(s))$ with $\mathrm{dep}([\mathrm{B}]^+_H\psi) > D$.

Then $T^G_D \leq \overline{T}^{|G|}_D$, where

$$\overline{T}^n_m = \begin{cases} 2n+1, & \text{if } m = 0 \\ 4 + \sum_{i=1}^{i} \overline{T}^i_{m-1}, & \text{if } m > 0 \end{cases} \tag{6.5}$$

Arguments justifying this inequality are similar to those used for analogous fact in proof of Lemma 6.13. The only new thing is 4 in the formula for the case of $m > 0$. In this case, by point (iii) of Claim 6.15, starting from the second state in the sequence under consideration, property **PB2** is satisfied for the remaining subsequence, for all $d \geq D$. Thus, by point (ii) of Claim 6.15, any subsequence of the remaining sequence with $\mathrm{dep}(\mathrm{Ind}(L(s)))$ remaining unchanged for its every state s, can have length at most 3.

By Fact 6.10 from (6.5) we get (for $D > 0$)

$$T^G_D \leq 4\frac{(|G|+D-1)!}{|G|!(D-1)!} + (|G|+2)\frac{(|G|+D-1)!}{(|G|-1)!D!} = \mathcal{O}\Big(D^{|G|}\Big).$$

Let now S^G_D denote the maximal length of a sequence of R^{B}_G-Successors in the pre-tableau constructed by Algorithm 6.1 such that points $1-3$ are satisfied for it. Then $S^G_D \leq \overline{S}^{|G|}_D$, where

$$\overline{S}^n_m = \begin{cases} 2n+1, & \text{if } m = 0 \\ 4 + \overline{T}^n_{m-1} + \sum_{i=1}^{n} \overline{S}^i_{m-1}, & \text{if } m > 0. \end{cases} \tag{6.6}$$

Explanation for this equation is again analogous to that used in proof of Lemma 6.6. By Fact 6.10 from (6.6) we get (for $D > 0$)

$$S_D^G \leq \left(4 + T_{D-1}^G\right) \frac{(|G| + D - 1)!}{|G|!(D-1)!} + (|G| + 2) \frac{(|G| + D - 1)!}{(|G| - 1)!D!} = \mathcal{O}\Big(D^{2|G|}\Big).$$

Thus the maximal length of a sequence of R^{B}-Successors with the same modal depth of labels is $\leq S_{\mathrm{dep}(\varphi)+1}^{\mathcal{A}} = \mathcal{O}\big(\mathrm{dep}(\varphi)^{2|\mathcal{A}|}\big)$. □

Now we are ready to prove the lemma on bounds of the state height of the pre-tableau constructed by Algorithm 6.1 for an input formula satisfying modal context restriction $\mathbf{R}_2$.

Lemma 6.18. *The state height of the pre-tableau constructed by Algorithm 6.1 for an input $\varphi \in \mathcal{L}_{\mathbf{R_2}}^{\mathrm{T}}$ is $\leq \mathcal{O}(\mathrm{dep}(\varphi)^{2|\mathcal{A}|+1})$ and its height is $\leq \mathcal{O}(|\varphi|\mathrm{dep}(\varphi)^{2|\mathcal{A}|+1})$.*

Proof. For any node n in a pre-tableau constructed by the algorithm $|L(n)| \leq (2|\mathcal{A}| + 1)|\varphi|$, as $L(n) \subseteq \neg\mathrm{Cl}(\varphi)$. Thus the path between any subsequent states s and t can contain at most $(2|\mathcal{A}| + 1)|\varphi| - 1$ internal nodes. Moreover, for any states s and t such that t is a descendant of s it must be that $\mathrm{dep}(L(t)) \leq \mathrm{dep}(L(s))$.

If s and t are states, such that t is an R^{G}-Successor of s, then $\mathrm{dep}(L(t)) < \mathrm{dep}(L(s))$. Thus any sequence of states in the pre-tableau can contain at most $\mathrm{dep}(\varphi)$ R^{G}-Successors. Also, if s, t and u are states such that t is an R_j^{B}-Successor of s and u is an R_k^{I}-Successor of t with $j \neq k$, then it holds that $\mathrm{dep}(L(u)) < \mathrm{dep}(L(s))$. By Lemma 6.2 and construction of the algorithm, if s, t and u are states such that t is an R_j^{B}-Successor of s and u is an R_k^{O}-Successor of t, where $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$, then it must hold that $j \neq k$. By Lemma 6.6, maximal length of a sequence of R^{I}-Successors with the same modal depth of labels in a pre-tableau constructed by the algorithm is $\leq \mathcal{O}(\mathrm{dep}(\varphi)^{2|\mathcal{A}|})$. Similarly, by Lemma 6.13, maximal length of a sequence of R^{B}-Successors with the same modal depth of labels in a pre-tableau constructed by the algorithm is $\leq \mathcal{O}(\mathrm{dep}(\varphi)^{2|\mathcal{A}|})$. Hence any sequence of nodes in the pre-tableau must be of length $\leq \mathcal{O}(|\varphi|\mathrm{dep}(\varphi)^{2|\mathcal{A}|+1})$ and contains $\leq \mathcal{O}(\mathrm{dep}(\varphi)^{2|\mathcal{A}|+1})$ states. □

Since the height of the pre-tableau constructed by the algorithm for an input formula $\varphi \in \mathcal{L}_{\mathbf{R_2}}^{\mathrm{T}}$ is bounded and the number of successor of any state is bounded as well so we have the following lemma as a corollary of Lemma 6.18.

Lemma 6.19. *For any input formula $\varphi \in \mathcal{L}_{\mathbf{R_2}}^{\mathrm{T}}$ Algorithm 6.1 terminates.*

What remains to be shown is the validity of the algorithm. Before proving the validity we need to introduce the following useful notions. Given a formula $\neg[O]_G^+\psi$ and a model $\mathcal{M}$ with a world v in it such that $(\mathcal{M}, v) \vDash \neg[O]_G^+\psi$, we call any sequence of worlds $v_0, \ldots, v_k$ such that $v_0 = v$, for all $0 < l \leq k$ it holds that $(v_{l-1}, v_l) \in R_{j_l}^{O}$ and $j_l \in G$, for all $0 < l < k$ it holds that $(\mathcal{M}, v) \vDash \psi$ and $(\mathcal{M}, v_k) \vDash \neg\psi$, a *satisfying sequence* for $\neg[O]_G^+\psi$ in $(\mathcal{M}, v)$. A satisfying sequence for $\neg[O]_G^+\psi$ in $(\mathcal{M}, v)$ that has minimal length is called a *minimal satisfying sequence* for $\neg[O]_G^+\psi$ in $(\mathcal{M}, v)$. Given a formula φ we call any pair $(\mathcal{M}, v)$ such that $(\mathcal{M}, v) \vDash \varphi$ *a satisfying pair* for φ. Given a set of formulas Φ with $\neg[O]_G^+\psi \in \Phi$ we say that $(\mathcal{M}, v)$ is a *satisfying pair for $\bigwedge \Phi$ with minimal satisfying sequence for $\neg[O]_G^+\psi$*, if $(\mathcal{M}, v) \vDash \bigwedge \Phi$ and a minimal satisfying sequence for $\neg[O]_G^+\psi$ in $(\mathcal{M}, v)$ is minimal over all satisfying pairs for $\bigwedge \Phi$.

Firstly, we show the following extension of Lemma 4.7.

Lemma 6.20. *Let n be an internal node in the pre-tableau constructed by Algorithm 6.1 for some input formula $\varphi \in \mathcal{L}^{\mathrm{T}}$. For any Kripke model $\mathcal{M}$ and a world v in it such that $(\mathcal{M}, v) \vDash \bigwedge L(n)$, there exists a successor m of n such that $(\mathcal{M}, v) \vDash \bigwedge L(m)$.*

Proof. Let $\mathcal{M}$ be a Kripke model with a world v in it such that $(\mathcal{M}, v) \vDash \bigwedge L(n)$. In the case of successors of n being created during propositional tableau formation or [B]-expanded tableau formation, the lemma can be shown by analogous arguments to those used in proof of Lemma 4.7. The remaining two cases are shown below, addressing the steps of $[\mathrm{B}]^+$-expanded and $[\mathrm{I}]^+$-expanded tableau formation and closed propositional tableau formation are shown below.

Suppose that successors of n were created during $[O]^+$-expanded tableau formation for a witness of the form $\neg[O]^+_G\xi \in L(n)$, where $O \in \{\mathrm{B}, \mathrm{I}\}$. Let M be the set of these successors. Since it must be that $(\mathcal{M}, v) \vDash \neg[O]^+_G\xi$, so there must be a satisfying sequence $v_0, \ldots, v_k$ for $\neg[O]^+_G\xi$ in $(\mathcal{M}, v)$. If $k = 1$, then $(\mathcal{M}, v) \vDash \neg[O]_{j_1}\xi$. Since for any $\zeta \in \bigcup_{j \in G \setminus \{j_1\}} \{[O]_j\xi\} \cup \bigcup_{j \in G} [O]_j[O]^+_G\xi\}$, either $(\mathcal{M}, v) \vDash \zeta$ or $(\mathcal{M}, v) \vDash \neg\zeta$, so there must be $m \in M$ with $\neg[O]_{j_1}\xi \in L(m)$ such that $(\mathcal{M}, v) \vDash \bigwedge L(m)$. Assume that $k > 1$. Since $(\mathcal{M}, v_1) \vDash \neg[O]^+_G\psi$, so $(\mathcal{M}, v) \vDash \neg[O]_{j_1}[O]^+_G\psi$ and, by analogous arguments to those used in the case of $k = 1$, there must be $m \in M$ with $\neg[O]_{j_1}[O]^+_G\psi \in L(m)$ such that $(\mathcal{M}, v) \vDash \bigwedge L(m)$.

Secondly, suppose that n has a successor created during closed tableau formation for a witness of the form $[O]^+_G\xi \in L(n)$ with $O \in \{\mathrm{B}, \mathrm{I}\}$. Let m be the successor of n. Since it must be that $(\mathcal{M}, v) \vDash [O]^+_G\psi$, so for any $u \in R^{O^+}_G(v)$ it must be that $(\mathcal{M}, u) \vDash \psi$ and (by transitivity of $R^{O^+}_G$) $(\mathcal{M}, u) \vDash [O]^+_G\psi$. Thus $(\mathcal{M}, v) \vDash [O]_j\psi$ and $(\mathcal{M}, v) \vDash [O]_j[O]^+_G\psi$. Hence $(\mathcal{M}, v) \vDash \bigwedge L(m)$. $\square$

We will also use the following lemma (the proof is moved to the Appendix).

Lemma 6.21. *Let $\Phi \subseteq \mathcal{L}^{\mathrm{T}}$ be a* [B]*-expanded tableau. Then the following hold*

1. *if $\neg[\mathrm{I}]^+_G\varphi \in \Phi$ and $(\mathcal{M}, w)$ is a satisfying pair for $\bigwedge \Phi$ with minimal satisfying sequence $v_0, \ldots, v_n$ for $[\mathrm{I}]^+_G\varphi$ such that $n \geq 2$, then $(\mathcal{M}, v_1)$ is a satisfying pair for $\bigwedge \Phi^{\neg[\mathrm{I}]_{j_1}}([\mathrm{I}]^+_G\varphi)$ with minimal satisfying sequence $v_1, \ldots, v_n$ for $\neg[\mathrm{I}]^+_G\varphi$.*
2. *if $\neg[\mathrm{B}]^+_G\varphi \in \Phi$ and $(\mathcal{M}, w)$ is a satisfying pair for $\bigwedge \Phi$ with minimal satisfying sequence $v_0, \ldots, v_n$ for $[\mathrm{B}]^+_G\varphi$ such that $n \geq 2$, then $(\mathcal{M}, v_1)$ is a satisfying pair for $\bigwedge \Phi^{\neg[\mathrm{B}]_{j_1}}([\mathrm{B}]^+_G\varphi)$ with minimal satisfying sequence $v_1, \ldots, v_n$ for $\neg[\mathrm{B}]^+_G\varphi$.*

One of the main differences in the proof of validity of the algorithm, as compared to analogous proofs for modal logics without iterated modalities, comes from the possibility of existence of nodes marked `undec` in a pre-tableau constructed by the algorithm. The following lemma is crucial for dealing with nodes that are marked `unsat` because of an unresolved formula in their label and existence of successors marked `undec`.

Lemma 6.22. *Let n be a node in the pre-tableau constructed by Algorithm 6.1 for some input formula $\varphi \in \mathcal{L}^{\mathrm{T}}$, with a formula $\neg[O]^+_G\psi \in L(n)$ (where $O \in \{\mathrm{B}, \mathrm{I}\}$) unresolved in n. Suppose also that for any descendant r of n it holds that if r is marked* **`unsat`**, *then $\bigwedge L(r)$ is not satisfiable. If $\bigwedge L(n)$ is satisfiable, then $B(n) \neq \{n\}$.*

Proof. We will show first that if n is a node of the pre-tableau, $\bigwedge L(n)$ is satisfiable, $\neg[O]^+_G\psi$ with $O \in \{\mathrm{B}, \mathrm{I}\}$ is unresolved in n and for each successor r of n, r being marked `unsat` implies that $\bigwedge L(r)$ is not satisfiable, then for any satisfying pair $(\mathcal{M}, v)$ for $\bigwedge L(n)$ with minimal satisfying sequence $v_0, \ldots, v_k$ for $\neg[O]^+_G\psi$, there exists $m \in B(n) \setminus \{n\}$ and $0 < l < k$ such that $(\mathcal{M}, v_l) \vDash \bigwedge L(m)$. To show that we will use induction over the maximal distance from n to a descendant leaf of the pre-tableau. Suppose that n is a leaf of the pre-tableau. Since $\bigwedge L(n)$ is satisfiable so $L(n)$ cannot be blatantly inconsistent and n must be a state. Moreover n must be a $\neg[O]_j[O]^+_G\psi$-Successor, for some $j \in G$, and must be marked `undec`, as $\neg[O]^+_G\psi$ is unresolved in n. Take any satisfying pair $(\mathcal{M}, v)$ for $\bigwedge L(n)$ with minimal satisfying sequence $v_0, \ldots, v_k$ for $\neg[O]^+_G\psi$. Since n is a state and $\neg[O]^+_G\psi$ is unresolved in it so, for all $j \in G$, $[O]_j\psi \in L(n)$. Thus it must be that $(\mathcal{M}, v_1) \vDash \psi$ and $(\mathcal{M}, v_1) \vDash \neg[O]^+_G\psi$. This, together with the fact that $L(n)$ is a $[O]^+$-expanded tableau, implies $\neg[O]_{j_1}[O]^+_G\psi \in L(n)$. Since n is a leaf of the pre-tableau so creation of a successor for $\neg[O]_{j_1}[O]^+_G\psi$ must be blocked by some node m and since n is a $\neg[O]_j[O]^+_G\psi$-successor, for some $j \in G$, so $m \in B(n)$. By construction of the algorithm it must hold that $L^{\neg[O]_{j_1}}(n, [O]^+_G\psi) = L(m)$ which, together with Lemma 4.3 and the fact that $(\mathcal{M}, v_1) \vDash \neg[O]^+_G\psi$, implies $(\mathcal{M}, v_1) \vDash \bigwedge L(m)$. Notice that since $[O]_{j_1}\psi \in L(n)$ so $(\mathcal{M}, v_1) \vDash \psi$ and so it must be that $k > 1$. To see that $m \neq n$, assume the opposite. Then $(\mathcal{M}, v_1)$ is a satisfying pair for $\bigwedge L(n)$ with $v_1, \ldots, v_k$ being a satisfying sequence for $\neg[O]^+_G\psi$. Thus we get a contradiction with the assumption of minimality of $v_0, \ldots, v_k$ and so it must be that $m \neq n$.

For the induction step, suppose that n is not a leaf of the pre-tableau. Take any satisfying pair $(\mathcal{M}, v)$ for $\bigwedge L(n)$ with minimal satisfying sequence $v_0, \ldots, v_k$ for $\neg[O]^+_G\psi$. Suppose first that n is an internal node. By Lemma 6.20 there must exist a successor r of n such that $(\mathcal{M}, v) \vDash \bigwedge L(r)$. Since n is marked `undec` and $\bigwedge L(r)$ is satisfiable, so r must be marked `undec` as well and $\neg[O]^+_G\psi$ must be unresolved in r. Moreover, by construction of the algorithm, it must be that $B(r) \subseteq B(n)$. Thus, by the induction hypothesis, there is $m \in B(n)$ and $0 < l < k$ such that $(\mathcal{M}, v_l) \vDash \bigwedge L(m)$. Moreover, it must be that $m \neq n$ as otherwise, by similar arguments to those used for the induction basis, we get a contradiction with the assumption of minimality of $v_0, \ldots, v_k$.

Suppose now that n is a state. As we argued for the induction basis, n is marked `undec`, for all $j \in G$ it holds that $[O]_j\psi \in L(n)$ and $\neg[O]_{j_1}[O]^+_G\psi \in L(n)$. By the fact that $\neg[O]^+_G\psi$ is unresolved in n, either a $\neg[O]_{j_1}[O]^+_G\psi$-successor r of n is marked `undec` or its creation is blocked by some ancestor m. In the latter case the claim holds by analogous arguments to those used for the induction basis. Suppose that the first case holds. Then $\neg[O]^+_G\psi$ is unresolved in r. Moreover, since $(\mathcal{M}, v)$ is a satisfying pair for $\bigwedge L(n)$ with a minimal satisfying sequence $v_0, \ldots, v_k$ for $\neg[O]^+_G\psi$ and $[O]_j\psi \in L(n)$, for all $j \in G$, so $k \geq 2$ and, by Lemma 6.21, $(\mathcal{M}, v_1)$ must be a satisfying pair for $\bigwedge L(r)$ with minimal satisfying sequence $v_1, \ldots, v_k$ for $\neg[O]^+_G\psi$. Hence, by the induction hypothesis, there must exist $m \in B(r) \setminus \{r\}$ and $1 < l < k$ such that $(\mathcal{M}, v_l) \vDash \bigwedge L(m)$. By construction of the algorithm $m \in B(n)$ and, by minimality of $v_0, \ldots, v_k$, it must be that $m \neq n$.

Now, suppose that n is a node of the pre-tableau with $\neg[O]^+_G\psi \in L(n)$ and satisfying all the assumptions stated in the lemma. Suppose also that $\bigwedge L(n)$ is satisfiable. Then there exists a satisfying pair for $\bigwedge L(n)$ with minimal satisfying sequence for $\neg[O]^+_G\psi$ and, by what was shown above, there must exist $m \in B(n) \setminus \{n\}$ and $0 < l < k$ such that $(\mathcal{M}, v_l) \vDash \bigwedge L(m)$. Hence it must be $B(n) \neq \{n\}$. □

Now we are ready to prove validity of the algorithm. We show that the algorithm is valid for any formula $\varphi \in \mathcal{L}^{\mathrm{T}}$ for which it terminates. Since, as we showed above, it terminates on any input from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$, this implies its full validity on $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$.

Lemma 6.23. *Suppose that Algorithm 6.1 terminates on a formula $\varphi \in \mathcal{L}^{\mathrm{T}}$. Then φ is satisfiable iff Algorithm 6.1 returns `sat` on the input φ.*

Proof. For the left to right implication suppose that Algorithm 6.1 terminates on the input $\varphi \in \mathcal{L}^{\mathrm{T}}$. Then it must have constructed a finite pre-tableau for that input. We start by showing, for any node n of the pre-tableau constructed by the algorithm, that if n is marked `unsat`, then $\bigwedge L(n)$ is unsatisfiable. The proof is by induction on the maximal length of paths from a node to one of its descendant leaves. If n is a leaf and it is marked `unsat`, then, by analogous arguments to those used in proof of Lemma 4.8, $\bigwedge L(n)$ must be unsatisfiable. For the induction step, suppose that n is a node that is marked `unsat` and that has at least one successor. Let n be an internal node. If all successors of n are marked `unsat`, then, by the induction hypothesis, for any successor m of n, $\bigwedge L(m)$ must be unsatisfiable. Suppose that $\bigwedge L(n)$ is satisfiable. By Lemma 6.20 there exists a successor m of n such that $\bigwedge L(m)$ is satisfiable and we get a contradiction. Thus $\bigwedge L(n)$ must be unsatisfiable in this case. Suppose that there exists a successor of n which is not marked `unsat`. Then it must be that all successors of n are marked either `unsat` or `undec`, there exists $\psi \in L(n)$ such that ψ is unresolved in n and $B(n) = \{n\}$. Hence, by Lemma 6.22, $\bigwedge L(n)$ must be unsatisfiable. Let n be a state. If there exists a successor of n which is marked `unsat`, than showing that $\bigwedge L(n)$ is unsatisfiable can be done like in proof of Lemma 4.8. Suppose then that none of the successors of n is marked `unsat`. Then there must exist a formula $\neg[O]^+_G\psi \in L(n)$ which is unresolved in n and it must hold that $B(n) = \{n\}$. Hence, by Lemma 6.22, $\bigwedge L(n)$ must be unsatisfiable.

Observe that `root` of any pre-tableau constructed by the algorithm is marked either `unsat` or `sat`, as it cannot block creation of a successor for any formula of the form $\neg[O]_j[O]^+_G\psi$ with $j \in G$, and so there cannot be any formula which is unresolved in `root`. Thus if `root` in the pre-tableau is not marked `sat`, then it is marked `unsat` and φ must be unsatisfiable. Hence if φ is satisfiable, then `root` node must be marked `sat` and the algorithm returns `sat`.

For the right to left implication, assume that Algorithm 6.1 returned `sat` on the input φ. Then it constructed a finite the pre-tableau

$$(N, \text{root}, \mathit{succ}, \{R^O_j\text{-succ} : j \in \mathcal{A},\ O \in \{\text{B}, \text{G}, \text{I}\}\}, L)$$

for φ. We will show how to construct, on the basis of this pre-tableau, a TeamLog tableau for φ such that the number of states of the tableau is $\leq ((2|\mathcal{A}|+1)|\varphi|)^s$, where s is the state height of the pre-tableau. Consider a model graph

$$\mathcal{T} = (W, \{R^O_j : j \in \mathcal{A},\ O \in \{\text{B}, \text{G}, \text{I}\}\}, L|_W)$$

where W is constructed as follows. We start with W consisting of a state marked `sat` from $\mathit{SS}(\texttt{root})$ (where $\mathit{SS}(\cdot)$ is defined like in proof of Lemma 4.8). Then, for each state $w \in W$ whose R^O-Successors were not added to the set yet, we take, for each R^O-successor node n of w, a state $v \in \mathit{SS}(n)$ which is marked `sat` (if there is such) or which is marked `undec` (otherwise). We proceed like that until leaves of the pre-tableau are reached. Since each state of the pre-tableau has at most $(2|\mathcal{A}|+1)|\varphi|$ R^O-successors (as the number of elements in its label is bounded by $(2|\mathcal{A}|+1)|\varphi|$), so W has $\leq ((2|\mathcal{A}|+1)|\varphi|)^s$ elements. Labelling function L is like in the pre-tableau but restricted to W. Before defining the accessibility relations, we need to define the set of states associated with nodes blocking creation of $\neg[O]_j[O]^+_G\xi$-successors of a given state. Let v be a state with a formula $\neg[O]_j[O]^+_G\xi \in L(v)$ and suppose that n is a node that blocks creation of a $\neg[O]_j[O]^+_G\xi$-successor of v. That is $n \in B(v)$ and there exists a $\neg[O]^+_G\xi$-Ancestor t of v with n being its $\neg[O]_j[O]^+_G\xi$-successor and such that $L^{[O]_j}(v) = L^{[O]_j}(t)$. In such a case we will call any state $u \in \mathit{SS}(n)$ a *R^O_j-loop-back state for v*. Given a state $v \in W$, $j \in \mathcal{A}$ and $O \in \{\text{B}, \text{I}\}$ let

- $\mathit{LB}^O_j(v) = \Big\{u \in W : u \text{ is a } R^O_j\text{-loop-back state for } v\Big\}$.

When constructing the accessibility relations, we will need to properly extend them with loop back connections. The accessibility relations of $\mathcal{T}$ are defined as follows:

- $R^{\text{B}}_j = (R^{\text{B}}_j\text{-Succ} \cap W \times W) \cup \{(v, u) \in W \times W : \text{there exists } w \in W \text{ such that } \{v, u\} \subseteq R^{\text{B}}_j\text{-Succ}(w)\} \cup \{(v, u) \in W \times W : u \in \mathit{LB}^{\text{B}}_j(v)\}$,

- $R_j^{\mathrm{G}} = (R_j^{\mathrm{G}}\text{-Succ} \cap W \times W) \cup \{(v,u) \in W \times W : \text{there exists } w \in W \text{ such that } v \in R_j^{\mathrm{B}}\text{-Succ}(w) \text{ and } u \in R_j^{\mathrm{G}}\text{-Succ}(w)\}$,
- $R_j^{\mathrm{I}} = (R_j^{\mathrm{I}}\text{-Succ} \cap W \times W) \cup \{(v,u) \in W \times W : \text{there exists } w \in W \text{ such that } v \in R_j^{\mathrm{B}}\text{-Succ}(w) \text{ and } u \in R_j^{\mathrm{I}}\text{-Succ}(w) \cup LB_j^{\mathrm{I}}(w)\} \cup \{(v,u) \in W \times W : u \in LB_j^{\mathrm{I}}(v)\}$,

Since there exists $w \in W$ such that $w \in SS(\mathtt{root})$ and $\varphi \in L(w)$ so it is enough to show that $\mathcal{T}$ is a TeamLog tableau. Before we show that, notice that the construction above guarantees that for any state $u \in W \setminus \{w\}$ and any $j \in \mathcal{A}$ it holds that $w \notin R_j^{\mathrm{B}}(u)$ (this is because w is not an R^{B}-Successor of any state). If we show that $\mathcal{T}$ is a TeamLog tableau, then, by Theorem 4.14, it will follow that φ is satisfiable. Since all elements of W are states, so they must be [B]-expanded tableaux. Since the construction of TeamLog tableau above is an extension of the construction of TeamLog$^{\mathrm{ind}}$ tableau used in proof of Lemma 4.8, so in showing that the properties of TeamLog tableau that are present in the definition of TeamLog$^{\mathrm{ind}}$ tableau are satisfied, we will focus on those affected by the extensions of the construction. We will mostly concentrate on condition **TC** which is related to iterated modalities.

Conditions **T1**, **T2**, **T4**, **TG4**, **TI4**, **T5**, **TG5**, **TI5** and **TIG** can be shown by arguments similar to those used in proof of Lemma 4.8, as for all $u \in LB_j^O(v)$ it holds that $L^{[O]_j}(v) \subseteq L(u)$ and $L^{\neg[O]_j}(v, [O]_G^+\xi) \subseteq L(u)$, where $\neg[O]_G^+\xi$ it the formula associated with u in $LB_j^O(v)$. Condition **TD** for $O \in \{\mathrm{B}, \mathrm{I}\}$ can also be shown by arguments similar to those used in Lemma 4.8 with additional argument that if a $[O]_j[O]_G^+\xi$-successor of some state $v \in W$ is not created, then it holds that $L^{[O]_j}(v) \subseteq L(v)$, so that $[O]_G^+\xi \in L(v)$.

For condition **TC**, suppose that $v \in W$ and suppose that $\neg[O]_G^+\psi \in L(v)$ with $O \in \{\mathrm{B}, \mathrm{I}\}$. We will show first that the condition holds for the states that are marked `sat` and are $\neg[O]_j[O]_G^+\psi$-Successors, with some $j \in G$. To show this we will use induction on the maximum distance from the state to its descendant leafs. Suppose that v is a leaf of the pre-tableau. Since v is a state so $L(v)$ is a $[O]^+$-expanded tableau. By condition **CE**, for all $j \in G$ it holds that $[O]_j\psi \in \neg L(v)$. If there was no $j \in G$ such that $\neg[O]_j\psi \in L(v)$ then $\neg[O]_G^+\psi$ would be unresolved in v and v would be marked `undec`. Hence there must exist such $j \in G$, in which case condition **TC** follows from condition **T2**. For the induction step notice that if there is $j \in G$ such that $\neg[O]_j\psi \in L(v)$ then the condition holds, by condition **T2**. Otherwise, by condition **CE** and by the fact that v is marked `sat`, there must exist $j \in G$ such that $\neg[O]_j[O]_G^+\psi \in L(v)$ and $\neg[O]_j[O]_G^+\psi$-successor of v is created and marked `sat`. Hence condition **TC** holds by the induction hypothesis.

Secondly we will show that condition **TC** is satisfied for states that are marked `undec` and are $\neg[O]_j[O]_G^+\psi$-Successors with some $j \in G$. To show this we will use induction on $M(v) = \min_{n \in B(v)} \mathrm{sheight}(n)$, starting from the minimal value. Let $V \subseteq W$ be the set of states such that for each $v \in V$, $M(v)$ is minimal. To show that the condition is satisfied for all $v \in V$ we will use induction on the maximum distance from the state to its descendant leafs. Suppose that v is a leaf. By the fact that v is marked `undec` and by condition **CE**, for all $j \in G$ it holds that $[O]_j\xi \in L(v)$ and there exists $j \in G$ such that $\neg[O]_j[O]_G^+\psi \in L(v)$. Let $m(v) = \mathrm{argmin}_{n \in B(v)}\mathrm{sheight}(n)$ and let $j \in G$ be such that creation of $\neg[O]_j[O]_G^+\psi$-successor of v is blocked by $m(v)$. Then there exists $u \in SS(m(v))$ such that $u \in R_j^O(v)$. By construction of the algorithm, $m(v)$ must be a $\neg[O]_k[O]_G^+\psi$-successor with $k \in G$. Moreover, by minimality of $M(v)$, $B(m(v)) = \{m(v)\}$ and so $m(v)$ must be marked `sat`, as it would be marked `unsat` otherwise. Hence u must be marked `sat` as well. Moreover it holds that $\neg[O]_G^+\psi \in L(u)$ and, by what we have shown above, condition **TC** is satisfied for it. Hence condition **TC** is satisfied for v as well. If v is not a leaf of the pre-tableau, then, by condition **CE** and by the fact that v is marked `undec`, there must exist $j \in G$ such that $\neg[O]_j[O]^+_G\psi \in L(v)$. Again we take $m(v)$. If there is $j \in G$ such that creation of $\neg[O]_j[O]^+_G\psi$-successor of v is blocked by $m(v)$, then condition **TC** is satisfied by arguments analogous to those used above. Otherwise there must be $j \in G$ such that there is a $\neg[O]_j[O]^+_G\psi$-Successor u of v with $m(v) \in B(u)$ and condition **TC** holds by the induction hypothesis. For the induction step (of the main induction) suppose that $M(v)$ is not minimal. Consider the set of states $V \subseteq W$ with the same value of $M(v)$. Arguments here are analogous to those used for the induction basis. The difference lies in the fact that $B(m(v)) \neq \{m(v)\}$ this time, as $M(v)$ is not minimal. However, this implies that there is $m' \in B(m(v))$ such that $\mathrm{sheight}(m') < \mathrm{sheight}(m(v))$ and the induction hypothesis applies.

Lastly we will show that condition **TC** is satisfied for states that are not $\neg[O]_j[O]^+_G\psi$-Successors with any $j \in G$. By condition **CE** there must exist $j \in G$ such that either $\neg[O]_j\psi \in L(v)$ or $\neg[O]_j[O]^+_G\psi \in L(v)$. In the first case, the condition holds by condition **T2**. Similarly in the second case, if $\neg[O]_j[O]^+_G\psi$-successor of v was created. If it was not, then v must be an R^{B}_j-Successor of some state $w \in W$ and there there exists $u \in R^O_j(w)$ such that u is a $\neg[O]_k[O]^+_G\psi$-Successor with $k \in G$ and $\neg[O]^+_G\psi \in L(u)$. By what we have shown above, condition **TC** is satisfied for u and $\neg[O]^+_G\psi$ and, consequently, it is satisfied for v and $\neg[O]^+_G\psi$ as well. Hence we have shown that $\mathcal{T}$ is a TeamLog tableau for φ and that φ is satisfiable. □

The following theorem states lower and upper bounds for complexity of the TeamLog satisfiability problem for formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$.

Theorem 6.24. *The* TeamLog *satisfiability problem for formulas from* $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$ *is PSPACE complete.*

Proof. By Lemmas 6.19 and 6.23 and arguments similar to those used in proof of Theorem 4.9, the problem is in PSPACE. On the other hand the problem of TeamLog satisfiability of formulas from $\mathcal{L}^{\mathrm{Tind}} \subseteq \mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$ is PSPACE hard. Hence the problem of TeamLog satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$ is PSPACE complete. □

As Lemma 6.18 and proof of Lemma 6.23 suggest, bounding modal depth of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$ makes the TeamLog satisfiability problem NPTIME complete.

Theorem 6.25. *For any fixed k, if modal depth of formulas from* $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$ *is bounded by k, then the* TeamLog *satisfiability problem for them is NPTIME complete.*

Proof. By Lemma 6.18 and the construction of TeamLog tableau based on the pre-tableau constructed by Algorithm 6.1 presented in Lemma 6.23, the size of the tableau for a satisfiable formula φ is bounded by $\mathcal{O}\Big(((2|\mathcal{A}|+1)|\varphi|)^{\mathrm{dep}(\varphi)^{2|\mathcal{A}|+1}}\Big)$. Hence, if modal depth of φ is bounded by k, then the size of the tableau is bounded by $\mathcal{O}\Big(((2|\mathcal{A}|+1)|\varphi|)^{k^{2|\mathcal{A}|+1}}\Big)$. This means that the satisfiability of φ with bounded modal depth can be checked by the following non-deterministic Algorithm 6.6.

Algorithm 6.6: DecideSatisfiabilityNonDeterministic

Input: a formula $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$
Output: a decision whether φ is satisfiable or not

Guess a TeamLog tableau $\mathcal{T}$ satisfying φ;
if *$\mathcal{T}$ is a tableau for φ* **then**
    return `satisfiable`;

Since tableau $\mathcal{T}$ constructed by Algorithm 6.6 is of polynomial size, so checking if it is a tableau for φ can be realized in polynomial time. This shows that satisfiability of φ can be checked in NPTIME. The problem is also NPTIME complete, as the satisfiability problem for propositional logic is NPTIME hard. □

6.3.2 The algorithm for $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$

The algorithm for checking TeamLog satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ requires a different approach since, as Theorem 6.1 shows, a model for such formulas may contain an exponentially long path. Algorithm 6.7 presented below is a modification of Algorithm 6.1 designed for checking the TeamLog satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ using polynomial space. The main difference between the two algorithms lies in the procedure of R^{B}-successors creation, specifically for the formulas of the form $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$ with $j \in G$. Since the satisfying sequence for such a formula may have exponential length with respect to the size of the set of formulas, the algorithm using a polynomial space cannot attempt to construct such a sequence storing it fully in the memory, as it was done in the case of Algorithms 4.1 and 6.1. For this reason Algorithm 6.7, presented below, constructs a pre-tableau just like Algorithm 6.1, creating R^{G}- and R^{I}-successors in the same way, but stopping creation of R^{B}-successors for formulas of the form $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$ when certain condition is satisfied. In such case Function 6.9 is used for checking if the label of the $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-successor is satisfiable. If it is decided by Function 6.9 that the label of the $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-successor is not satisfiable, then the state is marked `unsat`. Otherwise, the decision on how the state should be marked depends on the other successors and the same procedure of marking nodes as the one used in Algorithm 6.1 is applied.

In the algorithm we are referring to the following sets, defined for a given set of formulas $\Phi \subseteq \mathcal{L}^{\mathrm{T}}$ and $G \subseteq \mathcal{A}$:

$$\begin{aligned}
\Phi/[\mathrm{B}]^+_G &= \{\psi : [\mathrm{B}]^+_H\psi \in \Phi \text{ and } G \subseteq H\},\\
\Phi \sqcap [\mathrm{B}]^+_G &= \{[\mathrm{B}]^+_H\psi : [\mathrm{B}]^+_H\psi \in \Phi \text{ and } G \subseteq H\},\\
\Phi^{[\mathrm{B}]^+_G} &= (\Phi/[\mathrm{B}]^+_G) \cup (\Phi \sqcap [\mathrm{B}]^+_G).
\end{aligned}$$

Given a formula ψ, $G \subseteq \mathcal{A}$, $j \in G$ and a set of formulas Φ such that $\{[\mathrm{B}]_j\psi, \psi\} \subseteq \Phi$ as an input, Function 6.9 decides whether the set $\Phi^{\neg[\mathrm{B}]_j}([\mathrm{B}]^+_G\psi)$ is satisfiable or not. To describe the idea of this algorithm, let Ψ_1 and Ψ_2 be sets of formulas. Given $k \in \mathcal{A}$, we say that Ψ_1 and Ψ_2 are *connected with* k if $\Psi_1 \sqcap k = \Psi_2 \sqcap k$. Moreover, given a set $H \subseteq \mathcal{A}$, we say that Ψ_1 and Ψ_2 are *H-connected* if they are connected with some $k \in H$. Let Γ be a set of formulas and let $\mathcal{SS}(\Gamma)$ be the set of all minimal sets of formulas containing Γ as a subset that are [B]-expanded tableaux. Given $H \subseteq \mathcal{A}$, let $\mathcal{G}_H(\Gamma) = (V, E)$ be an undirected graph such that V consists of all elements $\Psi \in \mathcal{SS}(\Gamma)$ such that Algorithm 6.7 returns `sat` on input $\bigwedge \Psi$ and for all $(\Psi_1, \Psi_2) \in V \times V$, $(\Psi_1, \Psi_2) \in E$ iff they are H-connected. A *path* in $\mathcal{G}_H(\Gamma)$ is a sequence $\Gamma_0, \dots, \Gamma_n$ of elements of V such that for all $1 \le i \le n$, Γ_{i-1} and Γ_i are H-connected. The *length of path* $\Gamma_0, \dots, \Gamma_n$ is n. Given a path $\Gamma_0, \dots, \Gamma_n$ of length $n \ge 1$ in $\mathcal{G}_H(\Gamma)$ we call a sequence $j_1, \dots, j_n$ of elements from H such that for each $1 \le i \le n$, Γ_{i-1} and Γ_i are connected with j_i, a *sequence associated with path* $\Gamma_0, \dots, \Gamma_n$. If $n = 0$, then the sequence associated with the path is the empty sequence ε. Given two sets of formulas Ψ_0 and Ψ_1, we say that Ψ_1 is *reachable from* Ψ_0 in $\mathcal{G}_H(\Gamma)$ (in n steps) iff there exists a path $\Gamma_0, \dots, \Gamma_n$ in $\mathcal{G}_H(\Gamma)$ such that $\Psi_0 = \Gamma_0$ and $\Psi_1 = \Gamma_n$.

To decide the satisfiability of $\bigwedge \Phi^{\neg[\mathrm{B}]_j}([\mathrm{B}]^+_G\psi)$, Function 6.9 checks whether there exist two sets of formulas $\{\Psi_0, \Psi_1\} \subseteq \mathcal{SS}((\Phi/[\mathrm{B}]^+_H) \cup \{\psi\})$, with $H = \mathrm{ag}\Big(\big(\Phi \sqcap [\mathrm{B}]^+_{\{j\}}\big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$, such that

Algorithm 6.7: DecideSatisfiability3

Input: a formula φ
Output: a decision whether φ is satisfiable or not

```
/* Pre-tableau construction */
Construct a pre-tableau consisting of single node root, with L(root) = {φ} and all
successor relations being empty;
repeat
    Let Z be the set of all leaves of the pre-tableau with labelling sets that are not
    blatantly inconsistent;
    if there is n ∈ Z such that n is not a state and ψ ∈ L(n) is a witness to that then
        FormState2(n,ψ);
    else if there is s ∈ Z then
        foreach ψ ∈ L(s) do
            CreateSuccessorsB3(s,ψ);
            CreateSuccessorsG(s,ψ);
            CreateSuccessorsI2(s,ψ);
until no change occurred;
/* Marking nodes and deciding satisfiability */
repeat
    MarkNodes2;
until no new node marked;
if root is marked sat then
    return sat;
else
    return unsat;
```

Procedure 6.8: CreateSuccessorsB3

Input: a state s and a formula $\psi \in L(s)$

if *ψ is of the form $\neg[\mathrm{B}]_j\xi$* **then**
  if *there is an R_j^{B}-Predecessor t of s such that $\neg[\mathrm{B}]_j\xi \in L(t)$ and $L^{\neg[\mathrm{B}]_j}(t,\xi) = L^{\neg[\mathrm{B}]_j}(s,\xi)$* **then**
    if *$\xi = [\mathrm{B}]_G^+\zeta$ with $j \in G$ and s is a $\neg[\mathrm{B}]_j\xi$-Successor of t* **then**
      For every descendant m of t on the path from t to s set $B(m) := B(m) \cup \{n\}$, where n is an R_j^{B}-successor of t on the path from t to s;
  else if *$\xi = [\mathrm{B}]_G^+\zeta$ with $j \in G$ and there is a $\neg[\mathrm{B}]_k[\mathrm{B}]_G^+\zeta$-Predecessor t of s such that $L(s)/[\mathrm{B}]_j \cup \{\zeta\} = L(t)/[\mathrm{B}]_k \cup \{\zeta\}$* **then**
    if $[\mathrm{B}]_j\zeta \in L(s)$ **then**
      if $\zeta \notin L(s)$ **then**
        Create an R_j^{B}-Successor v of s with $L(v) = L^{\neg[\mathrm{B}]_j}(s,\xi)$;
      else if `DecideSatisfiabilityAux(`$L(s)$, G, ζ, j`)` $=$ `unsat` **then**
        Mark s `unsat`;
  else Create an R_j^{B}-Successor v of s with $L(v) = L^{\neg[\mathrm{B}]_j}(s,\xi)$;

else if *ψ is of the form $[\mathrm{B}]_j\xi$ and there are no formulas of the form $\neg[\mathrm{B}]_j\zeta \in L(s)$* **then**
  If there is no R_j^{B}-Predecessor t of s such that $[\mathrm{B}]_j\xi \in L(t)$ and $L^{[\mathrm{B}]_j}(t) = L^{[\mathrm{B}]_j}(s)$ and $L^{[\mathrm{B}]_j}(s) \nsubseteq L(s)$, then create an R_j^{B}-successor u of s with $L(u) = L^{[\mathrm{B}]_j}(s)$;

- $(\Phi \sqcap j) \setminus ((\Phi \sqcap [\mathrm{B}]_j) \cup (\Phi \sqcap \neg[\mathrm{B}]_j)) \subseteq \Psi_0$ and
- either there exists $k \in H$ such that Algorithm 6.7 returns `sat` on the input $\bigwedge\left(\Psi_1^{[\mathrm{B}]_k} \cup (\Phi/[\mathrm{B}]_H^+) \cup \{\sim\psi\}\right)$ and Ψ_1 is reachable from Ψ_0 in $\mathcal{G}_H((\Phi/[\mathrm{B}]_H^+) \cup \{\psi\})$ with path $\Gamma_0, \dots, \Gamma_n$ such that if $n = 0$, then $j \neq k$, and if $n \geq 1$, then there exists $j_n \in H \setminus \{k\}$ such that Γ_{n-1} and Γ_n are connected with j_n.
- or Ψ_1 is reachable from Ψ_0 in $\mathcal{G}_H((\Phi/[\mathrm{B}]_H^+) \cup \{\psi\})$ and there exists $k \in G \setminus H$ such that either Algorithm 6.7 returns `sat` on the input $\bigwedge\left(\Psi_1^{[\mathrm{B}]_k} \cup \Phi^{[\mathrm{B}]^+_{H\cup\{k\}}} \cup \{\sim\psi\}\right)$ or Algorithm 6.7 returns `sat` on $\bigwedge\left(\Psi_1^{[\mathrm{B}]_k} \cup \Phi^{[\mathrm{B}]^+_{H\cup\{k\}}} \cup \{\psi, \neg[\mathrm{B}]_G^+\psi, [\mathrm{B}]_k\psi, \neg[\mathrm{B}]_k[\mathrm{B}]_G^+\psi\}\right)$

To check reachability, Function 6.10 is used. Given sets of formulas Φ_1, Ψ and Φ_2, sets $H \subseteq \mathcal{A}$ and $F \subseteq H$, $p \in H$ and $K \geq 0$, Function 6.10 checks if there exists a set of formulas $\Gamma \in \mathcal{SS}(\Phi_1)$ such that Algorithm 6.7 returns `sat` on input $\bigwedge\Gamma$ and Φ_2 is reachable from Γ in $\mathcal{G}_H(\Phi_1)$ in at most $2^K - 1$ steps with a path $\Gamma_0, \dots, \Gamma_n$ such that if $n = 0$, then $p \notin F$ and if $n \geq 1$, then there exists $j_n \in H \setminus F$ such that Γ_{n-1} and Γ_n are connected with j_n. The set F with which the algorithm is called will always be either $\varnothing$ or a singleton. It is used to forbid, in certain situations, one of the possible connections between the last two sets in the constructed sequence.

The idea of the algorithm is based on the idea of Savitch's algorithm for checking reachability in graph that uses quadratic logarithmic space with respect to $|V|$ (c.f. [88]). Notice that all the sets in $\mathcal{SS}((\Psi/[\mathrm{B}]_H^+) \cup \{\psi\})$ have the same number of elements and if $\Gamma \in \mathcal{SS}((\Psi/[\mathrm{B}]_H^+) \cup \{\psi\})$, then $|\mathcal{SS}((\Psi/[\mathrm{B}]_H^+) \cup \{\psi\})| \leq 2^{|\Gamma|}$. Thus to check reachability in $\mathcal{G}_H((\Phi/[\mathrm{B}]_H^+) \cup \{\psi\})$ it is enough to check whether there is reachability in at most $2^{|\Gamma|} - 1$ steps, where $\Gamma \in$

$\mathcal{SS}((\Psi/[\mathrm{B}]^+_H) \cup \{\psi\})$.

The procedure of marking nodes remains like in Algorithm 6.1, however the notion of unresolved formula used by it is different in the case of formulas of the form $\neg[\mathrm{B}]^+_G\psi$. The modification is related to the fact that in the case of any $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-Successor state s and any formula $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi \in L(s)$ with $k \neq j$, either the $\neg[\mathrm{B}]_k[\mathrm{B}]^+_k\psi$-successor of s is created or Function 6.9 is used to check the satisfiability of $L^{\neg[\mathrm{B}]_k}(s, [\mathrm{B}]^+_G\psi)$. Hence the only situation in which such a formula can be unresolved is when $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(s)$ and all the other formulas from $\neg\widetilde{\mathrm{Cl}}([\mathrm{B}]^+_G\psi)$ appear positively in $L(s)$. Unresolved formula of the form $\neg[\mathrm{B}]^+_G\psi$ is defined as follows.

Definition 22 (Unresolved formula)**.** *Let n be a node in a pre-tableau and let $\neg[\mathrm{B}]^+_G\psi \in L(n)$. A formula $\neg[\mathrm{B}]^+_G\psi$ is* unresolved *in n if one of the following holds:*

- *n is an internal node and a $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-descendant with $j \in G$, none of its successors is marked* `sat`*, there exists a successor of n marked* `undec` *and $B(n) \neq \varnothing$,*
- *n is a state and a $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-Successor with $j \in G$, $B(n) \neq \varnothing$, $[\mathrm{B}]_k[\mathrm{B}]^+_G\psi \in L(n)$, for all $k \in G \setminus \{j\}$, and $[\mathrm{B}]_k\psi \in L(n)$, for all $k \in G$.*

We show first that for any input $\Phi \subseteq \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ Algorithm 6.7 stops. Notice that Lemma 6.6 stating the bounds on the length of the sequence of R^{I}-successors in the pre-tableau holds for Algorithm 6.7 as well, as it uses the same procedure of R^{I}-successors creation as Algorithm 6.1. The procedure of R^{B}-successors creation is changed in Algorithm 6.7 and the following lemma, stating the bounds on the length of a sequence of R^{B}-successors, can be shown.

Lemma 6.26. *The maximal length of sequence of R^{B}-Successors with unchanged modal depth of labels in the pre-tableau constructed by Algorithm 6.7 for an input $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ is $\leq \mathcal{O}\big(\mathrm{dep}(\varphi)^{2|\mathcal{A}|}\big)$.*

Proof. Notice that Claims 6.12 – 6.16 shown in proof of Lemma 6.13 hold in the case of Algorithm 6.7 as well and they require modal context restriction $\mathbf{R}_1$ only.

Consider a sequence of states $s_0, \ldots, s_m$ in the pre-tableau such that for any $0 < k \leq m$, s_k is an $R^{\mathrm{B}}_{j_k}$-Successor of s_{k-1}. Suppose that for any $0 < k \leq m$ it holds that $\mathrm{Ind}(L(s_{k-1})) \sqcap j_k = \varnothing$ and the sequence satisfies properties **PB1**, **PI1** and **PB2** for all $d \geq 0$. If $\widehat{\mathrm{Gr}}(L(s_0)) \sqcap \neg[\mathrm{B}]^+ = \varnothing$ or there is more than one formula of the form $\neg[\mathrm{B}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(s_0))$, then the length of such sequence must be ≤ 2, by the same arguments as those used in proof of Lemma 6.13. If $\widehat{\mathrm{Gr}}(L(s_0)) \sqcap \neg[\mathrm{B}]^+ = \{\neg[\mathrm{B}]^+_G\psi\}$, then the length of the sequence must be ≤ 2. This is because, by point (iii) of Claim 6.15, for all $0 < k \leq m$ it must be that s_k is a $\neg[\mathrm{B}]_{j_k}[\mathrm{B}]^+_G\psi$-Successor of s_{k-1} with $j_k \in G$. Suppose that the length of the sequence is > 2. By Lemma 6.2 it must be that $j_1 \neq j_2$ and $j_2 \neq j_3$. Claim 6.16 implies that $(L(s_0)/[\mathrm{B}]_{j_1}) \cup \{\psi\} = (L(s_1)/[\mathrm{B}]_{j_2}) \cup \{\psi\}$, $(L(s_1)/[\mathrm{B}]_{j_2}) \cup \{\psi\} = (L(s_2)/[\mathrm{B}]_{j_3}) \cup \{\psi\}$ and $(L(s_2)/[\mathrm{B}]_{j_3}) \cup \{\psi\} = (L(s_3)/[\mathrm{B}]_{j_3}) \cup \{\psi\}$. Since, for all $1 \leq i \leq 4$, s_{i-1} is a state so $L(s_{i-1})$ is a $[\mathrm{B}]^+$-expanded tableau and so $[\mathrm{B}]_{j_i}\psi \in \neg L(s_{i-1})$. Notice that if $\neg[\mathrm{B}]_{j_2}\psi \in L(s_1)$, then $\neg[\mathrm{B}]_{j_2}[\mathrm{B}]^+_G\psi$-Successor of s_1 would not be created and s_2 would not be in the sequence. Hence it must be that $[\mathrm{B}]_{j_2}\psi \in L(s_1)$. But then $\psi \in L(s_2)$ and either $[\mathrm{B}]_{j_3}\psi \in L(s_2)$ or $\neg[\mathrm{B}]_{j_3}\psi \in L(s_2)$. In any of these cases $\neg[\mathrm{B}]_{j_2}[\mathrm{B}]^+_G\psi$-Successor of s_2 would not be created and so s_3 cannot be in the sequence, which contradicts our assumptions. Hence the length of the sequence must be ≤ 2. Using arguments similar to those used in Lemma 6.13 it can be shown that the maximal length of a sequence of R^{B}-Successors with the same modal depth of labels is $\mathcal{O}\big(\mathrm{dep}(\varphi)^{2|\mathcal{A}|}\big)$. □

The following lemma states bounds on the state height of a pre-tableau constructed by Algorithm 6.7 for an input formula satisfying modal context restriction $\mathbf{R}_1$.

Function 6.9: DecideSatisfiabilityAux

Input: A formula ψ, $G \subseteq \mathcal{A}$, $j \in G$ and a set of formulas Φ with $\{[\mathrm{B}]_j\psi, \psi\} \subseteq \Phi$
Output: A decision whether $\Phi^{\neg[\mathrm{B}]_j}([\mathrm{B}]^+_G\psi)$ is satisfiable or not

Let $H = \mathrm{ag}\Big(\big(\Phi \sqcap [\mathrm{B}]^+_{\{j\}}\big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$;
Construct a pre-tableau consisting of single node `root`, with $L(\texttt{root}) = \big(\Phi/[\mathrm{B}]^+_H\big) \cup \{\psi\}$ and all successor relations being empty;
Let Z be the set of all leaves of the pre-tableau with labelling sets that are not blatantly inconsistent;
repeat
    if *there is* $n \in Z$ *such that* n *is not a state and* $\xi \in L(n)$ *is a witness to that* **then**
        `FormState2`(n,ξ);
until *no change occurred*;
foreach $n \in Z$ *such* n *is a state* **do**
    if `DecideSatisfiability3`($\bigwedge L(n)$) = `sat` **then**
        foreach $k \in H$ **do**
            if `Reachable`($(\Phi/[\mathrm{B}]^+_H) \cup \{\psi\}$, $(\Phi \sqcap j) \setminus ((\Phi \sqcap [\mathrm{B}]_j) \cup (\Phi \sqcap \neg[\mathrm{B}]_j))$, $L(n)$, H, $\{k\}$, j, $|L(n)|$) **then**
                if `DecideSatisfiability3`($\bigwedge \big(L^{[\mathrm{B}]_k}(n) \cup \Phi/[\mathrm{B}]^+_H \cup \{\sim\psi\}\big)$) = `sat` **then**
                    return `sat`;
        if `Reachable`($(\Phi/[\mathrm{B}]^+_H) \cup \{\psi\}$, $(\Phi \sqcap j) \setminus ((\Phi \sqcap [\mathrm{B}]_j) \cup (\Phi \sqcap \neg[\mathrm{B}]_j))$, $L(n)$, H, $\varnothing$, j, $|L(n)|$) **then**
            foreach $k \in G \setminus H$ **do**
                if `DecideSatisfiability3`($\bigwedge \Big(L^{[\mathrm{B}]_k}(n) \cup \Phi^{[\mathrm{B}]^+_{H\cup\{k\}}} \cup \{\sim\psi\}\Big)$) = `sat` **then**
                    return `sat`;
            foreach $k \in G \setminus H$ **do**
                if `DecideSatisfiability3`($\bigwedge \big(L^{[\mathrm{B}]_k}(n) \cup \Phi^{[\mathrm{B}]^+_{H\cup\{k\}}} \cup \{\psi, \neg[\mathrm{B}]^+_G\psi, [\mathrm{B}]_k\psi, \neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi\}\big)$) = `sat` **then**
                    return `sat`;
return `unsat`;

Function 6.10: Reachable

Input: Three sets of formulas Φ_1, Ψ and Φ_2, $H \subseteq \mathcal{A}$, $F \subseteq H$, $p \in H$, and $K \geq 0$

Output: A decision whether there exists a satisfiable set of formulas $\Gamma \in \mathcal{SS}(\Phi_1)$ such that $\Psi \subseteq \Gamma$ and Φ_2 is reachable from Γ in $\mathcal{G}_H(\Phi_1)$ in at most $2^K - 1$ steps with a path $\Gamma_0, \ldots, \Gamma_n$ such that if $n = 0$, then $p \notin F$ and if $n \geq 1$, then there exists $j_n \in H \setminus F$ such that Γ_{n-1} and Γ_n are connected with j_n. The algorithm is always used with F being either $\varnothing$ or containing exactly one element.

```
if p ∉ F then
    Construct a pre-tableau consisting of single node root with L(root) := Φ1 ∪ Ψ and
    all successor relations being empty;
    Let Z denote the set of all leaves of the pre-tableau with labelling sets that are not
    blatantly inconsistent;
    repeat
        if there is n ∈ Z such that n is not a state and ξ ∈ L(n) is a witness to that then
            FormState2(n,ξ);
    until all nodes of Z are states;
    foreach n ∈ Z such that n is a state do
        if L(n) = Φ2 then
            return
        true;
if K = 0 then
    return false;
else
    Construct a pre-tableau consisting of single node root with L(root) := Φ1 and all
    successor relations being empty;
    Let Z be the set of all leaves of the pre-tableau with labelling sets that are not
    blatantly inconsistent;
    repeat
        if there is n ∈ Z such that n is not a state and ξ ∈ L(n) is a witness to that then
            FormState2(n,ξ);
    until all nodes of Z are states;
    foreach n ∈ Z do
        if DecideSatisfiability3(⋀ L(n)) = sat then
            if Reachable(Φ1,Ψ,L(n),H,∅,p,K − 1) then
                foreach j ∈ H do
                    if Reachable(Φ1,L(n) ⊓ j,Φ2,H,F,j,K − 1) then
                        return true;
    return false;
```

Lemma 6.27. *State height of the pre-tableau constructed by Algorithm 6.7 for an input* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ *is* $\leq \mathcal{O}\big(\mathrm{dep}(\varphi)^{2|\mathcal{A}|+1}\big)$ *and its height is* $\leq \mathcal{O}\big(|\varphi|\mathrm{dep}(\varphi)^{2|\mathcal{A}|+1}\big)$.

Proof. The proof is by analogous arguments to those used in proof of Lemma 6.18, using Lemma 6.26 instead of Lemma 6.13. □

Now we are ready to show that Algorithm 6.7 terminates, for any input satisfying modal context restriction $\mathbf{R}_1$. In the proof we will use the notion of $[\mathrm{B}]^+$-depth of a formula defined below.

Definition 23 ($[\mathrm{B}]^+$-depth)**.** *The* $[\mathrm{B}]^+$-depth of a formula φ, *denoted by* $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$, *is defined inductively as follows:*

- $\mathrm{dep}_{[\mathrm{B}]^+}(p) = 0$, *where* $p \in \mathcal{P}$,
- $\mathrm{dep}_{[\mathrm{B}]^+}(\neg\varphi) = \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$,
- $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi_1 \wedge \varphi_2) = \max\{\mathrm{dep}_{[\mathrm{B}]^+}(\varphi_1), \mathrm{dep}_{[\mathrm{B}]^+}(\varphi_2)\}$,
- $\mathrm{dep}_{[\mathrm{B}]^+}([O]_j\varphi) = \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$, *where* $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ *and* $j \in \mathcal{A}$,
- $\mathrm{dep}_{[\mathrm{B}]^+}([\mathrm{I}]^+_G\varphi) = \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$, *where* $G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}$,
- $\mathrm{dep}_{[\mathrm{B}]^+}([\mathrm{B}]^+_G\varphi) = \mathrm{dep}_{[\mathrm{B}]^+}(\varphi) + 1$, *where* $G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}$.

Let Φ *be a finite set of formulas, then*

$$\mathrm{dep}_{[\mathrm{B}]^+}(\Phi) = \begin{cases} 0 & \textit{if } \Phi = \varnothing \\ \max\{\mathrm{dep}_{[\mathrm{B}]^+}(\varphi) : \varphi \in \Phi\} & \textit{otherwise} \end{cases}$$

Lemma 6.28. *For any input* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ *Algorithm 6.7 terminates.*

Proof. We will show that the algorithm terminates for any input $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ using induction on the pair $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ and $\mathrm{ag}\Big(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\Big)$ (in lexicographic order). For the induction basis, suppose that $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi) = 0$ (in this case $\mathrm{ag}\Big(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\Big) = \mathcal{A} \cup \{\omega\}$). Then there are no formulas of the form $[\mathrm{B}]^+_G\psi \in \neg\mathrm{Sub}(\varphi)$ and Algorithm 6.7 works like Algorithm 6.1 on the input φ. Thus, by Lemma 6.19, Algorithm 6.7 terminates on the input φ. For the induction step, suppose that $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi) > 0$. By Lemma 6.27 either Algorithm 6.7 terminates or Function 6.9 is called. Since any call to Algorithm 6.7 in Function 6.9 is made with an input $\xi = \bigwedge \Xi$ with $\Xi \subseteq \neg\mathrm{Sub}(\varphi)$ such that either $\mathrm{dep}_{[\mathrm{B}]^+}(\xi) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ or $\mathrm{dep}_{[\mathrm{B}]^+}(\xi) = \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ and $\mathrm{ag}\Big(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\Big) \subsetneq \mathrm{ag}\Big(\mathrm{Sub}(\xi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\Big) \subseteq \mathcal{A}$ so, by the induction hypothesis, each such call terminates. Thus each call to Function 6.10 in Function 6.9 terminates and since the number of calls to Algorithm 6.7 and Function 6.10 made in Function 6.9 is finite, so Function 6.9 terminates. Hence Algorithm 6.7 must terminate on the input φ, as Function 6.9 is called there finitely many times (this is because it is called for the states of the pre-tableau constructed by Algorithm 6.7 and the number of these states is finite, as the number of successors of any node is finite and, by Lemma 6.27, the depth of the pre-tableau is also finite). □

What remains to be shown is the validity of the algorithm. We first show two auxiliary lemmas that will be used in proof of validity. These lemmas show, essentially, that when the Function 6.9 is invoked for some state s and its $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-successor, then the labels of this successor and subsequent $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-successors, with $k \in \mathrm{ag}\Big(\big(\Phi \sqcap [\mathrm{B}]^+_{\{j\}}\big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$, that could follow, would differ on elements from $\neg((L(s) \sqcap j) \setminus ((L(s) \sqcap [\mathrm{B}]_j) \cup (L(s) \sqcap \neg[\mathrm{B}]_j)))$ only. More precisely, each of them would include a maximal subset of $\neg((L(s) \sqcap j) \setminus ((L(s) \sqcap [\mathrm{B}]_j) \cup (L(s) \sqcap \neg[\mathrm{B}]_j)))$. This justifies the use of Function 6.9 in such cases.

Lemma 6.29. *Let s be a state in the pre-tableau constructed by Algorithm 6.7 for an input $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$. Let there be a formula of the form $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(s)$ with $j \in G$ and such that a $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-successor of s was not created because of a $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-Predecessor t of s with $k \neq j$ and such that $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$. Then for any $l \in \mathrm{ag}\Big(\big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$ it holds that $L(s) \sqcap [\mathrm{B}]_l \subseteq \widetilde{\mathrm{Cl}}\Big(\big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$.*

Proof. Suppose that s is like stated in the lemma. We will start by showing that $L(s) \sqcap [\mathrm{B}]_j \subseteq \widetilde{\mathrm{Cl}}\Big(\big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$. To show that we will use induction on the modal depth of a formula, starting from maximal values. Take any formula of the form $[\mathrm{B}]_j\xi \in L(s)$ and suppose its modal depth is maximal in $L(s) \sqcap [\mathrm{B}]_j$. Since $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$ and $j \neq k$ so either $[\mathrm{B}]_j\xi \in \widetilde{\mathrm{Cl}}\Big(\big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$ or there is a formula $[\mathrm{B}]_k\zeta \in L(t)$ such that $[\mathrm{B}]_j\xi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\zeta)$. If the first case holds that the claim is satisfied. Suppose that the second case holds. Since $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$ so either $[\mathrm{B}]_j\zeta \in L(s)$ or $\zeta = \psi$. The first case is impossible, as it would contradict the assumption of maximality of $\mathrm{dep}([\mathrm{B}]_j\xi)$. On the other hand, if $\zeta = \psi$, then $[\mathrm{B}]_k\zeta \in \widetilde{\mathrm{Cl}}(\{\neg[\mathrm{B}]^+_G\psi\})$ and since $[\mathrm{B}]_j\xi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\zeta)$ so this violates modal context restriction $\mathbf{R}_1$. Hence this case is impossible as well.

For the induction step, suppose that modal depth of $[\mathrm{B}]_j\xi$ is not maximal. Like in the case of the induction basis, $[\mathrm{B}]_j\xi \in L(s)$ implies that either $[\mathrm{B}]_j\xi \in \widetilde{\mathrm{Cl}}\Big(\big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$ or there is a formula $[\mathrm{B}]_k\zeta \in L(t)$ such that $[\mathrm{B}]_j\xi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\zeta)$. If the first case holds that the claim is satisfied. Suppose that the second case holds. Since $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$ so either $[\mathrm{B}]_j\zeta \in L(s)$ or $\zeta = \psi$. Suppose that the first case holds. Then, by the induction hypothesis, $[\mathrm{B}]_j\zeta \in \mathrm{Cl}\Big(\big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$. But then $[\mathrm{B}]_j\xi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\zeta)$ means that we have a violation of modal context restriction $\mathbf{R}_1$. Hence this case is not possible. On the other hand, if $\zeta = \psi$, then, as we argued for the induction basis, $[\mathrm{B}]_k\zeta \in \widetilde{\mathrm{Cl}}(\{\neg[\mathrm{B}]^+_G\psi\})$ and we get a violation of modal context restriction $\mathbf{R}_1$ again. Hence this case is impossible as well.

Secondly, we will show that $L(t) \sqcap [\mathrm{B}]^+_{\{k\}} \subseteq L(s) \sqcap [\mathrm{B}]^+_{\{j\}}$. To see this, take any formula of the form $[\mathrm{B}]^+_H\xi \in L(t)$ with $k \in H$. Since t is a state, so $L(t)$ is a closed tableau and so $[\mathrm{B}]_k[\mathrm{B}]^+_H\xi \in L(t)$. Thus $[\mathrm{B}]^+_H\xi \in L(t)/[\mathrm{B}]_k$ and, consequently, $[\mathrm{B}]^+_H\xi \in L(s)$ (as $L(t)/[\mathrm{B}]_k \subseteq L(s)$). Hence we need to show that $j \in H$. Since $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$, so either $[\mathrm{B}]^+_H\xi = \psi$ or there is a formula $[\mathrm{B}]_j[\mathrm{B}]^+_H\xi \in L(s)$. The first case is not possible, as $k \in G$ and it would lead to violation of modal context restriction $\mathbf{R}_1$. Suppose that the second case holds. Then, by what we have shown above, it must be that $[\mathrm{B}]_j[\mathrm{B}]^+_H\xi \in \widetilde{\mathrm{Cl}}\Big(\big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$. Suppose that $j \notin H$. Then it would have to be that either $[\mathrm{B}]^+_H\xi = \psi$ or there is a formula $[\mathrm{B}]^+_{H'}[\mathrm{B}]^+_H\xi \in L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}$. As we have already shown, the first case would lead to violation of modal context restriction $\mathbf{R}_1$. The second case would lead to violation of modal context restriction $\mathbf{R}_1$ as well, as $k \in H'$ and $k \in H$. Thus it must be that $j \in H$ and so $[\mathrm{B}]^+_H\xi \in L(s) \sqcap [\mathrm{B}]^+_{\{j\}}$.

Now take any formula of the form $[\mathrm{B}]_l\xi \in L(s)$ with $l \in \mathrm{ag}\Big(\Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$. We will show that it must be that $[\mathrm{B}]_l\xi \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{l,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$.

Since $[\mathrm{B}]_l\xi \in L(s)$, so either

1. there exists a formula $[\mathrm{B}]_k\zeta \in L(t)$ with $[\mathrm{B}]_l\xi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\zeta)$, or
2. $[\mathrm{B}]_l\xi \in L(t) \sqcap k$, or
3. $[\mathrm{B}]_l\xi \in \widetilde{\mathrm{Cl}}(L(s))$.

Case 1
Suppose that the first case holds. Since $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$, so either $[\mathrm{B}]_j\zeta \in L(s)$ or $\zeta = \psi$. If $[\mathrm{B}]_j\zeta \in L(s)$, then, by what we have shown above, $[\mathrm{B}]_j\zeta \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$. Thus either there is a formula $[\mathrm{B}]^+_H\zeta \in L(t)$ with $\{j,k\} \subseteq H$, or there is a formula $[\mathrm{B}]^+_H\eta \in L(t)$ with $\zeta = [\mathrm{B}]^+_H\eta$ and $\{j,k\} \subseteq H$. Since $[\mathrm{B}]_l\xi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\zeta)$, so it must be that the first of these cases holds, that is $[\mathrm{B}]^+_H\zeta \in L(t)$ with $\{j,k\} \subseteq H$. Now, since $(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}) \cup \{\neg[\mathrm{B}]^+_G\psi\} \subseteq L(s)$, so $(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}) \cup \{\neg[\mathrm{B}]^+_G\psi\} \subseteq \Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}$. Moreover, since $l \in \mathrm{ag}\Big(\Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$, so it must be that $l \in \mathrm{ag}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$. Thus $l \in H$ and we get a violation of modal context restriction $\mathbf{R}_1$. If $\zeta = \psi$, then we get a violation of modal context restriction $\mathbf{R}_1$ again. This is because, by the assumption that $l \in \mathrm{ag}\Big(\Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$, it holds that $l \in G$.
Case 2.
Suppose that the second case holds, that is $[\mathrm{B}]_l\xi \in L(t) \sqcap k$. Then it must be that $l = k$. Since $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$, so either $[\mathrm{B}]_j\xi \in L(s)$ or $\xi = \psi$. If $[\mathrm{B}]_j\xi \in L(s)$, then, by what we have shown above, $[\mathrm{B}]_j\xi \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$ and, since $l = k$ and $L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}} \subseteq L(t) \sqcap [\mathrm{B}]^+_{\{k\}}$, so $[\mathrm{B}]_l\xi \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{l,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$. If $\xi = \psi$, then we also have $[\mathrm{B}]_l\xi \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{l,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$.
Case 3.
Suppose that the third case holds, that is $[\mathrm{B}]_l\xi \in \widetilde{\mathrm{Cl}}(L(s))$. Then either there is a formula $[\mathrm{B}]^+_H\xi \in L(s)$ with $l \in H$ or ξ is of the form $[\mathrm{B}]^+_H\zeta$, with $l \in H$, and $[\mathrm{B}]^+_H\zeta \in L(s)$.

Suppose that the first of these cases, that is $[\mathrm{B}]^+_H\xi \in L(s)$ with $l \in H$. Then there must be a formula $[\mathrm{B}]_k\zeta \in L(t)$ such that $[\mathrm{B}]^+_H\xi \in \neg\mathrm{PT}(\zeta)$. Since $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$, so either $[\mathrm{B}]_j\zeta \in L(s)$ or $\zeta = \psi$. If $\zeta = \psi$, then we get a violation of modal context restriction $\mathbf{R}_1$ again. This is because, by the assumption that $l \in \mathrm{ag}\Big(\Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$, it holds that $l \in G$. If $[\mathrm{B}]_j\zeta \in L(s)$, then, by what we have shown above, $[\mathrm{B}]_j\zeta \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$. Thus either there is a formula $[\mathrm{B}]^+_F\zeta \in L(t)$ with $\{j,k\} \subseteq F$, or there is a formula $[\mathrm{B}]^+_F\eta \in L(t)$ with $\zeta = [\mathrm{B}]^+_F\eta$ and $\{j,k\} \subseteq F$. Consider the first of these cases, that is $[\mathrm{B}]^+_F\zeta \in L(t)$ with $\{j,k\} \subseteq F$. Since $(L(t) \sqcap [\mathrm{B}]_{\{j,k\}}) \cup \{\neg[\mathrm{B}]^+_G\psi\} \subseteq L(s)$, so $(L(t) \sqcap [\mathrm{B}]_{\{j,k\}}) \cup \{\neg[\mathrm{B}]^+_G\psi\} \subseteq \Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}$. Moreover, since $l \in \mathrm{ag}\Big(\Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$, so it must be that $l \in \mathrm{ag}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$. Thus $l \in F$ and we get a violation of modal context restriction $\mathbf{R}_1$. Consider the second of these cases. Since $[\mathrm{B}]^+_H\xi \in \neg\mathrm{PT}(\zeta)$, so the only possibility in this case is $F = H$ and $\xi = \eta$, which means that $[\mathrm{B}]^+_H \xi \in L(t)$ and $\{l, j, k\} \subseteq H$. Thus $[\mathrm{B}]_l \xi \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{l,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G \psi\}\Big)$ in this case.

Suppose that the second of these cases holds, that is ξ is of the form $[\mathrm{B}]^+_H \zeta$, with $l \in H$, and $[\mathrm{B}]^+_H \zeta \in L(s)$. Then there must be a formula $[\mathrm{B}]_k \eta \in L(t)$ such that $[\mathrm{B}]^+_H \zeta \in \neg\mathrm{PT}(\eta)$. Since $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$, so either $[\mathrm{B}]_j \eta \in L(s)$ or $\eta = \psi$. If $\eta = \psi$, then we get a violation of modal context restriction $\mathbf{R}_1$ again. This is because, by the assumption that $l \in \mathrm{ag}\Big(\Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G \psi\}, [\mathrm{B}]^+\Big)$, it holds that $l \in G$. If $[\mathrm{B}]_j \eta \in L(s)$, then, by what we have shown above, $[\mathrm{B}]_j \eta \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G \psi\}\Big)$. Thus either there is a formula $[\mathrm{B}]^+_F \eta \in L(t)$ with $\{j, k\} \subseteq F$, or there is a formula $[\mathrm{B}]^+_F \chi \in L(t)$ with $\eta = [\mathrm{B}]^+_F \chi$ and $\{j, k\} \subseteq F$. By arguments analogous to those used above, the first of these cases leads to violation of modal context restriction $\mathbf{R}_1$. Consider the second of these cases. Since $[\mathrm{B}]^+_H \zeta \in \neg\mathrm{PT}(\eta)$, so the only possibility in this case is $F = H$ and $\zeta = \chi$, which means that $[\mathrm{B}]^+_H \zeta \in L(t)$ and $\{l, j, k\} \subseteq H$. Thus $[\mathrm{B}]_l \xi \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{l,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G \psi\}\Big)$ in this case.

We have shown that if $[\mathrm{B}]_l \xi \in L(s)$ with $l \in \mathrm{ag}\Big(\Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G \psi\}, [\mathrm{B}]^+\Big)$, then $[\mathrm{B}]_l \xi \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{l,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G \psi\}\Big)$. Since $L(t) \sqcap [\mathrm{B}]^+_{\{l,k\}} \subseteq L(t) \sqcap [\mathrm{B}]^+_{\{k\}}$ and, as we have shown above, $L(t) \sqcap [\mathrm{B}]^+_{\{k\}} \subseteq L(s) \sqcap [\mathrm{B}]^+_{\{j\}}$, so $[\mathrm{B}]_l \xi \in \widetilde{\mathrm{Cl}}\Big(\Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G \psi\}\Big)$. □

Lemma 6.30. *Let s be a state in the pre-tableau constructed by Algorithm 6.7 for an input $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$. Let there be a formula of the form $\neg[\mathrm{B}]_j [\mathrm{B}]^+_G \psi \in L(s)$ with $j \in G$ and such that a $\neg[\mathrm{B}]_j [\mathrm{B}]^+_G \psi$-successor of s was not created because of a $\neg[\mathrm{B}]_k [\mathrm{B}]^+_G \zeta$-Predecessor t of s with $k \neq j$ and such that $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$. Suppose that $\psi \in L(s)$ and let $H = \mathrm{ag}\Big(\Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G \psi\}, [\mathrm{B}]^+\Big)$. Then for any $\Gamma \in \mathcal{SS}\big((L(s)/[\mathrm{B}]^+_H) \cup \{\psi\}\big)$ it holds that $\neg((L(s) \sqcap j) \setminus ((L(s) \sqcap [\mathrm{B}]_j) \cup (L(s) \sqcap \neg[\mathrm{B}]_j))) = \neg(\Gamma \sqcap j)$.*

Proof. Suppose that s is like stated in the lemma. Take any $\Gamma \in \mathcal{SS}\big((L(s)/[\mathrm{B}]^+_H) \cup \{\psi\}\big)$. For the left to right inclusion, take any $\xi \in \neg((L(s) \sqcap j) \setminus ((L(s) \sqcap [\mathrm{B}]_j) \cup (L(s) \sqcap \neg[\mathrm{B}]_j)))$. Then $\xi \in \neg L(s)$ and either $\xi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\psi)$ or there exists a formula $[\mathrm{B}]_k \zeta \in L(t)$ such that $\xi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\zeta)$. If the first case holds, then $\xi \in \neg\Gamma$. Suppose that the first case does not hold and that the second case holds. Since $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$ so there exists a formula $[\mathrm{B}]_j \zeta \in L(s)$ and, by Lemma 6.29, $\zeta \in L(s)/[\mathrm{B}]^+_H$. Thus $\xi \in \neg\Gamma$.

For the right to left inclusion, take any $\xi \in \neg(\Gamma \sqcap j)$. Then either $\xi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\psi)$ or there exists $\zeta \in L(s)/[\mathrm{B}]^+_H$ such that $\xi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\zeta)$. If the first case holds, then $\xi \in \neg(L(s) \sqcap j)$, as $\psi \in L(s)$, and, by modal context restriction $\mathbf{R}_1$, ξ cannot be of the form $[\mathrm{B}]_j \eta$ nor of the form $\neg[\mathrm{B}]_j \eta$. Hence $\xi \in \neg((L(s) \sqcap j) \setminus ((L(s) \sqcap [\mathrm{B}]_j) \cup (L(s) \sqcap \neg[\mathrm{B}]_j)))$. Suppose that the first case does not hold and that the second case holds. Then there must exist a formula $[\mathrm{B}]^+_H \zeta \in L(s)$ and since $j \in H$ and $L(s)$ is a closed tableau, so it must be that $[\mathrm{B}]_j \zeta \in L(s)$. Then, by the fact that $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$ and $L(t)/[\mathrm{B}]_k \subseteq L(s)$, it holds that $\zeta \in L(s)$ and, consequently, $\xi \in \neg L(s)$. Hence $\xi \in L(s) \sqcap j$ and since, by modal context restriction $\mathbf{R}_1$, ξ cannot be of the form $[\mathrm{B}]_j \eta$ nor of the form $\neg[\mathrm{B}]_j \eta$ (as $j \in H$), so $\xi \in \neg((L(s) \sqcap j) \setminus ((L(s) \sqcap [\mathrm{B}]_j) \cup (L(s) \sqcap \neg[\mathrm{B}]_j)))$. □

Lemma 6.31. *A formula $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ is satisfiable iff Algorithm 6.7 returns `sat` on the input φ.*

Proof. For the left to right implication we will use induction on the pair $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ and $\mathrm{ag}\Big(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\Big)$ (in lexicographic order). For the induction basis, suppose that

$\mathrm{dep}_{[\mathrm{B}]^+}(\varphi) = 0$ (in this case $\mathrm{ag}\Big(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\Big) = \mathcal{A} \cup \{\omega\}$). Then there are no formulas of the form $[\mathrm{B}]^+_G\psi \in \neg\mathrm{Sub}(\varphi)$ and Algorithm 6.7 works like Algorithm 6.1 on the input φ. Thus, by Lemma 6.23, if φ is satisfiable, then Algorithm 6.7 returns `sat` on the input φ. For the induction step, suppose that $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi) > 0$. In this case, like in proof of Lemma 6.23, we will show, for any node n of the pre-tableau constructed by the algorithm for input φ, that if n is marked `unsat`, then $\bigwedge L(n)$ is unsatisfiable. Like in the case of proof of Lemma 6.23 we will use induction on the maximal length of paths from a node to one of its descendant leaves. Arguments for most of the cases are like in the aforementioned proof, apart from the case of the nodes with a formula of the form $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$ in their label, for which a $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-successor was not created and Function 6.9 was used to check whether the label of such a successor is satisfiable.

So suppose that n is a state of the pre-tableau with a formula of the form $\neg[\mathrm{B}]^+_G\psi \in L(n)$ and suppose that a $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-successor of n, with $j \in G$, was not created and that n was marked `unsat` because Function 6.9 returned `unsat` on the input $L^{\neg[\mathrm{B}]_j}(n, [\mathrm{B}]^+_G\psi)$. Since Function 6.9 was used to check the satisfiability of $L^{\neg[\mathrm{B}]_j}(n, [\mathrm{B}]^+_G\psi)$, so it must be that $\{[\mathrm{B}]_j\psi, \psi\} \subseteq L(n)$. Suppose that $\bigwedge L(n)$ is satisfiable and let $(\mathcal{M}, u)$ be such that $(\mathcal{M}, u) \vDash \bigwedge L(n)$. Since $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(n)$ and $(\mathcal{M}, u) \vDash \bigwedge L(n)$, so there must exist $v \in R^{\mathrm{B}}_j$ such that $(\mathcal{M}, v) \vDash \neg[\mathrm{B}]^+_G\psi$. Moreover, by Lemma 4.3, it must be that $(\mathcal{M}, v) \vDash \bigwedge L^{\neg[\mathrm{B}]_j}(n, [\mathrm{B}]^+_G\psi)$. Since $(\mathcal{M}, v) \vDash \neg[\mathrm{B}]^+_G\psi$ so there must exist a satisfying sequence for $\neg[\mathrm{B}]^+_G\psi$ in $(\mathcal{M}, v)$. Let $v_0, \ldots, v_k$ be minimal such sequence. Let $H = \mathrm{ag}\Big(\Big(L(n) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$. Notice that since $j \in G$, so $H \neq \varnothing$, as $j \in H$. Let $0 \leq l < k$ be maximal such that for all $1 \leq i \leq l$ it holds that $j_i \in H$. Observe that for all $0 \leq i \leq l$, $(\mathcal{M}, v_i) \vDash \bigwedge(L(n)/[\mathrm{B}]^+_H)$ and $(\mathcal{M}, v_i) \vDash \psi$ (in the case of v_0 this follows from the fact that $[\mathrm{B}]_j\psi \in L(n)$). Thus, for all $0 \leq i \leq l$, $(\mathcal{M}, v_i) \vDash \bigwedge\big((L(n)/[\mathrm{B}]^+_H) \cup \{\psi\}\big)$ and, by Lemma 6.20, for each $0 \leq i \leq l$ there exists a set of formulas $\Gamma_i \in \mathcal{SS}\big((L(n)/[\mathrm{B}]^+_H) \cup \{\psi\}\big)$ such that $(\mathcal{M}, v_i) \vDash \bigwedge \Gamma_i$. Since $\mathrm{dep}_{[\mathrm{B}]^+}\big((L(n)/[\mathrm{B}]^+_H) \cup \{\psi\}\big) < \mathrm{dep}_{[\mathrm{B}]^+}(L(n))$ so, for all $0 \leq i \leq l$, $\mathrm{dep}_{[\mathrm{B}]^+}(\Gamma_i) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$. Hence, by the induction hypothesis, Algorithm 6.7 cannot return `unsat` on the input $\bigwedge \Gamma_i$. Moreover, by transitivity, generalized transitivity, Euclideanity and generalized Euclideanity of accessibility relations $R^{\mathrm{B}}_{j_i}$, for all $0 < i \leq l$ it holds that $\Gamma_{i-1} \sqcap j_i = \Gamma_i \sqcap j_i$. Hence Γ_l is reachable from Γ_0 in $\mathcal{G}_H\big((L(n)/[\mathrm{B}]^+_H) \cup \{\psi\}\big)$. Moreover, by Lemma 6.30 it holds that $\neg((L(n) \sqcap j) \setminus ((L(n) \sqcap [\mathrm{B}]_j) \cup (L(n) \sqcap \neg[\mathrm{B}]_j))) = \neg(\Gamma_0 \sqcap j)$ and, by transitivity, generalized transitivity, Euclideanity and generalized Euclideanity of accessibility relation R^{B}_j it must be that $(L(n) \sqcap j) \setminus ((L(n) \sqcap [\mathrm{B}]_j) \cup (L(n) \sqcap \neg[\mathrm{B}]_j)) = \Gamma_0 \sqcap j$. Hence $(L(n) \sqcap j) \setminus ((L(n) \sqcap [\mathrm{B}]_j) \cup (L(n) \sqcap \neg[\mathrm{B}]_j)) \subseteq \Gamma_0$.

Now, if $l + 1 = k$, then it must be that $(\mathcal{M}, v_{l+1}) \vDash \neg\psi$ and so $(\mathcal{M}, v_{l+1}) \vDash {\sim}\psi$. Moreover, $(\mathcal{M}, v_{l+1}) \vDash \bigwedge\Big(L(n)/[\mathrm{B}]^+_{H \cup \{j_{l+1}\}}\Big)$ and, by Lemma 4.3, $(\mathcal{M}, v_{l+1}) \vDash \bigwedge \Gamma_l^{[\mathrm{B}]_{j_k}}$. If $j_{l+1} \in H$, then $(\mathcal{M}, v_{l+1}) \vDash \xi$, where $\xi = \bigwedge\Big(\Gamma_l^{[\mathrm{B}]_{j_k}} \cup \big(L(n)/[\mathrm{B}]^+_H\big) \cup \{{\sim}\psi\}\Big)$ and since $\mathrm{dep}_{[\mathrm{B}]^+}(\xi) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ so, by the induction hypothesis, Algorithm 6.7 cannot return `unsat` on the input ξ. Notice that if $k = 1$, then it must be that $j_k \neq j$, as $(\mathcal{M}, v_0) \vDash [\mathrm{B}]_j\psi$. Notice also that if $k \geq 2$, then, by minimality of $v_0, \ldots, v_k$, it must be that $j_k \neq j_{k-1}$. Thus Function 6.9 must return `sat` on the input $L^{\neg[\mathrm{B}]_j}(n, [\mathrm{B}]^+_G\psi)$, which contradicts our assumptions. If $j_{l+1} \in G \setminus H$, then, by simple induction on i, for all $0 \leq i \leq l + 1$ it holds that $(\mathcal{M}, v_i) \vDash \bigwedge\Big(L(n) \sqcap [\mathrm{B}]^+_{H \cup \{j_{l+1}\}}\Big)$. Hence $(\mathcal{M}, v_{l+1}) \vDash \xi$, where $\xi = \bigwedge\left(\Gamma_l^{[\mathrm{B}]_{j_{l+1}}} \cup L^{[\mathrm{B}]^+_{H \cup \{j_{l+1}\}}}(n) \cup \{{\sim}\psi\}\right)$. Since either $\mathrm{dep}_{[\mathrm{B}]^+}(\xi) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ or $\mathrm{dep}_{[\mathrm{B}]^+}(\xi) = \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ and $\mathrm{ag}(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)) \subseteq H \subsetneq H \cup \{j_{l+1}\} = \mathrm{ag}(\mathrm{Sub}(\xi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\xi))$ so, by the induction hypothesis, Algorithm 6.7 cannot return `unsat` on the input ξ. Thus Function 6.9 must return `sat` on the input $L^{\neg[\mathrm{B}]_j}(n, [\mathrm{B}]^+_G\psi)$, which contradicts our assumptions.

Otherwise, if $l+1<k$, then it must be that $j_{l+1} \in G \setminus H$. By minimality of $v_0, \ldots, v_k$, it must be that $(\mathcal{M}, v_l) \vDash [\mathrm{B}]_{j_{l+1}}\psi$, $(\mathcal{M}, v_{l+1}) \vDash \psi$ and $(\mathcal{M}, v_{l+1}) \vDash \neg[\mathrm{B}]^+_G\psi$. Hence, it must be that $(\mathcal{M}, v_l) \vDash \neg[\mathrm{B}]_{j_{l+1}}[\mathrm{B}]^+_G\psi$ and, by transitivity of $R^{\mathrm{B}}_{j_{l+1}}$, $(\mathcal{M}, v_{l+1}) \vDash [\mathrm{B}]_{j_{l+1}}\psi$ and $(\mathcal{M}, v_{l+1}) \vDash \neg[\mathrm{B}]_{j_{l+1}}[\mathrm{B}]^+_G\psi$. As we observed above, $(\mathcal{M}, v_i) \vDash \bigwedge\Big(L(n)/[\mathrm{B}]^+_{H\cup\{j_{l+1}\}}\Big)$ and $(\mathcal{M}, v_i) \vDash \bigwedge\Big(L(n) \sqcap [\mathrm{B}]^+_{H\cup\{j_{l+1}\}}\Big)$. Moreover, by Lemma 4.3, $(\mathcal{M}, v_{l+1}) \vDash \bigwedge \mathit{\Gamma}_l^{[\mathrm{B}]_{j_{l+1}}}$. Thus $(\mathcal{M}, v_{l+1}) \vDash \xi$, where $\xi = \bigwedge\Big(\mathit{\Gamma}_l^{[\mathrm{B}]_{j_{l+1}}} \cup L^{[\mathrm{B}]^+_{H\cup\{j_{l+1}\}}}(n) \cup \{\psi, \neg[\mathrm{B}]^+_G\psi, [\mathrm{B}]_{j_{l+1}}\psi, \neg[\mathrm{B}]_{j_{l+1}}[\mathrm{B}]^+_G\psi\}\Big)$. Since either $\mathrm{dep}_{[\mathrm{B}]^+}(\xi) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ or $\mathrm{dep}_{[\mathrm{B}]^+}(\xi) = \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ and $\mathrm{ag}\Big(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\Big) \subseteq H \subsetneq H \cup \{j_{l+1}\} = \mathrm{ag}\Big(\mathrm{Sub}(\xi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\xi)\Big)$ so, by the induction hypothesis, Algorithm 6.7 cannot return `unsat` on the input ξ. Thus Function 6.9 must return `sat` on the input $L^{\neg[\mathrm{B}]_j}(n, [\mathrm{B}]^+_G\psi)$, which contradicts our assumptions.

As was pointed out in proof of Lemma 6.23, `root` of any pre-tableau constructed by the algorithm must be marked either `unsat` or `sat` and if `root` is not marked `sat`, then it must be marked `unsat` and φ must be unsatisfiable. Hence if φ is satisfiable, then `root` node must be marked `sat` and the algorithm must return `sat`.

For the right to left implication we will show that if Algorithm 6.7 returns `sat` on the input φ, then a TeamLog tableau for φ can be constructed. More precisely, we will show that if Algorithm 6.7 returns `sat` on the input φ, then a TeamLog tableau for φ can be constructed, which has a state w such that $\varphi \in L(w)$ and for no other state u of this tableau there exists $j \in \mathcal{A}$ such that $w \in R^{\mathrm{B}}_j(u)$. To show that, we will again use induction on the pair $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ and $\mathrm{ag}\Big(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\Big)$ (in lexicographic order).

For the induction basis, suppose that $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi) = 0$. Then there are no formulas of the form $[\mathrm{B}]^+_G\psi \in \neg\mathrm{Sub}(\varphi)$, Algorithm 6.7 works like Algorithm 6.1 on the input φ and TeamLog tableau for φ can be constructed like in proof of Lemma 6.23. Recall that that construction guarantees that there exists a state w in that tableau such that $\varphi \in L(w)$ and there is no other state u in that tableau and no $j \in \mathcal{A}$ such that $w \in R^{\mathrm{B}}_j(u)$.

For the induction step, suppose that $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi) > 0$ and let

$$(N, \mathrm{root}, \mathit{succ}, \{R^O_j\text{-succ} : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, L)$$

be the pre-tableau constructed by Algorithm 6.7 for φ. Let

$$\mathcal{T} = (W, \{R^O_j : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, L|_W)$$

be a model graph constructed on the basis of this pre-tableau like in proof of Lemma 6.23. Let $V \subseteq W$ be the set of states such that for each $v \in V$ there is a formula of the form $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(v)$ with $j \in G$ and such that a $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-successor of v was not created because of a $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\zeta$-Predecessor u of v, with $k \in G$, such that $k \neq j$ and $(L(v)/[\mathrm{B}]_j) \cup \{\psi\} = (L(u)/[\mathrm{B}]_k) \cup \{\psi\}$.

As we remarked above, the construction from Lemma 6.23 guarantees that there exists $w \in W$ such that $\varphi \in L(w)$ and for all $u \in W \setminus \{w\}$ and $j \in \mathcal{A}$, $w \notin R^{\mathrm{B}}_j(u)$.

Notice that conditions **T1**, **T4**, **T5**, **TBG4** , **TBI4**, **TBG5**, **TBI5** and **TIG** are satisfied for all the states of $\mathcal{T}$, as the same argumentation as the one used in proof of Lemma 6.23 will work here. Similarly it can be shown like in proof of Lemma 6.23 that conditions **T2** and **TD** are satisfied for all the states of $\mathcal{T}$ that are not in V. Also, in the case of states from V and formulas of the form $[\mathrm{G}]_k\xi$ it can be shown like in proof of Lemma 6.23 that the condition **T2** holds for them. Similarly with states from V, formulas of the form $[\mathrm{I}]_k\xi$ and conditions **T2** and **TD**. Condition **TC** also holds for all states of $\mathcal{T}$ and formulas of the form $[\mathrm{I}]^+_H\xi$. The problem are conditions **T2** and **TD** for states from V and formulas of the form $[\mathrm{B}]_k\xi$, as well as conditions **TC** for all states of $\mathcal{T}$ and formulas of the form $[\mathrm{B}]^+_H\xi$. To satisfy these conditions, the model graph $\mathcal{T}$ has to be extend at the states from V, so that a TeamLog tableau for φ is created. The extension is by adding R^{B}-successors of states from V for the formulas of the form $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$ for which the successors were not created for the reasons described above. We will describe the extension for a given state $v \in V$ and a formula of the form $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(v)$ for which a successor was not created for the reasons described above, showing that this extension sustains the conditions of TeamLog tableau listed above, while making the unsatisfied conditions **T2**, **TD** and **TC** satisfied.

Take any $v \in V$ and $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(v)$ with $j \in G$ for which a $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-successor was not created for the reasons described above. Now two cases are possible: either $[\mathrm{B}]_j\psi \in L(v)$ or not. Suppose first that $[\mathrm{B}]_j\psi \notin L(v)$. Then, by the fact that v, being a state, is a $[\mathrm{B}]^+$-expanded tableau, it holds that $\neg[\mathrm{B}]_j\psi \in L(v)$. Let $\mathcal{T}$ be extended as follows:

- $R'^{\mathrm{B}}_j(v) = R^{\mathrm{B}}_j(v) \cup \{v\}$.

Notice that since $[\mathrm{B}]_j\psi \notin L(v)$, so $\psi \notin L(v)/[\mathrm{B}]_j$ and so $L(v)/[\mathrm{B}]_j \subseteq L(u)/[\mathrm{B}]_k \subseteq L(v)$. Since it also holds that $L(v) \sqcap j \subseteq L(v)$ and $\neg[\mathrm{B}]^+_G\psi \in L(v)$, so $L^{\neg[\mathrm{B}]_j}(v, [\mathrm{B}]^+_G\psi) \subseteq L(v)$. Hence the conditions **T1**, **T4**, **T5**, **TBG4** , **TBI4**, **TBG5**, **TBI5** and **TIG** are still satisfied for v after the extension. Also, conditions **T2**, **TD** and **TC** are still satisfied for those states and formulas for which they were satisfied before the extension. Notice also that after this extension it still holds that there exists $w \in W$ such that $\varphi \in L(w)$ and for all $u \in W \setminus \{w\}$ and $j \in \mathcal{A}$, $w \notin R^{\mathrm{B}}_j(u)$.

Secondly, suppose that $[\mathrm{B}]_j\psi \in L(v)$. Then Function 6.9 must have been used to check the satisfiability of $\bigwedge L^{\neg[\mathrm{B}]_j}(v, [\mathrm{B}]^+_G\psi)$. Since v, being in W, must be marked `sat` so Function 6.9 must have returned `sat` on the input $\bigwedge L^{\neg[\mathrm{B}]_j}(v, [\mathrm{B}]^+_G\psi)$. Thus sets of formulas Ψ_0 and Ψ_1 were found such that $\{\Psi_0, \Psi_1\} \subseteq \mathcal{SS}\big((L(v)/[\mathrm{B}]^+_H) \cup \{\psi\}\big)$, Ψ_1 is reachable from Ψ_0 in $\mathcal{G}_H\big((L(v)/[\mathrm{B}]^+_H) \cup \{\psi\}\big)$ (where $H = \mathrm{ag}\Big(\Big(L(v) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$) with path $(\Gamma_0, \ldots, \Gamma_n)$ and associated sequence $j_1, \ldots, j_n$ such that Algorithm 6.7 returned `sat` on each input $\bigwedge \Gamma_i$, with $0 \le i \le n$, and

1. either there exists $k \in H$ such that Algorithm 6.7 returned `sat` on the input $\bigwedge \Xi$, where $\Xi = \Psi_1^{[\mathrm{B}]_k} \cup \big(L(v)/[\mathrm{B}]^+_H\big) \cup \{\sim\psi\}$, in which case the sequence $k \neq j$, if $n = 0$, and $k \neq j_n$, if $n \ge 1$,

2. or there exists $k \in G \setminus H$ such that Algorithm 6.7 returned `sat` on the input $\bigwedge \Xi$, where $\Xi = \Psi_1^{[\mathrm{B}]_k} \cup L^{[\mathrm{B}]^+_{H\cup\{k\}}}(v) \cup \{\sim\psi\}$,

3. or there exists $k \in G \setminus H$ such that Algorithm 6.7 returned `sat` on the input $\bigwedge \Xi$, where $\Xi = \Psi_1^{[\mathrm{B}]^+_k} \cup L^{[\mathrm{B}]^+_{H\cup\{k\}}}(v) \cup \{\psi, \neg[\mathrm{B}]^+_G\psi, [\mathrm{B}]_k\psi, \neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi\}$.

Since Algorithm 6.7 returned `sat` on the input $\bigwedge \Gamma_i$, for each $0 \le i \le n$, and $\mathrm{dep}_{[\mathrm{B}]^+}(\Gamma_i) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ so, by the induction hypothesis, a sequence $(\mathcal{T}_0, \ldots, \mathcal{T}_n)$ of TeamLog tableaux can be created for each of the subsequent elements of $(\Gamma_0, \ldots, \Gamma_n)$. Also, in each of the cases 1 – 3 above, by the induction hypothesis, a TeamLog tableau $\mathcal{T}_{n+1}$ can be created for $\bigwedge \Xi$. In the case 1 this is because $\mathrm{dep}_{[\mathrm{B}]^+}(\Xi) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$, in the cases 2 and 3 this is because either

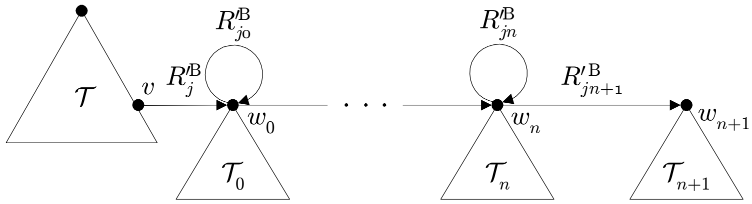

Figure 6.2: Extension of model graph $\mathcal{T}$ at $v \in V$ with tableaux $\mathcal{T}_0, \dots, \mathcal{T}_{n+1}$ constructed by Function 6.9.

$\mathrm{dep}_{[\mathrm{B}]^+}(\Xi) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ or $\mathrm{dep}_{[\mathrm{B}]^+}(\Xi) = \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ and $\mathrm{ag}(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)) \subseteq H \subsetneq H \cup \{k\} = \mathrm{ag}(\mathrm{Sub}(\Xi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\Xi))$.

Let $W_0, \dots, W_{n+1}$ be the sets of states in the subsequent tableaux $\mathcal{T}_0, \dots, \mathcal{T}_{n+1}$. Also let R^O_k, with $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $k \in \mathcal{A}$, be the accessibility relations in those tableaux and let L be the labelling function in those tableaux. Let $w_0, \dots, w_{n+1}$ be the sequence of states in those subsequent tableaux such that $\bigwedge \Gamma_i \in L(w_i)$, for each $0 \le i \le n$, $\bigwedge \Xi \in L(w_{n+1})$, and for each $0 \le i \le n+1$ it holds that for all $u \in W_i \setminus \{w_i\}$ and $j \in \mathcal{A}$, $w_i \notin R^{\mathrm{B}}_j(u)$. By the induction hypothesis such sequence of states exists. Let $\mathcal{T}$ be extended as follows (where $j_0 = j$, $j_{n+1} = k$ and $X_j(\neg[\mathrm{B}]^+_G\psi) = \{[\mathrm{B}]_j\psi, \neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi, \neg[\mathrm{B}]^+_G\psi\}$ for $j \in G$) (see Figure 6.2 for illustration of this extension):

- $R'^{\mathrm{B}}_j(v) = R^{\mathrm{B}}_j(v) \cup \{w_0\}$.
- $R'^{\mathrm{B}}_{j_i}(w_i) = R^{\mathrm{B}}_{j_i}(w_i) \cup \{w_i\}$, for $0 \le i \le n$.
- $R'^{\mathrm{B}}_{j_{i+1}}(w_i) = R^{\mathrm{B}}_{j_{i+1}}(w_i) \cup \{w_{i+1}\}$, for $0 \le i \le n$.
- $R'^{\mathrm{B}}_{j_{i+1}}(w_i) = R^{\mathrm{B}}_{j_{i+1}}(w_i) \cup \{w_i, w_{i+1}\}$, for $0 \le i \le n$.
- $L'(w_i) = L(w_i) \cup (L(v) \sqcap [\mathrm{B}]^+_H) \cup \widetilde{\mathrm{Cl}}(L(v) \sqcap [\mathrm{B}]^+_H) \cup X_{j_i}(\neg[\mathrm{B}]^+_G\psi) \cup X_{j_{i+1}}(\neg[\mathrm{B}]^+_G\psi)$, for $0 \le i \le n-1$.
- $L'(w_n) = L(w_n) \cup (L(v) \sqcap [\mathrm{B}]^+_H) \cup \widetilde{\mathrm{Cl}}(L(v) \sqcap [\mathrm{B}]^+_H) \cup X_{j_n}(\neg[\mathrm{B}]^+_G\psi)$, if the case 1 or 2 holds.
- $L'(w_n) = L(w_n) \cup (L(v) \sqcap [\mathrm{B}]^+_H) \cup \widetilde{\mathrm{Cl}}(L(v) \sqcap [\mathrm{B}]^+_H) \cup X_{j_n}(\neg[\mathrm{B}]^+_G\psi) \cup X_{j_{n+1}}(\neg[\mathrm{B}]^+_G\psi)$, if the case 3 holds.
- $L'(w_{n+1}) = L(w_{n+1}) \cup \Big(L(v) \sqcap [\mathrm{B}]^+_{H\cup\{k\}}\Big) \cup \widetilde{\mathrm{Cl}}\Big(L(v) \sqcap [\mathrm{B}]^+_{H\cup\{k\}}\Big)$.

Like in the case of the previous extension, conditions **T1**, **T4**, **T5**, **TBG4** , **TBI4**, **TBG5**, **TBI5** and **TIG** are still satisfied for v after the extension described above. Condition **TIG** is not affected by the extension, as it adds an R^{B}_j-successor of state v only. Condition **T1** could be affected in the case of formulas of the form $[\mathrm{B}]_j\xi \in L(v)$ only. Take any such formula. By Lemma 6.29 it holds that $L(v) \sqcap [\mathrm{B}]_j \subseteq \widetilde{\mathrm{Cl}}\Big(\Big(L(v) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$. Hence if $[\mathrm{B}]_j\xi \in L(v)$, then either $[\mathrm{B}]_j\xi = [\mathrm{B}]_j\psi$ or there exists a formula $[\mathrm{B}]^+_T\zeta \in L(v)$ such that $[\mathrm{B}]_j\xi \in \widetilde{\mathrm{Cl}}([\mathrm{B}]^+_T\zeta)$. Thus $\xi \in \big(L(v)/[\mathrm{B}]^+_H\big) \cup \{\psi\} \cup \{[\mathrm{B}]^+_T\zeta\}$ and since $L(w_0) \in \mathcal{SS}\big(\big(L(v)/[\mathrm{B}]^+_H\big) \cup \{\psi\}\big)$, so $\xi \in L'(w_0)$ and so condition **T1** is satisfied for v and $[\mathrm{B}]_j\xi$. Like in the case of Condition **T1**, Condition **T4** could be affected in the case of formulas of the form $[\mathrm{B}]_j\xi \in L(v)$ only. Take any such formula. As we argued above either $[\mathrm{B}]_j\xi = [\mathrm{B}]_j\psi$ or there exists a formula $[\mathrm{B}]^+_T\zeta \in L(v)$ such that

$[\mathrm{B}]_j\xi \in \widetilde{\mathrm{Cl}}([\mathrm{B}]^+_T\zeta)$. In either case $[\mathrm{B}]_j\xi \in L'(w_0)$ and condition **T4** is satisfied for v and $[\mathrm{B}]_j\xi$. Also Condition **T5** could be affected in the case of formulas of the form $[\mathrm{B}]_j\xi \in L'(w_0)$ only. Take any such formula. By modal context restriction $\mathbf{R}_1$, $[\mathrm{B}]_j\xi \notin \neg\mathrm{OT}_{[\mathrm{B}]}(\zeta)$ for any $\zeta \in \big(L(v)/[\mathrm{B}]^+_H\big) \cup \{\psi\}$. Hence $[\mathrm{B}]_j\xi \notin L(w_0)$ and it must be that $[\mathrm{B}]_j\xi \in L'(w_0) \setminus L(w_0)$. Since $L'(w_0) \setminus L(w_0) \subseteq L(v)$, so $[\mathrm{B}]_j\xi \in L(v)$ and condition **T5** is satisfied for v and $[\mathrm{B}]_j\xi$. Conditions **TBG4** and **TBI4** are satisfied because, by construction of the algorithm, it holds that $(L(v)\sqcap j)\setminus((L(v)\sqcap[\mathrm{B}]_j)\cup(L(v)\sqcap\neg[\mathrm{B}]_j)) \subseteq L(w_0)$. For conditions **TBG5** and **TBI5** notice that since, by construction of the algorithm, $(L(v)\sqcap j)\setminus((L(v)\sqcap[\mathrm{B}]_j)\cup(L(v)\sqcap\neg[\mathrm{B}]_j)) \subseteq L(w_0)$ so, by Lemma 6.30, $(L(v) \sqcap j) \setminus ((L(v) \sqcap [\mathrm{B}]_j) \cup (L(v) \sqcap \neg[\mathrm{B}]_j)) = L(w_0) \sqcap j$. Thus these conditions are satisfied for v as well. Notice also that after this extension it still holds that there exists $w \in W$ such that $\varphi \in L(w)$ and for all $u \in W \setminus \{w\}$ and $j \in \mathcal{A}$, $w \notin R^{\mathrm{B}}_j(u)$.

All the newly added states satisfy conditions of TeamLog tableau. To see this take any $\mathcal{T}_i$ with $0 \le i \le n+1$. Notice first that the extended label $L'(w_i)$ is not blatantly inconsistent. This is because $L(w_i) \in \mathcal{SS}\big(\big(L(v)/[\mathrm{B}]^+_H\big) \cup \{\psi\}\big)$ and by modal context restriction $\mathbf{R}_1$ it cannot contain any of the formulas extending it to $L'(w_i)$. Moreover, $L'(w_i)$ is a closed propositional tableau, as $L(w_i)$ and $L'(w_i) \setminus L(w_i)$ are closed propositional tableaux. The labels of all the other states of $\mathcal{T}_i$ remain unchanged. For the remaining conditions of TeamLog tableau, notice first that the only state of $\mathcal{T}_i$ that could be affected by the extension is w_i. This is because, by the induction hypothesis, there is no other state u of $\mathcal{T}_i$ such that $w_i \in R^O_l(u)$, for any $O \in \{\mathrm{B},\mathrm{G},\mathrm{I}\}$ and $l \in \mathcal{A}$. Hence we need to show that the remaining conditions of TeamLog tableau are satisfied for states w_i, for all $0 \le i \le n+1$. In showing the conditions, the following observation will be useful: for all $0 \le i \le n$ and any formula of the form $[\mathrm{B}]_l\xi \in \neg L'(w_i)$ with $l \in H$ it must be that $[\mathrm{B}]_l\xi \notin \neg L(w_i)$. For take any such formula and suppose that $[\mathrm{B}]_l\xi \in \neg L(w_i)$. Then there must be a formula $\zeta \in \big(L(v)/[\mathrm{B}]^+_H\big) \cup \{\psi\}$ such that $[\mathrm{B}]_l\xi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\zeta)$, which is impossible by modal context restriction $\mathbf{R}_1$.

We will consider two cases separately: $i = n+1$ and $0 \le i \le n$. Suppose first that $i = n+1$. Notice that $L'(w_{n+1}) \ne L(w_{n+1})$ only in the case 1, when $j_{n+1} \in H$. Thus if $j_{n+1} \notin H$, then the conditions of TeamLog tableau are satisfied for w_{n+1} in $\mathcal{T}_{n+1}$, as it is not affected by the extension. Suppose that $j_{n+1} \in H$. Then the only formulas that could be affected by the extension are formulas from $L'(w_{n+1}) \setminus L(w_{n+1}) = \widetilde{\mathrm{Cl}}\Big(L(v) \sqcap [\mathrm{B}]^+_{H\cup\{k\}}\Big) = \widetilde{\mathrm{Cl}}\big(L(v) \sqcap [\mathrm{B}]^+_H\big)$. Notice that for any formula of the form $[\mathrm{B}]_l\xi \in \neg L'(w_{n+1})$ with $l \in H$ it must be that $[\mathrm{B}]_l\xi \notin \neg L(w_{n+1})$. For take any such formula and suppose that $[\mathrm{B}]_l\xi \in \neg L(w_{n+1})$. Then there must be a formula $\zeta \in L^{[\mathrm{B}]j_{n+1}}(w_n) \cup \{\sim\psi\}$ such that $[\mathrm{B}]_l\xi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\zeta)$. This is impossible, as, by the observation above, $L^{[\mathrm{B}]j_{n+1}}(w_n) = \varnothing$ and $[\mathrm{B}]_l\xi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\sim\psi)$ would violate modal context restriction $\mathbf{R}_1$. Since there are no formulas of the form $[\mathrm{B}]_l\xi \in L(w_{n+1})$ so, by construction of tableau $\mathcal{T}_{n+1}$, $R^{\mathrm{B}}_l(w_{n+1}) = \varnothing$, for all $l \in H$, as no successor of a state can be created for a formula that is not in the label of the state. Thus condition **T1** is satisfied for all formulas from $L'(w_{n+1}) \setminus L(w_{n+1})$. Condition **TD** is satisfied for all the formulas from $L'(w_{n+1}) \setminus L(w_{n+1})$, as $L'(w_{n+1}) \setminus L(w_{n+1}) \subseteq L'(w_{n+1})$. Conditions **T4** and **T5** are satisfied for all the formulas from $L'(w_{n+1}) \setminus L(w_{n+1})$, as $R^{\mathrm{B}}_l(w_{n+1}) = \varnothing$, for all $l \in H$. The remaining conditions of TeamLog tableau are not applicable to any formula from $L'(w_{n+1}) \setminus L(w_{n+1})$, and so all the conditions of TeamLog tableau are satisfied for w_{n+1}.

Secondly, suppose that $0 \le i \le n$. Condition **T1** is satisfied for w_i and any formula from $L(w_i)$ in $\mathcal{T}_i$ and the only formulas for which it could be affected after the extension are formulas from $L'(w_i) \setminus L(w_i)$ and formulas the form $[\mathrm{B}]_{j_l}\xi \in L'(w_i)$ with $l \in \{i, i+1\}$. As we observed above, for any formula of the form $[\mathrm{B}]_l\xi \in \neg L'(w_i)$ with $l \in H$ it must be that $[\mathrm{B}]_l\xi \notin \neg L(w_i)$. Hence, by construction of tableau $\mathcal{T}_i$, $R^{\mathrm{B}}_l(w_i) = \varnothing$, for all $l \in H$, as no successor of a state can be created for a formula that is not in the label of the state. This, together with the fact that $(L'(w_i) \setminus L(w_i))/[\mathrm{B}]_l \subseteq L'(w_i)$, for all $l \in H$, implies that condition **T1** is satisfied for w_i, for all $0 \leq i \leq n-1$. In the case of $i = n$, condition **T1** is satisfied by the fact that $L^{[\mathrm{B}]_k}(w_n) \subseteq L(w_{n+1})$, $\widetilde{\mathrm{Cl}}(L(v) \sqcap [\mathrm{B}]^+_H)/[\mathrm{B}]_k = L^{[\mathrm{B}]^+_{H\cup\{k\}}}(v)$ and, in the case 3, $\psi \in L(w_{n+1})$. Condition **TD** is satisfied for w_i and any formula from $L(w_i)$ in $\mathcal{T}_i$ and the only formulas for which it could be affected after the extension are formulas from $L'(w_i) \setminus L(w_i)$. In the case of formulas of the form $[\mathrm{B}]_{j_{i+1}}\xi \in L'(w_i) \setminus L(w_i)$ the condition is satisfied by the fact that $w_{i+1} \in R^{\mathrm{B}}_{j_{i+1}}(w_i)$ and by condition **T1**. In the case of the remaining formulas of the form $[\mathrm{B}]_l\xi \in L'(w_i) \setminus L(w_i)$, it must be that $l \in H$ and since for any $l \in H$ it holds that $\widetilde{\mathrm{Cl}}(L(v) \sqcap [\mathrm{B}]^+_H)/[\mathrm{B}]_l \subseteq L^{[\mathrm{B}]^+_H}(v) \subseteq L(w_i)$ and $\psi \in L(w_i)$ so the condition is satisfied for all $0 \leq i \leq n-1$. Then only formulas in $L'(w_i)$ for which conditions **T4** and **T5** could be affected after the extension are formulas of the form $[\mathrm{B}]_{j_{i+1}}\xi$. If $0 \leq i \leq n-1$, then such formulas must be elements of $L'(w_i) \setminus L(w_i)$ and $L'(w_{i+1}) \setminus L(w_{i+1})$, as $j_i \in H$, and since $(L'(w_i) \setminus L(w_i)) \sqcap [\mathrm{B}]_{j_{i+1}} = (L'(w_{i+1}) \setminus L(w_{i+1})) \sqcap [\mathrm{B}]_{j_{i+1}}$ in this case, so the condition is satisfied. Suppose that $i = n$ and $j_{n+1} \in H$. Then $L'(w_n)$ extends $L(w_n)$ according to the case 1 and $L(w_{n+1})$ is like in this case as well. Hence $L'(w_n) \setminus L(w_n) = L'(w_{n+1}) \setminus L(w_{n+1})$ and the conditions are satisfied. Suppose that $i = n$ and $j_{n+1} \notin H$. Then the conditions are satisfied for formulas from $L(w_n)$ and $L^{[\mathrm{B}]_{j_{n+1}}}(w_n)$, by the same arguments as those used in proof of Lemma 4.8. For the remaining formulas, take any formula of the form $[\mathrm{B}]_{j_{n+1}}\xi \in L'(w_n) \setminus L(w_n)$. If $[\mathrm{B}]_{j_{n+1}}\xi \in \widetilde{\mathrm{Cl}}(L(v) \sqcap [\mathrm{B}]^+_H)$, then $[\mathrm{B}]_{j_{n+1}}\xi \in \widetilde{\mathrm{Cl}}\Big(L(v) \sqcap [\mathrm{B}]^+_{H\sqcap\{j_{n+1}\}}\Big)$ and, consequently, $[\mathrm{B}]_{j_{n+1}}\xi \in L'(w_{n+1})$, as $L(v) \sqcap [\mathrm{B}]^+_{H\sqcap\{j_{n+1}\}} \subseteq L(w_{n+1})$ and $L(w_{n+1})$ is a closed tableau. Otherwise it must be that $[\mathrm{B}]_{j_{n+1}}\xi = [\mathrm{B}]_{j_{n+1}}\psi$ and $L'(w_n)$ is constructed according to the case 3. Then $[\mathrm{B}]_{j_{n+1}}\xi \in L'(w_{n+1})$, as $[\mathrm{B}]_{j_{n+1}}\psi \in L(w_{n+1})$ in the case 3. On the other hand, take any formula of the form $[\mathrm{B}]_{j_{n+1}}\xi \in L'(w_{n+1}) \setminus L^{[\mathrm{B}]_{j_{n+1}}}(w_n)$. If $[\mathrm{B}]_{j_{n+1}}\xi \in L^{[\mathrm{B}]^+_{H\cup\{j_{n+1}\}}}(v)$, then $[\mathrm{B}]_{j_{n+1}}\xi \in L'(w_n)$, as $L^{[\mathrm{B}]^+_{H\cup\{j_{n+1}\}}}(v) \subseteq L'(w_n)$. Otherwise, it must be that $[\mathrm{B}]_{j_{n+1}}\xi = [\mathrm{B}]_{j_{n+1}}\psi$ and $L'(w_n)$ is constructed according to the case 3, in which case $[\mathrm{B}]_{j_{n+1}}\xi \in L'(w_n)$. Hence conditions **T4** and **T5** are satisfied for w_n. Conditions **TBG4**, **TBI4**, **TBG4** and **TBG5** are applicable at w_i to formulas from $L(w_i)$ only and since $L(w_i) \sqcap j_{i+1} = L(w_{i+1}) \sqcap j_{i+1}$, as Γ_i and Γ_{i+1} are connected with j_{i+1}, so the conditions are satisfied. Condition **TIG** holds at w_i as it cannot be affected by the extension. Condition **T2** holds for all the formulas from $L(w_i)$ and the only formulas from $L'(w_i) \setminus L(w_i)$ to which it is applicable are $\neg[\mathrm{B}]_{j_i}[\mathrm{B}]^+_G\psi$ and $\neg[\mathrm{B}]_{j_{i+1}}[\mathrm{B}]^+_G\psi$. If $0 \leq i \leq n-1$, then the condition is satisfied for both of these formulas as $\neg[\mathrm{B}]^+_G\psi \in L'(w_i)$, $\neg[\mathrm{B}]^+_G\psi \in L'(w_{i+1})$, $w_i \in R'^{\mathrm{B}}_{j_i}(w_i)$ and $w_{i+1} \in R'^{\mathrm{B}}_{j_{i+1}}(w_i)$. If $i = n$, then the condition is satisfied for $\neg[\mathrm{B}]_{j_i}[\mathrm{B}]^+_G\psi$, as $\neg[\mathrm{B}]^+_G\psi \in L'(w_i)$ and $w_i \in R'^{\mathrm{B}}_{j_i}(w_i)$. If $\neg[\mathrm{B}]_{j_{n+1}}[\mathrm{B}]^+_G\psi \in L'(w_n)$, then the case 3 must hold and the condition is satisfied, as in this case $\neg[\mathrm{B}]_{j_{n+1}}[\mathrm{B}]^+_G\psi \in L(w_{n+1})$. Condition **TC** holds for all formulas from $L(w_i)$ and the only formula from $L'(w_i) \setminus L(w_i)$ to which it is applicable is $\neg[\mathrm{B}]^+_G\psi$. Since $w_{n+1} \in R'^{\mathrm{B}^+}_G(w_i)$ and either $\sim\psi \in L'(w_{n+1})$ or $\neg[\mathrm{B}]^+_G\psi \in L(w_{n+1})$ and since condition **TC** is satisfied for w_{n+1}, so there must exist $u \in R'^{\mathrm{B}^+}_G(w_{n+1})$ with $\sim\psi \in L(u)$ and $u \in R'^{\mathrm{B}^+}_G(w_i)$. Thus condition **TC** is satisfied for w_i.

Let $\mathcal{T}'$ be the tableau extending $\mathcal{T}$ at all states from V in the way described above. In particular the set W' of worlds of $\mathcal{T}'$ extends W with all new worlds added for each state $v \in V$ and each formula of the form $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(v)$ for which the extension described above was made. We will show that the extended model graph $\mathcal{T}'$ is a TeamLog tableau for φ. As we argued above, the extension from $\mathcal{T}$ to $\mathcal{T}'$ sustains all the conditions of TeamLog tableau that were already satisfied for states of $\mathcal{T}$. Also, the conditions of TeamLog tableau are satisfied for all newly added states. Hence what remains to be shown are conditions **T2** and **TD** at states from V as well as condition **TC** for formulas of the form $[\mathrm{B}]^+_G\psi$.

For conditions **T2** and **TD** take any state $v \in V$. The only formulas from $L(v)$ for which condition **T2** is not satisfied in $\mathcal{T}$ are formulas of the form $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$ for which successors were not created by the algorithm for the reasons described above. Since for any such formula $R'^{\mathrm{B}}_j(v)$ extends $R^{\mathrm{B}}_j(v)$ with a new state u such that $\neg[\mathrm{B}]^+_G\psi \in L(u)$, so condition **T2** must be satisfied for that formula and v in $\mathcal{T}'$. For condition **TD**, take any formula of the form $[\mathrm{B}]_j\xi \in L(v)$ such that condition **TD** is not satisfied for it and for v in $\mathcal{T}$. Then there must be a formula of the form $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$ for which successors were not created for the reasons described above, in which case $R'^{\mathrm{B}}_j(v)$ extends $R^{\mathrm{B}}_j(v)$ with a new state u and since condition **T1** is satisfied for v, so condition **TD** is satisfied for v and $[\mathrm{B}]_j\xi$.

What remains to be shown is that condition **TC** is satisfied in $\mathcal{T}'$ for all states from W and formulas of the form $\neg[\mathrm{B}]^+_G\psi$. So take any state $v \in W$ and any formula of the form $\neg[\mathrm{B}]^+_G\psi \in L(v)$. If there is $j \in G$ such that $\neg[\mathrm{B}]_j\psi \in L(v)$, then condition **TC** is satisfied for $\neg[\mathrm{B}]^+_G\psi$ and v by the fact that condition **T2** is satisfied for v. Suppose then that for all $j \in G$, $\neg[\mathrm{B}]_j\psi \notin L(v)$. We will show first that condition **TC** holds for those of such states which are $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-Successors with some $k \in G$. Notice first that no state of $\mathcal{T}$ which is a $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-Successor can be marked `undec`. For assume the opposite and suppose that $v \in W$ is such a state. Suppose also that v is a $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-Successor of some state u with $k \in G$. Let n be the $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-successor of u on the path from u to v. Then, by construction of the algorithm, it must be that $\neg[\mathrm{B}]_k[\mathrm{B}]^+_k\psi \in L(v)$, a successor could not be created for it and for each descendant m of u on the path from u to v, $n \in B(m)$. Moreover, for all $l \in G \setminus \{k\}$ it must be that $[\mathrm{B}]_l[\mathrm{B}]^+_G\psi \in L(v)$. Since this is true for any $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-Successor of u which is marked `undec` and which is a descendant of n, so it must be that $B(n) = \{n\}$. Moreover, since $v \in W$, so there cannot be any state in $SS(n)$ which is marked `sat`. But then, by construction of the algorithm, n would have to be marked `unsat` and, consequently, u would have to marked `unsat` as well and it could not be that $u \in W$. This contradicts the assumption that $v \in W$. Hence v must be marked `sat`. Now, to show that condition **TC** holds for $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-Successors $v \in W$ such that for all $j \in G$, $\neg[\mathrm{B}]_j\psi \notin L(v)$, we will use induction on the maximal distance from states to descendant leaves of $\mathcal{T}$. For the induction basis suppose that v is a leaf of $\mathcal{T}$ and a $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-Successor with $k \in G$. Since v is a state, so $L(v)$ must be a $[\mathrm{B}]^+$-expanded tableau and since for all $j \in G$, $\neg[\mathrm{B}]_j\psi \notin L(v)$, so there must exist $j \in G$ such that $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(v)$. Since v is a leaf of $\mathcal{T}$, so it must be that a successor for $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$ could not be created, and since v cannot be marked `undec`, so it must be that $v \in V$. Then, by construction of $\mathcal{T}'$, there must exist $u \in R'^{\mathrm{B}}(v)$ with $\neg[\mathrm{B}]^+_G\psi \in L(u)$. Moreover, since $\neg[\mathrm{B}]_j\psi \notin L(v)$, so $u \notin W$ and, as we showed above, condition **TC** is satisfied for it and for $\neg[\mathrm{B}]^+_G\psi$. Hence there must exist $t \in R'^{\mathrm{B}^+}_G(u)$ such that $\sim\psi \in L(t)$. Since $t \in R'^{\mathrm{B}^+}_G(v)$, so condition **TC** is satisfied for v and $[\mathrm{B}]^+_G\psi$. For the induction step suppose that v is not a leaf of $\mathcal{T}$. If $v \in V$, then the condition **TC** is satisfied for v and $[\mathrm{B}]^+_G\psi$ by the same arguments as those used for the induction basis. Otherwise, there must exist $j \in G$ such that $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(v)$ and a successor of v was created for it (recall that v cannot be marked `undec`). By construction of $\mathcal{T}$, there must exists $u \in R^{\mathrm{B}}_j(v)$ such that $\neg[\mathrm{B}]^+_G\psi \in L(u)$ and, by the induction hypothesis, condition **TC** is satisfied for u and $\neg[\mathrm{B}]^+_G\psi$. Hence, by analogous arguments to those used for the induction basis, condition **TC** is satisfied for v and $\neg[\mathrm{B}]^+_G\psi$ as well. Thus we have shown that show that condition **TC** holds for $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-Successors $v \in W$ such that for all $j \in G$, $\neg[\mathrm{B}]_j\psi \notin L(v)$. For the case of states of $\mathcal{T}$ which are not $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-Successors with any $k \in G$, arguments analogous to those used in proof of Lemma 6.23 can be used to show that condition **TC** is satisfied for them and $\neg[\mathrm{B}]^+_G\psi$ as well.

Thus we have shown that if Algorithm 6.7 returns `sat` on input φ, then a TEAMLOG tableau for φ can be constructed which implies, by Theorem 4.14, that φ is satisfiable. $\square$

The following theorem states lower and upper bounds on the complexity of the TeamLog satisfiability problem for formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$.

Theorem 6.32. *The* TeamLog *satisfiability problem for formulas from* $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ *is PSPACE complete.*

Proof. The problem of TeamLog satisfiability of formulas from $\mathcal{L}^{\mathrm{Tind}} \subseteq \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ is PSPACE hard. Hence the problem of TeamLog satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ is PSPACE hard.

To show that the problem is in PSPACE, we will show that Algorithm 6.7 can be run by a deterministic Turing machine using polynomial space with respect to $|\varphi|$. To show that we will use induction on the pair $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ and $\mathrm{ag}\Big(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\Big)$ (in lexicographic order). For the induction basis, suppose that $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi) = 0$ (in this case $\mathrm{ag}\Big(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\Big) = \mathcal{A} \cup \{\omega\}$). Then there are no formulas of the form $[\mathrm{B}]^+_G\psi \in \neg\mathrm{Sub}(\varphi)$ and Algorithm 6.7 works like Algorithm 6.1 on the input φ. Thus, by Theorem 6.24, it can be run by a deterministic Turing machine using polynomial space with respect to $|\varphi|$. For the induction step, suppose that $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi) > 0$ (in which case $\mathrm{ag}\Big(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\Big) \subseteq \mathcal{A}$). To check the satisfiability of φ a pre-tableau is constructed and the decision with regard to the satisfiability is made on the basis of how the root of this pre-tableau is marked. Since the decision on how each node is marked in this pre-tableau depends on the nodes on the path from this node to the root of the pre-tableau and how descendants of this node are marked, so for deciding how the root node should be marked, the pre-tableau could be traversed in depth first search like manner. By Lemma 6.27, the depth of the pre-tableau constructed by Algorithm 6.7 is $\leq \mathcal{O}(|\varphi|\mathrm{dep}(\varphi)^{2|\mathcal{A}|+1})$. To mark some of the leaves of the pre-tableau constructed by Algorithm 6.7, Function 6.9 may be called. Each such call requires polynomial space. To see this, let ψ, G, j and Φ be the input for such call. Then $\Phi \subseteq \neg\mathrm{Cl}(\mathrm{Sub}(\varphi))$ and so $|\Phi| \leq (2|\mathcal{A}| + 1)|\varphi| = \mathcal{O}(\Phi)$. The algorithm enumerates the elements of $\mathcal{SS}(\Phi/[\mathrm{B}]^+_H)$, where $H = \mathrm{ag}\Big(\Big(\Phi \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$ is a subset of G. To enumerate these elements a pre-tableau is constructed with the root labelled with $\Phi/[\mathrm{B}]^+_H$ and then all the states that could be obtained from this set are enumerated. To enumerate these states depth first search method could be used, so that $\leq \mathcal{O}(|\varphi|)$ memory would be needed to remember each path leading from the root to a state. For each such a state it is checked whether its label is satisfiable, which can be done in polynomial space, as $\mathrm{dep}_{[\mathrm{B}]^+}(\xi) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$, where ξ is the conjunction of formulas in the states, and the induction hypothesis applies. Next for each such a state reachability to some other element of a graph $\mathcal{G}\big((\Phi/[\mathrm{B}]^+_H) \cup \{\psi\}\big)$ is checked, and this requires using Function 6.10 recursively, with depth of recursion $\leq \mathcal{O}(|\varphi|)$. Function 6.10 enumerates elements of $\mathcal{SS}\big((\Phi/[\mathrm{B}]^+_H) \cup \{\psi\}\big)$, which requires $\leq \mathcal{O}(|\varphi|)$ memory. Additionally, the satisfiability of the labels of these states is checked and since $\mathrm{dep}_{[\mathrm{B}]^+}(\xi) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$, where ξ is the conjunction of formulas in the label of a state, so, by the induction hypothesis, each such call requires polynomial space. Notice also that the upper bound on the number of steps of reachability can be stored using $\mathcal{O}(|\varphi|)$ space. Thus Function 6.10 uses polynomial space with respect to $|\varphi|$. Lastly, after checking reachability for a given state from $\mathcal{SS}\big(\Phi/[\mathrm{B}]^+_H\big)$, Function 6.9 checks the satisfiability of some properly constructed sets of formulas obtained from the label of the state. Since for each such a set of formulas $\mathrm{dep}_{[\mathrm{B}]^+}(\xi) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ or $\mathrm{dep}_{[\mathrm{B}]^+}(\xi) = \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ and $\mathrm{ag}\Big(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\Big) \subsetneq \mathrm{ag}\Big(\mathrm{Sub}(\xi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\xi)\Big)$, where ξ is the conjunction of the formulas in the set (c.f. proofs of Lemma 6.28 and Lemma 6.31), so, by the induction hypothesis, each such call uses at most polynomial space with respect to $|\varphi|$. Hence each call to Function 6.9 requires polynomial space with respect to $|\varphi|$ Thus the satisfiability of φ can be decided by a deterministic Turing machine using space of polynomial size with respect to $|\varphi|$. Hence the problem of TeamLog satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ is in PSPACE. $\square$

The tableau constructed in proof of Lemma 6.31 can have exponential depth with respect to the input formula. For that reason an algorithm similar to that used in proof of Theorem 6.25 for formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$ with modal depth bounded by a constant that works in polynomial time cannot be used in the case $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ with modal depth of formulas bounded by a constant. In fact finding such an algorithm may be very difficult, as the satisfiability problem for formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ with modal depth bounded by 2 is PSPACE hard, as stated below (proof is moved to the Appendix).

Theorem 6.33. *The problem of checking* TeamLog *satisfiability of formulas from* $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ *with modal depth bounded by* 2 *is PSPACE complete.*

6.3.3 Restriction $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}(c)}$

Since bounding modal depth of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ by a constant ≥ 2 leaves the TeamLog satisfiability problem PSPACE hard, we propose yet another restriction of the language, that, when combined with restricting modal depth, makes the TeamLog satisfiability problem NPTIME complete. The restriction is a refinement of the restriction $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ and it puts constraints on the number of formulas of the form $[\mathrm{I}]_j\xi$, $[\mathrm{G}]_j\xi$ and $[\mathrm{I}]^+_H\xi$ within a direct context of modal operators $[\mathrm{B}]^+_G$ with $j \in G$ or $H \cap G \neq \varnothing$. That is whenever a formula from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ has a subformula which violates modal context restriction $\mathbf{R}_2$, then this formula must satisfy this additional restriction. The restriction is defined formally below.

Definition 24 (Restriction $\mathbf{R}_{1(c)}$)**.** *Let $c \geq 0$. A formula φ satisfies the restriction $\boldsymbol{R}_{1(c)}$ if $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ and either $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$ or one of the following holds:*

- *φ is of the form $\neg\psi$ and ψ satisfies restriction $\boldsymbol{R}_{1(c)}$,*
- *φ is of the form $\psi_1 \wedge \psi_2$ and psi_1 and ψ_2 satisfies restriction $\boldsymbol{R}_{1(c)}$,*
- *φ is of the form $[O]_j\psi$, with $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $j \in \mathcal{A}$, and ψ satisfies restriction $\boldsymbol{R}_{1(c)}$,*
- *φ is of the form $[\mathrm{I}]^+_G\psi$, with $G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}$, and ψ satisfies restriction $\boldsymbol{R}_{1(c)}$,*
- *φ is of the form $[\mathrm{B}]^+_G\psi$, with $G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}$, ψ satisfies restriction $\boldsymbol{R}_{1(c)}$ and $\left|\{[O]_j\xi : [O]_j\xi \in \neg\mathrm{PT}(\psi) \text{ and } j \in G\} \cup \{[\mathrm{I}]^+_H\xi : [\mathrm{I}]^+_H\xi \in \neg\mathrm{PT}(\psi) \text{ and } H \cap G \neq \varnothing\}\right| \leq c$.*

The set of formulas in $\mathcal{L}^{\mathrm{T}}$ satisfying restriction $\boldsymbol{R}_{1(c)}$ will be denoted by $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}(c)}$.

Example 2. *The following formulas satisfy restriction $\boldsymbol{R}_{1(1)}$.*

$$[\mathrm{B}]^+_{\{1,2\}}[\mathrm{I}]^+_{\{1,2\}}p, \qquad [\mathrm{B}]^+_{\{1,2\}}\left([\mathrm{I}]_2 p \vee q\right).$$

The following formulas violate restriction $\boldsymbol{R}_{1(1)}$ and satisfy restriction $\boldsymbol{R}_{1(2)}$

$$[\mathrm{B}]^+_{\{1,2\}}\left([\mathrm{I}]_1 p \wedge [\mathrm{I}]_2 q\right), \qquad [\mathrm{B}]^+_{\{1,2\}}\left([\mathrm{I}]_2 p \vee q\right).$$

To check TeamLog satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}(c)}$ Algorithm 6.1 can be used. We will show that the algorithm stops on any input from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}(c)}$ and that its state height is bounded by a polynomial depending on modal depth of the input formula. We start by showing the bounds of the length of a sequence of R^{B}-Successors with unchanged modal depth of labels in the pre-tableau.

Lemma 6.34. *The maximal length of sequence of R^{B}-Successors with unchanged modal depth of labels in the pre-tableau constructed by Algorithm 6.1 for an input $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}(c)}$ is $\leq \mathcal{O}\big(\mathrm{dep}(\varphi)^{2|\mathcal{A}|}\big)$.*

Proof. Claims 6.12 – 6.16 shown in proof of Lemma 6.13 hold in the case of $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}(n)}$, as they require modal context restriction $\mathbf{R}_1$ only.

Consider a sequence of states $s_0, \ldots, s_m$ in the pre-tableau such that for any $0 < k \leq m$, s_k is an $R^{\mathrm{B}}_{j_k}$-Successor of s_{k-1}. Suppose that for any $0 < k \leq m$ it holds that $\mathrm{Ind}(L(s_{k-1})) \sqcap j_k = \varnothing$ and the sequence satisfies properties **PB1**, **PI1** and **PB2** for all $d \geq 0$. If $\widehat{\mathrm{Gr}}(L(s_0)) \sqcap \neg[\mathrm{B}]^+ = \varnothing$ or there is more than one formula of the form $\neg[\mathrm{B}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(s_0))$, then the length of such a sequence must be ≤ 2, by the same arguments as those used in proof of Lemma 6.13. If $\widehat{\mathrm{Gr}}(L(s_0)) \sqcap \neg[\mathrm{B}]^+ = \{\neg[\mathrm{B}]^+_G\psi\}$, then the length of the sequence must be $\leq 2^{c+1}|\mathcal{A}| + 1$. Arguments here are similar to those used in proof of 6.13 for analogous case. Suppose that the length of the sequence $m > 2^{c+1}|\mathcal{A}|+1$. Then there exists $0 < k_1 < \ldots < k_{2^{c+1}+1} \leq m$ such that $j_{k_1} = \ldots = j_{k_{2^{c+1}+1}}$. By Claim 6.16 the subsequent sets $L^{\neg[\mathrm{B}]_{j_{k_i}}}(s_{k_i-1}, [\mathrm{B}]^+_G\psi)$ differ on the elements from $Z_{j_{k_1}} \cup \{\psi\}$ only, where $Z_{j_{k_1}} = \neg\Big(\Big(L(s_{k_1-1}) \sqcap [\mathrm{I}]_{j_{k_1}}\Big) \cup \Big(L(s_{k_1-1}) \sqcap [\mathrm{G}]_{j_{k_1}}\Big) \cup \{[\mathrm{B}]_{j_{k_1}}\psi\}\Big)$. Moreover, each of these sets contains a maximal subset of $Z_{j_{k_1}}$ as a subset and contains ψ if and only if it contains $[\mathrm{B}]_{j_{k_1}}\psi$. Hence there are $2^{\left|Z_{j_{k_1}}\right|/2}$ such different sets and, by modal context restriction $\mathbf{R}_{1(c)}$, $\left|Z_{j_{k_1}}\right| \leq 2(c+1)$. Thus there must exist $1 \leq i' < i \leq 2^{c+1}$ such that $L^{\neg[\mathrm{B}]_{j_{k_{i'}}}}(s_{k_{i'}-1}, [\mathrm{B}]^+_G\psi) = L^{\neg[\mathrm{B}]_{j_{k_i}}}(s_{k_i-1}, [\mathrm{B}]^+_G\psi)$. But then, by construction of the algorithm, the $\neg[\mathrm{B}]_{j_{k_i}}[\mathrm{B}]^+_G\psi$-successor of s_{k_i-1} cannot be created and so s_{k_i} cannot be in the sequence, which contradicts our assumptions.

Using arguments similar to those used in Lemma 6.13 it can be shown that the maximal length of a sequence of R^{B}-Successors with the same modal depth of labels is $\mathcal{O}\big(\mathrm{dep}(\varphi)^{2|\mathcal{A}|}\big)$. Moreover, 2^{c+1} contributes to the factor of $\mathrm{dep}(\varphi)^{2|\mathcal{A}|}$. ☐

Bounds on the state height of the pre-tableau constructed by Algorithm 6.1 for an input formula satisfying modal context restriction $\mathbf{R}_{1(c)}$ are stated in the lemma below.

Lemma 6.35. *State height of the pre-tableau constructed by Algorithm 6.1 for an input $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}(c)}$ is $\leq \mathcal{O}\big(\mathrm{dep}(\varphi)^{2|\mathcal{A}|+1}\big)$ and its height is $\leq \mathcal{O}\big(|\varphi|\mathrm{dep}(\varphi)^{2|\mathcal{A}|+1}\big)$.*

Proof. Proof is by analogous arguments to those use in proof of Lemma 6.18, using Lemma 6.34 instead of Lemma 6.13. ☐

Since the size of a pre-tableau constructed by the algorithm for an input formula from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}(c)}$ is bounded so Algorithm 6.1 terminates on any input $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}(c)}$.

Lemma 6.36. *For any input formula $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}(c)}$ Algorithm 6.1 terminates.*

Since Algorithm 6.1 terminates on any input from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}(c)}$ so, by Lemma 6.23, it is also valid for checking the TeamLog satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}(c)}$. Moreover, since the state height of the pre-tableau constructed by Algorithm 6.1 for an input formula $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}(c)}$ is bounded by a polynomial depending on $\mathrm{dep}(\varphi)$ and c, so the size of the pre-tableau constructed for φ in proof of Lemma 6.35 on the basis of this pre-tableau has the size which is bounded by a polynomial depending on $|\varphi|$ with the degree depending on $\mathrm{dep}(\varphi)$. Thus the following theorem, analogous to Theorem 6.25 for $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$. Proof of the theorem is analogous to proof of Theorem 6.25.

Theorem 6.37. *For any fixed k, if modal depth of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}(c)}$ is bounded by k, then the* TeamLog *satisfiability for them is NPTIME complete.*

6.4 Restricting the number of propositional symbols of TeamLog

If the number of propositional symbols is bounded by 1, then the complexity of the satisfiability problem for logic TeamLog remains EXPTIME complete. Since the problem is in EXPTIME so it is enough to show that is is EXPTIME hard. To show that we will show that the problem of checking KD_2^+-satisfiability of formulas with the number of propositional symbols bounded by 1 is EXPTIME hard. Then, by Fact 4.18, it will follow that the problem of checking TeamLog satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}[\{p\}]$ is EXPTIME hard as well.

Lemma 6.38. *The problem of checking KD_2^+-satisfiability of formulas with the number of propositional symbols bounded by* 1 *is EXPTIME hard.*

Proof. To show the result we will show how to construct an infinite family $\pi_1, \pi_2, \ldots$ of formulas that can be used as replacements for propositional symbols. Then we could take a formula $\varphi(G)$ constructed in proof of Theorem 4.19 to reduce an EXPTIME hard problem of deciding if for a given corridor tiling game G there is a winning strategy for player A to $\mathrm{KD45}_2^+$ satisfiability problem and translate it to KD_1^+. The translation is done by replacing all the $[\mathrm{B}]_1$ and $[\mathrm{B}]_2$ operators $\varphi(G)$ with $[\mathrm{I}]_1$ operators and all $[\mathrm{B}]^+_{\{1,2\}}$ operators in $\varphi(G)$ with $[\mathrm{I}]^+_{\{1\}}$ operators. It can be shown, using similar arguments to those used in proof of Theorem 4.19, that thus obtained formula reduces the problem of deciding if there is a winning strategy for player A in G to the problem of checking KD_1^+ satisfiability. We will denote the translated formula by $\varphi'(G)$. Replacing all the propositional symbols in $\varphi'(G)$ with formulas from the family $\pi_1, \pi_2, \ldots$ will provide a reduction of the problem to checking KD_2^+ satisfiability of formulas with the number of propositional symbols bounded by 1.

The family $\pi_1, \pi_2, \ldots$ will be constructed to be a pp-like family of formulas, as defined in [56]. The notion of pp-like infinite family of formulas is defined for a given modal logic S with set of operators Ω. To define this notion, some additional notions are needed. Let $\{\pi_1, \ldots, \pi_n\}$ be a finite non-empty set of formulas. An *atom over* $\{\pi_1, \ldots, \pi_n\}$ is any conjunction of the form $\hat{\pi}_1 \wedge \ldots \wedge \hat{\pi}_n$, where $\hat{\pi}_i$ is either π_i or $\neg\pi_i$, for each $1 \leq i \leq n$. A *tree formula* over $\{\pi_1, \ldots, \pi_n\}$ is defined inductively as follows: it is either an atom over $\{\pi_1, \ldots, \pi_n\}$, or a formula of the form $\neg\Box\neg\tau$, where τ is a tree-like formula and $\Box \in \Omega$, or a conjunction of the form $\tau_1 \wedge \tau_2$, where τ_1 and τ_2 are tree-like formulas. Formulas $\pi_1, \ldots, \pi_n$ are *completely independent with respect to modal logic S* if every tree-formula over $\{\pi_1, \ldots, \pi_n\}$ is S-satisfiable. An infinite family of formulas $\pi_1, \pi_2, \ldots$ is *pp-like in a modal logic S* if every finite non-empty subset of this family is completely independent with respect to S.

In [56] it is shown how to construct a pp-like family of formulas for the logic KD_n. We will use the same construction here. Let $\pi_i = \neg[\mathrm{I}]_2\neg(p \wedge \neg[\mathrm{I}]_2\neg p)$. Notice that π_i is satisfied in a given model $\mathcal{M}$ at a world w in it if and only if there is a sequence of worlds $v_0, \ldots, v_{i+1}$ in $\mathcal{M}$

such that $w = v_0$, $v_j \in R_2^{\mathrm{I}}(v_{j-1})$ (for all $1 \leq j \leq i+1$), $(\mathcal{M}, v_1) \models \neg p$ and $(\mathcal{M}, v_{i+1}) \models p$. In other words there is a path in $\mathcal{M}$ starting from w, consisting of R_2^{I}-successors and having $\neg p$ satisfied in the second world and p satisfied in the last world. It is easy to see that the formula is pp-like for KD_n. Since it is constructed with modal operator $[\mathrm{I}]_2$, which does not appear in the formula $\varphi'(G)$, so replacing different propositional symbols in $\varphi'(G)$ with different formulas from the family $\pi_1, \pi_2, \ldots$ reduces the problem of deciding if there is a winning strategy for player A in G to the problem of checking KD_2^+ satisfiability of formulas with the number of propositional symbols bounded by 1. $\square$

As a consequence of Lemma 6.38 and Fact 4.18 we get the following theorem.

Theorem 6.39. *The* TeamLog *satisfiability problem for formulas from* $\mathcal{L}^{\mathrm{T}}[\{p\}]$ *is EXPTIME complete.*

Similarly to the case of TeamLog$^{\mathrm{ind}}$ satisfiability, we can show that if bounding the number of propositional symbols is combined with bounding modal depth of formulas, the complexity is reduced to linear time. The proof is analogous to the one for Theorem 5.3 and is omitted here.

Theorem 6.40. *For any fixed* $k, l \geq 1$*, if the number of propositional atoms is bounded by* l *and modal depth of formulas is bounded by* k*, then* TeamLog *satisfiability problem can be solved in linear time.*

Chapter 7

Discussion

In this chapter we discuss the modal context restrictions in the standard notation of TeamLog as presented in Section 2.1.

In Chapter 6 two modal context restrictions have been proposed, denoted by $\mathbf{R}_1$ and $\mathbf{R}_2$. Each of them results in PSPACE completeness of the satisfiability problem for TeamLog, however only $\mathbf{R}_2$ results in NPTIME completeness of this problem, when combined with restricting modal depth of formulas. On the other hand $\mathbf{R}_1$ can be refined to restriction $\mathbf{R}_1(c)$, where c is a constant, so that combining it with bounding modal depth of formulas results in NPTIME completeness of the satisfiability problem as well.

Let us start the discussion of the two restrictions with formulas specifying beliefs of groups of agents. When interpreted in the context of BDI agents, $\mathbf{R}_1$ can be seen as forbidding common introspection of beliefs within a group of agents. In other words, it forbids any formula of the form $\text{C-BEL}_G(\varphi)$ where φ contains, within the scope of propositional operators, any formulas referring to beliefs of agents from G. For example the following formula specifies that group G commonly beliefs that group H generally beliefs that φ holds:

$$\text{C-BEL}_G(\text{E-BEL}_H(\varphi)).$$

If $G \cap H = \varnothing$, then this formula satisfies restriction $\mathbf{R}_1$. However, if $G \cap H \neq \varnothing$, then this formula does not satisfy the restriction. Restriction $\mathbf{R}_2$ forbids, additionally, common introspection of goals and intentions within a group of agents. Hence the following formula

$$\text{C-BEL}_G(\text{E-INT}_H(\varphi)).$$

with $H \cap G \neq \varnothing$ is forbidden by $\mathbf{R}_2$, while it is allowed by $\mathbf{R}_1$.

The differences in the complexity of the satisfiability problem under the two restrictions are strictly related to the axioms of positive and negative introspection of goals and intentions of individual agents. Because of these axioms, for any TeamLog model $\mathcal{M}$ and a world w and any formula $\text{BEL}(j, \text{GOAL}(j, \psi))$ it holds that $(\mathcal{M}, w) \vDash \text{BEL}(j, \text{GOAL}(j, \psi))$ if and only if $(\mathcal{M}, w) \vDash \text{GOAL}(j, \psi)$. Similarly with formulas $\text{BEL}(j, \text{INT}(j, \psi))$ and $\text{INT}(j, \psi)$.

Now let us consider formulas specifying group intentions. In this case there is no difference between the two restrictions and both of them forbid mutual intentions towards intentions and beliefs about intentions within a group of agents. For example the formula

$$\text{M-INT}_G(\text{E-INT}_H(\varphi)),$$

does not satisfy neither $\mathbf{R}_1$ nor $\mathbf{R}_2$, if $G \cap H \neq \varnothing$. In this case specifying that G mutually intends that some of the agents from this group have certain intentions is forbidden.

The formula

$$\text{M-INT}_G\left(\text{E-BEL}_H\left(\text{E-INT}_H\left(\varphi\right)\right)\right)$$

does not satisfy the restrictions as well, if $G \cap H \neq \varnothing$. In this case specifying that G mutually intends that some group H containing agents from G generally beliefs that it generally intends something is forbidden. This results from the axioms of positive and negative introspection of intentions of individual agents.

The main difference between the two restrictions is seen in the context of group beliefs about motivational attitudes. For example the fundamental definition of collective intention:

$$\text{M-INT}_G\left(\varphi\right) \wedge \text{C-BEL}_G\left(\text{M-INT}_G\left(\varphi\right)\right)$$

does not satisfy $\mathbf{R}_2$, while it satisfies $\mathbf{R}_1$ (as long as it is satisfied by formulas φ and $\text{M-INT}_G\left(\varphi\right)$). Similarly, definition of social commitment of agent i towards agent j with respect to some action α:

$$\text{INT}\left(j,\alpha\right) \wedge \text{GOAL}\left(j, \mathit{done}(i,\alpha)\right) \wedge \mathit{awareness}_{\{i,j\}}(\text{INT}\left(j,\alpha\right) \wedge \text{GOAL}\left(j, \mathit{done}(i,\alpha)\right)),$$

does not satisfy $\mathbf{R}_2$ if the awareness part is set to common belief, i.e. $\mathit{awareness}_{\{i,j\}}$ is replaced by $\text{C-BEL}_{\{i,j\}}$. However, in the restriction $\mathbf{R}_1$ awareness expressed by common belief is allowed. Notice that these two formulas satisfy $\mathbf{R}_1(c)$ with $c = 2$.

In the case of specifying collective commitments even restriction $\mathbf{R}_1$ can be too strong. Consider for example the strongest two forms of collective commitment, robust and strong collective commitment, that assume the highest levels of awareness:

$$\text{R-COMM}_{G,P}\left(\varphi\right) \leftrightarrow \text{C-INT}_G\left(\varphi\right) \wedge \mathit{constitutes}(P,\varphi) \wedge \text{C-BEL}_G\left(\mathit{constitutes}(P,\varphi)\right) \wedge \bigwedge_{\alpha\in P}\bigvee_{i,j\in G} \text{C-BEL}_G\left(\text{COMM}\left(i,j,\alpha\right)\right).$$

$$\text{S-COMM}_{G,P}\left(\varphi\right) \leftrightarrow \text{C-INT}_G\left(\varphi\right) \wedge \mathit{constitutes}(P,\varphi) \wedge \text{C-BEL}_G\left(\mathit{constitutes}(P,\varphi)\right) \wedge \text{C-BEL}_G\left(\bigwedge_{\alpha\in P}\bigvee_{i,j\in G} \text{COMM}\left(i,j,\alpha\right)\right).$$

In both cases the last component, expressing team awareness about the distribution of social commitments within the group involves common beliefs within the group about beliefs of agents from this group (which are contained in the definition of $\text{COMM}\left(i,j,\alpha\right)$). Hence, both definitions of commitments do not satisfy the restriction $\mathbf{R}_1$. One way of dealing with this problem could be lowering the level of awareness about the distribution of the social commitments. In this case the awareness component may remain at the highest level of common belief. Another possibility is to consider an alternative definition of bilateral commitment, where the awareness component is removed. Let *responsibility* of agent i towards agent j with respect to action α be defined as follows:

$$\text{RESP}\left(i,j,\alpha\right) \leftrightarrow \text{INT}\left(j,\alpha\right) \wedge \text{GOAL}\left(j, \mathit{done}(i,\alpha)\right).$$

Notice that social commitment of one agent towards another is responsibility plus awareness about this responsibility. Consider now two definitions of collective commitments:

$$\text{R-COMM}'_{G,P}\left(\varphi\right) \leftrightarrow \text{C-INT}_G\left(\varphi\right) \wedge \mathit{constitutes}(P,\varphi) \wedge \text{C-BEL}_G\left(\mathit{constitutes}(P,\varphi)\right) \wedge \bigwedge_{\alpha\in P}\bigvee_{i,j\in G} \text{C-BEL}_G\left(\text{RESP}\left(i,j,\alpha\right)\right)$$

and

$$\text{S-COMM}'_{G,P}(\varphi) \leftrightarrow \text{C-INT}_G(\varphi) \wedge constitutes(P,\varphi) \wedge \text{C-BEL}_G(constitutes(P,\varphi)) \wedge \text{C-BEL}_G\left(\bigwedge_{\alpha\in P}\bigvee_{i,j\in G}\text{RESP}(i,j,\alpha)\right).$$

obtained by replacing social commitments by responsibilities. In both definitions bilateral awareness associated with each social commitment is 'moved' to the group level of team awareness about responsibilities within the group, therefore they satisfy modal context restriction $\mathbf{R}_1$.

Methodologies of agent oriented modelling and design, like KGR [68, 67, 66], often divide the process of agents modelling by separating construction of belief model, goal model and plan model. Similarly, in specification of a multiagent systems using formalisms like TeamLog, separate parts could be distinguished, where purely informational and purely motivational aspects of individual agents and groups of agents are specified and parts where interrelations between these parts are specified. In such cases restriction $\mathbf{R}_2$ can be applied to the purely informational or purely motivational parts, while restriction $\mathbf{R}_1$ can be applied to the mixed parts.

Chapter 8

Conclusions

In this thesis we studied the complexity of the satisfiability problem for a multiagent logic called TeamLog. We showed that the satisfiability problem for the individual fragment of TeamLog, TeamLog$^{\text{ind}}$, is PSPACE complete, while the full TeamLog it is EXPTIME complete.

These results suggest that practical tasks involving reasoning in TeamLog or TeamLog$^{\text{ind}}$ cannot be realized efficiently in general cases. For this reason it is important to study restrictions of the language that would lead to reduction of the complexity of the problem, or at least would lead to classes of complexity for which effective heuristics based methods could be applied. In the case of TeamLog$^{\text{ind}}$ we considered two restrictions: restricting modal depth of formulas by a constant and restricting the number of propositional symbols by a constant.

We showed that in the case of the first restriction the satisfiability problem for TeamLog$^{\text{ind}}$ becomes NPTIME complete. In the case of the second restriction it remains PSPACE complete. We showed also that combining both the restrictions makes the problem solvable in linear time. In this case, however, the constant multiplier depends exponentially on the number of propositional symbols.

It is instructive to compare these results with those obtained by Halpern and Moses in [54] and Halpern in [56]. In those works normal multimodal logics generated by subsets of axioms **T**, **D**, **4** and **5** are studied. TeamLog and other multimodal formalisms for multiagent systems combine some of these multimodal logics by adding additional axioms interrelating modalities from different groups.[1] Our results show that axioms associated with positive and negative introspection as well as the axiom of goals and intentions compatibility (called also strong realism axiom in the formalism of Rao & Georgeff) do not affect the complexity of the satisfiability problem of the combined multimodal logics.

In the case of TeamLog we showed that the satisfiability problem remains EXPTIME complete even if the modal depth of formulas is bounded by 2. For this reason restricting modal depth of TeamLog formulas is not promising and we introduced a new restriction, called modal context restriction, which generalises modal depth restriction. Two such restrictions where proposed, both resulting in PSPACE completeness of the satisfiability problem. In the case of the less restrictive one of them, called $\mathbf{R}_1$, the problem remains PSPACE hard even if modal depth of formulas is bounded by 2. In the case of the more restrictive one of them, called $\mathbf{R}_2$, combining it with restricting modal depth of formulas by a constant results in NPTIME completeness of the satisfiability problem. Since restriction $\mathbf{R}_2$ is too strong, at least in situations when aspects of multiagent systems combining informational and motivational attitudes are specified, like for example collective intentions and collective commitments, we

[1] See Appendix B for an overview of selected formalisms.

proposed a refinement of restriction $\mathbf{R}_1$. This refinement, called $\mathbf{R}_1(c)$, where c is a constant, results in NPTIME completeness of the satisfiability problem, when combined with bounding modal depth of formulas. Apart from modal context restrictions, we also investigated the effect of bounding the number of propositional symbols by a constant. We showed that in this case the problem remains EXPTIME complete (it remains such even in the case of multimodal logic KD_n^+ (for $n \geq 2$)). Combining this restriction with restricting modal depth of formulas results in the satisfiability problem for TeamLog being solvable in linear time, but again, like in the case of $\text{TeamLog}^{\text{ind}}$, the constant multiplier depends exponentially on the number of propositional symbols.

The restrictions of the language studied in this thesis do not lead to tractable fragments of the formalisms considered.[2] However, we were able to find NPTIME complete fragments of them, even in the case of full TeamLog, which originally has EXPTIME complete satisfiability problem. Two possible approaches could be undertaken to address this issue: reducing the satisfiability of the NPTIME complete fragments to some other NPTIME complete problems for which well performing, heuristics based algorithms exist, or studying further restrictions of the language that could lead to PTIME solvable satisfiability problem.

The first of these approaches was successfully used by Kacprzak, Lomuscio and Penczek in [64, 65], where model checking of temporal modal logic is studied. Authors reduce this problem to the problem of satisfiability of propositional calculus (SAT) and apply existing SAT-solvers to it. Using similar approach to the NPTIME complete fragments of TeamLog found in this thesis is a possible direction for further research.

For the second approach different language restrictions that were already studied in the literature could be considered. Firstly, it would be interesting to investigate the Horn fragment of TeamLog. In [83, 84] Linh Nguyen studied Horn fragments of various basic multimodal logics and he found out that when modal depth of formulas is bounded by a constant, then the satisfiability problem is PTIME complete in the case of multimodal logics generated by sets of axioms containing axiom **D** and, additionally, axiom **5**, whenever they contain axiom **4**. In the case of multimodal logics generated by sets of axioms that do not contain axiom **D** or contain axiom **4** without axiom **5**, the satisfiability problem is NPTIME complete, if modal depth of formulas is bounded by a constant.

An interesting research question is, what is the complexity of the satisfiability problem for the Horn fragment of TeamLog. In particular, is there a modal context restriction which leads to PTIME complete satisfiability problem of Horn fragment of TeamLog?

Another possibility would be to look at restrictions of propositional operators used in formulas. This approach was taken by Bauland et al. in [8]. To study different sets of propositional operators used in formulas they facilitate Post lattice [90], an algebraic tool proposed by Post that fully classifies boolean functions. Each node of the Post lattice corresponds to a different set of Boolean operators, closed under superposition (such sets are called clones). Post classified the lattice of all clones and provided a finite basis for each of them. Post lattice has been successfully used to classify the complexity problems for propositional calculus. In [74] Lewis used it to show a dichotomy of satisfiability problems for propositional calculus, proving that depending on the propositional operators used they are either NPTIME complete or PTIME solvable. In [8] Bauland et al. showed a trichotomy in the case of basic normal modal system K (depending on the boolean operators used the satisfiability problem is

[2] To be more precise, a combination of modal depth restriction and restricting the number of propositional symbols by a constant results in the satisfiability problem being solvable in linear time. However, from practical point of view this is not very useful, as the multiplier depends exponentially on the number of propositional symbols used, and this number may be large, especially in the case of multiagent systems designed to operate in complex domains.

either PSPACE complete, coNPTIME complete or PTIME solvable) and dichotomy in the case of normal modal system KD (in this case the satisfiability problem is either PSPACE complete or PTIME solvable). Almost complete characterization was also obtained for modal systems T, S4 and S5. Similar approach was also applied to LTL in [9] and to CTL* and CTL in [78].

Another question for further research is whether the modal context restrictions proposed can be relaxed to obtain similar complexity results. In other words, are the restrictions proposed in this thesis minimal. Is it true, that for any modal context restriction $\mathbf{R}$ applied to language of TeamLog, if the satisfiability problem is PSPACE solvable with it, then $\mathbf{R} \subseteq \mathbf{R_1}$?[3] Similarly, it it true, that for any modal context restriction $\mathbf{R}$ applied to TeamLog, if the satisfiability problem is NPTIME solvable when $\mathbf{R}$ is combined with bounding modal depth by a constant, then $\mathbf{R} \subseteq \mathbf{R_2}$? Trying to answer these questions is yet another possibility for further research.

[3]Recall that the modal context restriction is the set of modal contexts that are allowed under it. Hence the restriction is less restrictive if it allows for more.

Appendix A

Proofs

Proof of Theorem 6.1, page 61. Similarly to proof of Theorem 4.19 we will use propositional variables $q_1, \dots, q_N$ to enumerate worlds of the model. Each world v receives two numbers of length N in binary representation, that are encoded by the valuation of the formulas $[\mathrm{I}]_1 q_j$ and $[\mathrm{I}]_2 q_j$. Bits of the first number, $M_1(v)$, are encoded by the valuations of formulas $[\mathrm{I}]_1 q_j$, with $[\mathrm{I}]_1 q_1$ corresponding to the least significant bit and $[\mathrm{I}]_1 q_j$ being satisfied in $(\mathcal{M}, v)$ encoding the value 1 and $\neg[\mathrm{I}]_1 q_j$ being satisfied in $(\mathcal{M}, v)$ encoding the value 0 of bit j of $M_1(v)$. Value of $M_2(v)$ is encoded in analogous way with formulas $[\mathrm{I}]_2 q_j$.

Let

$$\varphi = \mathsf{INIT} \wedge [\mathrm{B}]^{+}_{\{1,2\}} \left(\mathsf{INC}_0 \wedge \bigwedge_{j=1}^{N-1} \mathsf{INC}_1(j) \right) \wedge \neg [\mathrm{B}]^{+}_{\{1,2\}} \left(\bigvee_{j=1}^{N} \neg [\mathrm{I}]_1 q_j \right),$$

where $[\mathrm{B}]'^{+}$ is defined as in proof of Theorem 4.19 and

$$\mathsf{INIT} = \left(\bigwedge_{j=1}^{N} \neg [\mathrm{I}]_1 q_j \right) \wedge \left([\mathrm{I}]_2 q_1 \wedge \bigwedge_{j=2}^{N} \neg [\mathrm{I}]_2 q_j \right) \tag{A.1}$$

$$\mathsf{INC}_0 = \neg [\mathrm{I}]_1 q_1 \rightarrow \left(\bigwedge_{j=2}^{N} ([\mathrm{I}]_1 q_j \leftrightarrow [\mathrm{I}]_2 q_j) \right) \tag{A.2}$$

$$\begin{aligned} \mathsf{INC}_1(i) = \left(\neg [\mathrm{I}]_1 q_{i+1} \wedge \bigwedge_{j=1}^{i} [\mathrm{I}]_1 q_j \right) \rightarrow \Bigg(\bigwedge_{j=i+2}^{N} ([\mathrm{I}]_1 q_j \leftrightarrow [\mathrm{I}]_2 q_j) \wedge \\ \left(\left(\neg [\mathrm{I}]_2 q_{i+1} \wedge \bigwedge_{j=1}^{i} [\mathrm{I}]_2 q_j \right) \vee \left([\mathrm{I}]_2 q_{i+1} \wedge \bigwedge_{j=1}^{i} \neg [\mathrm{I}]_2 q_j \right) \right) \Bigg) \end{aligned} \tag{A.3}$$

Notice that $|\varphi| = \mathcal{O}(N^2)$.[1] Take any $(\mathcal{M}, w)$ such that $(\mathcal{M}, w) \vDash \varphi$. The formula INIT enforces that the value of M_1 at the initial world w is 0, that is $M_1(w) = (0, \dots, 0)_2$, and the value of M_2 at the initial world w is 1, that is $M_2(w) = (0, \dots, 0, 1)_2$. The formulas INC_0 and $\mathsf{INC}_1(i)$, for $1 \leq i < N$, enforce that at each world $v \in {R^{\mathrm{B}}_{\{1,2\}}}^{+}(w)$, $M_1(v) \leq M_2(v) \leq M_1(v)+1$. More precisely, the formula INC_0 enforces that in the case of the least significant bit of $M_1(v)$

[1] Formula φ could be also constructed with use of operators $[\mathrm{G}]_1$ and $[\mathrm{G}]_2$ instead of $[\mathrm{I}]_1$ and $[\mathrm{I}]_2$.

being 0, while the formula $\mathsf{INC}_1(i)$ enforced that in the case of the $i+1$ bit being 0 and the bits from i to 1 being 1.

Mixed axioms **BI4** and **BI5** guarantee that for any world $v \in {R^{\mathrm{B}}_{\{1,2\}}}^{+}(w)$ and any world $u \in R^{\mathrm{B}}_1(v)$, $M_1(v) = M_1(u)$ and for any world $u \in R^{\mathrm{B}}_2(v)$, $M_2(v) = M_2(u)$. Thus if there exists a world $u \in {R^{\mathrm{B}}_{\{1,2\}}}^{+}(w)$ such that $M_1(u) = (1, \dots, 1)_2$, then for each $0 < x \leq 2^N - 1$ there must exist a world $v \in {R^{\mathrm{B}}_{\{1,2\}}}^{+}(w)$ such that $M_1(v) = x$ and $u \in {R^{\mathrm{B}}_{\{1,2\}}}^{+}(v)$. Hence if φ is satisfied in $(\mathcal{M}, w)$, then $(\mathcal{M}, w)$ must contain exponentially long, with respect to $|\varphi|$, sequence of pairwise different worlds.

To see that $|\varphi|$ is satisfiable, take the following model $\mathcal{M} = (W, \{R^O_j : j \in \{1,2\},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}, Val\})$, where

- $W = \{s_1, \dots, s_K\} \cup \{t_0, \dots, t_K\} \cup \{v_1, \dots, v_N\} \cup \{u_1, \dots, u_N\}$, where $K = 2^N - 1$,
- $R^{\mathrm{B}}_1(s_k) = \{s_k\}$ and $R^{\mathrm{B}}_2(t_k) = \{t_k\}$, for $1 \leq k \leq K$,
- $R^{\mathrm{B}}_1(t_k) = \{s_{k+1}\}$ and $R^{\mathrm{B}}_2(s_k) = \{t_k\}$, for $1 \leq k \leq K-1$,
- $R^{\mathrm{B}}_1(t_K) = \{t_K\}$, $R^{\mathrm{B}}_1(v_k) = R^{\mathrm{B}}_2(v_k) = \{v_k\}$, $R^{\mathrm{B}}_1(u_k) = R^{\mathrm{B}}_2(u_k) = \{u_k\}$, for $1 \leq k \leq N$,
- $R^{\mathrm{I}}_1(s_1) = \{w_1(0), \dots, w_N(0)\}$ and $R^{\mathrm{I}}_1(t_k) = R^{\mathrm{I}}_1(s_{k+1}) = \{w_1(k), \dots, w_N(k)\}$, for all $1 \leq k \leq K$,
- $R^{\mathrm{I}}_2(s_k) = R^{\mathrm{I}}_2(t_k) = \{w_1(k), \dots, w_N(k)\}$, for all $1 \leq k \leq K$,
- $R^{\mathrm{I}}_1(v_k) = R^{\mathrm{B}}_2(v_k) = \{v_k\}$, $R^{\mathrm{I}}_1(u_k) = R^{\mathrm{I}}_2(u_k) = \{u_k\}$, for $1 \leq k \leq N$,
- $R^{\mathrm{G}}_j = \varnothing$, for $j \in \{1,2\}$,
- $Val(q_j, v_j) = 1$, $Val(q_j, u_j) = 0$, for all $1 \leq j \leq N$, and $Val(q, v) = 0$ on all the remaining arguments,

and

$$w_i(k) = \begin{cases} v_i & \text{if the value of } i\text{-th bit (counting from the least significant bit) in binary} \\ & \text{representation of } k \text{ is } 1 \\ u_i & \text{otherwise.} \end{cases}$$

It is easy to see that $\mathcal{M}$ is a TeamLog model and that $(\mathcal{M}, v_0) \vDash \varphi$. □

Proof of Fact 6.10, page 78. To proof the fact we will use induction over m. Suppose that $m = 1$. Since

$$X^n_1 = B + \sum_{i=1}^{n} X^i_0 = B + \sum_{i=1}^{n} (2i+1) = B + (n+2)n = B\binom{n}{0} + (n+2)\binom{n}{1},$$

so the claim holds for any $n \geq 1$. For the induction take $m > 1$ and suppose that the claim holds for $m-1$ and any $n \geq 1$. Then

$$\begin{aligned} X^n_m = B + \sum_{i=1}^{n} X^i_{m-1} &= B + \sum_{i=1}^{n} \left[B\binom{i+m-2}{m-2} + (n+2)\binom{i+m-2}{m-1} \right] \\ &= B\sum_{i=0}^{n} \binom{i+m-2}{m-2} + (n+2)\sum_{i=1}^{n} \binom{i+m-2}{m-1} = \\ &= B\sum_{j=m-2}^{n+m-2} \binom{j}{m-2} + (n+2)\sum_{j=m-1}^{n+m-2} \binom{j}{m-1} \end{aligned}$$

From the properties of binomial coefficients we know that

$$\sum_{j=k}^{n} \binom{j}{k} = \binom{n+1}{k+1},$$

thus

$$X_m^n = B\binom{n+m-1}{m-1} + (n+2)\binom{n+m-1}{m},$$

for all $n \geq 1$. $\square$

Proof of Claim 6.14, page 81. Take any $d \geq D$. Notice that if $d > \mathrm{dep}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$, then points (i) and (ii) hold for it. Also if $d \geq \mathrm{dep}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$, then, since $\mathrm{dep}\Big(\widehat{\mathrm{Gr}}(L(s))\Big) = \mathrm{dep}(\mathrm{Gr}(L(s)))$ and $\mathrm{dep}\Big(\widehat{\mathrm{Gr}}(L(t))\Big) = \mathrm{dep}(\mathrm{Gr}(L(t)))$, so point (iii) holds for it as well.

For $d \leq \mathrm{dep}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$ we will use induction, starting with maximal value of d. So suppose that $d = \mathrm{dep}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$. As we observed above, point (iii) holds for d and we need to show points (i) and (ii) only. For point (i), assume that $j \notin \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d\Big)$. Then there must be a formula $[\mathrm{B}]_G^+\psi \in \neg\widehat{\mathrm{Gr}}(L(t))$ with $\mathrm{dep}([\mathrm{B}]_G^+\psi) = d$ and $j \notin G$. Hence either there is a formula $[\mathrm{B}]_j\xi \in \neg L(s)$, such that $[\mathrm{B}]_G^+\psi \in \neg\mathrm{Sub}(\xi)$ or there is a formula $[\mathrm{B}]_j\xi \in \neg L(t)$ such that $[\mathrm{B}]_G^+\psi \in \neg\mathrm{Sub}(\xi)$ and ξ was added during [B]-expanded tableau formation. The first case is impossible, as $\mathrm{dep}([\mathrm{B}]_j\xi) > \mathrm{dep}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$ and $\mathrm{dep}([\mathrm{B}]_j\xi) > \mathrm{dep}(\mathrm{Ind}(L(s)) \sqcap j)$ (c.f. proof of Claim 6.7). The second case is impossible as well, for either $[\mathrm{B}]_j\xi \in \neg L(s)$ which, as we shown above, is not possible, or there is a formula $[\mathrm{B}]_j\zeta \in \neg L(s)$ such that $[\mathrm{B}]_j\xi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\zeta)$. This case is impossible by analogous arguments to those used for the previous case. Thus it must be that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d\Big)$. For point (ii), assume that $\omega \notin \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d\Big)$. Then there must be a formula $[\mathrm{I}]_G^+\psi \in \neg\widehat{\mathrm{Gr}}(L(t))$ with $\mathrm{dep}([\mathrm{I}]_G^+\psi) = d$. Hence there must be a formula $[\mathrm{B}]_j\xi \in \neg L(s)$ such that $[\mathrm{I}]_G^+\psi \in \neg\mathrm{Sub}(\xi)$. Again this is impossible, as $\mathrm{dep}([\mathrm{B}]_j\xi) > \mathrm{dep}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$ and $\mathrm{dep}([\mathrm{B}]_j\xi) > \mathrm{dep}(\mathrm{Ind}(L(s)) \sqcap j)$. Thus it must be that $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d\Big)$.

For the induction step, suppose that $d < \mathrm{dep}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$. For point (iii) notice that if $[\mathrm{B}]_G^+\xi \in \neg L(s)$ and $\zeta \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$ and $\mathrm{dep}(\psi) \geq D$, then $\mathrm{dep}([\mathrm{B}]_G^+\xi) \geq d+1$. Moreover since, by point (i), $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d+1\Big)$ so, by property **PB1** it holds that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, d+1\Big)$ and so $j \in G$. Thus if $[\mathrm{B}]_G^+\xi \in \widehat{\mathrm{Gr}}(L(s))$, then $[\mathrm{B}]_G^+\xi \in L(t)$ and if $\neg[\mathrm{B}]_G^+\xi \in \widehat{\mathrm{Gr}}(L(s))$, then $[\mathrm{B}]_G^+\xi \in \neg L(t)$, by condition **PB2**. Since $j \in G$ and $\mathrm{dep}([\mathrm{B}]_G^+\xi) > d$ so, by point (i) of Lemma 6.11, $[\mathrm{B}]_G^+\xi \in \neg\widehat{\mathrm{Gr}}(L(t))$. Hence it must be that $\psi \notin \mathrm{Ind}(L(t))$ and $\psi \notin \widehat{\mathrm{Gr}}(L(t))$.

For point (i) assume that $j \notin \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d\Big)$. Then there must be a formula $[\mathrm{B}]_G^+\psi \in \neg\widehat{\mathrm{Gr}}(L(t))$, with $\mathrm{dep}([\mathrm{B}]_G^+\psi) = d$ and $j \notin G$. By the induction hypothesis it holds that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d'+1\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d'+1\Big)$, for all $d' \geq d$. Moreover, by properties **PI1** and **PB1** it holds that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, d'+1\Big)$ and

$\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, d'+1\Big)$, for all $d' \geq d$. Thus, by Corollary 6.12 there exists a formula $[\mathrm{B}]^+_H\xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ such that $[\mathrm{B}]^+_G\psi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$ (notice that since $j \notin G$, so neither point (iv) nor point (v) of Corollary 6.12 can apply here). Then, by point (iii) it holds that $[\mathrm{B}]^+_G\psi \notin \neg\widehat{\mathrm{Gr}}(L(t))$ which contradicts our assumptions. Hence it must be that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d\Big)$.

For point (ii) assume that $\omega \notin \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d\Big)$. Then there must be a formula $[\mathrm{I}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(t))$ with $\mathrm{dep}([\mathrm{I}]^+_G\psi) = d$. By arguments similar to those used above, it can be shown that there must be a formula $[\mathrm{B}]^+_H\xi \in \neg\widehat{\mathrm{Gr}}(L(t))$ such that $[\mathrm{I}]^+_G\psi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$, which contradicts the assumption that $[\mathrm{I}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(t))$. Hence it must be that $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d\Big)$. $\square$

Proof of Claim 6.15, page 82. *Point (i)*

For the fact that $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, D\Big) \subseteq \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, D\Big)$ assume that the opposite holds. Then there must exist a formula $[\mathrm{B}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(t))$ such that $\mathrm{dep}([\mathrm{B}]^+_G\psi) = D$ and $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, D\Big) \nsubseteq G$. Notice that by point (i) of Claim 6.14 it holds that $j \in G$.

By points (i) and (ii) of Claim 6.14 and properties **PB1** and **PI1** it holds that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, d\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, d\Big)$, for all $d > D$. Thus, by Corollary 6.12, either there exists a formula $[\mathrm{B}]^+_H\xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ such that $[\mathrm{B}]^+_G\psi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$, or $[\mathrm{B}]^+_G\psi \in \neg L(s)$.

The first case is impossible, as it implies that $j \in H$ and so it violates modal context restriction $\mathbf{R}_1$. Thus it must be that the second case holds and, by the fact that $j \in G$ and by point (i) of Lemma 6.11 it must be that $[\mathrm{B}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(s))$. But then it must hold that $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, D\Big) \subseteq G$, which contradicts our assumptions. Hence this case is impossible as well and it must be that $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, D\Big) \subseteq \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, D\Big)$.

For the fact that $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, D\Big) \subseteq \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, D\Big)$ notice that, by point (ii) of Claim 6.14 it holds that $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, D\Big)$ and so $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, D\Big) = \mathcal{A} \cup \{\omega\}$. Hence it holds that $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, D\Big) \subseteq \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, D\Big)$.

Point (ii)

Assume that the opposite holds. Then there exists a formula $\psi \in \mathrm{Ind}(L(t)) \setminus (L(t) \sqcap j)$ with $\mathrm{dep}(\psi) \geq D$. By points (i) and (ii) of Claim 6.14 and properties **PB1** and **PI1** it holds that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, d\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, d\Big)$, for all $d > D$. Thus, by Corollary 6.12, there exists a formula $[\mathrm{B}]^+_H\xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ with $\psi \in \neg\mathrm{OT}_{[\mathrm{B}]}(\xi)$. This is impossible as, by point (iii) of Claim 6.14, it implies that $\psi \notin \mathrm{Ind}(L(t))$ which contradicts our assumptions. Hence it must be that either $\mathrm{dep}(\mathrm{Ind}(L(t)) \setminus (L(t) \sqcap j)) < D$ or $\mathrm{Ind}(L(t)) \setminus (L(t) \sqcap j) = \varnothing$.

Point (iii)

Take any formula $\neg[\mathrm{B}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(t))$ with $\mathrm{dep}([\mathrm{B}]^+_G\psi) \geq D$. By point (i) of Claim 6.14 it must be that $j \in G$. By points (i) and (ii) of Claim 6.14 and properties **PB1** and **PI1**, it holds that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, d\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, d\Big)$, for all $d > D$. Thus, by Corollary 6.12, either there exists a formula $[\mathrm{B}]^+_H\xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ such that $[\mathrm{B}]^+_G\psi \in \neg\mathrm{PT}(\xi)$, or $\neg[\mathrm{B}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(s))$, $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(s)$ and t is a $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-Successor of t. The first case is impossible as $j \in H$ and it would violate modal context restriction $\mathbf{R}_1$. Thus it must be that the second case holds. This implies, in particular, that there can be at most one formula of the form $\neg[\mathrm{B}]^+_G\psi$ in $\widehat{\mathrm{Gr}}(L(t)) \sqcap \neg[\mathrm{B}]^+$ with $\mathrm{dep}(\neg[\mathrm{B}]^+_G\psi) \geq D$. $\square$

Proof of Claim 6.16, page 82. Let $k \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\Big)$. If $j = k$, then, by Lemma 6.2, it holds that $L^{[\mathrm{B}]_j}(s) = L^{[\mathrm{B}]_k}(t)$ and so the claim holds in this case. Suppose that $j \neq k$.

For point (i), let $\psi \in L(s)/[\mathrm{B}]_j$. Then there exists a formula $[\mathrm{B}]_j\psi \in L(s)$ and, by point (ii) of Lemma 6.11, $[\mathrm{B}]_j\psi \in \neg\widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$. Thus either there exists a formula $[\mathrm{B}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(s))$ or ψ is of the form $[\mathrm{B}]^+_G\xi$ and $\psi \in \neg\widehat{\mathrm{Gr}}(L(s))$.

Suppose that the first case holds. If $\neg[\mathrm{B}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(s))$, then $\neg[\mathrm{B}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(t))$, by property **PB2**. Otherwise $[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(s)$, as s is a state and $L(s)$ is a closed propositional tableau. Thus $[\mathrm{B}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(t))$, as $j \in G$, $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\Big)$ and point (i) of Lemma 6.11 applies. Since $[\mathrm{B}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(t))$, $k \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\Big)$ and $L(t)$ is a [B]-expanded closed propositional tableau, so either $\psi \in L(t)/[\mathrm{B}]_k$ or $\neg[\mathrm{B}]_k\psi \in \widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(t))\Big)$.

Suppose that the second case holds, that is ψ is of the form $[\mathrm{B}]^+_G\xi$ and $\psi \in \neg\widehat{\mathrm{Gr}}(L(s))$. Then, by arguments analogous to those used for the first case, it holds that $[\mathrm{B}]^+_G\xi \in \neg\widehat{\mathrm{Gr}}(L(t))$ and the point holds by the fact that t is a state and $L(t)$ is a [B]-expanded tableau.

For point (ii) we will show first that $[\mathrm{B}]_k\psi \in \neg L(t)$ implies $[\mathrm{B}]_j\psi \in \neg\widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$. Notice that, by point (ii) of Claim 6.15, it must be that $\mathrm{Ind}(L(t)) \setminus (L(t) \sqcap j) = \varnothing$ and since $j \neq k$, so $\mathrm{Ind}(L(t)) \sqcap k = \varnothing$. Now, suppose that $[\mathrm{B}]_k\psi \in \neg L(t)$. Since $k \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\Big)$ so, by point (ii) of Lemma 6.11, is holds that $[\mathrm{B}]_k\psi \in \widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(t))\Big)$. By points (i) and (ii) of Claim 6.14 it holds that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+\Big)$. Hence, since t is a state and $L(t)$ is a closed, fully expanded and $[O]^+$-expanded tableau, so it holds that $[\mathrm{B}]_j\psi \in \neg\widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(t))\Big)$. Then, by Lemma 6.2, it holds that $[\mathrm{B}]_j\psi \in \neg L(s)$. Moreover, by the fact that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t), [\mathrm{B}]^+)\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+\Big)$ and by properties **PB1** and **PI1** it holds that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+\Big)$. Thus, by Lemma 6.11, it holds that $[\mathrm{B}]_j\psi \in \neg\widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$. Now, let $\psi \in L(t)/[\mathrm{B}]_k$. Then there is a formula $[\mathrm{B}]_k\psi \in L(t)$ and, by what we shown above, either $[\mathrm{B}]_j\psi \in \widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$ and, consequently, $\psi \in L^{[\mathrm{B}]_j}(s)$ or $\neg[\mathrm{B}]_j\psi \in \widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$. $\square$

Proof of Claim 6.17, page 82. Before we start showing points (i) and (ii), notice that, by Claim 6.14 we have $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+\Big)$. Moreover, by properties **PB1** and **PI1**, we have also $\{j, k\} \subseteq \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+\Big)$.

For point (i), let $[O]_j\psi \in L(s)$. Since $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+\Big)$, so, by point (iii) of Lemma 6.11, $[O]_j\psi \in \neg L(s)$ implies $O = \mathrm{B}$ and $[O]_j\psi \in \neg\widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(s))\Big)$. Using arguments similar to those used in proof of Claim 6.16 it follows that $[O]_j\psi \in \neg\widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(t))\Big)$. Now, since $k \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\Big)$ so $[O]_k\psi \in \widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(t))\Big)$ and so $[O]_k\psi \in \neg L(t)$.

For point (ii), let $[O]_k\psi \in L(t)$. As we observed in proof of Claim 6.16, by point (ii) of Claim 6.15, it must be that $\mathrm{Ind}(L(t)) \sqcap k = \varnothing$. Since $k \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+\Big)$, so, by point (iii) of Lemma 6.11, $[O]_k\psi \in \neg L(t)$ implies $O = \mathrm{B}$ and $[O]_k\psi \in \neg\widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(t))\Big)$. Since $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\Big)$, so $[O]_j\psi \in \neg\widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(L(t))\Big)$. Thus $[O]_j\psi \in \neg L(t)$ and, by Lemma 6.2, $[O]_j\psi \in \neg L(s)$. □

Proof of Lemma 6.21, page 85. For point 1, let $[\mathrm{I}]^+_G\varphi \in \Phi$ and let $(\mathcal{M}, w)$ be a satisfying pair for $\bigwedge \Phi$ with minimal satisfying sequence $v_0, \ldots, v_n$ for $[\mathrm{I}]^+_G\varphi$ such that $n \geq 2$. Since $n \geq 2$ so $(\mathcal{M}, v_1) \vDash \neg[\mathrm{I}]^+_G\varphi$ and, by Lemma 4.3, $(\mathcal{M}, v_1)$ is a satisfying pair for $\bigwedge \Phi^{\neg[\mathrm{I}]_{j_1}}([\mathrm{I}]^+_G\varphi)$. What remains to be shown is that $(\mathcal{M}, v_1)$ is a satisfying pair for $\bigwedge \Phi^{\neg[\mathrm{I}]_{j_1}}([\mathrm{I}]^+_G\varphi)$ with minimal satisfying sequence for $\neg[\mathrm{I}]^+_G\varphi$ and that $v_1, \ldots, v_n$ is a minimal satisfying sequence for $\neg[\mathrm{I}]^+_G\varphi$ in $(\mathcal{M}, v_1)$. Assume that there exists a model $(\mathcal{M}', v_1')$ for $\bigwedge \Phi^{\neg[\mathrm{I}]_{j_1}}([\mathrm{I}]^+_G\varphi)$ with satisfying sequence $v_1', \ldots, v_m'$ (where $v_i' \in R^{\mathrm{I}}_{j_i'}(v_{i-1}')$) for $\neg[\mathrm{I}]^+_G\varphi$ shorter than $n-1$. Then a satisfying pair for $\bigwedge \Phi$ can be constructed with shorter satisfying sequence for $\neg[\mathrm{I}]^+_G\varphi$ than $v_0, \ldots, v_n$. In the construction we will assume that $W' \cap W = \varnothing$. In the case of $W' \cap W \neq \varnothing$ a copy of $\mathcal{M}'$ could be considered instead of $\mathcal{M}'$. Let $\mathcal{M}'' = (W'', \{R''^O_j : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}, Val''\})$ be defined as follows:

- $W'' = W \cup W' \cup W^{\mathrm{B}}_{j_1} \cup \{w''\}$, where $W^{\mathrm{B}}_{j_1}$ is the set of copies of worlds from $R^{\mathrm{B}}_{j_1}(w)$. Given $u \in R^{\mathrm{B}}_{j_1}(w)$ we will use u^{c} to refer to the element in $W^{\mathrm{B}}_{j_1}$ which is the copy of u. Also, given $u \in W^{\mathrm{B}}_{j_1}$ we will use u^{o} to refer to the element in $R^{\mathrm{B}}_{j_1}(w)$ which u is the copy of.
- $$Val''(p,u) = \begin{cases} Val(p,u), & \text{if } u \in W, \\ Val(p,w), & \text{if } u = w'', \\ Val'(p,u), & \text{if } u \in W', \\ Val(p,u^{\mathrm{o}}), & \text{if } u \in W^{\mathrm{B}}_{j_1}, \end{cases}$$

  for all $p \in \mathcal{P}$ and $u \in W''$.
- $R''^O_j(u) = R^O_j(u)$, for all $u \in W$, $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $j \in \mathcal{A}$.
- $R''^O_j(u) = R'^O_j(u)$, for all $u \in W'$, $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $j \in \mathcal{A}$.
- $R''^O_j(u) = R^O_j(u^{\mathrm{o}})$, for all $u \in W^{\mathrm{B}}_{j_1}$, $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $j \in \mathcal{A} \setminus \{j_1\}$.
- $R''^O_j(w'') = R^O_j(w)$, for all $u \in W^{\mathrm{B}}_{j_1}$, $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $j \in \mathcal{A} \setminus \{j_1\}$.
- $R''^{\mathrm{B}}_{j_1}(u) = W^{\mathrm{B}}_{j_1}$, for all $u \in W^{\mathrm{B}}_{j_1} \cup \{w''\}$.
- $R''^{\mathrm{G}}_{j_1}(u) = R^{\mathrm{G}}_{j_1}(w)$, for all $u \in W^{\mathrm{B}}_{j_1} \cup \{w''\}$.
- $R''^{\mathrm{I}}_{j_1}(u) = R^{\mathrm{I}}_{j_1}(w) \cup \{v_1'\}$, for all $u \in W^{\mathrm{B}}_{j_1} \cup \{w''\}$.

Roughly speaking, the construction extends model $\mathcal{M}$ with $\mathcal{M}'$ adding a new world w'' and copies of worlds from $R^{\mathrm{B}}_{j_1}(w)$, so that accessibility relations of $\mathcal{M}$ and $\mathcal{M}'$ remain unchanged within $\mathcal{M}''$. A diagramatic sketch of the construction is presented in Figure A.1.

It is easy to check that $\mathcal{M}''$ is a TeamLog model and that $(\mathcal{M}'', v_1') \vDash \bigwedge \Phi^{\neg[\mathrm{I}]_{j_1}}([\mathrm{I}]^+_G\varphi)$. Thus what remains to be shown is that $(\mathcal{M}'', w'') \vDash \bigwedge \Phi$, and it will follow that $w'', v_1', \ldots, v_m'$ is a satisfying sequence for $\neg[\mathrm{I}]^+_G\varphi$ in $(\mathcal{M}'', w'')$. To show that $(\mathcal{M}'', w'') \vDash \bigwedge \Phi$ we will show that, for all $\psi \in \Phi$, $(\mathcal{M}'', w'') \vDash \psi$. If $\psi \in \neg\mathcal{P}$ or ψ is of the form $[O]_j\xi$ or $\neg[O]_j\xi$ with either $O = \mathrm{G}$ or $O \in \{\mathrm{B}, \mathrm{I}\}$ and $j \neq j_1$, then it is easy to see that $(\mathcal{M}'', w'') \vDash \psi$, as $Val''(w'') = Val(w)$,

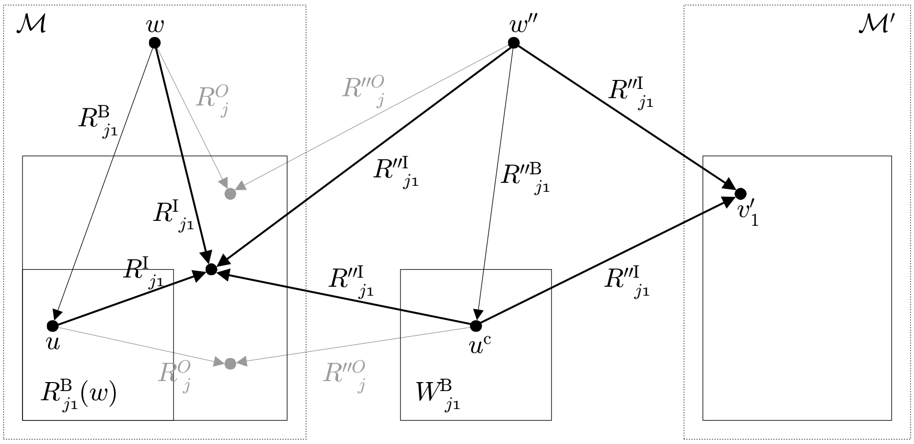

Figure A.1: Construction of a satisfying pair $(\mathcal{M}'', w'')$ for $\bigwedge \Phi$ with satisfying sequence for $\neg[\mathrm{I}]^+_G\varphi$ shorter than $v_0, \dots, v_n$.

for all $p \in \mathcal{P}$, and $R''^{O}_j(w'') = R^O_j(w)$. If ψ is of the form $\neg[\mathrm{I}]_{j_1}\xi$, then there must exist $u \in R^{\mathrm{I}}_{j_1}(w)$ such that $(\mathcal{M}, u) \vDash \neg\xi$ and since $u \in R''^{\mathrm{I}}_{j_1}(w'')$ and worlds and relations of $\mathcal{M}$ are not changed within $\mathcal{M}''$, so $(\mathcal{M}'', w'') \vDash \neg[\mathrm{I}]_{j_1}\xi$. If ψ is of the form $[\mathrm{I}]_{j_1}\xi$, then $(\mathcal{M}', v'_1) \vDash \xi$, as $\xi \in \Phi^{\neg[\mathrm{I}]_{j_1}}([\mathrm{I}]^+_G\varphi)$, and $(\mathcal{M}, u) \vDash \xi$, for all $u \in R^{\mathrm{I}}_{j_1}(w)$. Moreover worlds and relations of $\mathcal{M}$ and $\mathcal{M}'$ are not changed within $\mathcal{M}''$, so $(\mathcal{M}'', v'_1) \vDash \xi$ and $(\mathcal{M}'', u) \vDash \xi$, for all $u \in R^{\mathrm{I}}_{j_1}(w)$. Thus $(\mathcal{M}'', w'') \vDash [\mathrm{I}]_{j_1}\xi$.

Suppose now that ψ is of the form $\neg[\mathrm{B}]_{j_1}\xi$ or $[\mathrm{B}]_{j_1}\xi$. We will show first that, for all $\zeta \in \neg\mathrm{OT}_{[\mathrm{B}]}(\Phi/[\mathrm{B}]_{j_1})$ and $u \in W^{\mathrm{B}}_{j_1}$, $(\mathcal{M}, u^{\mathrm{o}}) \vDash \zeta$ implies $(\mathcal{M}'', u) \vDash \zeta$. If $\zeta \in \neg\mathcal{P}$ or ζ is of the form $[O]_j\eta$ or $\neg[O]_j\eta$ with $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $j \neq j_1$, then it is easy to see that the claim holds, by similar arguments to those used in the case of $(\mathcal{M}'', w'')$. Suppose that ζ is of the form $\neg[O]_{j_1}\eta$ or $[O]_{j_1}\eta$ with $O \in \{\mathrm{G}, \mathrm{I}\}$. For the first case, suppose that $(\mathcal{M}, u^{\mathrm{o}}) \vDash \neg[O]_{j_1}\eta$. Then there must exist $t \in R^O_{j_1}(u^{\mathrm{o}})$ such that $(\mathcal{M}, t) \vDash \neg\eta$. Moreover, $(\mathcal{M}'', t) \vDash \neg\eta$, as worlds and relations of $\mathcal{M}$ are not changed within $\mathcal{M}''$. Thus, since $R^O_{j_1}(u^{\mathrm{o}}) \subseteq R''^{O}_{j_1}(u)$, so $(\mathcal{M}'', u) \vDash \neg[O]_{j_1}\eta$. In the second case, it must be that $[O]_{j_1}\zeta \in \Phi$. This is because, by the fact that Φ is a [B]-expanded tableau, $[O]_{j_1}\zeta \in \neg\Phi$ and, consequently, $[O]_{j_1}\zeta \in \Phi^{[\mathrm{B}]_{j_1}}$. Now, since $(\mathcal{M}, u^{\mathrm{o}}) \vDash [O]_{j_1}\eta$ and since, by Lemma 4.3, $(\mathcal{M}, u^{\mathrm{o}}) \vDash \bigwedge \Phi^{[\mathrm{B}]_{j_1}}$, so it must be that $[O]_{j_1}\eta \in \Phi$. As was shown above, this implies that $(\mathcal{M}'', w'') \vDash [O]_{j_1}\eta$ and so, for all $t \in R''^{O}_{j_1}(w'')$, $(\mathcal{M}'', t) \vDash \eta$. Thus, by the fact that $R''^{O}_{j_1}(u) \subseteq R''^{O}_{j_1}(w'')$, it holds that $(\mathcal{M}'', u) \vDash [O]_{j_1}\eta$. Suppose that ζ is of the form $[O]^+_G\eta$ or $\neg[O]^+_G\eta$ with $O \in \{\mathrm{B}, \mathrm{I}\}$ and $j_1 \notin G$ in the case of $O = \mathrm{B}$. In the first case it must be that, for all $j \in G$, $(\mathcal{M}, u^{\mathrm{o}}) \vDash [O]_j\eta$ and $(\mathcal{M}, u^{\mathrm{o}}) \vDash [O]_j[O]^+_G\eta$. Thus, by the fact that $\{[O]_j\eta, [O]_j[O]^+_G\eta\} \subseteq \neg\mathrm{OT}_{[\mathrm{B}]}(\Phi/[\mathrm{B}]_{j_1})$ and by what was shown above, $(\mathcal{M}'', u) \vDash [O]_j\eta$ and $(\mathcal{M}'', u) \vDash [O]_j[O]^+_G\eta$. Hence for all $t \in R''^{O+}_{G}(u)$ it holds that $(\mathcal{M}'', t) \vDash \eta$ and so $(\mathcal{M}'', u) \vDash [O]^+_G\eta$. In the second case, there must exists $j \in G$ such that either $(\mathcal{M}, u^{\mathrm{o}}) \vDash \neg[O]_j\xi$ or $(\mathcal{M}, u^{\mathrm{o}}) \vDash \neg[O]_j[O]^+_G\xi$. Thus, by the fact that $\{\neg[O]_j\eta, \neg[O]_j[O]^+_G\eta\} \subseteq \neg\mathrm{OT}_{[\mathrm{B}]}(\Phi/[\mathrm{B}]_{j_1})$ and by what was shown above, either $(\mathcal{M}'', u) \vDash \neg[O]_j\eta$ or $(\mathcal{M}'', u) \vDash \neg[O]_j[O]^+_G\eta$, respectively, so there must exist $t \in R''^{O+}_{G}(u)$ such that $(\mathcal{M}'', t) \vDash \neg\eta$ and so $(\mathcal{M}'', u) \vDash \neg[O]^+_G\eta$. For the remaining forms of ζ we will use induction on the structure of the formula. Cases of ζ for the induction basis are covered above. For the induction step, suppose that ζ is of either of the form $\neg\neg\eta$, $\eta_1 \wedge \eta_2$ or $\neg(\eta_1 \wedge \eta_2)$. Then $\neg\mathrm{PT}(\zeta) \subseteq \neg\mathrm{OT}_{[\mathrm{B}]}(\Phi/[\mathrm{B}]_{j_1})$, as Φ is a [B]-expanded tableau. Hence it is easy to see, by the induction hypothesis, that if $(\mathcal{M}, u^{\mathrm{o}}) \vDash \zeta$, then $(\mathcal{M}'', u) \vDash \zeta$. Suppose that ζ is of the form $\neg[\mathrm{B}]_{j_1}\eta$ or $[\mathrm{B}]_{j_1}\eta$. In the first case, there must exist $t \in R^{\mathrm{B}}_{j_1}(u^{\mathrm{o}})$ such that $(\mathcal{M}, t) \vDash \neg\eta$. Moreover, by transitivity of $R^{\mathrm{B}}_{j_1}$, $t \in R^{\mathrm{B}}_{j_1}(w)$. Thus $t^{\mathrm{c}} \in W^{\mathrm{B}}_{j_1}$ and, by the induction hypothesis, $(\mathcal{M}'', t^{\mathrm{c}}) \vDash \neg\eta$. Since $t^{\mathrm{c}} \in R''^{\mathrm{B}}_{j_1}(u)$, so $(\mathcal{M}'', u) \vDash \neg[\mathrm{B}]_{j_1}\eta$. In the second case, it must be that for all $t \in R^{\mathrm{B}}_{j_1}(u^{\mathrm{o}})$, $(\mathcal{M}, t) \vDash \eta$. Moreover, by Euclideanity of $R^{\mathrm{B}}_{j_1}$, $R^{\mathrm{B}}_{j_1}(w) \subseteq R^{\mathrm{B}}_{j_1}(u^{\mathrm{o}})$. Thus, by the induction hypothesis, for all $t \in W^{\mathrm{B}}_{j_1}$, $(\mathcal{M}'', t) \vDash \eta$ and so $(\mathcal{M}'', u) \vDash [\mathrm{B}]_{j_1}\eta$. Suppose that ζ is of the form $\neg[\mathrm{B}]^{+}_{G}\eta$ or $[\mathrm{B}]^{+}_{G}\eta$, with $j_1 \in G$. In the first case, there must exist $t \in R^{\mathrm{B}+}_{G}(u^{\mathrm{o}})$ such that $(\mathcal{M}, t) \vDash \neg\eta$. Suppose first that $t \in R^{\mathrm{B}}_{j_1}(w)$. Then there exists $t^{\mathrm{c}} \in W^{\mathrm{B}}_{j_1}$ and, by the induction hypothesis, $(\mathcal{M}'', t^{\mathrm{c}}) \vDash \neg\eta$. Since $t^{\mathrm{c}} \in R''^{\mathrm{B}}_{j_1}(u)$, so $t^{\mathrm{c}} \in R''^{\mathrm{B}+}_{G}(u)$ and $(\mathcal{M}'', u) \vDash \neg[\mathrm{B}]^{+}_{G}\eta$. Secondly, suppose that $t \notin R^{\mathrm{B}}_{j_1}(w)$. Then there must exist $s \in R^{\mathrm{B}}_{j_1}(u^{\mathrm{o}})$ such that $t \in R^{\mathrm{B}}_{j}(s)$ with $j \neq j_1$. Since $s^{\mathrm{c}} \in W^{\mathrm{B}}_{j_1}$, $s^{\mathrm{c}} \in R''^{\mathrm{B}}_{j_1}(u)$ and $t \in R''^{\mathrm{B}}_{j}(s^{\mathrm{c}})$, so $(\mathcal{M}'', u) \vDash \neg[\mathrm{B}]^{+}_{G}\eta$. In the second case, it must be that for all $t \in R^{\mathrm{B}}_{G}(u^{\mathrm{o}})^{+}$, $(\mathcal{M}, t) \vDash \eta$. Since, by Euclideanity of $R^{\mathrm{B}}_{j_1}$, $R^{\mathrm{B}}_{j_1}(w) \subseteq R^{\mathrm{B}}_{j_1}(u^{\mathrm{o}})$ so, by the induction hypothesis, for all $t \in W^{\mathrm{B}}_{j_1}$, $(\mathcal{M}'', t) \vDash \eta$. Moreover, for all $t \in W^{\mathrm{B}}_{j_1}$ and $j \in G \setminus \{j_1\}$, $R''^{\mathrm{B}}_{j}(t) = R^{\mathrm{B}}_{j}(t^{\mathrm{o}})$. Thus for all $s \in R''^{\mathrm{B}+}_{G}(u)$ it holds that $(\mathcal{M}'', s) \vDash \eta$ and so $(\mathcal{M}'', u) \vDash [\mathrm{B}]^{+}_{G}\eta$.

Now, if ψ is of the form $\neg[\mathrm{B}]_{j_1}\xi$, then there must exist $u \in R^{\mathrm{B}}_{j_1}(w)$ such that $(\mathcal{M}, u) \vDash \neg\xi$. Thus, by what was shown above, $(\mathcal{M}'', u^{\mathrm{c}}) \vDash \neg\xi$ and so $(\mathcal{M}'', w'') \vDash \neg[\mathrm{B}]_{j_1}\xi$. If ψ is of the form $[\mathrm{B}]_{j_1}\xi$, then, for all $u \in R^{\mathrm{B}}_{j_1}(w)$, it holds that $(\mathcal{M}, u) \vDash \xi$ and, by what was shown above, for all $u \in W^{\mathrm{B}}_{j_1}$ it holds that $(\mathcal{M}'', u) \vDash \xi$. Thus $(\mathcal{M}'', w'') \vDash [\mathrm{B}]_{j_1}\xi$. Suppose that ψ is of the form $[O]^{+}_{G}\xi$ or $\neg[O]^{+}_{G}\xi$ with $O \in \{\mathrm{B}, \mathrm{I}\}$. In the first case, it holds that, for all $j \in G$, $[O]_j\xi \in \Phi$ and $[O]_j[O]^{+}_{G}\psi \in \Phi$, as Φ is a closed tableau. Moreover, by what was shown above, for all $j \in G$, $(\mathcal{M}'', w'') \vDash [O]_j\xi$ and $(\mathcal{M}'', w'') \vDash [O]_j[O]^{+}_{G}\psi$. Hence for all $u \in R''^{O+}_{G}(w'')$ it holds that $(\mathcal{M}'', u) \vDash \xi$ and so $(\mathcal{M}'', w'') \vDash [O]^{+}_{G}\xi$. In the second case, there must exists $j \in G$ such that either $\neg[O]_j\xi \in \Phi$ or $\neg[O]_j[O]^{+}_{G}\xi \in \Phi$, as Φ is a $[O]^{+}$-expanded tableau. Since, by what was shown above, either $(\mathcal{M}'', w'') \vDash \neg[O]_j\xi$ or $(\mathcal{M}'', w'') \vDash \neg[O]_j[O]^{+}_{G}\psi$, respectively, so there must exist $u \in R''^{O+}_{G}(w'')$ such that $(\mathcal{M}'', u) \vDash \neg\xi$ and so $(\mathcal{M}'', w'') \vDash \neg[O]^{+}_{G}\xi$. For the remaining forms of ψ, $(\mathcal{M}'', w'') \vDash \psi$ can be shown by simple induction on the length of the formula using the fact that Φ is a propositional tableau and that cases for the induction basis are already covered above. Thus we have shown that $(\mathcal{M}'', w'') \vDash \bigwedge \Phi$, which contradicts the assumption that $(\mathcal{M}, w)$ is a satisfying pair for Φ with minimal satisfying sequence for $\neg[\mathrm{I}]^{+}_{G}\varphi$. Hence it must be that $(\mathcal{M}, v_1)$ is a satisfying pair for $\Phi^{\neg[\mathrm{I}]_{j_1}}([\mathrm{I}]^{+}_{G}\varphi)$ with minimal satisfying sequence $v_1, \dots, v_n$ for $\neg[\mathrm{I}]^{+}_{G}\psi$.

For point 2, let $[\mathrm{B}]^{+}_{G}\varphi \in \Phi$ and let $(\mathcal{M}, w)$ be a satisfying pair for $\bigwedge \Phi$ with minimal satisfying sequence $v_0, \dots, v_n$ for $[\mathrm{B}]^{+}_{G}\varphi$ such that $n \geq 2$. Since $n \geq 2$ so $(\mathcal{M}, v_1) \vDash \neg[\mathrm{B}]^{+}_{G}\varphi$ and, by Lemma 4.3, $(\mathcal{M}, v_1)$ is a satisfying pair for $\bigwedge \Phi^{\neg[\mathrm{B}]_{j_1}}([\mathrm{B}]^{+}_{G}\varphi)$. What remains to be shown is that $(\mathcal{M}, v_1)$ is a satisfying pair for $\bigwedge \Phi^{\neg[\mathrm{B}]_{j_1}}([\mathrm{B}]^{+}_{G}\varphi)$ with minimal satisfying sequence for $\neg[\mathrm{B}]^{+}_{G}\varphi$ and that $v_1, \dots, v_n$ is a minimal satisfying sequence for $\neg[\mathrm{B}]^{+}_{G}\varphi$ in $(\mathcal{M}, v_1)$. Assume that there exists a model $(\mathcal{M}', v'_1)$ for $\bigwedge \Phi^{\neg[\mathrm{B}]_{j_1}}([\mathrm{B}]^{+}_{G}\varphi)$ with satisfying sequence $v'_1, \dots, v'_m$ (where $v'_i \in R^{\mathrm{B}}_{j'_i}(v'_{i-1})$) for $\neg[\mathrm{B}]^{+}_{G}\varphi$ shorter than $n - 1$. Then a satisfying pair for $\bigwedge \Phi$ can be constructed with shorter satisfying sequence for $\neg[\mathrm{B}]^{+}_{G}\varphi$ than $v_0, \dots, v_n$. Like in the case of point 1, we will assume that $W \cap W' = \varnothing$. Let $\mathcal{M}'' = (W'', \{R''^{O}_{j} : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}, Val''\})$ be defined as follows:

- $W'' = W \cup W' \cup W^{\mathrm{B}}_{j_1} \cup W'^{\mathrm{B}}_{j_1} \cup \{w''\}$, where $W^{\mathrm{B}}_{j_1}$ is the set of copies of worlds from $R^{\mathrm{B}}_{j_1}(w)$ and $W'^{\mathrm{B}}_{j_1}$ is the set of copies of elements from $R'^{\mathrm{B}}_{j_1}(v'_1) \cup \{v'_1\}$. Like in the case of point 1, given $u \in R^{\mathrm{B}}_{j_1}(w) \cup R'^{\mathrm{B}}_{j_1}(v'_1) \cup \{v'_1\}$, we will use u^{c} to refer to the element in $W^{\mathrm{B}}_{j_1} \cup W'^{\mathrm{B}}_{j_1}$ which is the copy of u. Also, given $u \in W^{\mathrm{B}}_{j_1} \cup W'^{\mathrm{B}}_{j_1}$ we will use u^{o} to refer to the element

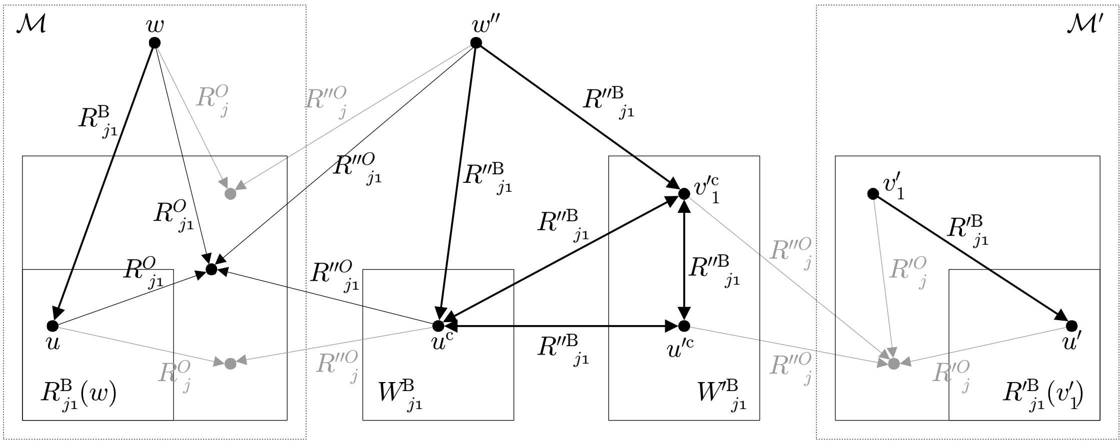

Figure A.2: Construction of a satisfying pair $(\mathcal{M}'', w'')$ for $\bigwedge \Phi$ with satisfying sequence for $\neg[\mathrm{B}]^{+}_{G}\varphi$ shorter than $v_0, \ldots, v_n$.

in $R^{\mathrm{B}}_{j_1}(w) \cup R'^{\mathrm{B}}_{j_1}(v'_1) \cup \{v'_1\}$ which u is the copy of.

- $$Val''(p,u) = \begin{cases} Val(p,u) & \text{if } u \in W, \\ Val(p,w) & \text{if } u = w'', \\ Val'(p,u) & \text{if } u \in W', \\ Val(p,v'_1) & \text{if } u = v''_1, \\ Val(p,u^{\mathrm{o}}) & \text{if } u \in W^{\mathrm{B}}_{j_1}, \\ Val'(p,u^{\mathrm{o}}) & \text{if } u \in W'^{\mathrm{B}}_{j_1}, \end{cases}$$

  for all $p \in \mathcal{P}$ and $u \in W''$.

- $R''^{O}_{j}(u) = R^{O}_{j}(u)$, for all $u \in W$, $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $j \in \mathcal{A}$.
- $R''^{O}_{j}(u) = R'^{O}_{j}(u)$, for all $u \in W'$, $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $j \in \mathcal{A}$.
- $R''^{O}_{j}(u) = R^{O}_{j}(u^{\mathrm{o}})$, for all $u \in W^{\mathrm{B}}_{j_1}$, $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $j \in \mathcal{A} \setminus \{j_1\}$.
- $R''^{O}_{j}(u) = R'^{O}_{j}(u^{\mathrm{o}})$, for all $u \in W'^{\mathrm{B}}_{j_1}$, $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $j \in \mathcal{A} \setminus \{j_1\}$.
- $R''^{O}_{j}(w'') = R^{O}_{j}(w)$, for all $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $j \in \mathcal{A} \setminus \{j_1\}$.
- $R''^{\mathrm{B}}_{j_1}(u) = W^{\mathrm{B}}_{j_1} \cup W'^{\mathrm{B}}_{j_1}$, for all $u \in W^{\mathrm{B}}_{j_1} \cup W'^{\mathrm{B}}_{j_1} \cup \{w''\}$.
- $R''^{O}_{j_1}(u) = R^{O}_{j_1}(w) \cup R'^{O}_{j_1}(v'_1)$, for all $O \in \{\mathrm{G}, \mathrm{I}\}$ and $u \in W^{\mathrm{B}}_{j_1} \cup W'^{\mathrm{B}}_{j_1} \cup \{w''\}$.

Roughly speaking, the construction extends model $\mathcal{M}$ with $\mathcal{M}'$ adding a new world w'' and copies of worlds from $R^{\mathrm{B}}_{j_1}(w)$ and $R'^{\mathrm{B}}_{j_1}(v'_1)$, so that accessibility relations of $\mathcal{M}$ and $\mathcal{M}'$ remain unchanged within $\mathcal{M}''$. A diagramatic sketch of the construction is presented in Figure A.2.

It is easy to check that $\mathcal{M}''$ is a TeamLog model and that $(\mathcal{M}'', v'_1) \models \bigwedge \Phi^{\neg[\mathrm{B}]_{j_1}}([\mathrm{B}]^{+}_{G}\varphi)$. Thus what remains to be shown is that $(\mathcal{M}'', w'') \models \bigwedge \Phi$, and it will follow that $w'', v'^{\mathrm{c}}_1, \ldots, v'_m$ is a satisfying sequence for $\neg[\mathrm{B}]^{+}_{G}\varphi$ in $(\mathcal{M}'', w'')$. To show that $(\mathcal{M}'', w'') \models \bigwedge \Phi$ we will show that, for all $\psi \in \Phi$, $(\mathcal{M}'', w'') \models \psi$. If $\psi \in \neg\mathcal{P}$ or ψ is of the form $[O]_j\xi$ or $\neg[O]_j\xi$ with $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $j \neq j_1$, then it is easy to see that $(\mathcal{M}'', w'') \models \psi$, as $Val''(w'') = Val(w)$, for all $p \in \mathcal{P}$, and $R''^{O}_{j}(w'') = R^{O}_{j}(w)$. If ψ is of the form $\neg[O]_{j_1}\xi$, with $O \in \{\mathrm{G}, \mathrm{I}\}$, then there must exist $u \in R^O_{j_1}(w)$ such that $(\mathcal{M}, u) \vDash \neg\xi$ and since $u \in R''^O_{j_1}(w'')$ and worlds and relations of $\mathcal{M}$ are not changed within $\mathcal{M}''$, so $(\mathcal{M}'', w'') \vDash \neg[O]_{j_1}\xi$. If ψ is of the form $[O]_{j_1}\xi$, with $O \in \{\mathrm{G}, \mathrm{I}\}$, then $(\mathcal{M}, u) \vDash \xi$, for all $u \in R^O_{j_1}(w)$, and $(\mathcal{M}'', u) \vDash \xi$, as worlds and relations of $\mathcal{M}$ are not changed in $\mathcal{M}''$. Moreover, $(\mathcal{M}', v'_1) \vDash [O]_{j_1}\xi$, as $[O]_{j_1}\xi \in \Phi^{\neg[\mathrm{B}]_{j_1}}([\mathrm{B}]^+_G\psi)$, and so $(\mathcal{M}', u) \vDash \xi$, for all $u \in R^O_{j_1}(v'_1)$. Since worlds and relations of $\mathcal{M}'$ are not changed in $\mathcal{M}''$, so $(\mathcal{M}'', u) \vDash \xi$ and so $(\mathcal{M}'', w'') \vDash [O]_{j_1}\xi$.

Suppose now that ψ is of the form $\neg[\mathrm{B}]_{j_1}\xi$ or $[\mathrm{B}]_{j_1}\xi$. We will show first, for all $\zeta \in \neg\mathrm{OT}_{[\mathrm{B}]}(\Phi/[\mathrm{B}]_{j_1})$, that

(i). for all $u \in W^{\mathrm{B}}_{j_1}$, $(\mathcal{M}, u^{\mathrm{o}}) \vDash \zeta$ implies $(\mathcal{M}'', u) \vDash \zeta$,

(ii). for all $u \in W'^{\mathrm{B}}_{j_1}$, $(\mathcal{M}', u^{\mathrm{o}}) \vDash \zeta$ implies $(\mathcal{M}'', u) \vDash \zeta$,

If $\zeta \in \neg\mathcal{P}$ or ζ is of the form $[O]_j\eta$ or $\neg[O]_j\eta$, with $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $j \neq j_1$, then it is easy to see that both points hold, by similar arguments to those used in the case of $(\mathcal{M}'', w'')$. Suppose that ζ is of the form $\neg[O]_{j_1}\eta$ with $O \in \{\mathrm{G}, \mathrm{I}\}$. For point (i), suppose that $(\mathcal{M}, u^{\mathrm{o}}) \vDash \neg[O]_{j_1}\eta$. Then there must exist $t \in R^O_{j_1}(u^{\mathrm{o}})$ such that $(\mathcal{M}, t) \vDash \neg\eta$. Moreover, $(\mathcal{M}'', t) \vDash \neg\eta$, as worlds and relations of $\mathcal{M}$ are not changed within $\mathcal{M}''$. This, together with the fact that $R^O_{j_1}(u^{\mathrm{o}}) \subseteq R''^O_{j_1}(u)$, implies $(\mathcal{M}'', u) \vDash \neg[O]_{j_1}\eta$. Similarly, for point (ii), suppose that $(\mathcal{M}', u^{\mathrm{o}}) \vDash \neg[O]_{j_1}\eta$. Then there must exist $t \in R'^O_{j_1}(u^{\mathrm{o}})$ such that $(\mathcal{M}', t) \vDash \neg\eta$. Moreover, $(\mathcal{M}'', t) \vDash \neg\eta$, as worlds and relations of $\mathcal{M}'$ are not changed within $\mathcal{M}''$. This, together with the fact that $R'^O_{j_1}(u^{\mathrm{o}}) \subseteq R''^O_{j_1}(u)$ implies $(\mathcal{M}'', u) \vDash \neg[O]_{j_1}\eta$. Suppose that ζ is of the form $[O]_{j_1}\eta$ with $O \in \{\mathrm{G}, \mathrm{I}\}$. Then it must be that $[O]_{j_1}\zeta \in \Phi$. This is because, by the fact that Φ is a [B]-expanded tableau, $[O]_{j_1}\zeta \in \neg\Phi$ and, consequently, $[O]_{j_1}\zeta \in \neg\Phi^{[\mathrm{B}]_{j_1}}$. In the case of point (i), since $(\mathcal{M}, u^{\mathrm{o}}) \vDash [O]_{j_1}\eta$ and since, by Lemma 4.3, $(\mathcal{M}, u^{\mathrm{o}}) \vDash \bigwedge \Phi^{[\mathrm{B}]_{j_1}}$, so it must be that $[O]_{j_1}\eta \in \Phi$. Similarly, in the case of point (ii), it holds that $(\mathcal{M}', u^{\mathrm{o}}) \vDash [O]_{j_1}\eta$ and since $(\mathcal{M}', v'_1) \vDash \bigwedge \Phi^{\neg[\mathrm{B}]_{j_1}}([\mathrm{B}]^+_G\psi)$ so, by Lemma 4.3, $(\mathcal{M}', u^{\mathrm{o}}) \vDash \bigwedge(\Phi^{\neg[\mathrm{B}]_{j_1}}([\mathrm{B}]^+_G\psi))^{[\mathrm{B}]_{j_1}}$ and so it must be that $[O]_{j_1}\eta \in \Phi$. Now, as was shown above, $(\mathcal{M}'', w'') \vDash [O]_{j_1}\eta$ and so, for all $t \in R''^O_{j_1}(w'')$, $(\mathcal{M}'', t) \vDash \eta$. Thus, by the fact that $R''^O_{j_1}(u) \subseteq R''^O_{j_1}(w'')$, it holds that $(\mathcal{M}'', u) \vDash [O]_{j_1}\eta$. Suppose that ζ is of the form $[O]^+_G\eta$ or $\neg[O]^+_G\eta$ with $O \in \{\mathrm{B}, \mathrm{I}\}$ and $j_1 \notin G$ in the case of $O = \mathrm{B}$. Arguments for points (i) and (ii) are analogous here and we present only those for point (i). In the case of ζ being of the form $[O]^+_G\eta$, it must be that, for all $j \in G$, $(\mathcal{M}, u^{\mathrm{o}}) \vDash [O]_j\eta$ and $(\mathcal{M}, u^{\mathrm{o}}) \vDash [O]_j[O]^+_G\eta$. Thus, by the fact that $\{[O]_j\eta, [O]_j[O]^+_G\eta\} \subseteq \neg\mathrm{OT}_{[\mathrm{B}]}(\Phi/[\mathrm{B}]_{j_1})$ and by what was shown above, $(\mathcal{M}'', u) \vDash [O]_j\eta$ and $(\mathcal{M}'', u) \vDash [O]_j[O]^+_G\eta$. Hence for all $t \in R''^{O+}_G(u)$ it holds that $(\mathcal{M}'', t) \vDash \eta$ and so $(\mathcal{M}'', u) \vDash [O]^+_G\eta$. In the second case, there must exists $j \in G$ such that either $(\mathcal{M}, u^{\mathrm{o}}) \vDash \neg[O]_j\xi$ or $(\mathcal{M}, u^{\mathrm{o}}) \vDash \neg[O]_j[O]^+_G\xi$. Thus, by the fact that $\{\neg[O]_j\eta, \neg[O]_j[O]^+_G\eta\} \subseteq \neg\mathrm{OT}_{[\mathrm{B}]}(\Phi/[\mathrm{B}]_{j_1})$ and by what was shown above, either $(\mathcal{M}'', u) \vDash \neg[O]_j\eta$ or $(\mathcal{M}'', u) \vDash \neg[O]_j[O]^+_G\eta$, respectively. Hence there must exist $t \in R''^{O+}_G(u)$ such that $(\mathcal{M}'', t) \vDash \neg\eta$ and so $(\mathcal{M}'', u) \vDash \neg[O]^+_G\eta$. For the remaining forms of ζ we will use induction on the structure of the formula. Cases of ζ for the induction basis are covered above. For the induction step, suppose that ζ is of either of the form $\neg\neg\eta$, $\eta_1 \wedge \eta_2$ or $\neg(\eta_1 \wedge \eta_2)$. Then $\neg\mathrm{PT}(\zeta) \subseteq \neg\mathrm{OT}_{[\mathrm{B}]}(\Phi/[\mathrm{B}]_{j_1})$, as Φ is a [B]-expanded tableau. Hence it is easy to see, by the induction hypothesis, that points (i) and (ii) hold. Suppose that ζ is of the form $\neg[\mathrm{B}]_{j_1}\eta$. In the case of point (i), there must exist $t \in R^{\mathrm{B}}_{j_1}(u^{\mathrm{o}})$ such that $(\mathcal{M}, t) \vDash \neg\eta$. Moreover, by transitivity of $R^{\mathrm{B}}_{j_1}$, $t \in R^{\mathrm{B}}_{j_1}(w)$. Thus $t^{\mathrm{c}} \in W^{\mathrm{B}}_{j_1}$ and, by the induction hypothesis, $(\mathcal{M}'', t^{\mathrm{c}}) \vDash \neg\eta$. Since $t^{\mathrm{c}} \in R''^{\mathrm{B}}_{j_1}(u)$, so $(\mathcal{M}'', u) \vDash \neg[\mathrm{B}]_{j_1}\eta$. In the case of point (ii), there must exist $t \in R'^{\mathrm{B}}_{j_1}(u^{\mathrm{o}})$ such that $(\mathcal{M}', t) \vDash \neg\eta$. Moreover, by transitivity of $R'^{\mathrm{B}}_{j_1}$, $t \in R'^{\mathrm{B}}_{j_1}(v'_1)$. Thus $t^{\mathrm{c}} \in W'^{\mathrm{B}}_{j_1}$ and, by the induction hypothesis, $(\mathcal{M}'', t^{\mathrm{c}}) \vDash \neg\eta$. Since $t^{\mathrm{c}} \in R''^{\mathrm{B}}_{j_1}(u)$, so $(\mathcal{M}'', u) \vDash \neg[\mathrm{B}]_{j_1}\eta$. Suppose that ζ is of the form $[\mathrm{B}]_{j_1}\eta$. By the arguments analogous to those used in the case of ζ being of the form $[O]_{j_1}\eta$ with $O \in \{\mathrm{G},\mathrm{I}\}$, each of the points (i) and (ii) implies that $[\mathrm{B}]_{j_1}\eta \in \Phi$ and $\{[\mathrm{B}]_{j_1}\eta, \eta\} \subseteq \Phi^{[\mathrm{B}]_{j_1}}$. Thus in the case of each of the points it must hold that $(\mathcal{M}, w) \vDash [\mathrm{B}]_{j_1}\eta$, $(\mathcal{M}', v_1') \vDash [\mathrm{B}]_{j_1}\eta$ and $(\mathcal{M}', v_1') \vDash \eta$. Hence for all $t \in R^{\mathrm{B}}_{j_1}(w)$ it must hold that $(\mathcal{M}, t) \vDash \eta$ and for all $t \in R'^{\mathrm{B}}_{j_1}(v_1') \cup \{v'\}$ it must hold that $(\mathcal{M}', t) \vDash \eta$. Thus, by the induction hypothesis, for all $t \in W^{\mathrm{B}}_{j_1} \cup W'^{\mathrm{B}}_{j_1}$ it holds that $(\mathcal{M}'', t) \vDash \eta$ and so in the case of both point (i) and point (ii) it holds that $(\mathcal{M}'', u) \vDash [\mathrm{B}]_{j_1}\eta$. Suppose that ζ is of the form $\neg[\mathrm{B}]^+_G\eta$, with $j_1 \in G$. Arguments for points (i) and (ii) are analogous here and we present those for point (i) only. Since $(\mathcal{M}, u^{\mathrm{o}}) \vDash \neg[\mathrm{B}]^+_G\eta$, so there must exist $t \in R^{\mathrm{B}^+}_G(u^{\mathrm{o}})$ such that $(\mathcal{M}, t) \vDash \neg\eta$. Suppose first that $t \in R^{\mathrm{B}}_{j_1}(w)$. Then there exists $t^{\mathrm{c}} \in W^{\mathrm{B}}_{j_1}$ and, by the induction hypothesis, $(\mathcal{M}'', t^{\mathrm{c}}) \vDash \neg\eta$. Since $t^{\mathrm{c}} \in R''^{\mathrm{B}}_{j_1}(u)$, so $t^{\mathrm{c}} \in R''^{\mathrm{B}^+}_G(u)$ and $(\mathcal{M}'', u) \vDash \neg[\mathrm{B}]^+_G\eta$. Secondly, suppose that $t \notin R^{\mathrm{B}}_{j_1}(w)$. Then there must exist $s \in R^{\mathrm{B}}_{j_1}(u^{\mathrm{o}})$ such that $t \in R^{\mathrm{B}}_j(s)$ with $j \neq j_1$. Since $s^{\mathrm{c}} \in W^{\mathrm{B}}_{j_1}$, $s^{\mathrm{c}} \in R''^{\mathrm{B}}_{j_1}(u)$ and $t \in R''^{\mathrm{B}}_j(s^{\mathrm{c}})$, so $(\mathcal{M}'', u) \vDash \neg[\mathrm{B}]^+_G\eta$. Suppose that ζ is of the form $[\mathrm{B}]^+_G\eta$, with $j_1 \in G$. Then $[\mathrm{B}]_{j_1}\eta \in \neg\mathrm{OT}_{[\mathrm{B}]}(\Phi/[\mathrm{B}]_{j_1})$ and $[\mathrm{B}]_{j_1}[\mathrm{B}]^+_G\eta \in \neg\mathrm{OT}_{[\mathrm{B}]}(\Phi/[\mathrm{B}]_{j_1})$. By the arguments analogous to those used in the case of ζ being of the form $[O]_{j_1}\eta$, each of the points (i) and (ii) implies that $\{[\mathrm{B}]_{j_1}\eta, [\mathrm{B}]_{j_1}[\mathrm{B}]^+_G\eta\} \subseteq \Phi$ and $\{[\mathrm{B}]_{j_1}\eta, [\mathrm{B}]_{j_1}[\mathrm{B}]^+_G\eta, \eta\} \subseteq \Phi^{[\mathrm{B}]_{j_1}}$. Thus in the case of each of the points (i) and (ii) it must hold that $(\mathcal{M}, w) \vDash [\mathrm{B}]_{j_1}\eta$, $(\mathcal{M}', v_1') \vDash [\mathrm{B}]_{j_1}\eta$ and $(\mathcal{M}', v_1') \vDash \eta$ which, as was shown in the case of ζ being of the form $[\mathrm{B}]_{j_1}\eta$, implies that $(\mathcal{M}'', t) \vDash \eta$, for all $t \in W^{\mathrm{B}}_{j_1} \cup W'^{\mathrm{B}}_{j_1}$. Moreover, in the case of each of the points (i) and (ii), $(\mathcal{M}, w) \vDash [\mathrm{B}]_{j_1}[\mathrm{B}]^+_G\eta$, $(\mathcal{M}', v_1') \vDash [\mathrm{B}]_{j_1}[\mathrm{B}]^+_G\eta$ and $(\mathcal{M}', v_1') \vDash [\mathrm{B}]^+_G\eta$. Hence for all $t \in R^{\mathrm{B}}_{j_1}(w)$ it holds that $(\mathcal{M}, t) \vDash [\mathrm{B}]^+_G\eta$ and for all $t \in R'^{\mathrm{B}}_{j_1}(v_1') \cup \{v_1'\}$ it holds that $(\mathcal{M}', t) \vDash [\mathrm{B}]^+_G\eta$. Thus, for all $j \in G \setminus \{j_1\}$, it holds that for all $t \in R^{\mathrm{B}}_{j_1}(w)$, $(\mathcal{M}, t) \vDash [\mathrm{B}]_j[\mathrm{B}]^+_G\eta$ and for all $t \in R'^{\mathrm{B}}_{j_1}(v_1') \cup \{v_1'\}$ it holds that $(\mathcal{M}', t) \vDash [\mathrm{B}]_j[\mathrm{B}]^+_G\eta$. Hence, as was shown above, for all $t \in W^{\mathrm{B}}_{j_1} \cup W'^{\mathrm{B}}_{j_1}$ it holds that $(\mathcal{M}'', t) \vDash [\mathrm{B}]_j[\mathrm{B}]^+_G\eta$. This, together with the fact that for all $t \in W^{\mathrm{B}}_{j_1} \cup W'^{\mathrm{B}}_{j_1}$, $(\mathcal{M}'', t) \vDash \eta$ implies that for all $t \in W^{\mathrm{B}}_{j_1} \cup W'^{\mathrm{B}}_{j_1}$ and for all $s \in R^{\mathrm{B}^+}_G(t)$, $(\mathcal{M}'', s) \vDash \eta$ and, consequently, $(\mathcal{M}'', t) \vDash [\mathrm{B}]^+_G\eta$. Thus, in particular, $(\mathcal{M}'', u) \vDash [\mathrm{B}]^+_G\eta$, for both points (i) and (ii).

Now, if ψ is of the form $\neg[\mathrm{B}]_{j_1}\xi$, then there must exist $u \in R^{\mathrm{B}}_{j_1}(w)$ such that $(\mathcal{M}, u) \vDash \neg\xi$. Thus, by what was shown above, $(\mathcal{M}'', u^{\mathrm{c}}) \vDash \neg\xi$ and so $(\mathcal{M}'', w'') \vDash \neg[\mathrm{B}]_{j_1}\xi$. If ψ is of the form $[\mathrm{B}]_{j_1}\xi$, then, for all $u \in R^{\mathrm{B}}_{j_1}(w)$, it holds that $(\mathcal{M}, u) \vDash \xi$ and, by what was shown above, for all $u \in W^{\mathrm{B}}_{j_1}$ it holds that $(\mathcal{M}'', u) \vDash \xi$. Also, if $[\mathrm{B}]_{j_1}\xi \in \Phi$, then $\{[\mathrm{B}]_{j_1}\xi, \xi\} \subseteq \Phi^{\neg[\mathrm{B}]_{j_1}}([\mathrm{B}]^+_G\varphi)$ and so $(\mathcal{M}', v_1') \vDash [\mathrm{B}]_{j_1}\xi$ and $(\mathcal{M}', v_1') \vDash \xi$. Thus for all $u \in R'^{\mathrm{B}}_{j_1}(v_1') \cup \{v_1'\}$, it holds that $(\mathcal{M}', u) \vDash \xi$ and, by what was shown above, for all $u \in W'^{\mathrm{B}}_{j_1}$ it holds that $(\mathcal{M}'', u) \vDash \xi$. Hence $(\mathcal{M}'', w'') \vDash [\mathrm{B}]_{j_1}\xi$. Suppose that ψ is of the form $[O]^+_G\xi$ or $\neg[O]^+_G\xi$ with $O \in \{\mathrm{B},\mathrm{I}\}$. In the first case, it holds that, for all $j \in G$, $[O]_j\xi \in \Phi$ and $[O]_j[O]^+_G\psi \in \Phi$, as Φ is a closed tableau. Moreover, by what was shown above, for all $j \in G$, $(\mathcal{M}'', w'') \vDash [O]_j\xi$ and $(\mathcal{M}'', w'') \vDash [O]_j[O]^+_G\psi$. Hence for all $u \in R''^{O^+}_G(w'')$ it holds that $(\mathcal{M}'', u) \vDash \xi$ and so $(\mathcal{M}'', w'') \vDash [O]^+_G\xi$. In the second case, there must exists $j \in G$ such that either $\neg[O]_j\xi \in \Phi$ or $\neg[O]_j[O]^+_G\xi \in \Phi$, as Φ is a $[O]^+$-expanded tableau. Since, by what was shown above, either $(\mathcal{M}'', w'') \vDash \neg[O]_j\xi$ or $(\mathcal{M}'', w'') \vDash \neg[O]_j[O]^+_G\psi$, respectively, so there must exist $u \in R''^{O^+}_G(w'')$ such that $(\mathcal{M}'', u) \vDash \neg\xi$ and so $(\mathcal{M}'', w'') \vDash \neg[O]^+_G\xi$. For the remaining forms of ψ, $(\mathcal{M}'', w'') \vDash \psi$ can be shown by simple induction on the length of the formula and using the fact that Φ is a propositional tableau and that cases for the induction basis are already covered above. Thus we have shown that $(\mathcal{M}'', w'') \vDash \bigwedge \Phi$. Now, if $j_2' \neq j_1$, then $w'', v_1', \ldots, v_m'$ is a minimal satisfying sequence for $[\mathrm{B}]^+_G\varphi$ in $(\mathcal{M}'', w'')$ shorter than n. Otherwise, if $j_2' = j_1$, then $w'', v_2'^{\mathrm{c}}, v_3', \ldots, v_m'$ (or w'', v_2' in the case of $m = 2$) is a satisfying minimal satisfying sequence for $[\mathrm{B}]^+_G\varphi$ in $(\mathcal{M}'', w'')$ shorter than n.[2] In either case we have a contradiction with the assumption that $(\mathcal{M}, w)$ is a satisfying pair for Φ with minimal satisfying sequence for $\neg[\mathrm{B}]^+_G\varphi$. Hence it must be that $(\mathcal{M}, v_1)$ is a satisfying pair for $\Phi^{\neg[\mathrm{B}]_{j_1}}([\mathrm{B}]^+_G\varphi)$ with minimal satisfying sequence $v_1, \ldots, v_n$ for $\neg[\mathrm{B}]^+_G\psi$. □

Proof of Theorem 6.33, page 108. The problem is in PSPACE by Theorem 6.32. To show hardness we will construct a formula $\varphi^I_T \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ with $\mathrm{dep}(\varphi) = 2$ whose models encode the computation of a given polynomial space bounded deterministic Turing machine T on the input I. The constructed formula φ^I_T will be satisfiable if and only if the computation of T on I terminates in accepting state. A deterministic Turing machine is a tuple $T = (Q, \Sigma, \Gamma, \delta, \mathrm{B}, q_0, q_{\mathrm{A}}, q_{\mathrm{R}})$, where

- Q is a finite set of states,
- $q_0 \in Q$ is the starting state,
- $q_{\mathrm{A}} \in Q$ is the accepting state,
- $q_{\mathrm{R}} \in Q$ is the rejecting state,
- Γ is a finite worktape alphabet,
- $\Sigma \subseteq \Gamma$ is a finite input alphabet,
- $\mathrm{B} \in \Gamma \setminus \Sigma$ is the blank symbol,
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{-1, 0, 1\}$ is the transition function.

Let T be a deterministic Turing machine and I be its input and suppose that during its computation on I, T uses $\leq M(|I|)$ cells of the worktape, where $|I|$ denotes the size of I. To define the formula φ^I_T we will first define three formulas that describe the initial configuration of the machine, a valid configuration of the machine and a valid transition of the machine. To define the formulas we will use the following propositional symbols:

- a^x_i, where $x \in \Gamma$ and $1 \leq i \leq M(n)$, to indicate that the symbol in the i'th cell of the worktape is x;
- s^q_i, where $q \in Q$ and $1 \leq i \leq M(n)$, to indicate that the current state is q and the head of the machine is at the i'th cell of the worktape.

A configuration of the machine will be encoded by valuations of formulas $[\mathrm{I}]_k a^x_i$ and $[\mathrm{I}]_k s^q_i$, where $k \in \{1, 2\}$, $x \in \Gamma$, $q \in Q$ and $1 \leq i \leq M(|I|)$. This way two configurations of the machine are encoded at any world v of a TeamLog model, one by valuations of the formulas with operator $[\mathrm{I}]_1$ and another one by the formulas with operator $[\mathrm{I}]_2$ at this world. We will refer to them by $C_1(v)$ and $C_2(v)$, respectively.

Firstly, the initial configuration is when the head is at the first cell of the machine and the input is written in the first $|I|$ cells of the machine, while the remaining $M(|I|) - |I|$ cells are filled in with blanks. The formula describing the initial configuration is called INIT_I. At

[2] Notice that since $v'_1, \ldots, v'_m$ is a minimal satisfying sequence for $[\mathrm{B}]^+_G\varphi$, so it must be that $j'_2 \neq j'_3$ (in the case of $m \geq 3$), as otherwise a shorter sequence could by constructed using transitivity of R'^{B}_j.

any given world v of a TeamLog model it holds that if INIT_I is satisfied there, then $C_1(v)$ encodes the initial configuration of the machine T on input I.

$$\mathsf{INIT}_I = [\mathrm{I}]_1 s_1^{q_0} \wedge \bigwedge_{i=1}^{n} [\mathrm{I}]_1 a_i^{I_i} \wedge \bigwedge_{i=n+1}^{M(|I|)} [\mathrm{I}]_1 a_i^{\mathrm{B}}. \tag{A.4}$$

Secondly, a formula stating that both C_1 and C_2 encode valid configurations is defined. The formula is called CONFIG_T and it is a conjunction of the two formulas below. The first of these formulas states that in each cell from 1 to $M(|I|)$ exactly one symbol from Γ is put. The second of these formulas states that that the machine is in exactly one state and the head is positioned at exactly one cell from 1 to $M(|I|)$. At any given world of a TeamLog model it holds that if CONFIG_T is satisfied there, then $C_1(v)$ and $C_2(v)$ represent valid configurations of T.

$$\bigwedge_{k\in\{1,2\}} \bigwedge_{i=1}^{M(|I|)} \bigvee_{x\in\Gamma} \left([\mathrm{I}]_k a_i^x \wedge \bigwedge_{y\in\Gamma\setminus\{x\}} \neg [\mathrm{I}]_k a_i^y \right) \tag{A.5}$$

$$\bigwedge_{k\in\{1,2\}} \left(\bigvee_{i=1}^{M(|I|)} \bigvee_{q\in Q} [\mathrm{I}]_k s_i^q \wedge \bigwedge_{i=1}^{M(|I|)} \bigwedge_{q\in Q} \left([\mathrm{I}]_k s_i^q \rightarrow \bigwedge_{r\in Q\setminus\{q\}} \neg [\mathrm{I}]_k s_i^r \right) \right) \tag{A.6}$$

Thirdly, transitions of the machine are described by a formula TRANS_T, which is a conjunction of the two formulas below. At any given world v of a TeamLog model it holds that if TRANS_T is satisfied there, then either $C_1(v)$ and $C_2(v)$ encode the same configuration of T or $C_2(v)$ encodes the configuration succeeding the configuration encoded by $C_1(v)$ in the run of the machine T on the input I.

$$\begin{gathered} \bigwedge_{i=1}^{M(|I|)} \bigwedge_{x\in\Gamma} \bigwedge_{q\in Q} ([\mathrm{I}]_1 a_i^x \wedge [\mathrm{I}]_1 s_i^q) \rightarrow \\ \left(\bigwedge_{j=1, j\neq i}^{M(|I|)} \bigwedge_{z\in\Gamma} ([\mathrm{I}]_1 a_j^z \leftrightarrow [\mathrm{I}]_2 a_j^z) \wedge \left(([\mathrm{I}]_2 a_i^x \wedge [\mathrm{I}]_2 s_i^q) \vee \left([\mathrm{I}]_2 a_i^{\delta_2(q,x)} \wedge [\mathrm{I}]_2 s_{i+\delta_3(q,x)}^{\delta_1(q,x)} \right) \right) \right) \end{gathered} \tag{A.7}$$

$$\begin{gathered} \bigwedge_{i=1}^{M(|I|)} \bigwedge_{x\in\Gamma} \bigwedge_{q\in Q} \left([\mathrm{I}]_2 a_i^{\delta_2(q,x)} \wedge [\mathrm{I}]_2 s_{i+\delta_3(q,x)}^{\delta_1(q,x)} \right) \rightarrow \\ \left(\bigwedge_{j=1, j\neq i}^{M(|I|)} \bigwedge_{z\in\Gamma} ([\mathrm{I}]_1 a_j^z \leftrightarrow [\mathrm{I}]_2 a_j^z) \wedge \left(\left([\mathrm{I}]_1 a_i^{\delta_2(q,x)} \wedge [\mathrm{I}]_1 s_{i+\delta_3(q,x)}^{\delta_1(q,x)} \right) \vee ([\mathrm{I}]_1 a_i^x \wedge [\mathrm{I}]_1 s_i^q) \right) \right) \end{gathered} \tag{A.8}$$

Let φ_T^I be defined as follows:

$$\varphi_T^I = \mathsf{INIT}_I \wedge [\mathrm{B}]'^{+}_{\{1,2\}} (\mathsf{CONFIG}_T \wedge \mathsf{TRANS}_T) \wedge \neg [\mathrm{B}]^{+}_{\{1,2\}} \left(\neg \bigvee_{i=1}^{M(|I|)} [\mathrm{I}]_1 s_i^{q_{\mathrm{A}}} \right), \tag{A.9}$$

where $[\mathrm{B}]'^{+}$ is defined as in proof of Theorem 4.19.[3] Notice that the size of φ_T^I is polynomial with respect to $|I|$. To see that the if φ_T^I is satisfiable, then T accepts the input I, suppose

[3]Formula φ_T^I could be also constructed with use of operators $[\mathrm{G}]_1$ and $[\mathrm{G}]_2$ instead of $[\mathrm{I}]_1$ and $[\mathrm{I}]_2$.

that $(\mathcal{M}, w) \vDash \varphi_T^I$. Then $C_1(w)$ encodes the initial configuration of T on the input I, at all worlds $v \in {R^{\mathrm{B}}_{\{1,2\}}}^+(w)$, $C_1(v)$ and $C_2(v)$ encode valid configurations of T and either $C_1(v)$ and $C_2(v)$ encode the same configuration or the configuration encoded by $C_2(v)$ succeeds the configuration encode by $C_1(v)$ in the run of the machine T on the input I. Moreover, there exists $u \in {R^{\mathrm{B}}_{\{1,2\}}}^+(w)$ such that $C_1(u)$ encodes a configuration at accepting state of T. The computation of T on I that leads to that configuration can be read from the path leading from w to u. For any two subsequent states v_1 and v_2 on this path, such that $v_2 \in R^{\mathrm{B}_j}(v_1)$ with $j \in \{1, 2\}$, by generalized transitivity it holds that $C_j(v_1)$ and $C_j(v_2)$ encode the same configuration. Hence the configurations encoded by C_1 at the subsequent worlds on the path represent states of the run of T on the input I with the possibility that at some worlds not transitions is performed or preceding states of the machine are restored. After removing such states from the sequence, a sequence of configurations C_1 can be obtained that represents the whole run accepting run of T on the input I. On the other hand suppose T accepts the input I. Then we could construct a TeamLog model containing a sequence of worlds connected alternately by accessibility relations R_2^{B} and R_1^{B} such that at every second state of this sequence the configurations of the machine encoded by C_1 are the subsequent configuration of the run of T on the input I. This shows that T accepts I if and only if φ_T^I is satisfiable. Thus we have shown that the problem of TeamLog satisfiability for formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ with modal depth bounded by 2 is PSPACE hard and since it is also in PSPACE, so it is PSPACE complete. $\square$

Appendix B

Overview of selected multiagent formalisms

B.1 Formalism of Cohen & Levesque

Arguably the most influential early formalization of individual agents based on the BDI concept is the formalism proposed by Cohen and Levesque in [26]. In [73] this formalism was extended to concepts related to multiple agents such as mutual beliefs, joint goals, commitments and intentions. The formalism is based on linear time logic, with standard temporal operators F (finally), G (globally), X (next), U (until) and B (before) (c.f. [43]), extending it with the following modal operators: $\mathrm{BEL}\,(j,\cdot)$, $\mathrm{MB}\,(i,j,\cdot)$, $\mathrm{GOAL}\,(j,\cdot)$, $\mathrm{MG}\,(i,j,\cdot)$, HAPPENS and a DONE. Operators $\mathrm{BEL}\,(j,\cdot)$ are used to represent beliefs of agents and the axiom system $\mathrm{KD45}_n$ of multiagent doxastic logic is adopted for them. Operators $\mathrm{MB}\,(i,j,\cdot)$ are used to represent mutual beliefs of pairs of agents. Operators $\mathrm{GOAL}\,(j,\cdot)$ are used to represent goals (choices) of agents, i.e. $\mathrm{GOAL}\,(j,\varphi)$ means that agent j has a goal that φ holds. It is assumed that goals of an agent are consistent and so the axioms system KD_n is adopted for them. Additionally, it is assumed that if an agent believes that φ holds, then it has to also adopt φ as a goal. This assumption, called *realism*, is associated with the following axiom:

$$\mathrm{BEL}\,(j,\varphi) \rightarrow \mathrm{GOAL}\,(j,\varphi)\,.$$

Operators $\mathrm{MG}\,(i,j,\cdot)$ are used to represent *mutual goals* of two agents. They are defined analogically to mutual beliefs but on the basis of operators $\mathrm{GOAL}\,(i,\cdot)$ and $\mathrm{GOAL}\,(j,\cdot)$.

Operators HAPPENS and DONE are related to actions that agents can perform. Predicate $\mathrm{HAPPENS}(\alpha)$ means that a sequence of events compatible with action expression α starts in the current state. Predicate $\mathrm{DONE}(\alpha)$ means that a sequence of events compatible with action expression α has just ended in the given state. Operators $\mathrm{HAPPENS}(\mathrm{j},\cdot)$ and $\mathrm{DONE}(j,\cdot)$ referring to actions performed by given agent are also defined. Formulas of the formalism are interpreted in Kripke models with accessibility relations corresponding to operators $\mathrm{BEL}\,(j,\cdot)$ and $\mathrm{GOAL}\,(j,\cdot)$ defined on the set of worlds, where each world consists of a set of states representing a time line, with accessibility relations connecting subsequent states being labelled with agents and atomic actions.

Introduction of time related operators allow for expressing how motivational attitudes of the agent change with time. For example the assumption of *no persistence* stating that agents cannot try to achieve their goals forever is associated with the following axiom:

$$\mathsf{F}\,(\neg\mathrm{GOAL}\,(j,\neg\varphi \wedge \mathsf{F}\varphi))\,.$$

Using the basic operators described above important notions related to motivational attitudes of the agent as well as properties of these attitudes can be expressed. Firstly, the notion of *achievement goals* is defined, that is those goals that the agent believes currently to be false (as opposed to *maintenance goals*, that the agent believes currently to be true):

$$\text{A-GOAL}(j, \varphi) \stackrel{\text{def}}{=} \text{BEL}\,(j, \neg\varphi) \wedge \text{GOAL}\,(j, \mathsf{F}\varphi)\,.$$

A strong form of commitment to a goal is expressed as a *persistent goals*, that is a goal that the agent keeps until either it believes that the goal is achieved or it believes that the goal is unachievable. This is defined as follows:[1]

$$\text{P-GOAL}(j, \varphi) \stackrel{\text{def}}{=} \text{A-GOAL}(j, \varphi) \wedge ((\text{BEL}\,(j, \varphi) \vee \text{BEL}\,(j, \mathsf{G}\,(\neg\varphi)))\ \mathsf{B}\ (\neg\text{GOAL}\,(j, \mathsf{F}\varphi)))\,,$$

The condition of dropping the goal could be extended to include a formula expressing additional conditions of dropping the goal. This leads to a weaker form of commitment expressed as a *persistent relativized goal* defined as follows:[2]

$$\begin{aligned}\text{P-R-GOAL}(j, \varphi, \psi) \stackrel{\text{def}}{=} {} & \text{A-GOAL}(j, \varphi) \wedge \\ & ((\text{BEL}\,(j, \varphi) \vee \text{BEL}\,(j, \mathsf{G}\,(\neg\varphi)) \vee \text{BEL}\,(j, \neg\psi))\ \mathsf{B}\ (\neg\text{GOAL}\,(j, \mathsf{F}\varphi)))\,.\end{aligned}$$

Cohen and Levesque define intentions of agents as choices to which agents are committed in some way. The commitment could be strong, as in the case of persistent goals. Then intentions of agents are defined as persistent goals. In the case of intentions towards actions described by action expression α the definition is as follows:

$$\text{INT}\,(j, \varphi) \stackrel{\text{def}}{=} \text{P-GOAL}(j, \text{DONE}(j, \text{BEL}\,(j, \text{HAPPENS}(\alpha))\,?;\alpha)).$$

In [73] the notion of *joint persistent goal* is defined, which formalizes a form of commitment of two agents towards a goal φ. The definition is based on the notion of *weak goal of one agent towards another with respect to a given proposition.* This notion describes the commitments that an agent has when trying to achieve a goal for some other agent. An agent i has a weak goal towards an agent j with respect to φ if either i does not believe that φ holds and has a goal that φ is eventually true or i believes that φ holds and has a goal that eventually i and j mutually believe that φ holds or i believes that φ can never be true, in which case i has a goal that eventually i and j will mutually believe it as well. This is expressed formally as follows:

$$\begin{aligned}\text{WG}\,(i, j, \varphi) \stackrel{\text{def}}{=} {} & \text{A-GOAL}(i, \mathsf{F}\varphi) \vee \\ & (\text{BEL}\,(i, \varphi) \wedge \text{GOAL}\,(i, \mathsf{F}\,(\text{MB}\,(i, j, \varphi)))) \vee \\ & (\text{BEL}\,(i, \mathsf{G}\,(\neg\varphi)) \wedge \text{GOAL}\,(i, \mathsf{F}\,(\text{MB}\,(i, j, \mathsf{G}\,(\neg\varphi)))))\,.\end{aligned}$$

Two agents i and j have a *weak mutual goal* to achieve φ if they mutual believe that each of them has a weak goal towards another with respect to φ. This is expressed formally as follows:

$$\text{WMG}\,(i, j, \varphi) \stackrel{\text{def}}{=} \text{MB}\,(i, j, \text{WG}\,(i, j, \varphi) \wedge \text{WG}\,(j, i, \varphi))\,.$$

For joint persistent goal the relativized form of commitment is adopted. Two agents i and j have a joint persistent goal towards φ with additional dropping condition ψ if they mutually believe that φ does not hold, they have a mutual goal that φ holds and they keep a weak mutual goal towards φ until either they believe that φ is achieved, or they believe that it

[1] We present a version of this definition based on [73].

[2] We present a version of this definition given in [73].

cannot be achieved or the dropping condition ψ becomes true. This is expressed formally as follows:

$$\begin{aligned}\mathrm{JPG}(i,j,\varphi,\psi) \stackrel{\mathrm{def}}{=} &\ \mathrm{MB}\,(i,j,\neg\varphi) \wedge \mathrm{MG}\,(i,j,\varphi) \wedge \\ &((\mathrm{MB}\,(i,j,\varphi) \vee \mathrm{MB}\,(i,j,\mathsf{G}\,(\neg\varphi)) \vee \mathrm{MB}\,(i,j,\neg\psi))\ \mathsf{B}\ (\neg\mathrm{WMG}\,(i,j,\varphi)))\,.\end{aligned}$$

Similarly to the case of individual agents intentions, joint intentions of two agents are defined as a form of joint choice with joint commitment expressed as a joint persistent goal. See [73] for details.

B.2 Formalism of Rao & Georgeff and $\mathcal{LORA}$

In [96, 97] Rao and Georgeff developed a formalism designed to model individual BDI agents. It was further used as a basis for a multiagent formalism called $\mathcal{LORA}$ (*Logic of Rational Agents*) proposed by Wooldridge in [119]. Below we present both formalisms, starting with the one for individual agents and then presenting some of the extensions introduced in $\mathcal{LORA}$.[3] The formalism proposed in [96, 97] and further extended in [119] could be seen as consisting of four components: (c.f. [119]):

Domain component This component is used to characterise the domain in which agents operate. In [96, 119] this component is the first-order logic. In [97] it is a propositional calculus.

BDI component This component is used to characterise informational and motivational aspects of individual agents [96, 97] as well as groups of agents [119].

Dynamic component This component is used to characterise dynamic aspects of the multi-agent system and the domain. For this component a branching time logic CTL* [43] is adopted.

Action component This component characterizes the actions and effects of actions that agents perform and it is introduced in [119].

Assuming a finite and non-empty set of agents, there are three groups of modalities introduced in the BDI component:

$$\begin{aligned} &\mathrm{BEL}\,(j,\varphi)\,, && \text{agent } j \text{ believes that } \varphi \text{ holds,}\\ &\mathrm{DES}\,(j,\varphi)\,, && \text{the desired state of the world of agent } j \text{ is one where } \varphi \text{ holds,}\\ &\mathrm{INT}\,(j,\varphi)\,, && \text{the intended state of the world of agent } j \text{ is one where } \varphi \text{ holds.}\end{aligned}$$

For beliefs the multimodal system $\mathrm{KD45}_n$ is adopted, while for intentions and desires the multimodal system KD_n is adopted. Additionally, several axioms are proposed that could be added to the formalism to reflect different important relations between individual agents beliefs, desires and intentions. The axioms state different forms of *realism constraints* that may be put on attitudes of different agents.[4]

[3] Across the papers were these formalisms are presented slightly different notation is used, we will adopt one of them for the sake of uniformity.

[4] In [96] Rao and Georgeff discuss additional important relations between individual agents attitudes associated with agent's awareness of his attitudes. However this were dropped in their later publication [97] and were also not adopted in [119], where informational and motivational attitudes are added to the formalism. These relations are important in TeamLog formalism, presented in Section 2.1.

The constraint of *strong realism* is formally expressed as follows:

$$\begin{aligned}\mathrm{DES}\,(j,\varphi) &\rightarrow \mathrm{BEL}\,(j,\varphi)\,,\\ \mathrm{INT}\,(j,\varphi) &\rightarrow \mathrm{DES}\,(j,\varphi)\,.\end{aligned}$$

In the basic version of this constraint φ is assumed to be a so called *optional formula*, i.e. a formula of the form $\mathsf{E}\psi$. Then the first of these axioms states that if an agent has a goal that ψ is optionally true, then it also believes that ψ is optionally true.

The constraint of *realism* is based on the axiom of realism proposed by Cohen and Levesque and extends it to intentions and desires. It is formally expressed as follows:

$$\begin{aligned}\mathrm{DES}\,(j,\varphi) &\rightarrow \mathrm{INT}\,(j,\varphi)\,,\\ \mathrm{BEL}\,(j,\varphi) &\rightarrow \mathrm{DES}\,(j,\varphi)\,.\end{aligned}$$

The constraint of *weak realism* is formally expressed as follows:

$$\begin{aligned}\mathrm{DES}\,(j,\varphi) &\rightarrow \neg\mathrm{BEL}\,(j,\varphi)\,,\\ \mathrm{INT}\,(j,\varphi) &\rightarrow \neg\mathrm{DES}\,(j,\varphi)\,,\\ \mathrm{INT}\,(j,\varphi) &\rightarrow \neg\mathrm{BEL}\,(j,\varphi)\,.\end{aligned}$$

In the case of beliefs and desires, this axiom states that if an agent desires that φ is true, then it does not believe that φ is already true.

The dynamic component introduces modalities of branching time logic and allows for specifying different forms of commitment that are associated with intentions of agents. Rao and Georgeff discuss three kinds of such forms of commitment, called *commitment strategies*: blind commitment (the strongest one), singe-minded commitment (the intermediate one) and open-minded commitment (the weakest one) [96]. Each of these commitments strategies specifies what conditions of dropping the goal are entailed by an agent intending to achieve that goal. For example the open-minded commitment is formally expressed as follows:

$$\mathrm{INT}\,(j,\mathsf{AF}\varphi) \rightarrow \mathsf{A}\left((\mathrm{INT}\,(j,\mathsf{AF}\varphi))\ \mathsf{U}\ (\mathrm{BEL}\,(j,\varphi) \vee \neg\mathrm{DES}\,(j,\mathsf{EF}\varphi))\right),$$

which states that intention of an agent towards a future goal φ may be dropped only if either the agent believes that φ already holds or the agent does not have a goal that φ is optionally true.

Wooldridge in [119] extends the framework proposed by Rao and Georgeff, introducing to it group modalities that express informational and motivational attitudes of groups of agents: mutual beliefs ($\mathrm{M\text{-}BEL}_G\,(\cdot)$), mutual desires ($\mathrm{M\text{-}DES}_G\,(\cdot)$) and mutual intentions ($\mathrm{M\text{-}INT}_G\,(\cdot)$). The latter two are defined on the basis of operators $\mathrm{DES}\,(j,\cdot)$ and $\mathrm{INT}\,(j,\cdot)$ using fixpoint constructs, similarly to how $\mathrm{M\text{-}BEL}_G\,(\cdot)$ is defined.

On the basis of these group modalities, joint commitments within a cooperating group of agents can be expressed. The conditions under which commitments can be abandoned are given by a finite set of rules called *convention*. Each such rule is a pair (ρ,γ) specifying conditions under which the commitment could be abandoned (a *reevaluation condition*) ρ and responsibilities of an agent associated with dropping the commitment, that is a goal γ that the agent has to adopt dropping the commitment. The general schema of joint commitment of group G to goal φ with respect to convention C and precondition ψ is defined as follows:

$$\mathrm{TEAM}_C(G,\varphi,\psi) \leftrightarrow \psi \wedge \bigwedge_{j\in G}\left(\mathsf{A}\left(\left(\mathrm{INT}\,(j,\varphi)\wedge \mathrm{FC}_C^j\right)\ \mathsf{U}\ (\mathrm{TC}_C)\right)\right),$$

where

$$\mathrm{FC}_C^j \stackrel{\text{def}}{=} \bigwedge_{(\rho,\gamma)\in C} (\mathrm{BEL}\,(j,\varrho) \rightarrow \mathsf{A}\,((\mathrm{INT}\,(j,\gamma))\ \mathsf{U}\ (\mathrm{TC}_C)))$$

states that j *follows the convention* C and

$$\mathrm{TC}_C \stackrel{\text{def}}{=} \bigvee_{(\rho,\gamma)\in C} \gamma$$

is the *termination condition* under convention C. Thus joint commitment is formed when some precondition ψ is satisfied and under the joint commitment, each agent in group G sustains his intention towards the joint goal φ until termination condition is satisfied. Moreover, until the termination condition is satisfied, each agent follows the convention by adopting a goal γ whenever it believes that the reevaluation condition associated with it holds. A group of agents with a joint commitment towards some goal is called a *team*. Using the schema of joint commitment different forms of such commitments can be defined. For example a *blind social commitment* of group G to goal φ is defined by adopting a condition $\bigwedge_{j\in G} \neg\mathrm{BEL}\,(j,\varphi)$ as the precondition and $\{(\varphi, \text{M-BEL}_G\,(\varphi))\}$ as the convention. Thus such a commitment holds if no agent in the group believes that the goal is already satisfied and the goal can be dropped by a group member only if it believes that the goal is satisfied, in which case it has to adopt the intention to attain mutual belief in the team about this fact.

$\mathcal{LORA}$ formalism can be also used to formalise the process of cooperative problem solving as well as communication between agents, as presented in [122, 119].

B.3 KARO

KARO (*logic for Knowledge, Ability, Results, and Opportunities*) is a logical framework developed by van der Hoek, van Linder and Meyer [114, 115, 80, 113] as a formalism for specifying individual agents and further extended by Aldewereld [4] to cover group aspects of multiagent systems. The framework allows for describing agents and groups of agents on five levels:

- informational level,
- action level,
- dynamic level,
- motivational level,
- social level.

For specifying informational aspects of individual agents operators $\mathbf{B}_j$ are introduced, where $\mathbf{B}_j\varphi$ means that agent j believes that φ holds. The standard doxastic axiom system $\mathrm{KD45}_n$ is assumed for these operators.[5] On this basis the operators $\mathbf{EB}_G$ and $\mathbf{MB}_G$ are defined, to represent general and mutual beliefs of agents.

Action level introduces actions to the framework and dynamic level describes how formulas related to actions are interpreted. Actions can be either atomic or complex. Complex actions

[5] To be more precise, in [113] four different kinds of operators are considered, representing knowledge $\mathbf{B}_j^k$(axioms system $\mathrm{S5}_n$), beliefs obtained by observation $\mathbf{B}_j^o$ ($\mathrm{S5}_n$), beliefs obtained by communication $\mathbf{B}_j^c$ ($\mathrm{KD45}_n$) and beliefs adopted by default $\mathbf{B}_j^d$ ($\mathrm{KD45}_n$). However, in the last work [4] only one operator $\mathbf{B}_j$ is adopted.

can be either individual or team actions. For individual complex actions strict deterministic propositional dynamic logic (SDPDL) [55] is adopted. This means that that atomic actions have deterministic effects and that non-determinism is removed from complex actions by adopting strict programs constructs (c.f. [55]). Since actions can be performed by different agents, given a complex action α, $do_j(\alpha)$ denotes action α performed by agent j. Given an action α, a SDPDL formula $\langle do_j(\alpha)\rangle\top$ expresses the fact that agent j has an *opportunity* to perform α. A formula $[do_j(\alpha)]\,\varphi$ states that φ is among the results of α, if opportunity to perform α is present. Abilities of individual agents are represented by operators $\mathbf{A}_j$. If j is an agent and α an action, then $\mathbf{A}_j\alpha$ means that j is *capable* of performing α. The operators $\mathbf{A}_j$ are factually non-modal operators with semantics determined by a function c which, for a given agent j and action α, provides a function that yields the ability of j to perform α in different worlds of the model. Details of how this function is defined as well as full deduction system for a fragment of KARO framework covering knowledge, abilities and opportunities of agents together with results of actions performed by them are given in [115, 113]. In [115] soundness and completeness results are also provided. Primitives representing opportunities and abilities of agents allow for expressing important notion of *practical possibility* to bring about (truth of) a given proposition φ by performing a given action α:

$$\mathbf{PracPoss}_j\,(\alpha,\varphi) \stackrel{\text{def}}{=} \langle do_j(\alpha)\rangle\varphi \wedge \mathbf{A}_j\alpha.$$

Apart from primitives allowing for expressing opportunities and abilities of individual agents, action level introduces one more primitive concept – *implementability.* Given a formula φ and an agent j, $\mathbf{Impl}_j\varphi$ states that φ is implementable by j. The operator $\mathbf{Impl}_j$ is a modal operator defined semantically as follows:

$$(\mathcal{M},w) \models \mathbf{Impl}_j\varphi \text{ iff there exists a sequence of atomic actions } a_1,\ldots,a_n \text{ such that } (\mathcal{M},w) \models \mathbf{PracPoss}_j\,(a_1;\ldots;a_n,\varphi)\,.$$

Thus φ is implementable by j if j has practical possibility to bring about φ by performing some sequence of atomic actions.

Group complex actions extend the constructs of individual actions by allowing parallel executions of atomic actions and verification actions. They are defined on the basis of *basic multi-actions*. A basic multi-action is a tuple consisting of atomic actions and, possibly, `skip` actions, that stand for 'doing nothing' and do not change the world. The `skip` action is defined as `confirm` $\top$. Complex group actions may be either joint verification actions, i.e. `joint-confirm` $\varphi = ($`confirm` $\varphi,\ldots,$ `confirm` $\varphi)$, or be obtained using sequential composition, conditional choice or loop constructs of strict programs. Joint execution of a basic multi-action or a joint confirmation action $\alpha=(\alpha_1,\ldots,\alpha_n)$ by group of agents $G=\{j_1,\ldots,j_n\}$ is a parallel execution of individual actions from α by corresponding agents from G: $do_G(\alpha)=do_{j_1}(\alpha_1)\parallel\ldots\parallel do_{j_n}(\alpha_n)$. An opportunity of group of agents G to perform a group action α is expressed by a formula$\langle do_G(\alpha)\rangle\top$. Group abilities are defined on the basis of individual abilities in the following way. The ability of group of agents G to perform a group action α is defined as

$$\mathbf{A}_G\alpha \stackrel{\text{def}}{=} \bigwedge_{j\in G}\mathbf{A}_j\alpha.$$

Thus the group G has ability to perform α if and only if every agent in the group is capable of performing α. Practical possibility of group G to bring about a given proposition φ by performing a given group action α is defined analogously to its individual counterpart. On this basis the modal operator $\mathbf{Impl}_G\varphi$ of implementability by groups of agents is defined:

$$\mathbf{PracPoss}_j\,(\alpha,\varphi) \stackrel{\text{def}}{=} \langle do_j(\alpha)\rangle\varphi \wedge \mathbf{A}_j\alpha.$$

$(\mathcal{M}, w) \models \mathbf{Impl}_G\varphi$ iff there exists a sequence of basic multi-actions $a_1, \ldots, a_n$ such that
$$(\mathcal{M}, w) \models \mathbf{PracPoss}_G(a_1; \ldots ;a_n, \varphi).$$

The basic motivational attitude introduced in KARO framework is that of a *wish* of an agent. Wishes of agents are specified with modal operator $\mathbf{W}_j$ for which axiom system K_n is assumed. Given an agent j and a formula φ, $\mathbf{W}_j\varphi$ means that j has a *wish* that φ holds. Goals of an agent are then defined to be selected, unfulfilled and implementable wishes:

$$\mathbf{Goal}_j\varphi \stackrel{\text{def}}{=} \mathbf{W}_j\varphi \wedge \mathbf{C}_j\varphi \wedge \neg\varphi \wedge \mathbf{Impl}_j\varphi.$$

The operator $\mathbf{C}_j$ represents choices of agent j. It is a non-modal operator whose semantics in different worlds is determined by a syntactic assignment function C that yields, for a given agent j and a given world w, the set of formulas $C(j, w)$ that are the formulas that j chooses in w. No restrictions are put on the choice function C and it is possible, in particular, that $C(j, w)$ contains both a formula and its negation.[6]

Wishes of groups of agents are defined on the basis individual wishes. Given a non-empty group of agents G and a proposition φ, the wish of group G that φ holds is defined as

$$\mathbf{W}_G\varphi \stackrel{\text{def}}{=} \bigwedge_{j\in G} \mathbf{W}_j\varphi.$$

Thus the group G has a wish that φ holds if and only if every agent in the group has a wish that φ holds. On the basis of group wishes, weak goals of an agent with respect to a group of agents and joint goals of groups of agents can be defined (c.f. [4]).

On the social level, operators that allow for specifying commitments of agents to perform actions are introduced. On this basis the notion of joint commitment to complex group action is defined. Given an agent j and an (individual) action α, $\mathbf{Com}_j\alpha$ states that agent j is committed to performing α. Similarly to operators $\mathbf{C}_j$, representing choices of agents, operators $\mathbf{Com}_j$ are non-modal operators with semantics in different worlds determined by a syntactic assignment function *Agenda* that yields, for a given agent j and a given world w, the set of actions $Agenda(j, w)$ that the agent is committed to. In the case of $\mathbf{Com}_j$, however, this semantics is more restrictive so that in a given world an agent is committed to any action that is an 'initial part' of the actions written in its agenda in all world accessible from the current world with the accessibility relation associated with the operator $\mathbf{B}_j$ (c.f. [80]). Such a definition results in the following formula being true in the class of models allowed by the KARO framework:

$$\mathbf{Com}_j\varphi \rightarrow \mathbf{B}_j\mathbf{Com}_j\varphi.$$

Thus the definition ensures that agents are aware of their commitments. Further properties of commitments, beliefs and results of actions are provided in [80]. In [80] actions `commit_to` and `uncommit` that change agents commitments are also discussed.

Given a basic multi-action or a joint verification action $\alpha = (\alpha_1, \ldots, \alpha_n)$ and a group of agents $G = \{j_1, \ldots, j_n\}$, the joint commitment of G to perform α is defined as:

$$\mathbf{Joint\text{-}Commit}_G\varphi \stackrel{\text{def}}{=} \bigwedge_{i=1}^{n} (\mathbf{Com}_{j_i}\alpha_i \wedge \mathbf{MB}_G\mathbf{Com}_{j_i}\alpha_i),$$

that is each agent in G is (individually) committed to performing its corresponding action in α and there is a proper awareness about these commitments in group G.

[6] In [80, 113] action `select` updating choices of agents is discussed and its semantics puts some restrictions on agents choices in worlds that this action transforms between.

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