University of Warsaw,
Faculty of Mathematics, Informatics and Mechanics

Doctoral Thesis

Applications of various compactness methods in the context of compressible fluid mechanics

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Supervisors' statements
Hereby I confirm that the presented thesis was prepared under my supervision and that it fulfils the requirements for the degree of Doctor in the field of Natural Sciences in the discipline of mathematics

February 5th, 2024

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Piotr. B. Mucha

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Author's statement
Hereby I declare that the presented thesis was prepared by me and none of its contents was obtained by means that are against the law. The thesis has never before been a subject of any procedure of obtaining an academic degree. Moreover, I declare that the present version of the thesis is identical to the attached electronic version.

February 5th, 2024

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Maja Szlenk

Abstract

The topic of the thesis is focused on different compactness methods used to construct weak solutions to equations describing the dynamics of viscous compressible fluids. The problems considered in the dissertation include the question of existence of solutions to three different systems. The first part concerns the compressible Stokes system, where existence and uniqueness of weak solutions are obtained. The structure of this equation allows to perform the analysis at the level of the Lagrangian coordinates. This brings our original problem to the investigation of the properties of the transformation from the Lagrangian to the Eulerian coordinates. The existence of such transformation is shown using the Crippa–De Lellis stability results for the transport equation. In order to show uniqueness, we improve the logarithmic inequality for BMO functions, developed by Mucha and Rusin (2008).

The second outcome of the thesis consists of the existence result for the special case of a non-Newtonian fluid. It is shown that there exists a weak solution to the compressible non-Newtonian Stokes system for the case where the shear viscosity is a singular function of a shear rate. Due to the nonlinear structure of the stress tensor, the previous approach cannot be applied. Instead, using the Calderón–Zygmund estimates we extract the BMO regularity for the quantity $\operatorname{div} u - p(\varrho)$, where u is a velocity vector and $p(\varrho)$ denotes the pressure. This information allows to adapt the compactness method, developed by Feireisl, Liao and Málek (2015) for a different class of non-Newtonian fluids.

The last result concerns the compressible, pressureless Navier–Stokes equations with the nonlocal attraction–repulsion forces. We consider the case of the density-dependent viscosity, which causes the degeneracy in the stress tensor. The higher regularity of the density are obtained via the Bresch–Desjardins estimates. We first obtain a weak solution to the system on a torus, with a suitable truncation of the nonlocal term, and then extend the spatial domain to get the result in the whole space. The construction of solutions follows the approach of Vasseur and Yu (2016). In order to show the compactness of solutions, we obtain the Mellet–Vasseur estimates, which provide the uniform integrability of a certain logarithmic function of the velocity. To incorporate the nonlocal term, we apply the generalized version of the Young inequality for convex functions.

Keywords: weak solutions, compressible flow, Stokes equation, Navier–Stokes equation, non-Newtonian fluid, nonlocal interaction forces, density-dependent viscosity

Streszczenie

Tematyka pracy doktorskiej skupia się na różnych metodach zwartościowych, stosowanych do konstrukcji słabych rozwiązań równań opisujących lepkie płyny ściśliwe. Zagadnienia rozpatrywane w rozprawie obejmują kwestię istnienia rozwiązań dla trzech różnych układów równań. Pierwszy otrzymany wynik dotyczy ściśliwego równania Stokesa, dla którego pokazane są istnienie oraz jednoznaczność słabych rozwiązań. Struktura tego równania pozwala na przeprowadzenie analizy na poziomie współrzędnych Lagrange'a. Sprowadza to nasz pierwotny problem do badania właściwości transformacji ze współrzędnych Lagrange'a do Eulera. Istnienie takiego przekształcenia zostało pokazane wykorzystując rezultaty Crippy i De Lellisa dotyczące stabilności dla równania transportu. W celu pokazania jednoznaczności, korzystamy z logarytmicznej nierówności dla funkcji z przestrzeni BMO opracowanej przez Muchę i Rusina (2008), osłabiając jednocześnie jej założenia.

Drugi wynik otrzymany w dysertacji dotyczy istnienia rozwiązań dla szczególnego przypadku płynu nienewtonowskiego. Pokazujemy istnienie słabych rozwiązań ściśliwego równania Stokesa dla cieczy, w której lepkość jest singularną funkcją szybkości ścinania. Ze względu na nieliniową strukturę tensora naprężeń, metoda opisana powyżej nie może być zastosowana. Zamiast tego, przy użyciu teorii Calderóna–Zygmunda dla operatorów singularnych, pokazujemy ograniczoność w przestrzeni BMO dla wyrażenia $\operatorname{div} u - p(\varrho)$, gdzie u to wektor prędkości a $p(\varrho)$ oznacza ciśnienie. Otrzymana regularność pozwala na zaadaptowanie metody zwartościowej, opracowanej przez przez Feireisla, Liao i Malka (2015) w kontekście innego rodzaju płynów nienewtonowskich.

Ostatnia część pracy pochyla się nad ściśliwym, bezciśnieniowym równaniem Naviera–Stokesa z nielokalnymi siłami przyciągania-odpychania. Rozważamy tu przypadek lepkości zależnej od gęstości płynu, co powoduje degenerację w tensorze naprężeń. Za pomocą oszacowań Brescha–Desjardina otrzymujemy wyższą regularność gęstości. W pierwszym kroku uzyskujemy słabe rozwiązanie na torusie z odpowiednio dostosowanym nielokalnym członem, a następnie rozszerzamy dziedzinę i w konsekwencji dostajemy wynik na całej przestrzeni. Konstrukcja rozwiązań korzysta z podejścia Vasseura i Yu (2016). W celu pokazania zwartości, otrzymujemy oszacowania Melleta–Vasseura, które zapewniają jednostajną całkowalność pewnej logarytmicznej funkcji prędkości. Nielokalny człon jest oszacowany za pomocą uogólnionej nierówności Younga dla funkcji wypukłych.

Słowa kluczowe: słabe rozwiązania, przepływ ściśliwy, równanie Stokesa, równanie Naviera–Stokesa, płyn nienewtonowski, nielokalne siły interakcji, lepkość zależna od gęstości

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Contents

Chapter 1

Introduction

1.1 Description of the problem and overview of the theory

The behaviour of the compressible, isothermal fluid is described by a system of Partial Differential Equations (PDEs), for the evolution of the density of the fluid ϱ and the velocity field u. The first equation – called the continuity equation – provides the conservation of mass:

$$\partial_t \varrho + \operatorname{div}(\varrho u) = 0. \tag{1.1}$$

In the barotropic case, it is coupled with the momentum equation

$$\partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u) - \operatorname{div} \mathbb{S}(\mathbb{D}u) + \nabla p(\varrho) = 0. \tag{1.2}$$

Above, the function $p(\varrho)$ denotes the pressure and $\mathbb{S}$ is the viscous stress tensor, depending on the symmetric gradient of u, $\mathbb{D}u = \frac{1}{2}(\nabla u + \nabla^T u)$.

The system (1.1)–(1.2) has several modifications, depending on further properties of the fluid. In the case of a Newtonian fluid it becomes the classical Navier-Stokes equations with the linear stress tensor given by

$$\mathbb{S}(\mathbb{D}u) = \mu \mathbb{D}u + \lambda \operatorname{div} u \mathbb{I}, \tag{1.3}$$

where $\mathbb{I}$ is an identity matrix, for some viscosity coefficients μ, λ. Some other considered models involve the Euler equations, used to model the ideal, inviscid gas. In that case, the stress tensor disappears and the momentum equation has the form

$$\partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u) + \nabla p(\varrho) = 0.$$

One can also mention the Navier-Stokes-Fourier system, which covers the case of non-constant temperature.

The existence results for these types of systems divide into two categories. Concerning regular solutions, satisfying the equation in the ponitwise sense, one can show only local existence, meaning that either the solution exists only on some finite time interval, or the initial conditions has to be close to the stationary state. The early results in this topic were obtained in the 1960s and 1970s in particular by Nash [82] (for the system including dependence of the temperature) and Solonnikov [95]. After that, this problem was thoroughly examined with different types of boundary conditions in the frameworks of different functional spaces for example by Matsumura and Nishida [74], Tani [101], Mucha, Valli, Zajączkowski [77, 104], Danchin [37] and many others. Since it is not the objective of this thesis, we cite only a small part of the extensive literature on this topic. An illustrative example of a solution with a finite time blow-up was also shown by Vaigant in [103].

In the case of weak solutions, the picture is very different. Since the solution satisfies the equation only in the distributional sense, the required conditions on its regularity are relaxed. Because of that, one can usually expect the global in time existence for arbitrarily large initial data. On the other hand, the low regularity of solutions becomes problematic in the construction, in particular due to the nonlinear terms. In this theory, the significant breakthrough was due to Lions [67] and Feireisl [48], who constructed weak solutions to compressible Navier-Stokes equations for the pressure in the form $p(\varrho) = \varrho^\gamma$ for $\gamma > 9/5$ (in three space dimensions) and $\gamma > 3/2$ respectively. This approach was then subsequently applied to other related systems of equations and is a core for the present studies on this topic. Some of the modifications include generalization of the pressure, in particular allowing the lack of monotonicity on a finite interval [47]. Another related results were obtained for heat conducting fluids for example by Feireisl, Mucha, Novotný and Pokorný [49, 50]. For the steady case we refer to series of papers by Mucha and Pokorný [79, 92, 76].

The way of showing existence of weak solutions involves first introducing an approximated equation, usually by adding suitable regularizing terms, and then applying chosen fixed point theorem. One of the widely used strategies is based on adding the dissipation $\varepsilon\Delta\varrho$ in the continuity equation (1.1). It changes the structure of the equation from hyperbolic into a parabolic one and greatly improves the regularity of the density. For a given u, it can be solved using the Galerkin approximation. In the case of the classical Navier-Stokes equations with linear stress tensor (1.3), this is a starting point of the construction. It is based on a fixed point argument on a velocity field, where u is again obtained by the Galerkin method from the momentum equation. For a more detailed description of this approach we refer to Chapters 7.6-7.7 in the book of Novotný and Štraskraba [85]. Another regularization technique involves mollification of the velocity in the continuity equation. In the context of the Navier-Stokes equations it was used recently in [29].

The next step is to show that the approximating sequence converges, and that the limit satisfies the target system of equations. This is usually the biggest challenge. Using the Banach-Alaoglu theorem, one can show weak compactness of the sequence of approximate solutions in some suitable spaces. However, it is not enough to pass to the limit in the nonlinear terms. The regularity of solutions is also too low to apply the well-known compactness tools like Aubin-Lions lemma. Instead, one has to either find a way to derive some better estimates for the solutions, or apply more complex tools to show the desired convergence. As mentioned earlier, in the context of classical Navier-Stokes equations (1.1)-(1.3) this problem was resolved by Lions in [67]and Feireisl in [48]. The strong convergence of the sequence of approximate densities, necessary to pass to the limit in the nonlinear pressure term, is deduced from the compactness of the effective viscous flux

$$(2\mu + \lambda)\mathrm{div}\, u - p(\varrho).$$

It is observed that this quantity has better compactness properties than $\mathrm{div}\, u$ and the pressure separately. Denoting by ϱ and u the weak limits of the approximate sequence, one can show that

$$\overline{\varrho p(\varrho)} - (2\mu + \lambda)\overline{\varrho \mathrm{div}\, u} = \varrho\overline{p(\varrho)} - (2\mu + \lambda)\varrho \mathrm{div}\, u,$$

where by $\overline{(\cdot)}$ we denote the weak limit of the respective sequence. From that relation it is concluded that $\overline{\varrho \log \varrho} = \varrho \log \varrho$, and in consequence the sequence of densities converges strongly. A different approach to show strong convergence of the density sequence, based on the Kolmogorov compactness criterion, was recently implemented by D. Bresch & P.-E. Jabin in [10], which allowed to treat non-monotone pressure term and anisotropy in the viscous stress tensor. The main idea of this approach is that the compactness of the density sequence in L^p is equivalent to the fact that

$$\limsup_{k\to\infty} \sup_{t\geq 0} \frac{1}{\|\mathcal{K}_h\|_{L^1}} \iint_{\Omega\times\Omega} \mathcal{K}_h(x-y)|\varrho(t,x) - \varrho(t,y)|^p \mathrm{d}x\mathrm{d}y \to 0$$

as $h \to 0$ for a suitably chosen family of kernels $\mathcal{K}_h$.

The crucial feature in the above methods is the $W^{1,2}$ regularity of the density, coming from the viscous stress tensor. Because of that, in the context of different types of equations and complex fluids, the question of existence of global weak solutions still remains open. An important example is the Euler equation, for which there are no global existence results so far (in the multidimensional case). On the other hand, it turns out that the system is ill-posed, i. e. admits infinitely many solutions (see e. g. [30, 31]). Another problem are the systems with additional nonlinearities in viscous stress tensor, which do not cooperate with the Lions–Feireisl technique. This is for example the case for the non-Newtonian fluids, which is one of the topic of this thesis.

1.2 Main results of the thesis

The main objective of the dissertation is to investigate different methods for showing global existence of weak solutions to equations emerging from modelling of the compressible fluids. The thesis consists of three main results, each oriented around weak solutions to distinct equations and their properties. Each of the considered systems involve completely different methods of analysis.

The first result, presented in **Chapter 2** and published in [100], concerns the compressible Stokes system

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho u) &= 0, \\ -\mu \Delta u - (\mu+\lambda)\nabla \operatorname{div} u + \nabla p(\varrho) &= 0, \end{aligned} \tag{1.4}$$

being the approximation of the Navier-Stokes system (1.1)-(1.3) in the low Reynolds number regime, where the viscous forces dominate the convective ones. Because of that, the term $\partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u)$ (which is related to the material derivative $\frac{Du}{Dt} = u_t + u \cdot \nabla u$) is neglected in the momentum equation. The spacial domain is the torus and as a initial condition we put $\varrho_{|t=0} = \varrho_0 \in L^\infty(\mathbb{T}^d)$. The main result concerns the existence and uniqueness of solutions. It is shown that if the pressure satisfy

$$0 \le p(\varrho) \le CP(\varrho) := C\left(\varrho \int_{\bar\varrho}^{\varrho} \frac{p(s)}{s^2}\, \mathrm{d}s + C_1 \varrho + C_2\right)$$

for some C, C_1, C_2, then there exists a unique global in time weak solution to (1.4), satisfying

$$\varrho, u \in L^\infty([0,\infty) \times \mathbb{T}^d), \quad \operatorname{rot} u = 0,$$

and

$$\nabla u \in L^\infty([0,\infty); BMO), \ \operatorname{div} u \in L^\infty([0,\infty) \times \mathbb{T}^d).$$

Although the existence of weak solutions to system (1.4) has been already established (see Chapter 8 in the book of Lions [67]), the method we use allows to additionally show the L^∞ bound on the density and uniqueness of solutions, which were not established before. Another feature is the required condition on the pressure. It forms a very general class of admissible functions, which in particular can be non-monotone and dropping to zero for arbitrary large arguments.

The main tool to treat the equation is to rewrite the system in the Lagrangian coordinates. As $\operatorname{rot} u = 0$, by taking the divergence of the momentum equation the system (1.4) is equivalent to

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho u) &= 0, \\ \operatorname{div} u &= p(\varrho) - \frac{1}{\mathbb{T}^d}\int_{\mathbb{T}^d} p(\varrho)\, \mathrm{d}x. \end{aligned} \tag{1.5}$$

Then we solve the continuity equation using the method of characteristics. By putting $\eta(t,y) = \varrho(t, x(t,y))$ and $\sigma(t,y) = \operatorname{div} u(t, x(t,y))$ for

$$\dot{x}(t,y) = u(t, x(t,y)), \quad x(0,y) = y, \tag{1.6}$$

we rewrite (1.5) again as

$$\begin{aligned} \partial_t \eta + \eta\sigma = 0, \\ \sigma = p(\eta) - \frac{1}{\mathbb{T}^d} \int_{\mathbb{T}^d} p(\eta) e^{\int_0^t \sigma(s,\cdot)\,\mathrm{d}s}\,\mathrm{d}y. \end{aligned}$$

For this new system, a simple ODE-like argument allows to show the L^∞ bound for the density (roughy speaking, it follows from the fact that the time derivative of η has to be negative when $p(\eta)$ becomes too large). By the elliptic estimates and the irrotational assumption on u, the L^∞ bound on $\operatorname{div} u$ also provides $\nabla u \in L^\infty(0,T;BMO)$.

The most interesting and novel part of this result is the uniqueness of solutions. It is shown that the transformation to Lagrangian coordinates is reversible. In other words, for a given σ there exists a unique solution u to equation

$$\operatorname{div} u(t, x(t,y)) = \sigma(t,y) \tag{1.7}$$

for x being the flow generated by u, defined as in (1.6). Note that if u was assumed to be Lipschitz continuous, this question would be obvious, since the flow generated by u would be invertible, the inverse being also Lipschitz continuous. In our case the situation becomes more complicated, nevertheless the improved BMO regularity for ∇u turns out to be sufficient. The strategy is to take two solutions and show that they coincide. It is done by defining a certain weighted flow between two velocities. For $u_1, u_2 \in L^\infty([0,T]\times\mathbb{T}^d)$, $\nabla u_1, \nabla u_2 \in L^\infty(0,T;BMO)$ and $s \in [0,1]$ we define x_s by

$$\dot{x}_s = s u_1(t, x_s) + (1-s) u_2(t, x_s), \quad x_s(0,y) = y.$$

One can show that

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(\mathbb{T}^d)} \le C \int_0^1 \left\| \frac{\mathrm{d}x_s}{\mathrm{d}s} \right\|_{L^2(\mathbb{T}^d)} \mathrm{d}s.$$

On the other hand, from the equation on x_s one also gets

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^d} \left| \frac{\mathrm{d}x_s}{\mathrm{d}s} \right|^2 \mathrm{d}y \le 2 \left| \int_{\mathbb{T}^d} \frac{\mathrm{d}x_s}{\mathrm{d}s} \nabla v(t,x_s) \frac{\mathrm{d}x_s}{\mathrm{d}s} \mathrm{d}y \right| \\ + \int_{\mathbb{T}^d} |u_1(t,x_s) - u_2(t,x_s)|^2 \mathrm{d}y + \int_{\mathbb{T}^d} \left| \frac{\mathrm{d}x_s}{\mathrm{d}s} \right|^2 \mathrm{d}y, \end{aligned} \tag{1.8}$$

where $v = su_1 + (1-s)u_2$. Note again that in the classical case, when u_1, u_2 are Lipschitz continuous, the function ∇v in (1.8) belongs to $L^\infty([0,T]\times\mathbb{T}^d)$ and straight from the Gronwall's lemma we get $\frac{\mathrm{d}x_s}{\mathrm{d}s} = 0$ and in consequence $u_1 = u_2$.

In the framework of BMO functions instead, the key point is to use the logarithmic inequality, proved in Lemma B.6:

$$\begin{aligned} \left| \int_{\mathbb{T}^d} fg\mathrm{d}x \right| \le & C\|f\|_{BMO}\|g\|_{L^1} \\ & \times \left(|\ln \|g\|_{L^1}| + \ln(1 + \|g\|_{L^q}) + (1 + |\ln \|g\|_{L^1}|)\|g\|_{L^q}^{\frac{q-2}{2}} \right). \end{aligned} \tag{1.9}$$

In our setting, $\nabla v \in L^\infty(0,T;BMO)$ and thus we can apply the logarithmic inequality (1.9) to the first term on the right hand side of (1.8). In consequence, we get an ODE

$$\dot{\alpha} \leq C\alpha(1 + |\ln \alpha|) + C\alpha \quad \text{for} \quad \alpha(t) = \int_0^1 \left\| \frac{\mathrm{d}x_s}{\mathrm{d}s} \right\|_{L^2(\mathbb{T}^d)} \mathrm{d}s$$

and we conclude that $\alpha(t) \equiv 0$ from Osgood's lemma and comparison criterion for ODEs.

Let us also mention the proof of existence of solutions itself. We present the alternative approach to the one presented in [67]. The approximation scheme also relays on the Lagrangian formulation and thus is consistent with the rest of the chapter. It was used before in [12] in the context of the compressible Stokes equation for multiphase flows. Here, the key feature is the stability estimate for regular Lagrangian flows, proved in [35].

The content of **Chapter 3** is the second main result in the thesis, recently published in [93] in collaboration with Milan Pokorný from Charles University in Prague. We again deal with the question of existence of weak solutions to compressible Stokes system, however in the non-Newtonian regime. In particular, it means that the viscosity parameters μ and λ depend on ∇u in a nonlinear way. Our system states

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho u) &= 0, \\ -\operatorname{div} \mathbb{S} + \nabla p(\varrho) &= 0, \end{aligned} \tag{1.10}$$

where the viscous stress tensor is given by

$$\mathbb{S}(u,\varrho) = (\mu_0(|\mathbb{D}u|) + 2\mu_1)\mathbb{D}u + (\lambda(|\operatorname{div} u|)\operatorname{div} u)\mathbb{I}$$

for $\mu_1 > 0$ constant and μ_0, λ satisfying the following growth conditions

$$0 \leq \mu_0(z), \lambda(z) \leq \frac{C}{z},$$

together with certain monotonicity assumption. Such form of $\mathbb{S}$ in particular involves the special case of Hershel-Bulkley fluid (see e. g. [42]). Similarly as before, our spacial domain is the torus $\mathbb{T}^d$ and the initial condition ϱ_0 belongs to $L^\infty(\mathbb{T}^d)$. However, this time we restrict the pressure form only to the typical barotropic case $p(\varrho) \sim \varrho^\gamma$ for some $\gamma > 1$. It is shown that there exists a global weak solution to the system (1.10), satisfying

$$\nabla u \in L^2((0,\infty) \times \mathbb{T}^d), \quad \varrho \in L^\infty(0,\infty; L^\gamma),$$

and moreover for any $T > 0$ and $1 \leq p < \infty$

$$\|\operatorname{div} u\|_{L^\infty(0,T;L^p)} + \|\varrho\|_{L^\infty(0,T;L^p)} \leq C(p,T),$$

where C approaches ∞ if p or T do so.

Although equation (1.10) has a similar structure to (1.4), the methods of analysis become completely different. The form of (1.4) in the Lagrangian coordinates strongly relies on the linear structure of the stress tensor and decomposition of the velocity field into the potential and rotational part. Here, the additional nonlinearity does not cooperate with the (linear) decomposition, and thus the previous approach cannot be applied. In particular, the L^∞ estimate for ϱ and $\operatorname{div} u$ seems to be out of reach. Instead, we adapt the compactness method developed by Feireisl, Liao and Málek in [45]. In that paper, the authors showed the existence of weak solutions to the compressible non-Newtonian Navier-Stokes equations with the assumption that $\operatorname{div} u$ is bounded, which is not the case in our situation. However, using the special form of our stress tensor, we can again work in the framework of BMO space. The starting point are the Calderón-Zygmund estimates, which provide that

$$\mu_1 \operatorname{div} u - p(\varrho) \in L^\infty(0,T;BMO). \tag{1.11}$$

It is worth pointing out that estimate (1.11) does not bring any higher regularity separately for ϱ and $\operatorname{div} u$, which are still only L^p-integrable in space for $p < \infty$. Nevertheless, it allows to apply the method from [45]). The idea lays in comparing different energy equalities. First, as usual we introduce a suitable approximation of the system and show the existence of solution via Schauder fixed point theorem. Then, calculating the energy at the level of approximated equation, by the weakly lower semicontinuity of the norm one obtains

$$\int_0^t \int \overline{\mu_0(|\mathbb{D}u|)\mathbb{D}u} : \mathbb{D}u + \mu_1|\nabla u|^2 + \mu_1(\operatorname{div} u)^2 + \overline{\lambda(|\operatorname{div} u|)\operatorname{div} u}\, \operatorname{div} u \,\mathrm{d}x\mathrm{d}s \\ + \frac{1}{\gamma-1}\int \overline{\varrho^\gamma}(t,\cdot)\mathrm{d}x \le \frac{1}{\gamma-1}\int \varrho_0^\gamma \mathrm{d}x,$$

where the monotonicity assumption on the stress tensor provided that

$$\overline{\mu_0(|\mathbb{D}u|)|\mathbb{D}u|^2} \ge \overline{\mu_0(|\mathbb{D}u|)\mathbb{D}u} : \mathbb{D}u$$

and analogously for $\lambda(|\operatorname{div} u|)$. On the other hand, using weak compactness of the approximating sequence, by passing to the limit in and then testing by the limiting u, we derive

$$\int_0^t \int \overline{\mu_0(|\mathbb{D}u|)\mathbb{D}u} : \mathbb{D}u + \mu_1|\nabla u|^2 + \mu_1(\operatorname{div} u)^2 \\ + \overline{\lambda(|\operatorname{div} u|)\operatorname{div} u}\, \operatorname{div} u \,\mathrm{d}x\mathrm{d}s = \int_0^t \int \overline{\varrho^\gamma} \operatorname{div} u \,\mathrm{d}x\mathrm{d}s.$$

Comparing these two relations, using the continuity equation we obtain

$$\frac{1}{\gamma-1}\int \big(\overline{\varrho^\gamma}(t,\cdot) - \varrho^\gamma(t,\cdot)\big)\mathrm{d}x \le -\int_0^t \int \big(\overline{\varrho^\gamma} - \varrho^\gamma\big)\operatorname{div} u\mathrm{d}x\mathrm{d}s.$$

In the case when $\operatorname{div} u$ is bounded, from Gronwall's lemma it immediately follows that $\overline{\varrho} = \varrho^\gamma$. In our case, we write

$$-\int_0^t \int \big(\overline{\varrho^\gamma} - \varrho^\gamma\big)\operatorname{div} u \,\mathrm{d}x\mathrm{d}s = -\int_0^t \int \big(\overline{\varrho^\gamma} - \varrho^\gamma\big)(\operatorname{div} u - \overline{\varrho^\gamma})\mathrm{d}x\mathrm{d}s \\ -\int_0^t \int \big(\overline{\varrho^\gamma} - \varrho^\gamma\big)\overline{\varrho^\gamma}\mathrm{d}x\mathrm{d}s.$$

Since $\overline{\varrho^\gamma} \ge \varrho^\gamma$ by the convexity of the function $z \mapsto z^\gamma$, in the end we have

$$\frac{1}{\gamma-1}\int \big(\overline{\varrho^\gamma}(t,\cdot) - \varrho^\gamma(t,\cdot)\big)\mathrm{d}x \le -\int_0^t \int \big(\overline{\varrho^\gamma} - \varrho^\gamma\big)(\operatorname{div} u - \varrho^\gamma)\,\mathrm{d}x\mathrm{d}s.$$

Finally, applying inequality (1.9) we derive the logarithmic integral inequality on the quantity $\int_{\mathbb{T}^d} (\overline{\varrho^\gamma} - \varrho^\gamma)\,\mathrm{d}x$, and the equality of weak and strong limits of the pressure follows again by the argument based on Osgood's Lemma and comparison criterion.

Having proven the strong convergence of the pressure (and in consequence also density), the strong convergence of the gradients of velocities, required to pass to the limit with the nonlinear term in the stress tensor, is then a simple consequence of the monotonicity assumptions.

The topic of **Chapter 4** concerns the more complicated system and is undoubtedly the most technical. This time, we keep the convective term in (1.2) and include the attraction-repulsion nonlocal interactions, which replace the standard pressure term. The spacial domain is the whole space $\mathbb{R}^3$. Additionally, we assume that the viscosity coefficients depend on the density. Overall, the analysed system has the form

$$\begin{aligned} \partial_t\varrho + \operatorname{div}(\varrho u) = 0, \\ \partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u) - \operatorname{div}(\varrho \mathbb{D}u) + \varrho\nabla(K * \varrho) = 0, \end{aligned} \tag{1.12}$$

where for the kernel K we assume that

$$K(x) = \frac{c_1}{|x|^\alpha} + \frac{c_2}{2}|x|^2, \quad \alpha \in (0,2).$$

The motivation to study system (1.12) comes from the models of collective behaviour, where the movement of particular species depends on other around them. The singular term provides the particles from colliding, whereas the quadratic term controls their spread in space. It is also worth pointing out that we do not require an additional (pointwise) pressure term.

The weak solutions to system (1.12) on the torus, in the case of a standard barotropic pressure ϱ^γ instead of the nonlocal term, were recently constructed by Vasseur and Yu in [106, 105]. Let us now describe their approach, and then present the modifications necessary for the nonlocal case, which is the original contribution of the thesis. First, note that the dependence of the viscosity coefficients on the density causes some degeneracy in the system. Since we allow the case of vacuum (i. e. the density might not be strictly positive), on the set where $\varrho = 0$ any information on the velocity field and its gradient is lost. It can be however compensated to some extent by virtue of an inequality derived by Bresch and Desjardins in [6], which provides higher regularity on the density. Without the nonlocal term, assuming the solutions are sufficiently regular, by testing the momentum equation by $\nabla \log \varrho$ and combining it with the energy estimate, one obtains the estimate

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} |\nabla\sqrt{\varrho}|^2 \,\mathrm{d}x + \int_0^T \int_{\mathbb{R}^3} \varrho|\nabla u - \nabla^T u|^2 \,\mathrm{d}x\mathrm{d}t \leq C. \tag{1.13}$$

In particular, the estimates on $\nabla\varrho$ provide strong compactness of the sequence of approximated densities in a suitable L^p space. To deal with the lack of estimates on the velocity, one needs to show the compactness of $\sqrt{\varrho}u$ instead. This would allow to pass to the limit and to derive system (1.12), where u is defined up to the set $\{\varrho = 0\}$. However, in order to do that, the regularity of u still needs to be improved. This is provided by the estimate

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} \varrho\frac{1+|u|^2}{2} \ln(1+|u|^2) \leq C, \tag{1.14}$$

which in turn allows to show the strong convergence of $\sqrt{\varrho}u$ in L^2 and pass to the limit in the nonlinear terms of (1.12). The above estimate was derived by Mellet and Vasseur in [75] (again in the pointwise case). The idea is based on testing the momentum equation by $(1 + \ln(1 + |u|^2))u$. For the construction of the approximate solutions, Vasseur and Yu introduced several regularizing terms which greatly improve the regularity of solutions, so that the derivation of Bresch–Desjardins inequality is justified. This allows to construct weak solutions on a periodic domain, with the additional damping terms $r_0 u + r_1\varrho|u|^2 u$ and the quantum potential $\kappa\varrho\nabla\left(\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\right)$. The Mellet–Vasseur estimate is then formally derived by using a suitable bounded truncation of a function $F(|u|) = \frac{1+|u|^2}{2}\ln(1+|u|^2)$, and by approximating the velocity field by $v = \phi(\varrho)u$, where ϕ is a certain cut-off function on the set where ϱ is close to zero or infinity. By passing to the limit with all approximations one derives the estimate (1.14), which allows to drop the remaining damping terms.

Let us now present the way how to incorporate the nonlocal term in the above approach. In the Bresch-Desjardins inequality, the term coming from the nonlocal pressure can be written as

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \nabla\varrho(t,x)\nabla\varrho(t,y)K(x-y)\,\mathrm{d}x\mathrm{d}y.$$

By computing the Fourier transform, one can show that the contribution from the singular part of K has a good sign, i. e.

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \nabla\varrho(t,x)\nabla\varrho(t,y)\frac{1}{|x-y|^\alpha}\,\mathrm{d}x\mathrm{d}y \geq 0.$$

On the other hand, for the quadratic part we get

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \nabla\varrho(t,x)\nabla\varrho(t,y)\frac{|x-y|^2}{2}\,\mathrm{d}x\mathrm{d}y = -3\|\varrho(t,\cdot)\|_{L^1}^2.$$

Overall, the estimate (1.13) still closes, however the bound will depend on time.

The derivation of the Mellet-Vasseur estimate turns out to be more complicated. The biggest issue is the fact that the kernel K goes to infinity as $|x|\to\infty$, and thus one has to carefully control the behaviour of ϱ far from the origin. The main idea to close the estimate (1.14) is based on the following Young inequality for convex functions

$$ab \leq F(a) + F^*(b)$$

for F convex, where F^* given by

$$F^*(s) = \sup\{sz - F(z) : z\in\mathbb{R}\}$$

is a convex conjugate of F. Denoting $F(z) = \frac{1+z^2}{2}\ln(1+z^2)$, we (formally) test the momentum equation by $F'(|u|)\frac{u}{|u|}$. Then, from the quadratic part of K one needs to estimate

$$\begin{aligned}\iint_{\mathbb{R}^3\times\mathbb{R}^3} F'(|u(t,x)|)|x-y|\varrho(t,x)\varrho(t,y)\,\mathrm{d}x\mathrm{d}y \leq& \|\varrho\|_{L^1}\int_{\mathbb{R}^3} F^*(F'(|u(t,x)|))\varrho(t,x)\,\mathrm{d}x\\ &+\iint_{\mathbb{R}^3\times\mathbb{R}^3} F(|x-y|)\varrho(t,x)\varrho(t,y)\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^3\times\mathbb{R}^3} F(|x-y|)\varrho(t,x)\varrho(t,y)\,\mathrm{d}x\mathrm{d}y \leq& 2\iint_{\mathbb{R}^3\times\mathbb{R}^3} F'(|x-y|)|u(t,x)\varrho(t,x)\varrho(t,y)\,\mathrm{d}x\mathrm{d}y\\ \leq& 2\iint_{\mathbb{R}^3\times\mathbb{R}^3} F^*(F'(|x-y|))\varrho(t,x)\varrho(t,y)\,\mathrm{d}x\mathrm{d}y\\ &+2\|\varrho\|_{L^1}\int_{\mathbb{R}^3} F(|u(t,x)|)\varrho(t,x)\,\mathrm{d}x.\end{aligned}$$

Since the properties of F provide that $F^*(F'(z)) \leq CF(z)$ (see also Proposition 4.21), in the end we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^3} \varrho(t,x)F(|u(t,x)|)\,\mathrm{d}x + \iint_{\mathbb{R}^3\times\mathbb{R}^3} F(|x-y|)\varrho(t,x)\varrho(t,y)\,\mathrm{d}x\mathrm{d}y\right)$$
$$\leq C\left(\int_{\mathbb{R}^3} \varrho(t,x)F(|u(t,x)|)\,\mathrm{d}x + \iint_{\mathbb{R}^3\times\mathbb{R}^3} F(|x-y|)\varrho(t,x)\varrho(t,y)\,\mathrm{d}x\mathrm{d}y\right)$$

and we close the estimate by Gronwall's lemma.

Of course the expected regularity of solutions does not allow to perform the above calculations, analogously as for the "local" case. To construct the solutions, we closely follow the approach of Vasseur and Yu. First, by introducing a suitable truncations of K and initial conditions, we restrict the system to a periodic domain. Then, the Vasseur–Yu approximation from [106] allows to derive the Bresch–Desjardins inequality also in the nonlocal case. In order to get the Mellet-Vasseur estimate, we follow the arguments from [105]. However, since the arguments to close the estimate differs from the ones by Mellet–Vasseur in [75], we need to make significant adjustments in the Vasseur–Yu approach. In particular, we modify the approximation of F, so that it remains strictly convex. Since all the calculations are performed at the level of distributional formulation of (1.12), we also use the weak version of Gronwall's lemma (see Lemma C.1) to close the estimate. Once we derive all the necessary estimates to show compactness of solutions, we pass to the limit with the size of the torus and in the end obtain solutions on the whole space. In this last step we lose the compactness properties of the nonlocal term. However, the convergence follows from the control of the double second moment

$$\sup_{t\in[0,T]} \iint_{\mathbb{R}^3\times\mathbb{R}^3} |x-y|^2\varrho(t,x)\varrho(t,y)\,\mathrm{d}x\mathrm{d}y,$$

which provides a sufficient decay of the density at infinity.

Concluding remarks. In the end, let me sum up my contributions in each of these results:

- In Chapter 2, the novel contribution to the theory is obtaining the L^∞ estimates for the compressible Stokes system and proving uniqueness of solutions. I also improved the Mucha–Rusin inequality from [80], which turns out to be a very useful tool to analyse these types of systems.

- In Chapter 3, I invented the way of adapting the method of Feireisl et al. from [45] to the particular case with unbounded divergence. I extracted the $L^\infty(0,T;BMO)$ estimate for the effective viscous flux using the Calderón–Zygmund theory for singular integrals and then used it to show compactness of the approximating sequence.

- In Chapter 4, I was resposible for adapting the Vasseur and Yu approach from [106, 105] for the nonlocal system. I defined the suitable truncations to the periodic domain and performed the construction of solutions. Compared to the literature, my most novel contribution in this part of the thesis was to find a way to derive the Mellet–Vasseur estimates. In particular, I observed that the application of the Young inequality allows to close the estimate.

Notation.

- Throughout the thesis, by C we will denote the generic positive constants.

- By $L^p(0,T;X(\Omega))$ for a Banach space $X(\Omega)$, we denote the Bochner space of functions on $[0,T]\times\Omega$ with the norm

$$\|f\|^p_{L^p(0,T;X(\Omega))} = \int_0^T \|f(t,\cdot)\|^p_{X(\Omega)}\,\mathrm{d}t,\ 1\le p<\infty,$$
$$\|f\|_{L^\infty(0,T;X(\Omega))} = \operatorname{ess}\sup_{t\in[0,T]} \|f(t\cdot)\|_{X(\Omega)},\ p=\infty.$$

  In our case, Ω will be either the torus or the whole space $\mathbb{R}^d$. For simplicity we will write X instead of $X(\Omega)$, when from the context it is clear what the spatial domain is.

- By $C(0,T;X(\Omega))$ we denote the space of functions on $[0,T]\times\Omega$, such that $t\mapsto f(t,\cdot)$ is continuous with respect to the strong topology on X. Similarly, the space $C_{\text{weak}}(0,T;X(\Omega))$ denotes the space of functions continuous in time with respect to the weak topology.

Chapter 2

Weak solutions for the Stokes system for compressible fluids with general pressure

The content of this chapter was published in [100]. We prove existence and uniqueness of global in time weak solutions for the Stokes system for compressible fluids with a general, non-monotone pressure. First, we find the unique solution at the level of Lagrangian formulation and then define the transformation to the original Eulerian coordinates. For a nonnegative and bounded initial density, the solution is nonnegative for all $t > 0$ as well, and belongs to $L^\infty([0,\infty) \times \mathbb{T}^d)$. A key point of our considerations is to show that transformation from Lagrangian to Eulerian coordinates is unique. Since the velocity might not be Lipschitz continuous, we develop a method which relies on the results of Crippa & De Lellis, concerning regular Lagriangian flows. The uniqueness is obtained thanks to the application of a certain weighted flow and detailed analysis based on the properties of the BMO space.

2.1 Introduction

The Stokes system is an approximation of the Navier-Stokes equations for small Reynolds number. In such cases, the advective intertial forces are relatively small and explicit dependence on time and convective term can be omitted. This is a typical situation for highly viscous fluids, or when the velocities are very small. The flows satisfying these conditions are called Stokes or creeping, and they occur in numerous biological and physical problems, e.g. to describe dynamics of the blood in a process of sedimentation [88], or to model swimming of microorganisms [43, 52, 63]. Other applications include also engineering, where the Stokes flow is used in the process of designing microfluids and microdevices [56, 98]. The Stokes model is also connected to the Darcy law, which describes the flow of a fluid through porous media. Such phenomena are observed in biological tissues [14, 41] and have many applications in petroleum engineering [65, 81]. The other situation, where the fluid motion is governed by the Stokes equation is a laminar flow. In this case, the fluid particles move in adjacent layers, with little mixing between them.

We consider the compressible Stokes flow on the d-dimensional torus $\mathbb{T}^d$

$$\begin{cases} \varrho_t + \operatorname{div}(\varrho u) = 0, \\ -\mu\Delta u - \nabla(\lambda+\mu)\operatorname{div} u + \nabla p(\varrho) = 0, \end{cases} \tag{2.1}$$

where $\varrho\colon [0,T] \times \mathbb{T}^d \to \mathbb{R}$ and $u\colon [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ are the sought fluid density and velocity field. The function $p(\varrho)$ denotes the pressure term, and the parameters μ, λ represent the shear and bulk viscosity.

We further assume that the flow is potential, that is $\mathrm{rot} u = 0$. It is equivalent to velocity having the form $u(t,x) = \nabla\phi(t,x)$ for some $\phi\colon [0,T]\times\mathbb{T}^d \to \mathbb{R}$. In consequence, we obtain the condition $\int_{\mathbb{T}^d} u(t,x)\,\mathrm{d}x = 0$. The second equation of (2.1) can be then rewritten in terms of the effective viscous flux, which turns out to be constant. Therefore instead of the second equation of (2.1) we obtain

$$(\lambda + 2\mu)\mathrm{div}\, u = p(\varrho) - \{p(\varrho)\},$$

where $\{f\} = \frac{1}{|\mathbb{T}^d|}\int_{\mathbb{T}^d} f(x)\,\mathrm{d}x$. As the qualitative properties of solutions do not depend of the values of λ and μ, without loss of generality we take $\lambda + 2\mu = 1$. Under these assumptions, the system (2.1) can be transformed into

$$\begin{aligned} \varrho_t + \mathrm{div}(\varrho u) &= 0, \\ \mathrm{div}\, u = p(\varrho) - \{p(\varrho)\}, &\quad \mathrm{rot} u = 0. \end{aligned} \tag{2.2}$$

The system (2.2) is coupled with the initial condition on the density, which is assumed to be bounded and nonnegative, namely

$$\varrho_{|t=0} := \varrho_0 \in L^\infty(\mathbb{T}^d), \quad \varrho_0 \geq 0.$$

It is worth emphasizing that we do not require the density to be strictly positive. In particular ϱ_0 can be equal $\mathbb{1}_A$ for some $A \subset \mathbb{T}^d$.

Our method allows the pressure to be in a quite general form. We require $p(\varrho)$ to be of class C^1 and unbounded, so that in particular we can choose a sequence $\varrho_n \to \infty$ such that $p(\varrho_n) \to \infty$. Moreover, we assume that there exist constants $C, C_1, C_2, \bar{\varrho}$ such that p satisfies the inequality

$$0 \leq p(\varrho) \leq CP(\varrho) := C\left(\varrho\int_{\bar{\varrho}}^{\varrho} \frac{p(s)}{s^2}\,\mathrm{d}s + C_1\varrho + C_2\right). \tag{2.3}$$

The properties of functions satisfying (2.3) are discussed in Subsection 2.1.2.

2.1.1 Statement of the main theorem

The mathematical theory of weak solutions to the compressible fluid equations has been widely developing in the last twenty years, since the groundbreaking results of Lions in 1998 [67] and Feireisl [48, 46] in 2001. They proved the existence of weak solutions to the compressible Navier-Stokes equations, provided that the pressure term is of the form $p(\varrho) = \varrho^\gamma$ with $\gamma > \frac{9}{5}$ and $\gamma > \frac{3}{2}$ respectively. In [47] and [99], this method was also adjusted to the pressure which is non-monotone on some finite interval. In particular, it allows to deal with the pressures expressed via equations of state, which are of more complex form than ideal gas, the model example being van der Waals' equation of state. The Lions & Feireisl approach can be also adjusted to more complex systems, for example Navier-Stokes-Fourier system [49, 50] and other including entropy transport [69] or heat conductivity [8].

The important results concerning non-monotone pressure laws are due to Bresch and Jabin [10, 11]. Their method, based on the Kolmogorov compactness criterion, allows to deal with the pressure satisfying

$$C^{-1}\varrho^\gamma - C \leq p(\varrho) \leq C\varrho^\gamma + C \quad \text{and} \quad |p'(\varrho)| \leq \bar{p}\varrho^{\gamma-1} \tag{2.4}$$

for $\gamma > \frac{9}{5}$. In context of our work, we refer the reader especially to [11], where this approach was also presented for a modification of the Stokes system. In this case Bresch and Jabin proved the existence of global weak solutions for any $\gamma > 1$ with the same regularity as in the isentropic case, namely $\varrho \in L^\infty(0,T;L^\gamma)$.

Besides the relaxed conditions on the pressure, the Bresch & Jabin compactness criterion can be also applied to various classes of equations, where the Lions and Feireisl method was insufficient. One of the examples are the systems with the additional term in the continuity equation. Such models can be obtained from the multi-fluid systems [12] or appear in the mathematical modelling of tumor growth [107]. In this case the additional term results in lack of compensated compactness between $\operatorname{div} u$ and the pressure and therefore the classical method fails. However, the Bresch & Jabin criterion allows to omit this problem. The problem with the convergence of effective viscous flux arise also in the anisotropic case. The recent result [5], concerning the anisotropic compressible Stokes system, resolves this problem by controlling a certain defect measure associated to the pressure.

The other related topic are steady flows, where the behaviour of the fluid does not depend on time. The equations describing such flows are the classical equations of fluid mechanics with the time derivative set as zero, for example the compressible Navier-Stokes [89, 91] or the Navier-Stokes-Fourier systems [79, 57]. Another analyzed system is the steady Oseen flow [90], which is a linearization of the Navier-Stokes system with partial consideration of the convective forces. Note that in our case the explicit dependence on time is removed only in the momentum equation, therefore this system can be considered as an intermediate step between steady and unsteady flows.

As the Stokes system has a simpler structure than the Navier-Stokes system, the analysis can be carried out in the more general setting. If ϱ_0 has higher regularity, then the solutions to the Stokes problem exist and are unique for the general p (see [67], Remark 8.14). However, in case of $\varrho_0 \in L^\infty(\mathbb{T}^d)$, the classical method requires the monotonicity condition on the pressure and the uniqueness was not established. In this paper we obtain the uniqueness of solutions to the Stokes system in case of the low regularity of the initial density and under very general pressure laws. Our main theorem states as follows:

Theorem 2.1. *Let $\varrho_0 \in L^\infty(\mathbb{T}^d)$, $\varrho_0 \geq 0$ and the pressure satisfy (2.3). Then, there exists a unique global in time weak solution to (2.2), satisfying*

$$\varrho, u \in L^\infty([0,\infty) \times \mathbb{T}^d)$$

and

$$\nabla u \in L^\infty([0,\infty); BMO),\ \operatorname{div} u \in L^\infty([0,\infty) \times \mathbb{T}^d),$$

that is for each $\varphi \in C_0^\infty([0,\infty) \times \mathbb{T}^d)$

$$-\int_0^\infty \int_{\mathbb{T}^d} \varrho \partial_t \varphi \,\mathrm{d}x\mathrm{d}t - \int_0^\infty \int_{\mathbb{T}^d} \varrho u \cdot \nabla \varphi \,\mathrm{d}x\mathrm{d}t = \int_{\mathbb{T}^d} \varrho_0 \varphi(0,\cdot)\,\mathrm{d}x$$

and the second equation of (2.2) is satisfied a. e.

Remark 2.2. Note that from the definition of a weak solution in Theorem 2.1, it follows that the solution (ϱ, u) satisfies the system (2.1) in the distributional sense, and thus it corresponds to a conventional definition of a weak solution to the Stokes system.

Our approach is based on the Lagrangian reformulation of the system, which allows us to obtain a global L^∞ estimate on the density. Having that estimate, we can straightforwardly apply Bresch and Jabin method to obtain compactness, and in consequence existence of solutions for the relaxed conditions on the pressure. In addition, using the results from the theory of transport equations and classical harmonic analysis, we were able to establish also uniqueness of solutions.

The first step is to reformulate the system in the Lagrangian coordinates. Assuming that (ϱ, u) is a solution and u is sufficiently smooth, let $x\colon [0,T]\times\mathbb{T}^d \to \mathbb{T}^d$ be a solution to the ODE

$$\dot{x} = u(t,x), \quad x(0,y) = y.$$

Rewriting the system in the new variables $\eta(t,y) = \varrho(t, x(t,y))$ and $\sigma(t,y) = \operatorname{div} u(t,x(t,y))$, we obtain the set of equations in a simpler form, for which we are able to find a unique solution. Then, it suffices to define a transformation back to the original Eulerian coordinates. However, one encounters some difficulties in the construction, resulting from the low regularity of the solution, namely $\nabla u \in L^\infty(0,T;BMO)$. In particular, ∇u may not be bounded with respect to space variable and in consequence the flow $x(t,y)$ generated by u may not be invertible on the whole torus. Nevertheless, the divergence of u remains bounded, therefore $x(t,y)$ is a regular Lagrangian flow and we can treat it using the properties from [34] and [35]. The key tool needed in our analysis is the recent result of Crippa and De Lellis [35], concerning the stability of regular Lagrangian flows in L^1, which allows us to pass to the limit with the smooth approximation of the system.

The above reasoning, however, does not preserve uniqueness. We prove the latter by taking two solutions (ϱ_1, u_1) and (ϱ_2, u_2) satisfying the same Lagrangian formulation, and showing that $u_1 = u_2$ (the equality of ϱ_1 and ϱ_2 follows then from the uniqueness of solutions to continuity equation). For this purpose we introduce the flow x_s, which for $s \in [0,1]$ is generated by a convex combination of u_1 and u_2:

$$\dot{x}_s = s u_1(t, x_s) + (1-s) u_2(t, x_s), \quad x_s(0,y) = y.$$

The key point is to estimate $\|u_1 - u_2\|_{L^2(\mathbb{T}^d)}$ by a suitable norm of $\frac{\mathrm{d}x_s}{\mathrm{d}s}$. Then, having the BMO regularity of ∇u_i, $i = 1,2$, we use the John-Nirenberg inequality and the integral inequality for functions of bounded mean oscillation from [80, 78] to obtain the necessary estimates for $\frac{\mathrm{d}x_s}{\mathrm{d}s}$. Finally, using the above tools we are able to show that x_s does not depend on s and in consequence $u_1 = u_2$.

The rest of the chapter is divided into sections, which contain the main steps of the proof of Theorem 2.1. The structure of the proof is as follows:

- In Section 2.2 we present the a priori estimates and results at the level of Lagrangian coordinates, namely the L^∞ bounds and unique existence of a solution in the Lagrangian reformulation.

- Section 2.3 contains the necessary tools and definitions, together with the proof of uniqueness of solutions to system (2.2).

- In Section 2.4 we define the transformation from Lagrangian to Eulerian coordinates using the construction from [12], and therefore prove the existence of solutions to (2.2). Note that the estimates obtained in Section 2.2 provide the BMO regularity of the gradient, necessary to obtain uniqueness The existence of solutions could be done independently, using for example standard Lions method and the Bresch & Jabin compactness criterion. Nevertheless, we present here an alternative approach, which is set in the framework of Lagrangian regular flows and therefore is more consistent with the rest of this paper.

Notation remarks: For the notational simplicity, we omit the subscript while integrating over the torus, namely

$$\int \mathrm{d}x \; := \; \int_{\mathbb{T}^d} \mathrm{d}x.$$

By $\{\cdot\}_Q$ we denote the mean integral over Q, while in the case of the whole torus we again omit the subscript.
Moreover, to distinguish between the norms in $L^\infty(\mathbb{T}^d)$ and $L^\infty([0,T]\times\mathbb{T}^d)$, we denote

$$\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\mathbb{T}^d)} \quad \text{and} \quad \|\cdot\|_{\infty,T} := \|\cdot\|_{L^\infty([0,T]\times\mathbb{T}^d)}.$$

2.1.2 Discussion on the pressure term

First, observe that our condition (2.3) in particular includes the assumptions from [11]: if p satisfies (2.4), then $P(\varrho) \geq c\varrho^\gamma - c\varrho$ and for a sufficiently large C we obtain (2.3). However, our assumptions also allow the pressure to drop to 0 for arbitrary large ϱ and we do not require the bounds on the derivative.

Such class of possible pressures includes many physical situations. It contains the cases which were covered before, for example van der Waals' fluid. However, our assumptions also allow the pressure to be expressed via virial expansion, namely defined as a power series of the density:

$$p(\varrho) = \sum_{k=1}^{\infty} B_k \varrho^k,$$

where coefficients B_k depend on the temperature and are derived from statistical mechanics. The virial equation of state was also considered in [10], but our result allows wider range of pressures of this type. The other case, where our result may be applicable, is the use in biological models, where the pressure term is responsible for interactions between different biological agents and therefore can take form other than resulting from physical constitutive laws.

Let us present some further properties of p satisfying (2.3):

- Condition (2.3) implies that in particular $p(\varrho) \leq \tilde{C}\varrho^\gamma + \tilde{C}$ for $\varrho \geq \bar{\varrho}$ and some $\gamma > 1$: Let p satisfy (2.3) and define $\alpha(\varrho) = \int_{\bar{\varrho}}^{\varrho} \frac{p(s)}{s^2}\mathrm{d}s$, $\alpha(\bar{\varrho}) = 0$. Then

  $$\alpha'(\varrho) = \frac{p(\varrho)}{\varrho^2} \leq \frac{C\alpha(\varrho)}{\varrho} + \frac{C}{\varrho}\left(1 + \frac{1}{\varrho}\right)$$

  and by the comparison criterion $\alpha(\varrho) \leq \bar{C}\varrho^C - \frac{C}{C+1}\frac{1}{\varrho} - 1$ for $\varrho \geq \bar{\varrho}$, where $\bar{C}$ depends on $\bar{\varrho}$. Therefore

  $$p(\varrho) \leq C(\varrho\alpha(\varrho) + \varrho + 1) \leq \bar{C}C\varrho^{C+1} + \frac{C^2}{C+1} \leq \tilde{C}\varrho^\gamma + \tilde{C}$$

  for $\gamma = C + 1$ and a suitable $\tilde{C}$.

- On any finite interval we can estimate p by sufficiently large constant, hence in particular (2.3) is fulfilled. Therefore to check if indeed p satisfies (2.3) for all $\varrho \geq 0$, it suffices to analyse the behaviour of P when $\varrho \to \infty$. It is immediate to check that if $p(\varrho) \leq C\varrho^\gamma + C$ for some $\gamma > 1$, and P satisfies

  $$\liminf_{\varrho\to\infty} \frac{P(\varrho)}{\varrho^\gamma} = \liminf_{\varrho\to\infty} \frac{1}{\varrho^{\gamma-1}} \int_{\bar{\varrho}}^{\varrho} \frac{p(s)}{s^2}\mathrm{d}s \geq c > 0,$$

  then p satisfies (2.3), however these conditions are not equivalent.

- The most significant difference between our class of admissible pressures and the cases considered before is that we allow the pressure to drop to 0 even for large ϱ. Moreover, the derivative of p may grow arbitrarily fast. For example, let $f\colon [0,\infty) \to [0,\infty)$ be a smooth, increasing function such that f' is also increasing and define

$$p(\varrho) = \varrho^2(1+\cos(f(\varrho))).$$

By the alternating series test, the integral

$$\int_{\bar\varrho}^{\infty} \cos(f(x))\mathrm{d}x = \int_{f(\bar\varrho)}^{\infty} \frac{\cos(y)}{f'(f^{-1}(y))}\mathrm{d}y$$

is convergent in the sense of Riemann. Therefore we have

$$\lim_{\varrho\to\infty} \frac{1}{\varrho}\int_{\bar\varrho}^{\varrho} \frac{p(s)}{s^2}\mathrm{d}s = \lim_{\varrho\to\infty} \frac{1}{\varrho}\int_{\bar\varrho}^{\varrho} 1+\cos(f(s))\mathrm{d}s = 1$$

and p satisfies (2.3). Moreover, it periodically drops to 0 and the derivative of p depends on f', which can be arbitrarily large.

- The condition $p(\varrho) \le C\varrho^\gamma + C$ is not sufficient to obtain (2.3). For example, let η be a smooth function supported in $[-1,1]$ such that $0 \le \eta \le 1$ and $\eta(0) = 1$. Define

$$p(\varrho) = \begin{cases} \varrho^2\eta(2^k(\varrho-k)) & \text{for } k-2^{-k} \le \varrho \le k+2^{-k}, k = 1,2,\ldots \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

Then

$$\begin{aligned}\int_0^k \frac{p(s)}{s^2}\mathrm{d}s &= \sum_{i=1}^{k-1}\int_{i-2^{-i}}^{i+2^{-i}} \eta(2^i(s-i))\mathrm{d}s + \int_{k-2^{-k}}^{k} \eta(2^k(s-k))\mathrm{d}s \\ &= \sum_{i=1}^{k-1} 2^{-i}\int_{-1}^{1}\eta\,\mathrm{d}x + 2^{-k}\int_{-1}^{0}\eta\,\mathrm{d}x \\ &= (1-2^{-k+1})\int_{-1}^{1}\eta\,\mathrm{d}x + 2^{-k}\int_{-1}^{0}\eta\,\mathrm{d}x\end{aligned}$$

and therefore for any C we can choose sufficiently large k such that

$$p(k) = k^2 \ge C\left(k\int_0^k \frac{p(s)}{s^2}\mathrm{d}s + k + 1\right) \approx \tilde{C}k.$$

In Figure 2.1 one can find an illustrative comparison between a typical example of a function satysfying (2.3) and not.

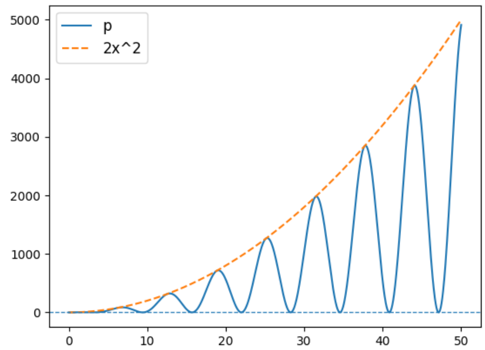
2.2 The L^∞ bound on the density

2.2.1 Energy estimates

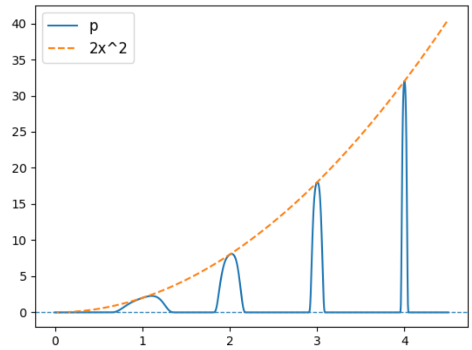
First, we obtain the a priori estimates for our solutions.

Lemma 2.3. *Let (ϱ, u) be a sufficiently smooth solution to (2.2). Then it satisfies the estimate*

$$\int_0^T\int (\operatorname{div} u)^2\,\mathrm{d}x\mathrm{d}t + \sup_{t\in[0,T]}\int P(\varrho)\,\mathrm{d}x \le C\int P(\varrho_0)\,\mathrm{d}x.$$



(A) An example of a function satisfying (2.3).



(B) A function which does not satisfy (2.3).

FIGURE 2.1: Figure 2.1a shows the plot of a function $p(\varrho) = \varrho^2(1+\cos\varrho)$, which satisfies (2.3). Figure 2.1b shows the plot of p given by (2.5), which is an example of a function having similar properties as the function in Figure 2.1a, but not satisfying (2.3).

Proof. By multiplying $(2.2)_2$ by $\operatorname{div} u$ and integrating by parts over the torus, we get

$$\int (\operatorname{div} u)^2 \,\mathrm{d}x - \int p(\varrho)\operatorname{div} u \,\mathrm{d}x = 0.$$

Moreover,

$$\begin{aligned} -\int p(\varrho)\operatorname{div} u \,\mathrm{d}x &= \int \nabla p \cdot u \,\mathrm{d}x = \int \frac{p'(\varrho)}{\varrho}\nabla\varrho \cdot (\varrho u)\,\mathrm{d}x = -\int P'(\varrho)\cdot \operatorname{div}(\varrho u)\,\mathrm{d}x \\ &= \int P'(\varrho)\varrho_t \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int P(\varrho)\,\mathrm{d}x. \end{aligned}$$

After integration over time, we get the desired estimate. □

The above a priori bounds also provide that $p(\varrho) \in L^\infty(0,T;L^1)$. Indeed, from assumption (2.3) we have

$$\sup_{t\in[0,T]} \int p(\varrho)\mathrm{d}x \le C \sup_{t\in[0,T]} \int P(\varrho)\mathrm{d}x \le C. \tag{2.6}$$

2.2.2 The Lagrangian formulation

To prove the global L^∞ estimate on the density, we need to rewrite the system (2.2) in the Lagrangian coordinates. That is, we carry out a certain change of variables, which allows us to reduce the continuity equation to a simple ordinary differential equation.

If u is a velocity field, then the trajectory of a single fluid parcel moves along the integral curve of $\dot{x} = u(t,x)$. Therefore, if at the starting time the particle was at a point y, then after the time t it would be at the point $x(t,y)$, where $x(t,y)$ is the solution to the Cauchy problem

$$\begin{aligned} \dot{x}(t,y) &= u(t,x(t,y)), \\ x(0,y) &= y. \end{aligned}$$

The above ODE defines the flow $x\colon [0,T]\times\mathbb{T}^d \to \mathbb{T}^d$ generated by u. Note that if u is sufficiently smooth, from the classical theory of ODEs it follows that the solution exists locally. Since in our case x lays on the torus, it cannot blow up in finite time, which gives the

global existence of the flow.
Differentiating our ODE with respect to y and using the Liouville formula, we obtain the equation for the Jacobian $J = \det \frac{\mathrm{d}x}{\mathrm{d}y}$:

$$\begin{aligned} \dot{J}(t,y) &= \operatorname{div} u(t,x(t,y))J(t,y), \\ J(0,y) &= 1, \end{aligned}$$

and in consequence $J(t,y) = \exp\left(\int_0^t \operatorname{div} u(s,x(s,y))\mathrm{d}s\right)$. This in particular means that in the classical setting $x(t,\cdot)$ is a diffeomorphism on $\mathbb{T}^d$ for any t.

We rewrite the unknowns of the system (2.2) in terms of y instead of x, so at a time t the space variable is the position of a parcel starting from y. Let

$$\eta(t,y) = \varrho(t,x(t,y)) \quad \text{and} \quad \sigma(t,y) = \operatorname{div} u(t,x(t,y)).$$

Then the system (2.2) has the form

$$\begin{aligned} \partial_t \eta + \eta\sigma &= 0, \\ \sigma &= p(\eta) - \{p(\eta)\}_\sigma, \end{aligned} \tag{2.7}$$

where $\{\cdot\}_\sigma$ is the mean integral in the new coordinates given by

$$\{f\}_\sigma = \frac{1}{|\mathbb{T}^d|}\int f(t,y)\exp\left(\int_0^t \sigma(s,y)\mathrm{d}s\right)\mathrm{d}y.$$

Now for the system (2.7) we obtain the following result:

Theorem 2.4. *For $\varrho_0 \in L^\infty(\mathbb{T}^d)$ and any $T > 0$ there exists a unique solution*

$$(\eta,\sigma) \in L^\infty([0,T]\times\mathbb{T}^d) \times L^\infty([0,T]\times\mathbb{T}^d)$$

to the system (2.7) with the initial condition $\eta(0,y) = \varrho_0(y)$. Moreover, there exists a constant r, independent of T, such that

$$\|\eta\|_{\infty,T} \le r.$$

The immediate consequence of Theorem 2.4 is the similar uniform bound on σ. As T is arbitrary, we hence obtain the existence of a solution on the whole half-line.

The proof of the existence of a unique solution is a standard application of the Banach Fixed Point Theorem and is presented in the Appendix A.1. Below, we show only the second part of Theorem 2.4, namely the L^∞ bound on η.

Proposition 2.5. *If $(\eta,\sigma) \in L^\infty([0,T]\times\mathbb{T}^d) \times L^\infty([0,T]\times\mathbb{T}^d)$ is a solution to (2.7), then*

$$\|\eta\|_{\infty,T} \le r,$$

where the constant r does not depend on T.

Proof. Let us solve the first equation of (2.7). We get the identity

$$\eta(t,y) = \varrho_0(y)\exp\left(-\int_0^t \sigma(s,y)\mathrm{d}s\right). \tag{2.8}$$

Note that this explicit formula for η provides that in particular if σ is bounded and the initial density ϱ_0 is strictly positive, then for any t the density is strictly positive as well.

Using (2.8), we see that η is also continuous with respect to time. For a fixed y, we have

$$\begin{aligned}|\eta(t+\varepsilon,y)-\eta(t,y)| &= \varrho_0(y)\exp\left(-\int_0^t \sigma(s,y)\mathrm{d}s\right)\left|\exp\left(-\int_t^{t+\varepsilon}\sigma(s,y)\mathrm{d}s\right)-1\right| \\ &\le \varrho_0(y)e^{t\|\sigma\|_{\infty,T}}\left|e^{\varepsilon\|\sigma\|_{\infty,T}}-1\right|.\end{aligned}$$

Hence $|\eta(t+\varepsilon,y)-\eta(t,y)|$ goes to 0 as $\varepsilon\to 0$ and indeed $\eta(\cdot,y)$ is continuous.

The continuity of η allows us to show global boundedness. Recall that by virtue of (2.6), the mean value $\{p(\eta)\}_\sigma$ is bounded. Let

$$\sup_{t\in[0,T]}\{p(\eta)\}_\sigma = M.$$

From (2.6) M is finite and does not depend on T. As p is unbounded, we can choose $r > \|\varrho_0\|_{L^\infty(\mathbb{T}^d)}$ such that $p(r) > M$. Then, at the point $\eta = r$ we get

$$\partial_t\eta|_{\eta=r} = -r(p(r)-\{p(\eta)\}_\sigma) < -r(p(r)-M) < 0.$$

However, as $t\mapsto\eta(t,y)$ is continuous for a fixed y and $\eta(0,y)=\varrho_0(y)<r$, if it exceeds the value r, it must have a nonnegative derivative at that point, which gives a contradiction. Hence for any $y\in\mathbb{T}^d$ the function $\eta(\cdot,y)$ is also bounded by r. □

2.3 Uniqueness of solutions

Using Lagrangian coordinates introduced in the previous section, we are able to show that the solutions to (2.2) are unique.

Theorem 2.6. *If (ϱ_i,u_i), $i=1,2$ are solutions to (2.2) with the regularity from Theorem 2.1, then $(\varrho_1,u_1)=(\varrho_2,u_2)$.*

First, let us show that if (ϱ,u), is a solution to (2.2) satisfying

$$\varrho,u,\operatorname{div}u\in L^\infty([0,T]\times\mathbb{T}^d),\quad \nabla u\in L^\infty(0,T;BMO),$$

then in the Lagrangian coordinates it satisfies (2.7). Under that regularity of u the classical theory of ODEs does not apply, however from the results from transport theory of DiPerna & Lions [40] it follows that there exists a unique flow $x(t,y)$ generated by u, such that $x\in C(0,T;L^p)$ for any $1<p<\infty$ and

$$\dot{x}=u(t,x),\quad x(0,y)=y.$$

Moreover, if ϱ is a solution to the continuity equation

$$\varrho_t+\operatorname{div}(\varrho u)=0,\quad \varrho(0,\cdot)=\varrho_0, \tag{2.9}$$

then the function $\varrho(t,x(t,y))$ is given by

$$\varrho(t,x(t,y))=\varrho_0(y)\exp\left(-\int_0^t \operatorname{div}u(s,x(s,y))\mathrm{d}s\right)=\varrho_0(y)\exp\left(-\int_0^t\sigma(s,y)\mathrm{d}s\right)$$

and therefore $\eta(t,y) = \varrho(t,x(t,y))$ satisfies the first equation of (2.7) with $\sigma(t,y) = \operatorname{div} u(t,x(t,y))$. Furthermore, taking the second equation of (2.2) at a point $x(t,y)$, we obtain

$$\sigma(t,y) = p(\eta(t,y)) - \frac{1}{\mathbb{T}^d}\int p(\varrho(t,x))\mathrm{d}x.$$

However, by Lemma 3.1. from [34], for any $f \in L^1(\mathbb{T}^d)$ we have

$$\int f(x)\mathrm{d}x = \int f(x(t,y))e^{\int_0^t \operatorname{div} u(s,x(s,y))\mathrm{d}s}\mathrm{d}y = \int f(x(t,y))e^{\int_0^t \sigma(s,y)\mathrm{d}s}\mathrm{d}y.$$

Thus

$$\int p(\varrho(t,x))\mathrm{d}x = \int p(\varrho(t,x(t,y))e^{\int_0^t \sigma(s,y)\mathrm{d}s}\mathrm{d}y = \int p(\eta(t,y))e^{\int_0^t \sigma(s,y)\mathrm{d}s}\mathrm{d}y$$

and σ satisfies the second equation of (2.7).

From the uniqueness of solutions in the Lagrangian formulation, we conclude that if (ϱ_i,u_i), $i=1,2$ are solutions to (2.2), then they are equal at the level of Lagrangian coordinates. In particular,

$$\operatorname{div} u_1(t,x_1(t,y)) = \operatorname{div} u_2(t,x_2(t,y)) = \sigma(t,y).$$

Therefore the uniqueness of the solutions to (2.2) is equivalent to the uniqueness of solutions to the equation

$$\operatorname{div} u(t,x(t,y)) = \sigma(t,y), \tag{2.10}$$

where $\sigma \in L^\infty([0,T]\times\mathbb{T}^d)$ is given and $x(t,y)$ satisfies the ODE

$$\begin{aligned} \dot{x}(t,y) &= u(t,x), \\ x(0,y) &= y. \end{aligned} \tag{2.11}$$

Having the unique u, the uniqueness of ϱ follows then again from the classical results from transport theory (see [40]). The regularity of u provides that in particular $u \in L^1(0,T;W^{1,p})$ for some $p \geq 1$ and $\operatorname{div} u \in L^1(0,T;L^\infty)$. Therefore there exists a unique $\varrho \in L^\infty([0,T]\times\mathbb{T}^d)$, which is a solution to the continuity equation (2.9).

2.3.1 Definition of the flow x_s

For u_1 and u_2 being the solutions to (2.10)-(2.11), we introduce the *weighted flow between* u_1 *and* u_2, that is a function $x_s(t,y)$, $s \in [0,1]$ such that x_s for $s=0$ is the flow generated by u_2 and for $s=1$ the flow generated by u_1. Such x_s is defined by the ordinary differential equation

$$\begin{aligned} \dot{x_s}(t,y) &= su_1(t,x_s) + (1-s)u_2(t,x_s) \\ x_s(0,y) &= y \end{aligned} \tag{2.12}$$

for $s \in [0,1]$. Note that since for every $s \in [0,1]$

$$\|s\operatorname{div} u_1(t,x_s) + (1-s)\operatorname{div} u_2(t,x_s)\|_\infty \leq \|\sigma\|_\infty,$$

the Jacobian $J_s = \det\frac{\mathrm{d}x_s}{\mathrm{d}y}$ of x_s satisfies the same bounds as the Jacobian J of x_1 and x_2, namely

$$e^{-L} \leq J_s(t,y) \leq e^L \quad \text{with} \quad L = \int_0^T \|\sigma\|_\infty \mathrm{d}t. \tag{2.13}$$

The first step to show uniqueness is to obtain certain L^p estimates for the derivative of x_s with respect to s.

Lemma 2.7. *If x_s is defined by (2.12) for $u_1, u_2 \in L^\infty([0,T]\times\mathbb{T}^d)$ such that $\nabla u_i \in L^\infty(0,T;BMO)$, $i=1,2$, then for sufficiently small t, $\frac{\mathrm{d}x_s}{\mathrm{d}s}(t,\cdot)\in L^p$ for some $p>4$.*

Proof. Differentiating $\dot{x}_s$ with respect to s, we get

$$\frac{\mathrm{d}\dot{x}_s}{\mathrm{d}s} = u_1(t,x_s) - u_2(t,x_s) + (s\nabla u_1(t,x_s) + (1-s)\nabla u_2(t,x_s))\frac{\mathrm{d}x_s}{\mathrm{d}s}, \tag{2.14}$$

hence from the Gronwall's lemma

$$\left|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right| \le \exp\left(\int_0^t |s\nabla u_1(\tau,x_s) + (1-s)\nabla u_2(\tau,x_s)|\mathrm{d}\tau\right)\int_0^t |u_1(\tau,x_s)-u_2(\tau,x_s)|\mathrm{d}\tau.$$

Let $\nabla v = s\nabla u_1 + (1-s)\nabla u_2$. As $u_1,u_2\in L^\infty([0,T]\times\mathbb{T}^d)$, we have

$$\int_0^t |u_1(\tau,x_s)-u_2(\tau,x_s)|\,\mathrm{d}\tau \le T(\|u_1\|_{\infty,T}+\|u_2\|_{\infty,T}).$$

Therefore

$$\int\left|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right|^p \mathrm{d}y \le CT^p\int\exp\left(p\int_0^t|\nabla v(\tau,x_s)|\mathrm{d}\tau\right)\mathrm{d}y.$$

We will now estimate

$$I = \int\exp\left(p\int_0^t|\nabla v(\tau,x_s(\tau,y))\mathrm{d}\tau\right)\mathrm{d}y.$$

By Jensen's inequality and the bounds on Jacobian J_s, we have

$$\begin{aligned}
\int e^{p\int_0^t|\nabla v(\tau,x_s(\tau,y))|\mathrm{d}\tau}\mathrm{d}y &\le \int\frac{1}{t}\int_0^t e^{pt|\nabla v(\tau,x_s(\tau,y))|}\,\mathrm{d}\tau\mathrm{d}y\\
&\le \left\|\frac{1}{J_s}\right\|_{\infty,T}\int\frac{1}{t}\int_0^t e^{pt|\nabla v(\tau,x_s(\tau,y))|}J_s(\tau,y)\,\mathrm{d}\tau\mathrm{d}y\\
&\le e^L\frac{1}{t}\int_0^t\int e^{pt|\nabla v(\tau,x)|}\,\mathrm{d}x\mathrm{d}\tau.
\end{aligned}$$

From the fact that $\sup_{t\in[0,T]}\|\nabla u_i\|_{BMO}\le C$ for $i=01,2$, we know that $\nabla v\in L^\infty(0,T;BMO)$ and from the Corollary B.4

$$\int e^{pt|\nabla v(\tau,x)|}\mathrm{d}x\le C \quad\text{for all}\quad p\le\frac{C}{t\|\nabla v\|_{L^\infty(0,T;BMO)}}.$$

In particular, for sufficiently small t we have $\frac{\mathrm{d}x_s}{\mathrm{d}s}\in L^p(\mathbb{T}^d)$ for some $p>4$. □

Lemma 2.8. *Let x_s be as in Lemma 2.7 and let T_1 be such that $\frac{\mathrm{d}x_s}{\mathrm{d}s}\in L^p$ for $p>4$ and $t\in[0,T_1]$. Then $\left\|\frac{\mathrm{d}x_s}{\mathrm{d}s}(t,\cdot)\right\|_2^2$ satisfies the inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right\|_2^2 \le C\left\|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right\|_2^2\left(1+\left|\ln\left\|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right\|_2^2\right|\right)+C\|u_1(t,\cdot)-u_2(t,\cdot)\|_2^2 \tag{2.15}$$

for $t\in[0,T_1]$.

Proof. Multiplying both sides of (2.14) by $\frac{\mathrm{d}x_s}{\mathrm{d}s}$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right|^2 = \frac{\mathrm{d}x_s}{\mathrm{d}s}\nabla v(t,x_s)\frac{\mathrm{d}x_s}{\mathrm{d}s} + \big(u_1(t,x_s) - u_2(t,x_s)\big)\frac{\mathrm{d}x_s}{\mathrm{d}s}. \tag{2.16}$$

Integrating (2.16) over torus, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\left|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right|^2\mathrm{d}y &\le 2\left|\int\frac{\mathrm{d}x_s}{\mathrm{d}s}\nabla v(t,x_s)\frac{\mathrm{d}x_s}{\mathrm{d}s}\mathrm{d}y\right| + 2\int|u_1(t,x_s)-u_2(t,x_s)|\left|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right|\mathrm{d}y\\ &\le 2\left|\int\frac{\mathrm{d}x_s}{\mathrm{d}s}\nabla v(t,x_s)\frac{\mathrm{d}x_s}{\mathrm{d}s}\mathrm{d}y\right| + \int|u_1(t,x_s)-u_2(t,x_s)|^2\mathrm{d}y + \int\left|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right|^2\mathrm{d}y.\end{aligned}$$

From the bound (2.13) on J_s, we have

$$\int|u_1(t,x_s)-u_2(t,x_s)|^2\mathrm{d}y \le C\|u_1(t,\cdot)-u_2(t,\cdot)\|_2^2.$$

By the regularity of $\frac{\mathrm{d}x_s}{\mathrm{d}s}$ from Lemma 2.7, $\left|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right|^2 \in L^q$ for some $q > 2$. Therefore we can apply Corollary B.7 to estimate $\left|\int\frac{\mathrm{d}x_s}{\mathrm{d}s}\nabla v(t,x_s)\frac{\mathrm{d}x_s}{\mathrm{d}s}\mathrm{d}y\right|$. In consequence we obtain the inequality (2.15), where C depends on $\|\nabla v\|_{BMO}$, $\left\|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right\|_q$ and $\|J\|_\infty$. ∎

2.3.2 The final argument

Having the results from the previous subsection, we can now prove the uniqueness:

Theorem 2.9. *The solution to system (2.10)-(2.11) with regularity from Theorem 2.1 is unique.*

Proof. Let u_1, u_2 be the solutions to (2.10)-(2.11), $u_i = \nabla\phi_i$ and x_1, x_2 are the flows generated by u_1, u_2 respectively. We will show that $\|u_1 - u_2\|_2 = 0$ for all $t \in [0,T]$. By the weak formulation of $\operatorname{div} u(t,x(t,y)) = \sigma(t,y)$, for any $\xi \in C^\infty([0,T]\times\mathbb{T}^d)$ we have

$$\begin{aligned}\int(u_1(t,x)-u_2(t,x))\nabla\xi(t,x)\,\mathrm{d}x &= -\int(\operatorname{div} u_1(t,x) - \operatorname{div} u_2(t,x))\xi(t,x)\,\mathrm{d}x\\ &= -\int\sigma(t,y)J(t,y)(\xi(t,x_1(t,y)) - \xi(t,x_2(t,y)))\,\mathrm{d}y.\end{aligned}$$

Using the definition of the flow x_s, we can rewrite the last integral as

$$\begin{aligned}&\int\sigma(t,y)J(t,y)\int_0^1\frac{\mathrm{d}}{\mathrm{d}s}\xi(t,x_s(t,y))\mathrm{d}s\mathrm{d}y\\ &\qquad = \int\sigma(t,y)J(t,y)\int_0^1\nabla\xi(t,x_s(t,y))\frac{\mathrm{d}x_s(t,y)}{\mathrm{d}s}\mathrm{d}s\mathrm{d}y.\end{aligned}$$

By the density of smooth functions in $W^{1,2}$, we can choose $\xi = \phi_1 - \phi_2$. Then $\nabla\xi = u_1 - u_2$ and we obtain

$$\begin{aligned}\int|u_1-u_2|^2\,\mathrm{d}x &= -\int\sigma(t,y)J(t,y)\int_0^1(u_1(t,x_s(t,y)) - u_2(t,x_s(t,y)))\frac{\mathrm{d}x_s}{\mathrm{d}s}(t,y)\,\mathrm{d}s\mathrm{d}y\\ &\le \|\sigma\|_\infty\|J\|_\infty\|u_1-u_2\|_2\int_0^1\left\|\frac{\mathrm{d}x_s}{\mathrm{d}s}\right\|_2\,\mathrm{d}s.\end{aligned}$$

Hence

$$\|u_1 - u_2\|_2 \leq C \int_0^1 \left\| \frac{\mathrm{d}x_s}{\mathrm{d}s} \right\|_2 \mathrm{d}s. \tag{2.17}$$

Substituting (2.17) into (2.15) and integrating over s, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \left\| \frac{\mathrm{d}x_s}{\mathrm{d}s} \right\|_2^2 \mathrm{d}s \leq C \int_0^1 \left\| \frac{\mathrm{d}x_s}{\mathrm{d}s} \right\|_2^2 \left(1 + \left| \ln \left\| \frac{\mathrm{d}x_s}{\mathrm{d}s} \right\|_2^2 \right| \right) \mathrm{d}s + C \left(\int_0^1 \left\| \frac{\mathrm{d}x_s}{\mathrm{d}s} \right\|_2 \mathrm{d}s \right)^2$$

for $t \in [0, T_1]$ for some $T_1 \leq T$.

Now let $\alpha(t) = \int_0^1 \|\frac{\mathrm{d}x_s}{\mathrm{d}s}(t,\cdot)\|_2^2 \mathrm{d}s$. As the function $x(1 + |\ln x|)$ is concave for $x < 1$ and x^2 is convex, we can estimate both terms in the right hand side from Jensen's inequality and obtain

$$\dot{\alpha} \leq C\alpha(1 + |\ln \alpha|) + C\alpha.$$

From Osgood's lemma, the problem

$$\dot{z} = Cz(1 + |\ln z|), \quad z(0) = 0$$

has a unique solution $z \equiv 0$. Therefore, as $\frac{\mathrm{d}x_s}{\mathrm{d}s}_{|t=0} = 0$, we have $\alpha(0) = 0$ and

$$\alpha(t) \leq 0 \text{ for all } t \in [0, T_1].$$

In conclusion, $\frac{\mathrm{d}x_s}{\mathrm{d}s} \equiv 0$ for all $t \in [0, T_1]$ and so is $\|u_1 - u_2\|_2$. Having that, we can perform analogous reasoning on the consecutive intervals $[nT_1, (n+1)T_1]$ to get $u_1 = u_2$ for all $t \in [0, T]$. □

2.4 The existence of solutions to (2.10)-(2.11)

In this section we prove that the transformation from the system (2.7) to (2.2) is well defined, which will end the proof of Theorem 2.1. Having the solution (η, σ) in the Lagrangian coordinates by Theorem 2.4, we define the transformation to Eulerian coordinates (x, t). In other words, we need to find u such that $\displaystyle\int u(t,x)\, \mathrm{d}x = 0$ and u satisfies (2.10)–(2.11). By virtue of the discussion at the beginning of Section 2.3, such u provides us also existence of the density ϱ.

Theorem 2.10. *Let $\sigma \in L^\infty([0,T] \times \mathbb{T}^d)$. There exists $u \in L^\infty([0,T] \times \mathbb{T}^d)$ such that u is a solution to system (2.10)-(2.11) and*

$$\operatorname{div} u \in L^\infty([0,T] \times \mathbb{T}^d), \ \nabla u \in L^\infty(0,T; BMO).$$

Proof. First, we prove the existence for smoothened σ, by putting

$$\sigma_\delta = \sigma * \kappa_\delta,$$

where κ_δ is a standard mollifier.

Lemma 2.11. *There exists a unique $u_\delta \in C(0,T; W^{1,\infty})$ satisfying*

$$\operatorname{div} u_\delta(t, x_\delta(t,y)) = \sigma_\delta(t,y),$$

where $x_\delta(t,y)$ is given by (2.11) with the flow u_δ.

Proof. We define the suitable map, and then apply the Banach fixed point theorem. Let

$$\Phi\colon C(0,T;W^{1,\infty}) \to C(0,T;W^{1,\infty})$$

be defined in the following way:

1. If $\bar{u} \in C(0,T;W^{1,\infty})$, then $\bar{u}$ is Lipschitz, so there exists a unique solution to system

$$\dot{x} = \bar{u}(t,x), \quad x(0) = y. \tag{2.18}$$

2. We can now invert $x(t,y)$ to get $y(t,x)$ instead. After differentiation of (2.18) with respect to y, we get an ODE for the matrix $H(t,y) = \frac{\partial x}{\partial y}(t,y)$:

$$\partial_t H = \nabla_x \bar{u}(t,x(t,y))H,\ H(0,y) = I. \tag{2.19}$$

Moreover, the equation for $J(t,y) = \det H(t,y)$ yields

$$\partial_t J(t,y) = \mathrm{div}_x \bar{u}(t,x(t,y))J(t,y). \tag{2.20}$$

Therefore we have the estimates

$$\exp\left(-\int_0^T \|\nabla\bar{u}\|_\infty \mathrm{d}s\right) \le \|H\|_\infty \le \exp\left(\int_0^T \|\nabla\bar{u}\|_\infty \mathrm{d}s\right),$$

and

$$\exp\left(-\int_0^T \|\mathrm{div}\bar{u}\|_\infty \mathrm{d}s\right) \le \|J\|_\infty \le \exp\left(\int_0^T \|\mathrm{div}\bar{u}\|_\infty \mathrm{d}s\right),$$

and H is invertible, which allows us to treat y as a function of x.

3. Finally, we put $\Phi(\bar{u}) = u$, where u is a unique solution to the system

$$\begin{aligned} u(t,x) &= \nabla\phi(t,x), \\ \Delta\phi(t,x) &= \sigma_\delta(t,y(t,x)) \end{aligned} \tag{2.21}$$

for $y(t,x)$ being the inverse flow associated with $\bar{u}$.

The function $\sigma_\delta(t,y(t,\cdot))$ is Lipschitz, as

$$\begin{aligned} |\sigma_\delta(t,y(t,x_1)) - \sigma_\delta(t,y(t,x_2))| &\le \|\nabla\sigma_\delta\|_\infty |y(t,x_1) - y(t,x_2)| \\ &\le \|\nabla\sigma_\delta\|_\infty \left\|\frac{\mathrm{d}y}{\mathrm{d}x}\right\|_\infty |x_1 - x_2|. \end{aligned}$$

Therefore $\phi(t,\cdot) \in C^{2,1}(\mathbb{T}^d)$ and in consequence $u(t,\cdot) \in C^{1,1}(\mathbb{T}^d)$ and we have the estimates

$$\begin{aligned} \sup_{0\le t\le T} \|u\|_\infty &\le C\|\sigma_\delta\|_{\infty,T}, \\ \sup_{0\le t\le T} \|\nabla u\|_{C^{0,1}} &\le C \sup_{0\le t\le T} \|\sigma_\delta(t,y(t,\cdot))\|_{C^{0,1}}. \end{aligned} \tag{2.22}$$

Moreover, as $\sigma \in L^\infty([0,T]\times\mathbb{T}^d)$, by the classical results for L^p regularity of strong solutions to Poisson equation (see e.g. [108]), for any $p \in (1,\infty)$ we have the estimate

$$\sup_{0\le t\le T} \|u\|_\infty + \sup_{0\le t\le T} \|u\|_{W^{1,p}} + \sup_{0\le t\le T} \|\nabla u\|_{BMO} \le C\|\sigma\|_{\infty,T}, \tag{2.23}$$

which are uniform with respect to δ.

2.4.1 Additional regularity of u

We will now show that the fixed points of Φ are uniformly bounded in $L^\infty(0,T;W^{2,p})$. By use of an appropriate logarithmic inequality, it implies that $\Phi(K) \subseteq K$ for some bounded and closed $K \subset C(0,T;W^{1,\infty})$.

Proposition 2.12. *If u_δ is a fixed point of Φ, then*

$$\sup_{0\le t\le T} \|\nabla u_\delta\|_{W^{1,p}} \le C(p,\delta,T). \tag{2.24}$$

Proof. After differentiation of (2.21) with respect to x_i, we get

$$\Delta_x \frac{\partial \phi}{\partial x_i} = \nabla_y \sigma_\delta \cdot \frac{\partial y}{\partial x_i}.$$

Hence for any $1 < p < \infty$ the standard elliptic estimate gives

$$\|\nabla \phi\|_{W^{2,p}} \le C(p)\|\nabla \sigma_\delta\|_\infty \left\|\frac{\mathrm{d}y}{\mathrm{d}x}\right\|_\infty,$$

which leads to

$$\sup_{0\le t\le T} \|\nabla u\|_{W^{1,p}} \le C(p)\|\nabla \sigma_\delta\|_{\infty,T} e^{T\|\nabla \bar{u}\|_{\infty,T}}. \tag{2.25}$$

Now take $\bar{u} = u = u_\delta$ in (2.25) and apply the following inequality for $p > d$:

$$\|\nabla f\|_\infty \le C(p)\left(1 + \|\nabla f\|_{BMO}\left(\ln^+(\|\nabla f\|_{W^{1,p}} + \|f\|_\infty)\right)^{1/2}\right). \tag{2.26}$$

The proof of the above estimate one can find in [86], Corollary 2.4. By the estimates (2.23) and Cauchy inequality, we have

$$\begin{aligned}
\|\nabla u_\delta\|_{L^\infty(0,T;W^{1,p})} &\le C(p)\|\nabla \sigma_\delta\|_{\infty,T} \exp\left(T \sup_{0\le t\le T}\|\nabla u_\delta\|_\infty\right)\\
&\le \exp\left(CT \sup_{0\le t\le T}\|\nabla u_\delta\|_{BMO}\left(\ln^+(\|\nabla u_\delta\|_{L^\infty(0,T;W^{1,p})} + \|u_\delta\|_{\infty,T})\right)^{1/2}\right)\\
&\le \exp\left(CT^2 + \frac{1}{2}\ln^+\left(\|\nabla u_\delta\|_{L^\infty(0,T;W^{1,p})} + \|u_\delta\|_{\infty,T}\right)\right)\\
&\le C\left(\|\nabla u_\delta\|_{L^\infty(0,T;W^{1,p})} + \|u_\delta\|_{\infty,T}\right)^{1/2}.
\end{aligned}$$

Therefore

$$\|\nabla u_\delta\|^2_{L^\infty(0,T;W^{1,p})} - C\|\nabla u_\delta\|_{L^\infty(0,T;W^{1,p})} - C \le 0$$

and in consequence

$$\|\nabla u_\delta\|_{L^\infty(0,T;W^{1,p})} \le C$$

for some C depending on p, δ and T. □

The analogous reasoning also provides that $\Phi(K_{T_1}) \subseteq K_{T_1}$ for

$$K_{T_1} = \{u \in C(0,T_1;W^{1,\infty}) : \ \|u\|_{\infty,T_1}, \|\nabla u\|_{L^\infty(0,T_1;BMO)} \le C_1 \ \text{ and } \ \|\nabla u\|_{\infty,T_1} \le C_2\},$$

where C_1 is the constant from estimates (2.23) and $C_2 = C(p)\left(1 + C_1\left(\ln^+(C_p + C_1)\right)^{1/2}\right)$ is the right hand side of (2.26) for C_p being the constant from (2.24).

2.4.2 The fixed point argument.

Local existence. We show that $\Phi\colon K \to K$ is a contraction for sufficiently small T_1. From the elliptic estimates, we have

$$\begin{aligned}\|u_1(t,\cdot) - u_2(t,\cdot)\|_\infty &\leq C\|\sigma_\delta(t, y_1(t,\cdot)) - \sigma_\delta(t, y_2(t,\cdot))\|_\infty \\ &\leq C\|\nabla\sigma_\delta\|_{\infty,T}\|y_1(t,\cdot) - y_2(t,\cdot)\|_\infty.\end{aligned}$$

By Lemma A.2 from the Appendix, for small t we have

$$\|y_1(t,\cdot) - y_2(t,\cdot)\|_\infty \leq Ct\|\bar{u}_1 - \bar{u}_2\|_{\infty,T}$$

and due to the uniform bound on $\|\nabla u\|_\infty$ for $u \in K$ the constant C does not depend on $\bar{u}_1$ and $\bar{u}_2$. Hence

$$\|u_1 - u_2\|_{\infty,T_1} \leq C\|\nabla\sigma_\delta\|_{\infty,T}(T_1 + o(T_1))\|\bar{u}_1 - \bar{u}_2\|_{\infty,T_1}. \tag{2.27}$$

Choosing T_1 such that $C\|\nabla\sigma_\delta\|_{\infty,T}T_1 < 1$, we get that Φ is a contraction on K. Therefore there exists a unique fixed point u_δ of Φ on the interval $[0, T_1]$.

Extension to $[0,T]$. Having the uniqueness on $[0, T_1]$, we are able to perform the same reasoning on $[T_1, 2T_1]$. Note that the estimate (2.27) on $[T_1, 2T_1]$ would still depend only on the length of the interval. Therefore again Φ is a contraction on $[T_1, 2T_1]$, which gives us the unique fixed point on $[0, 2T_1]$. Performing this procedure on the consecutive intervals $[nT_1, (n+1)T_1]$, we obtain the existence of a unique fixed point on the whole interval $[0,T]$, which completes the proof of Lemma 2.11. $\square$

2.4.3 Letting $\delta \to 0$.

In Lemma 2.11 we obtained the unique $u_\delta = \nabla\phi_\delta$, which satisfies the equation

$$\Delta\phi_\delta(t,x) = \sigma_\delta(t, y_\delta(t,x)) \tag{2.28}$$

and the uniform estimates (2.23). Moreover, it turns out that $\partial_t u_\delta$ is uniformly bounded in L^2. Indeed, from the weak formulation of (2.28), for any $\pi \in C^\infty(\mathbb{T}^d)$ we have

$$\int \Delta\phi_\delta(t,x)\pi(x)\,\mathrm{d}x = \int \sigma_\delta(t, y_\delta(t,x))\pi(x)\,\mathrm{d}x = \int \sigma_\delta(t,y)\pi(x_\delta(t,y))J_\delta(t,y)\,\mathrm{d}y,$$

where $J_\delta(t,y) = \exp\left(\int_0^t \sigma_\delta(s,y)\mathrm{d}s\right)$ is the Jacobian of x_δ. Differentiating this equality with respect to time, we obtain

$$\begin{aligned}\int \Delta\partial_t\phi_\delta(t,x)\pi(x)\,\mathrm{d}x = &\int \partial_t\sigma_\delta(t,y)\pi(x_\delta(t,y))J_\delta(t,y)\,\mathrm{d}y \\ &+ \int \sigma_\delta(t,y)\partial_t\pi(x_\delta(t,y))J_\delta(t,y)\,\mathrm{d}y \\ &+ \int \sigma_\delta(t,y)\pi(x_\delta(t,y))\partial_t J_\delta(t,y)\,\mathrm{d}y.\end{aligned} \tag{2.29}$$

Let us now estimate the terms on the right hand side of (2.29). First, observe that from equation (2.7) $\partial_t\sigma$ is bounded, and therefore

$$|\partial_t\sigma_\delta| = |(\partial_t\sigma) * \kappa_\delta| \leq |\partial_t\sigma| \in L^\infty([0,T] \times \mathbb{T}^d).$$

For the second term, we have

$$\partial_t \pi(x_\delta(t,y)) = \nabla\pi(x_\delta(t,y))\dot{x_\delta}(t,y) = \nabla\pi(x_\delta(t,y))u(t,x_\delta(t,y))$$

and then using uniform estimates on u_δ, we get

$$\int |\nabla\pi(x_\delta(t,y))|^2|u_\delta(t,x_\delta(t,y))|^2 \, \mathrm{d}y \le \left\|\frac{1}{J_\delta}\right\|_\infty \int |\nabla\pi(t,x)|^2|u_\delta(t,x)|^2 \, \mathrm{d}x \le C,$$

therefore $\partial_t\pi(x_\delta(t,y)) \in L^\infty(0,T;L^2)$. The third term is bounded as well, as $\partial_t J_\delta = \sigma_\delta J_\delta$ and both σ_δ and J_δ are bounded by some $C(\|\sigma\|_{\infty,T})$.
The above estimates imply that $\Delta\partial_t\phi_\delta$ is bounded in $L^\infty(0,T;W^{-1,2})$ uniformly in δ and therefore

$$\|\partial_t u_\delta\|_{L^\infty(0,T;L^2)} = \|\nabla\partial_t\phi_\delta\|_{L^\infty(0,T;L^2)} \le C. \tag{2.30}$$

We now let $\delta \to 0^+$ and therefore obtain the solution to equation (2.10). The estimates (2.23) give

$$\|\phi_\delta\|_{L^\infty(0,T;W^{2,p})} \le C,$$

so $\phi_\delta \rightharpoonup^* \phi$ in $L^\infty(0,T;W^{2,p})$ up to a subsequence. From the uniform estimates (2.23) and (2.30), Aubin-Lions Lemma implies that in particular u_δ is compact in $L^1([0,T]\times\mathbb{T}^d)$. Moreover, using Theorem 2.9 from [35], we get

$$\sup_{0\le t\le T} \|x(t,y) - x_\delta(t,y)\|_{L^1(\mathbb{T}^d)} \le C\left|\ln\left(\|u-u_\delta\|_{L^1([0,T]\times\mathbb{T}^d)}\right)\right|^{-1},$$

where $x(t,y)$ is the flow generated by this weak* limit $u = \nabla\phi$. Therefore $x_\delta \to x$ in $L^\infty(0,T;L^1)$.

The above convergence allows us to pass to the limit with $\delta \to 0$ in a weak formulation of (2.21). For any $\xi \in C^\infty([0,T]\times\mathbb{T}^d)$, we have

$$\begin{aligned}\int_0^T\int \Delta\phi_\delta(t,x)\xi(t,x)\mathrm{d}x\mathrm{d}t &= \int_0^T\int (\sigma*\kappa_\delta)(t,y_\delta(t,x))\xi(t,x)\mathrm{d}x\mathrm{d}t\\ &= \int_0^T\int (\sigma*\kappa_\delta)(t,y)\xi(t,x_\delta(t,y))J_\delta(t,y)\mathrm{d}y\mathrm{d}t.\end{aligned}$$

Letting $\delta \to 0$, we get

$$\int_0^T\int \Delta\phi(t,x)\xi(t,x)\mathrm{d}x\mathrm{d}t = \int_0^T\int \sigma(t,y)\xi(t,x(t,y))J(t,y)\mathrm{d}y\mathrm{d}t, \tag{2.31}$$

where $J = \exp\left(\int_0^t \sigma(s,y)\mathrm{d}s\right)$ is the Jacobian of the limit flow $x(t,y)$.

To deduce that indeed we have $\operatorname{div} u(t,x(t,y)) = \sigma(t,y)$, we need to change the variables in one of the sides in (2.31). Despite the fact that $x(t,\cdot)$ is not a diffeomorphism, Lemma 3.1 from [34] allows us to perform the change of variables in the left hand side of (2.31) and obtain

$$\int_0^T\int \operatorname{div} u(t,x(t,y))\xi(t,x(t,y))J(t,y)\mathrm{d}y\mathrm{d}t.$$

Therefore the equality (2.31) is transformed into

$$\int_0^T\int \left[\operatorname{div} u(t,x(t,y)) - \sigma(t,y)\right]\xi(t,x(t,y))J(t,y)\mathrm{d}y\mathrm{d}t = 0.$$

As $\operatorname{div} u \in L^\infty([0,T] \times \mathbb{T}^d)$, the Jacobian $J(t,y)$ is strictly positive. Hence, from the arbitrary choice of ξ, we have $\operatorname{div} u(t, x(t,y)) = \sigma(t,y)$ in the sense of distributions, which ends the proof of Theorem 2.10. $\square$

As by Theorem 2.4 the norms $\|\eta\|_{\infty,T}$ and $\|\sigma\|_{\infty,T}$ do not depend on T, so are $\|\varrho\|_{\infty,T}$, $\|\operatorname{div} u\|_{\infty,T}$ and the estimates given by (2.23). Then again from arbitrary choice of T we obtain the unique existence on the whole real half-line and hence the proof of Theorem 2.1 is completed.

Chapter 3

Weak solutions for the compressible non-Newtonian Stokes system with unbounded divergence

This chapter consists of result published in [93]. We investigate the existence of weak solutions to a system of equations modeling the behavior of a certain compressible non-Newtonian fluid for small Reynolds number. We construct the weak solutions despite the lack of the L^∞ estimate on the divergence of the velocity field. The result was obtained by combining the regularity theory for singular operators with the logarithmic integral inequality for BMO functions, which allowed us to adjust the method from Feireisl et al. [45] to more relaxed conditions on the velocity.

3.1 Introduction

Our aim is to investigate the existence of weak solutions to equations modelling a special case of compressible, non-Newtonian fluid. In the most general setting, the motion of such a fluid without the presence of the external forces is described by the system of partial differential equations

$$\begin{aligned} \varrho_t + \operatorname{div}(\varrho u) &= 0, \\ (\varrho u)_t + \operatorname{div}(\varrho u \otimes u) - \operatorname{div} \mathbb{S} + \nabla p(\varrho) &= 0, \end{aligned} \quad \text{in} \quad [0,T] \times \Omega, \tag{3.1}$$

where $\Omega \subset \mathbb{R}^d$, ϱ is the density, u is a velocity vector and $\mathbb{S}$ is the stress tensor; we assume that it is given by

$$\mathbb{S}(\mathbb{D}u) = \mu \mathbb{D}u + \lambda \operatorname{div} u \mathbb{I},$$

where $\mathbb{I}$ is an idetity matrix, $\mu > 0$ and λ are the viscosity coefficients, $\mathbb{D} = \frac{1}{2}(\nabla + \nabla^T)$ is the symmetric gradient and $p(\varrho)$ is the pressure. In the case of constant viscosity (i.e., the resulting system is called the compressible Navier–Stokes equations) $d\lambda + \mu \geq 0$, where d is the space dimension.

We will focus on the case where the Reynolds number $\mathrm{Re} \sim \frac{\varrho |u|}{\mu}$ is small. As in this situation the advective forces are small compared to the viscous ones, we can approximate system (3.1) by the compressible Stokes-like system

$$\begin{aligned} \varrho_t + \operatorname{div}(\varrho u) &= 0, \\ -\operatorname{div} \mathbb{S}(\mathbb{D}u) + \nabla p(\varrho) &= 0. \end{aligned} \tag{3.2}$$

Our aim is to obtain weak solutions to a special case of system (3.2). We assume that the shear viscosity μ is in the form

$$\mu = \mu_0(|\mathbb{D}u|) + 2\mu_1, \quad \mu_1 > 0 \ \text{ constant} \tag{3.3}$$

and the bulk viscosity $\lambda = \lambda(|\mathrm{div}\, u|)$, where

$$0 \leq \mu_0(z), \lambda(z) \leq \frac{C}{z}, \quad z > 0. \tag{3.4}$$

Furthermore, we impose the monotonicity condition on the functions $\mu_0(|\cdot|)\cdot$ and $\lambda(|\cdot|)\cdot$, i. e. for all $A, B \in \mathbb{R}^{d\times d}$ and $s, t \in \mathbb{R}$

$$\big(\mu_0(|A|)A - \mu_0(|B|)B\big) : (A - B) \geq 0 \quad \text{and} \quad \big(\lambda(|s|)s - \lambda(|t|)t\big)(s - t) \geq 0. \tag{3.5}$$

For the pressure we assume the barotropic case with $p(\varrho) = \varrho^\gamma$ for $\gamma \geq 1$[1]. For simplicity we consider the space-periodic boundary conditions, namely

$$u \colon [0, T] \times \mathbb{T}^d \to \mathbb{R}^d \quad \text{and} \quad \varrho \colon [0, T] \times \mathbb{T}^d \to \mathbb{R},$$

where $\mathbb{T}^d$ is the d-dimensional torus. In conclusion, the analysed system of equations reads

$$\begin{aligned} \varrho_t + \mathrm{div}\,(\varrho u) = 0, \\ -\mathrm{div}\,(\mu_0(|\mathbb{D}u|)\mathbb{D}u) - \mu_1\Delta u - \nabla((\mu_1 + \lambda(\mathrm{div}\, u))\mathrm{div}\, u) + \nabla\varrho^\gamma = 0, \end{aligned} \tag{3.6}$$

with the initial condition

$$\varrho_{|t=0} = \varrho_0 \in L^\infty(\mathbb{T}^d), \quad \varrho_0 \geq 0 \tag{3.7}$$

and the compatibilty constraint

$$\int_{\mathbb{T}^d} u(t, x)\, \mathrm{d}x = 0 \quad \forall_{t>0}.$$

Our system describes a type of the power-law fluid. They are characterized by the behavior of the shear viscosity, which satisfies the relation

$$\mu \sim |\mathbb{D}u|^{r-2} \tag{3.8}$$

for some exponent $r \geq 1$. Typically, it is assumed that

$$\mu = \mu_0|\mathbb{D}u|^{r-2} \quad \text{or} \quad \mu = \mu_0(a + |\mathbb{D}u|)^{r-2},\ a > 0,$$

to ensure that the viscosity is strictly positive and does not have singularities. For $r = 2$ the fluid becomes Newtonian, whereas it is shear-thinning for $r < 2$ and shear-thickening for $r > 2$. The power-law fluids are used in many fields, for example glaciology [72, 59] and to analyze the dynamics in the Earth's Mantle [110] or blood flow [32, 96]. For more information we refer the reader, e.g., to [4]. Our situation corresponds specifically to a Herschel-Bulkley fluid, where the shear viscosity is in the form

$$\mu = \begin{cases} \mu_0, & |\mathbb{D}u| < \delta, \\ \dfrac{\tau_0}{|\mathbb{D}u|} + k|\mathbb{D}u|^{n-1}, & |\mathbb{D}u| \geq \delta \end{cases}$$

for some $n \geq 1$ and the parameters μ_0, τ_0, k are chosen in such way that μ remains continuous. Fluids of this type were thoroughly analysed in the incompressible case, and have many industrial applications, see e.g. [3, 36, 42].

[1] We could also replace this precise form just by asymptotic growth conditions similarly as in [45], i.e.

$$p(0) = 0, \quad p'(z) > 0 \text{ for } z > 0, \quad \text{and} \quad \lim_{z\to\infty} \frac{p(z)}{z^\gamma} \in (0, \infty),$$

but we skip it to avoid unnecessary complications

The mathematical theory concerning weak solutions to systems describing *incompressible* non-Newtonian fluids has been thoroughly developed in the past. There is a large number of results for several aspects of these problems. As it turns out, the existence and regularity of solutions to incompressible Navier–Stokes equations with the power-law relation (3.8) for viscosity depends on the value of r. For $r > \frac{2d}{d+2}$ the existence of weak solutions for the problem

$$\begin{aligned} \operatorname{div} u &= 0, \\ u_t + \operatorname{div}(u \otimes u) - \operatorname{div} \mathbb{S} + \nabla p &= 0 \end{aligned} \tag{3.9}$$

with the Dirichlet boundary conditions was shown for the first time in [39]; its uniqueness is known for $r \geq \frac{3d+2}{d+2}$, see [15]. As a matter of fact, the problem for $r < \frac{2d}{d+2}$ is ill-posed, see [16]. However, existence of more general, dissipative solutions can be shown also in this case, see [1].

Contrary to the incompressible case, the current literature on the compressible non-Newtonian fluids is very limited. In [70, 71] Mamontov proved the existence of weak solutions to the system with linear pressure term and in the framework of Orlicz spaces with exponential growth, see also [2] for further properties of these solutions. The results for more general form of the stress tensor were obtained in [45], where the authors considered the system (3.1) with μ of the form (3.8) and a special form of λ, which provided the L^∞ bound on $\operatorname{div} u$. Using the classical Lions & Feireisl method [48, 67], the authors proved the existence of weak solutions for the same range of r's as in the uniqueness and regularity theory for incompressible fluids ($r \geq \frac{11}{5}$ in three dimensions). The additional bound on the divergence was crucial to obtain the strong convergence of the density in the final limit passage.

3.1.1 Main result and structure of the paper

Let us first define what we understand by a weak solution:

Definition 3.1. *We say that $(\varrho, u) \in L^\infty(0,T;L^\gamma) \times L^2(0,T;W^{1,2})$, $\gamma \geq 1$, is a weak solution to the system (3.2) on $[0,T] \times \mathbb{T}^d$ with the initial condition $\varrho_0 \in L^1(\mathbb{T}^d)$, if*

$$\varrho u, \ \mathbb{S}(\mathbb{D}u), \ p(\varrho) \in L^1((0,T) \times \mathbb{T}^d)$$

and for each $\varphi \in C_0^\infty([0,T) \times \mathbb{T}^d; \mathbb{R})$ and $\psi \in C_0^\infty([0,T) \times \mathbb{T}^d; \mathbb{R}^d)$ it holds

$$-\int_0^T \int_{\mathbb{T}^d} \varrho \partial_t \varphi \, \mathrm{d}x\mathrm{d}t - \int_0^T \int_{\mathbb{T}^d} \varrho u \cdot \nabla \varphi \, \mathrm{d}x\mathrm{d}t = \int_{\mathbb{T}^d} \varrho_0 \varphi(0, \cdot) \, \mathrm{d}x$$

and

$$\int_0^T \int_{\mathbb{T}^d} \mathbb{S}(\mathbb{D}u) : \mathbb{D}\psi \, \mathrm{d}x\mathrm{d}t - \int_0^T \int_{\mathbb{T}^d} p(\varrho) \operatorname{div} \psi \, \mathrm{d}x\mathrm{d}t = 0.$$

Our main result states:

Theorem 3.2. *Let $\varrho_0 \in L^\infty(\mathbb{T}^d)$, $\varrho_0 \geq 0$, $\gamma \geq 1$ and let (3.4) hold. Then for any $T > 0$ there exists a weak solution to the system (3.6), satisfying*

$$\|\nabla u\|_{L^2((0,T)\times\mathbb{T}^d)} + \|\varrho\|_{L^\infty(0,T;L^\gamma)} \leq C,$$

where C does not depend on T, and

$$\|\operatorname{div} u\|_{L^\infty(0,T;L^p)} + \|\varrho\|_{L^\infty(0,T;L^p)} \leq C(p,T),$$

for any $1 \leq p < \infty$, where C approaches ∞ if p or T do so.

The proof of Theorem 3.2 uses the technique from [45]. However, due to the absence of the convective term we are able to obtain the result for relaxed assumptions on $\operatorname{div} u$. In particular, we do not need to derive the L^∞ bound on $\operatorname{div} u$. Instead, we obtain the BMO regularity in space for the term

$$2\mu_1 \operatorname{div} u - \varrho^\gamma.$$

This allows us to replicate the main step in the limit passage by using the logarithmic integral inequality from the previous chapter, see Lemma B.6 and [100].

The rest of the article is devoted to the proof of Theorem 3.2. First, in Section 3.2 we derive the a priori estimates, in particular the crucial BMO estimate for the quantity $2\operatorname{div} u - \varrho^\gamma$. Then, in Section 3.3, we prove the existence of solutions to the approximate system with the regularized continuity and momentum equations. In Section 3.4 we finish the proof by passing to the limit in the weak formulation of the approximate system and in consequence we obtain the weak solution to the original one.

Preliminary remarks. Similarly as in the previous chapter, we omit the subscript while integrating over the torus, namely

$$\int \mathrm{d}x \;:=\; \int_{\mathbb{T}^d} \mathrm{d}x.$$

Furthermore, as the results do not depend on the values of μ_1, for simplicity we set $\mu_1 = 1$.

3.2 A priori estimates

Lemma 3.3. *Under the assumptions of Theorem 3.2, if the solution to (3.6) is sufficiently smooth, it satisfies*

$$\|\nabla u\|_{L^2((0,T)\times\mathbb{T}^d)} + \|\varrho\|_{L^\infty(0,T;L^\gamma)} \leq C$$

and

$$\|2\operatorname{div} u - \varrho^\gamma\|_{L^\infty(0,T;BMO)} \leq C$$

for C depending only on $\|\varrho_0\|_\infty$. Furthermore,

$$\|\operatorname{div} u\|_{L^\infty(0,T;L^p)} + \|\varrho\|_{L^\infty(0,T;L^p)} \leq C(p,T)$$

for any $p < \infty$, where $C(p) \to \infty$ as $p \to \infty$ or $T \to \infty$.

Proof. Multiplying the second equation of (3.6) by u and integrating over the torus, we obtain (if $\gamma = 1$, the last integral is replaced by $\int \varrho \ln \varrho \,\mathrm{d}x$)

$$\begin{aligned}\int \mu_0(|\mathbb{D}u|)|\mathbb{D}u|^2 \mathrm{d}x + \int |\nabla u|^2 \mathrm{d}x + \int (\operatorname{div} u)^2 \mathrm{d}x \\ + \int \lambda(|\operatorname{div} u|)(\operatorname{div} u)^2 \mathrm{d}x + \frac{1}{\gamma-1}\frac{\mathrm{d}}{\mathrm{d}t}\int \varrho^\gamma \mathrm{d}x = 0. \end{aligned} \tag{3.10}$$

Integrating the above equality from 0 to T, we obtain the first desired estimate.

To obtain the L^p estimate of the density, we use as a test function in $(3.6)_2$ the function $\psi = -(-\Delta)^{-1}\nabla\left(\varrho^\theta - \{\varrho^\theta\}\right)$ for a suitable $\theta > 1$. We have

$$\operatorname{div}\psi = \varrho^\theta - \{\varrho^\theta\}$$

and

$$\|\nabla\psi\|_{L^r((0,T)\times\mathbb{T}^d)} \leq C(r)\|\varrho^\theta\|_{L^r((0,T)\times\mathbb{T}^d)} \text{ for any } 1 < r < \infty.$$

The above estimate follows for example from the Marcinkiewicz Multiplier Theorem [73]. Note that $C(r) \to \infty$ if $r \to 1^+$ or $r \to \infty$. Moreover,

$$-\int \Delta u \cdot \psi \, \mathrm{d}x = -\int u \cdot \nabla \left(\varrho^\theta - \{\varrho^\theta\}\right) \mathrm{d}x = \int \varrho^\theta \mathrm{div}\, u \, \mathrm{d}x.$$

Then

$$\int \varrho^{\gamma+\theta} \, \mathrm{d}x - 2\int \varrho^\theta \mathrm{div}\, u \, \mathrm{d}x = \int \mu_0(|\mathbb{D}u|)\mathbb{D}u : \nabla\psi \, \mathrm{d}x + \int \varrho^\theta \lambda(|\mathrm{div}\, u|)\mathrm{div}\, u \, \mathrm{d}x$$
$$+ \frac{1}{|\mathbb{T}^d|} \int \varrho^\gamma \mathrm{d}x \int \varrho^\theta \mathrm{d}x.$$

Using the growth conditions on μ_0 and λ, we get

$$\int \varrho^\theta \lambda(|\mathrm{div}\, u|)\mathrm{div}\, u \, \mathrm{d}x \le C \int \varrho^\theta \, \mathrm{d}x$$

and

$$\int \mu_0(|\mathbb{D}u|)\mathbb{D}u : \nabla\psi \, \mathrm{d}x \le C \int |\nabla\psi| \, \mathrm{d}x \le C(\delta)\|\varrho^\theta\|_{L^{1+\delta}}.$$

Moreover, as

$$\int \varrho^\theta \mathrm{div}\, u \, \mathrm{d}x = \frac{-1}{\theta-1}\frac{d}{\mathrm{d}t}\int \varrho^\theta \mathrm{d}x,$$

in the end we obtain

$$\frac{2}{\theta-1}\frac{d}{\mathrm{d}t}\int \varrho^\theta \mathrm{d}x + \int \varrho^{\gamma+\theta}\mathrm{d}x \le C\left(\int \varrho^\theta \mathrm{d}x + \left(\int \varrho^{(1+\delta)\theta}\mathrm{d}x\right)^{\frac{1}{1+\delta}}\right);$$

whence, for a suitably chosen δ

$$\|\varrho\|_{L^\infty(0,T;L^p)} \le C(p, T, \varrho_0).$$

The bound in the BMO space comes from the Calderón–Zygmund estimates. By taking the divergence of the momentum equation, we get

$$-\Delta((2 + \lambda(\mathrm{div}\, u))\mathrm{div}\, u - \varrho^\gamma) = \mathrm{div}\,\mathrm{div}\, \left(\mu_0(|\mathbb{D}u|)\mathbb{D}u\right).$$

Therefore in consequence for $A(t) = \frac{1}{\mathbb{T}^d}\int \lambda(|\mathrm{div}\, u(t,y)|)\mathrm{div}\, u(t,y) - \varrho^\gamma(t,y) \, \mathrm{d}y$, we have

$$\begin{aligned}(2 + \lambda(|\mathrm{div}\, u(t,x)|))\mathrm{div}\, u(t,x) - \varrho^\gamma(t,x) - A(t) &= \int \mathrm{div}\,\mathrm{div}\, \left(\mu_0(|\mathbb{D}u|)\mathbb{D}u\right)\bar{K}(x-y)\mathrm{d}y \\ &= \int \mathrm{div}\, \left(\mu_0(|\mathbb{D}u|)\mathbb{D}u\right) \cdot \nabla\bar{K}(x-y)\mathrm{d}y \\ &= \int \mu_0(|\mathbb{D}u|)\mathbb{D}u : \nabla^2\bar{K}(x-y)\mathrm{d}y,\end{aligned} \tag{3.11}$$

where $\bar{K}$ is the fundamental solution to the Laplace equation on $\mathbb{T}^d$. Note that we can write $\bar{K}$ explicitly using Fourier series: if $-\Delta\varphi = f$ in $\mathbb{T}^d$ for f with mean value 0, then $|k|^2\hat{\varphi}(k) = \hat{f}(k)$. In consequence

$$\varphi(x) = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{1}{|k|^2} e^{2\pi i k\cdot x}\hat{f}(k) = \int f(y) \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{1}{|k|^2} e^{2\pi i k(x-y)}\mathrm{d}y$$

and thus

$$\bar{K}(x) = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{1}{|k|^2} e^{2\pi i k\cdot x}.$$

By Example 3.1.19 from [53], we know that

$$\bar{K}(x) = \frac{c_d}{|x|^{d-2}} + h_1(x)$$

for some $h_1 \in C^\infty([0,1)^d)$.

Therefore from the Calderón–Zygmund theorem we conclude that

$$\|(2+\lambda(|\operatorname{div} u|))\operatorname{div} u - \varrho^\gamma - A(t)\|_{L^\infty(0,T;BMO)} \le C\,\|\mu_0(|\mathbb{D}u|)\mathbb{D}u\|_{L^\infty((0,T)\times\mathbb{T}^d)} \le C$$

for some constant C independent of T (for the results concerning singular integrals in the periodic case we refer the reader to [18]). Note that by the condition (3.4)

$$\|\lambda(|\operatorname{div} u|)\operatorname{div} u\|_{L^\infty([0,T]\times\mathbb{T}^d)} \le C$$

and from (3.10)

$$\|A(t)\|_{L^\infty([0,T])} \le C.$$

The BMO regularity is thus satisfied for $2\operatorname{div} u - \varrho^\gamma$. Moreover, the $L^\infty(0,T;L^p)$ estimate on ϱ^γ also implies the same regularity for $\operatorname{div} u$, which finishes the proof. □

3.3 Existence of approximate solutions

In this section we construct the approximate solutions. To do that, we consider the system

$$\begin{aligned} \varrho_t + \operatorname{div}(\varrho u) + \delta\varrho^\beta &= \delta\Delta\varrho, \\ -\operatorname{div}(\mu_0(\mathbb{D}u)\mathbb{D}u) - \Delta u - \nabla(1+\lambda(\operatorname{div} u))\operatorname{div} u + \nabla\varrho^\gamma &= -\varepsilon\Delta^{2m}u, \\ \int u(t,x)\,\mathrm{d}x &= 0 \end{aligned} \tag{3.12}$$

for a sufficiently small $\delta,\varepsilon > 0$, sufficiently large $m\in\mathbb{N}$ and $\beta \ge \max\{\gamma+1, 4\}$ being an odd integer, with the initial condition

$$\varrho_{|t=0} = \varrho_{0,\delta} \in C^\infty(\mathbb{T}^d), \quad \varrho_{0,\delta} > 0, \quad \varrho_{0,\delta} \to \varrho_0 \text{ in any } L^p,\ p<\infty.$$

To prove the existence of solutions, we will employ the following version of the Schauder fixed point theorem:

Theorem 3.4. *Let X be a Banach space and $\Phi\colon X\to X$ be continuous and compact. If the set*

$$\{x\in X : \quad x = s\Phi(x) \quad \textit{for some} \quad s\in[0,1]\}$$

is bounded, then Φ has a fixed point.

Let us define the map $\Phi\colon C([0,T];L^{2\gamma}) \to C([0,T];L^{2\gamma})$ in the following way:

1. For $\tilde\varrho \in C([0,T];L^{2\gamma})$, let u be the unique solution to the equation

$$\begin{aligned} -\operatorname{div}(\mu_0(|\mathbb{D}u|)\mathbb{D}u) - \Delta u - \nabla(1+\lambda(\operatorname{div} u))\operatorname{div} u + \nabla(\tilde\varrho)^\gamma &= -\varepsilon\Delta^{2m}u, \\ \int u(t,x)\,\mathrm{d}x &= 0. \end{aligned} \tag{3.13}$$

2. Then, let ϱ be the solution to

$$\varrho_t + \operatorname{div}(\varrho u) + \delta\varrho^\beta = \delta\Delta\varrho, \quad \varrho_{|t=0} = \varrho_{0,\delta}. \tag{3.14}$$

We set $\Phi(\tilde\varrho) := \varrho$. It is easy to see that the fixed point ϱ and the corresponding u solve our problem (3.12).

First, let us show that the operator Φ is well-defined.

Proposition 3.5. *If $\tilde\varrho^\gamma \in L^\infty(0,T;L^2)$, then there exists a unique solution u to equation (3.13), satisfying*

$$\|\nabla u\|_{L^\infty(0,T;L^2)} + \sqrt{\varepsilon}\|\Delta^m u\|_{L^\infty(0,T;L^2)} \le C\|\tilde\varrho^\gamma\|_{L^\infty(0,T;L^2)}.$$

In particular, if m is large enough, then

$$\|u\|_{L^\infty(0,T;W^{1,\infty})} \le \frac{C}{\sqrt{\varepsilon}}\|\tilde\varrho^\gamma\|_{L^\infty(0,T;L^2)}.$$

Proof. By multiplying the equation by u and integrating over the torus, we get

$$\begin{aligned}\int \mu_0(|\mathbb{D}u|)|\mathbb{D}u|^2 + |\nabla u|^2 + (1+\lambda(\operatorname{div} u))(\operatorname{div} u)^2 + \varepsilon|\Delta^m u|^2 \,\mathrm{d}x &= \int \tilde\varrho^\gamma \operatorname{div} u \,\mathrm{d}x \\ &\le \eta\int(\operatorname{div} u)^2 \mathrm{d}x + \frac{C}{\eta}\|\tilde\varrho^\gamma\|^2_{L^\infty(0,T;L^2)},\end{aligned}$$

hence picking η small enough and taking supremum over time, we get the desired estimate.

For existence, we consider the functional I defined in

$$\overline{H^{2m}}(\mathbb{T}^d) = \left\{ v \in H^{2m}(\mathbb{T}^d) : \int v \,\mathrm{d}x = 0 \right\},$$

given by

$$I[v] := \int \left(F(\nabla v) + \frac{1}{2}|\nabla v|^2 + \Lambda(\operatorname{div} v) + \frac{\varepsilon}{2}|\Delta^m v|^2 - \tilde\varrho^\gamma(t,\cdot)\operatorname{div} v \right)\mathrm{d}x,$$

where F satisfies

$$\frac{\partial}{\partial b_{i,j}} F(B) = \mu_0(|B|) b_{i,j},$$

for $B = (b_{i,j})_{i,j} \in \mathbb{R}^{d\times d}$ and Λ is such that $\Lambda'(s) = s + \lambda(s)s$. In particular, the assumptions on μ_0 and λ imply that F and Λ are convex and bounded from below.

From the definitions of F and Λ it follows that any minimizer of I corresponds to a weak solution to (3.13). By the convexity of F and Λ, the functional I is convex. Moreover, for certain C and $\eta < C$,

$$I[v] \ge \varepsilon\|\Delta^m v\|^2_{L^2} + C\|\nabla v\|^2_{L^2} - \eta\|\nabla v\|^2_{L^2} - \frac{C}{\eta}\|\tilde\varrho^\gamma(t,\cdot)\|^2_{L^2} \ge C\|v\|^2_{H^{2m}} - C\|\tilde\varrho^\gamma\|^2_{L^\infty(0,T;L^2)},$$

and thus I is coercive. Therefore I has at a.e. time level a unique minimizer $v(t,\cdot) \in \overline{H^{2m}}(\mathbb{T}^d)$ and in consequence there exists a unique $u \in L^\infty(0,T;H^{2m})$ with zero mean value over the torus, solving (3.13). □

Now we use the following classical result for the heat equation (see e.g. Lemmas 7.37-38 in [85]):

Proposition 3.6. *Let $h \in L^2(0,T;L^q)$ for $1 < q < \infty$. Then the solution to*

$$\partial_t \varrho - \varepsilon \Delta \varrho = h, \quad \varrho_{|t=0} = \varrho_0$$

satisfies the estimate

$$\begin{aligned} \varepsilon^{1/2} \|\varrho\|_{L^\infty(0,T;W^{1,q})} + \|\partial_t \varrho\|_{L^2(0,T;L^q)} + \varepsilon \|\varrho\|_{L^2(0,T;W^{2,q})} \\ \leq C \left(\varepsilon^{1/2} \|\varrho_0\|_{W^{1,q}} + \|h\|_{L^2(0,T;L^q)} \right). \end{aligned} \tag{3.15}$$

Moreover, if $h = \operatorname{div} w$, $w \in L^2(0,T;L^q)$, then

$$\varepsilon^{1/2} \|\varrho\|_{L^\infty(0,T;L^q)} + \varepsilon \|\nabla \varrho\|_{L^2(0,T;L^q)} \leq C \left(\varepsilon^{1/2} \|\varrho_0\|_{L^q} + \|w\|_{L^2(0,T;L^q)} \right). \tag{3.16}$$

From the previous Proposition, we can also conclude

Proposition 3.7. *If $u \in L^\infty(0,T;W^{1,\infty})$, then for equation (3.14) there exists a unique nonnegative solution $\varrho \in L^\infty(0,T;W^{1,r})$ with $\partial_t \varrho \in L^2(0,T;W^{-1,r})$ for any $r < \infty$.*

Proof. We construct $\varrho \in L^\infty(0,T;W^{1,2})$ solving (3.14) by the Galerkin approximation. The nonnegativity of solutions is obtained by testing by negative part of ϱ and is a conclusion of the fact that the function $\varrho \mapsto \varrho^\beta$ is odd. By decomposing $\varrho = \varrho_+ - \varrho_-$ into positive and negative parts, $\varrho_+, \varrho_- \geq 0$, testing (3.14) by $-\varrho_-$ we get

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int \varrho_-^2 \, \mathrm{d}x + \delta \int \varrho_-^{\beta+1} \, \mathrm{d}x + \delta \int |\nabla \varrho_-|^2 \, \mathrm{d}x &= -\frac{1}{2} \int \varrho_-^2 \operatorname{div} u \, \mathrm{d}x \\ &\leq \frac{1}{2} \|\operatorname{div} u\|_{L^\infty} \int \varrho_-^2 \, \mathrm{d}x \end{aligned}$$

and $\varrho_- = 0$ from Gronwall's lemma.

Next, testing equation (3.14) by $p\varrho^{p-1}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varrho^p \mathrm{d}x \leq (p-1) \|\operatorname{div} u\|_{L^\infty} \int \varrho^p \mathrm{d}x$$

and therefore

$$\|\varrho\|_{L^\infty(0,T;L^p)} \leq \|\varrho_{0,\varepsilon}\|_{L^p} e^{\frac{p-1}{p} \|u\|_{L^1(0,T;W^{1,\infty})}}.$$

Taking $p \to \infty$ we have $\varrho \in L^\infty((0,T) \times \mathbb{T}^d)$. In consequence $\varrho u \in L^2(0,T;L^r)$ for any $r < \infty$ and we can use (3.16) to obtain

$$\nabla \varrho \in L^2(0,T;L^r).$$

Employing the fact that $u \in L^\infty(0,T;W^{1,\infty})$, we have $\operatorname{div}(\varrho u) \in L^2(0,T;L^r)$ and by (3.15) with $h = -\operatorname{div} u - \delta \varrho^\beta$, $\varrho \in L^\infty(0,T;W^{1,r})$ for any $r < \infty$, whereas the estimate for $\partial_t \varrho$ comes directly from the equation (3.14). The uniqueness is shown by taking two possibly distinct solutions ϱ_1, ϱ_2 and computing $\|\varrho_1 - \varrho_2\|_{L^\infty(0,T;L^2)}$. Similarly as before, we have

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int (\varrho_1 - \varrho_2)^2 \, \mathrm{d}x + \delta \int (\varrho_1^\beta - \varrho_2^\beta)(\varrho_1 - \varrho_2) \, \mathrm{d}x + \delta \int |\nabla(\varrho_1 - \varrho_2)|^2 \, \mathrm{d}x \\ = -\frac{1}{2} \int (\varrho_1 - \varrho_2)^2 \operatorname{div} u \, \mathrm{d}x \end{aligned}$$

and then we proceed analogously as in the proof of nonnegativity, using the fact that $(\varrho_1^\beta - \varrho_2^\beta)(\varrho_1 - \varrho_2) \geq 0$. □

We will now show the properties of Φ, which allow us to apply directly the Schauder fixed point theorem (Theorem 3.4).

Proposition 3.8. *The operator Φ is continuous and compact from $C([0,T];L^{2\gamma})$ to itself. Moreover, the set*

$$\{\varrho \in C([0,T];L^{2\gamma}) : \varrho = s\Phi(\varrho) \quad \textit{for some} \quad s \in [0,1]\}$$

is bounded.

Proof. Let $\tilde{\varrho}_1, \tilde{\varrho}_2 \in C([0,T];L^{2\gamma})$ and u_1, u_2 be the corresponding solutions to (3.13). As before, denote $\Phi(\tilde{\varrho}_i) = \varrho_i$, $i = 1,2$.

Compactness. From the previous propositions we know that $\varrho \in L^\infty(0,T;W^{1,2\gamma})$ and $\partial_t \varrho \in L^2(0,T;W^{-1,2\gamma})$ and the bounds are uniform for bounded sets of $\tilde{\varrho}$ in the given spaces. Therefore, the compactness of Φ in $C([0,T];L^{2\gamma})$ follows straight from a variant of the Aubin–Lions lemma from [94].

Continuity. We will estimate $u_1 - u_2$ in terms of $\tilde{\varrho}_1 - \tilde{\varrho}_2$. We have

$$\begin{aligned} -\operatorname{div}\Big(\mu_0(|\mathbb{D}u_1|)\mathbb{D}u_1 - \mu_0(|\mathbb{D}u_2|)\mathbb{D}u_2\Big) - \nabla\Big((1+\lambda(\operatorname{div} u_1))\operatorname{div} u_1 - (1+\lambda(\operatorname{div} u_2))\operatorname{div} u_2\Big) \\ - \Delta(u_1 - u_2) + \varepsilon\Delta^{2m}(u_1 - u_2) = -\nabla(\tilde{\varrho}_1^\gamma - \tilde{\varrho}_2^\gamma). \end{aligned}$$

Multiplying the above equality by $u_1 - u_2$ and integrating over $\mathbb{T}^d$, we get

$$\mathcal{A}(u_1,u_2) + \int \varepsilon|\Delta^m(u_1-u_2)|^2 + |\nabla(u_1-u_2)|^2 \,\mathrm{d}x = \int (\tilde{\varrho}_1^\gamma - \tilde{\varrho}_2^\gamma)(\operatorname{div} u_1 - \operatorname{div} u_2)\mathrm{d}x,$$

where

$$\begin{aligned} \mathcal{A}(u_1,u_2) = \int & (\mu_0(|\mathbb{D}u_1|)\mathbb{D}u_1 - \mu_0(|\mathbb{D}u_2|)\mathbb{D}u_2) : (\mathbb{D}u_1 - \mathbb{D}u_2) \\ & + \Big((1+\lambda(\operatorname{div} u_1))\operatorname{div} u_1 - (1+\lambda(\operatorname{div} u_2))\operatorname{div} u_2\Big)(\operatorname{div} u_1 - \operatorname{div} u_2)\mathrm{d}x \geq 0 \end{aligned}$$

from the monotonicity of the functions $B \mapsto \frac{B}{a+|B|}$ and $s \mapsto \lambda(s)s$. In consequence, we have

$$\begin{aligned} \|\nabla(u_1-u_2)\|^2_{L^2(\mathbb{T}^d)} &+ \varepsilon\|\Delta^m(u_1-u_2)\|^2_{L^2(\mathbb{T}^d)} \\ &\leq \|\tilde{\varrho}_1^\gamma - \tilde{\varrho}_2^\gamma\|_{L^2(\mathbb{T}^d)}\|\operatorname{div}(u_1-u_2)\|_{L^2(\mathbb{T}^d)} \\ &\leq C\left(\|\tilde{\varrho}_1\|^{\gamma-1}_{L^\infty(\mathbb{T}^d)} + \|\tilde{\varrho}_2\|^{\gamma-1}_{L^\infty(\mathbb{T}^d)}\right)\|\tilde{\varrho}_1 - \tilde{\varrho}_2\|_{L^2(\mathbb{T}^d)}\|\nabla(u_1-u_2)\|_{L^2(\mathbb{T}^d)} \\ &\leq C(\eta)\|\tilde{\varrho}_1 - \tilde{\varrho}_2\|^2_{L^2(\mathbb{T}^d)} + \eta\|\nabla(u_1-u_2)\|^2_{L^2(\mathbb{T}^d)}. \end{aligned}$$

Hence, choosing η small enough we get

$$\begin{aligned} \|u_1 - u_2\|_{L^\infty(0,T;W^{1,\infty})} &\leq C\|\Delta^m(u_1-u_2)\|_{L^\infty(0,T;L^2)} \\ &\leq C\|\tilde{\varrho}_1 - \tilde{\varrho}_2\|_{L^\infty(0,T;L^2)} \\ &\leq C\|\tilde{\varrho}_1 - \tilde{\varrho}_2\|_{L^\infty(0,T;L^{2\gamma})}. \end{aligned}$$

Moreover, $\varrho_1 - \varrho_2$ satisfy

$$\partial_t(\varrho_1 - \varrho_2) + \delta(\varrho_1^\beta - \varrho_2^\beta) - \delta\Delta(\varrho_1 - \varrho_2) = -\operatorname{div}(\varrho_1 u_1 - \varrho_2 u_2) \tag{3.17}$$

with

$$(\varrho_1 - \varrho_2)_{|t=0} = 0.$$

Let us now estimate $\|\varrho_1-\varrho_2\|_{L^\infty(0,T;L^p)}$. First, we write $\operatorname{div}(\varrho_1 u_1-\varrho_2 u_2)$ as

$$\operatorname{div}(\varrho_1 u_1-\varrho_2 u_2)=u_1\nabla(\varrho_1-\varrho_2)+\nabla\varrho_2(u_1-u_2)+\operatorname{div}u_1(\varrho_1-\varrho_2)+\varrho_2(\operatorname{div}u_1-\operatorname{div}u_2).$$

Then, multiplying (3.17) by $p|\varrho_1-\varrho_2|^{p-2}(\varrho_1-\varrho_2)$ and integrating, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int|\varrho_1-\varrho_2|^p\mathrm{d}x+&\delta p\int|\varrho_1-\varrho_2|^{p-2}(\varrho_1^\beta-\varrho_2^\beta)(\varrho_1-\varrho_2)\,\mathrm{d}x\\ &=-\delta p(p-1)\int|\varrho_1-\varrho_2|^{p-2}|\nabla(\varrho_1-\varrho_2)|^2\,\mathrm{d}x\\ &\quad-(p-1)\int|\varrho_1-\varrho_2|^p\operatorname{div}u_1\mathrm{d}x\\ &\quad-\int(\nabla\varrho_2(u_1-u_2)+\varrho_2(\operatorname{div}u_1-\operatorname{div}u_2))|\varrho_1-\varrho_2|^{p-2}(\varrho_1-\varrho_2)\mathrm{d}x.\end{aligned}$$

In consequence, as $(\varrho_1^\beta-\varrho_2^\beta)(\varrho_1-\varrho_2)\geq 0$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int|\varrho_1-\varrho_2|^p\mathrm{d}x\leq(p-1)\|u_1\|_{W^{1,\infty}(\mathbb{T}^d)}&\int|\varrho_1-\varrho_2|^p\mathrm{d}x\\ &+\|\varrho_2\|_{W^{1,p}(\mathbb{T}^d)}\|u_1-u_2\|_{W^{1,\infty}(\mathbb{T}^d)}\|\varrho_1-\varrho_2\|_{L^p(\mathbb{T}^d)}^{p-1}.\end{aligned}$$

Therefore from Gronwall's lemma

$$\|\varrho_1-\varrho_2\|_{L^\infty(0,T;L^p)}\leq C\|u_1-u_2\|_{L^\infty(0,T;W^{1,\infty})},$$

where C depends on T, $\|u_1\|_{L^1(0,T;W^{1,\infty})}$ and $\|\varrho_2\|_{L^2(0,T;W^{1,p})}$. In particular,

$$\|\varrho_1-\varrho_2\|_{L^\infty(0,T;L^{2\gamma})}\leq C\|u_1-u_2\|_{L^\infty(0,T;W^{1,\infty})}\leq C\|\tilde\varrho_1-\tilde\varrho_2\|_{L^\infty(0,T;L^{2\gamma})}.$$

Estimates for the fixed points. To complete the proof of the Proposition, we need to check if the points satisfying $\varrho=s\Phi(\varrho)$ are bounded in $L^\infty(0,T;L^{2\gamma})$ for any $s\in[0,1]$. Throughout the proof we will denote by C various constants independent on s. If $s=0$, the claim is trivial. For $s>0$, we have

$$\frac{1}{s}\partial_t\varrho+\frac{1}{s}\operatorname{div}(\varrho u)+\frac{\delta}{s^\beta}\varrho^\beta=\frac{1}{s}\delta\Delta\varrho$$

and

$$-\operatorname{div}\,(\mu_0(|\mathbb{D}u|)\mathbb{D}u)-\Delta u-\nabla(1+\lambda(\operatorname{div}u))\operatorname{div}u+\varepsilon\Delta^{2m}u+\nabla\varrho^\gamma=0.$$

Multiplying the momentum equation by u and integrating, we obtain analogously as for the a priori estimates

$$\begin{aligned}\int_0^T\int|\nabla u|^2\,\mathrm{d}x\mathrm{d}t+\varepsilon\int_0^T\int|\Delta^m u|^2\,\mathrm{d}x\mathrm{d}t+\sup_{t\in[0,T]}\frac{1}{\gamma-1}\int\varrho^\gamma\,\mathrm{d}x&\\ +\frac{\delta}{s^{\beta-1}}\frac{\gamma}{\gamma-1}\int_0^T\int\varrho^{\beta+\gamma-1}\mathrm{d}x\mathrm{d}t+\delta\gamma\int_0^T\int|\nabla\varrho|^2\varrho^{\gamma-2}\mathrm{d}x\mathrm{d}t&\leq\int\varrho_{0,\varepsilon}^\gamma\mathrm{d}x\leq C.\end{aligned}$$

Repeating the estimate from Proposition 3.7, we get again

$$\|\varrho\|_{L^\infty(0,T;L^{2\gamma})}\leq\|\varrho_{0,\varepsilon}\|_{L^\infty(\mathbb{T}^d)}e^{\|u\|_{L^1(0,T;W^{1,\infty})}}\leq C.$$

$\square$

In consequence, the assumptions of Theorem 3.4 are satisfied and there exists at least one solution to (3.12) on $[0,T]\times\mathbb{T}^d$ for arbitrary $T>0$.

3.4 Compactness

We will now prove that we can pass to the limit with $\delta,\varepsilon\to 0$ to obtain the solutions to system (3.6). First, we will pass to the limit with $\varepsilon\to 0$ and then we improve the estimates on ϱ uniform in δ and perform the second limit passage. Below, by $\overline{f}$ we will denote the weak limit of a sequence f_n.

3.4.1 Limit passage with $\varepsilon\to 0$

Let $(\varrho_{\delta,\varepsilon},u_{\delta,\varepsilon})$ be a solution to (3.12). We have the following estimates uniform in ε:

$$\begin{aligned}\|u_{\delta,\varepsilon}\|^2_{L^2(0,T;W^{1,2})}+\|\varrho_{\delta,\varepsilon}\|^\gamma_{L^\infty(0,T;L^\gamma)}+\delta\|\nabla\varrho_{\delta,\varepsilon}^{\gamma/2}\|^2_{L^2((0,T)\times\mathbb{T}^d)}&\\ +\delta\|\varrho_{\delta,\varepsilon}\|^{\gamma+\beta-1}_{L^{\gamma+\beta-1}((0,T)\times\mathbb{T}^d)}+\delta\|\nabla\varrho_{\delta,\varepsilon}\|^2_{L^2((0,T)\times\mathbb{T}^d)}&\le C.\end{aligned}$$

In particular, at least up to a subsequence,

$$u_{\delta,\varepsilon}\rightharpoonup u_\delta\quad\text{in}\quad L^2(0,T;W^{1,2}).$$

Moreover, as the lower bounds on β provide $\beta+\gamma-1\ge 2\gamma$, we know that

$$\|\varrho_{\delta,\varepsilon}^\gamma\|_{L^2((0,T)\times\mathbb{T}^d)},\|\varrho_{\delta,\varepsilon}u_{\delta,\varepsilon}\|_{L^p((0,T)\times\mathbb{T}^d)}\le C(\delta)$$

for some suitable $p<2$. In consequence,

$$\|\nabla\varrho_{\delta,\varepsilon}\|_{L^2((0,T)\times\mathbb{T}^d)},\ \|\varrho_{\delta,\varepsilon}\|_{L^p((0,T)\times\mathbb{T}^d)},\ \|\partial_t\varrho_{\delta,\varepsilon}\|_{L^p(0,T;W^{-1,p})}\le C(\delta).$$

Therefore from the Aubin–Lions lemma $\varrho_{\delta,\varepsilon}\to\varrho_\delta$ in $L^p((0,T)\times\mathbb{T}^d)$, (at least up to a subsequence). Then we also have $\varrho_{\delta,\varepsilon}^\gamma\to\varrho_\delta^\gamma$ and $\varrho_{\delta,\varepsilon}^\beta\to\varrho_\delta^\beta$ in suitable L^q spaces. In consequence, we are able to pass to the limit in the continuity equation. For the momentum equation, note that the regularizing term satisfies

$$\varepsilon^{1/2}\|\Delta^m u_{\delta,\varepsilon}\|_{L^2((0,T)\times\mathbb{T}^d)}\le C$$

and thus in the weak formulation

$$\varepsilon\int_0^T\int\Delta^m u_{\delta,\varepsilon}\cdot\Delta^m\phi\,\mathrm{d}x\mathrm{d}t\to 0$$

for $\phi\in C_0^\infty((0,T)\times\mathbb{T}^d)$. Therefore in the weak formulation we obtain

$$\begin{aligned}\int_0^T\int\overline{\mu_0(|\mathbb{D}u_\delta|)\mathbb{D}u_\delta}:\mathbb{D}\varphi+\nabla u_\delta:\nabla\varphi&\\ +\operatorname{div}u_\delta\operatorname{div}\varphi+\overline{\lambda(\operatorname{div}u_\delta)\operatorname{div}u_\delta}\operatorname{div}\varphi-\varrho_\delta^\gamma\operatorname{div}\varphi\,dx\mathrm{d}t&=0.\end{aligned}$$

Testing by u_δ, we get

$$\int_0^T \int \overline{\mu_0(|\mathbb{D}u_\delta|)\mathbb{D}u_\delta} : \mathbb{D}u_\delta + |\nabla u_\delta|^2 \\ + (\mathrm{div}\, u_\delta)^2 + \overline{\lambda(\mathrm{div}\, u_\delta)\mathrm{div}\, u_\delta}\mathrm{div}\, u_\delta - \varrho_\delta^\gamma \mathrm{div}\, u_\delta \, \mathrm{d}x\mathrm{d}t = 0. \quad (3.18)$$

On the other hand, testing first the momentum equation in (3.12), by $u_{\delta,\varepsilon}$, we get

$$\int_0^T \int \mu_0(|\mathbb{D}u_{\delta,\varepsilon}|)\mathbb{D}u_{\delta,\varepsilon} + |\nabla u_{\delta,\varepsilon}|^2 \\ + (\mathrm{div}\, u_{\delta,\varepsilon})^2 + \lambda(|\mathrm{div}\, u_{\delta,\varepsilon}|)(\mathrm{div}\, u_{\delta,\varepsilon})^2 - \varrho_{\delta,\varepsilon}^\gamma \mathrm{div}\, u_{\varepsilon,\delta} \, \mathrm{d}x\mathrm{d}t = 0$$

and thus passing to the limit with ε we get

$$\int_0^T \int \overline{\mu_0(|\mathbb{D}u_\delta|)|\mathbb{D}u_\delta|^2} + \overline{\lambda(\mathrm{div}\, u_\delta)(\mathrm{div}\, u_\delta)^2}\mathrm{d}x\mathrm{d}t \\ + \liminf_{\varepsilon\to 0} \int_0^T \int |\nabla u_{\delta,\varepsilon}|^2 + (\mathrm{div}\, u_{\delta,\varepsilon})^2 \mathrm{d}x\mathrm{d}t - \int_0^T \int \varrho_\delta^\gamma \mathrm{div}\, u \mathrm{d}x\mathrm{d}t \leq 0. \quad (3.19)$$

From the monotonicity assumptions on μ_0, we know that

$$(\mu_0(|\mathbb{D}u_{\delta,\varepsilon}|)\mathbb{D}u_{\delta,\varepsilon} - \mu_0(|\mathbb{D}u_\delta|)\mathbb{D}u_\delta) : (\mathbb{D}u_{\delta,\varepsilon} - \mathbb{D}u_\delta) \geq 0.$$

Therefore passing to the limit with $\varepsilon \to 0$ and using weak convergence of $\mathbb{D}u_{\delta,\varepsilon}$, we get

$$\overline{\mu_0(|\mathbb{D}u_\delta|)|\mathbb{D}u_\delta|^2} \geq \overline{\mu_0(|\mathbb{D}u_\delta|)\mathbb{D}u_\delta} : \mathbb{D}u_\delta. \quad (3.20)$$

Analogously, for λ we have

$$\overline{\lambda(|\mathrm{div}\, u_\delta|)(\mathrm{div}\, u_\delta)^2} \geq \overline{\lambda(|\mathrm{div}\, u_\delta|)\mathrm{div}\, u_\delta}\mathrm{div}\, u_\delta. \quad (3.21)$$

Therefore substracting (3.18) from (3.19) and using weak lower semicontinuity of the norm, we obtain the convergence $\nabla u_{\delta,\varepsilon} \to \nabla u_\delta$ in $L^2((0,T) \times \mathbb{T}^d)$, which allows us to pass to the limit in the remaining nonlinear terms.

3.4.2 Limit passage with $\delta \to 0$

Now, let $(\varrho_\delta, u_\delta)$ be the function obtained in the previous section, solving

$$\begin{aligned} \varrho_t + \mathrm{div}\,(\varrho u) + \delta\varrho^\beta &= \delta\Delta\varrho, \\ -\mathrm{div}\,((\mu_0(|\mathbb{D}u|) + 1)\mathbb{D}u) - \nabla(\lambda(\mathrm{div}\, u)\mathrm{div}\, u) + \nabla\varrho^\gamma &= 0. \end{aligned} \quad (3.22)$$

Note that repeating the calculations from Section 3.2, we get the estimate

$$\|2\mathrm{div}\, u_\delta - \varrho_\delta^\gamma\|_{L^\infty(0,T;BMO)} \leq C.$$

Moreover, using the uniform estimates on $\|u_\delta\|_{L^2(0,T;W^{1,2})}$ and $\|\varrho_\delta\|_{L^\infty(0,T;L^\gamma)}$, we will improve the integrability of ϱ_δ uniformly in δ.

Let $p > 1$ and let $T_k \in C^\infty([0,\infty))$ be the truncation operator, namely $T_k(z) = z$ for $z < k$, $T_k(z) = k+1$ for $z > 2k$, $T_k'(z) \geq 0$ as well as $T_k(z) \nearrow z$ as $k \to \infty$. We define the function $P_k(\varrho)$ as

$$P_k(\varrho) = \varrho \int_0^\varrho \frac{T_k(z)^p}{z^2} \mathrm{d}z.$$

The choice of P_k is motivated by the fact that

$$\varrho P_k'(\varrho) - P_k(\varrho) = T_k(\varrho)^p.$$

One can also observe that $P_k(\varrho) \nearrow \frac{1}{p-1}\varrho^p$ as $k \to \infty$.

Testing the continuity equation in (3.22) by $P_k'(\varrho)$, we get

$$\frac{\mathrm{d}}{\mathrm{dt}} \int P_k(\varrho_\delta)\,\mathrm{d}x + \delta \int \left(\varrho_\delta^\beta P_k'(\varrho_\delta) + \frac{pT_k(\varrho_\delta)^{p-1}T_k'(\varrho_\delta)}{\varrho_\delta} |\nabla \varrho_\delta|^2 \right) \mathrm{d}x$$
$$= - \int T_k(\varrho_\delta)^p \mathrm{div}\, u_\delta\,\mathrm{d}x. \quad (3.23)$$

Now, let us test the momentum equation by the function

$$\psi = \Delta^{-1}\nabla \left(T_k(\varrho_\delta)^p - \{T_k(\varrho_\delta)^p\} \right).$$

We have

$$\begin{aligned}
\int_0^T \int \varrho_\delta^\gamma T_k(\varrho_\delta)^p\,\mathrm{d}x\mathrm{dt}
&\le C\|\mu_0(|\mathbb{D}u_\delta|)\mathbb{D}u_\delta\|_{L^\infty((0,T)\times\mathbb{T}^d)} \|\nabla\psi\|_{L^{1+\frac{\gamma}{p}}((0,T)\times\mathbb{T}^d)} \\
&\quad + \|\lambda(|\mathrm{div}\, u_\delta|)\mathrm{div}\, u_\delta\|_{L^\infty((0,T)\times\mathbb{T}^d)} \|\nabla\psi\|_{L^{1+\frac{\gamma}{p}}((0,T)\times\mathbb{T}^d)} \\
&\quad + 2\int_0^T \int T_k(\varrho_\delta)^p \mathrm{div}\, u_\delta\,\mathrm{d}x\mathrm{dt} \\
&\quad + C\|\varrho_\delta\|_{L^\infty(0,T;L^\gamma)} \int_0^T \int T_k(\varrho_\delta)^p \mathrm{d}x\mathrm{dt} \\
&\le C\|T_k(\varrho_\delta)\|^p_{L^{p+\gamma}((0,T)\times\mathbb{T}^d)} + 2\int_0^T \int T_k(\varrho_\delta)^p \mathrm{div}\, u_\delta\,\mathrm{d}x\mathrm{dt}.
\end{aligned}$$

Then by Cauchy inequality,

$$\int_0^T \int \varrho_\delta^\gamma T_k(\varrho_\delta)^p \mathrm{d}x \le \eta \|T_k(\varrho_\delta)\|^{p+\gamma}_{L^{p+\gamma}((0,T)\times\mathbb{T})} + C(\eta) + 2\int_0^T \int T_k(\varrho_\delta)^p \mathrm{div}\, u_\delta \mathrm{d}x.$$

As $T_k(\varrho)^{p+\gamma} \le \varrho^\gamma T_k(\varrho)^p$, for sufficiently small η we get

$$\int_0^T \int \varrho_\delta^\gamma T_k(\varrho_\delta)^p \mathrm{d}x\mathrm{dt} - 2\int_0^T \int T_k(\varrho_\delta)^p \mathrm{div}\, u_\delta \mathrm{d}x \le C.$$

Therefore using (3.23), we get

$$\begin{aligned}
&\int_0^T \int \varrho_\delta^\gamma T_k(\varrho_\delta)^p \mathrm{d}x\mathrm{dt} + \sup_{t\in(0,T]} \int P_k(\varrho_\delta(t,\cdot))\mathrm{d}x \\
&+ \delta \int \left(\varrho_\delta^\beta P_k'(\varrho_\delta) + \frac{pT_k(\varrho_\delta)^{p-1}T_k'(\varrho_\delta)}{\varrho_\delta} |\nabla\varrho_\delta|^2 \right) \mathrm{d}x \le C(T,p).
\end{aligned}$$

Since $P_k(\varrho_\delta) \nearrow \frac{1}{p-1}\varrho_\delta^p$, we pass to the limit with $k \to \infty$ using monotone convergence theorem and in consequence we get

$$\|\varrho_\delta\|_{L^\infty(0,T;L^p)} \le C(T,p) \quad \text{for any} \quad p < \infty.$$

Having that estimate, we are ready to pass to the limit with $\delta \to 0$.

From the estimates uniform in δ, we know that in particular

$$\begin{aligned} u_\delta \rightharpoonup u \quad & \text{in} \quad L^2(0,T;W^{1,2}), \\ \varrho_\delta \rightharpoonup^* \varrho \quad & \text{in} \quad L^\infty(0,T;L^\gamma), \\ \varrho_\delta^\gamma \rightharpoonup^* \overline{\varrho^\gamma} \quad & \text{in} \quad L^\infty(0,T;L^p) \end{aligned}$$

and

$$\mu_0(\mathbb{D}u_\delta)\mathbb{D}u_\delta,\ \lambda(\operatorname{div} u_\delta)\operatorname{div} u_\delta \rightharpoonup^* \overline{\mu_0(|\mathbb{D}u|)\mathbb{D}u},\ \overline{\lambda(\operatorname{div} u)\operatorname{div} u} \quad \text{in} \quad L^\infty((0,T)\times\mathbb{T}^d).$$

Moreover, $\|2\operatorname{div} u - \overline{\varrho^\gamma}\|_{L^\infty(0,T;BMO)} \leq C$. Note that from the continuity equation it also follows that $\varrho_\delta \to \varrho$ in $C([0,T];W^{-1,r})$ for a suitable r, and in consequence $\overline{\varrho u} = \varrho u$. Having the above estimates and testing the continuity equation by ϱ_δ, we also obtain

$$\delta^{1/2}\|\nabla\varrho_\delta\|_{L^2((0,T)\times\mathbb{T}^d)} \leq C.$$

Then for $\phi \in C_0^\infty((0,T)\times\mathbb{T}^d)$, together with the estimate on $\|\varrho_\delta\|_{L^\infty(0,T;L^p)}$,

$$\delta\int_0^T\int\left(\varrho_\delta^\beta\phi + \nabla\varrho_\delta\cdot\nabla\phi\right)\mathrm{d}x\mathrm{d}t \to 0 \quad \text{with} \quad \delta\to 0.$$

In consequence, (ϱ,u) satisfies the continuity equation in (3.6) in the renormalized sense.

Next, we will pass to the limit in the momentum equation and apply an argument from [45]. Passing to the limit in the weak formulation, we get for any $\phi \in C_0^\infty((0,T)\times\mathbb{T}^d)$ and $t \leq T$

$$\int_0^t\int \overline{\mu_0(|\mathbb{D}u|)\mathbb{D}u} : \mathbb{D}\phi + \nabla u : \nabla\phi + \operatorname{div} u\ \operatorname{div}\phi + \overline{\lambda(\operatorname{div} u)\operatorname{div} u}\ \operatorname{div}\phi\ \mathrm{d}x\mathrm{d}s$$
$$= \int_0^t\int \overline{\varrho^\gamma}\operatorname{div}\phi\ \mathrm{d}x\mathrm{d}s. \quad (3.24)$$

The regularity of u allows us to put $\phi = u$ in (3.24) and then

$$\int_0^t\int \overline{\mu_0(|\mathbb{D}u|)\mathbb{D}u} : \mathbb{D}u + |\nabla u|^2 + (\operatorname{div} u)^2 + \overline{\lambda(\operatorname{div} u)\operatorname{div} u}\ \operatorname{div} u\ \mathrm{d}x\mathrm{d}s$$
$$= \int_0^t\int \overline{\varrho^\gamma}\operatorname{div} u\ \mathrm{d}x\mathrm{d}s. \quad (3.25)$$

On the other hand, the solutions to approximate equation (3.22) satisfy

$$\int_0^t\int \mu_0(|\mathbb{D}u_\delta|)|\mathbb{D}u_\delta|^2\mathrm{d}x\mathrm{d}s + \int_0^t\int |\nabla u_\delta|^2 + (\operatorname{div} u_\delta)^2 + \lambda(\operatorname{div} u_\delta)(\operatorname{div} u_\delta)^2\ \mathrm{d}x\mathrm{d}s$$
$$+ \frac{1}{\gamma-1}\int\varrho_\delta^\gamma(t,\cdot)\mathrm{d}x \leq \frac{1}{\gamma-1}\int\varrho_{0,\delta}^\gamma\mathrm{d}x. \quad (3.26)$$

Analogously as in the previous limit passage, by taking $\liminf_{\delta\to 0}$ in the energy inequality (3.26), we obtain

$$\int_0^t \int \overline{\mu_0(|\mathbb{D}u|)|\mathbb{D}u|^2} + \overline{\lambda(\operatorname{div} u)(\operatorname{div} u)^2} \,\mathrm{d}x\mathrm{d}s + \liminf_{\delta\to 0} \int_0^t \int |\nabla u_\delta|^2 + (\operatorname{div} u_\delta)^2 \,\mathrm{d}x\mathrm{d}s$$
$$+ \frac{1}{\gamma-1}\int \overline{\varrho^\gamma}(t,\cdot)\mathrm{d}x \le \frac{1}{\gamma-1}\int \varrho_0^\gamma \mathrm{d}x.$$

Using again the monotonicity of $\mu_0(|\mathbb{D}u|)\mathbb{D}u$ and $\lambda(|\operatorname{div} u|)\operatorname{div} u$, by virtue of (3.20) and (3.21) we obtain

$$\int_0^t \int \overline{\mu_0(|\mathbb{D}u|)\mathbb{D}u} : \mathbb{D}u + |\nabla u|^2 + (\operatorname{div} u)^2 + \overline{\lambda(\operatorname{div} u)\operatorname{div} u}\ \operatorname{div} u \,\mathrm{d}x\mathrm{d}s$$
$$+ \frac{1}{\gamma-1}\int \overline{\varrho^\gamma}(t,\cdot)\mathrm{d}x \le \frac{1}{\gamma-1}\int \varrho_0^\gamma \mathrm{d}x. \quad (3.27)$$

Comparing (3.27) with (3.25), we get

$$\frac{1}{\gamma-1}\int \overline{\varrho^\gamma}(t,\cdot)\,\mathrm{d}x - \frac{1}{\gamma-1}\int \varrho_0^\gamma\,\mathrm{d}x \le -\int_0^t\int \overline{\varrho^\gamma}\operatorname{div} u\,\mathrm{d}x\mathrm{d}s.$$

We would like to estimate $\displaystyle\int \overline{\varrho^\gamma}(t,\cdot) - \varrho^\gamma(t,\cdot)\mathrm{d}x$. As we already know that ϱ satisfies the continuity equation in the renormalized sense, we have

$$\frac{1}{\gamma-1}\int \varrho^\gamma(t,\cdot)\,\mathrm{d}x - \frac{1}{\gamma-1}\int \varrho_0^\gamma\,\mathrm{d}x = -\int_0^t\int \varrho^\gamma\operatorname{div} u\,\mathrm{d}x\mathrm{d}s. \quad (3.28)$$

Therefore

$$\frac{1}{\gamma-1}\int (\overline{\varrho^\gamma}(t,\cdot) - \varrho^\gamma(t,\cdot))\mathrm{d}x \le -\int_0^t\int (\overline{\varrho^\gamma} - \varrho^\gamma)\operatorname{div} u\mathrm{d}x\mathrm{d}s.$$

We now use the fact that $2\operatorname{div} u - \overline{\varrho^\gamma} \in L^\infty(0,T;BMO)$ and the logarithmic inequality (B.1). As $\overline{\varrho^\gamma} \ge \varrho^\gamma$ and $\varrho^\gamma, \overline{\varrho^\gamma} \in L^\infty(0,T;L^p)$ for any $p<\infty$, we have

$$\begin{aligned}
-\int_0^t\int (\overline{\varrho^\gamma} - \varrho^\gamma)\operatorname{div} u\,\mathrm{d}x\mathrm{d}s = & -\int_0^t\int (\overline{\varrho^\gamma} - \varrho^\gamma)\left(\operatorname{div} u - \frac{1}{2}\overline{\varrho^\gamma}\right)\mathrm{d}x\mathrm{d}s \\
& - \frac{1}{2}\int_0^t\int (\overline{\varrho^\gamma} - \varrho^\gamma)\overline{\varrho^\gamma}\mathrm{d}x\mathrm{d}s \\
\le & -\int_0^t\int (\overline{\varrho^\gamma} - \varrho^\gamma)\left(\operatorname{div} u - \frac{1}{2}\overline{\varrho^\gamma}\right)\mathrm{d}x\mathrm{d}s \\
\le & C\int_0^t\int (\overline{\varrho^\gamma} - \varrho^\gamma)\mathrm{d}x\left(\left|\ln\left(\int (\overline{\varrho^\gamma} - \varrho^\gamma)\mathrm{d}x\right)\right| + 1\right)\mathrm{d}s,
\end{aligned}$$

where C depends on $\|\operatorname{div} u - \frac{1}{2}\overline{\varrho^\gamma}\|_{L^\infty(0,T;BMO)}$ and $\|\overline{\varrho^\gamma} - \varrho^\gamma\|_{L^\infty(0,T;L^q)}$ for some $q>2$. Thus denoting $y(t) = \displaystyle\int (\overline{\varrho^\gamma} - \varrho^\gamma)\mathrm{d}x$, we have the inequality

$$y(t) \le C\int_0^t y(s)(|\ln y(s)|+1)\mathrm{d}s \quad \text{with} \quad y(0) = 0.$$

Let us now define $z(t) = \int_0^t y(s)(|\ln y(s)|+1)\mathrm{d}s$. Obviously $z(0)=0$. Since the function $y \mapsto y(|\ln y|+1)$ is increasing, we have

$$z' = y(t)(|\ln y(t)|+1) \leq Cz(t)(|\ln z(t)|+1).$$

Therefore by Osgood's lemma $z \equiv 0$ on $[0,T]$ and in consequence $y \equiv 0$ as well. From this, as $\overline{\varrho^\gamma} \geq \varrho^\gamma$, it follows that in fact $\overline{\varrho^\gamma} = \varrho^\gamma$. Now taking again the limit in (3.26) and subtracting (3.25), using $\varrho^\gamma = \overline{\varrho^\gamma}$ and (3.28) we get

$$\limsup_{\delta\to 0} \int_0^t \int \left(|\nabla u_\delta|^2 - |\nabla u|^2\right)\mathrm{d}x\mathrm{d}s \leq 0.$$

Since

$$|\nabla u_\delta|^2 - |\nabla u|^2 = |\nabla u_\delta - \nabla u|^2 + 2\nabla u : (\nabla u_\delta - \nabla u)$$

and $\nabla u_\delta \rightharpoonup \nabla u$ in $L^2((0,T)\times\mathbb{T}^d)$, we conclude that

$$\limsup_{\delta\to 0} \int_0^t \int |\nabla u_\delta - \nabla u|^2 \,\mathrm{d}x\mathrm{d}t \leq 0$$

and thus $\nabla u_\varepsilon \to \nabla u$ in $L^2((0,T)\times\mathbb{T}^d)$. Therefore (for possibly another subsequence) the sequence of velocities converges also a.e. In consequence, by virtue of the Lebesgue dominated convergence theorem,

$$\overline{\mu_0(|\mathbb{D}u|)\mathbb{D}u} = \mu_0(|\mathbb{D}u|)\mathbb{D}u$$

and

$$\overline{\lambda(\operatorname{div} u)\operatorname{div} u} = \lambda(\operatorname{div} u)\operatorname{div} u,$$

and thus (ϱ, u) satisfies the weak formulation of the system (3.6), which finishes the proof of Theorem 3.2.

Let us finish by the following remark concerning singular viscosities:

Remark 3.9. The assumptions on μ_0 and λ and the used method allows the situation when the viscosities are singular at 0, e.g. $\mu_0 = \frac{1}{|\mathbb{D}u|}$. Note, however, that in this case, while passing to the limit in the weak formulation, the term

$$\int \frac{\mathbb{D}u}{|\mathbb{D}u|} : \nabla\varphi \mathrm{d}x$$

is well defined by the values of $\mathbb{D}u$ provided $|\mathbb{D}u|>0$. For $|\mathbb{D}u|=0$ it is just defined as the corresponding limit, which is not necessarily equal to zero if $|\mathbb{D}u|$ is so, cf. e.g. [62] in a similar context. On can also define the limit stress tensor via the relation

$$\begin{aligned} \mathbb{S} = \mathbb{D}u + \frac{\mathbb{D}u}{|\mathbb{D}u|} \quad &\text{if} \quad |\mathbb{D}u| \neq 0, \\ |\mathbb{S}| \leq 1 \quad &\text{if} \quad \mathbb{D}u = 0. \end{aligned}$$

Such formulation was used before in the context of incompressible Hershel–Bulkley fluids, see e.g. [42, 68].

Chapter 4

An attraction–repulsion system in the framework of compressible viscous flows

We analyze the pressureless Navier-Stokes system with nonlocal attraction–repulsion forces. Such systems appear in the context of models of collective behavior. We prove the existence of weak solutions on the whole space $\mathbb{R}^3$ in the case of density-dependent degenerate viscosity, where for the nonlocal term it is assumed that the interaction kernel has the quadratic growth at infinity and almost quadratic singularity at zero. Under these assumptions, we derive the analog of the Bresch–Desjardins and Mellet–Vasseur estimates for the nonlocal system. In particular, we are able to adapt the approach of Vasseur and Yu [106, 105] to construct a weak solution. This part of the thesis so far remains unpublished, as the article is still in preparation.

4.1 Introduction

Hydrodynamic equations with nonlocal forces often arise from the modelling of collective behaviour. The applications of these types of systems involve in particular flocking and swarming phenomena, appearing in many animal species and bacteria (see e. g. [60, 87]). On the microscopic scale, the model consists of N agents, which in some way align their position and velocity in relation to others. On the macroscopic scale, assuming the number of agents is very large, by passing with $N \to \infty$ one derives the system of partial differential equations on the macroscopic density ϱ and the macroscopic velocity u. The main difference between the classical hydrodynamical equations and the systems arising as the limit of the collective agent-based systems is the presence of the nonlocal interactions. This is due to the fact that on the microscopic level, the velocity of the particular agent depends on the position of others. In the continuous model, this corresponds to a nonlocal pressure-like term in the momentum equation. The examples of the above mathematical models and performing the limit from discrete to continuous systems one can be found in [26, 17, 33], see also references therein.

The goal of this chapter is to construct a weak solution to the pressureless degenerate Navier-Stokes system

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho u) &= 0 \\ \partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u) - \mu \operatorname{div}(\varrho \mathbb{D} u) + \varrho \nabla (K * \varrho) &= 0 \end{aligned} \quad \text{in} \quad [0,T] \times \mathbb{R}^3, \tag{4.1}$$

where $\varrho \colon [0,T] \times \mathbb{R}^3 \to \mathbb{R}_+$ is the density of the particles and $u \colon [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ its velocity. The term $\varrho \nabla (K * \varrho)$ corresponds to the attractive-repulsive forces, coming from the collective nature of the model. Roughly speaking, it corresponds to the situation where the particular agents want to keep relatively close to each other, but at the same time avoid collision. More precisely, ∇K consists of two parts: the first forces the particles to keep some small distance, and the second controls its spread in the whole space. We consider the kernel K in the form

$$K(x) = \frac{c_1}{|x|^\alpha} + \frac{c_2}{2}|x|^2, \tag{4.2}$$

where $\alpha \in (0,2)$, $c_1, c_2 > 0$. Note that we can express the singular part in terms of the Riesz potential

$$I_s(f)(x) = (-\Delta)^{-s/2} f(x) = \frac{1}{\tilde{c}_s} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|^{3-s}} \, \mathrm{d}y, \quad \tilde{c}_s = \pi^{3/2} 2^s \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{3-s}{2}\right)}$$

for $s = 3 - \alpha$. That way

$$\frac{1}{|\cdot|^\alpha} * \varrho = \tilde{c}_{3-\alpha} I_{3-\alpha}(\varrho).$$

It is also worth to point out that the special case $\alpha = 1$ covers the Newtonian potential and then equation (4.1) becomes the Navier-Stokes-Poisson system. For simplicity, since it does not affect our result we put $\mu = c_1 = c_2 = 1$ in (4.1) and (4.2).

In recent years, different types of equations with nonlocal interactions were considered. In case of the potential proportional to $|x|^{-\alpha}$ for $\alpha \in (0,d)$, stationary solutions and their stability were thoroughly studied for example in [19, 20, 28]. In [27] it was shown that the nonlocal Euler system admits infinitely many weak solutions. Concerning the existence of weak solutions, in [24] the authors proved the existence of weak solutions to the compressible Navier-Stokes system with damping, when the nonlocal term is sufficiently integrable. The existence of weak solutions for Navier-Stokes-Poisson system on the torus in the same, degenerate setting as equation (4.1), was shown in [109].

The equation (4.1) is a special case of the system, where the viscous stress tensor depends on the density. In general such models have the form

$$\begin{aligned} \varrho_t + \operatorname{div}(\varrho u) &= 0, \\ \partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u) - \operatorname{div}(\mu(\varrho)\mathbb{D}u) - \nabla(\lambda(\varrho)\operatorname{div} u) + \nabla P &= 0. \end{aligned} \tag{4.3}$$

In two dimensions, equation (4.3) with $\mu = \varrho$, $\lambda = 0$ is used to describe shallow water flow. When $\mu(0) = 0$ the system is degenerate, in the sense that the stress tensor does not provide us the typical L^2 integrability of the velocity gradient. An important tool to deal with this problem is then the Bresch-Desjardins inequality, first established in [6]. It allows to show that when μ and λ satisfy the compatibility condition $\lambda(\varrho) = \varrho\mu'(\varrho) - \mu(\varrho)$, then testing the momentum equation by a certain function depending on ϱ, one can get the estimate on $\nabla\varphi(\varrho)$ in $L^\infty(0,T;L^2)$ for some φ depending on μ (in the case $\mu = \varrho, \lambda = 0$, the suitable test function is $\nabla \log \varrho$ and one gets the estimate on $\nabla\sqrt{\varrho}$). With that information at hand, the authors constructed weak solutions with the additional regularizing terms in the momentum equation in [7, 9]. Another interesting result, involving density-dependent viscosity, concerns the existence of weak solutions for quantum fluids, analysed for example in [58]. To construct weak solutions to the degenerate Navier-Stokes system (4.3) one needs another inequality, first used by Mellet and Vasseur in [75]. It provides the $L^\infty(0,T;L^1)$ estimate on the quantity $\varrho|u|^2\ln(1+|u|^2)$, which in consequence allows to derive compactness in L^2 of $\sqrt{\varrho}u$ (and in turn proves stability of solutions). Using this idea, Vasseur and Yu presented full, rigorous construction of global weak solutions to (4.3) with $\mu = \varrho, \lambda = 0$ in [106, 105]. Independently at the same time, using a different approximation scheme, the existence of weak solutions to (4.3) was also shown for $\mu \sim \varrho^\alpha$ in [64]. The case of more general viscosities was also covered in [13], where the authors showed the existence of renormalized solutions, following the definition from [61].

An important feature of the density-dependent viscosity case and the Bresch-Desjardins inequality is that it allows to derive a priori estimates for the density without the use of the pressure term. In the classical theory of weak solutions to Navier-Stokes equations with constant viscosities, developed by Lions [67] and Feireisl [48], the construction of solutions is possible when $P \sim \varrho^\gamma$ for $\gamma > 3/2$. Without the pressure term, or with too low value of γ, the density is not integrable enough to show the compactness of the approximating sequence. In the density-dependent viscosity case, by virtue of the Bresch-Desjardins inequality, we get the estimate on the gradient of the density, and then compactness follows straight from the Aubin-Lions lemma. The lack of pressure term is natural in the case of systems derived from the models of collective behaviour, since they describe the interactions of different nature than gases. The pressureless systems were obtained as a mean-field limit from the agent-based model for example in [51] in the presence of the nonlocal alignment forces. The Euler-Poisson system with quadratic confinement was also recently considered in a spherically symmetric multi-dimensional setting by Carrillo and Shu [22] and we refer to this paper for up to date overview of results on that system in the context of continuous collective behaviour models. In the context of our work, particularly interesting results were obtained even earlier in the one-dimensional setting [23] and [25], where similar form of the nonlocal kernel was considered. In [23] the authors analysed the asymptotics and critical thresholds for pressureless Euler system, whereas in [25] they showed that these solutions can be approximated by the solutions of the corresponding Navier-Stokes type system with degenerate viscosity. In higher dimensions without symmetry assumptions little is known about weak solutions to pressureless systems. In [54], a multidimensional version of result by Haspot and Zatorska [55] was proved, demonstrating that pressureless limit of (4.3) leads to the porous medium equation for "well prepared" data. However, according to our knowledge there are no corresponding results concerning the nonlocal systems.

4.1.1 The main result

We supplement problem (4.1) with the initial data

$$\varrho(0,x) = \varrho_0(x), \quad (\varrho u)(0,x) = m_0(x), \tag{4.4}$$

and we assume that

$$\varrho_0 \geq 0, \quad \sqrt{\varrho_0} \in H^1(\mathbb{R}^3). \tag{4.5}$$

Moreover, for F defined as

$$F(z) = \frac{1+z^2}{2}\ln(1+z^2), \tag{4.6}$$

we assume that

$$\int_{\mathbb{R}^3} \varrho_0 F(|u_0|)\,\mathrm{d}x + \iint_{\mathbb{R}^3\times\mathbb{R}^3} F(|x-y|)\varrho_0(x)\varrho_0(y)\,\mathrm{d}x\,\mathrm{d}y < \infty, \tag{4.7}$$

where we define $u_0 = \frac{m_0}{\varrho_0}$ on the set $\{x \in \mathbb{R}^3 : \varrho_0(x) > 0\}$. In particular, since $\frac{1}{2}z^2 \leq F(z)$ for large z, from (4.7) it follows that

$$\int_{\mathbb{R}^3} \frac{1}{2}\varrho_0|u_0|^2\,\mathrm{d}x + \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{1}{2}|x-y|^2\varrho_0(x)\varrho_0(y)\,\mathrm{d}x\,\mathrm{d}y < \infty.$$

Before we formulate the main result, let us precise what we mean by a weak solution in this case:

Definition 4.1. *We say that (ϱ, u) is a weak solution to (4.1) on $[0,T]\times\mathbb{R}^3$ with initial conditions (ϱ_0, m_0), if*

$$\begin{aligned} \varrho &\in L^\infty(0,T;L^1),\\ \sqrt{\varrho} &\in L^\infty(0,T;H^1),\\ \sqrt{\varrho}u &\in L^\infty(0,T;L^2),\\ \overline{\varrho\mathbb{D}u} &\in L^2(0,T;W^{-1,1}), \end{aligned}$$

$$\int_0^T\iint_{\mathbb{R}^3\times\mathbb{R}^3}|x-y|\varrho(t,x)\varrho(t,y)\,\mathrm{d}x\,\mathrm{d}y<\infty,$$

and for each $\varphi\in C_0^\infty([0,T)\times\mathbb{R}^3;\mathbb{R})$ and $\psi\in C_0^\infty([0,T)\times\mathbb{R}^3;\mathbb{R}^3)$ we have

$$-\int_0^T\int_{\mathbb{R}^3}\varrho\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_{\mathbb{R}^3}\sqrt{\varrho}\sqrt{\varrho}u\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t=\int_{\mathbb{R}^3}\varrho_0\varphi(0,\cdot)\,\mathrm{d}x$$

and

$$\begin{aligned} -\int_{\mathbb{R}^3}m_0\psi(0,\cdot)\,\mathrm{d}x-\int_0^T\int_{\mathbb{R}^3}\sqrt{\varrho}\sqrt{\varrho}u\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_{\mathbb{R}^3}(\sqrt{\varrho}u\otimes\sqrt{\varrho}u):\nabla\psi\,\mathrm{d}x\,\mathrm{d}t&\\ +\langle\overline{\varrho\mathbb{D}u},\nabla\psi\rangle+\int_0^T\int_{\mathbb{R}^3}\varrho\nabla(K*\varrho)\cdot\psi\,\mathrm{d}x&=0, \end{aligned}$$

where we define

$$\langle\overline{\varrho\mathbb{D}u},\nabla\psi\rangle:=-\int_0^T\int_{\mathbb{R}^3}\varrho u\cdot(\Delta\psi+\nabla\mathrm{div}\,\psi)\,\mathrm{d}x\,\mathrm{d}t-2\int_0^T\int_{\mathbb{R}^3}(\nabla\sqrt{\varrho}\otimes\sqrt{\varrho}u):\nabla\psi\,\mathrm{d}x\mathrm{d}t$$

Remark 4.2. Note that in the sense of Definition 4.1, the velocity itself is not defined on the set where $\varrho=0$. Because of that, we operate with the variable $\sqrt{\varrho}u$ instead, and u is defined only via $u(t,x)=\frac{(\sqrt{\varrho}u)(t,x)}{\sqrt{\varrho}(t,x)}$ for (t,x) such that $\varrho(t,x)\neq 0$. In particular the gradient $\mathbb{D}u$ is not well defined as well. Because of that, we denote the stress tensor by $\overline{\varrho\mathbb{D}u}$ instead, which is defined using the relation

$$\begin{aligned} \overline{\varrho\mathbb{D}u}&=\mathbb{D}(\varrho u)-\nabla\varrho\otimes u\\ &=\mathbb{D}(\varrho u)-2\nabla\sqrt{\varrho}\otimes\sqrt{\varrho}u. \end{aligned}$$

To avoid unnecessary complications of the notation, later on we will drop the bars and just write $\varrho\mathbb{D}u$, keeping in mind the above definition.

Under these assumptions, our main result states:

Theorem 4.3. *Let (ϱ_0, m_0) satisfy (4.5-4.7). Then there exists a global in time weak solution (ϱ, u) to (4.1), satisfying*

1. *the energy estimate*

$$\sup_{t\geq 0}\frac{1}{2}\int_{\mathbb{R}^3}\varrho|u|^2+\varrho(K*\varrho)\,\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}^3}\varrho|\mathbb{D}u|^2\,\mathrm{d}x\mathrm{d}t\leq\frac{1}{2}\int_{\mathbb{R}^3}\varrho_0|u_0|^2+\varrho_0(K*\varrho_0)\,\mathrm{d}x, \tag{4.8}$$

2. *the Bresch-Desjardins estimate*

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^3}|\nabla\sqrt{\varrho}|^2\,\mathrm{d}x+\int_0^T\int_{\mathbb{R}^3}\varrho|\nabla u-\nabla^T u|^2\,\mathrm{d}x\,\mathrm{d}t\leq C(T), \tag{4.9}$$

where $C(T) \to \infty$ as $T \to \infty$,

3. *the Meller-Vasseur estimate*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} \varrho F(|u|)\,\mathrm{d}x + \sup_{t\in[0,T]} \iint_{\mathbb{R}^3\times\mathbb{R}^3} F(|x-y|)\varrho(t,x)\varrho(t,y)\,\mathrm{d}x\mathrm{d}y \le C(T), \tag{4.10}$$

with $C(T) \to \infty$ as $T \to \infty$ as well.

Below, we explain the overall strategy of the proof and discuss the main difficulties. The starting point is to find a solution to a certain approximation of (4.1). In this construction, we follow the approach of Vasseur and Yu from [106]. It is a multi-level construction with many approximation parameters regularising the solutions. Additionally, we need to restrict the problem to the torus $\mathbb{T}_L^3 \sim [-L,L]^3$ and modify the interaction kernel to $K_L = K\phi_L$ for a suitable cut-off function ϕ_L. The final step of the construction is the expansion of the torus to the whole space and recovery of the solution to the original system (4.1) with (4.2). Similar approach to derive solutions on the whole space was proposed for the system (4.3) by Li and Xin in [64], and for quantum isothermal fluids by Carles, Carapatoso and Hillairet in [21]. Our approximate system reads as follows

$$\begin{aligned} &\partial_t\varrho + \operatorname{div}(\varrho u) = \varepsilon\Delta\varrho,\\ &\partial_t(\varrho u) + \operatorname{div}(\varrho u\otimes u) - \operatorname{div}(\varrho\mathbb{D}u) + \varrho\nabla(K_L * \varrho)\\ &= -r_0 u - r_1\varrho|u|^2 u + \kappa\varrho\nabla\left(\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\right) - \varepsilon\nabla\varrho\cdot\nabla u - \nu\Delta^2 u + \eta\nabla\varrho^{-6} + \delta\varrho\nabla\Delta^3\varrho. \end{aligned} \tag{4.11}$$

The outline of the paper follows the consecutive steps of the proof of Theorem 4.3, described below:

1. Construction of the solution to the approximated system (4.11) on $(0,T)\times\mathbb{T}_L^3$ via the Galerkin method and and the Schauder fixed point theorem. At this point, the artificial viscosity $\varepsilon\Delta\varrho$ in the continuity equation allows us to apply classical approach for the construction.

2. Derivation of the approximate version of the Bresch-Desjardins inequality (4.9). To this purpose, one needs to test the momentum equation by $\nabla\log\varrho$. The terms $\eta\nabla\varrho^{-6}$ and $\delta\varrho\nabla\Delta^3\varrho$ provide that the density is strictly positive and that $\nabla\log\varrho$ is sufficiently regular in space. On the other hand, the parameter ν allows to differentiate the continuity equation and to deduce that $\nabla\log\varrho$ is also sufficiently regular in time to be used as a test function.

3. Passage to the limit with ε,ν,η and δ. Having derived the estimate (4.9), the improved regularity of ϱ allows to pass to the limit with consecutive regularizing parameters. The proof of Theorem 4.3 up to this point is pretty standard, and is only sketched in Sections 4.3.5-4.3.6.

4. Derivation of the approximate Meller-Vasseur inequality (4.10), uniformly with the size of the torus. This is the key step of the proof. We employ the approximating procedure introduced in [105], however due to the presence of nonlocal terms, we need different arguments to close the estimates. The estimate is derived by renormalization of the momentum equation. The overall idea lays in using $F'(|u|)\frac{u}{|u|} = (1+\ln(1+|u|^2))u$ as a test function for the momentum equation. However, due to the lack of differentiability of u and the growth of F, this function does not belong to $L^2(0,T;W^{1,2})$ and thus is not an admissible test function. Because of that, we introduce suitable approximation of F, and in place of u we put $v = \phi_m^0(\varrho)\phi_k^\infty(\varrho)u$, where ϕ_m and ϕ_k cut off the density

at zero and infinity respectively. Then, passing with m and k to ∞, we derive the desired estimate in the limit. Bounding together the parameters κ and k, by deriving the estimate we simultaneously pass to the limit with κ as well.

The biggest challenge here is to deal with the attractive part of the kernel K, since on the whole space it is not integrable with any power. Because of that we are not able to follow the arguments from [105]. Instead, we apply the weak version of Gronwall's lemma and use generalized Young inequality for convex functions.

5. Passage to the limit with r_0 and r_1, contained in Section 4.5. This is the final limit passage on the torus. The main issue is that although the density-dependent viscous stress tensor provides extra regularity for the density (via the Bresch-Desjardins estimate), it gives no information on u itself on the set where $\varrho = 0$. Without the extra friction terms, we end up with very little regularity of the velocity. However, having the estimate (4.10), following the arguments from [75] we are able to show strong convergence of $\sqrt{\varrho}u$, which combined with compactness properties of the density allows to still perform the limit passage.

6. Expansion of the torus. In the previous steps we needed to restrict our domain to the torus $\mathbb{T}^3_L \sim [-L, L]^3$. The last part of the proof is to pass to the limit with $L \to \infty$ and in consequence to obtain the solutions on the whole space $\mathbb{R}^3$. The previously derived estimates are uniform in L and thus allow to extend our solution. During this limit passage we also lose the compactness of the nonlocal term. Nonetheless, the energy inequality provides the estimate on a double second moment

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x-y|^2 \varrho(t,x)\varrho(t,y)\, \mathrm{d}x\mathrm{d}y,$$

which allows to control the behaviour of the density far from the origin, and in consequence pass to the limit in the nonlocal term as well.

For the reader's convenience, Table 4.1 contains a list with all parameters, together with its short descriptions.

4.2 Convergence lemmas

Before we start the proof of Theorem 4.3, let us present useful convergence lemmas, which will be used repeatedly in Sections 4.3 and 4.4.

Lemma 4.4. *Assume the sequence of functions (ϱ_n, u_n) defined on $[0,T]\times\mathbb{T}^3_L$ satisfies*

$$\|\partial_t \varrho_n\|_{L^2(0,T;L^{6/5})} + \|\partial_t(\varrho_n u_n)\|_{L^2(0,T;H^{-m})} \le C$$

for some $m \ge 1$,

$$\sup_{t\in[0,T]} \int_{\mathbb{T}^3_L} \varrho_n |u_n|^2\,\mathrm{d}x + \int_0^T\int_{\mathbb{T}^3_L} \varrho_n |\nabla u_n|^2\,\mathrm{d}x\mathrm{d}t \\ + \int_0^T\int_{\mathbb{T}^3_L} |u_n|^2\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{T}^3_L} \varrho_n|u_n|^4\,\mathrm{d}x\mathrm{d}t \le C, \quad (4.12)$$

and

$$\|\sqrt{\varrho_n}\|_{L^\infty(0,T;H^1)} \le C \quad (4.13)$$

ε	Necessary to construct the approximate solution by the Faedo-Galerkin approximation
ν	Provides that $\partial_t \nabla \log \varrho \in L^2(0,T;L^2)$, which is needed for derivation of the Bresch-Desjardins estimate
η, δ	Provide that $\frac{1}{\varrho}$ is bounded and that $\nabla \log \varrho \in L^2(0,T;H^2)$, i.e. is a suitable test function (together with time regularity)
κ	Provides that $\sqrt{\varrho} \in L^2(0,T;H^2)$ and $\nabla \varrho^{1/4} \in L^4(0,T;L^4)$, which is necessary for successful renormalization of the momentum equation
r_0, r_1	Provide improved integrability of u; combined with estimates coming from κ enable to renormalize the momentum equation
m, k	The cutoff function $\phi_m^0(\varrho)$ cuts the area when $\varrho < \frac{1}{m}$ and $\phi_k^\infty(\varrho)$ when $\varrho > k$. They appear in the proof of the Mellet-Vasseur inequality and are the additional levels of approximation needed for renormalization of the momentum equation
L	indicates the size of the torus. By taking $L \to \infty$, we obtain the solution on the whole space

TABLE 4.1: "Cheat sheet" describing the parameters appearing in the paper

uniformly in n. Then up to a subsequence

$$\begin{aligned} \sqrt{\varrho_n} &\rightharpoonup \sqrt{\varrho} \quad \text{in} \quad L^2(0,T;H^1), \\ u_n &\rightharpoonup u \quad \text{in} \quad L^2(0,T;L^2) \end{aligned}$$

and

$$\begin{aligned} \varrho_n &\to \varrho \quad \text{in} \quad C(0,T;L^{3/2}), \\ \varrho_n u_n &\to \varrho u \quad \text{in} \quad L^2(0,T;L^{3/2}). \end{aligned}$$

Moreover, if additionally $\partial_t \sqrt{\varrho_n}$ is bounded in $L^2(0,T;L^2)$ and

$$\int_0^T \int_{\mathbb{T}_L^3} \varrho_n |\nabla^2 \log \varrho_n|^2 \, \mathrm{d}x\mathrm{d}t \le C,$$

then

$$\sqrt{\varrho_n} \rightharpoonup \sqrt{\varrho} \quad \text{in} \quad L^2(0,T;H^2)$$

and

$$\sqrt{\varrho_n} \to \sqrt{\varrho} \quad \text{in} \quad L^2(0,T;H^1).$$

Proof. The weak convergence of $\sqrt{\varrho_n}$ and u_n follows straight from the Banach-Alaoglu Theorem. To prove strong convergence, we use the Aubin-Lions lemma. Note that

$$\nabla \varrho_n = 2\sqrt{\varrho_n}\, \nabla \sqrt{\varrho_n}$$

and therefore

$$\|\nabla \varrho_n\|_{L^\infty(0,T;L^{3/2})} \le \|\sqrt{\varrho_n}\|_{L^\infty(0,T;L^6)} \|\nabla \sqrt{\varrho_n}\|_{L^\infty(0,T;L^2)} \le C$$

by (4.13) and the Sobolev embedding. Therefore from the Aubin-Lions-Simon lemma (see e.g. [94])

$$\varrho_n \to \varrho \quad \text{in} \quad C(0,T;L^{3/2}).$$

Since
$$\begin{aligned}\nabla(\varrho_n u_n) &= \nabla\varrho_n \otimes u_n + \varrho_n \nabla u_n\\ &= 2\varrho_n^{1/4}\nabla\sqrt{\varrho_n}\otimes \varrho_n^{1/4}u_n + \sqrt{\varrho_n}\sqrt{\varrho_n}\nabla u_n,\end{aligned}$$
we have
$$\begin{aligned}\|\nabla(\varrho_n u_n)\|_{L^2(0,T;L^{6/5})} \leq & C\|\varrho_n\|^{1/4}_{L^\infty(0,T;L^3)}\|\nabla\sqrt{\varrho_n}\|_{L^\infty(0,T;L^2)}\|\varrho_n^{1/4}u_n\|_{L^4(0,T;L^4)}\\ &+ C\|\sqrt{\varrho_n}\|_{L^\infty(0,T;L^6)}\|\sqrt{\varrho_n}\nabla u_n\|_{L^2(0,T;L^2)} \leq C.\end{aligned}$$
Moreover,
$$\|\varrho_n u_n\|_{L^\infty(0,T;L^{3/2})} \leq \|\sqrt{\varrho_n}\|_{L^\infty(0,T;L^6)}\|\sqrt{\varrho_n}u_n\|_{L^\infty(0,T;L^2)} \leq C$$
and thus again from the Aubin-Lions lemma
$$\varrho_n u_n \to \varrho u \quad \text{in} \quad L^2(0,T;L^{3/2}).$$

For the second part of the lemma, the estimate on $\varrho_n|\nabla^2\log\varrho_n|^2$ in particular yields
$$\|\sqrt{\varrho_n}\|_{L^2(0,T;H^2)} + \|\nabla\varrho_n^{1/4}\|_{L^4(0,T;L^4)} \leq C.$$
This is the consequence of the following Proposition, proved in [58]:

Proposition 4.5. *For smooth ϱ, we have*
$$\int_{\mathbb{T}^3_L}\varrho|\nabla^2\log\varrho|^2\mathrm{d}x \geq \frac{1}{7}\int_{\mathbb{T}^3_L}|\nabla^2\sqrt{\varrho}|^2\mathrm{d}x$$
and
$$\int_{\mathbb{T}^3_L}\varrho|\nabla^2\log\varrho|^2\mathrm{d}x \geq \frac{1}{8}\int_{\mathbb{T}^3_L}|\nabla\varrho^{1/4}|^4\mathrm{d}x.$$

Thus, having the $L^2(0,T;H^2)$ bound on $\sqrt{\varrho_n}$, again by Aubin-Lions lemma
$$\sqrt{\varrho_n} \to \sqrt{\varrho} \quad \text{in} \quad L^2(0,T;H^1).$$
□

Lemma 4.6 (Limit in the nonlinear damping). *If $\sqrt{\varrho_n}\to\sqrt{\varrho}$ in $L^2(0,T;H^1)$, $\varrho_n u_n \to \varrho u$ in $L^2(0,T;L^{3/2})$ and $u_n \rightharpoonup u$ in $L^2(0,T;L^2)$, and additionally*
$$\int_0^T\int_{\mathbb{T}^3_L}\varrho_n|u_n|^4\mathrm{d}x\mathrm{d}t \leq C$$
uniformly in n, then
$$\varrho_n|u_n|^2u_n \to \varrho|u|^2u \quad \textit{in} \quad L^1(0,T;L^1).$$

Proof. First, note that $\varrho_n^{1/4}u_n \rightharpoonup \varrho^{1/4}u$ in $L^4(0,T;L^4)$. Therefore, from the lower semicontinuity of the norm and Fatou's lemma,
$$\int_0^T\int_{\mathbb{T}^3_L}\varrho|u|^4\mathrm{d}x\mathrm{d}t \leq \int_0^T\int_\Omega \liminf_{n\to\infty}\varrho_n|u_n|^4\mathrm{d}x\mathrm{d}t \leq \liminf_{n\to\infty}\int_0^T\int_{\mathbb{T}^3_L}\varrho_n|u_n|^4\mathrm{d}x\mathrm{d}t \leq C.$$
From the strong convergence of $\sqrt{\varrho_n}$ and $\varrho_n u_n$, we know that
$$\varrho_n(t,x)\to\varrho(t,x) \quad \text{and} \quad (\varrho_n u_n)(t,x)\to(\varrho u)(t,x) \quad \text{a.e.},$$

up to a subsequence. Therefore, for almost every (t,x) such that $\varrho_n(t,x) \not\to 0$, we have

$$u_n = \frac{\varrho_n u_n}{\varrho_n} \to u.$$

For the points where $\varrho_n \to 0$, we write

$$\varrho_n |T_M(u_n)|^3 \le M^3 \varrho_n \to 0 = \varrho |T_M(u)|^3,$$

where T_M is the truncation operator defined as

$$T_M(u) = \begin{cases} u, & |u| \le M, \\ M\dfrac{u}{|u|}, & |u| > M, \end{cases} \tag{4.14}$$

Therefore from the dominated convergence theorem,

$$\varrho_n |T_M(u_n)|^2 T_M(u_n) \to \varrho |T_M(u)|^2 T_M(u) \quad \text{in} \quad L^1(0,T;L^1)$$

for any fixed $M > 0$. Moreover, we have

$$\begin{aligned}
\int_0^T \int_{\mathbb{T}_L^3} \left| \varrho_n |u_n|^2 u_n - \varrho |u|^2 u \right| \mathrm{d}x\mathrm{d}t \le & \int_0^T \int_{\mathbb{T}_L^3} \left| \varrho_n |T_M(u_n)|^2 T_M(u_n) - \varrho |T_M(u)|^2 T_M(u) \right| \mathrm{d}x\mathrm{d}t \\
& + 2 \int_0^T \int_{\mathbb{T}_L^3} \varrho_n |u_n|^3 \mathbb{1}_{|u_n|>M} \mathrm{d}x\mathrm{d}t \\
& + 2 \int_0^T \int_{\mathbb{T}_L^3} \varrho |u|^3 \mathbb{1}_{|u|>M} \mathrm{d}x\mathrm{d}t \\
\le & \int_0^T \int_{\mathbb{T}_L^3} \left| \varrho_n |u_n|^2 u_n \mathbb{1}_{|u_n|\le M} - \varrho |u|^2 u \mathbb{1}_{|u|\le M} \right| \mathrm{d}x\mathrm{d}t \\
& + \frac{2}{M} \int_0^T \int_{\mathbb{T}_L^3} \varrho_n |u_n|^4 \mathrm{d}x\mathrm{d}t + \frac{2}{M} \int_0^T \int_{\mathbb{T}_L^3} \varrho |u|^4 \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Therefore

$$\limsup_{n\to\infty} \int_0^T \int_{\mathbb{T}_L^3} \left| \varrho_n |u_n|^2 u_n - \varrho |u|^2 u \right| \mathrm{d}x\mathrm{d}t \le \frac{C}{M}.$$

Letting $M \to \infty$, we obtain the desired convergence. □

4.3 Fundamental level of approximation

The aim of this section is first to construct the solution (ϱ_L, u_L) to system (4.11) on the torus $\mathbb{T}_L^3 \sim [-L,L]^3$, and then pass to the limit with $\varepsilon, \nu, \delta, \eta \to 0$. The construction is done by means of the Galerkin approximation and the fixed point theorem. To perform the limit passages, we use the auxillary lemmas from Section 4.2.

4.3.1 Truncation to periodic domain

We begin by modification of kernel K in a way that allows $K * \varrho$ to be well-defined on the torus.

Let $\phi_L \geq 0$ be a radial, decreasing cut-off function such that supp $\phi_L \subset B(0, L)$, $\phi_L(x) \equiv 1$ for $|x| < \frac{L}{2}$ and

$$|\nabla \phi_L| \leq \frac{C}{L}, \quad |\Delta \phi_L| \leq \frac{C}{L^2}. \tag{4.15}$$

Then we simply put

$$K_L = K\phi_L.$$

In a similar way we prepare the initial conditions. We put

$$\sqrt{\varrho_{0,L}} = \sqrt{\varrho_0}\phi_L,$$

where ϕ_L is defined above, then periodize. In consequence we obtain the initial condition on the torus $\mathbb{T}^3_L$, satisfying:

Lemma 4.7. *The function* $\sqrt{\varrho_{0,L}}$ *satisfies the following properties:*

$$\|\nabla\sqrt{\varrho_{0,L}}\|_{L^2(\mathbb{T}^3_L)} \leq \|\nabla\sqrt{\varrho_0}\|_{L^2(\mathbb{R}^3)} + \frac{C}{L}\|\varrho_0\|^{1/2}_{L^1(\mathbb{R}^3)},$$

$$\iint_{\mathbb{T}^3_L\times\mathbb{T}^3_L} \varrho_{0,L}(x)\varrho_{0,L}(y)K_L(x-y)\,\mathrm{d}x\mathrm{d}y \leq \iint_{\mathbb{R}^3\times\mathbb{R}^3} \varrho_0(x)\varrho_0(y)K(x-y)\,\mathrm{d}x\mathrm{d}y$$

and

$$\varrho_{0,L} \to \varrho_0 \quad \text{in} \quad L^1(\mathbb{R}^3).$$

Proof. The proof follows straight from the definition of $\varrho_{0,L}$. First, we have

$$\nabla\sqrt{\varrho_{0,L}} = \nabla\sqrt{\varrho_0}\phi_L + \sqrt{\varrho_0}\nabla\phi_L$$

and thus

$$\|\nabla\sqrt{\varrho_{0,L}}\|_{L^2(\mathbb{T}^3_L)} \leq \|\nabla\sqrt{\varrho_0}\|_{L^2(\mathbb{R}^3)} + \frac{C}{L}\|\varrho_0\|^{1/2}_{L^1(\mathbb{R}^3)}.$$

The next estimate follows immediately, since

$$\iint_{\mathbb{T}^3_L\times\mathbb{T}^3_L} \varrho_0(x)\varrho_0(y)K(x-y)\phi_L^2(x)\phi_L^2(y)\phi_L(x-y)\,\mathrm{d}x\mathrm{d}y \\ \leq \iint_{\mathbb{R}^3\times\mathbb{R}^3} \varrho_0(x)\varrho_0(y)K(x-y)\,\mathrm{d}x\mathrm{d}y$$

by integrating over a larger domain and estimating ϕ_L by 1.

The convergence in L^1 follows immediately from the dominated convergence theorem, since $\phi_L \to 1$ pointwise. □

With the definition of $\varrho_{0,L}$, we can also define properly the initial conditions on u. Defining

$$u_{0,L}(x) = 0 \quad \text{for} \quad \varrho_{0,L}(x) = 0; \quad u_{0,L}(x) = u_0(x) \quad \text{otherwise},$$

we can periodize it in the same way as $\varrho_{0,L}$, and moreover

$$\int_{\mathbb{T}^3_L} \varrho_{0,L}|u_{0,L}|^2\,\mathrm{d}x \leq \int_{\mathbb{R}^3} \varrho_0|u_0|^2\,\mathrm{d}x.$$

The full approximated system on the torus is then given by (4.11) and it is supplemented with the initial condition:

$$u_{|t=0} = u_{0,L}, \quad \varrho_{|t=0} = \tilde{\varrho}_0 := \varrho_{0,L} * \xi_{\bar{\delta}} + \frac{1}{m_1}, \tag{4.16}$$

where $m_1 > 0$, $\xi_{\bar{\delta}}$ is the standard mollifier on the torus, and we choose $\bar{\delta}$ depending on δ such that $\delta\|\nabla\Delta\tilde{\varrho}_0\|^2_{L^2(\mathbb{T}^3_L)} \to 0$ as $\delta \to 0$.

4.3.2 The Galerkin method

We solve the system (4.11) using the Galerkin approximation. We present here only a sketch of the construction, and for the details we refer to the paper of Vasseur and Yu [106] and the book of Feireisl [44]. Let $(e_i)_{i\in\mathbb{N}}$ be a suitable basis of $H^2(\mathbb{T}^3_L)$ and set $X_N := \{e_1, \dots, e_N\}$. We put

$$u_N(t,x) = \sum_{i=1}^{N} \lambda_i(t) e_i(x).$$

Moreover, let $S\colon C(0,T;X_N) \to C(0,T;C^k)$ be such that $\varrho = S(u)$ solves

$$\varrho_t + \operatorname{div}(\varrho u) - \varepsilon\Delta\varrho = 0, \quad \varrho_{|t=0} = \varrho_0.$$

Then, we construct the solution by applying the Schauder fixed point theorem for the operator

$$\mathfrak{M}^{-1}[S(u_N)](t)\left(\mathfrak{M}[\varrho_0](u_0) + \int_0^T \mathfrak{N}(S(u_N), u_N)(s)\mathrm{d}s\right),$$

where $\mathfrak{M}[\varrho]\colon X_N \to X_N^*$ is given by

$$\langle \mathfrak{M}[\varrho]u, w\rangle = \int_{\mathbb{T}^3_L} \varrho u \cdot w \, \mathrm{d}x$$

and

$$\begin{aligned}\mathfrak{N}(\varrho, u) = &-\operatorname{div}(\varrho u \otimes u) + \operatorname{div}(\varrho \mathbb{D}u) - \varrho\nabla(K_L * \varrho)\\ &- r_0 u - r_1\varrho|u|^2 u + \kappa\varrho\nabla\left(\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\right) - \varepsilon\nabla\varrho\cdot\nabla u - \nu\Delta^2 u + \eta\nabla\varrho^{-6} + \delta\varrho\nabla\Delta^3\varrho.\end{aligned}$$

As a result, we obtain a smooth solution (ϱ_N, u_N) on some interval $[0, T_*]$, corresponding to initial conditions $\tilde{\varrho}_0$ and

$$u_{0,N} = \sum_{i=1}^{N} \langle u_0, e_i\rangle.$$

4.3.3 Energy estimates

Testing the momentum equation of (4.11) by u_N and using the approximate continuity equation, we get that the following equality is satisfied uniformly in N

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(\varrho_N,u_N)+\nu\int_{\mathbb{T}^3_L}|\Delta u_N|^2\mathrm{d}x+\int_{\mathbb{T}^3_L}\varrho_N|\mathbb{D}u_N|^2\mathrm{d}x+\varepsilon\delta\int_{\mathbb{T}^3_L}|\Delta^2\varrho_N|^2\mathrm{d}x\\+\frac{2}{3}\varepsilon\eta\int_{\mathbb{T}^3_L}|\nabla\varrho_N^{-3}|^2\mathrm{d}x+r_0\int_{\mathbb{T}^3_L}|u_N|^2\mathrm{d}x+r_1\int_{\mathbb{T}^3_L}\varrho_N|u_N|^4\mathrm{d}x\\+\kappa\varepsilon\int_{\mathbb{T}^3_L}\varrho_N|\nabla^2\log\varrho_N|^2\mathrm{d}x+\varepsilon\int_{\mathbb{T}^3_L}\nabla(K_L*\varrho_N)\cdot\nabla\varrho_N\mathrm{d}x=0,\end{aligned}\tag{4.17}$$

where

$$E(\varrho,u)=\int_{\mathbb{T}^3_L}\left(\frac{1}{2}\varrho|u|^2+\frac{1}{2}\varrho(K_L*\varrho)+\frac{\eta}{7}\varrho^{-6}+\frac{\kappa}{2}|\nabla\sqrt{\varrho}|^2\mathrm{d}x+\frac{\delta}{2}|\nabla\Delta\varrho|^2\right)\mathrm{d}x.$$

To deduce useful bounds from this equality, we extract certain estimates from the nonlocal term $\varepsilon\int_{\mathbb{T}^3_L}\nabla(K_L*\varrho_N)\cdot\nabla\varrho_N\mathrm{d}x$, which are a consequence of the following lemma:

Lemma 4.8. *For a sufficiently smooth ϱ, we have*

$$\int_{\mathbb{T}^3_L}\nabla(K_L*\varrho)\cdot\nabla\varrho\,\mathrm{d}x\geq -C\|\varrho\|^2_{L^1(\mathbb{T}^3_L)}\tag{4.18}$$

for some $C>0$ not depending on L.

Proof. We will consider two cases, depending on α:

1. If $\alpha\leq 1$, note that

$$\int_{\mathbb{T}^3_L}\nabla(K_L*\varrho)\cdot\nabla\varrho\,\mathrm{d}x=-\iint_{\mathbb{T}^3_L\times\mathbb{T}^3_L}\varrho(x)\varrho(y)\Delta(K\phi_L)(x-y)\mathrm{d}x\mathrm{d}y.$$

We have

$$\Delta(K\phi_L)=\Delta K\phi_L+2\nabla K\cdot\nabla\phi_L+K\Delta\phi_L$$

Further note that $\Delta\left(\frac{1}{2}|x|^2\right)=3$, and

$$\Delta\left(\frac{1}{|x|^\alpha}\right)=-\frac{\alpha(1-\alpha)}{|x|^{\alpha+2}},\quad\text{for}\quad\alpha<1\quad\text{and}\quad\Delta\left(\frac{1}{|x|}\right)=-4\pi\delta_0\quad\text{for}\quad\alpha=1.$$

Putting it all together, for $\alpha<1$ we get

$$
\begin{aligned}
&\int_{\mathbb{T}_L^3} \nabla(K_L * \varrho)\cdot\nabla\varrho\,\mathrm{d}x \\
&\qquad=\alpha(1-\alpha)\iint_{\mathbb{T}_L^3\times\mathbb{T}_L^3}\varrho(x)\varrho(y)\frac{\phi_L(x-y)}{|x-y|^{\alpha+2}}\,\mathrm{d}x\mathrm{d}y \\
&\qquad\quad-3\iint_{\mathbb{T}_L^3\times\mathbb{T}_L^3}\varrho(x)\varrho(y)\phi_L(x-y)\mathrm{d}x\mathrm{d}y \\
&\qquad\quad-2\iint_{T_L^3\times\mathbb{T}_L^3}\varrho(x)\varrho(y)\left[-\alpha\frac{x-y}{|x-y|^{\alpha+2}}+x-y\right]\nabla\phi_L(x-y)\mathrm{d}x\mathrm{d}y \\
&\qquad\quad-\iint_{\mathbb{T}_L^3\times\mathbb{T}_L^3}\varrho(x)\varrho(y)\left[\frac{1}{|x-y|^{\alpha}}+\frac{|x-y|^2}{2}\right]\Delta\phi_L(x-y)\mathrm{d}x\mathrm{d}y,
\end{aligned}
\tag{4.19}
$$

whereas if $\alpha=1$ the first term gets replaced by $4\pi\int_{\mathbb{T}_L^3}\varrho^2\,\mathrm{d}x$. From the assumptions (4.15) on ϕ_L, we have $|\phi_L|\le 1$,

$$\left|-\alpha\frac{x-y}{|x-y|^{\alpha+2}}+x-y\right||\nabla\phi_L(x-y)|\le C\mathbb{1}_{\{\frac{L}{2}<|x-y|<L\}}\left(\frac{1}{L^{\alpha+2}}+1\right)\le C$$

and

$$\left(\frac{1}{|x-y|^{\alpha}}+\frac{|x-y|^2}{2}\right)|\Delta\phi_L(x-y)|\le C\mathbb{1}_{\{\frac{L}{2}<|x-y|<L\}}\left(\frac{1}{L^{\alpha+2}}+1\right)\le C.$$

Applying these estimates to the last three terms in (4.19), we derive (4.18).

2. In the case $\alpha>1$, we use the fact that $\mathcal{F}(\frac{\phi_L}{|x|^{\alpha}})$ is positive, where by $\mathcal{F}$ we denote the Fourier transform of f on $\mathbb{R}^3$ or $\mathbb{T}_L^3$ respectively, i. e.

$$\mathcal{F}(f)(\xi)=\hat f(\xi)=\int_{\mathbb{R}^3}e^{-2\pi i\xi\cdot x}f(x)\,\mathrm{d}x,\ \xi\in\mathbb{R}^3\quad\text{or}\quad \hat f(k)=\int_{\mathbb{T}_L^3}e^{-2\pi k\cdot x}f(x)\,\mathrm{d}x,\ k\in\mathbb{Z}^3.$$

We have the following proposition:

Proposition 4.9. *Let $F\in L^p_{loc}(\mathbb{R}^3)$ be positive, such that $F(x)=f(|x|)$ with $rf(r)$ decreasing for $r>0$ and $\lim_{r\to\infty}rf(r)=0$. Then $\hat F$ is positive.*

Proof. Using spherical coordinates, we get

$$\hat F(\xi)=\int_0^\infty\int_0^{2\pi}\int_0^{\pi}e^{-ir|\xi|\cos\theta}r^2f(r)\sin\theta\,\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}r=\frac{4\pi}{|\xi|}\int_0^\infty rf(r)\sin(r|\xi|)\,\mathrm{d}r,$$

and the integral $\int_0^\infty rf(r)\sin(r|\xi|)\,\mathrm{d}r$ is convergent and positive from the assumptions on $rf(r)$. □

The positivity of Fourier transform allows us in turn to show the positivity of the integral operator.

Proposition 4.10. *If K is a radially symmetric kernel with support in $[-L,L]^d$ and positive Fourier transform, then for any sufficiently regular function with period $2L$ we have*

$$\int_{\mathbb{T}_L^3}f(x)\int_{\mathbb{R}^3}f(y)K(x-y)\,\mathrm{d}x\mathrm{d}y\ge 0.$$

Proof. The assertion follows straight from the identity

$$\int_{\mathbb{T}^3_L} f(x)g(x)\,\mathrm{d}x = (f(-\cdot)*g)(0) = \sum_{k\in\mathbb{Z}^3} \mathcal{F}(f(-\cdot)*g)(k) = \sum_{k\in\mathbb{Z}^3} \hat{f}(-k)\hat{g}(k).$$

In our case, it translates to

$$\begin{aligned}\int_{\mathbb{T}^3_L} f(x)\int_{\mathbb{R}^3} f(y)K(x-y)\,\mathrm{d}y\mathrm{d}x &= \sum_{k\in\mathbb{Z}^d} \hat{f}(-k)\mathcal{F}(K*f)(k)\\ &= \sum_{k\in\mathbb{Z}^d} \hat{K}(k)\hat{f}(-k)\hat{f}(k) = \sum_{k\in\mathbb{Z}^d} \hat{K}(k)|\hat{f}(k)|^2 \geq 0.\end{aligned}$$

□

By virtue of Proposition 4.9 and the assumptions on ϕ_L, the kernel $\frac{\phi_L(x)}{|x|^\alpha}$ has positive Fourier transform. Then in particular

$$\int_{\mathbb{T}^3_L} \nabla\varrho\cdot\nabla\left(\frac{\phi_L(\cdot)}{|\cdot|^\alpha}*\varrho\right)\,\mathrm{d}x = \int_{\mathbb{T}^3_L}\nabla\varrho(x)\int_{\mathbb{R}^3}\nabla\varrho(y)\frac{\phi_L(x-y)}{|x-y|^\alpha}\,\mathrm{d}y\,\mathrm{d}x \geq 0.$$

Dealing wht the quadratic part of K_L in the same way as in the case $\alpha\leq 1$, we end the proof of Lemma 4.8.

□

Thanks to Lemma 4.8, we get from (4.17) the following energy estimate:

$$\begin{aligned}\sup_{t\in[0,T]} E(\varrho_N,u_N) &+ \nu\int_0^T\int_{\mathbb{T}^3_L}|\Delta u_N|^2\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{T}^3_L}\varrho_N|\mathbb{D}u_N|^2\mathrm{d}x\mathrm{d}t\\ &+ \varepsilon\delta\int_0^T\int_{\mathbb{T}^3_L}|\Delta^2\varrho_N|^2\mathrm{d}x\mathrm{d}t + \frac{2}{3}\varepsilon\eta\int_0^T\int_{\mathbb{T}^3_L}|\nabla\varrho_N^{-3}|^2\mathrm{d}x\\ &+ r_0\int_0^T\int_{\mathbb{T}^3_L}|u_N|^2\mathrm{d}x\mathrm{d}t + r_1\int_0^T\int_{\mathbb{T}^3_L}\varrho|u_N|^4\mathrm{d}x\mathrm{d}t\\ &+ \kappa\varepsilon\int_0^T\int_{\mathbb{T}^3_L}\varrho_N|\nabla^2\log\varrho_N|^2\mathrm{d}x\mathrm{d}t\\ &\leq E(\varrho_0,u_0) + C\varepsilon T\|\varrho_0\|^2_{L^1(\mathbb{T}^3_L)}.\end{aligned} \tag{4.20}$$

As the estimates in (4.20) are satisfied for any $T<\infty$, we can extend the solution to the whole interval $[0,T]$ for any $T<\infty$. From (4.20) we also extract the estimates to pass to the limit with $N\to\infty$. Note that it in particular provides us the estimates needed in Lemmas 4.4 and 4.6, where for the time regularity we get the bound for $\partial_t(\varrho_N u_N)$ in $L^2(0,T;H^{-3})$ straight from the momentum equation. For the estimate on $\partial_t\varrho_N$ we further have

$$\begin{aligned}\partial_t\varrho_N &= -\varrho_N\mathrm{div}\,u_N - \nabla\varrho_N\cdot u_N + \varepsilon\Delta\varrho_N\\ &= -\sqrt{\varrho_N}\sqrt{\varrho_N}\mathrm{div}\,u_N - 4\nabla\varrho_N^{1/4}\cdot\varrho_N^{1/4}u_N\cdot\sqrt{\varrho_N} + \varepsilon\Delta\varrho_N,\end{aligned}$$

and therefore

$$\begin{aligned}\|\partial_t \varrho_N\|_{L^2(0,T;L^2)} \leq & \|\sqrt{\varrho_N}\|_{L^\infty(0,T;L^\infty)}\|\sqrt{\varrho_N}\nabla u_N\|_{L^2(0,T;L^2)} \\ & + 4\|\nabla \varrho^{1/4}\|_{L^4(0,T;L^4)}\|\varrho_N^{1/4} u_N\|_{L^4(0,T;L^4)}\|\sqrt{\varrho_N}\|_{L^\infty(0,T;L^\infty)} \\ & + \varepsilon\|\Delta \varrho_N\|_{L^2(0,T;L^2)} \\ \leq & C(\delta, \kappa\varepsilon, r_1).\end{aligned}$$

In consequence from Lemmmas 4.4 and 4.6 we have the following:

$$\begin{aligned}\varrho_N \rightharpoonup \varrho \quad & \text{in} \quad L^2(0,T;H^1), \\ \sqrt{\varrho_N} \rightharpoonup \sqrt{\varrho} \quad & \text{in} \quad L^2(0,T;H^2), \\ u_N \rightharpoonup u \quad & \text{in} \quad L^2(0,T;L^2)\end{aligned}$$

and

$$\begin{aligned}\varrho_N \to \varrho \quad & \text{in} \quad C(0,T;L^{3/2}), \\ \sqrt{\varrho_N} \to \sqrt{\varrho} \quad & \text{in} \quad L^2(0,T;H^1), \\ \varrho_N u_N \to \varrho u \quad & \text{in} \quad L^2(0,T;L^{3/2}), \\ \varrho_N |u_N|^2 u_N \to \varrho |u|^2 u \quad & \text{in} \quad L^1((0,T)\times \mathbb{T}_L^3).\end{aligned}$$

Moreover, the estimates on $\nabla\Delta\varrho_N$ and $\Delta^2\varrho_N$, together with the time regularity, provide that

$$\varrho_N \to \varrho \quad \text{in} \quad L^2(0,T;H^3) \quad \text{and} \quad \varrho_N \rightharpoonup \varrho \quad \text{in} \quad L^2(0,T;H^4).$$

To pass to the limit in the term $\eta\nabla\varrho_N^{-6}$, we use the following version of the Sobolev inequality:

Lemma 4.11. *For $\varrho \in H^3(\mathbb{T}_L^3)$, $\varrho^{-1} \in L^6(\mathbb{T}_L^3)$, it holds*

$$\|\varrho^{-1}\|_{L^\infty(\mathbb{T}_L^3)} \leq C(1+\|\varrho\|_{H^3(\mathbb{T}_L^3)})^2(1+\|\varrho^{-1}\|_{L^6(\mathbb{T}_L^3)})^3. \tag{4.21}$$

Proof. We have

$$\nabla^2\varrho^{-1} = -\frac{1}{\varrho^2}\nabla^2\varrho + \frac{2}{\varrho^3}\nabla\varrho\otimes\nabla\varrho.$$

Therefore

$$\begin{aligned}\|\nabla^2\varrho^{-1}\|_{L^2(\mathbb{T}_L^3)} &\leq \|\varrho^{-2}\|_{L^3(\mathbb{T}_L^3)}\|\nabla^2\varrho\|_{L^6(\mathbb{T}_L^3)} + 2\|\nabla\varrho\|^2_{L^\infty(\mathbb{T}_L^3)}\|\varrho^{-3}\|_{L^2(\mathbb{T}_L^3)} \\ &\leq C\|\varrho^{-1}\|^2_{L^6(\mathbb{T}_L^3)}\|\varrho\|_{H^3(\mathbb{T}_L^3)} + C\|\varrho\|^2_{H^3(\mathbb{T}_L^3)}\|\varrho^{-1}\|^3_{L^6(\mathbb{T}_L^3)} \\ &\leq C(1+\|\varrho\|_{H^3(\mathbb{T}_L^3)})^2(1+\|\varrho^{-1}\|_{L^6(\mathbb{T}_L^3)})^3,\end{aligned}$$

which ends the proof by Sobolev embedding. □

From (4.21) and (4.20), we get that

$$\varrho_N \geq C(\delta,\eta) > 0.$$

Since ϱ_N convergences strongly in $C(0,T;L^{3/2})$, it convergences almost everywhere up to a subsequence. Therefore $\varrho_N^{-6} \to \varrho^{-6}$ a. e. as well. Moreover, inequality (4.20) yields the estimate on ϱ_N^{-6} in $L^\infty(0,T;L^1)$ and $L^1(0,T;L^3)$, hence by interpolation we have

$$\|\varrho_N^{-6}\|_{L^{5/3}(0,T;L^{5/3})} \leq C.$$

In consequence,

$$\varrho_N^{-6} \to \varrho^{-6} \quad \text{in} \quad L^1(0,T;L^1).$$

The limit passage in the remaining terms follows immediately from the weak and strong convergences of ϱ_N and u_N in spaces. By the weak lower semicontinuity of convex functions, the limit also satisfies the energy estimate

$$\begin{aligned}\sup_{t\in[0,T]} E(\varrho,u)+&\nu\int_0^T\int_{\mathbb{T}_L^3}|\Delta u|^2\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{T}_L^3}\varrho|\mathbb{D}u|^2\mathrm{d}x\mathrm{d}t+\varepsilon\delta\int_0^T\int_{\mathbb{T}_L^3}|\Delta^2\varrho|^2\mathrm{d}x\mathrm{d}t\\ &+\varepsilon\eta\int_0^T\int_{\mathbb{T}_L^3}|\nabla\varrho^{-3}|^2\mathrm{d}x+r_0\int_0^T\int_{\mathbb{T}_L^3}|u|^2\mathrm{d}x\mathrm{d}t+r_1\int_0^T\int_{\mathbb{T}_L^3}\varrho|u|^4\mathrm{d}x\mathrm{d}t\\ &+\kappa\varepsilon\int_0^T\int_{\mathbb{T}_L^3}\varrho|\nabla^2\log\varrho|^2\mathrm{d}x\mathrm{d}t\\ &\le E(\varrho_0,u_0)+C\varepsilon T\|\varrho_0\|^2_{L^1(\mathbb{T}_L^3)}.\end{aligned}\tag{4.22}$$

4.3.4 The Bresch-Desjardins estimates

Before we pass to the limit with the approximating parameters, we derive the so-called Bresch–Desjardins inequality. To do that, we test the momentum equation by $\nabla\log\varrho$ and combine it with the energy inequality (4.22). In consequence, for

$$E_{BD}(\varrho,u)=\int_{\mathbb{T}_L^3}\left(\frac{1}{2}\varrho\left|u+\frac{1}{\varrho}\nabla\varrho\right|^2+\varrho(K_L*\varrho)+\frac{\delta}{2}|\nabla\Delta\varrho|^2+\frac{\kappa}{2}|\nabla\sqrt{\varrho}|^2+\frac{\eta}{7}\varrho^{-6}\right)\mathrm{d}x$$

we obtain

$$\begin{aligned}\sup_{t\in[0,T]} E_{BD}(\varrho,u)+&\frac{2}{3}\eta(1+\varepsilon)\int_0^T\int_{\mathbb{T}_L^3}|\nabla\varrho^{-3}|^2\mathrm{d}x\mathrm{d}t+\delta(1+\varepsilon)\int_0^T\int_{\mathbb{T}_L^3}|\Delta^2\varrho|^2\mathrm{d}x\mathrm{d}t\\ &+\frac{1}{4}\int_0^T\int_{\mathbb{T}_L^3}\varrho|\nabla u-\nabla^T u|^2\mathrm{d}x\mathrm{d}t+\nu\int_0^T\int_{\mathbb{T}_L^3}|\Delta u|^2\mathrm{d}x\mathrm{d}t\\ &+r_0\int_0^T\int_{\mathbb{T}_L^3}|u|^2\,\mathrm{d}x\,\mathrm{d}t+r_1\int_0^T\int_{\mathbb{T}_L^3}\varrho|u|^4\,\mathrm{d}x\,\mathrm{d}t\\ &+\frac{\kappa(1+\varepsilon)}{2}\int_0^T\int_{\mathbb{T}_L^3}\varrho|\nabla^2\log\varrho|^2\mathrm{d}x\mathrm{d}t\\ &+\varepsilon\int_0^T\int_{\mathbb{T}_L^3}\frac{|\Delta\varrho|^2}{\varrho}\,\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{T}_L^3}\nabla(K_L*\varrho)\cdot\nabla\varrho\,\mathrm{d}x\mathrm{d}t\\ \le& E_{BD}(\varrho_0,u_0)+C\varepsilon T\|\varrho_0\|^2_{L^1(\mathbb{T}_L^3)}\\ &+\varepsilon\int_0^T\int_{\mathbb{T}_L^3}\nabla\varrho\nabla u\nabla\log\varrho\,\mathrm{d}x\mathrm{d}t+\varepsilon\int_0^T\int_{\mathbb{T}_L^3}\Delta\varrho\frac{|\nabla\log\varrho|^2}{2}\,\mathrm{d}x\mathrm{d}t\\ &-\varepsilon\int_0^T\int_{\mathbb{T}_L^3}\mathrm{div}\,(\varrho u)\frac{1}{\varrho}\Delta\varrho\,\mathrm{d}x\mathrm{d}t-\nu\int_0^T\int_{\mathbb{T}_L^3}\Delta u\cdot\nabla\Delta\log\varrho\,\mathrm{d}x\mathrm{d}t\\ &-r_1\int_0^T\int_{\mathbb{T}_L^3}|u|^2u\nabla\varrho\,\mathrm{d}x\mathrm{d}t-r_0\int_0^T\int_{\mathbb{T}_L^3}\frac{u\cdot\nabla\varrho}{\varrho}\,\mathrm{d}x\mathrm{d}t\\ =&E_{BD}(\varrho_0,u_0)+C\varepsilon T\|\varrho_0\|^2_{L^1(\mathbb{T}_L^3)}+\sum_{i=1}^6 R_i.\end{aligned}\tag{4.23}$$

The necessary calculations to derive (4.23) are performed in the Appendix C.1.

The terms R_1–R_4 go to 0 as $\varepsilon, \nu \to 0$. For R_5, we have

$$R_5 = - r_1 \int_0^T \int_{\mathbb{T}_L^3} |u|^2 u \cdot \nabla \varrho \, \mathrm{d}x\mathrm{d}t = r_1 \int_0^T \int_{\mathbb{T}_L^3} \varrho \mathrm{div}\,(|u|^2 u) \, \mathrm{d}x\mathrm{d}t$$
$$\leq C r_1 \int_0^T \int_{\mathbb{T}_L^3} \varrho |u|^2 |\nabla u| \, \mathrm{d}x\mathrm{d}t \leq C r_1^2 \int_0^T \int_{\mathbb{T}_L^3} \varrho |u|^4 \mathrm{d}x\mathrm{d}t + \frac{1}{4} \int_0^T \int_{\mathbb{T}_L^3} \varrho |\nabla u|^2 \mathrm{d}x\mathrm{d}t.$$

Since

$$\int_0^T \int_{\mathbb{T}_L^3} \varrho |\nabla u|^2 \, \mathrm{d}x\mathrm{d}t \leq \int_0^T \int_{\mathbb{T}_L^3} \varrho |\mathbb{D}u|^2 \, \mathrm{d}x\mathrm{d}t + \frac{1}{2} \int_0^T \int_{\mathbb{T}_L^3} \varrho |\nabla u - \nabla^T u|^2 \, \mathrm{d}x\mathrm{d}t,$$

the last term is further estimated by

$$\frac{1}{8} \int_0^T \int_{\mathbb{T}_L^3} \varrho |\nabla u - \nabla^T u|^2 \, \mathrm{d}x\mathrm{d}t + E(\varrho_0, u_0) + C\varepsilon T \|\varrho_0\|^2_{L^1(\mathbb{T}_L^3)}.$$

For R_6, we write

$$R_6 = -r_0 \int_0^T \int_{\mathbb{T}_L^3} \frac{\mathrm{div}\,(\varrho u) - \varrho \mathrm{div}\, u}{\varrho} \, \mathrm{d}x\mathrm{d}t = r_0 \int_0^T \int_{\mathbb{T}_L^3} \partial_t \log \varrho \, \mathrm{d}x\mathrm{d}t - \varepsilon r_0 \int_0^T \int_{\mathbb{T}_L^3} \frac{\Delta \varrho}{\varrho} \, \mathrm{d}x\mathrm{d}t.$$

Since ϱ is bounded in $L^\infty(0, T; L^1)$, defining

$$\log_+ \varrho = \begin{cases} \log \varrho, \ \varrho > 1, \\ 0, \ \varrho \leq 1, \end{cases}$$

we have

$$r_0 \sup_{t \in [0,T]} \int_{\mathbb{T}_L^3} \log_+ \varrho \, \mathrm{d}x \leq C.$$

For the second term of R_6, we get

$$\left| \varepsilon r_0 \int_0^T \int_{\mathbb{T}_L^3} \frac{\Delta \varrho}{\varrho} \mathrm{d}x \right| \leq C T \varepsilon r_0 \|\varrho\|_{L^\infty(0,T;H^2)} \|\varrho^{-1}\|_{L^\infty([0,T] \times \mathbb{T}_L^3)},$$

which also tends to 0 as $\varepsilon \to 0$.

In consequence, using again Lemma 4.8, we get

$$
\begin{aligned}
E_{BD}(\varrho,u)-r_0\int_{\mathbb{T}_L^3}&\log\varrho\,\mathrm{d}x\\
&+\frac{2}{3}\eta(1+\varepsilon)\int_0^T\int_{\mathbb{T}_L^3}|\nabla\varrho^{-3}|^2\mathrm{d}x\mathrm{d}t+\delta(1+\varepsilon)\int_0^T\int_{\mathbb{T}_L^3}|\Delta^2\varrho|^2\mathrm{d}x\mathrm{d}t\\
&+\frac{1}{8}\int_0^T\int_{\mathbb{T}_L^3}\varrho|\nabla u-\nabla^T u|^2\mathrm{d}x\mathrm{d}t+\nu\int_0^T\int_{\mathbb{T}_L^3}|\Delta u|^2\mathrm{d}x\mathrm{d}t\\
&+r_0\int_0^T\int_{\mathbb{T}_L^3}|u|^2\,\mathrm{d}x\,\mathrm{d}t+r_1\int_0^T\int_{\mathbb{T}_L^3}\varrho|u|^4\,\mathrm{d}x\,\mathrm{d}t\\
&+\frac{\kappa(1+\varepsilon)}{2}\int_0^T\int_{\mathbb{T}_L^3}\varrho|\nabla^2\log\varrho|^2\mathrm{d}x\mathrm{d}t+\varepsilon\int_{\mathbb{T}_L^3}\frac{|\Delta\varrho|^2}{\varrho}\mathrm{d}x\\
&\le\sum_{i=1}^4 R_i+Cr_1^2\int_0^T\int_{\mathbb{T}_L^3}\varrho|u|^4\mathrm{d}x\mathrm{d}t+CT\varepsilon r_0\|\varrho\|_{L^\infty(0,T;H^2)}\|\varrho^{-1}\|_{L^\infty([0,T]\times\mathbb{T}_L^3)}\\
&+E_{BD}(\varrho_0,u_0)-r_0\int_{\mathbb{T}_L^3}\log\varrho_0\,\mathrm{d}x+E(\varrho_0,u_0)+CT\|\varrho_0\|^2_{L^1(\mathbb{T}_L^3)}.
\end{aligned}
\tag{4.24}
$$

4.3.5 Limit passage with $\nu,\varepsilon\to 0$

Now we pass to the limit with $\nu,\varepsilon\to 0$. Note that the inequality (4.24), together with the energy estimate (4.22) provides us the estimates required in Lemmas 4.4-4.6 uniformly in ε and ν, where the estimate on $\varrho|\nabla u|^2$ comes from the estimate on the symmetric gradient $\varrho|\mathbb{D}u|^2$ in (4.22) and the antisymmetric part $\varrho|\nabla u-\nabla^T u|^2$ in (4.24).

For the terms depending on ε and ν, in the weak formulation of (4.11) we have

$$
\varepsilon\int_0^T\int_{\mathbb{T}_L^3}\nabla\varrho_{\nu,\varepsilon}\cdot\nabla\varphi\,\mathrm{d}x\mathrm{d}t\le\sqrt{\varepsilon}\sqrt{\varepsilon}\|\nabla\varrho_{\nu,\varepsilon}\|_{L^2(0,T;L^2)}\|\nabla\varphi\|_{L^2(0,T;L^2)}\to 0,
$$

$$
\begin{aligned}
\varepsilon\int_0^T\int_{\mathbb{T}_L^3}&\nabla\varrho_{\nu,\varepsilon}\nabla u_{\nu,\varepsilon}\varphi\,\mathrm{d}x\mathrm{d}t\\
&\le\varepsilon\|\nabla\sqrt{\varrho_{\nu,\varepsilon}}\|_{L^\infty(0,T;L^2)}\|\sqrt{\varrho_{\nu,\varepsilon}}\nabla u_{\nu,\varepsilon}\|_{L^2(0,T;L^2)}\|\varphi\|_{L^2(0,T;L^\infty)}\to 0
\end{aligned}
$$

and

$$
\nu\int_0^T\int_{\mathbb{T}_L^3}\Delta u_{\nu,\varepsilon}\Delta\varphi\mathrm{d}x\mathrm{d}t\le\sqrt{\nu}\sqrt{\nu}\|\Delta u_{\nu,\varepsilon}\|_{L^2(0,T;L^2)}\|\Delta\varphi\|_{L^2(0,T;L^2)}\to 0.
$$

In consequence, performing limit passages in the same way as before, we obtain the solutions to the system

$$
\begin{aligned}
&\partial_t\varrho+\operatorname{div}(\varrho u)=0,\\
&\partial_t(\varrho u)+\operatorname{div}(\varrho u\otimes u)-\operatorname{div}(\varrho\mathbb{D}u)+\varrho\nabla(K_L*\varrho)\\
&=-r_0u-r_1\varrho|u|^2u+\kappa\varrho\nabla\left(\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\right)+\eta\nabla\varrho^{-6}+\delta\varrho\nabla\Delta^3\varrho.
\end{aligned}
\tag{4.25}
$$

Passing to the limit in (4.22) and (4.24), again by the lower semicontinuity of convex functions, we get

$$\sup_{t\in[0,T]} E(\varrho,u) + \int_0^T\int_{\mathbb{T}^3_L} \varrho|\mathbb{D}u|^2 \mathrm{d}x\mathrm{d}t$$
$$+ r_0 \int_0^T\int_{\mathbb{T}^3_L} |u|^2 \mathrm{d}x\mathrm{d}t + r_1 \int_0^T\int_{\mathbb{T}^3_L} \varrho|u|^4 \mathrm{d}x\mathrm{d}t \le E(\varrho_0,u_0) \tag{4.26}$$

and for the Bresch-Desjardins inequality

$$\begin{aligned} E_{BD}(\varrho,u) - r_0 \int_{\mathbb{T}^3_L} \log\varrho \,\mathrm{d}x + \eta \int_0^T\int_{\mathbb{T}^3_L} |\nabla\varrho^{-3}|^2 \,\mathrm{d}x\mathrm{d}t + 2\delta \int_0^T\int_{\mathbb{T}^3_L} |\Delta^2\varrho|^2 \mathrm{d}x\mathrm{d}t \\ + \frac{1}{8}\int_0^T\int_{\mathbb{T}^3_L} \varrho|\nabla u - \nabla^T u|^2 \mathrm{d}x\mathrm{d}t + \kappa \int_0^T\int_{\mathbb{T}^3_L} \varrho|\nabla^2\log\varrho|^2 \mathrm{d}x\mathrm{d}t \\ \le E_{BD}(\varrho_0,u_0) - r_0 \int_{\mathbb{T}^3_L} \log\varrho_0 \,\mathrm{d}x + E(\varrho_0,u_0) + CT\|\varrho_0\|^2_{L^1(\mathbb{T}^3_L)}. \end{aligned} \tag{4.27}$$

4.3.6 Limit passage with $\eta,\delta\to 0$.

We will first pass to the limit with η and then with δ. Note that the inequalities (4.26) and (4.27) provide us again the estimates required in Lemmas 4.4 and 4.6, this time uniformly in η and δ (provided that $\delta\|\nabla\Delta\tilde{\varrho}_0\|^2_{L^2(\mathbb{T}^3_L)} \to 0$). We need to pass to the limit only with the terms $\eta\nabla\varrho^{-6}$ and $\delta\varrho\nabla\Delta^3\varrho$, since the remaining terms are treated in the same way as before. Note that since we lose the information on $\mathbb{D}u$ on the set where $\varrho = 0$, we pass to the limit in the stress tensor using the relation

$$\varrho\mathbb{D}u = \mathbb{D}(\varrho u) - \nabla\varrho\otimes u.$$

From (4.26) and (4.27) we also have the estimates

$$\eta\|\varrho^{-6}_{\eta,\delta}\|_{L^\infty(0,T;L^1)},\ \sqrt{\eta}\|\varrho^{-3}_{\eta,\delta}\|_{L^2(0,T;H^1)} \le C(T). \tag{4.28}$$

In consequence, using the interpolation between $L^\infty(0,T;L^1)$ and $L^1(0,T;L^3)$, we have

$$\|\eta\varrho^{-6}_{\eta,\delta}\|_{L^{5/3}(0,T;L^{5/3})} \le C(T)$$

as well.

The above estimates allow us to show

Lemma 4.12. *If $\varrho_{\eta,\delta}$ satisfies the estimates following from (4.26) and (4.27), then*

$$\eta\int_0^T\int_{\mathbb{T}^3_L} \varrho^{-6}_{\eta,\delta} \mathrm{d}x\mathrm{d}t \to 0 \quad as \quad \eta\to 0.$$

Proof. From (4.27) we know that

$$r_0\int_{\mathbb{T}^3_L} \log_+\left(\frac{1}{\varrho_{\eta,\delta}}\right) \,\mathrm{d}x \le C(T).$$

As $\sqrt{\varrho_{\eta,\delta}} \to \sqrt{\varrho_\delta}$ in $L^2(0,T;H^1)$, then $\varrho_{\eta,\delta} \to \varrho_\delta$ a. e. and using the convexity of a function $y \mapsto \log_+\left(\frac{1}{y}\right)$ from Fatou's Lemma

$$\begin{aligned}\int_{\mathbb{T}^3_L} \log_+\left(\frac{1}{\varrho_\delta}\right) \mathrm{d}x &\leq \int_{\mathbb{T}^3_L} \liminf_{\eta\to 0} \log_+\left(\frac{1}{\varrho_{\eta,\delta}}\right) \mathrm{d}x \\ &\leq \liminf_{\eta\to 0} \int_{\mathbb{T}^3_L} \log_+\left(\frac{1}{\varrho_{\eta,\delta}}\right) \mathrm{d}x \leq C.\end{aligned}$$

Therefore

$$|\{x : \varrho_\delta(t,x) = 0\}| = 0 \quad \text{for almost every} \quad t.$$

Then, as $\varrho_{\eta.\delta} \to \varrho_\delta$ a. e.,

$$\eta\varrho_{\eta,\delta}^{-6} \to 0 \quad \text{a. e.}$$

As $\eta\varrho_{\eta,\delta}^{-6}$ is uniformly bounded in $L^{5/3}(0,T;L^{5/3})$, it follows that

$$\eta\varrho_{\eta,\delta}^{-6} \to 0 \quad \text{in} \quad L^1(0,T;L^1).$$

□

Now we pass to the limit with δ:

Lemma 4.13. *For any $\varphi \in C_0^\infty([0,T] \times \mathbb{T}^3_L)$ we have*

$$\delta \int_0^T \int_{\mathbb{T}^3_L} \varrho_\delta \nabla\Delta^3 \varrho_\delta \varphi \,\mathrm{d}x\mathrm{d}t \to 0$$

as $\delta \to 0$.

Proof. We have

$$\delta \int_0^T \int_{\mathbb{T}^3_L} \varrho_\delta \nabla\Delta^3 \varrho_\delta \varphi \,\mathrm{d}x\mathrm{d}t = -\delta \int_0^T \int_{\mathbb{T}^3_L} \Delta \mathrm{div}\,(\varrho_\delta\varphi)\Delta^2 \varrho_\delta \,\mathrm{d}x\mathrm{d}t.$$

The inequalities (4.26) and (4.27) give

$$\sqrt{\delta}\|\varrho_\delta\|_{L^\infty(0,T;H^3)}, \sqrt{\delta}\|\varrho_\delta\|_{L^2(0,T;H^4)} \leq C(T).$$

Moreover, from the uniform estimate on $\nabla\sqrt{\varrho_\delta}$ in $L^\infty(0,T;L^2)$, we also have

$$\|\varrho_\delta\|_{L^\infty(0,T;L^3)} \leq C(T).$$

Using the Gagliardo-Nirenberg inequality

$$\|\nabla^3\varrho_\delta\|_{L^3} \leq C\|\nabla^4\varrho_\delta\|_{L^2}^{\frac{6}{7}}\|\varrho_\delta\|_{L^3}^{\frac{1}{7}},$$

we get

$$\delta\int_0^T \|\nabla^3\varrho_\delta\|_{L^3}^{\frac{7}{3}}\mathrm{d}t \leq C \sup_{t\in[0,T]} \|\varrho_\delta\|_{L^3}^{\frac{1}{3}} \int_0^T \delta\|\nabla^4\varrho_\delta\|_{L^2}^2\mathrm{d}t$$

and in consequence

$$\delta^{\frac{3}{7}}\|\nabla^3\varrho_\delta\|_{L^{\frac{7}{3}}(0,T;L^3)} \leq C(T).$$

Then

$$\left|\delta\int_0^T\int_{\mathbb{T}_L^3}\Delta\nabla\varrho_\delta\cdot\Delta^2\varrho_\delta\varphi\,\mathrm{d}x\mathrm{d}t\right|$$
$$\leq C(\varphi)\delta^{\frac{1}{14}}\|\sqrt{\delta}\nabla^4\varrho_\delta\|_{L^2(0,T;L^2)}\|\delta^{\frac{3}{7}}\nabla^3\varrho_\delta\|_{L^{\frac{7}{3}}(0,T;L^3)}\to 0$$

as $\delta\to 0$. Applying the same arguments to the rest of the terms from

$$\delta\int_0^T\int_{\mathbb{T}_L^3}\Delta\mathrm{div}\,(\varrho_\delta\varphi)\Delta^2\varrho_\delta\mathrm{d}x\mathrm{d}t,$$

we finish the proof of the Lemma. □

Remark 4.14. Note that in the limit passage with η we lost any information on ∇u. However, from the uniform estimates we know that up to a subsequence

$$\sqrt{\varrho_{\eta,\delta}}\nabla u_{\eta,\delta}\rightharpoonup\overline{\sqrt{\varrho_\delta}\nabla u_\delta}\quad\text{in}\quad L^2((0,T)\times\mathbb{T}_L^3).$$

Using the relation

$$\sqrt{\varrho}\nabla u=\nabla(\sqrt{\varrho}u)-\nabla\sqrt{\varrho}\otimes u,$$

from the strong convergence of $\varrho_{\eta,\delta}$ and $\nabla\sqrt{\varrho_{\eta,\delta}}$ and weak convergence of $u_{\eta,\delta}$ we get

$$\overline{\sqrt{\varrho_\delta}\nabla u_\delta}=\nabla(\sqrt{\varrho_\delta}u_\delta)-\nabla\sqrt{\varrho_\delta}\otimes u_\delta.$$

Proceeding analogously, after passing to the limit with $\delta\to 0$ we get as well

$$\overline{\sqrt{\varrho}\nabla u}=\nabla(\sqrt{\varrho}u)-\nabla\sqrt{\varrho}\otimes u. \tag{4.29}$$

In the analogous way we can also define $\overline{\sqrt{\varrho}\mathbb{D}u}$, $\overline{\sqrt{\varrho}\mathrm{div}\,u}$ etc. In the next sections we will again omit the bars, keeping in mind the relation (4.29).

4.4 The Mellet - Vasseur estimates

Before we pass to the limit with the remaining parameters, we need to extract another estimate from the system. In the previous section, we showed the existence of a weak solution to the system

$$\begin{gathered}\partial_t\varrho+\mathrm{div}\,(\varrho u)=0\\ \partial_t(\varrho u)+\mathrm{div}\,(\varrho u\otimes u)-\mathrm{div}\,(\varrho\mathbb{D}u)+\varrho\nabla(K_L*\varrho)=-r_0u-r_1\varrho|u|^2u+\kappa\varrho\nabla\left(\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\right)\end{gathered}\tag{4.30}$$

on $[0,T]\times\mathbb{T}_L^3$ with the initial conditions

$$\varrho_{|t=0}:=\tilde{\varrho}_{0,L}=\varrho_{0,L}+\frac{1}{m_1},\qquad u_{|t=0}=u_{0,L},$$

where $m_1>0$ and $\varrho_{0,L},u_{0,L}$ are like in Section 4.3.1. Similarly as in the Definition 4.1, this means that for each $\varphi\in C_0^\infty([0,T)\times\mathbb{T}_L^3;\mathbb{R})$ and $\psi\in C_0^\infty([0,T)\times\mathbb{T}_L^3;\mathbb{R}^3)$ it holds

$$-\int_0^T\int_{\mathbb{T}_L^3}\varrho\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_{\mathbb{T}_L^3}\sqrt{\varrho}\sqrt{\varrho}u\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t=\int_{\mathbb{T}_L^3}\tilde{\varrho}_{0,L}\varphi(0,\cdot)\,\mathrm{d}x$$

and

$$
\begin{aligned}
-\int_{\mathbb{T}_L^3} \tilde{\varrho}_{0,L} u_{0,L} \psi(0,\cdot)\,\mathrm{d}x - \int_0^T\!\!\int_{\mathbb{T}_L^3} \sqrt{\varrho}\sqrt{\varrho} u \partial_t \psi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\!\!\int_{\mathbb{T}_L^3} (\sqrt{\varrho}u \otimes \sqrt{\varrho}u) : \nabla\psi\,\mathrm{d}x\,\mathrm{d}t \\
+\langle \varrho \mathbb{D}u, \nabla\psi\rangle + \int_0^T\!\!\int_{\mathbb{T}_L^3} \varrho\nabla(K_L * \varrho)\cdot\psi\,\mathrm{d}x \\
= -r_0 \int_0^T\!\!\int_{\mathbb{T}_L^3} u\cdot\psi\,\mathrm{d}x\mathrm{d}t - r_1 \int_0^T\!\!\int_{\mathbb{T}_L^3} \varrho|u|^2 u\cdot\psi\,\mathrm{d}x\mathrm{d}t \\
- \kappa \int_0^T\!\!\int_{\mathbb{T}_L^3} \Delta\sqrt{\varrho}\sqrt{\varrho}\operatorname{div}\psi\,\mathrm{d}x\mathrm{d}t - 2\kappa \int_0^T\!\!\int_{\mathbb{T}_L^3} \Delta\sqrt{\varrho}\nabla\sqrt{\varrho}\cdot\psi\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

The solution satisfies the following estimates:

$$
\begin{aligned}
\sup_{t\in[0,T]} \frac{1}{2}\int_{\mathbb{T}_L^3} \left(\varrho|u|^2 + \varrho(K_L*\varrho) + \kappa|\nabla\sqrt{\varrho}|^2\right)\mathrm{d}x + \int_0^T \int_{\mathbb{T}_L^3} \varrho|\mathbb{D}u|^2 \mathrm{d}x\mathrm{d}t \\
+ r_0 \int_0^T \int_{\mathbb{T}_L^3} |u|^2 \mathrm{d}x\mathrm{d}t + r_1 \int_0^T \int_{\mathbb{T}_L^3} \varrho|u|^4 \mathrm{d}x\mathrm{d}t \le E(\tilde{\varrho}_{0,L}, u_{0,L}),
\end{aligned} \tag{4.31}
$$

where

$$
E(\tilde{\varrho}_{0,L}, u_{0,L}) = \frac{1}{2}\int_{\mathbb{T}_L^3} \left(\tilde{\varrho}_{0,L}|u_{0,L}|^2 + \tilde{\varrho}_{0,L}(K_L*\tilde{\varrho}_{0,L}) + \kappa|\nabla\sqrt{\tilde{\varrho}_{0,L}}|^2\right)\,\mathrm{d}x,
$$

and

$$
\begin{aligned}
\int_{\mathbb{T}_L^3} \left(|\nabla\sqrt{\varrho}|^2 - r_0\log\varrho\right)\,\mathrm{d}x + \frac{1}{8}\int_0^T \int_{\mathbb{T}_L^3} \varrho|\nabla u - \nabla^T u|^2 \mathrm{d}x + \kappa \int_0^T \int_{\mathbb{T}_L^3} \varrho|\nabla^2 \log\varrho|^2 \mathrm{d}x \\
\le 2E(\tilde{\varrho}_{0,L}, u_{0,L}) + \int_{\mathbb{T}_L^3} \left(|\nabla\sqrt{\tilde{\varrho}_{0,L}}|^2 - r_0\log\tilde{\varrho}_{0,L}\right)\mathrm{d}x + CT\|\tilde{\varrho}_{0,L}\|_{L^1(\mathbb{T}_L^3)}^2.
\end{aligned} \tag{4.32}
$$

From (4.32) and Proposition 4.5, it also follows that

$$
\kappa^{1/2}\|\sqrt{\varrho}\|_{L^2(0,T;H^2)} + \kappa^{1/4}\|\nabla\varrho^{1/4}\|_{L^4(0,T;L^4)} \le C. \tag{4.33}
$$

For the time regularity we have

$$
\|\partial_t\sqrt{\varrho}\|_{L^2(0,T;L^2)} \le \frac{1}{2}\|\sqrt{\varrho}\operatorname{div}u\|_{L^2(0,T;L^2)} + \frac{1}{2}\|\nabla\varrho^{1/4}\|_{L^4(0,T;L^4)}\|\varrho^{1/4}u\|_{L^4(0,T;L^4)}
$$

and since

$$
\partial_t\varrho = -2\nabla\sqrt{\varrho}\cdot\varrho^{1/4}u\cdot\varrho^{1/4} - \sqrt{\varrho}\sqrt{\varrho}\operatorname{div}u,
$$

from (4.31) and (4.32) we get

$$
\|\partial_t\varrho\|_{L^2(0,T;L^{6/5})} \le C.
$$

In this section we perform the limit with $\kappa \to 0$ and simultaneously derive another estimate. The main result states

Lemma 4.15. *There exists a solution to system (4.30) with $\kappa = 0$, which satisfies the estimate*

$$\sup_{t\in[0,T]}\left(\int_{\mathbb{T}_L^3} \varrho F(|u|)\,\mathrm{d}x + \iint_{\mathbb{T}_L^3\times\mathbb{T}_L^3} F(|x-y|)\varrho(x)\varrho(y)\,\mathrm{d}x\mathrm{d}y\right)$$
$$\leq C\left(\int_{\mathbb{T}_L^3} \varrho_0 F(|u_0|) + \iint_{\mathbb{T}_L^3\times\mathbb{T}_L^3} F(|x-y|)\varrho_0(x)\varrho_0(y)\,\mathrm{d}x\mathrm{d}y\right) + C + \frac{C}{L^2} \tag{4.34}$$

for $F(z) = \frac{1+z^2}{2}\ln(1+z^2)$, where C does not depend on r_0 and r_1. Moreover, estimates (4.31) and (4.32) are valid with $\kappa = 0$.

Proof. The strategy of the proof is based on the approach from [105], however the need to incorporate the nonlocal term imposes some key differences in the method. Let

$$F(z) = \frac{1+z^2}{2}\ln(1+z^2), \qquad \psi(z) = \frac{1}{z}F'(z) = 1 + \ln(1+z^2)$$

for $z > 0$. To get (4.34), we would like to test the momentum equation by the function $F'(|u|^2)\frac{u}{|u|}$. However, the regularity of the solution coming from the estimates (4.8) and (4.9) does not allow that (in particular, they do not provide any Sobolev regularity on u itself). Instead, we will introduce a suitable approximation of the function $F'(|u|^2)\frac{u}{|u|}$, which will allow us to perform the needed renormalization of the momentum equation. Then (4.34) is obtained by passing to the limit.

Preparation of initial data. Note that our approximation of the initial data satisfies in particular

$$\tilde{\varrho}_{0,L} \geq \frac{1}{m_1}.$$

In order to get the suitable continuity of ϱ and u, we need to further truncate the initial data. Hence we will first derive the desired inequality assuming that

$$\tilde{\varrho}_{0,L}|u_{0,L}|^2 \in L^\infty(\mathbb{T}_L^3). \tag{4.35}$$

Under this additional assumption we show that

Proposition 4.16. *For (ϱ, u) solving (4.30), we have*

$$\varrho \in C(0,T;L^2) \quad \textit{and} \quad \sqrt{\varrho}u \in C(0,T;L^2).$$

Proof. Since $\varrho \in L^\infty(0,T;H^1)$,

$$\partial_t\sqrt{\varrho} \in L^2(0,T;L^2) \quad \text{and} \quad \sqrt{\varrho} \in L^2(0,T;H^2),$$

we have

$$\sqrt{\varrho} \in C(0,T;L^2) \quad \text{and} \quad \nabla\sqrt{\varrho} \in C(0,T;L^2).$$

(the last convergence is shown by computing $\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\sqrt{\varrho} - \nabla\sqrt{\varrho_0}\|_{L^2}^2$).
As $\sqrt{\varrho} \in L^\infty(0,T;L^6)$, we therefore get

$$\sqrt{\varrho} \in C(0,T;L^p) \quad \text{for} \quad 2 \leq p < 6.$$

In particular, it also follows that

$$\varrho \in C(0,T;L^2).$$

Now we show that
$$\sqrt{\varrho}u \in C(0,T;L^2).$$
By the estimates on $\partial_t(\varrho u)$ and ϱu, we know that
$$\varrho u \in C_{\mathrm{weak}}(0,T;L^{3/2}).$$
We will estimate $|\sqrt{\varrho}u - \sqrt{\tilde{\varrho}_{0,L}}u_{0,L}|^2$ using the continuity properties of energy. Since the function $t \mapsto \int_{\mathbb{T}^3_L} \frac{|m(t,x)|^2}{\varrho(t,x)} \mathbb{1}_{\{\varrho>0\}}\,\mathrm{d}x$ is lower-semicontinuous (see also Lemma 7.19 in [85]), we have
$$\int_{\mathbb{T}^3_L} \tilde{\varrho}_{0,L}|u_{0,L}|^2 + \tilde{\varrho}_{0,L}(K_L * \tilde{\varrho}_{0,L}) + \kappa|\nabla\sqrt{\tilde{\varrho}_{0,L}}|^2\,\mathrm{d}x$$
$$\leq \liminf_{t\to 0} \int_{\mathbb{T}^3_L} \varrho|u|^2 + \varrho(K_L * \varrho) + \kappa|\nabla\sqrt{\varrho}|^2\,\mathrm{d}x.$$
Combining that with the energy inequality, we get
$$\lim_{t\to 0}\int_{\mathbb{T}^3_L} \left(\varrho|u|^2 + \varrho(K_L * \varrho) + \kappa|\nabla\sqrt{\varrho}|^2\right)\mathrm{d}x$$
$$= \int_{\mathbb{T}^3_L} \left(\tilde{\varrho}_{0,L}|u_{0,L}|^2 + \tilde{\varrho}_{,L}(K_L * \tilde{\varrho}_{0,L}) + \kappa|\nabla\sqrt{\tilde{\varrho}_{0,L}}|^2\right)\mathrm{d}x$$
With this information at hand, we write
$$\begin{aligned}
\int_{\mathbb{T}^3_L} |\sqrt{\varrho}u - \sqrt{\tilde{\varrho}_{0,L}}u_{0,L}|^2\mathrm{d}x = &\int_{\mathbb{T}^3_L} \left(\varrho|u|^2 + \varrho(K_L * \varrho) + \kappa|\nabla\sqrt{\varrho}|^2\right)\mathrm{d}x \\
&- \int_{\mathbb{T}^3_L} \left(\tilde{\varrho}_{0,L}|u_{0,L}|^2 + \tilde{\varrho}_{0,L}(K_L * \tilde{\varrho}_{0,L}) + \kappa|\nabla\sqrt{\tilde{\varrho}_{0,L}}|^2\right)\mathrm{d}x \\
&+ 2\int_{\mathbb{T}^3_L} \sqrt{\tilde{\varrho}_{0,L}}u_{0,L}(\sqrt{\tilde{\varrho}_{0,L}}u_{0,L} - \sqrt{\varrho}u)\,\mathrm{d}x \\
&+ \int_{\mathbb{T}^3_L} (\tilde{\varrho}_{0,L}(K_L * \tilde{\varrho}_{0,L}) - \varrho(K_L * \varrho))\,\mathrm{d}x \\
&- \kappa\int_{\mathbb{T}^3_L} |\nabla\sqrt{\tilde{\varrho}_{0,L}} - \nabla\sqrt{\varrho}|^2\mathrm{d}x \\
&+ 2\kappa\int_{\mathbb{T}^3_L} \nabla\sqrt{\tilde{\varrho}_{0,L}}(\nabla\sqrt{\tilde{\varrho}_{0,L}} - \nabla\sqrt{\varrho})\mathrm{d}x.
\end{aligned}$$
From the continuity of $\nabla\sqrt{\varrho}$, we have
$$\lim_{t\to 0}\int_{\mathbb{T}^3_L} |\nabla\sqrt{\tilde{\varrho}_{0,L}} - \nabla\sqrt{\varrho}|^2\mathrm{d}x = 0$$
and
$$\lim_{t\to 0}\int_{\mathbb{T}^3_L} \nabla\sqrt{\tilde{\varrho}_{0,L}}(\nabla\sqrt{\tilde{\varrho}_{0,L}} - \sqrt{\varrho})\mathrm{d}x = 0.$$

Since $K_L \in L^p(\mathbb{T}^3_L)$ for $p < 3/\alpha$, from the continuity of ϱ it follows that $K_L * \varrho \in C(0,T;L^q)$ for $q < \frac{3}{\alpha-2}$ (or ∞ if $\alpha \le 2$) and therefore

$$\lim_{t\to 0} \int_{\mathbb{T}^3_L} (\tilde{\varrho}_{0,L}(K_L * \tilde{\varrho}_{0,L}) - \varrho(K_L * \varrho))\mathrm{d}x \to 0.$$

In consequence,

$$\begin{aligned}\operatorname{ess}\limsup_{t\to 0} \int_{\mathbb{T}^3_L} &|\sqrt{\varrho}u - \sqrt{\tilde{\varrho}_{0,L}}u_{0,L}|^2\mathrm{d}x \\ &= 2\operatorname{ess}\limsup_{t\to 0} \int_{\mathbb{T}^3_L} \sqrt{\tilde{\varrho}_{0,L}}u_{0,L}(\sqrt{\tilde{\varrho}_{0,L}}u_{0,L} - \sqrt{\varrho}u)\,\mathrm{d}x. \quad (4.36)\end{aligned}$$

Now, let ϕ_{m_1} be a smooth cutoff function such that $\phi_{m_1}(\varrho) = 1$ for $\varrho \ge \frac{1}{m_1}$ and $\phi_{m_1}(\varrho) = 0$ for $\varrho \le \frac{1}{2m_1}$. Then we write

$$\begin{aligned}\int_{\mathbb{T}^3_L} \sqrt{\tilde{\varrho}_{0,L}}u_{0,L}(\sqrt{\tilde{\varrho}_{0,L}}u_{0,L} - \sqrt{\varrho}u)\mathrm{d}x =& \int_{\mathbb{T}^3_L} \sqrt{\tilde{\varrho}_{0,L}}u_{0,L}(\sqrt{\tilde{\varrho}_{0,L}}u_{0,L} - \phi_{m_1}(\varrho)\sqrt{\varrho}u)\mathrm{d}x \\ &- \int_{\mathbb{T}^3_L} \sqrt{\tilde{\varrho}_{0,L}}u_{0,L}(1-\phi_{m_1}(\varrho))\sqrt{\varrho}u\,\mathrm{d}x \\ =& B_1(t) + B_2(t).\end{aligned}$$

For the term B_1, we use the relation

$$\begin{aligned}\sqrt{\tilde{\varrho}_{0,L}}u_{0,L}(\sqrt{\tilde{\varrho}_{0,L}}u_{0,L} - \phi_{m_1}(\varrho)\sqrt{\varrho}u) =& \sqrt{\tilde{\varrho}_{0,L}}u_{0,L}\frac{\phi_{m_1}(\varrho)}{\sqrt{\varrho}}(\tilde{\varrho}_{0,L}u_{0,L} - \varrho u) \\ &+ \tilde{\varrho}_{0,L}|u_{0,L}|^2\left(1 - \frac{\sqrt{\tilde{\varrho}_{0,L}}}{\sqrt{\varrho}}\phi_{m_1}(\varrho)\right).\end{aligned}$$

Since $\tilde{\varrho}_{0,L} \ge \frac{1}{m_1}$, we know that in particular $\phi_{m_1}(\tilde{\varrho}_{0,L}) = 1$ and then

$$\begin{aligned}\lim_{t\to 0} B_1(t) =& \lim_{t\to 0}\int_{\mathbb{T}^3_L} \sqrt{\tilde{\varrho}_{0,L}}u_{0,L}\frac{\phi_{m_1}(\varrho)}{\sqrt{\varrho}}(\tilde{\varrho}_{0,L}u_{0,L} - \varrho u)\mathrm{d}x \\ &+ \lim_{t\to 0}\int_{\mathbb{T}^3_L} \sqrt{\tilde{\varrho}_{0,L}}|u_{0,L}|^2\sqrt{\tilde{\varrho}_{0,L}}\left(\frac{\phi_{m_1}(\tilde{\varrho}_{0,L}}{\sqrt{\tilde{\varrho}_{0,L}}} - \frac{\phi_{m_1}(\varrho)}{\sqrt{\varrho}}\right)\mathrm{d}x = 0\end{aligned}$$

by the weak continuity of ϱu and strong continuity of ϱ. For B_2, by (4.35) we have

$$|B_2(t)| \le \|\sqrt{\tilde{\varrho}_{0,L}}u_{0,L}\|_{L^\infty(\mathbb{T}^3_L)}\|\sqrt{\varrho}u\|_{L^\infty(0,T;L^2)}\|1-\phi_{m_1}(\varrho)\|_{L^2(\mathbb{T}^3_L)},$$

which goes to zero as $t \to 0$ again from the strong continuity of ϱ.

In consequence, $\sqrt{\varrho}u \in C(0,T;L^2)$ as we needed to prove. □

Preparation of the test function. Let ϕ^0_m and ϕ^∞_k be smooth cutoff functions at zero at infinity respectively, such that

$$\phi^0_m(\varrho) = 1 \quad \text{for} \quad \varrho > \frac{1}{m}, \quad \phi^0_m(\varrho) = 0 \quad \text{for} \quad \varrho < \frac{1}{2m}, \quad |(\phi^0_m)'| \le 2m,$$

and

$$\phi_k^\infty(\varrho) = 1 \quad \text{for} \quad \varrho < k, \quad \phi_k^\infty(\varrho) = 0, \quad \text{for} \quad \varrho > 2k, \quad |(\phi_k^\infty)'| \le \frac{2}{k}.$$

Then we define

$$v_{m,k} = \phi_{m,k}(\varrho)u \quad \text{for} \quad \phi_{m,k}(\varrho) = \phi_m^0(\varrho)\phi_k^\infty(\varrho).$$

To simplify the notation, we will just write $v = v_{m,k}$ and $\phi = \phi_{m,k}$ when it does not raise confusion. It turns out that v has the $W^{1,2}$ regularity missing for u:

Proposition 4.17.

$$\nabla v \in L^2((0,T)\times \mathbb{T}_L^3).$$

Proof. By straightforward calculations,

$$\begin{aligned}\nabla v =& \phi'(\varrho)\nabla\varrho\otimes u + \phi(\varrho)\nabla u\\ =& 4\phi'(\varrho)\sqrt{\varrho}\nabla\varrho^{1/4}\otimes\varrho^{1/4}u + \frac{\phi(\varrho)}{\sqrt{\varrho}}\sqrt{\varrho}\nabla u.\end{aligned}$$

From the definition of ϕ,

$$\frac{\phi(\varrho)}{\sqrt{\varrho}} \le \sqrt{2m} \quad \text{and} \quad |\phi'(\varrho)\sqrt{\varrho}| \le \max\left(2m\cdot\frac{1}{\sqrt{m}}, \frac{2}{k}\cdot\sqrt{2k}\right)$$

and therefore

$$\begin{aligned}&\|\nabla v\|_{L^2((0,T)\times\mathbb{T}_L^3)}\\ &\quad\le C(m,k)\left(\|\nabla\varrho^{1/4}\|_{L^4((0,T)\times\mathbb{T}_L^3)}\|\varrho^{1/4}u\|_{L^4((0,T)\times\mathbb{T}_L^3)} + \|\sqrt{\varrho}\nabla u\|_{L^2((0,T)\times\mathbb{T}_L^3)}\right).\end{aligned}$$

□

In order to construct a suitable test function, we need to approximate the functions F and ψ as well. Let

$$F_n(z) = \begin{cases} \dfrac{1+z^2}{2}\ln(1+z^2), & z \le n,\\ \left(nz + \dfrac{1-n^2}{2}\right)\ln(1+z^2), & z > n\end{cases}$$

and

$$\psi_n(z) = \frac{1}{z}F_n'(z) = \begin{cases} 1+\ln(1+z^2), & z\le n,\\ \dfrac{n}{z}\ln(1+z^2) + \dfrac{2nz+1-n^2}{1+z^2}, & z>n\end{cases}$$

That way

$$F_n(z) \le C_n|z|^{1+\delta}$$

and

$$\psi_n(z)z = F_n'(z) \le C_n|z|^\delta$$

for any $\delta\in(0,1)$ and some $C_n > 0$. Since $F_n \le \frac{1+z^2}{2}\ln(1+z^2)$ and $\psi_n \le 1+\ln(1+z^2)$, we also have the estimates

$$F_n(z) \le C + C|z|^{2+\delta}, \quad \psi_n(z)z \le C + C|z|^{1+\delta},$$

where this time $C > 0$ does not depend on n.

Finally, we have the estimates for the second derivative:

$$F_n''(z) = \psi_n'(z)z + \psi_n(z) = \begin{cases} 1 + \ln(1+z^2) + \dfrac{2z^2}{1+z^2}, & z \le n, \\ \dfrac{2nz}{1+z^2} + \dfrac{(n^2-1)z^2 + 4nz - (n^2-1)}{(1+z^2)^2}, & z > n \end{cases}$$

and thus it is positive and bounded.

Having the suitable approximations, we can state the first step towards the proof of Lemma 4.15.

Lemma 4.18. *For any nonnegative $\xi(t) \in C_0^\infty(0,+\infty)$ we have*

$$-\int_0^T \int_{\mathbb{T}_L^3} \xi'(t)\varrho F_n(|v|)\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{T}_L^3} \xi(t)\psi_n(|v|)v \cdot G\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{T}_L^3} \xi(t) S : \nabla(\psi_n(|v|)v)\,\mathrm{d}x\mathrm{d}t = 0, \quad (4.37)$$

where

$$S = \varrho\phi(\varrho)\left(\mathbb{D}u + \kappa\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\mathbb{I}\right)$$

and

$$\begin{aligned} G =& \varrho^2 u\phi'(\varrho)\mathrm{div}\, u + \varrho\nabla\phi(\varrho)\mathbb{D}u + \phi(\varrho)\varrho\nabla(K_L * \varrho) + r_0 u\phi(\varrho) \\ &+ r_1\varrho|u|^2 u\phi(\varrho) + \kappa\sqrt{\varrho}\nabla\phi(\varrho)\Delta\sqrt{\varrho} + 2\kappa\phi(\varrho)\nabla\sqrt{\varrho}\Delta\sqrt{\varrho}. \end{aligned}$$

Proof. Testing the momentum equation by $\phi(\varrho)\varphi$ for $\varphi \in C_0^\infty((0,T)\times\mathbb{T}_L^3)$, we obtain

$$\begin{aligned} \partial_t(\varrho v) - \varrho u\phi'(\varrho)\partial_t\varrho &+ \mathrm{div}\,(\varrho u \otimes v) - \varrho u \otimes u\nabla\phi(\varrho) \\ &- \mathrm{div}\,(\phi(\varrho)\varrho\mathbb{D}u) + \varrho\mathbb{D}u\nabla\phi(\varrho) + \phi(\varrho)\varrho\nabla(K * \varrho) \\ =& - r_0 v - r_1\phi(\varrho)\varrho|u|^2 u \\ &- \kappa\Big(\sqrt{\varrho}\Delta\sqrt{\varrho}\nabla\phi(\varrho) + 2\phi(\varrho)\Delta\sqrt{\varrho}\nabla\sqrt{\varrho} - \nabla\big(\phi(\varrho)\sqrt{\varrho}\Delta\sqrt{\varrho}\big)\Big) \end{aligned} \quad (4.38)$$

in the sense of distributions. Since

$$-\varrho u\phi'(\varrho)\partial_t\varrho - \varrho u \otimes u\nabla\phi(\varrho) = \varrho^2 u\phi'(\varrho)\mathrm{div}\, u,$$

we can rewrite (4.38) as

$$\partial_t(\varrho v) + \mathrm{div}\,(\varrho u \otimes v) - \mathrm{div}\, S + G = 0. \quad (4.39)$$

Now, let us take $\xi \in C_0^\infty(0,+\infty)$. We test equation (4.39) by $\Phi = (\xi(t)\psi_n(|v_\varepsilon|)v_\varepsilon)_\varepsilon$, where $f_\varepsilon = f * \eta_\varepsilon$ denotes the mollification over time and space. Note that since ξ has compact support, for sufficiently small ε the function $\Phi(t,\cdot)$ is well defined on $(0,\infty)$. Then we get

$$\int_0^T \int_{\mathbb{T}_L^3} \xi(t)\psi_n(|v_\varepsilon|)v_\varepsilon \cdot \big(\partial_t(\varrho v) + \mathrm{div}\,(\varrho u \otimes v) - \mathrm{div}\, S + G\big)_\varepsilon\,\mathrm{d}x\mathrm{d}t = 0.$$

Let us rewrite the first two terms of the above. We have

$$\begin{aligned}\int_0^T \int_{\mathbb{T}_L^3} &\xi(t)\psi_n(|v_\varepsilon|)v_\varepsilon \cdot (\partial_t(\varrho v))_\varepsilon \,\mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{T}_L^3} \xi(t)\psi_n(|v_\varepsilon|)v_\varepsilon \partial_t(\varrho v_\varepsilon) \,\mathrm{d}x\mathrm{d}t + R_1 \\ &= \int_0^T \int_{\mathbb{T}_L^3} \xi(t)\Big(\partial_t\varrho\psi_n(|v_\varepsilon|)|v_\varepsilon|^2 + \varrho\partial_t F_n(|v_\varepsilon|)\Big) \,\mathrm{d}x\mathrm{d}t + R_1,\end{aligned}$$

where

$$R_1 = \int_0^T \int_{\mathbb{R}^3} \xi(t)\psi_n(|v_\varepsilon|)v_\varepsilon \Big(\big(\partial_t(\varrho v)\big)_\varepsilon - \partial_t(\varrho v_\varepsilon)\Big) \,\mathrm{d}x\mathrm{d}t.$$

Furthermore,

$$\begin{aligned}\int_0^T \int_{\mathbb{T}_L^3} &\xi(t)\psi_n(|v_\varepsilon|)v_\varepsilon \cdot (\operatorname{div}(\varrho u \otimes v))_\varepsilon \,\mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{T}_L^3} \xi(t)\psi_n(|v_\varepsilon|)v_\varepsilon \operatorname{div}(\varrho u \otimes v_\varepsilon) \,\mathrm{d}x\mathrm{d}t + R_2 \\ &= \int_0^T \int_{\mathbb{T}_L^3} \xi(t)\Big(- \partial_t\varrho\psi_n(|v_\varepsilon|)|v_\varepsilon|^2 + \partial_t\varrho F_n(|v_\varepsilon|)\Big) \,\mathrm{d}x\mathrm{d}t + R_2,\end{aligned}$$

where

$$R_2 = \int_0^T \int_{\mathbb{T}_L^3} \xi(t)\psi_n(|v_\varepsilon|)v_\varepsilon \Big(\big(\operatorname{div}(\varrho u \otimes v)\big)_\varepsilon - \operatorname{div}(\varrho u \otimes v_\varepsilon)\Big) \,\mathrm{d}x\mathrm{d}t.$$

In conclusion, we get

$$\begin{aligned}\int_0^T \int_{\mathbb{T}_L^3} &\xi(t)\partial_t(\varrho F_n(|v_\varepsilon|)) \,\mathrm{d}x\mathrm{d}t + R_1 + R_2 \\ &- \int_0^T \int_{\mathbb{T}_L^3} \xi(t)\psi_n(|v_\varepsilon|)v_\varepsilon \big(\operatorname{div} S\big)_\varepsilon \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{T}_L^3} \xi(t)\psi_n(|v_\varepsilon|)v_\varepsilon G_\varepsilon \,\mathrm{d}x\mathrm{d}t = 0.\end{aligned}$$

Since $v \in L^2(0,T;H^1)$, $v_\varepsilon \to v$ in $L^2(0,T;H^1)$. In particular, up to a subsequence $v_\varepsilon \to v$ almost everywhere. For $1 < p$ and $\delta > 0$ such that $\frac{p}{3} + \frac{p(1+\delta)}{2} \leq 1$ and $p(1+\delta) < 2$, by the definition of F_n we have

$$\begin{aligned}\|\xi'(t)\varrho F_n(|v_\varepsilon|)\|^p_{L^p((0,T)\times\mathbb{T}_L^3)} &\leq C_n\|\xi'\|_{L^\infty(0,T)} \int_0^T \int_{\mathbb{T}_L^3} \varrho^p |v_\varepsilon|^{p(1+\delta)} \,\mathrm{d}x\mathrm{d}t \\ &\leq C_n\|\xi'\|_{L^\infty(0,T)} \int_0^T \|\varrho\|^p_{L^3}\|v_\varepsilon\|^{p(1+\delta)}_{L^2} \,\mathrm{d}t \\ &\leq C(n,T)\|\varrho\|^p_{L^\infty(0,T;L^3)}\|v\|^{p(1+\delta)}_{L^2((0,T)\times\mathbb{T}_L^3)}.\end{aligned}$$

Therefore $\xi'(t)\varrho F_n(|v_\varepsilon|)$ converges in $L^1((0,T)\times\mathbb{T}_L^3)$ and we have

$$\lim_{\varepsilon\to 0} \int_0^T \int_{\mathbb{T}_L^3} \xi'(t)\varrho F_n(|v_\varepsilon|) \,\mathrm{d}x\mathrm{d}t = \int_0^T \int_{\mathbb{T}_L^3} \xi'(t)\varrho F_n(|v|) \,\mathrm{d}x\mathrm{d}t.$$

Since $G \in L^{4/3}((0,T)\times\mathbb{T}_L^3)$ and $\psi_n(|v_\varepsilon|)|v_\varepsilon| \le C_n|v_\varepsilon|^\delta$, we similarly have the convergence

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{T}_L^3}\xi(t)\psi_n(|v_\varepsilon|)v_\varepsilon G_\varepsilon\,\mathrm{d}x\mathrm{d}t = \int_0^T\int_{\mathbb{T}_L^3}\xi(t)\psi_n(|v|)v\cdot G\,\mathrm{d}x\mathrm{d}t.$$

Moreover, we have

$$\begin{aligned}\int_0^T\int_{\mathbb{T}_L^3}\xi(t)\psi_n(|v_\varepsilon|)v_\varepsilon(\mathrm{div}\,S)_\varepsilon\,\mathrm{d}x\mathrm{d}t &= -\int_0^T\int_{\mathbb{T}_L^3}\xi(t)S_\varepsilon:\nabla(\psi_n(|v_\varepsilon|)v_\varepsilon)\,\mathrm{d}x\mathrm{d}t\\ &= -\int_0^T\int_{\mathbb{T}_L^3}\xi(t)(\psi_n'(|v_\varepsilon|)|v_\varepsilon|+\psi_n(|v_\varepsilon|))S_\varepsilon:\nabla v_\varepsilon\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

By virtue of the estimates on $\sqrt{\varrho}\mathbb{D}u$ and $\Delta\sqrt{\varrho}$, the function S belongs to $L^2((0,T)\times\mathbb{T}_L^3$ and thus $S_\varepsilon\to S$ in $L^2((0,T)\times\mathbb{T}_L^3)$. Moreover ∇v_ε converges strongly in $L^2((0,T)\times\mathbb{T}_L^3)$ to ∇v, and $\psi_n'(|v_\varepsilon|)|v_\varepsilon|+\psi_n(|v_\varepsilon|)$ is uniformly bounded in $L^\infty((0,T)\times\mathbb{T}_L^3)$. Therefore we have the convergence

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{T}_L^3}\xi(t)\psi_n(|v_\varepsilon|)v_\varepsilon(\mathrm{div}\,S)_\varepsilon\,\mathrm{d}x\mathrm{d}t = -\int_0^T\int_{\mathbb{T}_L^3}\xi(t)S:\nabla(\psi_n(|v|)v)\,\mathrm{d}x\mathrm{d}t.$$

What is left is to show that $R_1, R_2\to 0$ as $\varepsilon\to 0$.

To do that, we use the following commutator lemmas (see e. g. Lemma 2.3 in [66]):

Lemma 4.19. *Let $f\in (W^{1,p}(\mathbb{R}^d))^d$ and $g\in L^q(\mathbb{R}^d)$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}<1$. Then*

$$\|(\mathrm{div}\,(fg))_\varepsilon - \mathrm{div}\,(fg_\varepsilon)\|_{L^r}\le C\|f\|_{W^{1,p}}\|g\|_{L^q}$$

for some $C>0$ independent of ε, and

$$(\mathrm{div}\,(fg))_\varepsilon - \mathrm{div}\,(fg_\varepsilon)\to 0 \quad in \quad L^r(\mathbb{R}^d).$$

Analogously with respect to time, we also have

Lemma 4.20. *Let $f_t\in L^p((0,T))$ and $g\in L^q(0,T)$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}<1$. Then*

$$\|(\partial_t(fg))_\varepsilon - \partial_t(fg_\varepsilon)\|_{L^r}\le C\|f_t\|_{L^p}\|g\|_{L^q}$$

and

$$(\partial_t(fg))_\varepsilon - \partial_t(fg_\varepsilon)\to 0 \quad in \quad L^r((0,T)).$$

Now we apply the above lemmas to R_1 and R_2. By Sobolev embedding, $v\in L^2(0,T;L^6)$. For R_2, we have

$$\nabla(\varrho u) = \varrho^{1/2}\nabla\varrho^{1/4}\otimes\varrho^{1/4}u + \sqrt{\varrho}\sqrt{\varrho}\nabla u\in L^2(0,T;L^{3/2}).$$

Therefore since $\psi_n(|v_\varepsilon|)|v_\varepsilon|\le C_n|v_\varepsilon|^{1/3}$, from Lemma 4.19 we get

$$\begin{aligned}&\Big|\int_{\mathbb{T}_L^3}\psi_n(|v_\varepsilon|)v_\varepsilon\big((\mathrm{div}\,(\varrho u\otimes v))_\varepsilon - \mathrm{div}\,(\varrho u\otimes v_\varepsilon)\big)\mathrm{d}x\Big|\\ &\qquad\le C_n\|v_\varepsilon\|_{L^2(\mathbb{T}_L^3)}^{1/3}\|(\mathrm{div}\,(\varrho u\otimes v))_\varepsilon - \mathrm{div}\,(\varrho u\otimes v_\varepsilon)\|_{L^{6/5}(\mathbb{T}_L^3)}\\ &\qquad\le C_n\|v\|_{L^\infty(0,T;L^2)}^{1/3}\|\nabla(\varrho u)\|_{W^{1,3/2}(\mathbb{T}_L^3)}\|v\|_{L^6(\mathbb{T}_L^3)}\end{aligned}$$

and the right hand side is integrable in time (note that the $L^\infty(0,T;L^2)$ estimate on v follows from the same regularity of $\sqrt{\varrho}u$). Thus from the Dominated Convergence Theorem

$$|R_2| \leq \|\xi\|_{L^\infty(0,T)} \int_0^T \int_{\mathbb{T}_L^3} \big|\psi_n(|v_\varepsilon|)v_\varepsilon\big((\mathrm{div}\,(\varrho u\otimes v))_\varepsilon - \mathrm{div}\,(\varrho u \otimes v_\varepsilon)\big)\big|\,\mathrm{d}x\mathrm{d}t \to 0$$

as $\varepsilon \to 0$. For R_1, first note that

$$v = \varrho^{-1/4}\phi(\varrho)\varrho^{1/4}u \in L^4((0,T)\times\mathbb{T}_L^3).$$

Moreover,

$$\partial_t\varrho = 4\sqrt{\varrho}\nabla\varrho^{1/4}\varrho^{1/4}u + \sqrt{\varrho}\sqrt{\varrho}\mathrm{div}\,u \in L^{3/2}((0,T)\times\mathbb{T}_L^3).$$

Then similarly as before we have

$$\begin{aligned}\left|\int_0^T \psi_n(|v_\varepsilon|)v_\varepsilon\big((\partial_t(\varrho v))_\varepsilon - \partial_t(\varrho v_\varepsilon)\big)\mathrm{d}t\right| &\leq C_n\|v_\varepsilon\|^{1/3}_{L^4(0,T)}\|(\partial_t(\varrho v))_\varepsilon - \partial_t(\varrho v_\varepsilon)\|_{L^{12/11}(0,T)}\\ &\leq C_n\|v\|^{4/3}_{L^4(0,T)}\|\partial_t\varrho\|_{L^{3/2}(0,T)}\end{aligned}$$

and from the Dominated Convergence Theorem

$$|R_1| \leq \|\xi\|_{L^\infty(0,T)}\int_0^T\int_{\mathbb{T}_L^3}\big|\psi_n(|v_\varepsilon|)v_\varepsilon\big((\partial_t(\varrho v))_\varepsilon - \partial_t(\varrho v_\varepsilon)\big)\big|\,\mathrm{d}x\mathrm{d}t \to 0.$$

Note that in this case we use the estimates in Lebesgue spaces on $(0,T)\times\mathbb{T}_L^3$ instead of Bochner spaces with different exponents over time and space, which allows us to change the order of integration.

In consequence, when $\varepsilon \to 0$ we derive (4.37) for any $\xi \in C_0^\infty(0,+\infty)$. □

The next step of the proof of Lemma 4.15 is based on application of the Weak Gronwall's Lemma (Lemma C.1). Let us rewrite (4.37) as

$$\begin{aligned}&-\int_0^T\int_{\mathbb{T}_L^3}\xi'(t)\varrho F_n(|v|)\,\mathrm{d}x\mathrm{d}t\\ &\qquad = -\int_0^T\int_{\mathbb{T}_L^3}\xi(t)\psi_n(|v|)v\cdot\phi(\varrho)\varrho\nabla(K_L*\varrho)\,\mathrm{d}x\mathrm{d}t - \int_0^T\xi(t)b(t)\,\mathrm{d}t,\end{aligned}$$

where $b(t)$ contains the rest of the terms from G and S, i. e.

$$\begin{aligned}b(t) = &\int_{\mathbb{T}_L^3}\psi_n(|v|)v\cdot\Big[\varrho^2u\phi'(\varrho)\mathrm{div}\,u + \varrho\nabla\phi(\varrho)\mathbb{D}u + \phi(\varrho)\varrho\nabla(K_L*\varrho)\\ &+ r_0u\phi(\varrho) + r_1\varrho|u|^2u\phi(\varrho) + \kappa\sqrt{\varrho}\nabla\phi(\varrho)\Delta\sqrt{\varrho} + 2\kappa\phi(\varrho)\nabla\sqrt{\varrho}\Delta\sqrt{\varrho}\Big]\,\mathrm{d}x\\ &+\int_{\mathbb{T}_L^3}\varrho\phi(\varrho)\left(\mathbb{D}u + \kappa\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\mathbb{I}\right):\nabla(\psi_n(|v|)v)\,\mathrm{d}x.\end{aligned}$$

We first focus on the nonlocal term. From the definition of K_L, we have

$$\begin{aligned}|\nabla(K_L*\varrho)(x)| &= \left|\frac{\phi_L(\cdot)}{|\cdot|^\alpha}*\nabla\varrho + \frac{1}{2}\int_{\mathbb{R}^3}\nabla\big(|x-y|^2\phi_L(x-y)\big)\varrho(y)\,\mathrm{d}y\right|\\ &\leq\left|\frac{\phi_L(\cdot)}{|\cdot|^\alpha}*\nabla\varrho\right| + C\int_{|x-y|<L}|x-y|\varrho(y)\,\mathrm{d}y,\end{aligned}$$

where in the last term we used (4.15). Therefore we get

$$
\begin{aligned}
&\left|\int_0^T \int_{\mathbb{T}_L^3} \xi(t)\psi_n(|v|)v\phi(\varrho)\varrho\nabla(K_L * \varrho)\,\mathrm{d}x\mathrm{d}t\right| \\
&\qquad\qquad \leq \int_0^T \xi(t)\int_{\mathbb{T}_L^3}\psi_n(|v|)|v|\varrho\left|\frac{\phi_L(\cdot)}{|\cdot|^\alpha} * \nabla\varrho\right|\,\mathrm{d}x\mathrm{d}t \\
&\qquad\qquad\quad + C\int_0^T \xi(t)\iint_{\mathbb{T}_L^{3\times 3}}\psi_n(|v(x)|)|v(x)|\varrho(x)\varrho(y)|x-y|\,\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&\qquad\qquad = A_1 + A_2.
\end{aligned}
$$

To estimate A_1, we use the estimates on Riesz potentials. For $f \in L^1(\mathbb{T}_L^3)$, let

$$
I_{3-\alpha}(f) = \int_{\mathbb{T}_L^3}\frac{f(y)}{|x-y|^\alpha}\,\mathrm{d}y.
$$

Then, in particular

$$
\|I_{3-\alpha}(f)\|_{L^{p^*}(\mathbb{T}_L^3)} \leq C\|f\|_{L^p(\mathbb{T}_L^3)} \quad \text{for} \quad p^* = \frac{3p}{3-(3-\alpha)p}.
$$

From (4.32), we have

$$
\|\nabla\varrho\|_{L^\infty(0,T;L^{3/2})} \leq 2\|\sqrt{\varrho}\|_{L^\infty(0,T;L^6)}\|\nabla\sqrt{\varrho}\|_{L^\infty(0,T;L^2)} \leq C
$$

and thus

$$
\|I_{3-\alpha}(\nabla\varrho)\|_{L^\infty(0,T;L^q)} \leq C\|\nabla\varrho\|_{L^\infty(0,T;L^{3/2})} \quad \text{for} \quad q = \frac{3\cdot 3/2}{3-(3-\alpha)\cdot 3/2} = \frac{3}{\alpha-1}
$$

(if $\alpha \leq 1$, then $q < \infty$).

Since $\psi_n(|v|)|v| \leq C + C|v|^{1+\delta}$, the integral in A_1 is estimated by

$$
\begin{aligned}
\int_{\mathbb{T}_L^3}|v|^{1+\delta}\varrho\left|\frac{\phi_L(\cdot)}{|\cdot|^\alpha} * \nabla\varrho\right|\,\mathrm{d}x &= \int_{\mathbb{T}_L^3}|\sqrt{\varrho}v|^{1+\delta}\varrho^{\frac{1-\delta}{2}}|I_{3-\alpha}(\nabla\varrho)|\,\mathrm{d}x \\
&\leq \|\sqrt{\varrho}v\|^{1+\delta}_{L^\infty(0,T;L^2)}\|\varrho\|^{\frac{1-\delta}{2}}_{L^\infty(0,T;L^3)}\|I_{3-\alpha}(\nabla\varrho)\|_{L^\infty(0,T;L^{\frac{3}{1-\delta}})} \\
&\leq C\|\sqrt{\varrho}v\|^{1+\delta}_{L^\infty(0,T;L^2)}\|\varrho\|^{\frac{1-\delta}{2}}_{L^\infty(0,T;L^3)}\|\nabla\varrho\|_{L^\infty(0,T;L^{3/2})},
\end{aligned}
$$

provided that $\frac{3}{1-\delta} \leq \frac{3}{\alpha-1}$ (which is vaild for $\alpha < 2$ and sufficiently small δ). In the end, we get $|A_1| \leq C$, where C depends on the right hand sides of (4.31) and (4.32) (in particular it does not depend on n, m, k, r_0, r_1 and κ).

For the term A_2, we use the following generalized Young inequality for convex functions:

$$
ab \leq F(a) + F^*(b), \quad a, b \in \mathbb{R}, \tag{4.40}
$$

where F^* is a convex conjugate of F, given by

$$
F^*(s) = \sup\{sz - F(z) : z \in \mathbb{R}\}.
$$

The proof of (4.40) is elementary, since straight from the definition of F^*

$$
ab - F(a) \leq \sup\{bz - F(z) :\ z \in \mathbb{R}\} = F^*(b).
$$

Applying this inequality to A_2, we get

$$\begin{aligned}A_2 =& C\int_0^T \xi(t)\iint_{\mathbb{T}_L^3\times\mathbb{T}_L^3} F_n'(|v(x)|)|x-y|\varrho(x)\varrho(y)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t\\ \leq& C\int_0^T \xi(t)\iint_{\mathbb{T}_L^3\times\mathbb{T}_L^3} F_n^*(F_n'(|v(x)|))\varrho(x)\varrho(y)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t\\ &+ C\int_0^T \xi(t)\iint_{\mathbb{T}_L^3\times\mathbb{T}_L^3} F_n(|x-y|)\varrho(x)\varrho(y)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t.\end{aligned}$$

To further simplify the estimate, we use the following Proposition:

Proposition 4.21. *If $F \in C^1(\mathbb{R})$ is strictly convex and such that*

$$zF'(z) \leq aF(z)$$

for some $a > 1$, then

$$F^*(F'(z)) \leq (a-1)F(z).$$

Proof. Fix $s \in F'(\mathbb{R})$ and let $g(z) = sz - F(z)$. Then

$$g'(z) = s - F'(z)$$

and as F' is increasing, g attains a maximum at $z^* = (F')^{-1}(s)$. In consequence, F^* is explicitly given by

$$F^*(s) = g(z^*) = s(F')^{-1}(s) - F((F')^{-1}(s)).$$

Therefore

$$F^*(F'(z)) = F'(z)(F')^{-1}(F'(z)) - F((F')^{-1}(F'(z))) = zF'(z) - F(z) \leq (a-1)F(z),$$

which finishes the proof. □

One can check that $zF_n'(z) \leq 4F_n(z)$ for sufficiently large n and thus it satisfies the assumptions of Proposition 4.21. Therefore finally we derive

$$A_2 \leq C\int_0^T \xi(t)\iint_{\mathbb{T}_L^3\times\mathbb{T}_L^3} \varrho F_n(|v|)\,\mathrm{d}x\mathrm{d}t + \int_0^T \xi(t)\iint_{\mathbb{T}_L^3\times\mathbb{T}_L^3} F_n(|x-y|)\varrho(x)\varrho(y)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t.$$

To close the estimate, we need to control the second term. To do this, we compute its derivative using the continuity equation and applying again the Young inequality. From the continuity equation, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{T}^3_L\times\mathbb{T}^3_L} & F_n(|x-y|)\varrho(x)\varrho(y)\,\mathrm{d}x\mathrm{d}y \\
&= \iint_{\mathbb{T}^3_L\times\mathbb{T}^3_L} F_n(|x-y|)(\partial_t\varrho(x)\varrho(y)+\varrho(x)\partial_t\varrho(y))\,\mathrm{d}x\mathrm{d}y \\
&= -\iint_{\mathbb{T}^3_L\times\mathbb{T}^3_L} F_n(|x-y|)\mathrm{div}_{\,x}(\varrho u)(x)\varrho(y)\,\mathrm{d}x\mathrm{d}y \\
&= 2\iint_{\mathbb{T}^3_L\times\mathbb{T}^3_L} F_n'(|x-y|)\frac{x-y}{|x-y|}\cdot u(x)\varrho(x)\varrho(y)\,\mathrm{d}x\mathrm{d}y \\
&= 2\iint_{\mathbb{T}^3_L\times\mathbb{T}^3_L} F_n'(|x-y|)\frac{x-y}{|x-y|}v(x)\varrho(x)\varrho(y)\,\mathrm{d}x\mathrm{d}y \\
&\quad + 2\iint_{\mathbb{T}^3_L\times\mathbb{T}^3_L} F_n'(|x-y|)\frac{x-y}{|x-y|}((1-\phi(\varrho))u)(x)\varrho(x)\varrho(y)\,\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Therefore applying Young inequality and Proposition 4.21, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{T}^3_L\times\mathbb{T}^3_L} & F_n(|x-y|)\varrho(x)\varrho(y)\,\mathrm{d}x\mathrm{d}y \\
&\le 2\iint_{\mathbb{T}^3_L\times\mathbb{T}^3_L} F_n^*(F_n'(|x-y|))\varrho(x)\varrho(y)\,\mathrm{d}x\mathrm{d}y \\
&\quad + 2\|\varrho\|_{L^1(\mathbb{T}^3_L)}\int_{\mathbb{R}^3}\varrho F_n(|v|)\,\mathrm{d}x \\
&\quad + 2\iint_{\mathbb{T}^3_L\times\mathbb{T}^3_L} F_n'(|x-y|)(1-\phi(\varrho(x)))|u(x)|\varrho(x)\varrho(y)\,\mathrm{d}x\mathrm{d}y \\
&\le C\iint_{\mathbb{T}^3_L\times\mathbb{T}^3_L} F_n(|x-y|)\varrho(x)\varrho(y)\,\mathrm{d}x\mathrm{d}y \\
&\quad + C\int_{\mathbb{T}^3_L}\varrho F_n(|v|)\,\mathrm{d}x \\
&\quad + C_n\iint_{\mathbb{T}^3_L\times\mathbb{T}^3_L} |x-y|^\delta(1-\phi(\varrho(x)))|u(x)|\varrho(x)\varrho(y)\,\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

In consequence we obtain

$$
\begin{aligned}
-\int_0^T \xi'(t)f(t)\,\mathrm{d}t \le{}& C\int_0^T \xi(t)f(t)\,\mathrm{d}t \\
&+ \int_0^T \xi(t)\left(-b(t)+C+C_n\iint_{\mathbb{T}^3_L\times\mathbb{T}^3_L}|u(x)||x-y|^\delta(1-\phi(\varrho(x)))\varrho(x)\varrho(y)\,\mathrm{d}x\mathrm{d}y\right),
\end{aligned}
$$

where

$$
f(t)=\int_{\mathbb{T}^3_L}\varrho F_n(|v|)\,\mathrm{d}x+\iint_{\mathbb{T}^3_L\times\mathbb{T}^3_L}F_n(|x-y|)\varrho(x)\varrho(y)\,\mathrm{d}x\mathrm{d}y
$$

and

$$
\begin{aligned}
b(t) =& \int_{\mathbb{T}_L^3} \psi_n(|v|)v \cdot \Big(\varrho^2 u\phi'(\varrho)\mathrm{div}\, u + \varrho\nabla\phi(\varrho)\mathbb{D}u + r_0 u\phi(\varrho) + r_1\varrho|u|^2 u\phi(\varrho) \\
&+ \kappa\sqrt{\varrho}\nabla\phi(\varrho)\Delta\sqrt{\varrho} + 2\kappa\phi(\varrho)\nabla\sqrt{\varrho}\Delta\sqrt{\varrho}\Big)\, \mathrm{d}x \\
&+ \int_{\mathbb{T}_L^3} \varrho\phi(\varrho)\left(\mathbb{D}u + \kappa\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\mathbb{I}\right) : \nabla(\psi_n(|v|)v)\, \mathrm{d}x \\
=& J_1(t) + J_2(t).
\end{aligned}
\tag{4.41}
$$

Now applying weak Gronwall's lemma (Lemma C.1) and using the continuity in time of $\sqrt{\varrho}$ and $\sqrt{\varrho}u$, we get for a. e. $t \in [0,T]$

$$
\begin{aligned}
\int_{\mathbb{T}_L^3} \varrho F_n(|v|)\, \mathrm{d}x +& \iint_{\mathbb{T}_L^3\times\mathbb{T}_L^3} F_n(|x-y|)\varrho(x)\varrho(y)\, \mathrm{d}x\mathrm{d}y \\
&\le e^{CT}\left(\int_{\mathbb{T}_L^3} \varrho_0 F_n(|v_0|)\, \mathrm{d}x + \iint_{\mathbb{T}_L^3\times\mathbb{T}_L^3} F_n(|x-y|)\varrho_0(x)\varrho_0(y)\, \mathrm{d}x\mathrm{d}y \right) \\
&\quad - e^{CT}\int_0^T b(t)\mathrm{d}t + CTe^{CT} \\
&\quad + C_n e^{CT}\int_0^T \iint_{\mathbb{T}_L^3\times\mathbb{T}_L^3} |u(x)||x-y|^\delta(1-\phi(\varrho(x)))\varrho(x)\varrho(y)\, \mathrm{d}x\mathrm{d}y\mathrm{d}t,
\end{aligned}
\tag{4.42}
$$

where the constant C depends on L, but does not depend on n, m, k, κ, r_0 and r_1.

4.4.1 Limit passage with $m \to \infty$

We now pass to the limit with m in (4.42), i. e. remove the truncation of ϱ at zero. Obviously

$$
v_m = \phi_m^0(\varrho)\phi_k^\infty(\varrho)u \to \phi_k^\infty(\varrho)u \quad \text{a. e.}
$$

and $|v_m| \le |u|$. Since $\varrho F_n(|u|) \le C_n\varrho|u|^{1+\delta}$ is integrable, from the dominated convergence theorem we have

$$
\int_{\mathbb{T}_L^3} \varrho F_n(|v_m|)\, \mathrm{d}x \to \int_{\mathbb{T}_L^3} \varrho F_n(\phi_k^\infty(\varrho)|u|)\, \mathrm{d}x
$$

as $m \to \infty$ and similarly

$$
\int_{\mathbb{T}_L^3} \varrho_0 F_n(|v_{0,m}|)\, \mathrm{d}x \to \int_{\mathbb{T}_L^3} \varrho_0 F_n(\phi_k^\infty(\varrho_0)|u_0|)\, \mathrm{d}x.
$$

Since for $\delta < 1$ the term $|u(x)||x-y|^\delta\varrho(x)\varrho(y)$ is integrable on $[0,T]\times\mathbb{T}_L^3\times\mathbb{T}_L^3$ by virtue of energy estimate (4.31), the last term in (4.42) converges to

$$
\int_0^T \iint_{\mathbb{T}_L^3\times\mathbb{T}_L^3} |u(x)||x-y|^\delta(1-\phi_k^\infty(\varrho))\varrho(x)\varrho(y)\, \mathrm{d}x\mathrm{d}y\mathrm{d}t.
$$

Now we deal with the terms J_1 and J_2 of $b(t)$ (as in (4.41)). The convergence in all the terms will be a consequence of the following Proposition:

Proposition 4.22. *If $\|a_m\|_{L^\infty((0,T)\times\Omega)} \leq C$, $a_m \to a$ a. e. and $f \in L^1((0,T)\times\Omega)$, then*

$$\int_0^T \int_\Omega \phi_m^0(\varrho) a_m f \; \mathrm{d}x\mathrm{d}t \to \int_0^T \int_\Omega af \; \mathrm{d}x\mathrm{d}t$$

and

$$\int_0^T \int_\Omega |\varrho(\phi_m^0)'(\varrho) a_m f| \; \mathrm{d}x\mathrm{d}t \to 0$$

as $m \to \infty$.

Proof. Note that $\phi_m^0(\varrho) \to 1$ a. e. as $m \to \infty$. Since $|\phi_m^0(\varrho)f - f| \leq 2|f|$ and $|a_m f - af| \leq |f|(\|a_m\|_{L^\infty} + \|a\|_{L^\infty})$, by Dominated Convergence Theorem,

$$\int_0^T \int_\Omega |\phi_m^0(\varrho)f - f| \; \mathrm{d}x\mathrm{d}t \to 0 \quad \text{and} \quad \int_0^T \int_\Omega |a_m f - af| \; \mathrm{d}x\mathrm{d}t \to 0.$$

Therefore

$$\begin{aligned}\left| \int_0^T \int_\Omega \phi_m^0(\varrho) a_m f \; \mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega af \; \mathrm{d}x\mathrm{d}t \right| & \\ \leq \|a_m\|_{L^\infty} \int_0^T \int_\Omega |\phi_m^0(\varrho)f - f| \; \mathrm{d}x\mathrm{d}t &+ \int_0^T \int_\Omega |a_m f - af| \; \mathrm{d}x\mathrm{d}t \to 0.\end{aligned}$$

For the second part of the Proposition, it is enough to notice that $|(\varrho\phi_m^0)'(\varrho)| \leq C$ and $(\phi_m^0)'(\varrho) \to 0$ a. e. Then again from the dominated convergence theorem,

$$\int_0^T \int_\Omega |\varrho(\phi_m^0)'(\varrho) a_m f| \; \mathrm{d}x\mathrm{d}t \to 0.$$

□

We apply the above Proposition to each of the terms in $b(t)$. First, note that

$$\begin{aligned}\nabla(\psi_n(|v_m|)v_m) &= \frac{\psi_n'(|v_m|)}{|v_m|} v_m \otimes v_m \nabla v_m + \psi_n(|v_m|)\nabla v_m \\ &= F_n''(|v_m|)\Big(\nabla\phi_m^0(\varrho) \otimes (\phi_k^\infty(\varrho)u_m) + \phi_m^0(\varrho)\big(\nabla\phi_k^\infty(\varrho) \otimes u_m + \phi_k^\infty(\varrho)\nabla u_m\big)\Big)\end{aligned}$$

and therefore

$$\begin{aligned}\int_0^T J_2(t) \; \mathrm{d}t &= \int_0^T \int_{\mathbb{R}^3} \varrho\phi(\varrho) \left(\mathbb{D}u + \kappa\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\mathbb{I}\right) : \nabla(\psi_n(|v_m|)v_m) \; \mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{T}_L^3} \phi_m^0(\varrho) a_m f_1 \; \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{T}_L^3} \varrho(\phi_m^0)'(\varrho) a_m f_2 \; \mathrm{d}x\mathrm{d}t,\end{aligned}$$

where

$$a_m = \phi_m^0(\varrho) F_n''(|v_m|)$$

and

$$\begin{aligned}f_1 &= \varrho\phi_k^\infty(\varrho)\left(\mathbb{D}u + \kappa\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\mathbb{I}\right) : \big(\nabla\phi_k^\infty(\varrho) \otimes u + \phi_k^\infty(\varrho)\nabla u\big), \\ f_2 &= \phi_k^\infty(\varrho)\left(\mathbb{D}u + \kappa\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\mathbb{I}\right) : (\nabla\varrho \otimes (\phi_k^\infty(\varrho)u)).\end{aligned}$$

By virtue of Proposition 4.22,

$$\lim_{m\to\infty}\int_0^T\int_{\mathbb{T}_L^3}\varrho\phi_{m,k}(\varrho)\left(\mathbb{D}u+\kappa\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\mathbb{I}\right):\nabla(\psi_n(|v_m|)v_m)\;\mathrm{d}x\mathrm{d}t$$
$$=\int_0^T\int_{\mathbb{T}_L^3}\varrho\phi_k^\infty(\varrho)\left(\mathbb{D}u+\kappa\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\mathbb{I}\right):\nabla(\psi_n(|\phi_k^\infty(\varrho)u|)\phi_k^\infty(\varrho)u)\;\mathrm{d}x\mathrm{d}t.$$

Now we deal with J_1. From (4.41), we see that

$$\begin{aligned}J_1=&\int_{\mathbb{T}_L^3}\psi_n(|v_m|)v_m\cdot\Big(\varrho^2u\phi'(\varrho)\operatorname{div}u+\varrho\nabla\phi(\varrho)\mathbb{D}u\\&+r_0u\phi(\varrho)+r_1\varrho|u|^2u\phi(\varrho)+\kappa\sqrt{\varrho}\nabla\phi(\varrho)\Delta\sqrt{\varrho}+2\kappa\phi(\varrho)\nabla\sqrt{\varrho}\Delta\sqrt{\varrho}\Big)\;\mathrm{d}x\\=&\int_{\mathbb{T}_L^3}\psi_n(|v_m|)\phi_m^0(\varrho)\cdot\phi_k^\infty(\varrho)u\cdot\Big(\varrho^2u\phi'(\varrho)\operatorname{div}u+\varrho\nabla\phi(\varrho)\mathbb{D}u\\&+r_0u\phi(\varrho)+r_1\varrho|u|^2u\phi(\varrho)+\kappa\sqrt{\varrho}\nabla\phi(\varrho)\Delta\sqrt{\varrho}+2\kappa\phi(\varrho)\nabla\sqrt{\varrho}\Delta\sqrt{\varrho}\Big)\;\mathrm{d}x.\end{aligned}$$

We will group all the terms in J_1 with respect to $\phi_m^0(\varrho)$ and $\varrho(\phi_m^0)'(\varrho)$. Let

$$b_m=\phi_m^0(\varrho)\psi_n(|v_m|),$$

$$g_1=\phi_k^\infty(\varrho)^2\left(\varrho|u|^2\operatorname{div}u+u\cdot\mathbb{D}u\cdot\nabla\varrho+\kappa u\cdot\frac{1}{\sqrt{\varrho}}\Delta\sqrt{\varrho}\right)$$

and

$$\begin{aligned}g_2=&\phi_k^\infty(\varrho)(\phi_k^\infty)'(\varrho)\varrho^2|u|^2\operatorname{div}u+\phi_k^\infty(\varrho)(\phi_k^\infty)'(\varrho)\varrho u\cdot\mathbb{D}u\cdot\nabla\varrho+r_0\phi_k^\infty(\varrho)^2|u|^2\\&+r_1\phi_k^\infty(\varrho)^2\varrho|u|^4+\kappa\phi_k^\infty(\varrho)u\cdot(\phi_k^\infty)'(\varrho)\sqrt{\varrho}\nabla\varrho\Delta\sqrt{\varrho}+2\kappa\phi_k^\infty(\varrho)^2u\cdot\nabla\sqrt{\varrho}\Delta\sqrt{\varrho}.\end{aligned}$$

Then

$$\int_0^T J_1(t)\;\mathrm{d}t=\int_0^T\int_{\mathbb{T}_L^3}\varrho(\phi_m^0)'(\varrho)b_mg_1\;\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{T}_L^3}\phi_m^0(\varrho)b_mg_2\;\mathrm{d}x\mathrm{d}t$$

and therefore from Proposition 4.22

$$\begin{aligned}\lim_{m\to\infty}\int_0^T J_1(t)\;\mathrm{d}t=&\int_0^T\int_{\mathbb{T}_L^3}\psi_n(|\phi_k^\infty(\varrho)u|)\phi_k^\infty(\varrho)u\cdot\Big(\varrho^2u(\phi_k^\infty)'(\varrho)\operatorname{div}u+\varrho\nabla\phi_k^\infty(\varrho)\mathbb{D}u\\&+r_0u\phi_k^\infty(\varrho)+r_1\varrho|u|^2u\phi_k^\infty(\varrho)\\&+\kappa\sqrt{\varrho}\nabla\phi_k^\infty(\varrho)\Delta\sqrt{\varrho}+2\kappa\phi_k^\infty(\varrho)\nabla\sqrt{\varrho}\Delta\sqrt{\varrho}\Big)\;\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Combining J_1 and J_2, in the end we get

$$\begin{aligned}\lim_{m\to\infty}\int_0^T b(t)\mathrm{d}t=&\int_0^T\int_{\mathbb{T}_L^3}\psi_n(|\phi_k^\infty(\varrho)u|)\phi_k^\infty(\varrho)u\cdot\Big(\varrho^2u(\phi_k^\infty)'(\varrho)\operatorname{div}u+\varrho\nabla\phi_k^\infty(\varrho)\mathbb{D}u\\&+r_0\phi_k^\infty(\varrho)u+r_1\phi_k^\infty(\varrho)\varrho|u|^2u\\&+\kappa\sqrt{\varrho}\nabla\phi_k^\infty(\varrho)\Delta\sqrt{\varrho}+2\kappa\phi_k^\infty(\varrho)\nabla\sqrt{\varrho}\Delta\sqrt{\varrho}\Big)\;\mathrm{d}x\mathrm{d}t\\&+\int_0^T\int_{\mathbb{T}_L^3}\varrho\phi_k^\infty(\varrho)\left(\mathbb{D}u+\kappa\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\mathbb{I}\right):\nabla(\psi_n(|\phi_k^\infty(\varrho)u|)\phi_k^\infty(\varrho)u)\;\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Since

$$\int_0^T \int_{\mathbb{T}_L^3} \psi_n(|\phi_k^\infty(\varrho)u|)(\phi_k^\infty)^2(\varrho)\left(r_0|u|^2 + r_1\varrho|u|^4\right)\,\mathrm{d}x\mathrm{d}t \geq 0,$$

after taking $m \to \infty$ we finally obtain the following estimate:

$$\begin{aligned}
\int_{\mathbb{T}_L^3} \varrho F_n(|v_k|)\,\mathrm{d}x + \iint_{\mathbb{T}_L^3\times\mathbb{T}_L^3} F_n(|x-y|)\varrho(x)\varrho(y)\,\mathrm{d}x\mathrm{d}y \\
\leq e^{CT}\left(\int_{\mathbb{T}_L^3} \varrho_0 F_n(|\phi_k^\infty(\varrho_0)u_0|)\,\mathrm{d}x + \iint_{\mathbb{T}_L^{3\times3}} F_n(|x-y|)\varrho_0(x)\varrho_0(y)\,\mathrm{d}x\mathrm{d}y\right) \\
- e^{CT}\int_0^T b(t)\mathrm{d}t + CTe^{CT} \\
+ C_n e^{CT}\int_0^T \iint_{\mathbb{T}_L^{3\times3}} |u(x)||x-y|^\delta(1-\phi_k^\infty(\varrho(x)))\varrho(x)\varrho(y)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t
\end{aligned} \tag{4.43}$$

for $v_k = \phi_k^\infty(\varrho)u$, where

$$\begin{aligned}
b(t) = &\int_{\mathbb{T}_L^3} \psi_n(|v_k|)v_k \cdot \Big(\varrho^2 u(\phi_k^\infty)'(\varrho)\operatorname{div} u + \varrho\nabla\phi_k^\infty(\varrho)\mathbb{D}u \\
&+ \kappa\sqrt{\varrho}\nabla\phi_k^\infty(\varrho)\Delta\sqrt{\varrho} + 2\kappa\phi_k^\infty(\varrho)\nabla\sqrt{\varrho}\Delta\sqrt{\varrho}\Big)\,\mathrm{d}x \\
&+ \int_{\mathbb{T}_L^3} \varrho\phi_k^\infty(\varrho)\left(\mathbb{D}u + \kappa\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\mathbb{I}\right) : \nabla(\psi_n(|v_k|)v_k)\,\mathrm{d}x \\
&= \tilde{J}_1 + \tilde{J}_2.
\end{aligned} \tag{4.44}$$

4.4.2 Limit passage with $\kappa \to 0$ and $k \to \infty$

We choose the parameters κ and k, so that we can pass to the limit in (4.43) and (4.30) with both of them at the same time. Fix $\delta \leq 2/3$ and let $k = \kappa^{-2/\delta}$. We have the following lemma:

Lemma 4.23. *If $(\varrho_\kappa, u_\kappa)$ is the solution to (4.30) and $v_\kappa = \phi_k^\infty(\varrho_\kappa)u_\kappa$, then $(\varrho_\kappa, u_\kappa)$ converges to a solution (ϱ, u) to (4.30) with $\kappa = 0$, and the limit satisfies for a. e. $\in [0,T]$*

$$\begin{aligned}
\int_{\mathbb{T}_L^3} \varrho F_n(|u|)\,\mathrm{d}x + \iint_{\mathbb{T}_L^3\times\mathbb{T}_L^3} F_n(|x-y|)\varrho(x)\varrho(y)\,\mathrm{d}x\mathrm{d}y \\
\leq C\left(\int_{\mathbb{T}_L^3} \varrho_0 F_n(|u_0|)\,\mathrm{d}x + \iint_{\mathbb{T}_L^3\times\mathbb{T}_L^3} F_n(|x-y|)\varrho_0(x)\varrho_0(y)\,\mathrm{d}x\mathrm{d}y\right) + C,
\end{aligned} \tag{4.45}$$

where C depends on T and on the right hand sides of (4.31) and (4.32).

Proof. Since $\|\partial_t(\varrho_\kappa, u_\kappa)\|_{L^2(0,T;W^{-1,4})} \leq C$, by Lemma 4.4, we have

$$\varrho_\kappa \to \varrho \quad \text{in} \quad \text{in} \quad C(0,T;L^{3/2})$$

and

$$\varrho_\kappa u_\kappa \to \varrho u \quad \text{in} \quad L^2(0,T;L^{3/2}).$$

In consequence

$$\varrho_\kappa u_\kappa \to \varrho u \quad \text{a. e.}$$

and for a. e. (t,x) such that $\varrho(t,x) \neq 0$, we get

$$u_\kappa = \frac{\varrho_\kappa u_\kappa}{\varrho_\kappa} \to u.$$

Therefore $v_\kappa(t,x) \to u(t,x)$ as well. On the other hand, for a. e. (t,x) where $\varrho(t,x) = 0$ we have

$$\varrho_\kappa F_n(|v_\kappa|) \leq C_n \varrho_\kappa^{1-\delta} |\varrho_\kappa u_\kappa|^\delta \to 0$$

as $\kappa \to 0$. In consequence $\varrho_\kappa F_n(|v_\kappa|) \to \varrho F_n(|u|)$ a. e. Then Fatou's Lemma yields

$$\int_{\mathbb{T}^3_L} \varrho F_n(|u|)\,\mathrm{d}x \leq \liminf_{\kappa\to 0} \int_{\mathbb{T}^3_L} \varrho_\kappa F_n(|v_\kappa|)\,\mathrm{d}x.$$

Let us pass to the limit with the terms on the right hand side of (4.43) one by one. Similarly as before,

$$\int_{\mathbb{T}^3_L} \varrho_0 F_n(|\phi_k^\infty(\varrho_0)u_0|)\,\mathrm{d}x \to \int_{\mathbb{T}^3_L} \varrho_0 F_n(|u_0|)\,\mathrm{d}x.$$

Since, similarly as in the previous limit passage,

$$|u_\kappa(x)||x-y|^\delta (1-\phi_k^\infty(\varrho_\kappa(x)))\varrho_\kappa(x)\varrho_\kappa(y)$$

is uniformly bounded in $L^p((0,T)\times\mathbb{T}^3_L\times\mathbb{T}^3_L)$ for some $p>1$, and convergent to 0 a. e., we have

$$\lim_{\kappa\to 0}\int_0^T \iint_{\mathbb{T}^3_L\times\mathbb{T}^3_L} |u_\kappa(x)||x-y|^\delta (1-\phi_k^\infty(\varrho_\kappa(x)))\varrho_\kappa(x)\varrho_\kappa(y)\,\mathrm{d}x\mathrm{d}y\mathrm{d}t = 0.$$

What is left is to estimate the terms $\tilde{J}_1, \tilde{J}_2$ defined in (4.44). For $\tilde{J}_1$, we respectively have the following bounds:

$$\begin{aligned}
&\Big|\int_0^T \int_{\mathbb{T}^3_L} \psi_n(|\phi_k^\infty(\varrho_\kappa)u_\kappa|)\phi_k^\infty(\varrho_\kappa)(\phi_k^\infty)'(\varrho_\kappa)\varrho_\kappa^2|u_\kappa|^2 \operatorname{div} u_\kappa\,\mathrm{d}x\mathrm{d}t\Big| \\
&\qquad\qquad \leq C_n k^{-\delta/4}\int_0^T\int_{\mathbb{T}^3_L} \varrho_\kappa^{1/4}|\varrho^{1/4}u_\kappa|^{1+\delta}\sqrt{\varrho_\kappa}\operatorname{div} u_\kappa\,\mathrm{d}x\mathrm{d}t \\
&\qquad\qquad \leq C(n,T)k^{-\delta/4}\|\varrho_\kappa\|^{1/4}_{L^\infty(0,T;L^{\frac{1}{1-\delta}})}\|\varrho_\kappa^{1/4}u_\kappa\|^{1+\delta}_{L^4(0,T;L^4)}\|\sqrt{\varrho_\kappa}\operatorname{div} u_\kappa\|_{L^2(0,T;L^2)},
\end{aligned}$$

$$\begin{aligned}
&\Big|\int_0^T \int_{\mathbb{T}^3_L} \psi_n(|\phi_k^\infty(\varrho_\kappa)u_\kappa|)\phi_k^\infty(\varrho_\kappa)u_\kappa\cdot\varrho_\kappa\nabla\phi_k^\infty(\varrho_\kappa)\mathbb{D}u_\kappa\,\mathrm{d}x\mathrm{d}t\Big| \\
&\leq C_n k^{-\delta/4}\kappa^{-1/4}\int_0^T\int_{\mathbb{T}^3_L} \varrho^{1/4}|\varrho_\kappa^{1/4}u_\kappa|^\delta|\kappa^{1/4}\nabla\varrho_\kappa^{1/4}||\sqrt{\varrho_\kappa}\mathbb{D}u_\kappa|\,\mathrm{d}x\mathrm{d}t \\
&\leq C(n,T)\kappa^{1/4}\|\varrho_\kappa\|^{1/4}_{L^\infty(0,T;L^{\frac{1}{1-\delta}})}\|\varrho_\kappa^{1/4}u_\kappa\|^\delta_{L^4(0,T;L^4)}\|\kappa^{1/4}\nabla\varrho_\kappa^{1/4}\|_{L^4(0,T;L^4)}\|\sqrt{\varrho_\kappa}\mathbb{D}u_\kappa\|_{L^2(0,T;L^2)},
\end{aligned}$$

$$\begin{aligned}
\kappa\Big|\int_0^T\int_{\mathbb{T}_L^3}&\psi_n(|\phi_k^\infty(\varrho_\kappa)u_\kappa|)\phi_k^\infty(\varrho_\kappa)u_\kappa\cdot\sqrt{\varrho_\kappa}\nabla\phi_k^\infty(\varrho_\kappa)\Delta\sqrt{\varrho_\kappa}\,\mathrm{d}x\mathrm{d}t\Big|\\
&\le C_n\frac{\kappa^{1/4}}{k^{\delta/4}}\int_0^T\int_{\mathbb{T}_L^3}\varrho_\kappa^{1/4}|\varrho_\kappa^{1/4}u_\kappa|^\delta|\kappa^{1/4}\nabla\varrho_\kappa^{1/4}||\kappa^{1/2}\Delta\sqrt{\varrho_\kappa}|\,\mathrm{d}x\mathrm{d}t\\
&\le C(n,T)\frac{\kappa^{1/4}}{k^{\delta/4}}\|\varrho_\kappa\|^{1/4}_{L^\infty(0,T;L^{\frac{1}{1-\delta}})}\|\varrho_\kappa^{1/4}u_\kappa\|^\delta_{L^4(0,T;L^4)}\times\\
&\quad\times\|\kappa^{1/4}\nabla\varrho_\kappa^{1/4}\|_{L^4(0,T;L^4)}\|\kappa^{1/2}\Delta\sqrt{\varrho_\kappa}\|_{L^2(0,T;L^2)}
\end{aligned}$$

and

$$\begin{aligned}
\kappa\Big|\int_0^T\int_{\mathbb{T}_L^3}&\psi_n(|v_\kappa|)v_\kappa\cdot\phi_k^\infty(\varrho_\kappa)\nabla\sqrt{\varrho_\kappa}\Delta\sqrt{\varrho_\kappa}\,\mathrm{d}x\mathrm{d}t\Big|\\
&\le C_n\kappa^{1/4}\int_0^T\int_{\mathbb{T}_L^3}|\varrho_\kappa^{1/4}u_\kappa|^\delta\varrho_\kappa^{1/4}|\kappa^{1/4}\nabla\varrho_\kappa^{1/4}||\kappa^{1/2}\Delta\sqrt{\varrho_\kappa}|\,\mathrm{d}x\mathrm{d}t\\
&\le C(n,T)\kappa^{1/4}\|\varrho_\kappa^{1/4}u_\kappa\|^\delta_{L^4(0,T;L^4)}\|\kappa^{1/4}\nabla\varrho_\kappa^{1/4}\|_{L^4(0,T;L^4)}\|\kappa^{1/2}\Delta\sqrt{\varrho_\kappa}\|_{L^2(0,T;L^2)}.
\end{aligned}$$

By the estimates in (4.31) and (4.32), all above terms converge to 0 with $k\to\infty,\kappa\to 0$ if $\delta\le 2/3$. To estimate $\tilde{J}_2$, we further write

$$\begin{aligned}
\tilde{J}_2&=\int_{\mathbb{T}_L^3}\varrho_\kappa\phi_k^\infty(\varrho_\kappa)\mathbb{D}u_\kappa:\nabla(\psi_n(|v_k|)v_k)\,\mathrm{d}x+\kappa\int_{\mathbb{T}_L^3}\varrho_\kappa\phi_k^\infty(\varrho_\kappa)\frac{\Delta\sqrt{\varrho_\kappa}}{\sqrt{\varrho_\kappa}}\mathrm{div}\,(\psi_n(|v_k|)v_k)\,\mathrm{d}x\\
&=S_1+S_2.
\end{aligned}$$

For S_1, we have

$$\begin{aligned}
\int_0^T S_1(t)\,\mathrm{d}t&=\int_0^T\int_{\mathbb{T}_L^3}\varrho_\kappa\phi_k^\infty(\varrho_\kappa)\mathbb{D}u_\kappa:\nabla(\psi_n(|v_\kappa|)v_\kappa)\,\mathrm{d}x\mathrm{d}t\\
&=\int_0^T\int_{\mathbb{T}_L^3}\frac{\psi_n'(|v_\kappa|)}{|v_\kappa|}\varrho_\kappa(\phi_k^\infty)^2(\varrho_\kappa)\mathbb{D}u_\kappa:(v_\kappa\otimes v_\kappa\nabla v_\kappa)\,\mathrm{d}x\mathrm{d}t\\
&\quad+\int_0^T\int_{\mathbb{T}_L^3}\psi_n(|v_\kappa|)\varrho_\kappa\phi_k^\infty(\varrho_\kappa)\mathbb{D}u_\kappa:\nabla v_\kappa\,\mathrm{d}x\mathrm{d}t\\
&=A_1+A_2.
\end{aligned}$$

The term A_1 is estimated as follows:

$$\begin{aligned}
|A_1|\le&\int_0^T\int_{\mathbb{T}_L^3}|\psi_n'(|v_\kappa|)||v_\kappa|\varrho_\kappa(\phi_k^\infty)^2(\varrho_\kappa)|\nabla u_\kappa|^2\,\mathrm{d}x\mathrm{d}t\\
&+\int_0^T\int_{\mathbb{T}_L^3}|\psi_n'(|v_\kappa|)||v_\kappa|^2\varrho_\kappa\phi_k^\infty(\varrho_\kappa)|\mathbb{D}u_\kappa||\nabla\phi_k^\infty(\varrho_\kappa)|\,\mathrm{d}x\mathrm{d}t\\
=&A_{1,1}+A_{1,2}.
\end{aligned}$$

Since $|\psi_n'(|v_\kappa|)||v_\kappa|\le 2$ independently of n, we have

$$A_{1,1}\le 2\int_0^T\int_{\mathbb{T}_L^3}\varrho_\kappa|\nabla u_\kappa|^2\,\mathrm{d}x\mathrm{d}t,$$

which is bounded uniformly by virtue of (4.31) and (4.32).

For $A_{1,2}$, from the definition of ψ we have

$$\psi_n'(z)z^2 = \begin{cases} \dfrac{2z^3}{1+z^2}, & z \le n, \\ -n\ln(1+z^2) + \dfrac{4nz^2 - 2z^3(1-n^2)}{(1+z^2)^2}, & z > n \end{cases}$$

Therefore

$$|\psi_n'(v_\kappa|)||v_\kappa|^2 \le C_n + C_n|v_\kappa|^\delta$$

for some $C_n > 0$ and thus

$$A_{1,2} \le C_n \int_0^T\int_{\mathbb{T}_L^3} \phi_k^\infty(\varrho_\kappa)(\phi_k^\infty)'(\varrho_\kappa)|\sqrt{\varrho_\kappa}\mathbb{D}u_\kappa||\nabla\varrho_\kappa^{1/4}|\left(\varrho_\kappa^{5/4} + \varrho_\kappa^{5/4-\delta/4}|\varrho_\kappa^{1/4}u_\kappa|^\delta\right)\,dxdt.$$

In consequence

$$\begin{aligned} A_{1,2} \le C_n\|\sqrt{\varrho_\kappa}\mathbb{D}u_\kappa\|_{L^2((0,T)\times\mathbb{T}_L^3)}\|\kappa^{1/4}\nabla\varrho_\kappa^{1/4}\|_{L^4((0,T)\times\mathbb{T}_L^3)}\Big(k^{-1/2}\kappa^{-1/4}\|\varrho_\kappa^{3/4}\|_{L^4((0,T)\times\mathbb{T}_L^3)} \\ + k^{-\delta/4}\kappa^{-1/4}\|\varrho_\kappa\|^{1/4}_{L^{\frac{1}{1-\delta}}((0,T)\times\mathbb{T}_L^3)}\|\varrho_\kappa^{1/4}u_\kappa\|_{L^4((0,T)\times\mathbb{T}_L^3)}\Big), \end{aligned}$$

which converges to 0 with $k\to\infty, \kappa\to 0$.

For A_2, we write

$$\begin{aligned} A_2 = &\int_0^T\int_{\mathbb{T}_L^3} \psi_n(|v_\kappa|)\varrho_\kappa\phi_k^\infty(\varrho_\kappa)|\mathbb{D}u_\kappa|^2\,dxdt \\ &+ \int_0^T\int_{\mathbb{T}_L^3} \psi_n(|v_\kappa|)\varrho_\kappa\phi_k^\infty(\varrho_\kappa)\mathbb{D}u_\kappa : (\nabla\phi_k^\infty(\varrho_\kappa)\otimes u_\kappa)\,dxdt \\ =&A_{2,1} + A_{2,2}. \end{aligned}$$

The term $A_{2,1}$ is positive and thus we can estimate it in (4.43) by 0. The term $A_{2,2}$ however, is estimated by

$$\begin{aligned} |A_{2,2}| \le &\int_0^T\int_{\mathbb{T}_L^3} \psi_n(|v_\kappa|)|v_\kappa|\varrho_\kappa|\mathbb{D}u_\kappa||\nabla\phi_k^\infty(\varrho_\kappa)|\,dxdt \\ \le &C(n,T)\frac{\kappa^{-1/4}}{k^{\delta/4}}\|\varrho_\kappa^{1/4}u_\kappa\|^\delta_{L^4(0,T;L^4)}\|\kappa^{1/4}\nabla\varrho_\kappa^{1/4}\|_{L^4(0,T;L^4)}\times \\ &\times\|\sqrt{\varrho_\kappa}\mathbb{D}u_\kappa\|_{L^2(0,T;L^2)}\|\varrho_\kappa\|^{1/4}_{L^\infty(0,T;L^{\frac{1}{1-\delta}})} \end{aligned}$$

which converges to 0 as well.

What is left is the term

$$\begin{aligned} \int_0^T S_2(t)\,dt =&\kappa\int_0^T\int_{\mathbb{T}_L^3} \phi_k^\infty(\varrho_\kappa)\sqrt{\varrho_\kappa}\Delta\sqrt{\varrho_\kappa}\mathbb{I} : \nabla(\psi_n(|v_\kappa|)v_\kappa)\,dxdt \\ =&\kappa\int_0^T\int_{\mathbb{R}^3} \phi_k^\infty(\varrho_\kappa)\sqrt{\varrho_\kappa}\Delta\sqrt{\varrho_\kappa}\frac{\psi_n'(|v_\kappa|)}{|v_\kappa|}\mathbb{I} : (v_\kappa\otimes v_\kappa\nabla v_\kappa)\,dxdt \\ &+\kappa\int_0^T\int_{\mathbb{R}^3} \psi_n(|v_\kappa|)\phi_k^\infty(\varrho_\kappa)\sqrt{\varrho_\kappa}\Delta\sqrt{\varrho_\kappa}\,\mathrm{div}\,v_\kappa\,dxdt \\ =&B_1 + B_2. \end{aligned}$$

Similarly as before,

$$
\begin{aligned}
|B_1| \leq & \kappa^{1/2} \|\kappa^{1/2} \Delta \sqrt{\varrho_\kappa}\|_{L^2(0,T;L^2)} \|\sqrt{\varrho_\kappa} \mathbb{D} u_\kappa\|_{L^2(0,T;L^2)} \\
& + \kappa^{1/4} \|\kappa^{1/2} \Delta \sqrt{\varrho_\kappa}\|_{L^2(0,T;L^2)} \|\varrho_\kappa^{1/4} u_\kappa\|_{L^4(0,T;L^4)} \|\kappa^{1/4} \nabla \varrho_\kappa^{1/4}\|_{L^4(0,T;L^4)}
\end{aligned}
$$

and

$$
\begin{aligned}
|B_2| \leq & C_n \kappa^{1/2} \|\kappa^{1/2} \Delta \sqrt{\varrho_\kappa}\|_{L^2(0,T;L^2)} \|\sqrt{\varrho_\kappa} \operatorname{div} u_\kappa\|_{L^2(0,T;L^2)} \\
& + C_n \kappa^{1/4} \|\kappa^{1/2} \Delta \sqrt{\varrho_\kappa}\|_{L^2(0,T;L^2)} \|\kappa^{1/4} \nabla \varrho_\kappa^{1/4}\|_{L^4(0,T;L^4)} \|\varrho_\kappa^{1/4} u_\kappa\|_{L^4(0,T;L^4)},
\end{aligned}
$$

which means that both $B_1, B_2 \to 0$.

In conclusion, after performing the limit passage $\kappa \to 0$ in (4.43), we end up with the estimate (4.45). $\square$

With the estimate (4.45) at hand, we are ready to finish the proof of Lemma 4.15. Since $F_n \nearrow F$, simply taking the limit $n \to \infty$ in (4.45), by monotone convergence theorem we derive (4.34). In order to show that the limit (ϱ, u) satisfies (4.30) with $\kappa = 0$, we focus only on $\varrho_\kappa \mathbb{D} u_\kappa$, since the limit passage in the remaining terms is performed in the same way as before. We will show that

$$
\sqrt{\varrho_\kappa} u_\kappa \to \sqrt{\varrho} u \quad \text{in} \quad L^2((0,T) \times \mathbb{T}_L^3).
$$

The proof is similar to the proof of Lemma 4.6. From the weak convergence of $\varrho_\kappa^{1/4} u_\kappa$ in $L^4((0,T) \times \mathbb{T}_L^3)$ it follows that

$$
\int_0^T \int_{\mathbb{T}_L^3} \varrho |u|^4 \, dx dt \leq \liminf_{\kappa \to 0} \int_0^T \int_{\mathbb{T}_L^3} \varrho_\kappa u_\kappa \, dx dt \leq C.
$$

From the pointwise convergence of $\varrho_\kappa u_\kappa$ it follows that $\sqrt{\varrho_\kappa} T_M(u_\kappa) \to \sqrt{\varrho} T_M(u)$ a.e. for the truncation T_M defined as in (4.14). Since $\sqrt{\varrho_\kappa} T_M(u_\kappa)$ is also uniformly bounded with respect to κ in $L^\infty(0,T;L^6)$, in particular it also converges in $L^2((0,T) \times \mathbb{T}_L^3)$ and therefore

$$
\begin{aligned}
\|\sqrt{\varrho_\kappa} u_\kappa - \sqrt{\varrho} u\|_{L^2((0,T)\times\mathbb{T}_L^3)} \leq & \|\sqrt{\varrho_\kappa} T_M(u_\kappa) - \sqrt{\varrho} T_M(u)\|_{L^2((0,T)\times\mathbb{T}_L^3)} \\
& + 2\|\sqrt{\varrho_\kappa} |u_\kappa| \mathbb{1}_{|u_\kappa|>M}\|_{L^2((0,T)\times\mathbb{T}_L^3)} + 2\|\sqrt{\varrho} |u| \mathbb{1}_{|u|>M}\|_{L^2((0,T)\times\mathbb{T}_L^3)}.
\end{aligned}
$$

The last two terms are estimated as follows:

$$
\int_0^T \int_{\mathbb{T}_L^3} \varrho_\kappa |u_\kappa|^2 \mathbb{1}_{|u_\kappa|>M} \, dx dt \leq \frac{1}{M^3} \int_0^T \int_{\mathbb{T}_L^3} \varrho_\kappa |u_\kappa|^4 \, dx dt \leq \frac{C}{M^3}
$$

and for the limit analogously. In conclusion,

$$
\limsup_{\kappa \to 0} \|\sqrt{\varrho_\kappa} u_\kappa - \sqrt{\varrho} u\|_{L^2((0,T)\times\mathbb{T}_L^3)} \leq \frac{C}{M^3}
$$

and the convergence follows by taking $M \to \infty$.

With the above convergence, we pass to the limit using the relation

$$
\varrho_\kappa \mathbb{D} u_\kappa = \mathbb{D}(\sqrt{\varrho_\kappa} \sqrt{\varrho_\kappa} u_\kappa) - \nabla \sqrt{\varrho_\kappa} \otimes \sqrt{\varrho_\kappa} u_\kappa
$$

and weak convergence of $\nabla \sqrt{\varrho_\kappa}$, coming from (4.32). $\square$

Remark 4.24. Note that at this point we cannot use (4.29) to define $\sqrt{\varrho}\nabla u$ as in the previous Section. However, from the uniform estimates we still have the convergence

$$\sqrt{\varrho_\kappa}\nabla u_\kappa \rightharpoonup \overline{\sqrt{\varrho}\nabla u} \quad \text{in} \quad L^2((0,T)\times\mathbb{T}^3_L).$$

Using the strong convergence of $\sqrt{\varrho_\kappa}$, we can then pass to the limit in the relation

$$\sqrt{\varrho_\kappa}\sqrt{\varrho_\kappa}\nabla u_\kappa = \nabla(\varrho_\kappa u_\kappa) - \nabla\sqrt{\varrho_\kappa}\otimes\sqrt{\varrho_\kappa}u_\kappa$$

and obtain

$$\sqrt{\varrho}\,\overline{\sqrt{\varrho}\nabla u} = \nabla(\varrho u) - \nabla\sqrt{\varrho}\otimes\sqrt{\varrho}u. \tag{4.46}$$

In particular, from (4.46) it follows that $\nabla(\varrho u)\in L^2(0,T;L^1)$. Similarly as before, we will drop the bars and define other differential operators of u analogously.

4.5 Limit passage with $r_0, r_1 \to 0$

In the previous section, we constructed the solutions to

$$\begin{aligned} \partial_t\varrho + \operatorname{div}(\varrho u) = 0 \\ \partial_t(\varrho u) + \operatorname{div}(\varrho u\otimes u) - \operatorname{div}(\varrho\mathbb{D}u) + \varrho\nabla(K_L*\varrho) = -r_0 u - r_1\varrho|u|^2u, \end{aligned} \tag{4.47}$$

defined on the torus $\mathbb{T}^3_L$, satisfying the estimates (4.31) and (4.32) with $\kappa = 0$, together with (4.34).

Our next goal is to perform the last limit passage $r_0, r_1 \to 0$ and in consequence obtain the solutions to (4.1) on the torus. The main tool to do so is the following lemma:

Lemma 4.25. *Let $\Omega = \mathbb{T}^3$ or $\mathbb{R}^3$. Assume the sequence (ϱ_n, u_n) satisfies uniformly the following estimates:*

$$\sup_{t\in[0,T]}\int_\Omega \varrho_n|u_n|^2\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega \varrho_n|\mathbb{D}u_n|^2\,\mathrm{d}x\mathrm{d}t \le C, \tag{4.48}$$

$$\sup_{t\in[0,T]}\int_\Omega |\nabla\sqrt{\varrho_n}|^2\,\mathrm{d}x + \int_0^T\int_\Omega \varrho_n|\nabla u_n - \nabla^T u_n|^2\,\mathrm{d}x\mathrm{d}t \le C, \tag{4.49}$$

$$\sup_{t\in[0,T]}\int_\Omega \varrho_n F(|u_n|)\,\mathrm{d}x \le C, \tag{4.50}$$

and

$$\|\partial_t\varrho_n\|_{L^\infty(0,T;W^{-1,3/2})}, \|\partial_t(\varrho_n u_n)\|_{L^2(0,T;W^{-2,4/3})} \le C.$$

Then up to a subsequence we have

$$\varrho_n \to \varrho \quad in \quad C(0,T;L^{3/2}_{\mathrm{loc}})$$

(in consequence also in particular $\sqrt{\varrho_n}\to\sqrt{\varrho}$ in $C(0,T;L^2_{\mathrm{loc}})$). Moreover, there exists a function m such that

$$\varrho_n u_n \to m \quad in \quad L^2(0,T;L^p_{\mathrm{loc}}), \quad p<3/2,$$

and $m(t,x) = 0$ a. e. on $\{(t,x) : \varrho(t,x) = 0\}$. In conclusion, there exists also a function u (defined uniquely on the set $\{(t,x) : \varrho(t,x)\neq 0\}$) such that $m = \varrho u$, and moreover

$$\sqrt{\varrho_n}u_n \to \sqrt{\varrho}u \quad in \quad L^2_{\mathrm{loc}}([0,T]\times\Omega).$$

Lemma 4.25 combines together the consecutive limit passages performed in [75] in order to show stability of solutions to (4.1) without the nonlocal term. However, for completeness and for the reader's convenience, we also present the proof below:

Proof. Since

$$\|\nabla \varrho_n\|_{L^\infty(0,T;L^{3/2})} \le 2\|\sqrt{\varrho_n}\|_{L^\infty(0,T;L^6)}\|\nabla\sqrt{\varrho_n}\|_{L^\infty(0,T;L^2)},$$

the strong convergence of ϱ_n follows straight from the Aubin-Lions-Simon lemma. For the momentum, we have

$$\nabla(\varrho_n u_n) = 2\nabla\sqrt{\varrho_n} \otimes \sqrt{\varrho_n} u_n + \sqrt{\varrho_n}\sqrt{\varrho_n}\nabla u_n$$

and thus

$$\|\nabla(\varrho_n u_n)\|_{L^2(0,T;L^1)} \le C.$$

Moreover,

$$\|\varrho_n u_n\|_{L^\infty(0,T;L^{3/2}_{\mathrm{loc}})} \le \|\sqrt{\varrho_n}\|_{L^\infty(0,T;L^6_{\mathrm{loc}})}\|\sqrt{\varrho_n} u_n\|_{L^\infty(0,T;L^2)},$$

and thus the desired convergence again follows from the Aubin-Lions lemma.

By the above strong convergences, we can extract the subsequence (indexed again by n) such that ϱ_n and $\varrho_n u_n$ converge a. e. Denoting $\varrho_n u_n = m_n$, we will now show that the limit m is zero whenever $\varrho = 0$. From the estimate (4.48) and Fatou's lemma,

$$\int_\Omega \liminf_{n\to\infty} \frac{|m_n|^2}{\varrho_n} \, \mathrm{d}x < \infty$$

and thus $m(t,x) = 0$ a.e. on the set $\{\varrho(t,x) = 0\}$. Denoting $u = \frac{m}{\varrho}$ on the set $\{\varrho(t,x) \neq 0\}$ and $u = 0$ on $\{\varrho(t,x) = 0\}$, we get $m = \varrho u$.

With the notion of u, we are now ready to prove the last part of the Lemma. Here, the reasoning is again similar to the proof of Lemma 4.6 and the end of the previous Section, however this time we will use (4.50) instead of the estimates for $\varrho^{1/4}u$. From a. e. convergence of ϱ_n and $\varrho_n u_n$, on the set $\{\varrho(t,x) \neq 0\}$ we have

$$u_n(t,x) = \frac{\varrho_n(t,x) u_n(t,x)}{\varrho_n(t,x)} \to u(t,x) \quad \text{a. e.}$$

Then from Fatou's lemma we also have

$$\int_\Omega \varrho F(|u|) \, \mathrm{d}x = \int_{\{\varrho(t,x)\neq 0\}} \varrho F(|u|) \, \mathrm{d}x \le \liminf_{n\to\infty} \int_\Omega \varrho_n F(|u_n|) \, \mathrm{d}x < \infty. \tag{4.51}$$

On the other hand, defining the truncation operator T_M as in (4.14), for (t,x) such that $\varrho(t,x) = 0$ we get

$$\sqrt{\varrho_n}|T_M(u_n)| \le M\sqrt{\varrho_n} \to 0 = \sqrt{\varrho}T_M(u).$$

In consequence $\sqrt{\varrho_n}T_M(u_n) \to \sqrt{\varrho}T_M(u)$ a. e. for any $M > 0$. Since $\sqrt{\varrho_n}T_M(u_n)$ is bounded uniformly with respect to n in $L^\infty(0,T;L^6)$, in particular it also converges in $L^2_{\mathrm{loc}}([0,T]\times\Omega)$. Then, fixing a compact set $V \subseteq \Omega$, we have

$$\begin{aligned} \|\sqrt{\varrho_n}u_n - \sqrt{\varrho}u\|_{L^2([0,T]\times V)} \le{}& \|\sqrt{\varrho_n}T_M(u_n) - \sqrt{\varrho}T_M(u)\|_{L^2([0,T]\times V)} \\ &+ 2\|\sqrt{\varrho_n}|u_n|\mathbb{1}_{|u_n|>M}\|_{L^2([0,T]\times V)} + 2\|\sqrt{\varrho}|u|\mathbb{1}_{|u|>M}\|_{L^2([0,T]\times V)}. \end{aligned}$$

We estimate the last two terms as follows:

$$\int_0^T \int_V \varrho_n |u_n|^2 \mathbb{1}_{|u_n|>M} \, dxdt \le \frac{1}{\ln(1+M^2)} \int_0^T \int_\Omega \varrho_n |u_n|^2 \ln(1+|u_n|^2) \, dxdt,$$

and analogously for the limit. Therefore using the estimates (4.50) and (4.51) we get

$$\limsup_{n\to\infty} \|\sqrt{\varrho_n} u_n - \sqrt{\varrho} u\|_{L^2_{\mathrm{loc}}([0,T]\times\Omega)} \le \frac{C}{\ln(1+M^2)}$$

and taking $M \to \infty$ we get the desired convergence. □

In our case, let $r = r_0 = r_1$ and (ϱ_r, u_r) be the corresponding solution to (4.47). From (4.31), (4.32) and (4.34), we have the required uniform estimates. Note that even though the term $\sqrt{\varrho}\nabla u$ is defined only via (4.46), by virtue of Remark 4.24 it still provides the bound on $\|\nabla(\varrho u)\|_{L^2(0,T;L^1)}$, which is enough to apply Lemma 4.25. For the time regularity, from the continuity equation, we have

$$\|\partial_t \varrho_r\|_{W^{-1,3/2}(\mathbb{T}^3_L)} \le \|\sqrt{\varrho_r}\|_{L^6(\mathbb{T}^3_L)} \|\sqrt{\varrho_r} u_r\|_{L^2(\mathbb{T}^3_L)}.$$

For $\partial_t(\varrho_r u_r)$, the term of the highest order is $\operatorname{div}(\varrho_r u_r \otimes u_r)$. Since for $\varphi \in W^{2,4}(\Omega)$

$$\int_\Omega \operatorname{div}(\varrho_r u_r \otimes u_r) \cdot \varphi \, dx = -\int_\Omega (\sqrt{\varrho_r} u_r \otimes \sqrt{\varrho_r} u_r) : \nabla\varphi \, dx \le \|\varrho_r |u_r|^2\|_{L^1(\Omega)} \|\nabla\varphi\|_{L^\infty(\Omega)},$$

we get the bound

$$\|\partial_t(\varrho_r u_r)\|_{L^2(0,T;W^{-2,4/3})} \le C.$$

By Lemma 4.25, we can pass to the limit in all terms in the weak formulation of (4.47). In the convective term we use strong convergence of $\sqrt{\varrho_r} u_r$. To pass to the limit in the viscous stress tensor, we proceed analogously as in the previous limit passage and the limit is defined via relation (4.46).

For the nonlocal term, similarly as before we use the fact that $K_L \in L^p(\mathbb{T}^3_L)$ for $p < 3/\alpha$. Since $\nabla\varrho_r$ is bounded in $L^\infty(0,T;L^{3/2})$, up to a subsequence we get

$$K_L * \nabla\varrho_r \to K_L * \nabla\varrho \quad \text{in} \quad L^\infty(0,T;L^{\frac{3}{\alpha-1}}).$$

The convergence of $\varrho_r \nabla(K_L * \varrho_r)$ follows thus from the strong convergence of ϱ_r. For the rest of the terms, we have

$$r_0 \left| \int_0^T \int_{\mathbb{T}^3_L} u_r \varphi \, dxdt \right| \le \sqrt{r_0} \|\sqrt{r_0} u_r\|_{L^2([0,T]\times\mathbb{T}^3_L)} \|\varphi\|_{L^2([0,T]\times\mathbb{T}^3_L)} \to 0$$

and

$$r_1 \left| \int_0^T \int_{\mathbb{T}^3_L} \varrho_r |u_r|^2 u_r \varphi \, dxdt \right| \le r_1^{1/4} \|\varrho_r\|^{1/4}_{L^\infty(0,T;L^3)} \|r_1^{1/4} \varrho_r^{1/4} u_r\|^3_{L^4(0,T;L^4)} \|\varphi\|_{L^\infty(0,T;L^6)} \to 0.$$

Using the weak lower semicontinuity of the norm, Fatou's lemma and (4.51), the limit solution also satisfies the inequalities (4.31), (4.32) and (4.34) with $r_0 = r_1 = \kappa = 0$.

Note that up to this point we obtained the solutions for initial conditions satisfying

$$\tilde{\varrho}_{0,L} = \varrho_{0,L} + \frac{1}{m_1} \quad \text{and} \quad \sqrt{\tilde{\varrho}_{0,L}} u_{0,L} \in L^\infty(\mathbb{T}^3_L).$$

However, Lemma 4.25 in particular provides the sequential stability of solutions. Repeating the above reasoning, we are able to pass to the limit with $m_1 \to \infty$ and with the truncation of initial data. In conclusion, we constructed weak solutions to the system

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho u) = 0, \\ \partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u) - \operatorname{div}(\varrho \mathbb{D} u) + \varrho \nabla (K_L * \varrho) = 0, \end{aligned} \tag{4.52}$$

with the initial data $(\varrho_{0,L}, u_{0,L})$ defined as in Section 4.3.1, satisfying the energy estimate

$$\sup_{t\in[0,T]} E(\varrho, u) + \int_0^T \int_{\mathbb{T}_L^3} \varrho |\mathbb{D} u|^2 \mathrm{d}x\mathrm{d}t \le E(\varrho_0, u_0) \tag{4.53}$$

for

$$E(\varrho, u) = \frac{1}{2} \int_{\mathbb{T}_L^3} \varrho |u|^2 + \varrho (K_L * \varrho)\, \mathrm{d}x,$$

the Bresch–Desjardins estimates

$$\begin{aligned} &\sup_{t\in[0,T]} \int_{\mathbb{T}_L^3} |\nabla \sqrt{\varrho}|^2\, \mathrm{d}x + \frac{1}{8} \int_0^T \int_{\mathbb{T}_L^3} \varrho |\nabla u - \nabla^T u|^2 \mathrm{d}x \\ &\qquad \le 3E(\varrho_{0,L}, u_{0,L}) + \int_{\mathbb{T}_L^3} |\nabla \sqrt{\varrho_{0,L}}|^2 \mathrm{d}x + CT \|\varrho_{0,L}\|^2_{L^1(\mathbb{T}_L^3)} \end{aligned} \tag{4.54}$$

and

$$\begin{aligned} &\sup_{t\in[0,T]} \left(\int_{\mathbb{T}_L^3} \varrho F(|u|)\, \mathrm{d}x + \iint_{\mathbb{T}_L^3 \times \mathbb{T}_L^3} F(|x-y|) \varrho(x) \varrho(y)\, \mathrm{d}x\mathrm{d}y \right) \\ &\qquad \le C + C \left(\int_{\mathbb{T}_L^3} \varrho_{0,L} F(|u_{0,L}|) + \iint_{\mathbb{T}_L^3 \times \mathbb{T}_L^3} F(|x-y|) \varrho_{0,L}(x) \varrho_{0,L}(y)\, \mathrm{d}x\mathrm{d}y \right), \end{aligned} \tag{4.55}$$

where C depends on $E(\varrho_{0,L}, u_{0,L})$ and the right hand side of (4.54).

4.6 Expansion of the torus

Having the solutions to (4.52) defined on the torus $\mathbb{T}_L^3$, together with estimates (4.53)–(4.55), we can now pass to the limit with $L \to \infty$ and in consequence obtain the solutions on the whole space $\mathbb{R}^3$. Let (ϱ_L, u_L) be the solutions to (4.52) and by $(\tilde{\varrho}_L, \tilde{u}_L)$ we will denote (ϱ_L, u_L) extended by zero outside the torus. Using the properties of $\varrho_{0,L}$ described in Lemma 4.7, we have the following estimates, uniform in L:

$$\begin{aligned} &\sup_{t\in[0,T]} \frac{1}{2} \int_{\mathbb{R}^3} \left(\tilde{\varrho}_L |\tilde{u}_L|^2 + \tilde{\varrho}_L (K_L * \tilde{\varrho}_L) \right) \mathrm{d}x + \int_0^T \int_{[-L,L]^3} \tilde{\varrho}_L |\mathbb{D} \tilde{u}_L|^2 \mathrm{d}x\mathrm{d}t \\ &\qquad \le \frac{1}{2} \int_{\mathbb{R}^3} \varrho_0 |u_0|^2 + \varrho_0 (K * \varrho_0)\, \mathrm{d}x \end{aligned} \tag{4.56}$$

and

$$\begin{aligned} &\sup_{t\in[0,T]} \int_{[-L,L]^3} |\nabla \sqrt{\tilde{\varrho}_L}|^2\, \mathrm{d}x + \frac{1}{8} \int_0^T \int_{[-L,L]^3} \tilde{\varrho}_L |\nabla \tilde{u}_L - \nabla^T \tilde{u}_L|^2 \mathrm{d}x \\ &\le \int_{\mathbb{R}^3} \varrho_0 |u_0|^2 + \varrho_0 (K * \varrho_0)\, \mathrm{d}x + \int_{\mathbb{R}^3} |\nabla \sqrt{\varrho_0}|^2 \mathrm{d}x + \frac{C}{L^2} \|\varrho_0\|^{1/2}_{L^1(\mathbb{R}^3)} + CT \|\varrho_0\|^2_{L^1(\mathbb{R}^3)}. \end{aligned} \tag{4.57}$$

From Lemma 4.7 it also follows that the right hand sides of (4.53) and (4.54) are independent of L. As a consequence, the constant C in (4.55) does not depend on L as well, and we get the uniform bound

$$\sup_{t\in[0,T]}\left(\int_{\mathbb{R}^3}\tilde\varrho_L F(|\tilde u_L|)\,\mathrm{d}x+\iint_{\mathbb{R}^3\times\mathbb{R}^3}F(|x-y|)\tilde\varrho_L(x)\tilde\varrho_L(y)\,\mathrm{d}x\mathrm{d}y\right)\le C \tag{4.58}$$

for C depending on T and ϱ_0, u_0.

Now, let $V\subset\mathbb{R}^3$ be compact. Then $V\subset[-L,L]^3$ for sufficiently large L and the estimates (4.56), (4.57) and (4.58) provide the uniform estimates on $(\tilde\varrho_L,\tilde u_L)$ needed in Lemma 4.15 for $\Omega=V$. Moreover, using analogous arguments as in the previous section,

$$\|\partial\tilde\varrho_L\|_{L^\infty(0,T;W^{-1,3/2}(V))},\|\partial_t(\tilde\varrho_L,\tilde u_L)\|_{L^2(0,T;W^{2,4/3}(V))}\le C.$$

Therefore up to a subsequence we get the convergence from Lemma 4.15 for $\Omega=V$. By the arbitrary choice of V and applying the diagonal method, we finally get

$$\begin{aligned}
\sqrt{\tilde\varrho_L}\to\sqrt{\varrho}\quad&\text{in}\quad C(0,T;L^2_{\mathrm{loc}}),\\
\tilde\varrho_L\to\varrho\quad&\text{in}\quad C(0,T;L^{3/2}_{\mathrm{loc}}),\\
\tilde\varrho_L\tilde u_L\to\varrho u\quad&\text{in}\quad L^2(0,T;L^{3/2}_{\mathrm{loc}}),\\
\sqrt{\tilde\varrho_L}u_L\to\sqrt{\varrho}u\quad&\text{in}\quad L^2_{\mathrm{loc}}([0,T]\times\mathbb{R}^3).
\end{aligned}$$

Similarly from the Banach-Alaoglu theorem we get

$$\nabla\sqrt{\tilde\varrho_L}\mathbb{1}_{[-L,L]^d}\rightharpoonup^*\nabla\sqrt{\varrho}\quad\text{in}\quad L^\infty(0,T;L^2_{\mathrm{loc}}).$$

Then for the term $\tilde\varrho_L\mathbb{D}u_L$ the relation (4.46) again provides convergence in the sense of distributions.

4.6.1 Convergence of the nonlocal term

The obtained convergence allows us to pass to the limit with $(\tilde\varrho_L,\tilde u_L)$ in the weak formulation of (4.52) in all terms except the nonlocal one. Note that since K is unbounded, as $L\to\infty$ we lose any compactness properties of K_L. However, due to (4.56) we are able to show that

Lemma 4.26. *We have*

$$\tilde\varrho_L(\nabla K_L*\tilde\varrho_L)\to\varrho(\nabla K*\varrho)\quad\textit{in}\quad L^1([0,T]\times\mathbb{R}^3)$$

Proof. First, note that from the strong convergence of $\tilde\varrho_L$ it follows that up to a subsequence

$$\tilde\varrho_L\to\varrho\quad\text{a. e. in}\quad[0,T]\times\mathbb{R}^3.$$

From (4.56) in particular we have that

$$\sup_{t\in[0,T]}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\tilde\varrho_L(t,x)\tilde\varrho_L(t,y)|x-y|^2\phi_L(x-y)\,\mathrm{d}x\mathrm{d}y\le C.$$

Therefore by Fatou's lemma

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \varrho(t,x)\varrho(t,y)|x-y|^2 \,\mathrm{d}x\mathrm{d}y \\ \leq \liminf_{L\to\infty} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \tilde{\varrho}_L(t,x)\tilde{\varrho}_L(t,y)|x-y|^2\phi_L(x-y)\,\mathrm{d}x\mathrm{d}y \leq C. \quad (4.59)$$

Since $\nabla\sqrt{\varrho_L}$ is bounded in $L^\infty(0,T;L^2_{loc})$, we also have the bound on $\|\varrho_L\|_{L^\infty(0,T;L^3_{loc})}$ and in consequence

$$\tilde{\varrho}_L \to \varrho \quad \text{in} \quad C(0,T;L^p_{\mathrm{loc}})$$

for any $3/2 < p < 3$.

Now fix $R > 0$ and denote by B_R the ball of radius R centered in zero. From the definition of K_L and the strong convergence of $\tilde{\varrho}_L$ we know that $\nabla K_L \to K$ and $\tilde{\varrho}_L \to \varrho$ a. e. Moreover, $\nabla K_L \mathbb{1}_{B_R}$ is uniformly bounded with respect to L in $L^p(\mathbb{R}^3)$ for $p < \frac{3}{\alpha+1}$. Therefore we also have

$$\|\tilde{\varrho}_L(\nabla K_L\mathbb{1}_{B_R} * \varrho_L)\|_{L^\infty(0,T;L^q)} \leq \|\nabla K_L\mathbb{1}_{B_R}\|_{L^p(\mathbb{R}^3)}\|\tilde{\varrho}_L\|^2_{L^\infty(0,T;L^3)}$$

for $q < \frac{3}{\alpha}$ and in consequence

$$\tilde{\varrho}_L(\nabla K_L\mathbb{1}_{B_R} * \tilde{\varrho}_L) \to \varrho(\nabla K\mathbb{1}_{B_R} * \varrho) \quad \text{in} \quad L^1([0,T]\times\mathbb{R}^3).$$

We will now estimate the rest. We have

$$I = \left|\int_{\mathbb{R}^3} \tilde{\varrho}_L\left((\nabla K_L\left((1-\mathbb{1}_{B_R}) * \tilde{\varrho}_L\right)\,\mathrm{d}x\right| \\ \leq \iint_{|x-y|>R} \tilde{\varrho}_L(t,x)\tilde{\varrho}_L(t,y)|\nabla K_L(x-y)|\,\mathrm{d}x\mathrm{d}y.$$

Using the definition of K_L and ϕ_L, we have

$$\begin{aligned}|\nabla K_L(x-y)| \leq& \phi_L(x-y)\left(\frac{1}{|x-y|^{\alpha+1}} + |x-y|\right) \\ &+ |\nabla\phi_L(x-y)|\left(\frac{1}{|x-y|^{\alpha}} + \frac{1}{2}|x-y|^2\right) \\ \leq& \frac{1}{|x-y|^{\alpha+1}} + \phi_L(x-y)|x-y| + \frac{C}{L}\left(\frac{1}{|x-y|^{\alpha}} + \frac{1}{2}|x-y|^2\right).\end{aligned}$$

Therefore

$$\begin{aligned}I \leq& \frac{1}{R^{\alpha+1}}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \tilde{\varrho}_L(t,x)\tilde{\varrho}_L(t,y)\,\mathrm{d}x\mathrm{d}y + \iint_{|x-y|>R} \phi_L(x-y)\tilde{\varrho}_L(t,x)\tilde{\varrho}_L(t,y)|x-y|\,\mathrm{d}x\mathrm{d}y \\ &+ \frac{C}{LR^\alpha}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \tilde{\varrho}_L(t,x)\tilde{\varrho}_L(t,y)\,\mathrm{d}x\mathrm{d}y + \frac{C}{L}\iint_{|x-y|>R} \tilde{\varrho}_L(t,x)\tilde{\varrho}_L(t,y)|x-y|^2\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Let us now estimate all terms on the right hand side. For the first and the third term, we have

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \tilde{\varrho}_L(t,x)\tilde{\varrho}_L(t,y)\,\mathrm{d}x\mathrm{d}y = \|\varrho_L\|^2_{L^1(\mathbb{T}^3_L)} = \|\varrho_{0,L}\|^2_{L^1(\mathbb{T}^3_L)} \leq \|\varrho_0\|^2_{L^1(\mathbb{R}^3)}.$$

For the second term, by (4.56) we get

$$\iint_{|x-y|>R} \phi_L(x-y)\tilde{\varrho}_L(t,x)\tilde{\varrho}_L(t,x)|x-y|\,\mathrm{d}x\mathrm{d}y \\ \leq \frac{1}{R}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \phi_L(x-y)\tilde{\varrho}_L(t,x)\tilde{\varrho}_L(t,y)|x-y|^2\,\mathrm{d}x\mathrm{d}y \leq \frac{C}{R}.$$

Finally we estimate the last term using (4.58) as

$$\iint_{\mathbb{R}^{3\times 3}} \tilde{\varrho}_L(t,x)\tilde{\varrho}_L(t,y)|x-y|^2\mathbb{1}_{|x-y|>R}\,\mathrm{d}x\mathrm{d}y \\ \leq \frac{1}{\ln(1+R^2)}\iint_{\mathbb{R}^{3\times 3}} \tilde{\varrho}_L(t,x)\tilde{\varrho}_L(t,y)F(|x-y|)\,\mathrm{d}x\mathrm{d}y \leq \frac{C}{\ln(1+R^2)}.$$

In consequence,

$$I \leq \frac{C}{R} + \frac{C}{L\ln(1+R^2)}.$$

Doing analogous estimates for K, using (4.59), we get

$$\left|\iint_{|x-y|>R} \varrho(t,x)\varrho(t,y)\nabla K(x-y)\,\mathrm{d}x\mathrm{d}y\right| \\ \leq \frac{1}{R^2}\iint_{\mathbb{R}^{3\times 3}} \varrho(t,x)\varrho(t,y)\,\mathrm{d}x\mathrm{d}y + \iint_{|x-y|>R} \varrho(t,x)\varrho(t,y)|x-y|\,\mathrm{d}x\mathrm{d}y \leq \frac{C}{R}.$$

Combining the above estimates, we obtain

$$\liminf_{L\to\infty}\int_0^T\int_{\mathbb{R}^3} |\tilde{\varrho}_L\nabla K_L * \tilde{\varrho}_L - \varrho\nabla K * \varrho)|\,\mathrm{d}x\mathrm{d}t \leq \frac{C}{R}.$$

Finally, we end the proof by taking $R\to\infty$. □

Lemma 4.26 finishes the limit passage in the weak formulation of (4.52), in the sense of Definition 4.1. In consequence, we obtain the weak solution to (4.1) on $[0,T]\times\mathbb{R}^3$. Since $\nabla\sqrt{\tilde{\varrho}_L} \rightharpoonup \nabla\sqrt{\varrho}$ in $L^2(0,T;L^2_{loc})$,

$$\int_{\mathbb{R}^3} |\nabla\sqrt{\varrho}|^2\,\mathrm{d}x = \lim_{R\to\infty}\int_{B(0,R)} |\nabla\sqrt{\varrho}|^2\,\mathrm{d}x \leq \lim_{R\to\infty}\liminf_{L\to\infty}\int_{B(0,R)} |\nabla\sqrt{\tilde{\varrho}_L}|^2\,\mathrm{d}x \leq C(T).$$

Doing analogously with $\sqrt{\varrho}\nabla u$, we finally show that the solution satisfies the estimates (4.8), (4.9) and (4.10).

4.6.2 Mass preservation

In the end, let us conclude that

Lemma 4.27. *The total mass is conserved, i.e.*

$$\int_{\mathbb{R}^3} \varrho\,\mathrm{d}x = \int_{\mathbb{R}^3} \varrho_0\,\mathrm{d}x.$$

Proof. By Fatou's lemma, we have

$$\int_{\mathbb{R}^3} \varrho\,\mathrm{d}x \leq \liminf_{L\to\infty}\int_{\mathbb{R}^3} \varrho_L\,\mathrm{d}x = \liminf_{L\to\infty}\int_{\mathbb{T}^3_L} \varrho_{L,0}\mathrm{d}x \leq \int_{\mathbb{R}^3} \varrho_0\,\mathrm{d}x.$$

On the other hand, for a smooth function ϕ_R such that $\phi_R(x) = 1$ for $|x| < R/2$, $\phi_R \in (0,1)$ and $\mathrm{supp}\phi_R \subset B_R$, $|\nabla \phi_R| \leq \frac{C}{R}$, for L large enough we have

$$\begin{aligned}\int_{\mathbb{R}^3} \tilde{\varrho}_L(t,x)\phi_R(x)\,\mathrm{d}x &= \int_{\mathbb{T}_L^3} \varrho_L(0,x)\phi_R(x)\,\mathrm{d}x + \int_{\mathbb{T}_L^3} \varrho_L(t,x)u_L(t,x)\cdot\nabla\phi_R(x)\,\mathrm{d}x \\ &\geq \int_{\mathbb{T}_L^3} \varrho_{0,L}(x)\phi_R(x)\,\mathrm{d}x - \frac{C}{R}.\end{aligned}$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}^3} \varrho\,\mathrm{d}x \geq \int_{|x|\leq R} \varrho\,\mathrm{d}x &= \lim_{L\to\infty} \int_{|x|\leq R} \tilde{\varrho}_L \mathrm{d}x \\ &\geq \lim_{L\to\infty} \int_{\mathbb{T}_L^3} \varrho_{L,0}\phi_R\,\mathrm{d}x - \frac{C}{R} \geq \int_{\mathbb{R}^3} \varrho_0\,\mathrm{d}x - \frac{C}{R}\end{aligned}$$

and since $R >$ is arbitrary, we obtain

$$\int_{\mathbb{R}^3} \varrho\,\mathrm{d}x = \int_{\mathbb{R}^3} \varrho_0\,\mathrm{d}x.$$

$\square$

Appendix A

Supplementary proofs for Chapter 2

A.1 Existence of solutions to (2.7)

We now prove that there exists a unique solution to (2.7), which completes the proof of Theorem 2.4.

Lemma A.1. *For $\varrho_0 \in L^\infty(\mathbb{T}^d)$ and any $T > 0$ there exists a unique global in time solution*

$$(\eta, \sigma) \in L^\infty([0,T] \times \mathbb{T}^d) \times L^\infty([0,T] \times \mathbb{T}^d)$$

to the equation (2.7) with the initial condition $\eta(0,y) = \varrho_0(y)$.

Proof. The proof relies on double application of the Banach fixed point theorem. First, observe that for a fixed $\eta \in L^\infty([0,T] \times \mathbb{T}^d)$, there exists a unique σ, satisfying

$$\sigma = p(\eta) - \frac{1}{|\mathbb{T}^d|} \int p(\eta(t,y)) \exp\left(\int_0^t \sigma(s,y) \mathrm{d}s \right) \mathrm{d}y. \tag{A.1}$$

To see that, take

$$Q = \left\{ \sigma \in L^\infty([0,T] \times \mathbb{T}^d) : \|\sigma\|_{\infty,T} \leq \|p(\eta)\|_{\infty,T} \right\}$$

and $\Psi \colon Q_0 \to Q_0$ such that

$$\Psi(\sigma) = p(\eta) - \frac{1}{|\mathbb{T}^d|} \int p(\eta(t,y)) \exp\left(\int_0^t \sigma(s,y)\ \mathrm{d}s \right)\ \mathrm{d}y.$$

Then

$$\begin{aligned}
|\Psi(\sigma_1) - \Psi(\sigma_2)| &\leq \frac{1}{|\mathbb{T}^d|} \int p(\eta(t,y)) \left| \exp\left(\int_0^t \sigma_1(s,y) \mathrm{d}s \right) - \exp\left(\int_0^t \sigma_2(s,y) \mathrm{d}s \right) \right| \mathrm{d}y. \\
&\leq \frac{1}{|\mathbb{T}^d|} \int p(\eta(t,y) e^{T\|p(\eta)\|_{\infty,T}} \int_0^t |\sigma_1(s,y) - \sigma_2(s,y)| \mathrm{d}s \mathrm{d}y \\
&\leq Ct \|\sigma_1 - \sigma_2\|_{\infty,\tau}.
\end{aligned}$$

Hence taking τ such that $C\tau < 1$, from the Banach fixed point theorem we get the existence of a unique solution on the interval $[0,\tau]$. However, as τ depends only on $\|p(\eta)\|_{\infty,T}$ and T, we are able to extend the solution to (A.1) to the whole interval $0,T]$.

Now define $\Phi \colon L^\infty([0,T] \times \mathbb{T}^d) \to L^\infty([0,T] \times \mathbb{T}^d)$ as

$$\Phi(\eta) = \varrho_0(y) - \int_0^t \eta(s,y)\sigma(s,y) \mathrm{d}s,$$

where σ is given by (A.1). Similarly as before, we want to obtain the estimate

$$\|\Phi(\eta_1) - \Phi(\eta_2)\|_{\infty,\tau} \leq C\tau\|\eta_1 - \eta_2\|_{\infty,\tau}$$

and choose τ such that $C\tau < 1$ and $\Phi \colon \{\eta : \|\eta\|_{\infty,\tau} < r\} \to \{\eta : \|\eta\|_{\infty,\tau} < r\}$ is a contraction. We have

$$|\Phi(\eta_1) - \Phi(\eta_2)| = \left|\int_0^t \eta_1\sigma_1 - \eta_2\sigma_2 \mathrm{d}s\right| \leq \int_0^t |\eta_1 - \eta_2||\sigma_1|\mathrm{d}s + \int_0^t \eta_2|\sigma_1 - \sigma_2|\mathrm{d}s.$$

The first integral can be bounded by $Ct\|\eta_1 - \eta_2\|_{\infty,\tau}$, so to complete the desired estimate we need to estimate the difference of σ_1 and σ_2. We have

$$|\sigma_1 - \sigma_2| \leq |p(\eta_1) - p(\eta_2)| + |\{p(\eta_1)\}_{\sigma_1} - \{p(\eta_2)\}_{\sigma_2}|.$$

As the derivative of p is bounded on an interval $[0, r]$, we can estimate the first element by $C\|\eta_1 - \eta_2\|_\infty$. Moreover,

$$\begin{aligned}
|\mathbb{T}^d||\{p(\eta_1)\}_{\sigma_1} - \{p(\eta_2)\}_{\sigma_2}| &= \left|\int p(\eta_1)\exp\left(\int_0^t \sigma_1 \mathrm{d}s\right) - p(\eta_2)\exp\left(\int_0^t \sigma_2 \mathrm{d}s\right)\mathrm{d}y\right| \\
&\leq \int \exp\left(\int_0^t \sigma_1 \mathrm{d}s\right)|p(\eta_1) - p(\eta_2)|\mathrm{d}y + \int p(\eta_2)\left|\exp\left(\int_0^t \sigma_1 \mathrm{d}s\right) - \exp\left(\int_0^t \sigma_2 \mathrm{d}s\right)\right|\mathrm{d}y \\
&\leq C\|\eta_1 - \eta_2\|_\infty + C\int\left|\int_0^t (\sigma_1 - \sigma_2)\mathrm{d}s\right|\mathrm{d}y \\
&\leq C\|\eta_1 - \eta_2\|_\infty + Ct\sup_{0\leq s\leq t}|\{p(\eta_1)\}_{\sigma_1} - \{p(\eta_2)\}_{\sigma_2}|.
\end{aligned}$$

Hence for τ small enough, we get

$$\sup_{0\leq t\leq\tau}|\{p(\eta_1)\}_{\sigma_1} - \{p(\eta_2)\}_{\sigma_2}| \leq C\sup_{0\leq t\leq\tau}\|\eta_1 - \eta_2\|_\infty,$$

which gives us the desired estimate and ends the proof. □

A.2 Estimate for the inverse flows.

Here we prove the useful lemma for estimating the difference of the inverse flows $y_i = x_i^{-1}(t, \cdot)$ by the difference of vector fields generating x_1, x_2.

Lemma A.2. *Consider two ordinary differential equations with the same initial value:*

$$\begin{aligned}
\dot{x}_1 &= u_1(t, x_1), \\
\dot{x}_2 &= u_2(t, x_2), \\
x_1(0) &= x_2(0) = y,
\end{aligned}$$

where $u_1, u_2 \in C(0, T; W^{1,\infty})$. Let $y_1(t, x)$ and $y_2(t, x)$ be the inversions of x_1 and x_2 with respect to y. Then for sufficiently small t

$$\|y_1(t, \cdot) - y_2(t, \cdot)\|_\infty \leq Ct\|u_1 - u_2\|_{\infty,T},$$

where $C = C\left(\|\nabla u_1\|_{\infty,T}\right)$.

Proof. Let $M = \|\nabla u_1\|_{\infty,T}$. We have

$$\begin{aligned}|u_1(t,x_1) - u_2(t,x_2)| &\le |u_1(t,x_1) - u_1(t,x_2)| + |u_1(t,x_2) - u_2(t,x_2)| \\ &\le \|\nabla u_1\|_\infty |x_1 - x_2| + \|u_1 - u_2\|_\infty.\end{aligned}$$

Substituting it into the difference of x_1 and x_2, we get

$$\begin{aligned}|x_1(t,y) - x_2(t,y)| &\le \int_0^t |u_1(s,x_1(s,y)) - u_2(s,x_2(s,y))| \mathrm{d}s \\ &\le \int_0^t \|\nabla u_1\|_\infty |x_1(s,y) - x_2(s,y)| \,\mathrm{d}s + \int_0^t \|u_1 - u_2\|_\infty \mathrm{d}s.\end{aligned}$$

Hence from the Gronwall's lemma,

$$\begin{aligned}\|x_1(t,\cdot) - x_2(t,\cdot)\|_\infty &\le \int_0^t \|u_1 - u_2\|_\infty \mathrm{d}s + \int_0^t \|\nabla u_1\|_\infty \exp\left(\int_s^t \|\nabla u_1\|_\infty \mathrm{d}\tau\right) \int_0^s \|u_1 - u_2\|_\infty \mathrm{d}\tau \mathrm{d}s \\ &\le \|u_1 - u_2\|_{\infty,T} \int_0^t 1 + Me^{M(t-s)} s \,\mathrm{d}s \\ &= \frac{1}{M}\left(e^{Mt} - 1\right) \|u_1 - u_2\|_{\infty,T} \\ &\le 2t\|u_1 - u_2\|_{\infty,T}\end{aligned}$$

for $Mt < \ln 2$, where we use the fact that $e^\theta - 1 \le 2\theta$ for $\theta \le \ln 2$. Analogously, we obtain the estimate

$$\begin{aligned}|x_1(t,y_1) - x_2(t,y_2)| &\le |x_1(t,y_1) - x_1(t,y_2)| + |x_1(t,y_2) - x_2(t,y_2)| \\ &\le \left\|\frac{\partial x_1}{\partial y}\right\|_\infty |y_1 - y_2| + \|x_1 - x_2\|_\infty \\ &\le \exp\left(\int_0^t \|\nabla u_1\|_\infty \mathrm{d}s\right) |y_1 - y_2| + 2t\|u_1 - u_2\|_{\infty,T}.\end{aligned}$$

Combining the above estimates, we get

$$\begin{aligned}|y_1(t,x) - y_2(t,x)| &\le \int_0^t |u_1(s,x_1(s,y_1(t,x))) - u_2(s,x_2(s,y_2(t,x)))| \mathrm{d}s \\ &\le \int_0^t \|\nabla u_1\|_\infty |x_1(s,y_1(t,x)) - x_2(s,y_2(t,x))| + \|u_1 - u_2\|_\infty \mathrm{d}s \\ &\le \int_0^t \|\nabla u_1\|_\infty \exp\left(\int_0^s \|\nabla u_1\|_\infty \mathrm{d}\tau\right) |y_1(t,x) - y_2(t,x)| \mathrm{d}s \\ &\quad + \int_0^t \|\nabla u_1\|_\infty 2s \|u_1 - u_2\|_{\infty,T} \mathrm{d}s + \int_0^t \|u_1 - u_2\|_\infty \mathrm{d}s \\ &\le \left(e^{Mt} - 1\right) |y_1(t,x) - y_2(t,x)| + (t + Mt^2)\|u_1 - u_2\|_{L^\infty([0,T]\times\mathbb{T}^d)}.\end{aligned}$$

For small t we have $e^{Mt} < 2$ and therefore

$$|y_1(t,x) - y_2(t,x)| \le \frac{t + Mt^2}{2 - e^{Mt}} \|u_1 - u_2\|_{\infty,T} \le Ct\|u_1 - u_2\|_{\infty,T}$$

what we needed to prove. $\square$

Appendix B

Properties of the BMO space

We present here the useful properties of the BMO functions, which can be found for example in [97] and [102].

Definition B.1. *A function $f \in L^1(\Omega)$ belongs to space of bounded mean oscillation $BMO(\Omega)$ iff*

$$\|f\|_{BMO} = \sup_{Q\subset\Omega} \frac{1}{|Q|}\int_Q |f - \{f\}_Q| \mathrm{d}x < \infty,$$

where the supremum is taken over all cubes in Ω.

Note that $\|\cdot\|_{BMO}$ is not a norm, as $\|f\|_{BMO} = 0$ for f constant. However, we can equip the space $BMO(\Omega)$ with the norm

$$\|\cdot\|_{L^1} + \|\cdot\|_{BMO}$$

and then it becomes the Banach space.

It is straightforward from the definition that the standard mollification is bounded in BMO:

Proposition B.2. *For $f \in BMO$ and κ_δ the standard mollifier we have*

$$\|f * \kappa_\delta\|_{BMO} \le \|f\|_{BMO}.$$

One of the important tools concerning the BMO spaces is the John-Nirenberg inequality:

Lemma B.3 (John-Nirenberg)**.** *There exist constants $c_1, c_2 > 0$ such that for any cube $Q \subset \Omega$ and $f \in BMO(\Omega)$*

$$|\{x \in Q : |f - \{f\}_Q| > \lambda\}| \le c_1 \exp\left(-\frac{c_2\lambda}{\|f\|_{BMO}}\right)|Q|.$$

The useful applications of the John-Nirenberg inequality are the following:

Corollary B.4. *Let $f \in BMO(\Omega)$. Then*

1. *$f \in L^p_{loc}(\Omega)$ for any $1 \le p < \infty$.*
2. $\displaystyle\sup_{Q\subset\Omega}\int_Q \exp\left(\frac{|f - \{f\}_Q|}{\|f\|_{BMO}}\right)\mathrm{d}x < \infty.$

B.1 The logarithmic inequality

We recall here the inequality from [80]:

Lemma B.5. *Let $f \in BMO(\mathbb{R}^d)$ with compact support and $g \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then*

$$\left|\int_{\mathbb{R}^d} fg \mathrm{d}x\right| \leq C\|f\|_{BMO}\|g\|_{L^1}\big(|\ln\|g\|_{L^1}| + \ln(e+\|g\|_{L^\infty})\big).$$

An analogous inequality was also recently shown in [38] for f in exponential Orlicz space $L^{\exp}$ instead of BMO.

It turns out that after slight modifications, the similar inequality holds for $g \in L^q(\mathbb{T}^d)$ for sufficiently large q.

Lemma B.6. *Let $f \in BMO(\mathbb{T}^d)$ and $g \in L^q(\mathbb{T}^d)$ for some $q > 2$. Then*

$$\begin{aligned}\left|\int_{\mathbb{T}^d} fg \mathrm{d}x\right| \leq & C\|f\|_{BMO}\|g\|_{L^1} \\ & \times \big(|\ln\|g\|_{L^1}| + \ln(1+\|g\|_{L^q}) + (1+|\ln\|g\|_{L^1}|)\|g\|_{L^q}^{\frac{q-2}{2}}\big).\end{aligned} \tag{B.1}$$

Proof: Assume $\int g\mathrm{d}x = 0$. Then $g \in \mathcal{H}^1$ and from duality of $\mathcal{H}^1$ and BMO we have

$$\left|\int fg \,\mathrm{d}x\right| \leq \|f\|_{BMO}\|g\|_{\mathcal{H}^1}.$$

By the characterization of $\mathcal{H}^1$ (see e.g. paragraph III.4 in [97]) we can write $\|g\|_{\mathcal{H}^1}$ as

$$\|g\|_{\mathcal{H}^1} = \|g\|_{L^1} + \sum_{k=1}^{d} \|R_k g\|_{L^1},$$

where R_k is the Riesz transform given as $\mathcal{F}(R_k g) = -i\frac{\xi_k}{|\xi|}\mathcal{F}(g)$. As the Riesz transform is an operator of weak-type $(1,1)$ and strong-type (p,p), we can apply Proposition V.3.2. from [102] and obtain

$$\|R_k g\|_{L^1} \leq C + C\int |g(x)| \ln^+ |g(x)| \,\mathrm{d}x. \tag{B.2}$$

By scaling, we can rewrite (B.2) as

$$\|R_k g\|_{L^1} \leq \lambda + C\int |g(x)| \ln^+(|g(x)|/\lambda)\mathrm{d}x$$

for any $\lambda > 0$. For $|g| \geq \lambda$, we have $\ln^+(|g|/\lambda) = \ln|g| - \ln\lambda$. Then

$$|\ln(|g|_{|_{|g|\geq\lambda}})| = \left|\ln\left(\frac{|g|_{|_{|g|\geq\lambda}}}{1+\|g\|_{L^q}}\right) + \ln(1+\|g\|_{L^q})\right| \leq \ln(1+\|g\|_{L^q}) + \left|\ln\left(\frac{|g|_{|_{|g|\geq\lambda}}}{1+\|g\|_{L^q}}\right)\right|.$$

Now assume $\lambda < 1+\|g\|_{L^q}$ and take x such that $\lambda \leq |g(x)| \leq \frac{(1+\|g\|_{L^q})^2}{\lambda}$. Then

$$\left|\ln\left(\frac{|g(x)|}{1+\|g\|_{L^q}}\right)\right| \leq \left|\ln\left(\frac{\lambda}{1+\|g\|_{L^q}}\right)\right|$$

and in consequence

$$|\ln|g(x)|| \leq 2\ln(1+\|g\|_{L^q}) + |\ln\lambda|.$$

Choose $\lambda = \|g\|_{L^1}$. Then

$$\begin{aligned}\int_{\left\{|g|\leq\frac{(1+\|g\|_{L^q})^2}{\lambda}\right\}} |g|\ln^+(|g|/\lambda)\mathrm{d}x &\leq \int_{\left\{|g|\leq\frac{(1+\|g\|_{L^q})^2}{\lambda}\right\}} |g|(2\ln(1+\|g\|_{L^q}) + |\ln\|g\|_{L^1}|)\mathrm{d}x \\ &\leq \|g\|_{L^1}(2\ln(1+\|g\|_{L^q}) + |\ln\|g\|_{L^1}|)\end{aligned}$$

What is left is the case $|g(x)| > \frac{(1+\|g\|_{L^q})^2}{\lambda}$. From the Chebyshev inequality, we have

$$\left|\left\{x : |g(x)| > \frac{(1+\|g\|_{L^q})^2}{\|g\|_{L^1}}\right\}\right| \leq \left(\frac{\|g\|_{L^1}}{1+\|g\|_{L^q}}\right)^2.$$

Therefore from the Hölder inequality

$$\int_{\left\{|g|>\frac{(1+\|g\|_{L^q})^2}{\lambda}\right\}} |g|\ln^+(|g|/\lambda)\mathrm{d}x \leq \frac{\|g\|_{L^1}}{1+\|g\|_{L^q}}\left(\int |g|^2\ln(|g|/\lambda)^2\mathrm{d}x\right)^{1/2}.$$

Using the fact that both $\int |g|^2\ln|g|\mathrm{d}x$ and $\int |g|^2\ln^2|g|\mathrm{d}x$ are bounded by $C(1+\|g\|_q^q)$ for any $q > 2$, we obtain

$$\begin{aligned}\int |g|^2\ln(|g|/\lambda)^2\mathrm{d}x &= \int |g|^2\ln^2|g|\mathrm{d}x - 2\ln\lambda\int |g|^2\ln|g|\mathrm{d}x + |\ln\lambda|^2\int |g|^2\mathrm{d}x \\ &\leq C(1+\|g\|_{L^q}^q)(1+2|\ln\lambda| + |\ln\lambda|^2) \\ &= C(1+\|g\|_{L^q}^q)(1+|\ln\lambda|)^2.\end{aligned}$$

As $\frac{(1+s^q)^{1/2}}{1+s} \sim 1 + s^{\frac{q-2}{2}}$, after combining the estimates we get

$$\int_{\left\{|g|>\frac{(1+\|g\|_{L^q})^2}{\lambda}\right\}} |g|\ln^+(|g|/\lambda)\mathrm{d}x \leq C\|g\|_{L^1}(1+|\ln\|g\|_{L^1})(1+\|g\|_{L^q}^{\frac{q-2}{2}}).$$

Putting all terms together,

$$\|R_k g\|_{L^1} \leq C\|g\|_{L^1}\left(1+\ln(1+\|g\|_{L^q}) + |\ln\|g\|_{L^1}| + (1+|\ln\|g\|_{L^1}|)\|g\|_{L^q}^{\frac{q-2}{2}}\right)$$

and therefore we obtain inequality (B.1).
If $\int g\ \mathrm{d}x \neq 0$, then we can apply this inequality to $\bar{g}(x) = g(x) - \frac{1}{|\mathbb{T}^d|}\int g\mathrm{d}x$ and use the fact that the L^p norms of $\bar{g}$ are bounded by norms of g up to a constant. □

The same result holds if we replace f by a composition of f and the flow $x(t,y)$:

Corollary B.7. *If f and g satisfy assumptions of Lemma B.6 and $x(t,y)$ is the regular Lagrangian flow of some $u(t,x)$ with bounded divergence, then*

$$\begin{aligned}\left|\int f(x(t,y))g(y)\mathrm{d}y\right| \leq & C\|f\|_{BMO}\|g\|_{L^1} \\ & \times\left(|\ln\|g\|_{L^1}| + \ln(e+\|g\|_{L^q}) + (1+|\ln\|g\|_{L^1}|)\|g\|_{L^q}^{\frac{q-2}{2}}\right).\end{aligned}$$

Proof: We will first approximate u with smooth vector fields, then make the change of variables and apply Lemma B.6, and at the end show the convergence to the non-smooth case.

Let $J(t,y)$ be the Jacobian of x. By the properties of Lagrangian flows we have

$$e^{-L} \leq J(t,y) \leq e^{L}, \text{ where } L = \int_0^T \|\mathrm{div} u\|_\infty \mathrm{d}t.$$

Now let us approximate u by convolution, defining u_ε as a convolution with standard convolution kernels in time and space. Then if x_ε is a flow of u_ε, then $x_\varepsilon(t,\cdot)$ is the diffeomorphism and the Jacobian J_ε of x_ε still satisfies the bounds

$$e^{-L} \leq J_\varepsilon(t,y) \leq e^{L}.$$

By the change of variables, we have

$$\int f(x_\varepsilon(t,y))g(y)\mathrm{d}y = \int f(x)\frac{g(y_\varepsilon(t,x))}{J_\varepsilon(t,y_\varepsilon(t,x))}\mathrm{d}x,$$

where $y_\varepsilon(t,\cdot) = x_\varepsilon(t,\cdot)^{-1}$. Applying Lemma B.6, we obtain inequality (B.1) but with L^1 and L^q norms of $\frac{g(y_\varepsilon(t,\cdot))}{J_\varepsilon(t,y_\varepsilon(t,\cdot))}$ instead of g. However, changing the variables again and using the bounds on J_ε, we obtain for any $p \geq 1$

$$\int \frac{|g(y_\varepsilon(t,x)|^p}{J_\varepsilon(t,y_\varepsilon(t,x))^p}\mathrm{d}x = \int |g(y)|^p J_\varepsilon(t,y)^{1-p}\mathrm{d}y \leq e^{(p-1)L}\int |g(y)|^p \mathrm{d}y$$

and we are done.
Now we will show that indeed

$$\int f(x_\varepsilon(t,y))g(y)\mathrm{d}y \to \int f(x(t,y))g(y)\mathrm{d}y \;\text{ with }\; \varepsilon \to 0. \tag{B.3}$$

By the stability of the flow, we have the pointwise convergence $x_\varepsilon(t,y) \to x(t,y)$ up to a subsequence. If $f \in C^\infty(\mathbb{T}^d)$, then (B.3) holds by the dominated convergence theorem. Let us approximate f by $f_\delta = f * \kappa_\delta$, where κ_δ is again the standard mollifier. As $f \in L^p$ for $p = q'$, we have

$$\left|\int (f_\delta(x(t,y)) - f(x(t,y))g(y)\mathrm{d}y\right| \leq \|f_\delta(x(t,\cdot)) - f(x(t,\cdot))\|_p \|g\|_q$$

and by the bounds on $J(t,y)$,

$$\int |f_\delta(x(t,y)) - f(x(t,y))|^p \,\mathrm{d}x \leq e^{pL}\int |f_\delta(x) - f(x)|^p \,\mathrm{d}x \to 0,$$

therefore we have the desired convergence. Moreover, by the Proposition B.2 the norms $\|f_\delta\|_{BMO}$ in the right hand side of (B.1) are bounded by $\|f\|_{BMO}$, which ends the proof of the Corollary. $\square$

Appendix C

Supplementary proofs and lemmas for Chapter 4

C.1 Proof of Bresch-Desjardins estimates

Below we show how to derive the inequality (4.23). We need to compute

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}_L^3}\left(\frac{1}{2}\varrho|u+\nabla\log\varrho|^2+\varrho(K_L*\varrho)+\frac{\delta}{2}|\nabla\Delta\varrho|^2+\frac{\kappa}{2}|\nabla\sqrt{\varrho}|^2+\frac{\eta}{7}\varrho^{-6}\right)\mathrm{d}x\\
&\qquad=\frac{\mathrm{d}}{\mathrm{d}t}E(\varrho,u)+\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}_L^3}\varrho u\cdot\nabla\log\varrho\;\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}_L^3}\varrho|\nabla\log\varrho|^2\;\mathrm{d}x.
\end{aligned}
\tag{C.1}
$$

For the first term on the right hand side of (C.1), we use the energy inequality (4.22). For the second term, we have

$$
\begin{aligned}
\int_{\mathbb{T}_L^3}\varrho u\cdot\partial_t\nabla\log\varrho\;\mathrm{d}x&=-\int_{\mathbb{T}_L^3}\mathrm{div}\,(\varrho u)\frac{1}{\varrho}\partial_t\varrho\;\mathrm{d}x\\
&=\int_{\mathbb{T}_L^3}\frac{1}{\varrho}(\mathrm{div}\,(\varrho u))^2\;\mathrm{d}x-\varepsilon\int_{\mathbb{T}_L^3}\frac{1}{\varrho}\Delta\varrho\mathrm{div}\,(\varrho u)\;\mathrm{d}x
\end{aligned}
\tag{C.2}
$$

and from the momentum equation

$$\begin{aligned}
\int_{\mathbb{T}_L^3} \partial_t(\varrho u)\cdot\nabla\log\varrho\,\mathrm{d}x =& -\int_{\mathbb{T}_L^3}\operatorname{div}(\varrho u\otimes u)\cdot\nabla\log\varrho\,\mathrm{d}x + \int_{\mathbb{T}_L^3}\operatorname{div}(\varrho\mathbb{D}u)\cdot\nabla\log\varrho\,\mathrm{d}x\\
&-\int_{\mathbb{T}_L^3}\nabla(K_L*\varrho)\cdot\nabla\log\varrho\,\mathrm{d}x + \kappa\int_{\mathbb{T}_L^3}\varrho\nabla\left(\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\right)\cdot\nabla\log\varrho\,\mathrm{d}x\\
&-r_0\int_{\mathbb{T}_L^3}u\cdot\nabla\log\varrho\,\mathrm{d}x - r_1\int_{\mathbb{T}_L^3}\varrho|u|^2u\cdot\nabla\log\varrho\,\mathrm{d}x\\
&-\varepsilon\int_{\mathbb{T}_L^3}\nabla\varrho\cdot\nabla u\cdot\nabla\log\varrho\,\mathrm{d}x - \nu\int_{\mathbb{T}_L^3}\Delta^2u\cdot\nabla\log\varrho\\
&+\eta\int_{\mathbb{T}_L^3}\nabla\varrho^{-6}\cdot\nabla\log\varrho\,\mathrm{d}x + \delta\int_{\mathbb{T}_L^3}\varrho\nabla\Delta^3\varrho\cdot\nabla\log\varrho\,\mathrm{d}x\\
=& -\int_{\mathbb{T}_L^3}\operatorname{div}(\varrho u\otimes u)\cdot\nabla\log\varrho\,\mathrm{d}x - \int_{\mathbb{T}_L^3}\varrho\mathbb{D}u:\nabla^2\log\varrho\,\mathrm{d}x\\
&-\int_{\mathbb{T}_L^3}\nabla(K_L*\varrho)\cdot\nabla\log\varrho\,\mathrm{d}x - \frac{\kappa}{2}\int_{\mathbb{T}_L^3}\varrho|\nabla^2\log\varrho|^2\,\mathrm{d}x\\
&-r_0\int_{\mathbb{T}_L^3}u\cdot\frac{1}{\varrho}\nabla\varrho\,\mathrm{d}x - r_1\int_{\mathbb{T}_L^3}|u|^2u\cdot\nabla\varrho\,\mathrm{d}x\\
&-\varepsilon\int_{\mathbb{T}_L^3}\nabla\varrho\cdot\nabla u\cdot\nabla\log\varrho\,\mathrm{d}x - \nu\int_{\mathbb{T}_L^3}\Delta u\cdot\nabla\Delta\log\varrho\,\mathrm{d}x\\
&-\frac{2}{3}\eta\int_{\mathbb{T}_L^3}|\nabla\varrho^{-3}|^2\,\mathrm{d}x - \delta\int_{\mathbb{T}_L^3}|\Delta^2\varrho|^2\,\mathrm{d}x.
\end{aligned}$$

Note that

$$\begin{aligned}
-\int_{\mathbb{T}_L^3}\varrho\mathbb{D}u:\nabla^2\log\varrho\,\mathrm{d}x =& -\frac{1}{2}\sum_{i,j}\int_{\mathbb{T}_L^3}\varrho(\partial_{x_i}u_j+\partial_{x_j}u_i)\partial_{x_ix_j}\log\varrho\,\mathrm{d}x\\
=&\sum_{i,j}\int_{\mathbb{T}_L^3}\Big(\partial_{x_j}(\varrho\partial_{x_i}u_j)\partial_{x_i}\log\varrho + \partial_{x_i}(\varrho\partial_{x_j}u_i)\partial_{x_j}\log\varrho\Big)\mathrm{d}x\\
=&\int_{\mathbb{T}_L^3}\nabla u:(\nabla\varrho\otimes\nabla\log\varrho)\,\mathrm{d}x + \int_{\mathbb{T}_L^3}\varrho\nabla\operatorname{div}u\cdot\nabla\log\varrho\,\mathrm{d}x\\
=&\int_{\mathbb{T}_L^3}\nabla u:(\nabla\varrho\otimes\nabla\log\varrho)\,\mathrm{d}x - \int_{\mathbb{T}_L^3}\Delta\varrho\operatorname{div}u\,\mathrm{d}x,
\end{aligned}$$

which gives

$$
\begin{aligned}
\int_{\mathbb{T}_L^3} \partial_t(\varrho u)\cdot\nabla\log\varrho\,\mathrm{d}x =& -\int_{\mathbb{T}_L^3} \mathrm{div}\,(\varrho u\otimes u)\cdot\nabla\log\varrho\,\mathrm{d}x + \int_{\mathbb{T}_L^3} \nabla u:(\nabla\varrho\otimes\nabla\log\varrho)\,\mathrm{d}x\\
&-\int_{\mathbb{T}_L^3}\Delta\varrho\mathrm{div}\,u\,\mathrm{d}x\\
&-\int_{\mathbb{T}_L^3}\nabla(K_L*\varrho)\cdot\nabla\log\varrho\,\mathrm{d}x-\frac{\kappa}{2}\int_{\mathbb{T}_L^3}\varrho|\nabla^2\log\varrho|^2\,\mathrm{d}x\\
&-r_0\int_{\mathbb{T}_L^3}u\cdot\frac{1}{\varrho}\nabla\varrho\,\mathrm{d}x-r_1\int_{\mathbb{T}_L^3}|u|^2u\cdot\nabla\varrho\,\mathrm{d}x\\
&-\varepsilon\int_{\mathbb{T}_L^3}\nabla\varrho\cdot\nabla u\cdot\nabla\log\varrho\,\mathrm{d}x-\nu\int_{\mathbb{T}_L^3}\Delta u\cdot\nabla\Delta\log\varrho\,\mathrm{d}x\\
&-\frac{2}{3}\eta\int_{\mathbb{T}_L^3}|\nabla\varrho^{-3}|^2\,\mathrm{d}x-\delta\int_{\mathbb{T}_L^3}|\Delta^2\varrho|^2\,\mathrm{d}x.
\end{aligned}
\tag{C.3}
$$

To compute the last term on the right hand side of (C.1), we see that from the continuity equation

$$
\begin{aligned}
\partial_t\frac{|\nabla\log\varrho|^2}{2} =&\nabla\log\varrho\cdot\partial_t\nabla\log\varrho\\
=&\nabla\log\varrho\cdot\nabla(-\mathrm{div}\,((\log\varrho)u)+(\log\varrho-1)\mathrm{div}\,u+\varepsilon\frac{1}{\varrho}\Delta\varrho)\\
=&-u\cdot\nabla^2\log\varrho\cdot\nabla\log\varrho-\nabla u:(\nabla\log\varrho\otimes\nabla\log\varrho)-\nabla\log\varrho\cdot\nabla\mathrm{div}\,u\\
&+\varepsilon\nabla\log\varrho\cdot\nabla\left(\frac{1}{\varrho}\Delta\varrho\right).
\end{aligned}
$$

The above calculations are justified, since $u\in L^2(0,T;H^2)$, $\varrho\in L^\infty(0,T;H^3)$ and ϱ is bounded away from zero. Using the above calculations, we derive

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}_L^3}\varrho\frac{|\nabla\log\varrho|^2}{2}\mathrm{d}x =& \int_{\mathbb{T}_L^3}\varrho\partial_t\frac{|\nabla\log\varrho|^2}{2}\mathrm{d}x-\int_{\mathbb{T}_L^3}\frac{|\nabla\log\varrho|^2}{2}\mathrm{div}\,(\varrho u)\mathrm{d}x\\
&+\varepsilon\int_{\mathbb{T}_L^3}\frac{|\nabla\log\varrho|^2}{2}\Delta\varrho\,\mathrm{d}x\\
=&-\int_{\mathbb{T}_L^3}\varrho u\cdot\nabla^2\log\varrho\cdot\nabla\log\varrho)\,\mathrm{d}x-\int_{\mathbb{T}_L^3}\varrho\nabla u:(\nabla\log\varrho\otimes\nabla\log\varrho)\,\mathrm{d}x\\
&-\int_{\mathbb{T}_L^3}\varrho\nabla\log\varrho\cdot\nabla\mathrm{div}\,u\,\mathrm{d}x+\int_{\mathbb{T}_L^3}\varrho u\cdot\nabla\frac{|\nabla\log\varrho|^2}{2}\mathrm{d}x\\
&+\varepsilon\int_{\mathbb{T}_L^3}\varrho\nabla\log\varrho\cdot\nabla\left(\frac{1}{\varrho}\Delta\varrho\right)\,\mathrm{d}x+\varepsilon\int_{\mathbb{T}_L^3}\frac{|\nabla\log\varrho|^2}{2}\Delta\varrho\,\mathrm{d}x\\
=&-\int_{\mathbb{T}_L^3}\varrho\nabla u:(\nabla\log\varrho\otimes\nabla\log\varrho)\mathrm{d}x+\int_{\mathbb{T}_L^3}\Delta\varrho\mathrm{div}\,u\,\mathrm{d}x\\
&-\varepsilon\int_{\mathbb{T}_L^3}\frac{|\Delta\varrho|^2}{\varrho}\,\mathrm{d}x+\varepsilon\int_{\mathbb{T}_L^3}\frac{|\nabla\log\varrho|^2}{2}\Delta\varrho\,\mathrm{d}x
\end{aligned}
\tag{C.4}
$$

Since

$$\nabla\varrho\cdot\nabla\log\varrho=\varrho|\nabla\log\varrho|^2$$

and

$$\nabla \varrho \otimes \nabla \log \varrho = \varrho \nabla \log \varrho \otimes \nabla \log \varrho,$$

combining (C.2), (C.3) and (C.4), we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}_L^3} \varrho \frac{|\nabla \log \varrho|^2}{2} + \varrho u \cdot \nabla \log \varrho \mathrm{d}x = & \int_{\mathbb{T}_L^3} \frac{1}{\varrho}(\mathrm{div}\,(\varrho u))^2 \,\mathrm{d}x - \varepsilon \int_{\mathbb{T}_L^3} \frac{1}{\varrho}\Delta \varrho \mathrm{div}\,(\varrho u)\,\mathrm{d}x \\
& - \int_{\mathbb{T}_L^3} \mathrm{div}\,(\varrho u \otimes u) \cdot \nabla \log \varrho \,\mathrm{d}x \\
& - \varepsilon \int_{\mathbb{T}_L^3} \frac{|\Delta \varrho|^2}{\varrho}\,\mathrm{d}x + \varepsilon \int_{\mathbb{T}_L^3} \frac{|\nabla \log \varrho|^2}{2}\Delta \varrho \,\mathrm{d}x \\
& - \int_{\mathbb{T}_L^3} \nabla (K_L * \varrho) \cdot \nabla \log \varrho \,\mathrm{d}x - \frac{\kappa}{2}\int_{\mathbb{T}_L^3} \varrho |\nabla^2 \log \varrho|^2 \,\mathrm{d}x \\
& - r_0 \int_{\mathbb{T}_L^3} u \cdot \frac{1}{\varrho}\nabla \varrho \,\mathrm{d}x - r_1 \int_{\mathbb{T}_L^3} |u|^2 u \cdot \nabla \varrho \,\mathrm{d}x \\
& - \varepsilon \int_{\mathbb{T}_L^3} \nabla \varrho \cdot \nabla u \cdot \nabla \log \varrho \,\mathrm{d}x - \nu \int_{\mathbb{T}_L^3} \Delta u \cdot \nabla \Delta \log \varrho \,\mathrm{d}x \\
& - \frac{2}{3}\eta \int_{\mathbb{T}_L^3} |\nabla \varrho^{-3}|^2 \,\mathrm{d}x - \delta \int_{\mathbb{T}_L^3} |\Delta^2 \varrho|^2 \,\mathrm{d}x.
\end{aligned}$$

We have the relations

$$\begin{aligned}
-\nabla \log \varrho \cdot \mathrm{div}\,(\varrho u \otimes u) &= -\frac{1}{\varrho}\sum_{i,j} \partial_{x_i}\varrho \Big(\partial_{x_j}\varrho u_i u_j + \varrho \partial_{x_j} u_i u_j + \varrho u_i \partial_{x_j} u_j\Big) \\
&= -\frac{1}{\varrho}\Big((u \cdot \nabla \varrho)^2 + \varrho u \cdot (\nabla u \nabla \varrho) + \varrho \mathrm{div}\, u u \cdot \nabla \varrho\Big)
\end{aligned}$$

and

$$\frac{1}{\varrho}(\mathrm{div}\,(\varrho u))^2 = \frac{1}{\varrho}\Big(\varrho^2 (\mathrm{div}\, u)^2 + 2\varrho \mathrm{div}\, u u \cdot \nabla \varrho + (u \cdot \nabla \varrho)^2\Big),$$

therefore

$$\begin{aligned}
\int_{\mathbb{T}_L^3} -\nabla \log \varrho \cdot \mathrm{div}\,(\varrho u \otimes u) + \frac{1}{\varrho}(\mathrm{div}\,(\varrho u))^2 \,\mathrm{d}x &= \int_{\mathbb{T}_L^3} \mathrm{div}\, u u \cdot \nabla \varrho - u \cdot (\nabla u \nabla \varrho) + \varrho (\mathrm{div}\, u)^2 \,\mathrm{d}x \\
&= \sum_{i,j} \int_{\mathbb{T}_L^3} \varrho \partial_{x_j} u_i \partial_{x_i} u_j \,\mathrm{d}x
\end{aligned}$$

Subtracting $\int_{\mathbb{T}_L^3} \varrho |\mathbb{D} u|^2 \,\mathrm{d}x$, we get

$$\begin{aligned}
\int_{\mathbb{T}_L^3} \varrho \sum_{i,j} \left(-\left(\frac{\partial_{x_j} u_i + \partial_{x_i} u_j}{2}\right)^2 + \partial_{x_j} u_i \partial_{x_i} u_j \right) \,\mathrm{d}x &= -\int_{\mathbb{T}_L^3} \varrho \sum_{i,j} \left(\frac{\partial_{x_j} u_i - \partial_{x_i} u_j}{2}\right)^2 \,\mathrm{d}x \\
&= -\frac{1}{4}\int_{\mathbb{T}_L^3} \varrho |\nabla u - \nabla^T u|^2 \,\mathrm{d}x.
\end{aligned}$$

Combining the above calculations with (4.22) and integrating in time, we finally obtain

$$\begin{aligned}
&E_{BD}(\varrho,u)+\frac{2}{3}\eta(1+\varepsilon)\int_0^T\int_{\mathbb{T}_L^3}|\nabla\varrho^{-3}|^2\mathrm{d}x\mathrm{d}t+\delta(1+\varepsilon)\int_0^T\int_{\mathbb{T}_L^3}|\Delta^2\varrho|^2\mathrm{d}x\mathrm{d}t\\
&\quad+\frac{1}{4}\int_0^T\int_{\mathbb{T}_L^3}\varrho|\nabla u-\nabla^T u|^2\mathrm{d}x\mathrm{d}t+\nu\int_0^T\int_{\mathbb{T}_L^3}|\Delta u|^2\mathrm{d}x\mathrm{d}t\\
&\quad+r_0\int_0^T\int_{\mathbb{T}_L^3}|u|^2\,\mathrm{d}x\,\mathrm{d}t+r_1\int_0^T\int_{\mathbb{T}_L^3}\varrho|u|^4\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\frac{\kappa(1+\varepsilon)}{2}\int_0^T\int_{\mathbb{T}_L^3}\varrho|\nabla^2\log\varrho|^2\mathrm{d}x\mathrm{d}t+\varepsilon\int_0^T\int_{\mathbb{T}_L^3}\frac{|\Delta\varrho|^2}{\varrho}\mathrm{d}x+\int_0^T\int_{\mathbb{T}_L^3}\nabla(K_L*\varrho)\cdot\nabla\varrho\,\mathrm{d}x\\
&\le E_{BD}(\varrho_0,u_0)+3\varepsilon T\left(\int_{\mathbb{T}_L^3}\varrho_0\,\mathrm{d}x\right)^2\\
&\quad+\varepsilon\int_0^T\int_{\mathbb{T}_L^3}\left(\nabla\varrho\cdot\nabla u\cdot\nabla\log\varrho+\Delta\varrho\frac{|\nabla\log\varrho|^2}{2}-\mathrm{div}\,(\varrho u)\frac{1}{\varrho}\Delta\varrho\right)\mathrm{d}x\\
&\quad-\nu\int_0^T\int_{\mathbb{T}_L^3}\Delta u\cdot\nabla\Delta\log\varrho\,\mathrm{d}x-r_1\int_0^T\int_{\mathbb{T}_L^3}|u|^2u\nabla\varrho\,\mathrm{d}x-r_0\int_0^T\int_{\mathbb{T}_L^3}\frac{u\cdot\nabla\varrho}{\varrho}\,\mathrm{d}x
\end{aligned}$$

for

$$E_{BD}(\varrho,u)=\int_{\mathbb{T}_L^3}\left(\frac{1}{2}\varrho\left|u+\frac{1}{\varrho}\nabla\varrho\right|^2+\varrho(K_L*\varrho)+\frac{\delta}{2}|\nabla\Delta\varrho|^2+\frac{\kappa}{2}|\nabla\sqrt{\varrho}|^2+\frac{\eta}{7}\varrho^{-6}\right)\mathrm{d}x.$$

C.2 Weak Gronwall's Lemma

Below, let us present the weak version of Gronwall's Lemma, which becomes useful in Section 4.4 of Chapter 4:

Lemma C.1 (Weak version of Gronwall's lemma)**.** *Let $f\in L^1(0,T)$ satisfy*

$$-\int_0^T\xi'(s)f(s)\mathrm{d}s\le\int_0^T\xi(s)(af(s)+b(s))\mathrm{d}s$$

for any $\xi\in C_0^\infty(0,T)$, $\xi\ge 0$, a constant $a\ge 0$ and nonnegative function $b\in L^1(0,T)$. Then for almost all $0\le s<t<T$ we have

$$f(t)\le f(s)e^{a(t-s)}+\int_s^t e^{a(t-\tau)}b(\tau)\,\mathrm{d}\tau$$

Proof. Let $f_\varepsilon=f*\eta_\varepsilon$, where η_ε is a standard mollifier. Fix $t\in(0,T)$ and let $\xi(s)=\eta_\varepsilon(t-s)$. Then f satisfies

$$\int_0^T f(s)\eta_\varepsilon'(t-s)\mathrm{d}s\le a\int_0^T f(s)\eta_\varepsilon(t-s)\mathrm{d}s+\int_0^T b(s)\eta_\varepsilon(t-s)\,\mathrm{d}s,$$

which is equivalent to

$$f_\varepsilon'(t)\le af_\varepsilon(t)+b_\varepsilon(t).$$

Then from Gronwall inequality on f_ε, we get

$$f_\varepsilon(t)\le f_\varepsilon(s)e^{a(t-s)}+\int_s^t e^{a(t-\tau)}b_\varepsilon(\tau)\mathrm{d}\tau.$$

for any $0 \leq s < t < T$. Choosing s, t such that $f_\varepsilon \to f$ pointwise in s, t and passing to the limit with $\varepsilon \to 0$, we get

$$f(t) \leq f(s)e^{a(t-s)} + \int_s^t e^{a(t-\tau)} b(\tau) \mathrm{d}\tau.$$

□

Bibliography

[1] A. Abbatiello and E. Feireisl. "On a class of generalized solutions to equations describing incompressible viscous fluids". In: *Ann. Mat. Pura Appl. (4)* 199.3 (2020), pp. 1183–1195. DOI: 10.1007/s10231-019-00917-x.

[2] R. Andrášik and R. Vodák. "Compressible nonlinearly viscous fluids: asymptotic analysis in a 3D curved domain". In: *J. Math. Fluid Mech.* 21.1 (2019), Paper No. 13, 27. DOI: 10.1007/s00021-019-0412-y.

[3] F. Belblidia, T. Haroon, and M. F. Webster. "The Dynamics of Compressible Herschel—Bulkley Fluids in Die-Swell Flows". In: *Damage and Fracture Mechanics: Failure Analysis of Engineering Materials and Structures*. Springer Netherlands, 2009, pp. 425–434. DOI: 10.1007/978-90-481-2669-9_45.

[4] J. Blechta, J. Málek, and K. R. Rajagopal. "On the classification of incompressible fluids and a mathematical analysis of the equations that govern their motion". In: *SIAM J. Math. Anal.* 52.2 (2020), pp. 1232–1289. DOI: 10.1137/19M1244895.

[5] D. Bresch and C. Burtea. "Global existence of weak solutions for the anisotropic compressible Stokes system". In: *Annales de L'Institut Henri Poincare Section (C) Non Linear Analysis* 37.6 (Nov. 2020), pp. 1271–1297. DOI: 10.1016/j.anihpc.2020.04.003.

[6] D. Bresch and B. Desjardins. "Existence of Global Weak Solutions for a 2D Viscous Shallow Water Equations and Convergence to the Quasi-Geostrophic Model". In: *Communications in Mathematical Physics* 238 (July 2003), pp. 211–223. DOI: 10.1007/s00220-003-0859-8.

[7] D. Bresch and B. Desjardins. "On the construction of approximate solutions for the 2D viscous shallow water model and for compressible Navier–Stokes models". In: *Journal de Mathématiques Pures et Appliquées* 86.4 (2006), pp. 362–368. DOI: https://doi.org/10.1016/j.matpur.2006.06.005.

[8] D. Bresch and B. Desjardins. "On the existence of global weak solutions to the Navier–Stokes equations for viscous compressible and heat conducting fluids". In: *Journal de Mathématiques Pures et Appliquées* 87 (Jan. 2007), pp. 57–90. DOI: 10.1016/j.matpur.2006.11.001.

[9] D. Bresch, B. Desjardins, and C.-K. Lin. "On Some Compressible Fluid Models: Korteweg, Lubrication, and Shallow Water Systems". In: *Communications in Partial Differential Equations* 28.3-4 (2003), pp. 843–868. DOI: 10.1081/PDE-120020499.

[10] D. Bresch and P.-E. Jabin. "Global existence of weak solutions for compressible Navier–Stokes equations: Thermodynamically unstable pressure and anisotropic viscous stress tensor". In: *Annals of Mathematics* 188.2 (2018), pp. 577 –684. DOI: 10.4007/annals.2018.188.2.4.

[11] D. Bresch and P. Jabin. "Global weak solutions of PDEs for compressible media: A compactness criterion to cover new physical situations". In: *Springer INdAM Series* 17 (2017), pp. 33–54. DOI: 10.1007/978-3-319-52042-1_2.

[12] D. Bresch, P. B. Mucha, and E. Zatorska. “Finite-Energy Solutions for Compressible Two-Fluid Stokes System”. In: *Archive for Rational Mechanics and Analysis* (Sept. 2017). DOI: 10.1007/s00205-018-01337-6.

[13] D. Bresch, A. Vasseur, and C. Yu. “Global existence of entropy-weak solutions to the compressible Navier–Stokes equations with non-linear density dependent viscosities”. In: *Journal of the European Mathematical Society* 24.5 (). DOI: 10.4171/JEMS/1143.

[14] D. Bresch et al. “A viscoelastic model for avascular tumor growth”. In: *Discrete and Continuous Dynamical Systems* (2009).

[15] M. Bulíček, P. Kaplický, and D. Pražák. “Uniqueness and regularity of flows of non-Newtonian fluids with critical power-law growth”. In: *Math. Models Methods Appl. Sci.* 29.6 (2019), pp. 1207–1225. DOI: 10.1142/S0218202519500209.

[16] J. Burczak, S. Modena, and L. Székelyhidi. “Non uniqueness of power-law flows”. In: *Comm. Math. Phys.* 388.1 (2021), pp. 199–243. DOI: 10.1007/s00220-021-04231-7.

[17] J. A. Cañizo, J. A. Carrillo, and J. Rosado. “A well-posedness theory in measures for some kinetic models of collective motion”. In: *Mathematical Models and Methods in Applied Sciences* 21.03 (2011), pp. 515–539. DOI: 10.1142/S0218202511005131.

[18] A. Calderón and A. Zygmund. “Singular integrals and periodic functions”. In: *Studia Mathematica* 14.2 (1954), pp. 249–271.

[19] V. Calvez, J. Carrillo, and F. Hoffmann. “Equilibria of homogeneous functionals in the fair-competition regime”. In: *Nonlinear Analysis* 159 (2017). Advances in Reaction-Cross-Diffusion Systems, pp. 85–128. DOI: https://doi.org/10.1016/j.na.2017.03.008.

[20] V. Calvez, J. A. Carrillo, and F. Hoffmann. “The Geometry of Diffusing and Self-Attracting Particles in a One-Dimensional Fair-Competition Regime”. In: *Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions: Cetraro, Italy 2016*. Springer International Publishing, 2017, pp. 1–71. DOI: 10.1007/978-3-319-61494-6_1.

[21] R. Carles, K. Carrapatoso, and M. Hillairet. “Global weak solutions for quantum isothermal fluids”. In: *Annales de l'Institut Fourier* 72.6 (2022), pp. 2241–2298. DOI: 10.5802/aif.3489.

[22] J. Carrillo and R. Shu. “Existence of Radial Global Smooth Solutions to the Pressureless Euler–Poisson Equations with Quadratic Confinement”. In: *Archive for Rational Mechanics and Analysis* 247 (Aug. 2023). DOI: 10.1007/s00205-023-01905-5.

[23] J. A. Carrillo, Y.-P. Choi, and E. Zatorska. “On the pressureless damped Euler–Poisson equations with quadratic confinement: Critical thresholds and large-time behavior”. In: *Mathematical Models and Methods in Applied Sciences* 26.12 (2016), pp. 2311–2340. DOI: 10.1142/S0218202516500548.

[24] J. A. Carrillo, A. Wróblewska-Kamińska, and E. Zatorska. “On long-time asymptotics for viscous hydrodynamic models of collective behavior with damping and nonlocal interactions”. In: *Mathematical Models and Methods in Applied Sciences* 29.01 (2019), pp. 31–63. DOI: 10.1142/S0218202519500027.

[25] J. A. Carrillo, A. Wróblewska-Kamińska, and E. Zatorska. “Pressureless Euler with nonlocal interactions as a singular limit of degenerate Navier-Stokes system”. In: *Journal of Mathematical Analysis and Applications* 492.1 (2020), p. 124400. DOI: https://doi.org/10.1016/j.jmaa.2020.124400.

[26] J. A. Carrillo et al. "Particle, kinetic, and hydrodynamic models of swarming". In: *Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences*. Birkhäuser Boston, 2010, pp. 297–336. DOI: 10.1007/978-0-8176-4946-3_12.

[27] J. A. Carrillo et al. "Weak solutions for Euler systems with non-local interactions". In: *Journal of the London Mathematical Society* 95.3 (2017), pp. 705–724. DOI: https://doi.org/10.1112/jlms.12027.

[28] J. A. Carrillo et al. "Ground states in the diffusion-dominated regime". In: *Calculus of Variations and Partial Differential Equations* 57.5 (2018), Art. No. 127.

[29] N. Chaudhuri, P. Mucha, and E. Zatorska. "A new construction of weak solutions to compressible Navier–Stokes equations". In: *Mathematische Annalen* (Jan. 2024), pp. 1–61. DOI: 10.1007/s00208-023-02730-7.

[30] E. Chiodaroli. "A counterexample to well-posedness of entropy solutions to the compressible Euler system". In: *Journal of Hyperbolic Differential Equations* 11 (Sept. 2014), p. 493. DOI: 10.1142/S0219891614500143.

[31] E. Chiodaroli, C. De Lellis, and O. Kreml. "Global Ill-Posedness of the Isentropic System of Gas Dynamics". In: *Communications on Pure and Applied Mathematics* 68.7 (2015), pp. 1157–1190. DOI: https://doi.org/10.1002/cpa.21537.

[32] Y. Cho and K. Kensey. "Effects of the non-Newtonian viscosity of blood on hemodynamics of diseased arterial flows". In: *Advances in Bioengineering* 15 (Jan. 1989), pp. 147–148.

[33] Y.-L. Chuang et al. "State transitions and the continuum limit for a 2D interacting, self-propelled particle system". In: *Physica D: Nonlinear Phenomena* 232.1 (2007), pp. 33–47. DOI: https://doi.org/10.1016/j.physd.2007.05.007.

[34] M. Colombo, G. Crippa, and S. Spirito. "Renormalized solutions to the continuity equation with an integrable damping term". In: *Calculus of Variations and Partial Differential Equations* 54 (2014), pp. 1831–1845.

[35] G. Crippa and C. De Lellis. "Estimates and regularity results for the DiPerna-Lions flow". In: *Journal fur die Reine und Angewandte Mathematik* 616 (2008), pp. 15–46. DOI: 10.1515/CRELLE.2008.016.

[36] Y. Damianou, G. C. Georgiou, and I. Moulitsas. "Combined effects of compressibility and slip in flows of a Herschel–Bulkley fluid". In: *Journal of Non-Newtonian Fluid Mechanics* 193 (2013). Viscoplastic Fluids: From Theory to Application, pp. 89–102. DOI: https://doi.org/10.1016/j.jnnfm.2012.09.004.

[37] R. Danchin. "Global existence in critical spaces for compressible Navier-Stokes equations". In: *Invent. Math.* 141.3 (2000), pp. 579–614.

[38] L. De Rosa, M. Inversi, and G. Stefani. "Weak-strong uniqueness and vanishing viscosity for incompressible Euler equations in exponential spaces". In: *Journal of Differential Equations* 366 (2023), pp. 833–861. DOI: https://doi.org/10.1016/j.jde.2023.05.019.

[39] L. Diening, M. Růžička, and J. Wolf. "Existence of weak solutions for unsteady motions of generalized Newtonian fluids". In: *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 9.1 (2010), pp. 1–46.

[40] R. Diperna and P. L. Lions. "Ordinary differential equations, transport theory and Sobolev spaces". In: *Inventiones mathematicae* 98 (1989), pp. 511–547.

[41] D. Donatelli and K. Trivisa. "On a Nonlinear Model for Tumor Growth: Global in Time Weak Solutions". In: *J. Math. Fluid Mech.* 16 (2014), 787–803.

[42] H. Eberlein and M. Ruzicka. "Existence of Weak Solutions for Unsteady Motions of Herschel–Bulkley Fluids". In: *Journal of Mathematical Fluid Mechanics* 14 (Sept. 2011), pp. 1–16. DOI: 10.1007/s00021-011-0080-z.

[43] A. Farutin et al. "Amoeboid Swimming: A Generic Self-Propulsion of Cells in Fluids by Means of Membrane Deformations". In: *Phys. Rev. Lett.* 111 (22 2013), p. 228102. DOI: 10.1103/PhysRevLett.111.228102.

[44] E. Feireisl. *Dynamics of viscous compressible fluids*. Oxford Lecture Series in Mathematics and Its Applications 26. Oxford University Press, 2004.

[45] E. Feireisl, X. Liao, and J. Málek. "Global weak solutions to a class of non-Newtonian compressible fluids". In: *Mathematical Methods in the Applied Sciences* 38.16 (2015). DOI: 10.1002/mma.3432.

[46] E. Feireisl, A. Novotný, and H. Petzeltová. "On the Existence of Globally Defined Weak Solutions to the Navier—Stokes Equations". In: *Journal of Mathematical Fluid Mechanics* 3 (2001), pp. 358–392.

[47] E. Feireisl. "Compressible Navier–Stokes Equations with a Non-Monotone Pressure Law". In: *Journal of Differential Equations* 184 (Sept. 2002), pp. 97–108. DOI: 10.1006/jdeq.2001.4137.

[48] E. Feireisl. "On compactness of solutions to the compressible isentropic Navier-Stokes equations when the density is not square integrable". In: *Comment. Math. Univ. Carolin.* 42.1 (2001), 83–98.

[49] E. Feireisl and A. Novotný. *Singular Limits in Thermodynamics of Viscous Fluids*. Birkhäuser-Verlag, Basel, Jan. 2009. DOI: 10.1007/978-3-7643-8843-0.

[50] E. Feireisl et al. "Time-Periodic Solutions to the Full Navier-Stokes-Fourier System". In: *Archive for Rational Mechanics and Analysis* 204.3 (June 2012), pp. 745–786. DOI: 10.1007/s00205-012-0492-9.

[51] A. Figalli and M.-J. Kang. "A rigorous derivation from the kinetic Cucker–Smale model to the pressureless Euler system with nonlocal alignment". In: *Analysis & PDE* 12.3 (2019), pp. 843 –866. DOI: 10.2140/apde.2019.12.843.

[52] M. Garcia et al. "Random walk of a swimmer in a low-Reynolds-number medium". In: *Phys. Rev. E* 83 (3 2011), p. 035301. DOI: 10.1103/PhysRevE.83.035301.

[53] L. Grafakos. *Classical Fourier Analysis*. Springer New York, NY, 2008.

[54] B. Haspot. "From the Highly Compressible Navier-Stokes Equations to Fast Diffusion and Porous Media Equations, Existence of Global Weak Solution for the Quasi-Solutions". In: *Journal of Mathematical Fluid Mechanics* 18.2 (June 2016), pp. 243–291. DOI: 10.1007/s00021-015-0226-5.

[55] B. Haspot and E. Zatorska. "From the highly compressible Navier-Stokes equations to the Porous Medium equation - rate of convergence". In: *Discrete and Continuous Dynamical Systems* 36 (Apr. 2015). DOI: 10.3934/dcds.2016.36.3107.

[56] C.-M. Ho and Y.-C. Tai. "Micro-electro-mechanical-systems (MEMS) and fluid flows". In: *Annual Review of Fluid Mechanics* 30.1 (1998), pp. 579–612. DOI: 10.1146/annurev.fluid.30.1.579.

[57] D. Jesslé, A. Novotný, and M. Pokorný. "Steady Navier- Stokes-Fourier system with slip boundary conditions". In: *Mathematical Models and Methods in Applied Sciences* 24 (Jan. 2014). DOI: 10.1142/S0218202513500668.

[58] A. Jüngel. "Global Weak Solutions to Compressible Navier–Stokes Equations for Quantum Fluids". In: *SIAM Journal on Mathematical Analysis* 42.3 (2010), pp. 1025–1045. DOI: 10.1137/090776068.

[59] B. H. Kjartanson et al. "The creep of ice measured with the pressuremeter". In: *Canadian Geotechnical Journal* 25.2 (1988), pp. 250–261. DOI: 10.1139/t88-029.

[60] A. L. Koch and D. White. "The social lifestyle of myxobacteria". In: *BioEssays* 20.12 (1998), pp. 1030–1038. DOI: https://doi.org/10.1002.

[61] I. Lacroix-Violet and A. Vasseur. "Global weak solutions to the compressible quantum Navier–Stokes equation and its semi-classical limit". In: *Journal de Mathématiques Pures et Appliquées* 114 (2018), pp. 191–210. DOI: https://doi.org/10.1016/j.matpur.2017.12.002.

[62] M. Łasica, S. Moll, and P. B. Mucha. "Total variation denoising in l^1 anisotropy". In: *SIAM J. Imaging Sci.* 10.4 (2017), pp. 1691–1723. DOI: 10.1137/16M1103610.

[63] E. Lauga and T. R. Powers. "The hydrodynamics of swimming microorganisms". In: *Reports on Progress in Physics* 72.9 (2009), p. 096601. DOI: 10.1088/0034-4885/72/9/096601.

[64] J. Li and Z. Xin. "Global Existence of Weak Solutions to the Barotropic Compressible Navier-Stokes Flows with Degenerate Viscosities". In: *arXiv:1504.06826* ().

[65] I. Ligaarden et al. "On the Stokes–Brinkman Equations for Modeling Flow in Carbonate Reservoirs". In: Sept. 2010. DOI: 10.3997/2214-4609.20144924.

[66] P.-L. Lions. *Mathematical Topics in Fluid Mechanics Vol. 1: Incompressible Models.* Oxford Lecture Series in Mathematics and Its Applications 3. Oxford University Press, 1996.

[67] P.-L. Lions. *Mathematical Topics in Fluid Mechanics, Vol. 2: Compressible Models.* Oxford Lecture Series in Mathematics and Its Applications 10. Oxford University Press, 1998.

[68] J. Málek, M. Růžička, and V. V. Shelukhin. "Herschel–Bulkley fluids: existence and regularity of steady flows". In: *Mathematical Models and Methods in Applied Sciences* 15.12 (2005), pp. 1845–1861. DOI: 10.1142/S0218202505000996.

[69] D. Maltese et al. "Existence of weak solutions for compressible Navier–Stokes equations with entropy transport". In: *Journal of Differential Equations* 261 (July 2016). DOI: 10.1016/j.jde.2016.06.029.

[70] A. E. Mamontov. "Existence of Global Solutions to Multidimensional Equations for Bingham Fluids". In: *Mat. Zametki* 82 (4 2007), 560–577.

[71] A. E. Mamontov. "Global solvability of the multidimensional equations of compressible non-Newtonian fluids, transport equation and the Orlicz spaces". In: *Sib. Elektron. Mat. Izv.* 6 (2009), 120–165.

[72] C.-S. Man and Q.-X. Sun. "On the Significance of Normal Stress Effects in the Flow of Glaciers". In: *Journal of Glaciology* 33.115 (1987). Publisher: Cambridge University Press, pp. 268–273. DOI: 10.3189/S0022143000008832.

[73] J. Marcinkiewicz. "Sur les multiplicateurs des séries de Fourier". In: *Studia Math.* 8 (1939), 78–91.

[74] A. Matsumura and T. Nishida. "The initial value problem for the equations of motion of viscous and heat-conductive gases". In: *Journal of Mathematics of Kyoto University* 20.1 (1980), pp. 67 –104. DOI: 10.1215/kjm/1250522322.

[75] A. Mellet and A. Vasseur. "On the Barotropic Compressible Navier–Stokes Equations". In: *Communications in Partial Differential Equations* 32.3 (2007), pp. 431–452. DOI: 10.1080/03605300600857079.

[76] P. B. Mucha and M. Pokorný. “On a new approach to the issue of existence and regularity for the steady compressible Navier–Stokes equations”. In: *Nonlinearity* 19.8 (2006), p. 1747. DOI: 10.1088/0951-7715/19/8/003.

[77] P. Mucha and W. Zajączkowski. “Global existence of solutions of the Dirichlet problem for the compressible Navier-Stokes equations”. In: *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik* 84.6 (2004), pp. 417–424. DOI: https://doi.org/10.1002/zamm.200310080.

[78] P. B. Mucha. “Transport equation: Extension of classical results for $\operatorname{div} b \in BMO$”. In: *Journal of Differential Equations* 249 (July 2008). DOI: 10.1016/j.jde.2010.07.015.

[79] P. B. Mucha and M. Pokorný. “Weak solutions to equations of steady compressible heat conducting fluids”. In: *Mathematical Models and Methods in Applied Sciences* 20 (Apr. 2012). DOI: 10.1142/S0218202510004441.

[80] P. B. Mucha and W. Rusin. “Zygmund Spaces, Inviscid Limit and Uniqueness of Euler Flows”. In: *Communications in Mathematical Physics* 280 (June 2008), pp. 831–841. DOI: 10.1007/s00220-008-0452-2.

[81] M. Muskat et al. “Flow of Gas-Liquid Mixtures through Sands”. In: *Transactions of the AIME* 123 (Apr. 2013), pp. 69–96. DOI: 10.2118/937069-G.

[82] J. Nash. “Le problème de Cauchy pour les équations différentielles d’un fluide général”. In: *Bulletin de la Société Mathématique de France* 90 (1962), pp. 487–497. DOI: 10.24033/bsmf.1586.

[83] A. Novotny and M. Pokorný. “Steady compressible Navier-Stokes-Fourier system for monoatomic gas and its generalizations”. In: *Journal of Differential Equations* 251 (2011). DOI: 10.1016/j.jde.2011.04.008.

[84] A. Novotny and M. Pokorný. “Weak and variational solutions to steady equations for compressible heat conducting fluids”. In: *SIAM J. Math. Anal.* 43 (2011).

[85] A. Novotný and I. Straškraba. *Introduction to the Mathematical Theory of Compressible Flow*. Oxford Lecture Series in Mathematics and Its Applications. Oxford University Press, 2004.

[86] T. Ogawa. “Sharp Sobolev Inequality of Logarithmic Type and the Limiting Regularity Condition to the Harmonic Heat Flow”. In: *SIAM J. Math. Anal.* 34 (2003), pp. 1318–1330.

[87] J. K. Parrish and L. Edelstein-Keshet. “Complexity, Pattern, and Evolutionary Trade-Offs in Animal Aggregation”. In: *Science* 284.5411 (1999), pp. 99–101. DOI: 10.1126/science.284.5411.99.

[88] M. Peltomäki and G. Gompper. “Sedimentation of single red blood cells”. In: *Soft Matter* 9 (Sept. 2013), p. 8346. DOI: 10.1039/c3sm50592h.

[89] T. Piasecki. “On an inhomogeneous slip-inflow boundary value problem for a steady flow of a viscous compressible fluid in a cylindrical domain”. In: *Journal of Differential Equations* 248.8 (2010), pp. 2171–2198. DOI: https://doi.org/10.1016/j.jde.2009.12.009.

[90] T. Piasecki. “Steady compressible Oseen flow with slip boundary conditions”. In: *Banach Center Publications* 86.1 (2009), pp. 247–264. DOI: 10.4064/bc86-0-16.

[91] P. Plotnikov and W. Weigant. “Steady 3D viscous compressible flows with adiabatic exponent”. In: *Journal de Mathématiques Pures et Appliquées* 104 (Dec. 2013). DOI: 10.1016/j.matpur.2015.02.001.

[92] M. Pokorný and P. B. Mucha. "3D steady compressible Navier–Stokes equations". In: *Discrete and Continuous Dynamical Systems - S* 1.1 (2008), pp. 151–163. DOI: 10.3934/dcdss.2008.1.151.

[93] M. Pokorný and M. Szlenk. "Weak solutions for the Stokes system for compressible non-Newtonian fluids with unbounded divergence". In: *Mathematical Methods in the Applied Sciences* 46.8 (2023), pp. 9736–9750. DOI: https://doi.org/10.1002/mma.9083.

[94] J. Simon. "Compact sets in the space $L^p(0,T;B)$". In: *Ann. Mat. Pura Appl. (4)* 146 (1987), pp. 65–96. DOI: 10.1007/BF01762360.

[95] V. A. Solonnikov. "On solvability of an initial-boundary value problem for the equations of motion of a viscous compressible fluid". In: *J. Sov. Math.* 14 (1980), 1120–1130. [Translated from Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V. A. Steklova AN SSSR, Vol. 56, pp. 128–142, 1976.]

[96] H. Steffan et al. "Comparison of Newtonian and Non-Newtonian Blood Flow in Stenotic Vessels using Numerical Simulation". In: *Biofluid Mechanics*. Ed. by D. W. Liepsch. Berlin, Heidelberg: Springer, 1990, pp. 479–485. DOI: 10.1007/978-3-642-52338-0_61.

[97] E. M. Stein. *Harmonic Analysis: Real-Variable Methods, Orthogonality and OScillatory Integrals*. Princeton, New Jersey: Princeton University Press, 1993.

[98] H. Stone, A. Stroock, and A. Ajdari. "Engineering Flows in Small Devices: Microfluidics Toward a Lab-on-a-Chip". In: *Annual Review of Fluid Mechanics* 36.1 (2004), pp. 381–411. DOI: 10.1146/annurev.fluid.36.050802.122124.

[99] I. Straškraba et al. "Global in time solutions for compressible barotropic self-gravitating fluids". In: *Discrete and Continuous Dynamical Systems* 11 (July 2004), pp. 113–130. DOI: 10.3934/dcds.2004.11.113.

[100] M. Szlenk. "Weak solutions for the Stokes system for compressible fluids with general pressure". In: *Journal of Differential Equations* 312 (2022), pp. 317–346. DOI: https://doi.org/10.1016/j.jde.2021.12.011.

[101] A. Tani. "On the First Initial-Boundary Value Problem of Compressible Viscous Fluid Motion". In: *Publications of The Research Institute for Mathematical Sciences* 13 (1977), pp. 193–253.

[102] A. Torchinsky. *Real-Variable Methods in Harmonic Analysis*. Vol. 123. Pure and Applied Mathematics. Orlando, FL: Academic Press, Inc., 1986.

[103] V. Vaigant. "An example of nonexistence globally in time of a solution of the Navier-Stokes equations for a compressible viscous barotropic fluid". In: *Russian Academy of Sciences. Doklady. Mathematics* 50 (Jan. 1993).

[104] A. Valli and W. Zajączkowski. "Navier-Stokes equations for compressible fluids: Global existence and qualitative properties of the solutions in the general case". In: *Communications in Mathematical Physics* 103 (June 1986), pp. 259–296. DOI: 10.1007/BF01206939.

[105] A. Vasseur and C. Yu. "Existence of global weak solutions for 3D degenerate compressible Navier–Stokes equations". In: *Inventiones mathematicae* 206 (Dec. 2016). DOI: 10.1007/s00222-016-0666-4.

[106] A. F. Vasseur and C. Yu. "Global Weak Solutions to the Compressible Quantum Navier–Stokes Equations with Damping". In: *SIAM Journal on Mathematical Analysis* 48.2 (2016), pp. 1489–1511. DOI: 10.1137/15M1013730.

[107] N. Vauchelet and E. Zatorska. “Incompressible limit of the Navier-Stokes model for tumor growth”. In: *Nonlinear Analysis* 163 (Apr. 2017). DOI: 10.1016/j.na.2017.07.003.

[108] Z. Wu, J. Yin, and C. Wang. *Elliptic & Parabolic Equations*. Oct. 2006. DOI: 10.1142/6238.

[109] Y. Ye and C. Dou. “Global weak solutions to 3D compressible Navier–Stokes–Poisson equations with density-dependent viscosity”. In: *Journal of Mathematical Analysis and Applications* 455.1 (2017), pp. 180–211. DOI: https://doi.org/10.1016/j.jmaa.2017.05.044.

[110] D. A. Yuen and A. V. Malevsky. “Strongly Chaotic Newtonian and Non-Newtonian Mantle Convection”. In: *Chaotic Processes in the Geological Sciences*. Ed. by D. A. Yuen. The IMA Volumes in Mathematics and its Applications. New York, NY: Springer US, 1992, pp. 71–88. DOI: 10.1007/978-1-4684-0643-6_4.