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A Torres formula for multivariable Levine–Tristram signature

PhD Dissertation

submitted by

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Warsaw
2023

Author's declaration:
I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

November 28, 2023 ....
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Supervisors' declaration:
The dissertation is ready to be reviewed.

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Abstract

The topic of this dissertation concerns the multivariable Levine–Tristram signature of links. In particular, we are interested at the limits of the multivariable signature function as some variable approaches 1. We compare the value of this limit to the multivariable signature of a suitable sublink. We also consider similar relations for the multivariable nullity function. The motivation to do so comes from a similar relation, expressed by the formula of Torres, that holds for the multivariable Alexander polynomial.

We consider this from two points of view. First, we look at the multivariable signature defined as a signature of a certain Hermitian matrix obtained from a C-complex, a 3-dimensional construction. We show that in the limit, the signature and nullity of this matrix can be expressed as the sum of signature and nullity of the matrix associated to sublink, and the signature and nullity of a correction term matrix. We show that the correction term matrix is invariant under link homotopy and we explain how it can be recovered from combinatorial data associated to the link in question. Finally, we consider the inequalities relating the limits of the signature and nullity functions to the signature and nullity of a sublink obtained from this decomposition.

Afterwards, we consider the multivariable signature and nullity defined as invariants of 4-dimensional manifolds associated to a link together with an auxiliary choice of a bounding surface. More precisely, these are defined then as the signature and nullity of twisted intersection forms of such manifolds. First, we modify a previous construction of such a manifold, to obtain one with desirable properties when some of the variables are equal to 1. Then, we use this definition to obtain a relation between the signature and nullity at 1 and the signature and nullity of a sublink, and we consider the inequality relating the limits of the signature and nullity obtained from these relation. We show that this inequality is different to the one obtained from 3-dimensional considerations and compare their strength. Finally, we use the 4-dimensional definition to show that the signature and nullity are in a suitable sense concordance invariants, even when some variable is equal to 1.

Keywords: Knot theory, multivariable Levine–Tristram signature, Torres formula, twisted homology, C-complexes.

AMS MSC 2020 classification: 57K10, 57N70, 55N25.

Acknowledgments

First and foremost, I would like to express my gratitude towards my supervisors, Maciej Borodzik and Wojciech Politarczyk for all their help and guidance during my PhD studies.
I also want to offer deep thanks to David Cimasoni for all the illuminating discussions and support in reseaching this topic. Moreover, I am indebted to Gaëtan Simian for the hospitality offered to me during my research stay in Geneva. Thanks to him and David my visit there made for a memory I will treasure for a very long time.

1. Introduction

The Levine–Tristram signature is a classical link invariant, introduced in the 1960s [19, 30]. To each link, it associates a function from the unit circle of $\mathbb{C}$, with 1 removed, to the integers. It has proven to be a useful tool of knot theory and in particular in determining whether a given knot is slice. Moreover, the Levine–Tristram signature provides a lower bound on the unlinking number and splitting number of a link. A detailed discussion of properties of it can be found in [7]. Later, a multivariable extension of the Levine–Tristram signature was introduced [11, 5]. This generalization has also proven useful in studying link genera [10]. Given a μ-component link L the multivariable signature can be defined as the signature of a certain matrix with coefficients in $\mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}]$ with $(t_1, \dots, t_\mu)$ specialized to $\omega \in \mathbb{T}_*^\mu := (S^1 \setminus 1)^\mu$. This definition does not yield an interesting invariant for ω with some coordinate equal to 1. Instead, the signature in that case will always be equal to 0.
We wish to instead consider the limits of the signature function $\sigma_L(\omega)$ and a closely related function - the link nullity $\eta_L(\omega)$, as some coordinate of ω approaches 1. The reason for that lies in the theorem of Torres [29] which relates the multivariable Alexander polynomial of a link at $t_1 = 1$ to the multivariable Alexander polynomial of the sublink $L \setminus L_1$. The determinant of the matrix used to define the multivariable signature agrees with the multivariable Alexander polynomial up to a suitable normalization. Thus, we expect a similar relation between the limit of the signature and the signature of a sublink to hold. By considering the limits of this matrix we will show the following relation:

Theorem 1.1. *Let L be a μ-component link. Then,*

$$\left| \lim_{\omega_1 \to 1^\pm} \sigma_L(\omega_1, \omega) - \sigma_{L \setminus L_1}(\omega) - \sigma^\pm_{L, L_1}(\omega) \right| \leq \eta_{L \setminus L_1}(\omega) + \eta_{L, L_1}(\omega) - \lim_{\omega_1 \to 1^\pm} \eta_L(\omega_1, \omega),$$

for all $\omega \in \mathbb{T}_*^{\mu-1}$, where the correction terms $\sigma^\pm_{L,L_1}, \eta_{L,L_1}$ are invariant under link homotopy.

Moreover, we will provide a way to calculate the correction terms appearing in the statement of the theorem from some combinatorial data associated to L. The precise formulation of this is given in Theorems 3.11 and 3.14, the statements of which are too complex for the introduction.

Tools of knot theory have proved to be useful in the study of 4-dimensional manifolds, for example via Kirby calculus [17]. The Levine–Tristram signature and its multivariable version admit an interpretation in terms of invariants of 4-dimensional spaces. This allows it to be used to study manifolds obtained by surgery on links. For example, the multivariable signature determines some of the Casson-Gordon invariants of a manifold obtained in that way [5, Theorem 6.7]. We therefore wish to find another definition of signature as an invariant of 4-dimensional manifold, one which will extend well to the case of $\omega_i = 1$. This construction will lead to different formulas for the relation between the signature of L with some coordinate equal to 1 and the signature of a corresponding sublink. These will depend only on the linking numbers between components of L and the *slope* of the suitable component relative to L.

Theorem 1.2. *Let $L = L_1 \cup \ldots \cup L_\mu$ be an ordered link and let $\omega' \in \mathbb{T}_*^\mu$. Then, the following equations hold:*

$$\sigma_L(1, \omega') = \begin{cases} \sigma_{L'}(\omega') + \operatorname{sgn}((L_1/L)(\omega')) & \textit{if } \operatorname{lk}(L_1, L_i) = 0 \textit{ for all } i \geq 2, \\ \sigma_{L'}(\omega') & \textit{else}, \end{cases}$$

where L_1/L denotes the slope *of L_1 (cf. Definition 2.35) and we use the convention that* $\operatorname{sgn}(\infty) = 0$.

Theorem 1.3. *For all $\omega = (1, \omega') \in \mathbb{T}^\mu$ with $\omega' \in \mathbb{T}_*^{\mu-1}$, we have*

$$\eta_L(\omega) = \eta_{L'}(\omega') + \sum_{i=2}^{\mu} |\operatorname{lk}(L_1, L_i)| - 1,$$

if the linking numbers $\operatorname{lk}(L_1, L_i)$ are not all 0. If they all vanish, then we have

$$\eta_L(\omega) = \begin{cases} \eta_{L'}(\omega') + 1 & \text{if } (L_1/L')(\omega) = 0, \\ \eta_{L'}(\omega') - 1 & \text{if } (L_1/L')(\omega) = \infty, \\ \eta_{L'}(\omega') & \text{otherwise.} \end{cases}$$

These theorems improve on the results of Degtyarev, Florens and Lecuona regarding signature and nullity at 1. The precise relation between these theorems and [15, Lemma 4.9] will be elaborated on in Section 4.

The tools developed in this thesis can be further generalized to study multivariable signature of *colored links*, that is links where to each link component a label in $\{1, \ldots, \mu\}$ is assigned. The case of ordered links, considered here is a special example of colored links, where each link component has a different color. On the other end of the spectrum, a colored link with only one color is just an oriented link. In that case, the generalization of methods here can be used to prove the following [6, Theorem 5.4].

Theorem 1.4. *For any oriented link L, we have*

$$\left| \lim_{\omega \to 1} \sigma_L(\omega) - \operatorname{sign}(Lk_L) \right| \leq \operatorname{null}(Lk_L) - 1 - \operatorname{rank} A(L)\,,$$

where Lk_L is a matrix of linking numbers of L and $A(L)$ is the Alexander module of L.

This theorem strengthens the results of Borodzik and Zarzycki [3].

The organization of this thesis is as follows. In Section 2, we will recall all the necessary results of algebra and topology and outline previous constructions of link signature in detail. In Section 3, we will derive the Torres formula for signature and nullity by considering them from the 3-dimensional point of view, by looking at limits of families of matrices. In Section 4 we will give a definition of the signature and nullity in terms of invariants of 4-dimensional manifolds and examine its properties. Below we provide a more detailed outline, together with basic definitions, of Sections 3 and 4.

1.1. Outline of Section 3.

The Levine–Tristram signature can be defined as

$$\sigma_L^{LT}(\omega) = \operatorname{sgn}[(1-\omega)A + (1-\overline{\omega})A^T],$$

where A is any choice of a Seifert matrix of L.
The generalization of this link signature to multivariable functions that take each link component into consideration separately was introduced first in the case of two-component links by Cooper [11] and in full generality by Cimasoni and Florens [5]. Their construction expresses the multivariable link signature as the signature of a suitable matrix constructed from a *C-complex* chosen for the link L. Briefly, a C-complex is a collection of Seifert surfaces, for each component of the link, which intersect each other in a prescribed way, that of *clasp intersections*. The aforementioned matrix can be written as

$$H(\omega_1, \ldots, \omega_\mu) = \prod_{i=1}^{\mu} (1 - \overline{\omega}_i) A(\omega_1, \ldots, \omega_\mu),$$

where μ is the number of components of L, $\omega_i \neq 1$ are complex numbers of norm 1, and $A(t_1, \dots, t_\mu)$ is a matrix obtained from the chosen C-complex. Now, a key observation is that

$$\det A(t_1, \dots, t_\mu) \doteq \Delta_L(t_1, \dots, t_\mu),$$

the *multivariable Alexander polynomial* of L, where the symbol $\doteq$ denotes equality up to a normalization.
From the definition of H it is clear that if one tries to substitute 1 as the value of any of the ω_i, the whole matrix will be equal to zero. We can, however, consider instead the limits

$$\lim_{\omega_1 \to 1^\pm} \sigma_L(\omega_1, \dots, \omega_\mu) := \lim_{\omega_1 \to 1^\pm} \operatorname{sgn}(H(\omega_1, \dots, \omega_\mu))$$

of the signature function and ask if they carry interesting information about the link L. A key motivation is that for the Alexander polynomial the evaluation at 1 is closely related to the Alexander polynomial of a suitable sublink. Namely, we have, by a result of Torres [29]:

$$\Delta_L(1, t_2, \dots, t_\mu) = \begin{cases} \frac{t_2^{\mathrm{lk}(L_1,L_2)}-1}{t_2-1} \Delta_{L_2}(t_2) & \text{if } \mu = 2, \\ (t_2^{\mathrm{lk}(L_1,L_2)} \cdot \ldots \cdot t_\mu^{\mathrm{lk}(L_1,L_\mu)} - 1)\Delta_{L \setminus L_1}(t_2, \dots, t_\mu) & \text{if } \mu > 2. \end{cases}$$

Since the matrix H is closely related to the matrix A, this suggests that a similar relation should hold for the signature function. The main question we investigate in this thesis is: How to express the value of the differences

$$\lim_{\omega_1 \to 1^\pm} \sigma_L(\omega_1, \dots, \omega_\mu) - \sigma_{L \setminus L_1}(\omega_2, \dots, \omega_\mu),$$

$$\lim_{\omega_1 \to 1^\pm} \eta_L(\omega_1, \dots, \omega_\mu) - \eta_{L \setminus L_1}(\omega_2, \dots, \omega_\mu),$$

in terms of properties of the link L.
The question of the limit of the single-variable signature and nullity for links has already been considered in [3]. However, the approach taken there is not easily applicable to the study of multivariable signature. Moreover, the proof there uses nonstandard techniques of Hermitian variation structures and this approach in turn impose certain conditions on the Alexander polynomial of the link for the results to hold.
Our approach is as follows: First, we identify these differences with the signature and nullity of a limit of a one-parameter family of matrices, which we denote $\pm i M_{L,L_1}$. Then, we show that performing certain moves on the link L together with the chosen C-complex leave this limit matrix unchanged. Therefore, it will suffice to calculate the correction term for a suitably chosen representative in each equivalence class of links induced by those moves.
Unfortunately, we cannot simply state that the signature and nullity of a link L at $(1, \omega')$ are simply equal to sum of the signature and nullity of the sublink $L \setminus L_1$ and the correction terms, as the signature of a family of matrices can "jump" in the limit. Instead, we have the inequality of Theorem 1.1. We can, however note that in the case where $\Delta_L(\omega) \neq 0$ the right-hand side of the inequality vanishes.
We will begin exact calculations by first considering the case of 2-component links. In that case it is easy enough to obtain the desired formula, as the equivalence classes coincide with link-homotopy classes, for which a complete set of representatives is given by the torus links $T(2, 2\ell), \ell \in \mathbb{Z}$. This leads us to the following formula:

Theorem 1.5. *Let $L = L_1 \cup L_2$ be a two-component link with $\ell = \mathrm{lk}(L_1, L_2)$. Then, for $\omega_2 = e^{2i\pi\theta_2}$ with $\theta_2 \in [0, 1]$ the signature and nullity correction terms are expressed by the following formulas:*

- *If $\ell \neq 0$, then*

$$\sigma^+_{L,L_1}(\omega_2) = \sigma^-_{L,L_1}(\overline{\omega_2}) = \begin{cases} \ell - \operatorname{sgn}(\ell)\cdot(2k+1), & \text{for } \frac{k}{|\ell|} < \theta_2 < \frac{k+1}{|\ell|},\ k = 0,1,\dots,|\ell|-1, \\ \ell - \operatorname{sgn}(\ell)\cdot 2k, & \text{for } \theta = \frac{k}{|\ell|},\ k = 1,2,\dots,|\ell|-1\,. \end{cases}$$

$$\eta_{L,L_1}(\omega_2) = \begin{cases} 1, & \text{if } \omega_2 = \exp\left(\frac{2\pi i s}{\ell}\right),\ s = 1,2,\dots,|\ell|-1, \\ 0, & \text{else}\,. \end{cases}$$

- *If $\ell = 0$, then $\sigma^\pm_{L,L_1}(\omega_2) = 0$ and $\eta_{L,L_1}(\omega_2) = 0$.*

Finally, in the general case, we will be able to give the following formula:

Theorem 1.6. *Let L be a μ-component link together with a choice of a totally-connected C-complex S. Let the clasp intersections with S_1 be numbered from 1 to n in a way which agrees with orientation of L_1. We denote by $s(i) \in \{-1,+1\}$ the sign of i-th clasp and by $c(i)$ the label of the other surface in the clasp intersection. Then, the signature and nullity correction terms, $\sigma^\pm_{L,L_1}, \eta_{L,L_1}$ are given by the signature and nullity of a tridiagonal matrix with nonzero entries equal to:*

$$m_{i+1,i} = -\overline{m_{i,i+1}} = \frac{1}{1-\omega_{c(i+1)}^{s(i+1)}}, \qquad m_{i,i} = \frac{\omega_{c(i)}^{s(i)}\omega_{c(i+1)}^{s(i+1)} - 1}{(1-\omega_{c(i)}^{s(i)})(1-\omega_{c(i+1)}^{s(i+1)})}, \tag{1}$$

multiplied by $\pm i$.

This can be further strengthened to provide a formula for the correction terms which depends only on the linking numbers between L_1 and other link components of L [6, Theorem 3.1].

A further motivation for our study is the relation of the link signature to various constructions of 3- and 4-dimensional topology. It is well-known that the study of links is closely related to the study of 3- and 4-dimensional manifolds, for example through the Lickorish-Wallace theorem [22, 34] and through the tools of Kirby calculus [17].

1.2. **Outline of Section 4.** In this section, we will investigate the properties of link signature in terms of 4-dimensional manifold invariants. It is known that the multivariable signature admits an interpretation in terms of topological invariants of 4-manifolds, given first for characters of finite order by Cimasoni and Florens [5] and later refined to all nonvanishing characters by Viro [32]. We give here a quick review of this construction. For that, we need to introduce the notion of a *bounding surface* for a link.

Definition 1.7. *Let L be a μ-component link in $S^3 = \partial B^4$. A bounding surface for L is a collection of surfaces $F_i, 1 \le i \le \mu$, such that:*

- *Each F_i is a locally flat connected and oriented surface, properly embedded in B^4, so that $F_i \cap \partial B^4 = \partial F_i = L_i$;*
- *For each $i \neq j$ the surfaces F_i, F_j intersect each other transversally;*
- *For each i, j, k pairwise distinct the set $F_i \cap F_j \cap F_k$ is empty.*

One way to find bounding surfaces for L is to push a chosen C-complex for L inside B^4. We will often assume that bounding surfaces under our consideration arise in that way.

To define multivariable signature and nullity we will use the notion of *twisted homology groups*. For a manifold X together with a homomorphism from $\pi_1(X)$ to $\mathbb{C}^*$ we can consider the chain complex

$$C_*(\widetilde{X};\mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{C},$$

where the structure of $\mathbb{Z}[\pi_1(X)]$-module on $\mathbb{C}$ is defined via the chosen homomorphism and $\widetilde{X}$ is the universal covering space of X. The homology groups of this complex are called the twisted homology groups of X and are denoted by $H_*(X;\mathbb{C}^\omega)$. We will be concerned with twisted homology groups defined by compositions

$$\pi_1(X) \to \mathbb{Z}^\mu \xrightarrow{\omega} \mathbb{C}^*,$$

where each $\omega \in \mathbb{T}^\mu$ defines a homomorphism from $\mathbb{Z}^\mu, \mu \in \mathbb{N}$ to $\mathbb{C}^*$ by mapping the i-th generator of $\mathbb{Z}^\mu$ to the i-th coordinate of ω.
Twisted homology groups satisfy a lot of properties of ordinary homology groups, and for a four dimensional manifolds there exists a Hermitian form λ_ω on $H_2(X;\mathbb{C}^\omega)$ and we define the twisted signature and nullity of X at ω as the signature and nullity of λ_ω.

We can now use these ideas to give an alternative definition of the multivariable signature and nullity of links [10, Definition 3.2]:

Lemma 1.8. *Let F be a bounding surface for a μ-component link and let V_F denote the closure of $B^4 \setminus \nu(F)$, where $\nu(F)$ is a tubular neighborhood of F. There is a homomorphism from $\pi_1(V_F)$ to $\mathbb{Z}^\mu$ sending the i-th meridian to the i-th generator of $\mathbb{Z}^\mu$. Then, for each $\omega \in \mathbb{T}^\mu_* := (S^1 \setminus \{1\})^\mu$ the following hold:*

$$\sigma_L(\omega) = \operatorname{sign}_\omega V_F$$
$$\eta_L(\omega) = \operatorname{null}_\omega V_F.$$

If we allow ω to be an element of the entire μ-dimensional torus $\mathbb{T}^\mu$, then these identities no longer need to hold. Even worse, the twisted signature and nullity of V_F will in general depend on the particular choice of F in that case, so they will not be link invariants. Therefore, our first aim is to re-express the signature and nullity as invariants of a different manifold than the one previously considered. To that end, we will prove the following:

Theorem 1.9. *Let L be a μ-components link together with a suitable choice of a bounding surface F. Then, there exists a 4-dimensional manifold with boundary W_F, together with a homomorphism $\varphi : W_F \to \mathbb{Z}^\mu$ such that for each $\omega \in \mathbb{T}^\mu_*$ the following hold:*

$$\sigma_L(\omega) = \operatorname{sign}_\omega W_F$$
$$\eta_L(\omega) = \operatorname{null}_\omega W_F$$

and for each $\omega \in \mathbb{T}^\mu, \omega \neq (1, \dots, 1)$, the first homology group $H_1(\partial W_L; \mathbb{C}^\omega)$ vanishes.

This construction starts by assigning to a pair of a link and a suitable bounding surface a *plumbed manifold* $\operatorname{Pb}(G)$. By modifying the proof of [10, Lemma 4.9] we will show that this manifold is cobordant (in a way which respects homomorphisms from fundamental groups to $\mathbb{Z}^\mu$) to a disjoint union of products of closed surfaces with a circle. These can be closed off by gluing in products of handlebodies with a circle. Finally, we will show that by performing surgeries on thus obtained space to obtain a simply connected manifold Y_F. Finally, we will obtain the desired manifold W_F by gluing Y_F to V_F along a part of their boundaries.
We will use this definition of signature and nullity to prove the following theorems:

Theorem 1.10. *Let $L = L_1 \cup \ldots \cup L_\mu$ be an ordered link and let $\omega' \in \mathbb{T}^\mu_*$. Then, the following equations hold:*

$$\sigma_L(1, \omega') = \begin{cases} \sigma_{L'}(\omega') + \operatorname{sgn}((L_1/L)(\omega')) & \text{if } \operatorname{lk}(L_1, L_i) = 0 \text{ for all } i \geq 2 \\ \sigma_{L'}(\omega') & \text{else,} \end{cases}$$

where L_1/L denotes the slope *of L_1 (cf. Definition 2.35) and we use the convention that* $\operatorname{sgn}(\infty) = 0$.

Theorem 1.11. *For all $\omega = (1, \omega') \in \mathbb{T}^\mu$ with $\omega' \in \mathbb{T}_*^{\mu-1}$, we have*

$$\eta_L(\omega) = \eta_{L'}(\omega') + \sum_{i=2}^{\mu} |\operatorname{lk}(L_1, L_i)| - 1$$

if the linking numbers $\operatorname{lk}(L_1, L_i)$ are not all 0. If they all vanish, then we have

$$\eta_L(\omega) = \begin{cases} \eta_{L'}(\omega') + 1 & \textit{if } (L_1/L')(\omega) = 0 \\ \eta_{L'}(\omega') - 1 & \textit{if } (L_1/L')(\omega) = \infty \\ \eta_{L'}(\omega') & \textit{otherwise.} \end{cases}$$

Here, $(L_1/L)(\omega')$ denotes the *slope* of L_1 relative to L, as defined in [15]. While these formulas at a first glance look similar to the ones in [15, Lemma 4.9], there is an essential difference. The link signature and nullity as defined here are genuine link invariants, as opposed to their naive extension to $\mathbb{T}^\mu$, which is considered in the aforementioned article.
These formulas for the signature and the nullity at 1 can be related to the limits of these functions via an inequality different then the one obtained through 3-dimensional methods. We will compare these approaches and show that either of them can provide stronger results, depending on the link in question.

We say that two μ-component links are *concordant* if there exists a collection of embedded annuli $A = A_1 \sqcup \ldots \sqcup A_\mu$ in $S^3 \times I$, such that

$$\partial A_i = A_i \cap S^3 \times \{0, 1\} = L_i^0 \sqcup -L_i^1,$$

for each i, where L^j lies in $S^3 \times \{j\}$.
By expressing the signature and nullity of a link in terms of invariants of 4-manifolds, we can prove the following theorem on invariance under the relation of concordance:

Theorem 1.12. *If two μ-component links L^0, L^1 are concordant, then for each $\omega = (1, \omega')$ with $\omega' \in \mathbb{T}_{*!}^{\mu-1}$, we have*

$$\sigma_{L^0}(\omega) = \sigma_{L^1}(\omega).$$

Here, $\mathbb{T}_{*!}^{\mu-1}$ denotes the set of *non-concordance roots*, defined as the set of $\omega' \in \mathbb{T}^{\mu-1}$ for which $p(\omega) \neq 0$, where $p \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_{\mu-1}^{\pm 1}]$ is a Laurent polynomial such that $p(1, \ldots, 1) = 1$. This result generalizes the statement of [10, Theorem 7.1] as applied to concordances.

2. Preliminaries

All of the manifolds in this paper are assumed to be compact and oriented unless noted otherwise. A *knot* K is an embedding of a circle S^1 into S^3, considered up to an isotopy. A $\mu-$component *link* $L = L_1 \cup \ldots \cup L_\mu$ is an embedding of μ copies of S^1 into S^3, again considered up to an isotopy. A *Seifert surface* Σ for a knot K is an oriented compact surface in S^3 such that $\partial\Sigma = K$. These always exist for any knot [26].
We also introduce an equivalence relation for links weaker than that of equivalence, which is *link homotopy.* We call two links L^0, L^1 link homotopic if one can be deformed into the other by a homotopy $H : (S^1 \sqcup \ldots \sqcup S^1) \times [0, 1] \to S^3$ where the images of each copy of S^1 are disjoint at each $t \in [0, 1]$. Intuitively, we allow for self-intersections in each knot during the homotopy, but we still require each component not to cross any other component. Note that this means that any two knots are link-homotopic and that any link is link-homotopic to a link in which every component is an unknot.
For two disjoint oriented curves α, β in S^3 we can consider their *linking number* $\operatorname{lk}(\alpha, \beta)$ defined as the intersection number of β with any 2-chain C such that $\partial C = \alpha$. It is well-known that the intersection number can also be determined from any link diagram of α, β.

2.1. Complex linear algebra.

For a complex-valued matrix V we denote by V^* its *complex conjugate*, defined by

$$V^*_{ij} = \overline{V_{ji}}.$$

We call a matrix H *Hermitian* if $H = H^*$. The eigenvalues of a Hermitian matrix are all real, and therefore we can consider the signature and nullity of H: $\operatorname{sign} H$ is defined as the number of positive minus the number of negative eigenvalues and $\operatorname{null} H$ is defined as the multiplicity of zero eigenvalue.
Now, we want to consider one-parameter families of Hermitian matrices. We have:

Lemma 2.1. *Let $(H_t)_{t\in[0,\epsilon)}$ be a continuous one-parameter family of Hermitian matrices. Then*

$$\left| \lim_{t\to 0^+} \operatorname{sign}(H_t) - \operatorname{sign}(H_0) \right| \le \eta(H_0) - \lim_{t\to 0^+} \eta(H_t).$$

Proof. Observe that $\operatorname{rank}(H_t$ is constant for $t \in (0, \epsilon)$ for ϵ small enough. This in turn implies that $\operatorname{sign} H_t$ and $\eta(H_t)$ are constant for $t \in (0, \epsilon)$. At $t = 0$, precisely $\eta(H_0) - \lim_{t\to 0^+} \eta(H_t)$ eigenvalues vanish, yielding the expected upper bound on the difference of signatures. □

2.2. Gaussian elimination.

Whenever we are working with chain complexes, we often wish to simplify them through a procedure called *Gaussian elimination* [2, Lemma 4.2]. This takes the form of

Lemma 2.2. *Let V_1, V_2, W_1, W_2 be vector spaces over some ground field $\mathbb{F}$. Let C_* be a chain complex such that for some k, $C_k = V_1 \oplus V_2, C_{k-1} = W_1 \oplus W_2$ and the differential ∂_k is given by a matrix of functions $f_{ij} : V_i \to W_j$. Then, if the map f_{22} is an isomorphism the complex C_* is homotopy equivalent to a complex C'_* such that*

- *$C_i = C'_i$ for $i \neq k, k-1$;*
- *$\partial_i = \partial'_i$ for $i \neq k+1, k, k-1$;*
- *$C'_k = V_1, C'_{k-1} = W_1$;*
- *$\partial'_k = f_{11} - f_{21} f_{22}^{-1} f_{12}, \partial'_{k+1} = \pi_{V_1} \circ \partial_{k+1}, \partial'_{k-1} = \partial_{k-1} \circ \iota_{W_1}$, where π, ι are the respectively projections and inclusions of direct summands.*

Proof. We have the following morphisms of complexes:

$$\varphi : C'_* \to C_*, \quad \varphi_i = \begin{cases} \mathrm{Id} & \text{for } i \neq k, k-1 \\ \iota_{V_1} - \iota_{V_2} f_{22}^{-1} f_{12} & \text{for } i = k \\ \iota_{W_1} & \text{for } i = k-1 \end{cases}$$

$$\psi : C_* \to C'_*, \quad \psi_i = \begin{cases} \mathrm{Id} & \text{for } i \neq k, k-1 \\ \pi_{V_1} & \text{for } i = k \\ \pi_{W_1} - f_{21} f_{22}^{-1} \pi_{W_2} & \text{for } i = k-1 \end{cases}$$

Then, $\psi\varphi$ is just the identity morphism and a contracting homotopy H for $\varphi\psi$ can be constructed by:

$$H_i : C_i \to C_{i+1} = \begin{cases} 0 & \text{for } i \neq k-1, \\ -\iota_{V_1} f_{22}^{-1} \pi_{W_2} & \text{otherwise.} \end{cases}$$

□

Note that if $f_{12} = 0$ or $f_{21} = 0$ then the complex C'_* simply has $\partial'_k = f_{11}$. This means that to determine the homotopy equivalence class of the complex in that case we can simply omit V_2 and W_2.

2.3. **Rings with involution.** Let R be a ring. An *involution* of R is a map $(-)^* : R \to R$ such that:

- $(a^*)^* = a$;
- $(a+b)^* = a^* + b^*$;
- $(ab)^* = b^* a^*$.

An example of a ring with involution is the field of complex numbers $\mathbb{C}$ together with the complex conjugate operation. If G is a group, then the group ring $\mathbb{Z}[G]$ has an involution, which is defined on the basis of group elements by $(g)^* = g^{-1}$ and extended linearly to the entire ring.
Now, let R be a ring with involution and let M be a right R-module. We will denote by M^{tr} the *transposed* module, that is a left module with the same underlying abelian group and with the action of R defined by

$$r \cdot m = (m) \cdot r^*,$$

where the operation on the left is the action of R on M.

2.4. **Homology with twisted coefficients.** In order to define signature through a 4-dimensional approach, first we need to define the notions of *manifolds over* $\mathbb{Z}^\mu$ and *homology with twisted coefficients.*

Definition 2.3. *We call a pair* (M, φ)*, where* M *is a connected compact manifold (possibly with boundary) and* $\varphi : \pi_1(M) \to \mathbb{Z}^\mu$ *a manifold over* $\mathbb{Z}^\mu$*.*
We say that (N, φ_N) *is a submanifold of* (M, φ_M) *over* $\mathbb{Z}^\mu$ *if*

$$\varphi_N = \varphi_M \circ \iota_* : \pi_1(N) \to \mathbb{Z}^\mu,$$

where $\iota : N \to M$ *is the inclusion of a submanifold.*
By abuse of notation we will denote a manifold over $\mathbb{Z}^\mu$ *simply by* M *if no confusion arises from suppressing the homomorphism. By further abuse of language, we will also say that a disconnected manifold is a manifold over* $\mathbb{Z}^\mu$ *if each connected component is a manifold over* $\mathbb{Z}^\mu$

Each $\omega \in \mathbb{T}^\mu$ defines a homomorphism from $\mathbb{Z}^\mu$ to $\mathbb{C}^*$, mapping the i-th generator of $\mathbb{Z}^\mu$ to ω_i. Therefore, if M is a manifold over $\mathbb{Z}^\mu$ with $\pi = \pi_1(M)$, a choice of such an ω endows $\mathbb{C}$ with the structure of a (left) $\mathbb{Z}[\pi]$-module. Now, since the group π acts on the universal covering space $\widetilde{M}$ of M, the (singular) chain complex $C_*(\widetilde{M};\mathbb{Z})$ also admits a structure of a (right) $\mathbb{Z}[\pi]$-module. Thus, we can consider the *twisted chain complex*

$$C_*(M;\mathbb{C}^\omega) \coloneqq C_*(\widetilde{M};\mathbb{Z}) \otimes_{\mathbb{Z}[\pi]} \mathbb{C}$$

The homology of $C_*(M,\mathbb{C}^\omega)$ is called the *twisted homology* (or equivalently, *homology with twisted coefficients*) of M and is denoted by $H_*(M;\mathbb{C}^\omega)$. In particular, taking $\omega = (1,\dots,1)$ we recover the untwisted homology groups with complex coefficients. We define the twisted cohomology groups as the cohomology of the cochain complex

$$C^*(M;\mathbb{C}^\omega) \coloneqq \mathrm{Hom}_{\mathrm{left}-\mathbb{Z}[\pi]}(C_*(\widetilde{M};\mathbb{Z})^{\mathrm{tr}},\mathbb{C}).$$

If (M,N) is a pair of manifolds over $\mathbb{Z}^\mu$ we define *relative homology groups* $H_*(M,N;\mathbb{C}^\omega)$ as the homology groups of the chain complex $C_*(M,N;\mathbb{C}^\omega)$ which fits into the short exact sequence

$$0 \to C_*(N,\mathbb{C}^\omega) \to C_*(M;\mathbb{C}^\omega) \to C_*(M,N;\mathbb{C}^\omega) \to 0,$$

where the map on the left is induced by the inclusion of N in M.
Assume now that $M = \sqcup_{i=1}^n M_i$ is a disjoint union of a finite number of manifolds M_i, each over some $\mathbb{Z}^{\mu_i}$. In this case we will take as definition

$$C_*(M;\mathbb{C}^\omega) \coloneqq \bigoplus C_*(M_i;\mathbb{C}^{\omega_i}),$$

where $\omega = (\omega_1,\dots,\omega_n)$ and $\omega_i \in \mathbb{T}^{\mu_i}$ and define its twisted homology groups as the homology of this chain complex. In this case we define twisted cohomology and twisted relative homology groups analogously.

Proposition 2.4. Let (M,N) be a pair of manifolds over $\mathbb{Z}^\mu$ and let ω be any element of $\mathbb{T}^\mu$. Then, there exists a collection of homomorphisms

$$\partial_i : H_{i+1}(M,N;\mathbb{C}^\omega) \to H_i(N,\mathbb{C}^\omega),$$

such that

$$\dots \to H_{i+1}(M,N;\mathbb{C}^\omega) \xrightarrow{\partial_i} H_i(N,\mathbb{C}^\omega) \xrightarrow{\iota_*} H_i(M;\mathbb{C}^\omega) \xrightarrow{\pi_*} H_i(M,N;\mathbb{C}^\omega) \xrightarrow{\partial_i} \dots$$

is a long exact sequence, where ι_*, π_* are induced by inclusion and projection respectively.

Proof. Since the chain complexes $C_*(N;\mathbb{C}^\omega), C_*(M;\mathbb{C}^\omega), C_*(M,N;\mathbb{C}^\omega)$ by definition form a short exact sequence, their homology groups form the desired long exact sequence by [16, Proposition A3.17]. □

We will often make use of the following simple proposition (which appears as a part of [10, Lemma 2.6]). We also present an elementary proof of it for the readers' convenience.

Proposition 2.5. Let (X,φ) be a connected manifold over $\mathbb{Z}^\mu$ and let $\omega \in \mathbb{T}^\mu$ be such that the composition

$$\Phi : \pi_1(X) \xrightarrow{\varphi} \mathbb{Z}^\mu \xrightarrow{\omega} \mathbb{C}^*$$

is non-trivial. Then,

$$H_0(X;\mathbb{C}^\omega) = 0.$$

Proof. We will prove this result for X a CW-complex. Let x be a point in X, which we will take to be the unique 0-cell in the CW-decomposition of X. Choose any lift $\tilde{x}_0$ of x to the universal covering space $\widetilde{X}$. Let σ be a loop defining an element of $\pi_1(X,x)$ that is mapped to some $\alpha \neq 1$ in $\mathbb{C}$ by Φ. Consider the point $\tilde{x}_1 = \sigma(\tilde{x}_0)$, obtained from the natural action of $\pi_1(X,x)$ on $\widetilde{X}$. Denote by γ the 1-chain defined by σ. Then, in the tensor product

$$C_*(\widetilde{X},\mathbb{Z}) \otimes \mathbb{C}^\omega$$

we have $\tilde{x}_1 = \alpha\tilde{x}_0$ and $\partial\gamma = (\alpha-1)\tilde{x}_0 \neq 0$. Since all of the 0-cells in the CW-decomposition of $\widetilde{X}$ induced from the decomposition of X are lifts of x and since $\tilde{x}_0$ was chosen as any lift, this means that

$$\partial : C_1(\widetilde{X};\mathbb{Z}) \otimes \mathbb{C}^\omega \to C_0(\widetilde{X};\mathbb{Z}) \otimes \mathbb{C}^\omega$$

is onto and therefore $H_0(X;\mathbb{C}^\omega) = 0$ as desired. □

For manifolds over $\mathbb{Z}^\mu$ there is a notion of a product, which behaves similarly to the untwisted case:

Proposition 2.6 (Künneth formula for twisted homology)**.** Let $(X,\varphi_X),(Y,\varphi_Y)$ be two manifolds over $\mathbb{Z}^\mu,\mathbb{Z}^\nu$ respectively. We make the product $X\times Y$ into a manifold over $\mathbb{Z}^{\mu+\nu}$ via the composition

$$\varphi_{X\times Y} : \pi_1(X\times Y) \xrightarrow{\cong} \pi_1(X)\oplus\pi_1(Y) \xrightarrow{\varphi_X\oplus\varphi_Y} \mathbb{Z}^\mu\oplus\mathbb{Z}^\nu \xrightarrow{\cong} \mathbb{Z}^{\mu+\nu}.$$

Then, for any $\omega = (\omega_X,\omega_Y) \in \mathbb{T}^\mu\times\mathbb{T}^\nu \cong \mathbb{T}^{\mu+\nu}$ we have

$$H_k(X\times Y;\mathbb{C}^\omega) \cong \bigoplus_{i+j=k} H_i(X;\mathbb{C}^{\omega_X})\otimes H_j(Y;\mathbb{C}^{\omega_Y}).$$

Proof. The usual Künneth formula tells us that

$$C_k(\widetilde{X\times Y};\mathbb{Z}) \cong C_k(\widetilde{X}\times\widetilde{Y};\mathbb{Z}) \bigoplus_{i+j=k} C_i(\widetilde{X};\mathbb{Z})\otimes H_j(\widetilde{Y};\mathbb{Z}).$$

Now, since the subgroups $\pi_1(X),\pi_1(Y) \subset \pi_1(X\times Y)$ act trivially on the chain complexes $C_*(\widetilde{X};\mathbb{Z}), C_*(\widetilde{X};\mathbb{Z})$ respectively we get that

$$C_*(\widetilde{X};\mathbb{Z})\otimes_{\pi_1(X\times Y)}\mathbb{C} \cong C_*(\widetilde{X};\mathbb{Z})\otimes_{\pi_1(X)}\mathbb{C},$$

$$C_*(\widetilde{Y};\mathbb{Z})\otimes_{\pi_1(X\times Y)}\mathbb{C} \cong C_*(\widetilde{Y};\mathbb{Z})\otimes_{\pi_1(Y)}\mathbb{C},$$

which leads us directly to the desired result. □

We will often use the following corollary of the Künneth formula [32, Corollary B.B]:

Corollary 2.7. *Let $(X\times S^1,\varphi)$ be a manifold over $\mathbb{Z}^\mu$ and let $\omega\in\mathbb{T}^\mu$ be such that the composition*

$$\Phi : \pi_1(X\times S^1) \xrightarrow{\varphi} \mathbb{Z}^\mu \xrightarrow{\omega} \mathbb{C}$$

is non-trivial on the circle factor, i.e. $\Phi(*\times S^1)\neq 1$. Then,

$$H_*(X\times S^1;\mathbb{C}^\omega) = 0.$$

Proof. The universal covering space of a circle S^1 is the real line $\mathbb{R}$. We have a description of S^1 as a CW-complex consisting of one 0-cell, one 1-cell and no others. This decomposition lifts to a CW-complex structure of the real line consisting of 0-cells e_0^i and 1-cells e_1^i, both indexed by the integers, such that the cell e_1^i is attached to e_0^{i-1} and e_0^i. Therefore, the cellular chain complex of $\mathbb{R}$ can be described as follows:

$$C_0 = \oplus\mathbb{Z}[e_0^i], C_1 = \oplus\mathbb{Z}[e_1^i], \partial e_1^i = e_0^i - e_0^{i-1}.$$

Then, the fundamental group $\pi_1(S^1) = \mathbb{Z}[\sigma]$ acts on $C(\widetilde{S^1};\mathbb{Z})$ by

$$\sigma(e_k^i) = e_k^{i+1},$$

for $k = 0, 1$. If $\alpha = \Phi(* \times S^1)$ is a complex number in $S^1 \setminus \{1\}$, then in the tensor product

$$C(\widetilde{S^1};\mathbb{Z}) \otimes_{\mathbb{Z}[\pi]} \mathbb{C}$$

the terms $e_k^i, \alpha e_k^{i-1}$ are identified with each other. Therefore, we have

$$\partial(e_1^i) = (1-\alpha)e_0^{i-1},$$

which is an isomorphism for $\omega \neq 1$, which means that the circle S^1 is $\mathbb{C}^\omega$-acyclic. Now, the acyclicity of $X \times S^1$ follows directly from the Künneth formula. $\square$

Unlike in the case of ordinary homologies with coefficients in a ring, a short exact sequence is not enough to relate the groups $H_*(X;\mathbb{C}^\omega)$ to the groups $H_*(X;\mathbb{C})$. Instead, we will need to use the machinery of spectral sequences. Specifically, we can use the following, see [20, Theorem 2.3] and [23, Theorem 2.20]:

Lemma 2.8 (Universal Coefficient Spectral Sequence)**.** *Let (X,φ) be a manifold over $\mathbb{Z}^\mu$ and let $\omega \in \mathbb{T}^\mu$ define a twisted coordinate system on it. Denote by $\widetilde{X}$ the covering space of X induced by φ. Then, there exists a (first quadrant, homological) spectral sequence $E_k^{i,j}$ such that:*

- $E^2_{i,j} = \operatorname{Tor}^j_{\mathbb{C}[\pi_1(X)]}(H_i(\widetilde{X};\mathbb{C}),\mathbb{C}^\omega)$;
- $\bigoplus_{i+j=n} E^\infty_{i,j} \cong H_n(X;\mathbb{C}^\omega)$.

Note that since all the homology groups considered are vector spaces over a field, we are justified in stating that the group $H_n(X;\mathbb{C}^\omega)$ is a direct sum of the $E^\infty_{i,j}$ groups.

For homology with twisted coefficients we have, as in the untwisted case, see for example [9, Section 2.4]:

Theorem 2.9 (Poincaré-Lefschetz duality with twisted coefficients)**.** *Let M be an n-dimensional compact manifold over $\mathbb{Z}^\mu$. Then, there is a natural isomorphism*

$$H_i(M;\mathbb{C}^\omega) \xrightarrow{\text{PD}} H^{n-i}(M,\partial M;\mathbb{C}^\omega)$$

for each i.

We will often make us of the following lemma, which is a consequence of Poincaré duality:

Lemma 2.10. *Let M be a 3-dimensional manifold over $\mathbb{Z}^\mu$ with boundary ∂M. Denote by ι the inclusion $\partial M \to M$. Then, the following equation holds:*

$$\dim\ker(H_1(\partial M;\mathbb{C}^\omega) \xrightarrow{\iota_*} H_1(M;\mathbb{C}^\omega)) = \frac{1}{2}\dim H_1(\partial M;\mathbb{C}^\omega).$$

In particular, by taking the coordinate system to be trivial the following also holds:

$$\dim\ker(H_1(\partial M;\mathbb{C}) \xrightarrow{\iota_*} H_1(M;\mathbb{C})) = \frac{1}{2}\dim H_1(\partial M;\mathbb{C}).$$

Proof. Denote this kernel by K. From the long exact sequence of a pair (Proposition 2.4), we have that the sequence

$$\begin{aligned} 0 \to K \to H_1(\partial M;\mathbb{C}^\omega) \to H_1(M;\mathbb{C}^\omega) \to H_1(M,\partial M;\mathbb{C}^\omega) \\ \to H_0(\partial M;\mathbb{C}^\omega) \to H_0(M;\mathbb{C}^\omega) \to H_0(M,\partial M;\mathbb{C}^\omega) \to 0 \end{aligned}$$

is exact, as is the sequence

$$\begin{aligned}0 \to H_3(\partial M;\mathbb{C}^\omega) \to H_3(M;\mathbb{C}^\omega) \to H_3(M,\partial M;\mathbb{C}^\omega)\\ \to H_2(\partial M;\mathbb{C}^\omega) \to H_2(M;\mathbb{C}^\omega) \to H_2(M,\partial M;\mathbb{C}^\omega) \to K \to 0.\end{aligned}$$

Denote by $\beta_i(M)$ the dimension of $H_i(M;\mathbb{C}^\omega)$, and do analogously for ∂M, with the twisted coefficients being understood implicitly for the remainder of this proof.
From Poincaré duality and since $\mathbb{C}$ is a field we have that

$$\beta_i(M) = \dim H_{3-i}(M,\partial M;\mathbb{C}^\omega)$$

and

$$\beta_i(\partial M) = \beta_{2-i}(\partial M).$$

Noting that $\beta_3(\partial M) = 0$ and knowing that the Euler characteristic of an exact sequence is zero, we get after substitutions

$$\dim K - \beta_1(\partial M) + \beta_1(M) - \beta_2(M) + \beta_0(\partial M) - \beta_0(M) + \beta_3(M) = 0,$$

and

$$\beta_3(M) - \beta_0(M) + \beta_0(\partial M) - \beta_2(M) + \beta_1(M) - \dim K = 0.$$

By subtracting the second equation from the first we get

$$2\dim K - \beta_1(\partial M) = 0,$$

which was to be proved. □

2.5. **Twisted intersection forms.** Now, let M be a $2n$-dimensional manifold over $\mathbb{Z}^\mu$. We have the following sequence of homomorphisms [10, Definition 2.7]:

$$\Psi : H_n(M;\mathbb{C}^\omega) \xrightarrow{\iota_*} H_n(M,\partial M;\mathbb{C}^\omega) \xrightarrow{\mathrm{PD}} H^n(M;\mathbb{C}^\omega) \xrightarrow{\mathrm{ev}} \mathrm{Hom}_{\mathrm{left}-\mathbb{C}}(H_n(M;\mathbb{C}^\omega),\mathbb{C})^{\mathrm{tr}}.$$

We introduce the following:

Definition 2.11 (Twisted intersection pairing)**.** *We define the* twisted intersection pairing λ_ω *as*

$$\lambda_\omega : H_n(M;\mathbb{C}^\omega) \times H_n(M,\mathbb{C}^\omega) \to \mathbb{C}$$

by $\lambda_\omega(x,y) = \Psi(x)(y)$. In the case where $\omega = (1,\dots,1)$ we will call λ_ω the untwisted signature and denote it simply by λ.

This form is Hermitian in the case when n is even and anti-Hermitian when n is odd, but it is not necessarily nonsingular. More precisely, we have:

Lemma 2.12. *Let M be a $2n-$manifold over $\mathbb{Z}^\mu$ and choose $\omega \in \mathbb{T}^\mu$. Then, the subspace annihilated by λ_ω is precisely the image of $H_n(\partial M;\mathbb{C}^\omega)$ under the homomorphism induced by inclusion of the boundary.*

Proof. We have that the homomorphism ev in the sequence defining Ψ is an isomorphism by [10, Proposition 2.3] since $\mathbb{C}$ is a field. Therefore, the kernel of Ψ is equal to the kernel of the only non-isomorphism in the sequence. Finally, from the long exact sequence of the pair $(M,\partial M)$,

$$\dots \to H_2(\partial M;\mathbb{C}^\omega) \to H_2(\partial M;\mathbb{C}^\omega) \xrightarrow{\iota_*} H_2(M,\partial M;\mathbb{C}^\omega) \to \dots$$

we know that $\ker(\iota_*)$ is equal to the image of $H_2(\partial M;\mathbb{C}^\omega)$. □

We will be mostly interested in intersection forms in the case where $n = 2$. In this case, we want to note that we can further rephrase the dimension of this kernel via an application of Poincaré duality:

Proposition 2.13. Let M be a 4-dimensional manifold over $\mathbb{Z}^\mu$. Then the following equality holds:

$$\dim\ker(\Psi) = \dim\operatorname{im}(H_2(\partial M;\mathbb{C}^\omega) \to H_2(M;\mathbb{C}^\omega)) = \dim\ker(H_1(\partial M;\mathbb{C}^\omega) \to H_1(M;\mathbb{C}^\omega)).$$

Proof. Denote the image of $H_2(\partial M;\mathbb{C}^\omega)$ by I and the kernel of the morphism from $H_1(\partial M;\mathbb{C}^\omega)$ by K. From the long exact sequence of a pair we have that the following sequence is exact:

$$0 \to K \to H_1(\partial M;\mathbb{C}^\omega) \to H_1(M;\mathbb{C}^\omega) \to H_1(M,\partial M;\mathbb{C}^\omega) \\ \to H_0(\partial M;\mathbb{C}^\omega) \to H_0(M;\mathbb{C}^\omega) \to H_0(M,\partial M;\mathbb{C}^\omega) \to 0$$

as is the sequence

$$0 \to H_4(\partial M;\mathbb{C}^\omega) \to H_4(M;\mathbb{C}^\omega) \to H_4(M,\partial M;\mathbb{C}^\omega) \to H_3(\partial M;\mathbb{C}^\omega) \to H_3(M;\mathbb{C}^\omega) \\ \to H_3(M,\partial M;\mathbb{C}^\omega) \to H_2(\partial M;\mathbb{C}^\omega) \to I \to 0.$$

Denote the dimension of $H_i(M;\mathbb{C}^\omega)$ by $\beta_i^\omega(M)$. We observe that the group $H_4(\partial M;\mathbb{C}^\omega)$ vanishes, since the boundary of M is a closed 3-dimensional manifold. Thus, by considering the Euler characteristic of these exact sequences and applying Poincaré duality we get

$$\dim K - \beta_1^\omega(\partial M) + \beta_1^\omega(M) - \beta_3^\omega(M) + \beta_0^\omega(\partial M) - \beta_0^\omega(M) + \beta_4^\omega(M) = 0,$$

and

$$\beta_4^\omega(M) - \beta_0^\omega(M) + \beta_0^\omega(\partial M) - \beta_3^\omega(M) + \beta_1^\omega(M) - \beta_1^\omega(\partial M) + \dim I = 0.$$

By comparing the two expressions, we arrive at the desired equality. □

Now, we can make the following definition:

Definition 2.14. *Let M be a 4-dimensional over $\mathbb{Z}^\mu$. Then, for each $\omega \in \mathbb{T}^\mu$ we define the twisted signature and nullity of M as*

$$\operatorname{sign}_\omega M = \operatorname{sign}(\lambda_\omega) \tag{2}$$

$$\operatorname{null}_\omega M = \operatorname{null}(\lambda_\omega) = \dim\ker(H_1(\partial M;\mathbb{C}^\omega) \to H_1(M,\mathbb{C}^\omega)) \tag{3}$$

We will also be interested in the signature defect *of M, defined as*

$$\operatorname{dsign}_\omega(M) = \operatorname{sign}_\omega M - \operatorname{sign} M,$$

where $\operatorname{sign} M$ *is the ordinary (untwisted) signature of M.*

Note that in particular, if $H_1(M;\mathbb{C}^\omega)$ vanishes, then the nullity of M is equal simply to the dimension of $H_1(\partial M;\mathbb{C}^\omega)$.

2.6. **Maslov indices.** Let V be a real vector space equipped with an anti-symmetric bilinear form Φ. We recall the following definition:

Definition 2.15. *Let W be a subspace of V, Φ. We denote by $W^\perp$ the subspace*

$$W^\perp := \{v \in V \mid \Phi(v,w) = 0,\ \forall w \in W\}.$$

We call a subspace W Lagrangian *if $W = W^\perp$.*

Now, let A, B, C be Lagrangian subspaces of V. Consider the space

$$W = \frac{A \cap (B+C)}{(A\cap B) + (A \cap C)}.$$

The roles of A, B, C in the definition of W are interchangeable as there is a canonical isomorphism between

$$\frac{A \cap (B+C)}{(A\cap B) + (A \cap C)}, \frac{B \cap (A+C)}{(B\cap A) + (B \cap C)} \text{ and } \frac{C \cap (A+B)}{(C\cap A) + (C \cap B)}.$$

We represent the elements of $A \cap (B + C)$ as triples a, b, c with $a \in A, b \in B, c \in C$ such that $a + b + c = 0$.
We can now introduce a new bilinear pairing Ψ:

$$\Psi : A \cap (B + C) \times A \cap (B + C) \to \mathbb{R},$$

by $\Psi(a, a') := \Phi(a, b')$ where $b' \in B$ is such that $a' + b' + c' = 0$. The value $\Psi(a, a')$ does not depend on the choice of such b' and Ψ is a symmetric pairing.
Now, we define the *Maslov index* associated to V and A, B, C as the signature of Ψ and denote it by $\sigma(V; A, B, C)$. Direct calculations show that even permutations of A, B, C leave the Maslov index unchanged and odd permutations reverse its sign [33].

2.7. **Novikov additivity and Wall non-additivity.**

Theorem 2.16 (Novikov additivity [1])**.** *Let M_+, M_- be two manifolds such that $\partial M_+ = -\partial M_- = N$ and let M denote the closed manifold obtained by gluing M_+, M_- along N. Then,*

$$\operatorname{sign}(M) = \operatorname{sign}(M_+) + \operatorname{sign}(M_-).$$

Now, consider the situation where the gluing is done only along a part of the boundary of the manifold M. In this case we can find the signature of M through the celebrated theorem of Wall.

Theorem 2.17 (Wall non-additivity [33])**.** *Let M be a 4-dimensional manifold with boundary which can be decomposed into a union of 4-dimensional manifolds with boundary*

$$M = M_- \cup M_+$$

so that

$$\partial(M_\pm) = N_0 \cup N_\pm$$
$$\partial N_+ = \partial N_- = \partial N_0 = Z.$$

Then,

$$\operatorname{sign}(M) = \operatorname{sign}(M_+) + \operatorname{sign}(M_-) + \sigma(H_1(Z); \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3),$$

where $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are the kernels of the maps induced on first homology groups by the inclusions of Z into N_-, N_0, N_+ respectively and $\sigma(H_1(Z); \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ is the Maslov index as defined in subsection 2.6, where the bilinear pairing on V is the intersection form of Z.

We often want to make use of the following simple corollary:

Corollary 2.18. *Let $M = M_+ \cup_N M_-$ be a 4-dimensional manifold and assume that at least two of the three maps induced by inclusions*

$$H_1(\partial N) \to H_1(M_+), H_1(M_-), H_1(N)$$

have the same kernel.
Then,

$$\operatorname{sign}(M) = \operatorname{sign}(M_+) + \operatorname{sign}(M_-).$$

Proof. Assume without loss of generality that the spaces L_2, L_3 in the notation of Wall non-additivity above coincide. Then, the Maslov index can be calculated as the signature of a symmetric pairing on the space

$$W = \frac{L_1 \cap (L_2 + L_3)}{(L_1 \cap L_2) + (L_1 \cap L_3)} = \frac{L_1 \cap L_2}{L_1 \cap L_2} = 0,$$

and therefore the term $\sigma(V; L_1, L_2, L_3)$ must be equal to zero, from which our claim follows. $\square$

2.8. **Alexander modules and Alexander polynomials.** Let K be a knot in S^3. Denote by X_K the exterior of K, defined as the closure in S^3 of $S^3 \setminus \nu(K)$, where $\nu(K)$ is a tubular neighborhood of K. By Alexander duality we know that

$$H_1(X_K;\mathbb{Z}) \cong \mathbb{Z},$$

generated by a meridian m of K. This means that we can consider the covering space $\widetilde{X_K}$ of X_K defined by the homomorphism $\pi_1(X_K) \to \mathbb{Z}$ obtained by precomposing this isomorphism with the Hurewicz homomorphism. Denote by t a generator of the group of deck transformations of $\widetilde{X_K}$. This endows the homology group $H_1(\widetilde{X_K})$ with the structure of a $\Lambda = \mathbb{Z}[t,t^{-1}]$-module. This module, called the *Alexander module*, turns out to be torsion and the annihilator of it is generated by a single element, denoted by $\Delta_K(t)$. This polynomial is well-defined up to a multiplication by powers of $\pm t$ and we call it the *Alexander polynomial* of the knot K. If L is a $\mu-$component link, we can again consider the covering space of X_L associated to the Hurewicz map

$$\pi_1(X_L) \to H_1(X_L;\mathbb{Z}) \cong \mathbb{Z}^\mu,$$

where $\mathbb{Z}^\mu$ is generated by meridians of components of L. The homology group $H_1(\widetilde{X_L})$ considered as a module over $\Lambda_\mu := \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]$ is again called the (multivariable) Alexander module of L.
Now, let P be a presentation matrix of $H_1(\widetilde{X}_L)$ as a Λ_μ-module, that is an $m \times n$ matrix with entries in Λ_μ such that

$$\Lambda_\mu^m \xrightarrow{\cdot P} \Lambda_\mu^n \to H_1(\widetilde{X}_L) \to 0$$

is an exact sequence. We may assume that $m \geq n$ without loss of generality.

Definition 2.19. *The k-th Alexander ideal of L, denoted $E_k(L)$ is the ideal of Λ_μ generated by $(n-k)$-minors of P, with the convention that for $E_k(L) = \Lambda_\mu$ if $k > n$.*

It is a standard fact that the Alexander ideals do not depend on the particular choice of a presentation matrix P [26, Definition 8.B.3]. We define the multivariable Alexander polynomial of a link L to be the greatest common divisor of all elements of $E_0(L)$. This always exists since Λ_μ is a unique factorization domain, and is defined up to multiplication by an element of the form $\pm t_1^{r_1} \ldots t_\mu^{r_\mu}$, the invertible elements of Λ_μ.
For a knot K with a Seifert matrix A, the matrix $A - tA^T$ is a presentation matrix of its Alexander module [26, Theorem 8.C.3]. Since this is a square matrix, its 0-th ideal is generated by its determinant and the relation

$$\Delta_K(t) = \det(A - tA^T)$$

holds.

2.9. **Levine–Tristram signature.**

2.9.1. *Definition.* Let K be a knot with a Seifert surface Σ. As this is an oriented surface, we can find an embedding

$$F : \Sigma \times [-1,1] \hookrightarrow S^3$$

such that F restricts to identity of $\Sigma \times \{0\}$. Then, for any curve $\alpha \subset \Sigma$ we define its *push-offs* $\alpha^\pm$ as $F(\alpha, \pm 1)$. The homotopy class of $\alpha^\pm$ is well defined and does not depend on the choice of F.
Now, let $\{\alpha_i\}_{1 \leq i \leq 2g}$ be a family of curves on Σ such that the homology classes $[\alpha_i]$ constitute a basis of $H_1(\Sigma)$. We can then define the *Seifert matrix* A with respect to this basis as

$$A_{ij} = \operatorname{lk}(\alpha_i^-, \alpha_j).$$

Now, for any $\omega \in \mathbb{T}^1_*$ we can consider the matrix

$$(1-\omega)A + (1-\overline{\omega})A^T,$$

Note that this matrix is Hermitian and therefore we can consider its signature and nullity. These do not depend on the choice of Σ and basis $\{\alpha_i\}$ and therefore we get knot invariants - the Levine–Tristram signature and nullity of K:

Definition 2.20. *Let K be a knot in S^3 with a Seifert surface Σ and let A be the Seifert matrix defined as above. Then, the signature σ_K and nullity η_K of K are defined as [7]:*

$$\sigma_K(\omega) = \operatorname{sign}((1-\omega)A + (1-\overline{\omega})A^T) \tag{4}$$

$$\eta_K(\omega) = \operatorname{null}((1-\omega)A + (1-\overline{\omega})A^T) \tag{5}$$

for each $\omega \in \mathbb{T}^1_*$.

2.9.2. *4-dimensional definition.* Here, we provide an alternative construction of the Levine–Tristram signature, following [7]: Let L be a link in $S^3 = \partial D^4$ and let F be a *bounding surface* for L, that is an oriented surface properly embedded in D^4 such that $\partial F = L$. Denote by V_F the complement of a tubular neighborhood $\nu(F)$ of F in D^4. We have the following [10, Lemma 3.1]

Lemma 2.21. *Let F be a bounding surface for a μ-component link L and let V_F denote its exterior. Then,*

$$H_1(V_F;\mathbb{Z}) \simeq \oplus\mathbb{Z}\langle m_i\rangle,$$

where m_i is an oriented meridian of the i-th component of F.

Proof. Pick a small ball B_x around each intersection point of F. We have that $V_F{=}B^4 \setminus (\bigcup B_x \cup \bigcup \mathring{F}_i)$, where $\mathring{F}_i$ are the surfaces F_i with small discs removed around each intersection point. The Mayer-Vietoris sequence associated to decomposition $B^4 \setminus (\bigcup B_x) = V_F \cup \bigcup \mathring{F}_i$ gives us

$$0 \to H_1(\bigcup \mathring{F}_i \times S^1) \to H_1(\bigcup \mathring{F}_i \times D^2) \oplus H_1(V_F) \to 0,$$

where the zeros at both ends arise from the homology of $B^4 \setminus (\bigcup B_x)$, which vanish in dimension $1,2$. Applying the Künneth theorem to the products $\mathring{F}_i \times S^1$ yields

$$0 \to \bigoplus_{1\le i\le\mu} (H_1(\mathring{F}_i) \oplus H_1(p_i \times S^1)) \to (\bigoplus_{1\le i\le\mu} H_1(\mathring{F}_i)) \oplus H_1(V_F) \to 0,$$

where p_i are chosen basepoints on $\mathring{F}_i$. This after applying Lemma 2.2 to remove the homology groups of $\mathring{F}_i$ gives the exact sequence

$$0 \to \bigoplus_{1\le i\le\mu} H_1(p_i \times S^1) \to H_1(V_F) \to 0,$$

which concludes the proof. $\square$

We can therefore consider for any $\omega \in \mathbb{T}^1_*$ the homomorphism

$$H_1(V_F;\mathbb{Z}) \to \mathbb{C}^*,$$

sending each meridian to ω. This allows us to define twisted homology groups $H_*(V_F;\mathbb{C}^\omega)$ and we have that

$$\sigma_L(\omega) = \operatorname{sign}(\lambda_\omega),$$

where λ_ω is the intersection form on $H_2(V_F;\mathbb{C}^\omega)$.

2.9.3. *Properties.*
Limits at 1. The following proposition, due to Levine [21] tells us that the limits of the Levine–Tristram signature as ω is close to 1 are not interesting as invariants for knots:

Proposition 2.22. Let K be a knot with a choice of a Seifert surface S and let $\sigma_K(\omega) = \operatorname{sign}((1-\omega)A + (1-\overline{\omega})A^T)$ be its Levine–Tristram signature function. Then, we have

$$\lim_{\omega \to 1^{\pm}} \sigma_K(\omega) = 0,$$

where the limit is taken from either the positive or negative imaginary side of $1 \in S^1$.

Proof. Consider the matrix $(1-\omega)A+(1-\overline{\omega})A^T$. We know that multiplying a Hermitian matrix by a positive real number does not change its signature or nullity. This means we can look at the matrix

$$\hat{A}(\omega) = \frac{1}{|1-\omega|}((1-\omega)A + (1-\overline{\omega})A^T).$$

We have that $\operatorname{sign}(\hat{A}(\omega)) = \sigma_L(\omega)$ for $\omega \neq 1$. Now however, we know that

$$\lim_{\omega \to 1^+} \frac{1-\omega}{|1-\omega|} = -\lim_{\omega \to 1^+} \frac{1-\overline{\omega}}{|1-\omega|} = i$$

when we approach 1 from the positive imaginary side, and

$$\lim_{\omega \to 1^-} \frac{1-\omega}{|1-\omega|} = -\lim_{\omega \to 1^-} \frac{1-\overline{\omega}}{|1-\omega|} = -i$$

for negative imaginary side.
This gives us

$$\lim_{\omega \to 1^{\pm}} \hat{A}(\omega) = \pm iA \mp iA^T.$$

This matrix is obviously anti-symmetric, and since the eigenvalues of a Hermitian matrix are all real, this means that all of them vanish and the signature of the limit is equal to 0.
Now, we know that $\Delta_K(\omega) = \det(A - \omega A^T)$. Since for any knot K we have that $\Delta_K(1) = \pm 1$ the matrix

$$\hat{A}(\omega) = \frac{1}{|1-\omega|}((1-\omega)A + (1-\overline{\omega})A^T) = \frac{1-\omega}{|1-\omega|}(A - \overline{\omega}A^T)$$

has determinant of norm one. Therefore, it is non-degenerate in some neighborhood of 1 and so the limit of the signature is equal to the signature of the limit. Ultimately we have

$$\lim_{\omega \to 1} \sigma_K(\omega) = \operatorname{sign}(\lim_{\omega \to 1} \hat{A}(\omega)) = 0.$$

□

Note that this does not hold for links L of more than one component. To the contrary, we have [3]:

Theorem 2.23. *Let L be a μ-component link with Levine–Tristram signature function $\sigma_L^{LT}(\omega)$ and Alexander polynomial $\Delta_L(t)$. Define the matrix of linking numbers Lk by*

$$Lk_{ij} = \begin{cases} \operatorname{lk}(L_i, L_j) & \text{if } i \neq j \\ -\sum_{k \neq i} \operatorname{lk}(L_i, L_k) & \text{if } i = j \end{cases}$$

Then,

$$\lim_{\omega \to 1} \sigma_L(\omega) = \operatorname{sign}(Lk),$$

provided that $(t-1)^{\mu}$ does not divide $\Delta_L(t)$.

The proof of the above statement uses non-standard techniques, namely *Hermitian variational structures* and it does not easily extend to the case of multivariable signature.

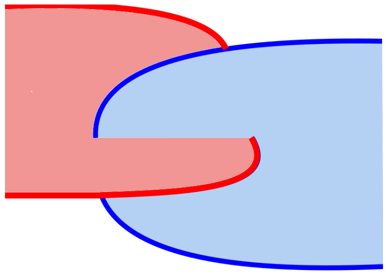


Figure 1. A clasp

2.10. **C-complexes.** Let L be a μ-component link. Although we can find an oriented surface Σ such that $\partial\Sigma = L_1 \cup \ldots \cup L_\mu$, this generalization of the concept of Seifert surface does not allow us to keep track on individual components. Instead, we wish to use the following definition:

Definition 2.24. *Let $L = L_1 \cup \ldots \cup L_\mu$ be a μ-component link. A C-complex S is a collection $S_1, \ldots, S_\mu$ of surfaces in S^3 such that:*

(1) *S_i is a Seifert surface for L_i,*
(2) *Any two S_i, S_j for $i \neq j$ intersect transversally in a collection of disjoint arcs, each of which has one endpoint on L_i and another on L_j,*
(3) *$S_i \cap S_j \cap S_k = \emptyset$ for any triple of distinct i, j, k.*

The intersections $S_i \cap S_j$ are called clasps*, see Figure 1. We also note that by* [11] *any link admits a C-complex.*

We say that a C-complex S is *totally connected* if the intersection of any two S_i, S_j is nonempty. Note that every link admits a totally connected C-complex [5, Lemma 2.2].
Let α be a curve in $S = S_1 \cup \ldots \cup S_\mu$ and $\varepsilon \in \{-1, +1\}$ be a choice of a sign for each component of S. We can then define the push-off α^ε by pushing α in the positive or negative direction from each component S_i according to ε_i. The condition (2) of 2.24 assures us that this is coherent defined whenever α goes over a clasp, in that the homology class of α^ε in $S^3 \setminus S$ is well-defined.

2.11. **Multivariable Levine–Tristram signature.** Here, we recall the definition of the generalization of Levine–Tristram signature to the case of links as defined by Cimasoni and Florens in [5].
Let L be a μ-component link with a choice of a C-complex S and let $\{\alpha_i\}$ be a set of curves in S such that their homology classes constitute a basis of $H_1(S)$. Now, we can consider matrices A^ε defined as

$$(A^\varepsilon)_{ij} = \mathrm{lk}(\alpha^\varepsilon, \alpha_j).$$

Note that $(A^\varepsilon)^T = A^{-\varepsilon}$.
Now, we can define the matrix

$$A(t_1, \ldots, t_\mu) = \sum_{\varepsilon \in \{-1,+1\}^\mu} \Big(\prod_{i=1}^{\mu} \varepsilon_i t_i^{\frac{1-\varepsilon_i}{2}}\Big) A^\varepsilon,$$

which allows us to keep track of all the possible choices of ε.
Finally, for $\omega \in \mathbb{T}_*^\mu$ we define

$$H(\omega_1, \ldots, \omega_\mu) = \prod_{i=1}^{\mu} (1 - \overline{\omega}_i) A(\omega_1, \ldots, \omega_\mu).$$

Note that this matrix is Hermitian.
The signature and nullity of this matrix turns out not to depend on the choice of a C-complex or the set of curves on S and therefore defines link invariants:

Definition 2.25. *Let L be a link in S^3 with a C-complex S and let H be the matrix defined as above. Then, the signature σ_L and nullity η_L of L are defined as*

$$\sigma_L(\omega) = \operatorname{sign} H(\omega) \tag{6}$$

$$\eta_L(\omega) = \operatorname{null} H(\omega) + \beta_0(S) - 1 \tag{7}$$

for each $\omega \in \mathbb{T}_*^\mu$.

2.12. **Torres formula for Alexander polynomial.** Let L be a link, S a C-complex for it and matrix $A(t_1, \dots, t_\mu)$ defined as in the previous subsection. Then, the multivariable Alexander polynomial of L is equal to $\det(A(t_1, \dots, t_\mu))$ up to multiplication by $(t_i - 1)$ [4].
Torres [29] proved there is a close relation between the value of Δ_L with t_1 evaluated at 1 and the multivariable Alexander polynomial of the link $L \setminus L_1$. Precisely, we have [11]:

$$\Delta_L(1, t_2, \dots, t_\mu) = \begin{cases} \frac{t_2^{\mathrm{lk}(L_1,L_2)}-1}{t_2-1}\Delta_{L_2}(t_2) & \text{if } \mu = 2 \\ (t_2^{\mathrm{lk}(L_1,L_2)} \cdot \ldots \cdot t_\mu^{\mathrm{lk}(L_1,L_\mu)} - 1)\Delta_{L\setminus L_1}(t_2, \dots, t_\mu) & \text{if } \mu > 2. \end{cases}$$

2.13. **Link concordance.** Let $L^i = L_1^i \cup \ldots \cup L_\mu^i, i = 0, 1$ be two ordered μ-component links in S^3. A *link concordance* between them is a collection of μ properly embedded disjoint annuli $A_1, \dots, A_\mu$ in $S^3 \times [0, 1]$ such that

$$\partial A_j = L_j^0 \sqcup -L_j^1,$$

with L_j^i lying in $S^3 \times \{i\}$. Note that our definition of concordance always preserves the ordering of components of L^i
The relation of concordance preserves linking numbers, in that

$$\mathrm{lk}(L_i^0, L_j^0) = \mathrm{lk}(L_i^1, L_j^1)$$

for any i, j.
If A is a concordance between L^0, L^1, we can consider its exterior, that is the complement of its tubular neighborhood in $S^3 \times I$, which we will denote as T_A. Then,

$$\partial T_A = X_{L^0} \sqcup X_{L^1}.$$

Definition 2.26. *Let W be a compact manifold, the boundary of which decomposes as a disjoint union of M and $-N$, so that W is a cobordism from M to N. We say that W is a* homology cobordism *if the inclusions of M and $-N$ into W induce isomorphisms on all homology groups. Equivalently, W is a homology cobordism if all the groups $H_*(W, M; \mathbb{Z})$ and $H_*(W, -N; \mathbb{Z})$ vanish.*

We have the following proposition about concordance exteriors.

Proposition 2.27. The relative homology groups $H_*(T_A, X_{L^i}; \mathbb{Z})$ all vanish.

Proof. We have the following relative Mayer-Vietoris exact sequence associated to the decomposition $(S^3 \times I, S^3 \times \{0\}) = (T_A, X_{L^0}) \cup (\nu(A), \nu(L^0))$:

$$\begin{aligned} \ldots \to H_i(\partial\nu(L^0) \times I, \partial\nu(L^0)) \to H_i(T_A, X_{L^0}) \oplus H_i(\nu(A), \nu(L^0)) \to \\ \to H_i(S^3 \times I, S^3 \times \{0\}) \to H_{i-1}(\partial\nu(L^0) \times I, \partial\nu(L^0)) \to \ldots \end{aligned}$$

Now, the product of any space X with an interval I homotopy retracts onto $X \times \{0\}$ and so their relative homology groups vanish. This means that we obtain the following short exact sequences

$$0 \to 0 \to H_i(T_A, X_{L^0}) \oplus H_i(\nu(A), \nu(L^0)) \to 0 \to 0$$

groups $H_i(T_A, X_{L^0})$ all vanish. The proof for the groups $H_i(T_A, X_{L^1})$ is analogous. □

Let us make the following definition:

Definition 2.28. *We call a tuple $\omega \in \mathbb{T}^\mu_*$ a* concordance root *if there exists a Laurent polynomial $p \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}]$ such that $p(1, \dots, 1) = \pm 1$ and $p(\omega) = 0$. We denote the set of $\omega \in \mathbb{T}^\mu_*$ that are* not *concordance roots by $\mathbb{T}^\mu_{*!}$.*

Note that the set of concordance roots is contained within the set of algebraic numbers lying in $\mathbb{T}^\mu$. Therefore, the subset $\mathbb{T}^\mu_{*!}$ is dense in $\mathbb{T}^\mu$, and in particular is nonempty.
The importance of the notion of non-concordance root lies in the following [10, Lemma 2.16]:

Lemma 2.29. *Let k be a non-negative integer and let ω lie in $\mathbb{T}^\mu_{*!}$. If (X, Y) is a pair of CW-complexes over $B\mathbb{Z}^\mu$ such that $H_i(X, Y; \mathbb{Z}) = 0$ for $0 \le i \le k$ the the groups $H_i(X, Y; \mathbb{C}^\omega)$ also vanish for $0 \le i \le k$.*

This applied to the pairs (X_L, T_{L^i}) shows us that for $\omega \in \mathbb{T}^\mu_{*!}$ we have

$$\sigma_{L^0}(\omega_1, \dots, \omega_\mu) = \sigma_{L^1}(\omega_1, \dots, \omega_\mu),$$

whenever L^0, L^1 are concordant to each other.

2.14. **Cobordism theory.** For manifolds over $\mathbb{Z}^\mu$ we have a notion of cobordism:

Definition 2.30. *We say two n-dimensional manifolds $(M_1, \varphi_1), (M_2, \varphi_2)$ over $\mathbb{Z}^\mu$ are cobordant over $\mathbb{Z}^\mu$ if there exists a (W, ψ) a $(n{+}1)$-dimensional manifold over $\mathbb{Z}^\mu$ such that $\partial W = M_1 \sqcup -M_2$ and both M_1 and $-M_2$ are submanifolds of W over $\mathbb{Z}^\mu$.*

Note that choosing a homomorphism $\varphi : \pi_1(M) \to \mathbb{Z}^\mu$ is equivalent to choosing a homotopy class of a map from M into $B\mathbb{Z}^\mu \cong \mathbb{T}^\mu = (S^1)^\mu$. Therefore, the set of n-dimensional manifolds over $\mathbb{Z}^\mu$ considered up to cobordism forms a group denoted by $\Omega_n(\mathbb{Z}^\mu)$, where the group operation is that of disjoint sum.
First, we want to note the following:

Proposition 2.31. Let M be a 4-dimensional closed manifold over $\mathbb{Z}^\mu$ which is a boundary of a 5-dimensional manifold N over $\mathbb{Z}^\mu$. Then,

$$\operatorname{sign} M = \operatorname{sign}_\omega M = 0$$

for each $\omega \in \mathbb{T}^\mu$.

Proof. The statement for the untwisted signature is well-known, see for example [17] and the twisted case is [32, Theorem D.B]. □

We have now the following fact, via a modification of [10, Proposition 2.10]:

Proposition 2.32. Let Z be a 4-dimensional manifold over $\mathbb{Z}^\mu$. If Z is closed, then

$$\operatorname{dsign}_\omega(Z) = 0$$

for any $\omega \in \mathbb{T}^\mu$

To prove this, we will want to have a description of the bordism group $\Omega_4(\mathbb{Z}^\mu)$. We will use the following proposition [12, Section 9.3]:

Proposition 2.33. Let G be a group with a classifying space BG. Then, there is the following isomorphism:

$$\Omega_4(G) \cong \Omega_4(\mathrm{pt}) \oplus H_4(BG;\mathbb{Z}),$$

where the image of a class of a manifold Z with a homomorphism $\psi : \pi_1(Z) \to G$ under this isomorphism is given by:

- The class of Z in the first factor;
- The image $\Psi_*([Z])$ in the second factor, where $\Psi : Z \to BG$ is the map to the classifying space of G determined by ψ.

Proof of Proposition 2.32. Since both the twisted and untwisted signatures vanish for any manifold which bounds over $\mathbb{Z}^\mu$, the signature defect defines a homomorphism from the group $\Omega_4(\mathbb{Z}^\mu)$ to integers. Since the classifying space of $\mathbb{Z}^\mu$ is the $\mu-$dimensional torus $\mathbb{T}^\mu$, we get the following isomorphism by Proposition 2.33:

$$\Omega_4(\mathbb{Z}^\mu) \cong \Omega_4(\mathrm{pt}) \oplus H_4(\mathbb{T}^\mu;\mathbb{Z}).$$

Now, it suffices to check that the signature defect vanishes on any set generators of the group on the right.
It is well-known that $\Omega_4(\mathrm{pt})$ is generated by the class of $\mathbb{C}P^2$. Since this space is simply-connected, any twisted signature is equal to the untwisted signature and the defect therefore vanishes.
The group $H_4(\mathbb{T}^\mu;\mathbb{Z})$ is generated by 4-dimensional subtori $\mathbb{T}^4 \subset \mathbb{T}^\mu$ given by inclusion of factors. Let T be such a torus considered as an element of $\Omega_4(\mathbb{Z}^\mu)$ and let ω be an arbitrary element of $\mathbb{T}^\mu$. The homomorphism $\pi_1(T) \to \mathbb{C}^*$ induced by ω is either trivial, or non-trivial. If it is trivial, then the twisted signature of T is equal to the untwisted one, and the defect vanishes. If it is non-trivial, it has to be non-trivial on some of the coordinate factors. However, we then have a decomposition $T \cong \mathbb{T}^3 \times \mathbb{T}^1$ with the coordinate system of the last circle induced by ω being properly twisted. By Corollary 2.7 this means that the twisted homology of T vanishes, and consequently the twisted signature is equal to zero. Since the ordinary signature of a 4-dimensional torus is zero, we get that the signature defect vanishes in this case as well.
Ultimately, the signature defect vanishes on all elements of $\Omega_4(\mathbb{Z}^\mu)$ for any $\omega \in \mathbb{T}^\mu$. □

Following [10, Corollary 2.11] and citing the proof here, we have now the following corollary, applicable to all $\omega \in \mathbb{T}^\mu$:

Corollary 2.34. *Let M be a 3-dimensional manifold over $\mathbb{Z}^\mu$ and let W, W' be two fillings of M over $\mathbb{Z}^\mu$. Then, $\operatorname{dsign}_\omega W = \operatorname{dsign}_\omega W'$ for each $\omega \in \mathbb{T}^\mu$.*

Proof. Define the closed oriented 4-manifold $Z := W \cup_M -W'$, and notice that the map to $\mathbb{Z}^\mu$ can be extended to Z. Thanks to Proposition 2.32, we have $\operatorname{sign}_\omega Z - \operatorname{sign} Z = 0$, and by Novikov additivity we get

$$0 = \operatorname{sign}_\omega Z - \operatorname{sign} Z = (\operatorname{sign}_\omega W - \operatorname{sign} W) - (\operatorname{sign}_\omega W' - \operatorname{sign} W').$$

□

2.15. **Slopes of links.** To obtain formulas for the signature and nullity at 1 when considered from a 4-dimensional point of view, we will need the notion of a *slope* associated to an ordered link. Here, we will review its definition and basic properties. The following results and terminology are due to Degtyarev, Florens and Lecuona [15, 13, 14] unless noted otherwise.

Definition 2.35. *Let $L = L_1 \cup \ldots \cup L_\mu$ be an ordered link and fix a $\omega' \in \mathbb{T}_*^{\mu-1}$. We further denote $(1, \omega')$ by ω. We call such an ω'* admissible *if*

$$\prod_{i=2}^{\mu} \omega_i^{\mathrm{lk}(L_1, L_i)} = 1.$$

Note that if we consider X_L as a manifold over $\mathbb{Z}^\mu$ in a standard way, with the $i-$th meridian being mapped to the $i-$th generator of $\mathbb{Z}^\mu$, the zero-framed longitude ℓ_1 of L_1 is mapped to the sum

$$\mathrm{lk}(L_1, L_2) \cdot [m_2] + \ldots + \mathrm{lk}(L_1, L_\mu) \cdot [m_\mu],$$

which then is mapped to $\prod_{i=2}^{\mu} \omega_i^{\mathrm{lk}(L_1, L_i)} = 1$ under the twisted coordinate system defined by ω. Therefore, the admissibility of ω' is equivalent to the the fact that coordinate system on $\partial\nu(L_1)$ induced by ω is trivial as in that case any longitude is also mapped to 1. In this case, we have

$$H_1(\partial X_L; \mathbb{C}^\omega) = \mathbb{C}[m_1] \oplus \mathbb{C}[\ell_1],$$

since ∂X_L is a disjoint union of 2-dimensional tori and each $\partial\nu(L)_i, i \neq 1$ is $\mathbb{C}^\omega$-acyclic, since the corresponding meridian m_i is mapped to $\omega_i \neq 1$. From Lemma 2.10 we know that the dimension of the kernel of

$$H_1(\partial X_L; \mathbb{C}^\omega) \to H_1(X_L; \mathbb{C}^\omega)$$

is equal to half the dimension of the domain, and therefore is equal to 1. Consequently, it is uniquely determined by the ratio

$$(L_1/L)(\omega') = -\frac{a}{b} \in \mathbb{C} \cup \infty,$$

where a, b are such that $am_1 + b\ell_1$ any element of the kernel. We call this quantity the slope *of L_1 relative to L at ω'.*

We remark here that our notation for the slope differs slightly from the one used in [15]. We have the following proposition [15, Proposition 3.6]:

Proposition 2.36. For any ordered link L and all $\omega' \in \mathbb{T}_*^{\mu-1}$, the slope takes real values, i.e., $(L_1/L)(\omega') \in \mathbb{R} \cup \infty$.

In particular, the slope of L_1 relative to L has a well-defined sign if it is finite. In this thesis, we will use a convention that $\mathrm{sgn}(\infty) = 0$.
Link slopes are often easy to calculate thanks to the following [15, Theorem 3.21], which we cite here without proof:

Theorem 2.37. *Let $\nabla_L(\omega)$ denote the Conway potential function and let $\nabla_L' = \frac{\partial}{\partial \omega_1} \nabla_L$ denote the derivative with respect to the first variable. Furthermore, choose a branch of the radical function in $\mathbb{C}$, so that $\sqrt{\omega'} = (\sqrt{\omega_2}, \ldots, \sqrt{\omega_\mu})$ is well-defined. Then, the following equality holds:*

$$(L_1/L)(\omega') = -\frac{\nabla_L'(1, \sqrt{\omega'})}{2\nabla_{L \setminus L_1}(\sqrt{\omega'})},$$

provided the right-hand side makes sense, that is the two expressions do not vanish simultaneously.

In general, the slope of a link at any $\omega' \in \mathbb{T}_*^{\mu-1}$ can be calculated from a diagram of L via Fox calculus, but that does not yield a closed formula.

3. Torres formula from the 3-dimensional perspective

3.1. **Limits of the matrix $H_L(\omega_1, \dots, \omega_\mu)$.** Let L be a μ-component link together with a fixed C-complex S. The goal of the section is to study the limit of the signature and nullity of L as one of the variables tends to 1. To that end we first want to choose a specific type of basis for the first homology group of S to calculate the matrix H with.

Definition 3.1. *Let $S = S_1 \cup \dots \cup S_\mu$ be a C-complex for L and let $\mathcal{A} = \{\alpha_1, \dots, \alpha_{s_a}\}$ be a choice of curves in $S \setminus S_1$ whose images form a basis of $H_1(S \setminus S_1)$. By an abuse of notation we will also denote by $\mathcal{A}$ the image of $\mathcal{A}$ under the inclusion homomorphism $H_1(S \setminus S_1) \to H_1(S)$. We can find a collection of curves $\mathcal{B} = \{\beta_1, \beta_2, \dots, \beta_{s_b}\}$ in S such that the collection $\mathcal{A} \cup \mathcal{B}$ is a basis of $H_1(S)$. We will call such a basis an* adapted basis.

The advantage of using an adapted basis is that it allows us to put the matrix $H(\omega)$ in a block form, with the blocks corresponding to either curves that necessarily go over go S_1 or ones that do not. We will use the following lemma:

Lemma 3.2. *Let L be a μ-component link with a choice of a C-complex S and a set of curves $\mathcal{A}$ which give a basis of $H_1(S)$, giving rise to a matrix $H_L(\omega)$. Then, for each $\alpha_i, \alpha_j \in \mathcal{A}$ such that α_i or α_j is a curve contained in $S \setminus S_1$, the ij-th (and therefore also the ji-th) entry of $H_L(\omega)$ is divisible by $(1 - \omega_1)$.*

Proof. Assume without the loss of generality that α_i lies in $S \setminus S_1$. Then, for each $\varepsilon \in \{-1, 1\}^{\mu - 1}$ we have that

$$\alpha_i^{+1,\varepsilon} = \alpha_i^{-1,\varepsilon}$$

since the first entry only governs the push-off from S_1, and so all linking numbers of these push-offs with any other curve are equal. This means that

$$(A^{+1,\varepsilon})_{ij} = (A^{-1,\varepsilon})_{ij}$$

in the notation of Subsection 2.11 and so the contributions to the sum in the definition of $A(t_1, \dots, t_\mu)$ differ only by the terms in the product $\prod_{i=1}^{\mu} \varepsilon_i t_i^{\frac{1-\varepsilon_i}{2}}$ and so we can factor out $1 - t_1$, from which our claim follows. Since the matrix is Hermitian, the same is true about $H_L(\omega)_{ji}$. □

We can now present the matrix H_L in block form:

$$H_L(\omega) = \begin{pmatrix} C & D \\ E & F \end{pmatrix},$$

where the blocks correspond to curves in $\mathcal{A}, \mathcal{B}$. The previous lemma tells us that the block matrices D, E, F are all divisible by $(1 - \omega_1)$. All of these matrices divisible by $(1 - \overline{\omega}_1)$ by the definition. We want to make the following observation.

Observation 3.3. *Let L, S and $\mathcal{A}$ be as above. Then, we have*

$$C = (1 - \omega_1)(1 - \overline{\omega}_1) H_{L \setminus L_1}(\omega_2, \dots, \omega_\mu),$$

where C is the block-matrix component of H_L corresponding to the curves in $\mathcal{A}$, that is curves that lie entirely in $S \setminus S_1$.

Now, we can define a new matrix

$$G_L = \frac{1}{|1 - \overline{\omega}_1|} P H_L P^*, \quad P = \begin{pmatrix} \frac{1}{(1-\overline{\omega}_1)^{1/2}} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix},$$

where P is presented in block form with blocks of sizes equal to the cardinalities of $\mathcal{A}, \mathcal{B}$ respectively. We now get the following

Lemma 3.4. *Let L be a μ-component link together with a C-complex S, let $H_L(\omega)$ be a matrix corresponding to an adapted basis of $H_1(S)$ and let G_L be the matrix $\frac{1}{|1-\overline{\omega}_1|}PH_LP^*$. Then,*

$$\lim_{\omega_1\to 1^\pm} G_L(\omega_1,\dots,\omega_\mu) = \begin{pmatrix} H_{L\setminus L_1}(\omega_2,\dots,\omega_\mu) & 0 \\ 0 & \pm iM \end{pmatrix},$$

where M is the matrix $F(1,\omega_2,\dots,\omega_\mu) - F(-1,\omega_2,\dots,\omega_\mu)$.

Proof. Follows immediately from Observation 3.3 and the facts that

$$\lim_{\omega_1\to 1^\pm} \frac{1-\overline{\omega}_1}{|1-\overline{\omega}_1|} = \mp i,$$

$$\lim_{\omega_1\to 1^\pm} \frac{1-\omega_1}{|1-\omega_1|} = \pm i,$$

$$\lim_{\omega_1\to 1^\pm} \frac{1-\omega_1}{(1-\omega_1)^{1/2}} = 0,$$

where the notation $\lim_{\omega\to 1^\pm}$ is taken to mean that ω_1 tends to 1 along the unit circle from the positive (resp. negative) imaginary side.

□

As the signature (nullity) of a Hermitian matrix in a block diagonal form is simply the sum of signatures (nullities) of the entries, we get that

$$\operatorname{sgn}[\lim_{\omega_1\to 1^\pm} G_L(\omega_1,\dots,\omega_\mu)] = \operatorname{sgn} H_{L\setminus L_1}(\omega_2,\dots,\omega_\mu) \pm \operatorname{sgn}(-iM(\omega_2,\dots,\omega_\mu)).$$

$$\operatorname{null}[\lim_{\omega_1\to 1^\pm} G_L(\omega_1,\dots,\omega_\mu)] = \operatorname{null}\ H_{L\setminus L_1}(\omega_2,\dots,\omega_\mu) \pm \operatorname{null}(-iM(\omega_2,\dots,\omega_\mu)).$$

We will denote the correction term $\operatorname{sgn}(-iM(\omega_2,\dots,\omega_\mu))$ by $\sigma^+_{L,L_1}(\omega_2,\dots,\omega_\mu)$ and $\sigma^-_{L,L_1} = -\sigma^+_{L,L_1}$ so that

$$\operatorname{sgn}[\lim_{\omega_1\to 1^\pm} G_L(\omega_1,\dots,\omega_\mu)] = \operatorname{sgn}\ H_L(\omega_2,\dots,\omega_\mu) + \sigma^\pm_{L,L_1}(\omega_2,\dots,\omega_\mu).$$

We also define the nullity correction term

$$\eta_{L,L_1}(\omega_2,\dots,\omega_\mu) \coloneqq \operatorname{null}\ iM(\omega_2,\dots,\omega_\mu).$$

Note that the nullity correction term does not depend on the direction from which we take the limit, as the nullity of a matrix and its negative are the same.

Finally, we can formulate our main theorem:

Theorem 3.5. *Let L be a μ-component link. Then, for all $\omega \in \mathbb{T}_*^{\mu-1}$*

$$\left|\lim_{\omega_1\to 1^\pm} \sigma_L(\omega_1,\omega) - \sigma_{L\setminus L_1}(\omega) - \sigma^\pm_{L,L_1}(\omega)\right| \le \eta_{L\setminus L_1}(\omega) + \eta_{L,L_1}(\omega) - \lim_{\omega_1\to 1^\pm} \eta_L(\omega_1,\omega),$$

where the correction terms $\sigma^\pm_{L,L_1}, \eta_{L,L_1}$ *are invariant under link homotopy.*

Proof. We wish to apply Lemma 2.1 to the family $A_t = G_L(e^{it},\omega_2,\dots,\omega_\mu)$. We obtain then

$$\left|\lim_{t\to 0^+} \operatorname{sign}(G_L(e^{it},\omega_2,\dots,\omega_\mu)) - \operatorname{sign}(G(1,\omega_2,\dots,\omega_\mu))\right|$$
$$\le \eta(G_L(1,\omega_2,\dots,\omega_\mu)) - \lim_{t\to 0^+} \eta(G_L(e^{it},\omega_2,\dots,\omega_\mu)).$$

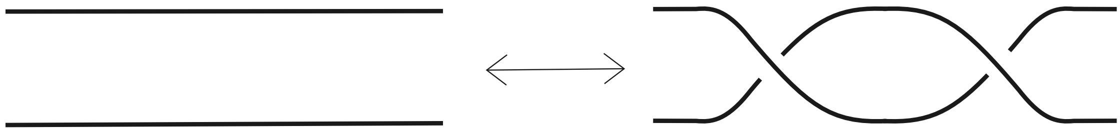


Figure 2. A band-twist move

By definition the signature and nullity of $G_L(\omega_1, \dots, \omega_\mu)$ are equal to, respectively, the signature and nullity of $H_L(\omega_1, \dots, \omega_\mu)$ for $\omega_1 \neq 1$. Therefore,

$$\left| \lim_{t \to 0^+} \operatorname{sign}(H_L(e^{it}, \omega_2, \dots, \omega_\mu)) - \operatorname{sign}(G_L(1, \omega_2, \dots, \omega_\mu)) \right| \\ \leq \eta(G_L(1, \omega_2, \dots, \omega_\mu)) - \lim_{t \to 0^+} \eta(H_L(e^{it}, \omega_2, \dots, \omega_\mu)).$$

As by definition the signature and nullity of H_L are equal to the signature and nullity of the link respectively, we get

$$\left| \lim_{t \to 0^+} \sigma_L(e^{it}, \omega_2, \dots, \omega_\mu) - \operatorname{sign}(G_L(1, \omega_2, \dots, \omega_\mu)) \right| \\ \leq \eta(G_L(1, \omega_2, \dots, \omega_\mu)) - \lim_{t \to 0^+} \eta_L(e^{it}, \omega_2, \dots, \omega_\mu)).$$

Finally, using Lemma 3.4 we arrive at

$$\left| \lim_{t \to 0^+} \sigma_L(e^{it}, \omega) - \sigma_{L \setminus L_1}(\omega) - \sigma^+_{L, L_1}(\omega) \right| \leq \eta_{L \setminus L_1}(\omega) + \eta_{L, L_1}(\omega)) - \lim_{t \to 0^+} \eta_L(e^{it}, \omega),$$

and since multiplying by a scalar does not change the nullity of a matrix, we have proven our claim. ☐

The matrix $M(\omega_2, \dots, \omega_\mu)$ appearing in the limit can be further written as $M_+(\omega_2, \dots, \omega_\mu) - M_-(\omega_2, \dots, \omega_\mu)$, where

$$M_\pm = \prod_{i=2}^{\mu} (1 - \bar{\omega_i}) \sum_{\varepsilon \in \{+1, -1\}^\mu, \varepsilon_1 = \pm 1} \varepsilon_2 \dots \varepsilon_\mu \omega_2^{\frac{1-\varepsilon_2}{2}} \dots \omega_\mu^{\frac{1-\varepsilon_\mu}{2}} M^\varepsilon,$$

and the matrix M^ε is defined by

$$m^\varepsilon_{ij} = \operatorname{lk}(\beta^\varepsilon_i, \beta_j).$$

Therefore, the matrix M depends only on the differences in linking numbers between positive and negative push-offs of curves from S_1. To make use of this fact, we will need the following result, adapted from [11, Theorem 2.1].

Lemma 3.6. *Let S be a C-complex and let S' be obtained from S by* twisting a band*, depicted pictorially in Figure 2. Then for each $\omega \in \mathbb{T}^{\mu-1}$ we have $M(\omega_2, \dots, \omega_\mu) = M'(\omega_2, \dots, \omega_\mu)$ under the natural identification of bases for $H_1(S)$ and $H_1(S')$, where M, M' are correction term matrices obtained from the C-complexes S, S' respectively.*

Proof. If the twisted band was contained in a component other than S_1 there is nothing to prove, as this move does not change any push-off from S_1 in that case. If the move was performed on S_1, then we use the fact that the linking number can be computed by counting signed crossings in a given link diagram. Since the band-twist move changes the link diagram only locally, this operation affects only the entries of M corresponding to a pair curves both going over this band. In this case both the positive and negative push-offs are affected in the same way, and therefore the total contribution to M vanishes. ☐

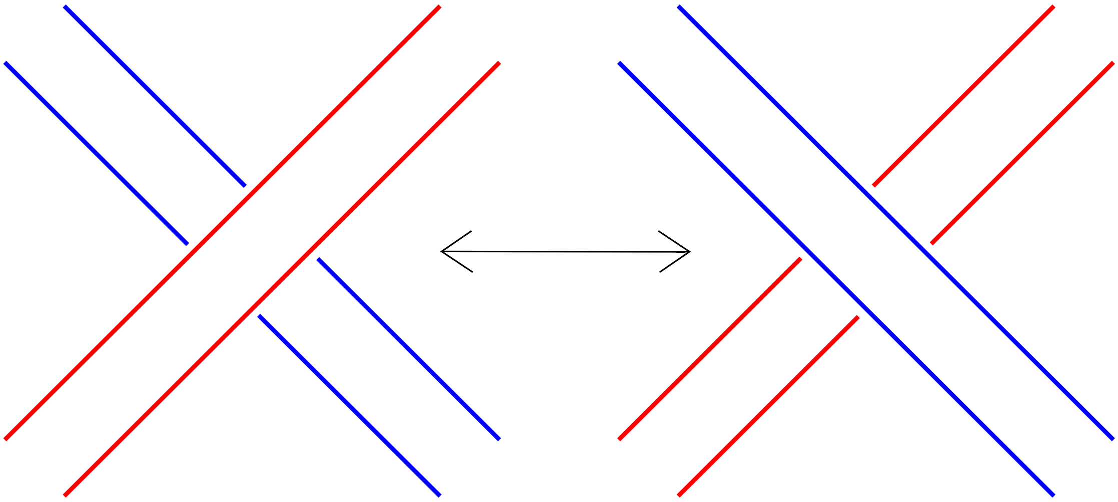


Figure 3. A band-pass move

We also make a second observation of a similar type:

Lemma 3.7. *Let S be a C-complex and let S' be obtained from S by making a* band-pass move*, depicted pictorially in Figure 3. Then for each $\omega \in \mathbb{T}^{\mu-1}$ we have $M(\omega_2, \dots, \omega_\mu) = M'(\omega_2, \dots, \omega_\mu)$ under the natural identification of bases for $H_1(S)$ and $H_1(S')$, where M, M' are correction term matrices obtained from the C-complexes S, S' respectively.*

Proof. Consider $\alpha, \beta \in \mathcal{A}$. Since the linking number can be computed by counting signed crossings in a given link diagram, it is clear that the only way a band-pass move can potentially change the linking number of a push-off of α with β is if α goes over one band and β over the other one. In this case, however, the push-offs of α from the positive and negative side both either lie below or above the other band and thus the difference $\operatorname{lk}(\alpha^{+1,\varepsilon}, \beta) - \operatorname{lk}(\alpha^{-1,\varepsilon}, \beta)$ is unchanged. □

3.2. **Two-component links.** If $L = L_1 \cup L_2$ is a two-component link the question of identifying the correction terms $\sigma^{\pm}_{L,L_1}(\omega_2), \eta_{L,L_1}(\omega_2)$ turns out to be simple. By the Lemma 3.6 we know that the correction term of the signature is an invariant of the homotopy type of a link. In the case of two-component links, we know that the linking number $\ell = \operatorname{lk}(L_1, L_2)$ is a complete invariant [24]. Therefore, an expression of the correction terms for a single two-component link with linking number $\ell \in \mathbb{Z}$ hold for each such a link. We will now restrict our attention to the family of torus links, $T(2, 2\ell)$ which we will use later as an example in calculating signature and nullity correction terms.

Example 3.8. Let $L = L_1 \cup L_2$ be the $T(2, 2\ell)$ torus link and let $\omega_i = e^{2i\pi\theta_i}$, with $\theta_i \in (0, 1)$. Then, its signature is given by

$$\sigma_{T(2,2\ell)}(\omega_1, \omega_2) = \operatorname{sgn}(\ell) \cdot f_{|\ell|}(\theta_1 + \theta_2),$$

where $f_n : [0, 2] \to \mathbb{Z}$ are functions defined by

$$f_2(\theta) = \begin{cases} n - 2k - 1 & \text{for } \frac{k}{n} < \theta < \frac{k+1}{n} \text{ with } k = 0, \dots, n-1, \\ n - 2k & \text{for } \theta = \frac{k+1}{n} \text{ with } k = 0, \dots, n-1, \\ 1 - n & \text{for } \theta = 1, \end{cases} \qquad \text{and} \quad f_n(2 - \theta) = f_n(\theta).$$

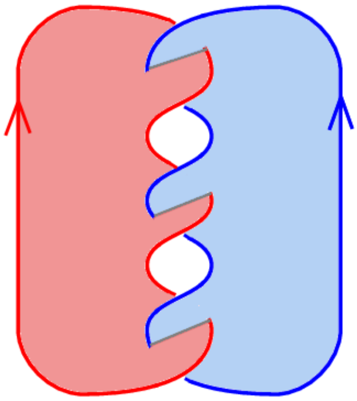


Figure 4. A C-complex for the link $T_{2,6}$; the gray lines denote the intersection of the two Seifert surfaces.

The nullity of $T(2,2\ell)$ is given by

$$\eta_{T(2,2\ell)}(\omega_1,\omega_2)=\begin{cases}1 & \text{for } \theta_1+\theta_2=\frac{k}{|\ell|}, \text{ where } k=1,\dots,|\ell|-1,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. The case where $\ell=0$ is obvious, as the signature and nullity functions of the two-component unlink are constant.
It is sufficient to consider the case $\ell>0$ since the following identities [5, Proposition 2.10] hold:

$$\sigma_{T(2,2\ell)}=-\sigma_{T(2,-2\ell)}$$
$$\eta_{T(2,2\ell)}=\eta_{T(2,-2\ell)}$$

Now, we consider a C-complex pictured in Figure 4. The matrices A^{ε} have size $(\ell-1)\times(\ell-1)$ and are given by the following formulas:

$$A^{--}=-\mathrm{sgn}(\ell)\begin{bmatrix}1 & 0 & \dots & 0\\ 1 & 1 & \dots & 0\\ \vdots & \ddots & \ddots & 0\\ 0 & \dots & 1 & 1\end{bmatrix},$$

A^{++} is the transpose of A^{--}, and $A^{+-}=A^{-+}=0$. Then, by [11, Theorem 2.1] we know that a presentation matrix for the Alexander module of L can be given by

$$P=\begin{bmatrix}t_1t_2+1 & 1 & \dots & 0\\ t_1t_2 & t_1t_2+1 & \dots & 0\\ \vdots & \ddots & \ddots & 1\\ 0 & \dots & t_1t_2 & t_1t_2+1\end{bmatrix}.$$

From this presentation, we easily calculate that

$$\Delta_{T(2,2\ell)}(t_1,t_2)=\frac{(t_1t_2)^{\ell}-1}{t_1t_2-1},$$

and the set of its zeroes in $\mathbb{T}^2$ is readily seen to be

$$\{(\omega_1,\omega_2)\mid \omega_1\omega_2=e^{\frac{2ki\pi}{\ell}}, k=1\dots,\ell-1\}.$$

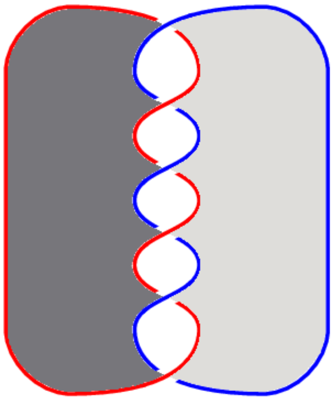


Figure 5. A Seifert surface for the link $T(2,6)$.

We can now consider matrices Q_k obtained from P by restricting it to first $\ell-1-k$ rows and columns 2 to $\ell-k$, where $1 \leq k \leq \ell-2$. The determinant of each of them is equal to 1, which shows that each Alexander ideal $E_k(L), k>0$ is equal to entire ring Λ_2. Thus, by [5, Theorem 4.1] we get that σ_L and η_L are constant on each diagonal $\omega_1\omega_2 = \text{const}$. We also know by [5, Proposition 2.5] that $\sigma_{T(2,2\ell)}(\omega,\omega) = \sigma^{LT}_{T(2,2\ell)}(\omega) + \ell$, where the second function is the ordinary Levine–Tristram signature of $T(2,2\ell)$. Therefore, we conclude that

$$\sigma_{T(2,2\ell)}(\omega_1,\omega_2) = \sigma^{LT}_{T(2,2\ell)}(\omega_1^{1/2}\omega_2^{1/2}) + \ell,$$

and similarly

$$\eta_{T(2,2\ell)}(\omega_1,\omega_2) = \eta^{LT}_{T(2,2\ell)}(\omega_1^{1/2}\omega_2^{1/2}),$$

for any choice of a branch of the complex root function. Finally, we need an explicit formula for the Levine–Tristram signature of a torus link. We can choose the following Seifert surface:

This gives us the Seifert matrix

$$A = -\begin{bmatrix} 1 & 0 & \dots & 0 \\ -1 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & -1 & 1 \end{bmatrix}$$

where the size of the matrix is $(2\ell-1)\times(2\ell-1)$. This means that the signature $\sigma^{LT}_{T(2,2\ell)}$ is equal to the signature of the matrix

$$-\begin{bmatrix} 2-2\mathrm{Re}(\omega) & \overline{\omega}-1 & \dots & \dots & 0 \\ \omega-1 & 2-2\mathrm{Re}(\omega) & \overline{\omega}-1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \overline{\omega}-1 \\ 0 & \dots & \dots & \omega-1 & 2-2\mathrm{Re}(\omega) \end{bmatrix}$$

The eigenvalues of this matrix are given by roots of the second type Chebyshev polynomial by [18, Theorem 2.2], and are given by:

$$2\mathrm{Re}(\omega) - 2 - 2|\overline{\omega}-1|\cos(\frac{k\pi}{2\ell}), \quad k = 1,\dots,2\ell-1\,.$$

Now, we want to make a change of variable, $\omega = e^{2i\pi\theta}$, with $\theta \in (0,1)$. Using the identities $1-\omega = -2i\sin(\pi\theta)e^{i\pi\theta}$ and $\mathrm{Re}(\omega) = \cos(2\pi\theta) = 1-2\sin^2(\pi\theta)$ we get

$$2\mathrm{Re}(\omega)-2-2|1-\omega|\cos(\frac{k\pi}{2\ell}) = -4\sin^2(\pi\theta)+4\sin(\pi\theta)\cos(\frac{k\pi}{2\ell}) = 4\sin(\pi\theta)\cdot(\cos(\frac{k\pi}{2\ell})-\sin(\pi\theta)).$$

Since $\theta \in (0,1)$, the factor $4\sin(\pi\theta)$ is positive. Therefore, we get that the whole expression for a given k is negative for $\theta \in (\frac{1}{2}-\frac{k}{2\ell}, \frac{1}{2}+\frac{k}{2\ell})$, zero for $\theta \in \{\frac{1}{2}-\frac{k}{2\ell}, \frac{1}{2}+\frac{k}{2\ell}\}$, and positive otherwise. Note that since θ is in $[0,1]$ this means that for $k > \ell$ this expression is always negative. This gives us the following formula for the signature function:

$$\sigma^{LT}_{T(2,2\ell)}(e^{2i\pi\theta}) = \begin{cases} -2k-1 & \text{for } \frac{k}{2\ell} < \theta < \frac{k+1}{2\ell} \text{ with } k = 0,\dots,\ell-1, \\ -2k & \text{for } \theta = \frac{k}{2\ell} \text{ with } k = 0,\dots,\ell-1, \\ -2\ell+1 & \text{for } \theta = \frac{1}{2}, \end{cases}$$

when $\theta \in [0,\frac{1}{2}]$ and $\sigma^{LT}_{T(2,2\ell)}(e^{2i\pi(1-\theta)}) = \sigma^{LT}_{T(2,2\ell)}(e^{2i\pi\theta})$.
We also obtain the formula for the single-variable nullity function:

$$\eta^{LT}_{T(2,2\ell)}(e^{2i\pi\theta}) = \begin{cases} 1 & \text{if } \theta = \frac{k}{2\ell} \text{ for } k = 1,\dots,\ell-1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, in the multivariable case we get the following expression for the multivariable signature of $T(2,2\ell)$:

$$\sigma_{T(2,2\ell)}(\omega_1,\omega_2) = \sigma_{T(2,2\ell)}(e^{2i\pi\theta_1}, e^{2i\pi\theta_2}) = \sigma^{LT}_{T(2,2\ell)}(e^{i\pi(\theta_1+\theta_2)}) + \ell = \mathrm{sgn}(\ell)\cdot f_{|\ell|}(\theta_1+\theta_2),$$

where $f_n : [0,2] \to \mathbb{Z}$ are functions defined by

$$f_2(\theta) = \begin{cases} n-2k-1 & \text{for } \frac{k}{n} < \theta < \frac{k+1}{n} \text{ with } k = 0,\dots,n-1, \\ n-2k & \text{for } \theta = \frac{k+1}{n} \text{ with } k = 0,\dots,n-1, \\ 1-n & \text{for } \theta = 1, \end{cases}$$

for $\theta \in [0,1]$ and $f_n(2-\theta) = f_n(\theta)$. We also get that

$$\eta_{T(2,2\ell)}(\omega_1,\omega_2) = \eta^{LT}_{T(2,2\ell)}(e^{i\pi(\theta_1+\theta_2)}) = \begin{cases} 1 & \text{for } \theta_1+\theta_2 = \frac{k}{|\ell|}, \text{ where } k = 1,\dots,|\ell|-1, \\ 0 & \text{otherwise,} \end{cases}$$

which was to be proven. □

We will return to this example at the end of this section to look at correction terms for these links.

3.3. **Links of three or more components.** In order to encode the information needed to recover the signature and nullity correction terms in the general case, we first want to make some observations. Note that a band-twist move introduces two adjacent crossings of the same sign on a suitable link diagram, or equivalently switches the sign of a single crossing. This means that the signature and nullity correction terms are invariants under link homotopy. Therefore to calculate them we can restrict our attention to links such that each individual component is an unknot. Moreover, since each unknot bounds a disk (that is, a Seifert surface of genus 0) and since any collection of Seifert surfaces can be deformed into a C-complex without altering their genera, we can choose a C-complex for the link in question, where each component S_i is a disk. This allows us to make the following:

Definition 3.9. *Let L be a μ-component link such that each L_i is an unknot, together with a choice of a C-complex S such that each S_i is a disk. The* intersection graph *of S, denoted by Γ_S is a decorated unoriented multigraph such that:*

- *The vertices correspond to the disks S_i;*
- *The edges correspond to intersection arcs between components of S_i. By an abuse of notation we will refer to both the edges and the intersection arcs with the same letter, as it should cause no confusion;*
- *Each edge e is decorated with a positive or negative sign corresponding to whether the intersection of surfaces at e contributes a $+1$ or -1 term to the linking number of the two links bounded by these surfaces;*
- *At each vertex v_i there is a cyclic ordering of the edges adjacent to this vertex. This means that there is a cyclic permutation $\sigma_i \in \Sigma_{E(v_i)}$ such that for each e, the edge $\sigma_i(e)$ corresponds to the next intersection arc encountered by following the boundary of S_i according to its orientation and starting at e.*

There is an obvious notion of an isomorphism for such decorated multigraphs.
We claim the following:

Theorem 3.10. *Let L, L' be two μ-component links such that each component L_i, L'_i is an unknot, together with choices of C-complexes S, S' respectively, with each consisting of only disks. Then, if $\Gamma_S \cong \Gamma_{S'}$, then*

$$\sigma_{L,L_1}(\omega_2,\dots,\omega_\mu) = \sigma_{L',L'_1}(\omega_2,\dots,\omega_\mu)$$
$$\eta_{L,L_1}(\omega_2,\dots,\omega_\mu) = \eta_{L',L'_1}(\omega_2,\dots,\omega_\mu)$$

for each $\omega \in \mathbb{T}_*^{\mu-1}$.

Proof. Let α, β be two curves on S going through S_1. Consider the push-offs $\alpha^{(-1,\varepsilon')}, \alpha^{(+1,\varepsilon')}$. We can isotope these curves so that they are identical outside of a neighborhood of S_1 in S^3. Since the linking number can be calculated locally, the difference $\mathrm{lk}(\alpha^{(+1,\varepsilon')},\beta) - \mathrm{lk}(\alpha^{(-1,\varepsilon')},\beta)$ depends only on the part of these curves lying in the neighborhood of S_1. However, since Γ_S encodes the cyclic ordering of the clasps around the boundary of S_1 we can recover the α curves up to an isotopy fixing endpoints through curves disjoint from S_1 (and such isotopies will preserve the linking number). Finally, since S is homotopy equivalent to Γ_S in an obvious way, we can recover a base for $H_1(S)$ from Γ_S. These two facts together mean that the signature and nullity of the matrix M_{L,L_1} depends only on the graph Γ_S. □

Now, we wish to give an explicit way to reconstruct the matrix $M_{L,L_1}(\omega_2,\dots,\omega_\mu)$ out of data contained in the graph Γ_S. First, we want to choose a collection of curves in $\mathcal{B}$ which we will use to calculate the suitable matrix. For that, we number the clasps from 1 to m in a way compatible with the cyclic orientation of the graph edges, where we choose the clasp to be labeled by 1 arbitrarily. Then, we choose β_i to be a curve joining the clasps labeled by $i, i+1$ on the disk S_1 and completed to a closed curve in any way in $S \setminus S_1$. For a totally connected C-complex S this is always possible.
Note that using this basis, the matrix M_{L,L_1} will be tri-diagonal, as β_i, β_j can be chosen disjoint from each other on S_1 for $|i-j| > 1$.

Theorem 3.11. *Let L be a μ-component link together with a choice of a totally-connected C-complex S. Let the clasp intersections with S_1 be numbered from 1 to n in a way which agrees with orientation of L_1. We denote by $s(i) \in \{-1,+1\}$ the sign of i-th clasp and by $c(i)$ the label of the other surface in the clasp intersection. Then, the correction term matrix $M_{L,L_1}(\omega)$ is a $n-1 \times n-1$ tri-diagonal antisymmetric matrix with the nonzero terms given by:*

$$m_{i+1,i} = -\overline{m_{i,i+1}} = \frac{1}{1-\omega_{c(i+1)}^{s(i+1)}}, \qquad m_{i,i} = \frac{\omega_{c(i)}^{s(i)}\omega_{c(i+1)}^{s(i+1)} - 1}{(1-\omega_{c(i)}^{s(i)})(1-\omega_{c(i+1)}^{s(i+1)})}, \tag{8}$$

multiplied by $\prod_{j=2}^{\mu} |1-\omega_j|^2$

Note that multiplication by strictly positive real scalar does not change the signature and nullity of a matrix, and so this factor can be omitted in calculations of the correction terms.

Proof. First, we look at the behavior of the push-offs for each pair of curves β_i, β_{i+1}. These are pictured in Figure 6 in the case where the sign of the clasp is negative; the case of a positive clasp differs only in replacing $\varepsilon_{c(i+1)}$ by $-\varepsilon_{c(i+1)}$. Here, the red curve is β_{i-1} which remains unmoved and the blue curve represents the possible push-offs of β_i near the clasp, with endpoints fixed. Denoting $\varepsilon = (\varepsilon_1, \varepsilon')$, we see that

$$\operatorname{lk}(\beta_{i+1}^{+1,\varepsilon'}, \beta_i) - \operatorname{lk}(\beta_{i+1}^{-1,\varepsilon'}, \beta_i) = \begin{cases} 0, & \text{if } \varepsilon_{c(i+1)} = +1 \\ 1, & \text{if } \varepsilon_{c(i+1)} = -1. \end{cases}$$

in the case of a negative claps and so

$$\operatorname{lk}(\beta_{i+1}^{+1,\varepsilon'}, \beta_i) - \operatorname{lk}(\beta_{i+1}^{-1,\varepsilon'}, \beta_i) = \begin{cases} 0, & \text{if } \varepsilon_{c(i+1)} = -s(i+1) \\ 1, & \text{if } \varepsilon_{c(i+1)} = s(i+1), \end{cases}$$

in general. Since the difference of the linking numbers does not depend on any ε_i for $i \neq 1, c(i)$ we see that

$$\begin{aligned} m_{i+1,i} &= \prod_{j=2}^{\mu}(1-\overline{\omega}_j) \sum_{\varepsilon \in \{+1,-1\}^\mu} \varepsilon_1 \ldots \varepsilon_\mu \cdot \omega_2^{\frac{1-\varepsilon_2}{2}} \ldots \omega_\mu^{\frac{1-\varepsilon_\mu}{2}} \operatorname{lk}(\beta_i^\varepsilon, \beta_{i-1}) \\ &= \prod_{j=2}^{\mu}(1-\overline{\omega}_j) \prod_{j=2, j\neq c(i+1)}^{\mu} (1-\omega_j) \sum_{\varepsilon_{c(i+1)}=\pm 1} \varepsilon_{c(i+1)} \omega_{c(i+1)}^{\frac{1-\varepsilon_{c(i+1)}}{2}} (\operatorname{lk}(\beta_{i+1}^{+1,\varepsilon'}, \beta_i) - \operatorname{lk}(\beta_{i+1}^{-1,\varepsilon'}, \beta_i)) \\ &= \prod_{j=2}^{\mu} |1-\omega_j|^2 \cdot \frac{1}{(1-\omega_{c(i+1)})} \cdot s(i+1)\omega_{c(i+1)}^{\frac{1-s(i+1)}{2}}. \end{aligned}$$

If $s(i+1) = 1$ we readily see that this agrees with the desired result. In the case of $s(i+1) = -1$, we have

$$\frac{1}{(1-\omega_{c(i+1)})} \cdot s(i+1)\omega_{c(i+1)}^{\frac{1-s(i+1)}{2}} = \frac{-\omega_{c(i+1)}}{(1-\omega_{c(i+1)})} = \frac{-1}{\omega_{c(i+1)}^{-1} - 1} = \frac{1}{1-\omega_{c(i+1)}^{-1}},$$

which was to be proved.

Figure 7 shows the push-off (in blue) of a curve in relation to the original curve (in red) where both clasps are negative. We see that a difference in the linking number for ε_1 changing from $+1$ to -1 only occurs if $\varepsilon_{c(i)} = \varepsilon_{c(i+1)}$. Note that if $c(i) = c(i+1)$ then this condition is always satisfied. Analogously, when the signs of the clasps differ, the condition for correction terms to appear is $\varepsilon_{c(i)} = -\varepsilon_{c(i+1)}$, and in the case of $c(i) = c(i+1)$ this will never be satisfied. In the end in the particular case of two negative clasps we have that

$$\operatorname{lk}(\beta_i^{+1,\varepsilon'}, \beta_i) - \operatorname{lk}(\beta_i^{-1,\varepsilon'}, \beta_i) = \begin{cases} -1, & \text{if } \varepsilon_{c(i)} = \varepsilon_{c(i+1)} = -1 \\ +1, & \text{if } \varepsilon_{c(i)} = \varepsilon_{c(i+1)} = +1 \\ 0, & \text{if } \varepsilon_{c(i)} = -\varepsilon_{c(i+1)} \end{cases}$$

and in general

$$\operatorname{lk}(\beta_i^{+1,\varepsilon'}, \beta_i) - \operatorname{lk}(\beta_i^{-1,\varepsilon'}, \beta_i) = \begin{cases} -1, & \text{if } \varepsilon_{c(i)} = s(i), \varepsilon_{c(i+1)} = s(i+1) \\ +1, & \text{if } \varepsilon_{c(i)} = -s(i), \varepsilon_{c(i+1)} = -s(i+1) \\ 0, & \text{otherwise.} \end{cases}$$

This gives us now, in the case of $c(i) \neq c(i+1)$:

$$
\begin{aligned}
m_{i,i} &= \prod_{j=2}^{\mu}(1-\overline{\omega}_j) \sum_{\varepsilon\in\{+1,-1\}^{\mu}} \varepsilon_1 \dots \varepsilon_\mu \cdot \omega_2^{\frac{1-\varepsilon_2}{2}} \dots \omega_\mu^{\frac{1-\varepsilon_\mu}{2}} \operatorname{lk}(\beta_i, \beta_i^{\varepsilon}) \\
&= \prod_{j=2}^{\mu}(1-\overline{\omega}_j) \prod_{j\neq c(i),c(i+1)}(1-\omega_j) \sum_{\substack{\varepsilon_{c(i)}=\pm 1\\ \varepsilon_{c(i+1)}=\pm 1}} \varepsilon_{c(i)}\varepsilon_{c(i+1)}\omega_{c(i)}^{\frac{1-\varepsilon_{c(i)}}{2}} \omega_{c(i+1)}^{\frac{1-\varepsilon_{c(i+1)}}{2}} (\operatorname{lk}(\beta_i^{+1,\varepsilon'}, \beta_i) - \operatorname{lk}(\beta_i^{-1,\varepsilon'}, \beta_i)) \\
&= \prod_{j=2}^{\mu} |1-\omega_j|^2 \cdot \frac{1}{(1-\omega_{c(i)})(1-\omega_{c(i+1)})} \cdot s(i)s(i+1)(\omega_{c(i)}^{\frac{1-s(i)}{2}} \omega_{c(i+1)}^{\frac{1-s(i+1)}{2}} + \omega_{c(i)}^{\frac{1+s(i)}{2}} \omega_{c(i+1)}^{\frac{1+s(i+1)}{2}}),
\end{aligned}
$$

and in the case of $c(i) = c(i+1)$ and $s(i) = s(i+1)$:

$$
\begin{aligned}
m_{i,i} &= \prod_{j=2}^{\mu}(1-\overline{\omega}_j) \sum_{\varepsilon\in\{+1,-1\}^{\mu}} \varepsilon_1 \dots \varepsilon_\mu \cdot \omega_2^{\frac{1-\varepsilon_2}{2}} \dots \omega_\mu^{\frac{1-\varepsilon_\mu}{2}} \operatorname{lk}(\beta_i, \beta_i^{\varepsilon}) \\
&= \prod_{j=2}^{\mu}(1-\overline{\omega}_j) \prod_{j\neq c(i)}^{\mu}(1-\omega_j) \sum_{\varepsilon_{c(i)}=\pm 1} \varepsilon_{c(i)}\omega_{c(i)}^{\frac{1-\varepsilon_{c(i)}}{2}} (\operatorname{lk}(\beta_i^{+1,\varepsilon'}, \beta_i) - \operatorname{lk}(\beta_i^{-1,\varepsilon'}, \beta_i)) \\
&= \prod_{j=2}^{\mu} |1-\omega_j|^2 \cdot \frac{1}{1-\omega_{c(i)}} \cdot s(i)(\omega_{c(i)}^{\frac{1+s(i)}{2}} + \omega_{c(i)}^{\frac{1-s(i)}{2}}),
\end{aligned}
$$

which agree with the statement of the theorem by arguments analogous to the first calculation. Finally, in the case where $c(i) = c(i+1)$ and $s(i) \neq s(i+1)$ the expression $\operatorname{lk}(\beta_i^{+1,\varepsilon'}, \beta_i) - \operatorname{lk}(\beta_i^{-1,\varepsilon'}, \beta_i)$ is always equal to zero, and so it agrees with the statement of the theorem. □

This result can be further used to obtain a closed formula for the correction terms, which depends only on the linking numbers of L, proven by David Cimasoni [6, Theorem 3.1]. The first step to do so is to make the following observation.

Proposition 3.12. Let L, S be as in Theorem 3.11. Assume that the $i, i+1$-th clasps have the same color and opposite signs. Let S' be the C-complex obtained by removing those two clasps and L' the link obtained by this operation. Then. the signature and nullity correction terms associated to L' are the same as those associated to L.

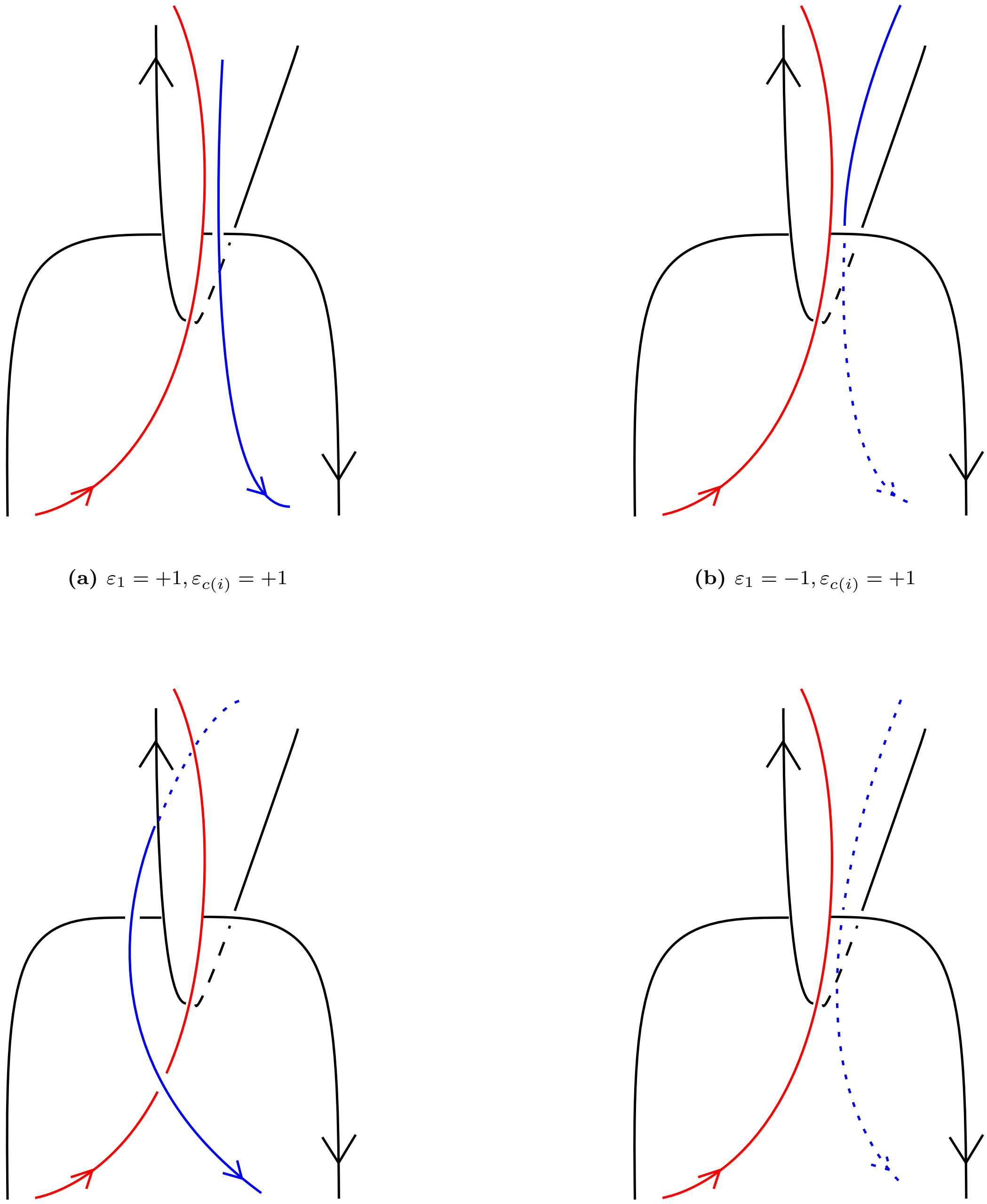
Proof. Renumber the clasps intersections with S_1 so that the $i, i+1$-th clasps become the $n-1, n$-th ones. Denote by M' the correction term matrix associated to the C-complex S'. Then, by Theorem 3.11 we see that

$$
\begin{aligned}
m_{n-1,n-1} &= 0, \\
m_{n-2,n-1} &\neq 0 \neq m_{n-1,n-2},
\end{aligned}
$$

and so the matrix M_{L,L_1} has the form

$$
M_{L,L_1} = \begin{bmatrix} M' & \xi & 0 \\ \xi^* & \alpha & \lambda \\ 0 & \overline{\lambda} & 0 \end{bmatrix},
$$

with $\alpha \in \mathbb{R}, \lambda \in \mathbb{C}^*$. Then, by the proof of [5, Theorem 2.1] we get that the signature and nullity of M_{L,L_1} and M' coincide. □



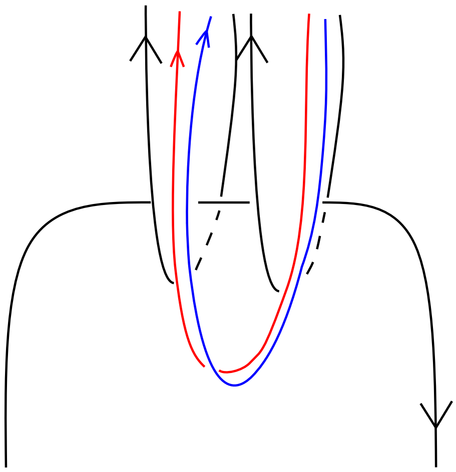
(a) $\varepsilon_1 = +1, \varepsilon_{c(i)} = +1$

(b) $\varepsilon_1 = -1, \varepsilon_{c(i)} = +1$

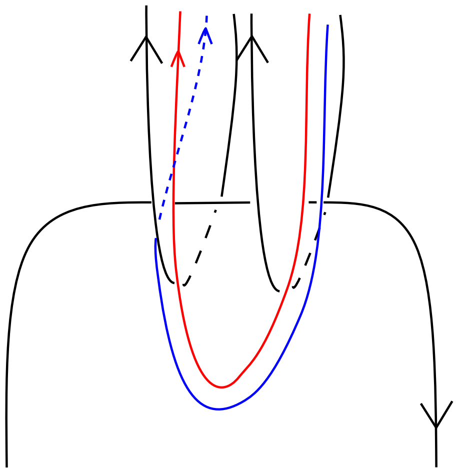
(c) $\varepsilon_1 = +1, \varepsilon_{c(i)} = -1$

(d) $\varepsilon_1 = -1, \varepsilon_{c(i)} = -1$

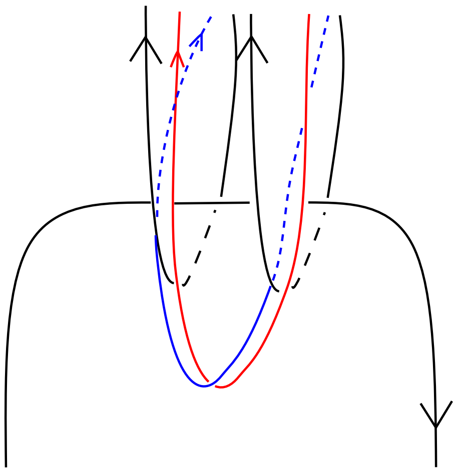
Figure 6. Behavior of push-offs at a negative clasp between S_1 and S_n.



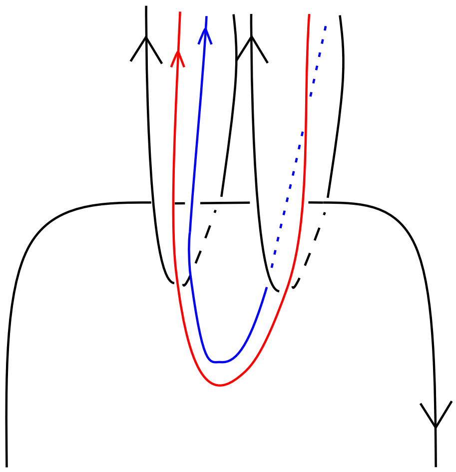
(a) $\varepsilon_1 = +1, \varepsilon_{c(i)} = \varepsilon_{c(i+1)} = +1$



(b) $\varepsilon_1 = +1, \varepsilon_{c(i)} = -\varepsilon_{c(i+1)} = -1$



(c) $\varepsilon_1 = +1, \varepsilon_{c(i)} = \varepsilon_{c(i+1)} = -1$



(d) $\varepsilon_1 = +1, \varepsilon_{c(i)} = -\varepsilon_{c(i+1)} = +1$

Figure 7. Behavior of push-offs of the same curve

Furthermore, it can also be shown that the correction terms are also unchanged under any permutation of the clasp intersections of S with S_1. This combined with Proposition 3.12 means that the correction terms depend only on the linking numbers of L_1 with other link components of L and can be expressed by the following formula.

Definition 3.13. *The function ρ is defined inductively by*

$$\rho(z_1,\dots,z_n) := \sum_{j=1}^{n-1} \rho(z_j, z_{j+1}\cdots z_n)\,,$$

and

$$\rho(z_1,z_2) := \operatorname{sgn}[i(z_1z_2-1)(\overline{z}_1-1)(\overline{z}_2-1)].$$

Theorem 3.14. [6, Theorem 3.1]
For a μ-component link $L = L_1\cup\cdots\cup L_\mu =: L_1\cup L'$ and all $\omega'\in\mathbb{T}_*^{\mu-1}$, we have

$$\left|\lim_{\omega_1\to 1^\pm} \sigma_L(\omega_1,\omega') - \sigma_{L'}(\omega') \mp \rho_\ell(\omega')\right| \le \eta_{L'}(\omega') + \tau_\ell(\omega') - \operatorname{rank} A(L)\,,$$

where $A(L)$ denotes the multivariable Alexander module of L, while ρ_ℓ and τ_ℓ are given by (9)

$$\rho_\ell(\omega') = \begin{cases} \rho(\underbrace{\omega_2^{s_2},\dots,\omega_2^{s_2}}_{|\ell_2|},\dots,\underbrace{\omega_\mu^{s_\mu},\dots,\omega_\mu^{s_\mu}}_{|\ell_\mu|}) & \text{if } |\ell|>0;\\ 0 & \text{else,}\end{cases} \qquad \tau_\ell(\omega') = \begin{cases} 1 & \text{if } \omega_2^{\ell_2}\cdots\omega_\mu^{\ell_\mu}=1;\\ 0 & \text{else,}\end{cases}$$

for $\omega' = (\omega_2,\dots,\omega_\mu)\in\mathbb{T}_*^{\mu-1}$, where $\ell = (\ell_2,\dots,\ell_\mu) = (\operatorname{lk}(L_1,L_2),\dots,\operatorname{lk}(L_1,L_\mu))$ and $s_i = \operatorname{sgn}(\ell_i)$.

Remark 3.15. The reason for why the rank of the Alexander module appears in the statement of the above theorem is that it can be shown to bound the nullity of L from below [6, Lemma 5.2]. Thus, this statement may be sometimes weaker than one using the limit of the nullity instead, but it may be easier to compute in specific cases.

We can now return to the example of two-component torus links, discussed in Example 3.8. We claim the following.

Theorem 3.16. *Let $L = L_1\cup L_2$ be a two-component link with $\ell = \operatorname{lk}(L_1,L_2)$. Then, for $\omega_2 = e^{2i\pi\theta_2}$ with $\theta_2\in[0,1]$ the signature and nullity correction terms are expressed by the following formulas:*

- *If $\ell\neq 0$, then*

$$\sigma^+_{L,L_1}(\omega_2) = \sigma^-_{L,L_1}(\overline{\omega_2}) = \begin{cases} \ell - \operatorname{sgn}(\ell)\cdot(2k+1), & \text{for } \frac{k}{|\ell|}<\theta_2<\frac{k+1}{|\ell|},\ k=0,1,\dots,|\ell|-1,\\ \ell-\operatorname{sgn}(\ell)\cdot 2k, & \text{for } \theta=\frac{k}{|\ell|},\ k=1,2,\dots,|\ell|-1\,.\end{cases}$$

$$\eta_{L,L_1}(\omega_2) = \begin{cases} 1, & \text{if } \omega_2 = \exp\left(\frac{2\pi i k}{\ell}\right), k=1,2,\dots,|\ell|-1,\\ 0, & \text{else}\,.\end{cases}$$

- *If $\ell=0$, then $\sigma^\pm_{L,L_1}(\omega_2)=0$ and $\eta_{L,L_1}(\omega_2)=1$.*

Proof. Since the signature and nullity correction terms are invariant under link homotopy, it is sufficient to consider the case when L is a torus link $T(2,2\ell)$. The case of $T(2,0)$, which is an unlink, is trivial and we can restrict our attention to the case $\ell>0$, as in the proof of Example 3.8. We will use Theorem 3.11 to calculate the correction terms in this case. From the same choice of a C-complex as in the proof of Example 3.8 we obtain a graph in which every edge between v_1 and v_2 has positive sign. We therefore obtain

$$\frac{1}{|1-\omega_2|^2}M_{L,L_1} = \begin{bmatrix} \frac{\omega_2^2-1}{(\omega_2-1)^2} & \frac{-1}{1-\overline{\omega_2}} & \cdots & \cdots & 0 \\ \frac{1}{1-\omega_2} & \frac{\omega_2^2-1}{(\omega_2-1)^2} & \frac{-1}{1-\overline{\omega_2}} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{-1}{1-\overline{\omega_2}} \\ 0 & \cdots & \cdots & \frac{1}{1-\omega_2} & \frac{\omega_2^2-1}{(\omega_2-1)^2} \end{bmatrix}.$$

We only need now to find the signature and nullity of iM_{L,L_1}. Since this is a tridiagonal Hermitian matrix we can again refer to [18, Theorem 2.2] to obtain an expression for its eigenvalues multiplied by $|1-\omega_2|^2$:

$$i\frac{\omega_2^2-1}{(\omega_2-1)^2} - 2(\frac{1}{|1-\omega_2|})\cos(\frac{k\pi}{\ell}),\ k = 1,\ldots,\ell-1,$$

which can be simplified to

$$\begin{aligned} &\frac{i(\omega_2+1)(\overline{\omega_2}-1) - 2|1-\omega_2|\cos(\frac{k\pi}{\ell})}{|1-\omega_2|^2} \\ &= \frac{2(\mathrm{Im}(\omega_2) - |1-\omega_2|\cos(\frac{k\pi}{\ell}))}{|1-\omega_2|^2}. \end{aligned}$$

Since $\mathrm{Im}(e^{2i\pi\theta_2}) = \sin(2\pi\theta_2) = 2\sin(\pi\theta_2)\cos(\pi\theta_2)$ and $|1-\omega_2| = 2\sin(\pi\theta_2)$ we get that the sign of the k-th eigenvalue is equal to the sign of

$$2\sin(\pi\theta_2)\cos(\pi\theta_2) - 2\sin(\pi\theta_2)\cos(\frac{k\pi}{\ell}),$$

which is further equal to the sign of

$$\cos(\pi\theta_2) - \cos(\frac{k\pi}{\ell}).$$

Since cosine is a decreasing function on $[0,\pi]$ this is just the sign of $\theta_2 - \frac{k}{\ell}$ and summing these signs over all k from 1 to $\ell-1$ yields the desired equalities. □

We now turn our attention to inequalities obtained from Theorem 3.5 by considering particular families of links.

Example 3.17. We wish to compare the formulas of Theorem 3.16 to the ones of Example 3.8 by applying Theorem 3.5. In this case, the sublink $L \setminus L_1$ is simply the unknot, which has signature and nullity equal to 0 everywhere. In the case of $\ell = 0$ all of the functions in the inequality are constant and for all ω_2 we obtain

$$\big|0-0-0\big| \leq 0+1-1,$$

which is true.

We now restrict our attention to the case of $\ell > 0$ and taking the limits from the positive imaginary side, as calculations in other situations are analogous.

First, consider the case when $\frac{k}{\ell} < \theta_2 < \frac{k+1}{\ell}$. Then, the inequality is as follows

$$\big|\lim_{\omega_1\to 1^{\pm}} \sigma_{T(2,2\ell)}(\omega_1,\omega_2) - 0 - \ell + 2k + 1\big| \leq 0 + 0 - \lim_{\omega\to 1^{\pm}} \eta_{T(2,2\ell)}(\omega_1,\omega_2).$$

In this case the signature and nullity of $T(2,2\ell)$ are constant for all (ω_1,ω_2) sufficiently close to $(0,\omega_2)$ with the imaginary part of ω_1 being positive, and we arrive at

$$\big|\ell - 2k - 1 - 0 - \ell + 2k + 1\big| \leq 0 + 0 - 0.$$

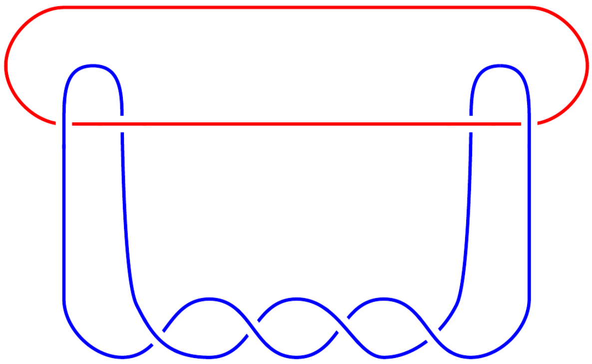


Figure 8. The twist link Θ_2

Note that here the right-hand side of the inequality is equal to zero, and so the limit of the signature is uniquely determined, even without knowing an explicit formula for $\sigma_{T(2,2\ell)}(\omega_1, \omega_2)$. This will no longer be the case when $\theta_2 = \frac{k}{\ell}$. Then, we obtain

$$\Big| \lim_{\omega_1 \to 1^\pm} \sigma_{T(2,2\ell)}(\omega_1, \omega_2) - 0 - \ell + 2k \Big| \leq 0 + 1 - \lim_{\omega \to 1^\pm} \eta_{T(2,2\ell)}(\omega_1, \omega_2),$$

and substituting in the limit of nullity with ω_2 fixed, which is equal to 0, we get

$$\Big| \lim_{\omega_1 \to 1^\pm} \sigma_{T(2,2\ell)}(\omega_1, \omega_2) - 0 - \ell + 2k \Big| \leq 1.$$

This of course is correct if we substitute in this limit of the signature, equal to $\ell - 2k - 1$. However, since the right-hand side is equal to 1, we no longer can say what the limit of the signature needs to be without the knowledge of a closed formula for it. This is not surprising; the inequality of Theorem 3.5 has to hold for all ways of approaching $(0, \omega_2)$ in the limit. If instead of keeping ω_2 fixed we would consider for example the family

$$\omega_t = (\exp(\pi i t), \exp(\omega_2 - 2\pi i t)),$$

the sum $\theta_1 + \theta_2$ would tend to θ from below and the expression

$$\lim \sigma_{T(2,2\ell)}(\omega_1, \omega_2) - 0 - \ell + 2k$$

would be equal to 1, instead of -1. In this sense, the bound obtained from Theorem 3.5 is here as good as an equality of its type could possibly get.

Example 3.18. Consider the family Θ_k of *twist links*. Figure 8 pictures the link Θ_2, and for a general k the link Θ_k has k full twists on the bottom (blue) strands. Note that Θ_0 is the two component unlink and $\Theta_{\pm 1}$ are the Whitehead links. All the links Θ_k have the linking numbers between the two components equal to 0. By taking a C-complex consisting of two disks with two claps of opposite, Theorem 3.11 gives us the correction terms.

$$\sigma^{\pm}_{L,L_1}(\omega_2) = 0,$$
$$\eta_{L,L_1}(\omega_2) = 1.$$

Thus, we obtain the inequality

$$\Big| \lim_{\omega_1 \to 1^\pm} \sigma_{\Theta_k}(\omega_1, \omega_2) - 0 - 0 \Big| \leq 0 + 1 - \lim_{\omega_1 \to 1^\pm} \eta_{\Theta_k}(\omega_1, \omega_2).$$

Now, from the same choice of a C-complex we get all Seifert matrices equal to $[k]$ and the matrix $H(\omega_1, \omega_2)$ is then a 1×1 matrix with the unique entry equal to

$$k|1-\omega_1|^2|1-\omega_2|^2.$$

Thus, for $k \neq 0$, the nullity of Θ_k is constant and equal to zero and the above inequality yields

$$\Big| \lim_{\omega_1 \to 1^{\pm}} \sigma_{\Theta_k}(\omega_1, \omega_2) \Big| \leq 1,$$

and so it is not enough to determine the limit of signature on its own. We will return to this example later, to contrast this with the inequality obtained from 4-dimensional considerations.

4. Signature and nullity from the 4-dimensional viewpoint

There are two main aims of this section. First, we want to extend the definition of the multivariable link signature and nullity functions to the full torus $\mathbb{T}^\mu$. Second, with that definition in hand we want to prove Torres formulas for those functions.
In order to do that, we will reframe the invariants in question as the twisted signature and nullity of a suitable 4-manifold. To begin, we will introduce the construction of plumbed manifolds and show basic facts about their twisted homology groups. Then, for each link L together with an auxiliary choice of a *bounding surface* F we will be able to construct a manifold W_F whose twisted signature and nullity extend the link signature and nullity to the full torus. Afterwards, the next two subsections will contain the proofs of the Torres formulas for link signature and nullity respectively.
Finally, we will also show that the signature and nullity defined as here are invariant under link concordance when restricted to a suitable subset of $\omega \in \mathbb{T}^\mu$, the set of *non-concordance roots*.
Throughout this subsection we will assume that whenever we consider a μ-component link L, the number of components is at least two. This is because while our arguments are applicable to choices of ω that do not lie in $\mathbb{T}^\mu_*$, they will usually fail when $\omega = (1, \dots, 1)$. Thus, we need to consider those $\omega \neq (1, \dots, 1)$ with $\omega_1 = 1$, which is possible only when $\mu \geq 2$.

First, we introduce the notion of a *bounding surface*:

Definition 4.1. *Let L be a μ-component link in $S^3 = \partial B^4$. A bounding surface for L is a collection of surfaces $F_i, 1 \leq i \leq \mu$, such that:*

- *Each F_i is a locally flat connected and oriented surface, properly embedded in B^4, so that $F_i \cap \partial B^4 = \partial F_i = L_i$;*
- *For each $i \neq j$ the surfaces F_i, F_j intersect each other transversally;*
- *For each i, j, k pairwise distinct the set $F_i \cap F_j \cap F_k$ is empty.*

Here, locally flat *means that for each interior point $x \in F_i$ there exists a neighborhood U of x in B^4 such that the pair $(U, U \cap F_i)$ is homeomorphic to (B^4, B^2), the standard embedding of a two dimensional disk in a four dimensional ball.*

Note that if S is a C-complex for L, we can obtain a bounding surface for L by pushing S slightly into the interior of B^4, transforming clasps into double points.

Now, consider the exterior of the bounding surface, $V_F := B^4 \setminus \nu(F)$. Lemma 2.21 shows that for an ordered link $L = L_1 \cup \ldots \cup L_\mu$ with a bounding surface F there is a natural structure of a manifold over $\mathbb{Z}^\mu$ on V_F. Viro [32, Theorem 2.A] showed that the signature and nullity of V_F for any $\omega \in \mathbb{T}^\mu_*$ with respect to this structure do not depend on the choice of a bounding surface. Finally, the following result relates invariants of V_F with those of L [10, Proposition 3.5]:

Proposition 4.2. Let L be a μ-component ordered link together with a choice of a bounding surface F. Then, for any $\omega \in \mathbb{T}^\mu_*$ we have

$$\operatorname{sign}_\omega V_F = \sigma_L(\omega)$$
$$\operatorname{null}_\omega V_F = \eta_L(\omega)$$

where $\sigma_L(\omega), \eta_L(\omega)$ are as in Definition 2.25.

We also have a further characterization of the nullity [5, Proof of Theorem 6.1].

Proposition 4.3. Let L be a μ-component ordered link together with a choice of a connected bounding surface F. Then, for any $\omega \in \mathbb{T}^\mu_*$ we have

$$\eta_L(\omega) = \dim(H_1(X_L; \mathbb{C}^\omega)).$$

Here we also note, that it might be beneficial to consider $\sigma_L(\omega)$ to be defined as the signature defect instead. This is possible due to the following proposition:

Proposition 4.4. Let L be a link in $S^3 = \partial B^4$ and let F be a bounding surface for L. Then,

$$\operatorname{sign} V_F = 0,$$

where V_F is the exterior of F in B^4.

Proof. The Mayer-Vietoris sequence associated to the decomposition $B^4 = V_F \cup \nu(F)$ gives us

$$H_3(B^4) = 0 \to H_2(\partial\nu(F)) \to H_2(\nu(F)) \oplus H_2(V_F) \to H_2(B^4) = 0$$

As $\nu(F)$ is homotopy equivalent to F, we get that $H_2(\nu(F)) = 0$ and so the homomorphism $H_2(\partial\nu(F)) \to H_2(V_F)$ is onto. Since $\partial\nu(F) \subset \partial V_F$, this subspace is annihilated by the intersection form and therefore $\operatorname{sign} V_F = 0$. □

Now, this definition of signature has very serious issues when some $\omega_i = 1$. In particular, it is not necessarily the case that the signature does not depend on the choice of F. A crucial step in the proof of Proposition 4.2 in [15] involves showing that for two choices of a bounding surface of L, F', F'' the twisted signature of the manifold $V := V_{F'} \cup_{X_L} -V_{F''}$ is equal to the difference of the twisted signatures of $V_{F'}, V_{F''}$. Consider now that the link L is such that $\operatorname{lk}(L_1, L_i) = 0$ and $F_1' \cap F_i' = \emptyset$ for all $i \geq 2$, but $F_1'' \cap F_2'' \neq \emptyset$. Such a link and a bounding surface exist, since we can always introduce two intersection points between any two components of a bounding surface, just as we can introduce two clasps of opposite sign in a C-complex. Then, as we will see by Proposition 4.24, the kernels of homomorphisms induced on homology by the inclusions $\partial\nu(L) \hookrightarrow V_{F'}, V_{F''}$ will not coincide when $\omega_1 = 1$. Therefore, there is no reason for the signature to be additive and as [15, Lemma 4.9] shows, the signature calculated with respect to F' will differ by the sign of the slope of L_1 relative to L from the one calculated with respect to F''. Similarly, the nullity of V_F also depends on the choice of the bounding surface F when $\omega \notin \mathbb{T}^\mu_*$. To see this, consider once again a bounding surface F such that $F_1 \cap F_i = \emptyset$ for $i \geq 2$, and so $H_1(F_1; \mathbb{C}^\omega) \cong H_1(F_1; \mathbb{C})$. Now, take a connected sum of F_1 with a torus $\mathbb{T}^2$ lying inside a small ball B disjoint from all F_i. Then, if $\omega_1 = 1$, a longitude of this torus is trivial in B and likewise in the exterior $V_{F'}$. This means that the dimension of the kernel of

$$H_1(\partial V_{F'}; \mathbb{C}^\omega) \to H_1(V_{F'}; \mathbb{C}^\omega)$$

is greater then that for V_F and the nullity is strictly bigger. This may seem to contradict [15, Lemma 4.9], since there the nullity does not depend on the choice of the bounding surface. This is because the authors of that article define η_L as the dimension of $H_1(X_L; \mathbb{C}^\omega)$. This, however, does not equal the nullity of the twisted intersection form of V_F when ω does not lie in $\mathbb{T}^\mu_*$. Furthermore, even if the nullity was independent of the choice of F, the group $H_1(V_F; \mathbb{C}^\omega)$ is not necessarily trivial when $\omega \notin \mathbb{T}^\mu_*$. Therefore, the equality

$$\operatorname{null}_\omega V_F = \dim H_1(\partial V_F; \mathbb{C}^\omega)$$

does not need to hold. This would result in undesirable behavior of the link nullity. For example. the proof of the concordance invariance of the nullity would fail in that case.

4.1. **Plumbed manifolds.** To get around the problems with the behavior of twisted signature at $\omega_1 = 1$, we construct a new manifold, W_F, whose signature and nullity at $\omega \in \mathbb{T}^\mu_*$ will also agree with the signature and nullity of L, for which F is a bounding surface.

First, we define a *plumbed 3-manifold* associated to a certain kind of a graph, which we cite here after [10, Construction 4.5]:

Definition 4.5. *Let $G = (V, E)$ be an unoriented graph with no loops. The set E is the set of oriented edges, and $s\colon E \to V$ and $t\colon E \to V$ are the source and the target maps. The involution $i\colon E \to E$ sends an oriented edge to the corresponding edge with the opposite orientation; see e.g.* [27, Section I.2]*. The graph is unoriented in the sense that for each edge, the set E also contains the edge with the opposite orientation. We shall sometimes also denote $i(e)$ by $\bar{e}$. Assume that to each vertex v is assigned an oriented, connected and compact surface F_v and that the edges $e \in E$ are labeled by weights $\varepsilon(e) = \varepsilon(\bar{e}) \in \{\pm 1\}$.*

For each edge e, we choose an embedded disc $D_e \subset F_{s(e)}$ in such a way that no two discs intersect. We then remove these discs, by defining for each surface F_v the complement

$$\mathring{F}_v = F_v \setminus \bigcup_{s(e)=v} D_e.$$

We define the plumbed 3-manifold $\mathrm{Pb}(G)$ *to be*

$$\mathrm{Pb}(G) := \left(\bigsqcup_{v \in V} \mathring{F}_v \times S^1 \right) / \sim,$$

where for all $e \in E$ the identifications are given by

$$(10) \qquad \begin{aligned} (-\partial D_e) \times S^1 &\to (-\partial D_{i(e)}) \times S^1 \\ (x, y) &\mapsto \begin{cases} (y^{-1}, x^{-1}), & \text{if } \varepsilon(e) = 1, \\ (y, x), & \text{if } \varepsilon(e) = -1. \end{cases} \end{aligned}$$

Since these identifications make use of orientation reversing homeomorphisms, the 3-manifold $\mathrm{Pb}(G)$ carries an orientation that extends the orientation of each $\mathring{F} \times S^1$.

Note here that usually the definition of a plumbed manifold involves choosing a framing for each constituent piece. Our approach is a special case of this more general definition, where we take each framing to be equal to zero, and so all the bundles considered are product bundles. We will also make use of meridional homomorphisms, see [10]:

Definition 4.6. *Let $\varphi : H_1(\mathrm{Pb}(G); \mathbb{Z}) \to \mathbb{Z}^\mu$ be a homomorphism. We call such a homomorphism meridional if, for each constituting piece $\mathring{F} \times S^1 \subset \mathrm{Pb}(G)$ with $F \in V$, the restriction of φ to $H_1(\mathring{F} \times S^1; \mathbb{Z})$ sends the class of $\{\mathrm{pt}\} \times S^1$ to one of the canonical generators $e_1, \ldots, e_\mu$ of $\mathbb{Z}^\mu$.*

If L is a link in $S^3 = \partial B^4$ with a bounding surface F we can associate to it the following plumbing graph:

Definition 4.7. *Let $L = L_1 \cup \ldots \cup L_\mu$ be an ordered link together with a choice of a bounding surface F. We then define the plumbing graph Γ_F in the following way:*

- *Each vertex v_i corresponds to a link component L_i and is assigned the surface F_i;*
- *The edges between v_i and v_j correspond to double points $F_i \cap F_j$ and each edge is decorated by the sign of this intersection point.*

The reason for this definition is the following:

Proposition 4.8. Let L be a link with a choice of a bounding surface F. Then,

$$\mathrm{Pb}(\Gamma_F) \simeq \partial \nu(F).$$

Proof. First, look at each surface F_i separately. The boundary of the tubular neighborhood, $\partial\nu(F_i)$ is a circle bundle over F_i. Since F_i is a surface with boundary, it retracts onto a wedge of circles and so any orientable two-dimensional vector bundle (and therefore also any circle bundle) over it is trivial. Now, to obtain $\partial\nu(F_i)$ from the trivial circle bundles $\partial\nu(F_i) \simeq F_i \times S^1$ we need to look at the intersection points of the surfaces. Since these intersect transversally, they look locally like the neighborhood of the subspace

$$\{x = y = 0\} \cup \{z = w = 0\} \subset \mathbb{R}^4,$$

taken with appropriate orientations and so we can look at the boundary of the neighborhood of that subspace of $\mathbb{R}^4$. That boundary can be described as

$$\{(x, y, w, z) \in \mathbb{R}^4 \mid \min(x^2 + y^2, z^2 + w^2) = 1\}.$$

Finally, we check easily that this description agrees with the result of gluing described in the plumbing construction, which end the proof. $\square$

To a link L we can associate the following plumbed manifold:

Definition 4.9. *Let $L = L_1 \cup \ldots \cup L_\mu$ be an ordered link. We define the graph Γ_L as follows:*

- *The vertices v_i correspond to the link components L_i and each of them is assigned a disk;*
- *The set of edges between v_i and v_j has $|\operatorname{lk}(L_i, L_j)|$ components and each edge in this set has the sign $\varepsilon(e) = \operatorname{sgn}(\operatorname{lk}(L_i, L_j))$.*

We call the manifold $\operatorname{Pb}(\Gamma_L)$ the plumbed 3-manifold associated to the link L, and we also denote it by $\operatorname{Pb}(L)$ if no confusion arises. Furthermore, if $-\Gamma$ denotes the graph with all the signs of edges multiplied by -1, we define $-\operatorname{Pb}(L)$ as the plumbed manifold associated to the graph $-\Gamma_L$.

We want to prove the following result concerning the twisted homology of plumbed 3-manifolds.

Lemma 4.10. *Let Γ be a plumbing graph with μ vertices and let $\pi_1(\operatorname{Pb}(\Gamma)) \to \mathbb{Z}^\mu$ be a meridional homomorphism. Then, for any $\omega \in \mathbb{T}^\mu_* = \prod_{i=1}^{\mu}(S^1 \setminus \{1\})$ the homology groups $H_i(\operatorname{Pb}(\Gamma))$ vanish.*

Proof. We prove this by induction on the number of vertices of Γ. If there is only one vertex to which F_1 is assigned, then $\operatorname{Pb}(\Gamma) = F_1 \times S^1$ with the circle factor being properly twisted since $\omega \in \mathbb{T}^\mu_*$ and therefore this manifold is acyclic.

Now, let Γ be a plumbing graph with μ vertices and denote by Γ' the graph obtained by deleting v_1 together with all edges adjacent to it. We have now a decomposition

$$\operatorname{Pb}(\Gamma) = \mathring{\operatorname{Pb}}(\Gamma') \cup (\mathring{F}_1 \times S^1),$$

where $\mathring{\operatorname{Pb}}(\Gamma')$ denotes the plumbed manifold with small disks removed around each intersection point corresponding to an edge between v_1 and another vertex and the gluing is done along a disjoint union of 2-dimensional tori. Since for each of the tori, the character induced by ω sends one coordinate circle to ω_1 and the other to ω_i, their twisted homology groups vanish.
Consequently, the Mayer-Vietoris sequence associated to this decomposition splits into sequences of the form

$$0 \to H_i(\operatorname{Pb}(\Gamma'); \mathbb{C}^\omega) \oplus H_i(\mathring{F}_1 \times S^1; \mathbb{C}^\omega) \to H_1(\Gamma; \mathbb{C}^\omega) \to 0.$$

Since $\omega \in \mathbb{T}^\mu_*$, the space $\mathring{F}_1 \times S^1$ is acyclic. This combined with the induction assumption on $\operatorname{Pb}(\Gamma')$ implies that all the groups $H_i(\Gamma; \mathbb{C}^\omega)$ must also vanish, which was to be proved. $\square$

We call a graph G *balanced*, if for any two vertices v, w the sum of signs of edges between v and w is equal to zero.
We will need the following lemma. It is a generalization of [10, Lemma 4.9], where instead of considering only $\omega \in \mathbb{T}^\mu_*$ we allow $\omega \in \mathbb{T}^\mu$, where at most one of the coordinates is equal to 1.

Lemma 4.11. *Suppose that $G = (V, E)$ is a balanced plumbing graph on μ vertices, all of which are closed connected surfaces. Consider the plumbed $\mathbb{Z}^\mu$-manifold $(\operatorname{Pb}(G), \psi)$, where ψ is meridional. Suppose that for every vertex $F_i \in V$ there exists a collection of curves on $\mathring{F}_i$*

$$L_{F_i} = \{\eta_{1,i}, \eta_{2,i}, \ldots, \eta_{g_i,i}\},$$

where g_i denotes the genus of F_i, with the following properties.

(1) *The image of L_{F_i} under the inclusion-induced map*

$$H_1(\mathring{F}_i) \to H_1(F_i)$$

is of dimension g_i and is a Lagrangian subspace with respect to the intersection form of F_i. We call such a subspace a Lagrangian half-basis.

(2) *Each curve in L_{F_i} is mapped to zero by the composition*

$$H_1(\mathring{F}_i) \to H_1(\mathrm{Pb}(G)) \xrightarrow{\psi} \mathbb{Z}^\mu.$$

Then, $\mathrm{Pb}(G)$ *bounds a* 4*-manifold Z over $\mathbb{Z}^\mu$ such that* $\operatorname{sign}(Z) = 0$ *and* $\operatorname{sign}_\omega(Z) = 0$ *for all $\omega \in \mathbb{T}^\mu$ such that at most one coordinate is equal to* 1*. Furthermore, if* $\mathrm{Pb}(G)$ *is connected, then so is Z.*

Proof. The proof is inductive with respect to the cardinality of the set E. First, let us consider the case $E = \emptyset$, i.e.,

$$\mathrm{Pb}(G) = \bigsqcup_{i=1}^{\mu} F_i \times S^1.$$

Define

$$X = \bigsqcup_{i=1}^{\mu} N_i \times S^1,$$

where N_i is a handlebody such that $\partial N_i = F_i$. Our assumptions imply that the map

$$\psi \colon H_1(\mathrm{Pb}(G)) \to \mathbb{Z}^\mu$$

extends to a map

$$\psi_X \colon H_1(X) \to \mathbb{Z}^\mu.$$

Furthermore, for all $\omega \in \mathbb{T}^\mu$, the inclusion induced map $H_2(\partial X; \mathbb{C}^\omega) \to H_2(X; \mathbb{C}^\omega)$ is surjective. Therefore by Lemma 2.12 intersection form vanishes, leading to $\operatorname{sign}_\omega(X) = 0$.

The general case can be reduced to the case $E = \emptyset$ with the aid of [10, Lemma 4.9]. In the proof of this statement, the authors construct a 4-dimensional bordism Δ over $\mathbb{Z}^\mu$ such that

(1) $\partial\Delta = -\mathrm{Pb}(G) \sqcup \mathrm{Pb}(G')$, where G' is a plumbing graph with $|V(G)| = |V(G')|$ and $E(G') = \emptyset$,
(2) The restriction of the map $H_1(\Delta) \to \mathbb{Z}^\mu$ to $H_1(\mathrm{Pb}(G'))$ is meridional,
(3) $\operatorname{sign}(\Delta) = 0$ and $\operatorname{sign}_\omega(\Delta) = 0$ for all $\omega \in \mathbb{T}^\mu_*$.

First, we will recall the construction of the bordism Δ from the proof of [10, Lemma 4.9]. Recall that to each edge e of the graph G there corresponds an embedded torus $T_e = -\partial D_e \times S^1$ in $\mathrm{Pb}(G)$. Choose vertices v_1, v_2 such that the set of edges between them is nonempty. Since the graph G is balanced, we can choose two edges e, e' with $\varepsilon(e) = 1 = -\varepsilon(e')$. Now, define $X_{e,e'}$ as $I \times I \times S^1 \times S^1$. We will call such a $X_{e,e'}$ a *toral handle* and denote it by TH when we are not interested in particular edges e, e'. Consider now the tori $T_e = (-\partial D_e) \times S^1$ and $T_{e'} = (-\partial D_{e'} \times S^1)$, with their oriented neighborhoods $T_e \times I, T_{e'} \times I$. We attach $X_{e,e'}$ to the trivial bordism $\mathrm{Pb}(G) \times I$ along $\mathrm{Pb}(G) \times \{1\}$ along the vertical boundary $\partial\,\mathrm{Pb}(G) \times I$ through homeomorphism f given by:

$$\begin{aligned} \{0\} \times I \times S^1 \times S^1 &\to I \times (-\partial D_e) \times S^1 \\ (0, t, x, y) &\mapsto (t, x, y), \end{aligned} \qquad \begin{aligned} \{1\} \times I \times S^1 \times S^1 &\to I \times (-\partial D_{e'}) \times S^1 \\ (1, t, x, y) &\mapsto (t, x^{-1}, y). \end{aligned}$$

The induced orientations are such that f is orientation reversing and so the orientations of $X_{e,e'}$ and $\mathrm{Pb}(G) \times I$ extend to

$$\Delta \coloneqq X_{e,e'} \cup_f \mathrm{Pb}(G) \times I.$$

We need to verify that the above construction can be performed in such a way that the assumptions of our lemma are satisfied at each step. For that purpose, for any vertex F_i of G choose a disk $D_{F_i} \subset F_i$ such that for any edge adjacent to e, $D_e \subset D_{F_i}$ (recall that D_e denotes the disk associated to the edge e, which is removed when we construct the plumbing). We can choose the curves in L_{F_i} so that they miss the disk D_{F_i}.

Let U denote the result of attaching a single toral handle TH to $\mathrm{Pb}(G)\times I$. The Mayer-Vietoris sequence gives

$$H_1(\mathrm{Pb}(G) \times I) \oplus H_1(TH) \to H_1(U) \to H_0(ATH) \cong \mathbb{Z}^2.$$

Arguing as in [10, Lemma 4.9], we can split the above sequence to obtain

$$H_1(\mathrm{Pb}(G) \times I) \oplus H_1(TH) \to H_1(U) \xrightarrow{p} \mathbb{Z}.$$

Conway, Nagel and Toffoli in the proof of [10, Lemma 4.9] show that the homomorphism $\psi\colon H_1(\mathrm{Pb}(G) \times I) \to \mathbb{Z}^\mu$ extends to a homomorphism

$$\psi'\colon \mathrm{Im}\,(H_1(\mathrm{Pb}(G) \times I) \oplus H_1(TH) \to H_1(U)) \to \mathbb{Z}^\mu.$$

Next, we can extend ψ' to a map

$$\psi_U\colon H_1(U) = \mathbb{Z} \oplus \mathrm{Im}\,(H_1(\mathrm{Pb}(G) \times I) \oplus H_1(TH) \to H_1(U)) \xrightarrow{0\oplus\psi'} \mathbb{Z}^\mu.$$

The right boundary of U is the plumbed manifold $\mathrm{Pb}(G)$, where

$$V(G') = (V(G) \setminus \{F_i, F_j\}) \cup \{F_i\#T^2, F_j\#T^2\},$$

and $|E(G')| = |E(G)| - 2$, i.e., we removed a pair of edges connecting F_i and F_j. We can choose a pair of curves α_1, β_1 in $F_i' = F_i\#T^2$, such that

(1) $H_1(F_i') = H_1(F_i) \oplus \langle\alpha_1, \beta_1\rangle$, where both curves lie within $D_{F_i}\#T^2 \subset F_i\#T^2 = F_i'$,
(2) β_1 is in the image of the map $H_1(TH) \to H_1(U)$,
(3) α_1 maps to a nontrivial element under the homomorphism $H_1(U) \to H_0(ATH)$ in the Mayer-Vietoris sequence.

By construction, $\psi_U(\alpha_1) = 0$, hence we can take $L_{F_i'} = L_{F_i} \cup \{\alpha_1\}$. Similarly, we can construct $L_{F_j'}$. By iterating the above procedure, we can remove all the edges of the plumbing graph, hence reduce to the base case.

It remains to show that $\mathrm{sign}(Z) = 0$ and $\mathrm{sign}_\omega(Z) = 0$ for all $\omega \in \mathbb{T}^\mu$ with at most one coordinate equal to 1. Since Z is obtained by gluing Δ to X along a closed 3-manifold, and since all the signatures of X vanish, we are left with the proof that $\mathrm{sign}(\Delta) = 0$ and that $\mathrm{sign}_\omega(\Delta) = 0$ for all $\omega \in \mathbb{T}^\mu$ with at most one coordinate equal to 1. The first statement is checked in the proof of [10, Lemma 4.9]. As for the twisted signature, we already mentioned that the 4-manifold $X = \bigsqcup_i N_i \times S^1$ satisfies the following property: for all $\omega \in \mathbb{T}^\mu$, the inclusion induced map $H_2(\partial X; \mathbb{C}^\omega) \to H_2(X; \mathbb{C}^\omega)$ is surjective, leading to its intersection form and signature vanishing. This property is also satisfied by the other 4-manifolds used to construct Y, namely $\mathrm{Pb}(G) \times I$ and the toral handles $I \times I \times T^2$. Moreover, the toral handle corresponding to a pair of edges connecting F_i and F_j is glued to $\mathrm{Pb}(G) \times I$ along the 3-manifold ATH whose boundary Σ is $\mathbb{C}^\omega$-acyclic as soon as $(\omega_i, \omega_j) \neq (1, 1)$. Since we assume that at most one coordinate is equal to 1, this is always the case, and the Novikov-Wall theorem once again implies that the signature is additive.

Finally, on easily checks that if $\mathrm{Pb}(G)$ is connected, then Z constructed above is connected as well. This concludes the proof. $\square$

Corollary 4.12. *Let Z be as in Lemma 4.11 and assume that it is connected. Then Z is $\mathbb{Z}^\mu$-bordant, rel boundary, to a compact connected oriented $\mathbb{Z}^\mu$-manifold (Y, φ) such that $\pi_1(Y) = \mathbb{Z}^\mu$, φ is an isomorphism, $\mathrm{sign}(Y) = 0$ and $\mathrm{sign}_\omega(Y) = 0$ for all $\omega \in \mathbb{T}^\mu$ with at most one coordinate equal to 1.*

Proof. Since Z is a $\mathbb{Z}^\mu$-manifold, it is equipped with a homomorphism

$$\psi_Z\colon \pi_1(Z) \to \mathbb{Z}^\mu\,.$$

Note that ψ_Z is surjective. Indeed, the homomorphism $\psi\colon \pi(\mathrm{Pb}(G)) \to \mathbb{Z}^\mu$ being meridional is surjective. Since it factors through ψ_Z, the latter homomorphism is surjective as well.

First observe that there exists a finite collection of group elements $g_1, g_2, \ldots, g_l \in \ker \psi_Z$ such that the smallest normal subgroup of $\pi_1(Z)$ containing these elements is equal to $\ker \psi_Z$. In other words, all conjugates of $g_1, g_2, \ldots, g_l$ in $\pi_1(Z)$ generate $\ker \psi_Z$. Indeed, let $p\colon \widetilde{Z} \to Z$ be the $\mathbb{Z}^\mu$-covering determined by ψ_Z. Observe that $(\ker \psi_Z)^{ab} = H_1(\widetilde{Z})$. Since Z is compact, it follows that $H_1(\widetilde{Z})$ is a finitely-generated $\mathbb{Z}[\mathbb{Z}^\mu]$-module. Let $x_1, x_2, \ldots, x_l$ denote the generators of $H_1(\widetilde{Z})$ as a $\mathbb{Z}[\mathbb{Z}^\mu]$-module. We can choose $g_1, g_2, \ldots, g_l$ to be the preimages of $x_1, x_2, \ldots, x_l$ under the quotient map

$$\ker \psi_Z \to (\ker \psi_Z)^{ab} = H_1(\widetilde{Z}).$$

The manifold Y will be constructed by performing surgery on loops representing $g_1, g_2, \ldots, g_l$. To be more precise, suppose that the map $f_1\colon S^1 \to Z$ represents g_1. Without loss of generality, we can assume that f_1 is a smooth embedding. Let N_1 denote a closed tubular neighborhood of $f_1(S^1)$, together with the identification $\alpha_1\colon N_1 \xrightarrow{\cong} S^1 \times D^3$, where α_1 maps $f_1(S^1)$ to $S^1 \times \{0\}$. Consider the manifold

$$Y_1 = \overline{Z \setminus N_1} \cup_{\partial N_1} (D^2 \times S^2),$$

where we use the map α_1 to identify the boundary of N_1 with the boundary of $D^2 \times S^2$. By the Seifert-van Kampen theorem, $\pi_1(Y_1)$ is isomorphic to the quotient of $\pi_1(Z)$ by the normal subgroup generated by g_1.

Since g_1 is in the kernel of ψ_Z, one easily shows that Y_1 is $\mathbb{Z}^\mu$-bordant to Z. In particular, Novikov additivity implies that $\operatorname{sign}(Y_1)$ coincides with $\operatorname{sign}(Z)$, which vanishes by the assumption. Similarly, for $\omega \in \mathbb{T}^\mu \setminus \{(1, 1, \ldots, 1)\}$, the fact that Y_1 and Z are $\mathbb{Z}^\mu$-bordant implies that

$$0 = \operatorname{sign}_\omega(Z \cup_\partial \overline{Y_1}) = \operatorname{sign}_\omega(Z) - \operatorname{sign}_\omega(Y_1) = -\operatorname{sign}_\omega(Y_1),$$

where the first equality follows from [32, Theorem D.B], the second inequality follows from Novikov additivity, and the last equality follows from our assumptions.

We can iterate the above procedure to obtain manifolds $Y_1, Y_2, \ldots, Y_l = Y$ with the desired properties. ☐

4.2. **Definition and properties of the manifold W_F.** Let L be a μ-component link and let F be a bounding surface of it. In this subsection we will construct a compact 4-manifold, depending on the pair (L, F) which will play a crucial role in the process of extending the signature to $\mathbb{T}^\mu$.

4.2.1. *Construction of W_F.*

Definition 4.13. *Let $L = L_1 \cup \ldots \cup L_\mu$ be a link in $S^3 = \partial B^4$ and let F be a bounding surface for L. Consider the manifolds $\partial\nu(F)$ and $-\mathrm{Pb}(L)$. The boundary of both of these is homomorphic to the boundary of a neighborhood of L in S^3, with different orientations. We can therefore consider the union*

$$\mathrm{Pb}(\Gamma_F) \cup_{\partial\nu(L)} -\mathrm{Pb}(L),$$

which is a plumbed manifold associated to the following graph G_F:

- *The vertices of G_F correspond to components of L and $\hat{F}_i = F_i \cup D$ is assigned to each vertex;*
- *The set of edges between v_i and v_j is the union of sets of edges between the appropriate vertices in Γ_F and $-\Gamma_L$, with the same signs as there.*

We now want to show that the plumbed manifold $\mathrm{Pb}(G_F)$ bounds over $\mathbb{Z}^\mu$:

Proposition 4.14. Let G_F be a plumbing graph associated to a link and a bounding surface F obtained pushing a C-complex into B^4, see Definition 4.13. Then, G_F satisfies the conditions of Lemma 4.11. Therefore there exists a manifold Y_F over $\mathbb{Z}^\mu$ with $\partial Y = \mathrm{Pb}(G_F)$ satisfying the conditions of Corollary 4.12.

Proof. Since by construction the graph G_F is balanced, it is sufficient to find a set of curves L_{F_i} satisfying the assumptions of Lemma 4.11. To that end, we first consider a C-complex S which we push into B^4 to obtain a bounding surface F. Let $A \subset S$ be a collar neighborhood of L in S such that all the clasp intersections of S are contained in A. Denote the subspace $S_i \setminus A$ by S_i^r. This is homotopy equivalent to S_i and is contained in $\mathring{S}_i := S_i \cap X_L$ as a subspace. Since the bounding surface F was obtained by pushing S in and the clasps of S correspond to double points of F, there is a homeomorphism $h_i : \mathring{S}_i \xrightarrow{\cong} \mathring{F}_i$ for each i. We denote by F_i^r the image of S_i^r under h_i. Consider the following commutative diagram of maps:

$$\begin{tikzcd}
H_1(F_i^r) \arrow[r] & H_1(\mathring{F}_i) \arrow[r] & H_1(\mathrm{Pb}(G)) \arrow[r, "\psi"] & \mathbb{Z}^\mu \\
H_1(S_i^r) \arrow[r] \arrow[u, "\cong", "h_{i*}^r"'] \arrow[rrr, bend right] & H_1(\mathring{S}_i) \arrow[rr] \arrow[u, "\cong", "h_{i*}"'] & & H_1(X_L)\,. \arrow[u, "\cong"]
\end{tikzcd}$$

Here, the unlabeled horizontal maps are induced by respective inclusions, and the bottom-most arrow is the composition of the two lower horizontal arrows. We wish to show that the bottom-most map is equal to zero. First, observe that the map $H_1(X_L) \to \mathbb{Z}^\mu$ is given by taking linking numbers of a curve with the components of L. For any class $[\gamma] \in H_1(S_i)$ we have $\mathrm{lk}([\gamma], [L_j]) = [\gamma] \cdot [S_j]$, where the dot indicates the intersection number. If $i \neq j$ then we can choose γ to be disjoint from S_j and therefore it gets mapped to zero in suitable coordinates in $\mathbb{Z}^\mu$. If $i = j$, then since the link is oriented and γ can be pushed-off of S_i we also have $[\gamma] \cdot S_i = 0$, which shows that this map is indeed equal to zero.
Now, since F_i^r is homotopy equivalent to F_i, their homology groups are isomorphic. Therefore, the commutativity of the above diagram implies that $H_1(F_i^r)$ is mapped to zero in $\mathbb{Z}^\mu$ for $r = 1, \dots, \mu$. In particular, any Lagrangian half-basis on $\mathring{F}_i$ satisfies the assumptions of Corollary 4.12, which ends the proof. □

The boundary of Y_F contains $\mathrm{Pb}(\Gamma_F)$ as a codimension zero submanifold, and it is homeomorphic to $\partial\nu(F)$ by Proposition 4.8. Therefore, we can define the manifold

$$W_F = V_F \cup Y_F,$$

where we glue the spaces by identifying respective copies of $\mathrm{Pb}(\Gamma_F)$ in ∂V_F and ∂Y_F.
To describe the structure of W_F as a manifold over $\mathbb{Z}^\mu$, we first wish to calculate its fundamental group.

Lemma 4.15. *Let W_F be the manifold associated to a μ-component link L and a totally connected bounding surface F. Then,*

$$\pi_1(W_F) \simeq \mathbb{Z}^\mu.$$

Proof. First, note that since F is totally connected, and so the space $\mathrm{Pb}(\Gamma_F)$ is connected. We can therefore apply the Seifert-Van Kampen theorem to the decomposition of W_F into V_F and Y_F. First, we know by [8, Proposition 3.1] that $\pi_1(V_F) = \mathbb{Z}^\mu$ since we assumed our bounding surface F to be totally connected. By Corollary 4.12 we also have that the meridional homomorphism from $\pi_1(Y_F)$ to $\mathbb{Z}^\mu$ is an isomorphism. Now, consider the following diagram of groups:

$$\begin{tikzcd}
\pi_1(V_F) \arrow[d, "\cong"] & \pi_1(\mathrm{Pb}(\Gamma_F)) \arrow[l] \arrow[d] \\
\mathbb{Z}^\mu & \pi_1(Y_F)\,. \arrow[l, "\cong"']
\end{tikzcd}$$

Since all of the homomorphisms are meridional, this diagram is commutative, and any of the compositions

$$\pi_1(\mathrm{Pb}(\Gamma_F)) \to \mathbb{Z}^\mu$$

are surjective. This means that that the diagram satisfies the universal property of a fibered product and so by the Seifert-van Kampen theorem we obtain that $\pi_1(W_F)$ is isomorphic to $\mathbb{Z}^\mu$. □

Now we define the structure of W_F as a manifold over $\mathbb{Z}^\mu$. The Seifert-Van Kampen theorem tells us that every element α of $\pi_1(W_F)$ is represented as an equivalence class of a path $[\tilde{\alpha}]$ in V_F in the amalgamated product. We then put

$$\varphi_{W_F}(\alpha) = \varphi_{V_F}([\tilde{\alpha}]),$$

and since $\pi_1(W_F) \cong \pi_1(V_F) \cong \mathbb{Z}^\mu$, this is well-defined.
Additionally $\partial W_F = X_L \cup_{\partial\nu(L)} -\mathrm{Pb}(L)$. In particular, ∂W_F can be thought of as a manifold obtained by performing a surgery on L, where each framing is the zero-framing, corresponding to the preferred longitude $\ell_i := \partial\nu(L_i) \cap S_i$. It is clear that it does not depend on the choice of bounding surface F. We will therefore be justified in denoting it further by M_L. This 3-dimensional manifold has already been considered in the context of link signatures by Toffoli [28].

4.2.2. *Properties of* W_F. The goal of this subsection is to prove that $\sigma_L(\omega) = \mathrm{dsign}_\omega W_F, \eta_L(\omega) = \mathrm{null}_\omega W_F$ for $\omega \in \mathbb{T}^\mu_*$ and that $\mathrm{dsign}_\omega W_F$ and $\mathrm{null}_\omega W_F$ depend on the link L but not the choice of F.

Proposition 4.16. Let L be a μ-component link together with a choice of a bounding surface F. Then, for any $\omega \in \mathbb{T}^\mu_*$, the following equations hold:

$$\mathrm{sign}_\omega W_F = \sigma_L(\omega),$$
$$\mathrm{null}_\omega W_F = \eta_L(\omega).$$

In order to prove this, we need first to prove the following lemma:

Lemma 4.17. *Let W_F be the manifold associated to a bounding surface F and fix an $\omega \in \mathbb{T}^\mu, \omega \neq (1, \dots, 1)$. Then,*

$$\mathrm{null}_\omega W_F = \dim H_1(M_L; \mathbb{C}^\omega).$$

Proof. First, recall that by Proposition 2.13 we know that

$$\mathrm{null}_\omega W_F = \dim\ker\left(H_1(M_L;\mathbb{C}^\omega) \xrightarrow{i_*} H_1(W_F;\mathbb{C}^\omega)\right)$$

for all $\omega \in \mathbb{T}^\mu$.
Now, we want to claim that $H_1(W_F;\mathbb{C}^\omega) = 0$. We have by Lemma 4.15 that $\pi_1(W_F) \simeq \mathbb{Z}^\mu$. This implies that for the universal cover $\widetilde{W_F}$ induced by the homomorphism to $\mathbb{Z}^\mu$ the following hold:

$$\pi_1(\widetilde{W_F}) \cong H_1(\widetilde{W_F}; G) = 0,$$

for any coefficient group G. We can apply the Universal Coefficient Spectral Sequence (see Lemma 2.8), to obtain

- $E^2_{1,0} = \mathrm{Tor}^0_{\mathbb{C}[\mathbb{Z}^\mu]}(H_1(\widetilde{W_F};\mathbb{C}), \mathbb{C}^\omega) = 0 \otimes \mathbb{C} \cong 0;$
- $E^2_{0,1} = \mathrm{Tor}^1_{\mathbb{C}[\mathbb{Z}^\mu]}(H_0(\widetilde{W_F};\mathbb{C});\mathbb{C}^\omega) = \mathrm{Tor}^1_{\mathbb{C}[\mathbb{Z}^\mu]}(\mathbb{C},\mathbb{C}^\omega).$

We need to show now that $\mathrm{Tor}^1_{\mathbb{C}[\mathbb{Z}^\mu]}(\mathbb{C},\mathbb{C}^\omega)$ vanishes. For that, we can use the technique of Koszul resolutions (see [35, Chapter 4.5]). Let $K(x)$ denote the chain complex

$$\mathbb{C}[\mathbb{Z}^\mu] \xrightarrow{\cdot x} \mathbb{C}[\mathbb{Z}^\mu]$$

concentrated in degrees 0 and 1. Let $t_1,\dots,t_\mu$ be elements of $\mathbb{C}[\mathbb{Z}^\mu]$ corresponding to the canonical generators of $\mathbb{Z}^\mu$. Then the complex

$$K_\mu := K(t_1-1)\otimes_{\mathbb{C}[\mathbb{Z}^\mu]} K(t_2-1)\otimes_{\mathbb{C}[\mathbb{Z}^\mu]} \dots \otimes_{\mathbb{C}[\mathbb{Z}^\mu]} K(t_\mu-1)$$

is a free resolution of $\mathbb{C}$ over $\mathbb{C}[\mathbb{Z}^\mu]$ [35, Corollary 4.5.5]. This means that

$$\mathrm{Tor}^1_{\mathbb{C}[\mathbb{Z}^\mu]}(\mathbb{C},\mathbb{C}^\omega) = H_1(K_\mu \otimes_{\mathbb{C}[\mathbb{Z}^\mu]} \mathbb{C}^\omega).$$

Since by our assumption $\omega \neq (1,\dots,1)$, there is a j such that $\omega_j \neq 1$. Then,

$$K(t_j-1)\otimes_{\mathbb{C}[\mathbb{Z}^\mu]} \mathbb{C}^\omega = \mathbb{C} \xrightarrow{\cdot(\omega_j-1)} \mathbb{C},$$

which is acyclic. Since a tensor product of an acyclic chain complex with any other chain complex is again acyclic, this implies that all the homology groups of $K_\mu \otimes_{\mathbb{C}[\mathbb{Z}^\mu]} \mathbb{C}^\omega$ vanish and indeed $\mathrm{Tor}^1_{\mathbb{C}[\mathbb{Z}^\mu]}(\mathbb{C},\mathbb{C}^\omega) = 0$.
Now, since both $E^2_{0,1}$ and $E^2_{1,0}$ are trivial, this means that the same is true for the spaces $E^k_{1,0}$ and $E^k_{0,1}$ on higher pages. Thus, $E^\infty_{0,1} = E^\infty_{1,0} = 0$ and we get that

$$H_1(W_F;\mathbb{C}^\omega) = 0.$$

Consequently,

$$\mathrm{null}_\omega W_F = \dim\ker(i_*) = \dim H_1(M_L;\mathbb{C}^\omega),$$

which was to be proved. $\square$

Proof of Proposition 4.16. We will prove the proposition by applying the Novikov-Wall formula to the decomposition $W_F = V_F \cup Z_F$, hence

$$\mathrm{sign}_\omega W_F = \mathrm{sign}_\omega V_F + \mathrm{sign}_\omega Z_F + \mathcal{M},$$

where $\mathcal{M}$ is the associated Maslov index.
We know by Proposition 4.2 that $\mathrm{sign}_\omega V_F = \sigma_L(\omega)$ for $\omega \in \mathbb{T}^\mu_*$. By Lemma 4.11, the signature of Z_F is zero for each $\omega \in \mathbb{T}^\mu_*$. Therefore, we are left with showing that the Maslov index $\mathcal{M}$ associated to the gluing of V_F and Z_F vanishes. However, since the common part of the boundary of V_F and Z_F is $\mathrm{Pb}(\Gamma_F)$, Lemma 4.10 tells us that it is acyclic, and therefore the Maslov index must also vanish.
For the nullity, we want to look at the decomposition

$$\partial W_F = M_L = X_L \cup -\mathrm{Pb}(L),$$

and the associated Mayer-Vietoris sequence. It takes the form of:

$$\dots \to H_i(\partial\nu(L);\mathbb{C}^\omega) \to H_i(X_L;\mathbb{C}^\omega)\oplus H_i(-\mathrm{Pb}(L);\mathbb{C}^\omega) \to H_i(M_L;\mathbb{C}^\omega) \to H_{i-1}(\partial\nu(L)) \to \cdots.$$

Now, $\partial\nu(L)$ is homeomorphic to a disjoint union of μ tori $S^1\times S^1$ and since each meridian $*\times S^1$ is mapped to $\omega_i \neq 1$, the coordinate systems on all of these tori are nontrivially twisted. Consequently, $H_*(\partial\nu(L);\mathbb{C}^\omega) = 0$ and setting $i=1$ we obtain

$$H_1(M_L;\mathbb{C}^\omega) \cong H_1(X_L;\mathbb{C}^\omega)\oplus H_1(-\mathrm{Pb}(L);\mathbb{C}^\omega).$$

However, by Lemma 4.10, $H_1(-\mathrm{Pb}(L);\mathbb{C}^\omega)$ vanishes and so we obtain

$$\dim H_1(M_L;\mathbb{C}^\omega) = \dim H_1(X_L;\mathbb{C}^\omega) = \eta_L(\omega).$$

$\square$

In order to use the signature and nullity of W_F as a definition of invariants of L we will need the following:

Lemma 4.18. *Let L be a link in $S^3 = \partial B^4$. Then, for any two choices of a bounding surface F, F' for L, the twisted invariants agree:*

$$\operatorname{sign}_\omega W_F = \operatorname{sign}_\omega W_{F'}$$

$$\operatorname{null}_\omega W_F = \operatorname{null}_\omega W_{F'}$$

for any $\omega \in \mathbb{T}^\mu$.

Proof. By Corollary 2.34 we know that the signature defects (cf. Definition 2.14) of W_F and $W_{F'}$ depend only on their boundaries (as manifolds over $\mathbb{Z}^\mu$). However, since the boundary of W_F is homeomorphic to $M_L = X_L \cup -\operatorname{Pb}(L)$ and does not depend on the choice of a bounding surface, we get that the signature defects coincide. Thus, it suffices to show that the untwisted signatures of W_F and $W_{F'}$ vanish.
From Novikov-Wall additivity we get

$$\operatorname{sign} W_F = \operatorname{sign} V_F + \operatorname{sign} Y_F + \mathcal{M}_F,$$

$$\operatorname{sign} W_{F'} = \operatorname{sign} V_{F'} + \operatorname{sign} Y_{F'} + \mathcal{M}_{F'}.$$

We have that $\operatorname{sign} V_F = 0$ by Proposition 4.4. By construction $\operatorname{sign} Y_F = 0$, so we are left to consider only the Maslov indices associated to these gluings.
We need to look at the maps induced on first homology groups by the inclusions $\partial\nu(L) \hookrightarrow X_L, \operatorname{Pb}(F), -\operatorname{Pb}(L)$. These can be explicitly computed, as in [25, Lemma 5.4] and they depend only on the linking numbers of components of L. Therefore, the untwisted signature of W_F does not depend on the choice of F. Similarly, since the nullity of W_F depends only on the first homology group of its boundary, which is $X_L \cup \partial(-\operatorname{Pb}(L))$, it does not depend on the choice of a bounding surface. Therefore, $\operatorname{sign}_\omega W_F$ indeed does not depend on the choice of a bounding surface. □

For the rest of this section we will be using the following definition of signature.

Definition 4.19. *Let L be a μ-component link and let F be a totally connected bounding surface of L obtained by pushing a C-complex into B^4. Then, the multivariable link signature of L are defined as*

$$\sigma_L(\omega) := \operatorname{sign}_\omega W_F \tag{11}$$

$$\eta_L(\omega) := \operatorname{null}_\omega W_F \tag{12}$$

for any $\omega \in \mathbb{T}^\mu$.

4.3. **Torres formula for the signature.**

The aim of this subsection is to prove the following:

Theorem 4.20. *Let $L = L_1 \cup \ldots \cup L_\mu$ be an ordered link and let $\omega' \in \mathbb{T}_*^{\mu-1}$. Then, the following equations hold:*

$$\sigma_L(1, \omega') = \begin{cases} \sigma_{L'}(\omega') + \operatorname{sgn}((L_1/L)(\omega')) & \textit{if } \operatorname{lk}(L_1, L_i) = 0 \textit{ for all } i \geq 2 \\ \sigma_{L'}(\omega') & \textit{else,} \end{cases}$$

where we use the convention that $\operatorname{sgn}(\infty) = 0$.

Note that if $\operatorname{lk}(L_1, L_i)$ vanish for all i then all $\omega \in \mathbb{T}_*^\mu$ are admissible in the sense of Definition 2.35 and so the slope of L_1 relative to L makes sense.
A result similar to this has already appeared as [15, Lemma 4.9]. The approach there, however, is less general then the one presented in this thesis. Primarily, the definition of link signature used in [15] does not extend directly to the full torus. Therefore, the authors are forced to use the literal extension of the signature, that is the twisted signature of the bounding surface exterior. This however, as the authors of [15] write themselves, is not well-defined unless there is a non-vanishing linking number of L_1 with some other component of L.
In order to prove Theorem 4.20, we will first examine the behavior of twisted homology groups of plumbed manifolds when one of coordinates is equal to 1:

Lemma 4.21. *Let F be a compact surface with nonempty boundary. Then, for any homomorphism $\varphi : \pi_1(F) \to \mathbb{Z}^n$ and each $\omega \in \mathbb{T}^n$ such that the image of the composition $\Phi : \pi_1(F) \xrightarrow{\varphi} \mathbb{Z}^n \xrightarrow{\omega} \mathbb{C}^*$ is nontrivial, we have*

$$H_1(F; \mathbb{C}^\omega) \cong \mathbb{C}^{-\chi(F)},$$

where $\chi(F)$ denotes the Euler characteristic of F.

Proof. Every surface with a nonempty boundary homotopy retracts onto a wedge sum of a number of circles,

$$F \simeq \bigvee_{i=1}^{-\chi(F)+1} S_i^1 \coloneqq Q_{-\chi(F)+1},$$

therefore it suffices to prove this lemma for such a space.
First, assume the homomorphism Φ is such that the image of $\Phi(S_i^1)$ is nontrivial for each i. We then proceed by induction on the number of circles. For a single circle, the result follows from our assumption, since by Corollary 2.7, $H_1(S^1; \mathbb{C}^\omega) = 0$. For general k we apply the Mayer-Vietoris sequence to the decomposition $Q_{k+1} = Q_k \vee S^1$:

$$H_1(\mathrm{pt}; \mathbb{C}^\omega) = 0 \to H_1(Q_k; \mathbb{C}^\omega) \oplus H_1(S^1; \mathbb{C}^\omega) \to H_1(Q_{k+1}; \mathbb{C}^\omega) \to H_0(\mathrm{pt}; \mathbb{C}^\omega) \to$$
$$\to H_0(Q_k; \mathbb{C}^\omega) \oplus H_0(S^1; \mathbb{C}^\omega) \to \cdots.$$

Since Φ is nontrivial on both the S^1 factor and on Q_k, their zeroth homology groups vanish by Proposition 2.5 and we obtain our desired result.
Finally, for a general $\Phi : \pi_1(Q_k) \to \mathbb{C}^*$, we form the decomposition

$$Q_k = Q_{k_1} \vee Q_{k_2},$$

where $k_1 + k_2 = k$ and Q_{k_1} is such that $\Phi(S_i^1) \neq 1$ for each S_i^1 in it, and Q_{k_2} is such that $\Phi(Q_{k_2}) = 1$. Then, the Mayer-Vietoris sequence gives us:

$$H_1(\mathrm{pt}; \mathbb{C}^\omega) = 0 \to H_1(Q_{k_1}; \mathbb{C}^\omega) \oplus H_1(Q_{k_2}; \mathbb{C}^\omega) \to H_1(Q_k; \mathbb{C}^\omega) \to$$
$$H_0(\mathrm{pt}; \mathbb{C}^\omega) \to H_0(Q_{k_1}; \mathbb{C}^\omega) \oplus H_0(Q_{k_2}; \mathbb{C}^\omega) \to H_0(Q_k).$$

We can remove the H_0 terms corresponding to spaces with nontrivially twisted coefficient systems since they vanish by Proposition 2.5. Then, we use the fact that $H_1(Q_{k_2}; \mathbb{C}^\omega) \cong H_1(Q_{k_2}; \mathbb{C}) = \mathbb{C}^{k_2}$ since its coefficient system is untwisted and we are left with the exact sequence

$$0 \to \mathbb{C}^{k_1-1} \oplus \mathbb{C}^{k_2} \to H_1(Q_k; \mathbb{C}^\omega) \to \mathbb{C} \to \mathbb{C} \to 0.$$

Since this sequence is exact the map $\mathbb{C} \to \mathbb{C}$ must be an isomorphism and we obtain that

$$H_1(Q_k; \mathbb{C}^\omega) \cong \mathbb{C}^{k_1-1} \oplus \mathbb{C}^{k_2},$$

which was to be proven. $\square$

Consequently, we can calculate twisted homology groups of $F \times S^1$;

Corollary 4.22. *Let (F, φ) be a compact surface over $\mathbb{Z}^\mu$ with $k > 0$ punctures, and let $(F \times S^1, \varphi')$ denote the product of (F, φ) and (S^1, ψ). Denote by $\Phi : \pi_1(F \times S^1) \to \mathbb{C}^*$ the appropriate composition given by a $\omega = (\omega', 1)$. Then,*

$$H_1(F \times S^1; \mathbb{C}^\omega) \simeq \begin{cases} H_1(F; \mathbb{C}) \oplus H_1(S^1; \mathbb{C}), & \text{if } \Phi(\pi_1(F \times S^1)) = 1; \\ H_1(F; \mathbb{C}^{\omega'}) \oplus 0, & \text{otherwise.} \end{cases}$$

Proof. If the image of φ is trivial, the coefficient system is untwisted. Therefore, by the Künneth theorem (Proposition 2.6),

$$H_1(F \times S^1; \mathbb{C}^\omega) \cong H_1(F \times S^1; \mathbb{C}) \cong H_1(F; \mathbb{C}) \oplus H_1(S^1; \mathbb{C}).$$

Otherwise, the homomorphism $\varphi : \pi_1(F) \to \mathbb{C}$ must be non-trivial, and so by Proposition 2.5 we have $H_0(F; \mathbb{C}^{\omega'}) = 0$. Applying the Künneth formula yields us then

$$H_1(F \times S^1; \mathbb{C}^\omega) = 0 \oplus (H_1(F; \mathbb{C}^{\omega'}) \otimes H_0(S^1; \mathbb{C})) = H_1(F; \mathbb{C}^{\omega'}),$$

hence the corollary follows. □

Finally, we look at the behavior of homology of plumbed manifolds:

Proposition 4.23. Let Γ be a plumbed manifold and let $\omega = (1, \omega')$ for $\omega' \in \mathbb{T}_*^{\mu-1}$ define a twisted coordinate system on $\mathrm{Pb}(\Gamma)$. Then, the following isomorphism holds:

$$H_*(\mathrm{Pb}(\Gamma); \mathbb{C}^\omega) \cong H_*(\mathring{F}_1; \mathbb{C}^\omega)$$

Proof. We look at the Mayer-Vietoris sequence associated to the decomposition

$$\mathrm{Pb}(\Gamma) = (\mathring{F}_1 \times S^1) \cup_{\sqcup T_e} \mathrm{Pb}(\Gamma'),$$

where Γ' is a graph obtained from Γ be deleting v_1 and all edges adjacent to it and the gluing is done along a union of tori corresponding to the intersection points.
We obtain

$$\ldots \to H_i(\sqcup T_e; \mathbb{C}^\omega) \to H_1(\mathring{F}_1 \times S^1; \mathbb{C}^\omega) \oplus H_1(\mathrm{Pb}(\Gamma'); \mathbb{C}^\omega)) \to \\ \to H_1(\mathrm{Pb}(\Gamma); \mathbb{C}^\omega) \to H_{i-1}(\sqcup T_e; \mathbb{C}^\omega) \to \ldots.$$

We know by Lemma 4.10 that the homology groups of $\mathrm{Pb}(\Gamma')$ must vanish. We now look at the spaces T_e. A class of $S^1 \times \mathrm{pt}$ gets identified with the meridian of $\mathring{F}_i \times S^1$ and as such is mapped to ω_i under the twisted coordinate system. Since $\omega' \in \mathbb{T}_*^{\mu-1}$, the twisting is nontrivial and therefore the whole product $\mathring{F}_i \times S^1$ is acyclic. Therefore, the sequence splits into sequences of the form

$$0 \to H_1(\mathring{F}_1 \times S^1; \mathbb{C}^\omega) \to H_1(\mathrm{Pb}(\Gamma); \mathbb{C}^\omega) \to 0,$$

which finishes the proof. □

In order to calculate Maslov indices or dimensions of spaces in a Mayer-Vietoris sequence, we need to know how the homomorphism induced on homology by the inclusion of a boundary of a plumbed manifold looks like. We can give an explicit description of the kernel of this morphism:

Proposition 4.24. Let Γ be a plumbing graph and let $\omega = (1, \omega')$ for $\omega' \in \mathbb{T}_*^{\mu-1}$ define a twisted coordinate system on $\mathrm{Pb}(\Gamma)$. Assume that ω' is admissible in the sense of Definition 2.35. Then, the kernel of the homomorphism induced by the inclusion of the boundary of $\mathrm{Pb}(\Gamma)$ on H_1 is given by:

$$\begin{cases} \mathbb{C}[\partial F_1 \times \mathrm{pt}], & \text{if the vertex } v_1 \text{ is isolated} \\ \mathbb{C}[m_1], & \text{otherwise} \end{cases}$$

Proof. Note that we need to assume the ω' is admissible, as otherwise the boundary would be acyclic. By Lemma 2.10 we know that the dimension of the kernel is one, consequently to describe the kernel it is enough to find one nonzero element of the kernel. We have a basis for $H_1(\partial\,\mathrm{Pb}(\Gamma);\mathbb{C}^\omega)$ given by the class of $\mathrm{pt}\times S^1 = m_1$ and $\partial F_1\times \mathrm{pt} = \ell_1$. Since by Proposition 4.23 we have $H_1(\mathrm{Pb}(\Gamma);\mathbb{C}^\omega)\cong H_1(\mathring{F}_1\times S^1;\mathbb{C}^\omega)$, we only need to look at the homology of the space $\mathring{F}_1\times S^1$.

If the vertex is isolated, then $\mathring{F}_1 = F_1$ and by definition of the structure of $\mathrm{Pb}(\Gamma)$ as a manifold over $\mathbb{Z}^\mu$ the coordinate system on $F_1\times S^1$ is untwisted. Therefore, $H_1(F_1\times S^1;\mathbb{C}^\omega)$ is isomorphic to $H_1(F_1\times S^1;\mathbb{C})\simeq H_1(F_1;\mathbb{C})\oplus H_1(S^1;\mathbb{C})$ by Corollary 4.22. For any surface with boundary F we have that the homology class defined by its boundary is equal to zero in $H_1(F;\mathbb{C})$, since the surface F itself has the chain ∂F as its boundary. Therefore in the case of v_1 being isolated the class of $\partial F\ \times \mathrm{pt}$ is equal to zero in $H_1(F_1\times S^1;\mathbb{C}^\omega)$ thus determines the kernel.

Assume now that there is an edge between v_1 and v_i. Then, the homology class of the puncture corresponding to this edge is nontrivial in $H_1(\mathring{F}_1;\mathbb{Z})$ and it gets mapped to $\omega_i\neq 1$ in the twisted coordinate system, and so the twisting is nontrivial. We have now by Corollary 4.22 that $m_1 = 0$ in $H_1(\mathring{F}_1\times S^1;\mathbb{C}^\omega)$, which finishes the proof. □

Proof of Theorem 4.20. Let $F = F_1\cup F'$ be a surface in B^4 bounding the ordered link $L = L_1\cup L'$, obtained by pushing a connected C-complex inside B^4. Writing $W_F = V_F\cup Y_F$ and $W_{F'} = V_{F'}\cup Y_{F'}$ for the corresponding 4-manifolds, we want to apply the Novikov-Wall theorem to the following decompositions:

(1) $W_F = V_F\cup Y_F$ and $W_{F'} = V_{F'}\cup Y_{F'}$,
(2) $V_{F'} = V_F\cup \nu(F_1)$,

and to apply Lemma 4.11 which yields $\mathrm{sign}_\omega(Y_F) = \mathrm{sign}_{\omega'}(Y_{F'}) = 0$.

Let us start with the first step, i.e. the application of the Novikov-Wall theorem to the decomposition $W_F = V_F\cup Y_F$ along $\mathrm{Pb}(\Gamma_F)$. Since the orientation on W_F induces an orientation on V_F and Y_F such that $\partial Y_F = \mathrm{Pb}(\Gamma_F)\cup -\mathrm{Pb}(L)$ and $\partial V_F = X_L\cup -\mathrm{Pb}(\Gamma_F)$, we have

$$\mathrm{sign}_\omega(W_F) = \mathrm{sign}_\omega(V_F) + \mathrm{sign}_\omega(Y_F) + \mathit{Maslov}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3)\,,$$

where $\mathcal{L}_1$ (resp. $\mathcal{L}_2,\mathcal{L}_3$) denotes the kernel of the inclusion induced maps from $H_1(\partial X_L;\mathbb{C}^\omega)$ to $H_1(\mathrm{Pb}(L);\mathbb{C}^\omega)$ (respectively $H_1(\mathrm{Pb}(\Gamma_F);\mathbb{C}^\omega)$, $H_1(X_L;\mathbb{C}^\omega)$). First note that $H_1(\partial X_L;\mathbb{C}^\omega)$ vanishes (and therefore, the Maslov index vanishes as well) unless $\lambda := \prod \omega_i^{\mathrm{lk}(L_1,L_i)} = 1$. In that case $H_1(\partial\nu(L_1);\mathbb{C}^\omega) = \mathbb{C}m\oplus\mathbb{C}\ell$ for $m = m_1, \ell = \ell_1$, the meridian and longitude of L_1. Since F is connected, we know from Proposition 4.24 that $\mathcal{L}_2$ is generated by m. We can also apply Proposition 4.24 to $\mathrm{Pb}(L)$ to obtain that if $\mathrm{lk}(L_1,L_i)\neq 0$ for some $i>1$ then $\mathcal{L}_1$ is also generated by m and so the Maslov index vanishes by Corollary 2.18. We are left then only with the case when $\mathrm{lk}(L_1,L_i) = 0$ for all $i\geq 2$, in which case the vertex corresponding to L_1 in Γ_L is isolated and we get by Proposition 4.24 that $\mathcal{L}_1$ is generated by ℓ.

The final Lagrangian, $\mathcal{L}_2$, is generated by some linear combination of m,ℓ, which we can denote by $\mathbb{C}\,(\alpha_L(\omega)m + \beta_L(\omega)\ell)$. We are then left with the computation of the Maslov index of the Lagrangians generated by ℓ, m and $\alpha_L(\omega)m+\beta_L(\omega)\ell$.

By definition (see e.g. [31, Chapter IV.3]), this Maslov index is given by the signature of the form Ψ on $(\mathcal{L}_1+\mathcal{L}_2)\cap\mathcal{L}_3$ given as follows: if $a = a_1+a_2\in(\mathcal{L}_1+\mathcal{L}_2)\cap\mathcal{L}_3$ with $a_1\in\mathcal{L}_1, a_2\in\mathcal{L}_2$ and $b\in(\mathcal{L}_1+\mathcal{L}_2)\cap\mathcal{L}_3$, then $\Psi(a,b) = a_2\cdot b$. In our case, we have that $\mathcal{L}_1+\mathcal{L}_2$ is the entire space $H_1(\partial\nu(L_1);\mathbb{C}^\omega)\cong H_1(\partial\nu(L);\mathbb{C}^\omega)$ and so $(\mathcal{L}_1+\mathcal{L}_2)\cap\mathcal{L}_3 = \mathcal{L}_3$. Also, the form Ψ in this case is defined on a 1-dimensional space, and therefore its signature can by given by the sign of $\Psi(a,a)$ for any $a\neq 0$.

Now, since the intersection form of the boundary torus $\partial\nu(L_1)$ has the form

$$(z_1m + z_2\ell)\cdot(z_3m+z_4\ell) = z_1\overline{z_4} - z_2\overline{z_3},$$

we obtain

$$
\begin{aligned}
\mathit{Maslov}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3) &= \operatorname{sgn}(\Psi(\alpha_L(\omega)m + \beta_L(\omega)\ell, \alpha_L(\omega)m + \beta_L(\omega)\ell)) \\
&= \operatorname{sgn}(\alpha_L(\omega)m \cdot (\alpha_L(\omega)m + \beta_L(\omega)\ell)) = \operatorname{sgn}(\alpha_L(\omega)\overline{\beta_L(\omega)}).
\end{aligned}
$$

We get therefore that

$$
\operatorname{sign}_\omega(W_F) = \begin{cases} \operatorname{sign}_\omega(V_F) + \operatorname{sign}_\omega(Y_F) + \operatorname{sgn}(\alpha_L(\omega)\overline{\beta_L(\omega)}) & \text{if } \operatorname{lk}(L_1, L_i) = 0 \text{ for all } i \geq 2 \\ \operatorname{sign}_\omega(V_F) + \operatorname{sign}_\omega(Y_F) & \text{else.} \end{cases} \tag{13}
$$

Note here that $\operatorname{sgn}(\alpha_L(\omega)\overline{\beta_L(\omega)})$ is nothing else than the sign of the slope provided that we use convention that $\operatorname{sgn}(\infty) = 0$.
Let us now consider the decomposition $W_{F'} = V_{F'} \cup Y_{F'}$. Since we take ω' to lie in $\mathbb{T}_*^{\mu-1}$ it follows that $H_1(\partial X_{L'}; \mathbb{C}^\omega) = 0$, all the spaces $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ vanish and the Maslov index is therefore equal to zero. Hence, the signature is additive

$$
\operatorname{sign}_{\omega'}(W_{F'}) = \operatorname{sign}_{\omega'}(V_{F'}) + \operatorname{sign}_{\omega'}(Y_{F'}). \tag{14}
$$

We now turn to the second step, i.e. the application of the Novikov-Wall theorem to the decomposition $V_{F'} = V_F \cup \nu(\mathring{F}_1)$. Since $\mathring{F}_1$ is a surface with boundary, the 4-manifold $\nu(\mathring{F}_1) \simeq \mathring{F}_1 \times D^2$ has the homotopy type of a 1-dimensional CW-complex, and its signature vanishes. To compute the correction term, first note that the 3-manifold $M_1 := V_F \cap \nu(\mathring{F}_1)$ is equal to $\mathring{F}_1 \times S^1$, with boundary $\Sigma := \partial\nu(L_1) \cup \bigsqcup_e T_e$, where $\{T_e\}$ denotes the tori corresponding to the intersections of F_1 with the other surfaces. Since ω' belongs to $\mathbb{T}_*^{\mu-1}$, these tori are $\mathbb{C}^\omega$-acyclic. Therefore, we are once again in the situation where $H_1(\Sigma; \mathbb{C}^\omega)$ vanishes (as well as the correction term), unless $\lambda := \prod \omega_i^{\operatorname{lk}(L_1, L_i)} = 1$. If $\lambda = 1$ it is given by $H_1(\partial\nu(L_1); \mathbb{C}^\omega) = \mathbb{C}m \oplus \mathbb{C}\ell$. Now, observe that since F is connected, F_1 intersects the rest of the bounding surface, which implies that $H_0(\mathring{F}_1; \mathbb{C}^\omega)$ vanishes by the calculations in the proof of Proposition 4.24. By the Künneth formula (Proposition 2.6), we get $H_1(M_1; \mathbb{C}^\omega) \simeq H_1(\mathring{F}_1; \mathbb{C}^\omega)$. This implies that the meridian m lies in the kernel of the inclusion induced map $H_1(\partial\nu(L_1); \mathbb{C}^\omega) \to H_1(M_1; \mathbb{C}^\omega)$. Since this kernel is 1-dimensional, it is generated by m. To determine the second Lagrangian, observe that since $\nu(F_1)$ is homeomorphic to $F_1 \times D^2$, we have

$$
M_2 := \partial\nu(F_1) \setminus M_1 \simeq \big(\nu(L_1) \cup (F_1 \times S^1)\big) \setminus (\mathring{F}_1 \times S^1) = \nu(L_1) \cup \bigsqcup_e (D^2 \times S^1)\,,
$$

where the solid tori are indexed by the intersection points of F_1 with other surfaces. Since ω' belongs to $\mathbb{T}_*^{\mu-1}$, these tori are $\mathbb{C}^\omega$-acyclic, and we have $H_1(M_2; \mathbb{C}^\omega) = H_1(\nu(L_1); \mathbb{C}^\omega) = \mathbb{C}\ell$. As a consequence, the Lagrangian given by the kernel of the inclusion induced map $H_1(\partial\nu(L_1); \mathbb{C}^\omega) \to H_1(M_2; \mathbb{C}^\omega)$ is also given by $\mathbb{C}m$. Therefore, the Maslov correction term vanishes by Corollary 2.18, leading to

$$
\operatorname{sign}_\omega(V_F) = \operatorname{sign}_{\omega'}(V_{F'})\,. \tag{15}
$$

Finally, we have that $\operatorname{sign}_\omega(Y_F) = \operatorname{sign}_{\omega'}(Y_{F'}) = 0$ by the construction of Lemma 4.11. Using this fact and plugging equation 15 into equation 14 yields us

$$
\operatorname{sign}_{\omega'}(W_{F'}) = \operatorname{sign}_\omega(V_F).
$$

Now we use this identity and substitute $\sigma_L(\omega) = \operatorname{sign}_\omega(W_F), \sigma_{L'}(\omega') = \operatorname{sign}_{\omega'}(W_{F'})$ into (13) to obtain the desired conclusion. $\square$

4.4. **Torres formula for the nullity.** The aim of this subsection is to prove the following.

Theorem 4.25. *For all $\omega = (1, \omega') \in \mathbb{T}^\mu$ with $\omega' \in \mathbb{T}_*^{\mu-1}$, we have*

$$\eta_L(\omega) = \eta_{L'}(\omega') + \sum_{i=2}^{\mu} |\operatorname{lk}(L_1, L_i)| - 1,$$

if the linking numbers $\operatorname{lk}(L_1, L_i)$ are not all 0. If they all vanish, then we have

$$\eta_L(\omega) = \begin{cases} \eta_{L'}(\omega') + 1 & \text{if } (L_1/L)(\omega') = 0, \\ \eta_{L'}(\omega') - 1 & \text{if } (L_1/L)(\omega') = \infty, \\ \eta_{L'}(\omega') & \text{otherwise.} \end{cases}$$

Again, this statement looks similar to the statement of [15, Lemma 4.9]. We see, however, that in the case of all linking numbers $\operatorname{lk}(L_1, L_i)$ vanishing our theorem is stronger. Indeed, if the slope $(L_1/L)(\omega')$ is not equal to zero, then the nullity as defined in this thesis is strictly smaller than the literal extension of nullity. This in turn implies that the bound provided by Theorem 3.5 is stronger than the one using the literal extensions, even if they are well-defined. The strategy of the proof is to compute the dimension of $H_1(M_L; \mathbb{C}^\omega)$, which by 4.17 is equal to $\eta_L(\omega)$. We will calculate $\dim H_1(M_L; \mathbb{C}^\omega)$ from the Mayer-Vietoris sequence associated to the decomposition

$$M_L = X_L \cup -\operatorname{Pb}(L)$$

Furthermore, we will compare the dimensions of the homology groups of M_L and $M_{L'}$. First, we prove the following:

Lemma 4.26. *Let $L = L_1 \cup L'$ be a μ-components link, where $\omega = (1, \omega')$ for $\omega' \in \mathbb{T}_*^{\mu-1}$ defines a twisted coefficient system on the plumbed manifold $-\operatorname{Pb}(L)$. Then,*

$$H_1(-\operatorname{Pb}(L); \mathbb{C}^\omega) \cong H_1(-\operatorname{Pb}(L'); \mathbb{C}^{\omega'}) \oplus \begin{cases} \mathbb{C}[m_1] & \text{if } \operatorname{lk}(L_1, L_i) = 0 \text{ for all } i, \\ \mathbb{C}^{k-1} & \text{otherwise,} \end{cases}$$

where $k = \sum_{i=2}^{\mu} |\operatorname{lk}(L_1, L_i)|$.

We have the following decomposition of $-\operatorname{Pb}(L)$:

$$-\operatorname{Pb}(L) = -\mathring{\operatorname{Pb}}(L') \cup_{\amalg T_e} (\mathring{D}^2 \times S^1),$$

where $\mathring{D}^2$ is a disk capping off F_1 with punctures arising from plumbings in $-\operatorname{Pb}(L)$ and for each puncture e, $T_e = \partial D_e \times S^1$ and similarly $-\mathring{\operatorname{Pb}}(L')$ is the plumbed manifold corresponding to the sublink punctured in an appropriate way. This decomposition give rise to a Mayer-Vietoris sequence:

$$H_1(\amalg T_e; \mathbb{C}^\omega) \to H_1(-\mathring{\operatorname{Pb}}(L'); \mathbb{C}^\omega) \oplus H_1(\mathring{D}^2 \times S^1; \mathbb{C}^\omega) \to H_1(-\operatorname{Pb}(L); \mathbb{C}^\omega) \to H_0(\amalg T_e; \mathbb{C}^\omega) \to$$
$$\to H_0(-\mathring{\operatorname{Pb}}(L'); \mathbb{C}^\omega) \oplus H_0(\mathring{D}^2 \times S^1; \mathbb{C}^\omega) \to H_0(-\operatorname{Pb}(L); \mathbb{C}^\omega).$$

Now, $H_0(-\operatorname{Pb}(L); \mathbb{C}^\omega)$ and $H_0(-\mathring{\operatorname{Pb}}(L'); \mathbb{C}^\omega)$ vanish by Proposition 2.5. The space $-\mathring{\operatorname{Pb}}(L')$ is $\mathbb{C}^\omega$-acyclic, as it arises from gluing acyclic spaces (products of punctured surfaces with acyclic circles) along acyclic tori. Let us look at the homomorphism

$$\psi : H_1(\mathring{D}^2 \times S^1; \mathbb{Z}) \to \mathbb{Z}^\mu.$$

The untwisted homology group $H_1(\mathring{D}^2 \times S^1; \mathbb{Z})$ is generated by classes of $\partial D_e \times \mathrm{pt}$ where D_e are neighborhoods of punctures in D^2 and by a class $m_1 := [\mathrm{pt} \times S^1]$. We now look at the homomorphism extending the structure of X_L as a manifold over $\mathbb{Z}^\mu$ to the plumbed manifold $-\operatorname{Pb}(L)$. Firstly, it maps m_1 to $\omega_1 = 1$. Secondly, for each puncture in $\mathring{D}^2$ which corresponds

to an edge e between v_1 and v_i in the graph Γ_L and is obtained by removing a disk D_e, the class of $\partial D_e \times *$ is mapped to ω_i. The group $H_0(\mathring{D}^2 \times S^1; \mathbb{C}^\omega)$ vanishes if and only if the twisting of the coordinate system is nontrivial by Proposition 2.5. By the description of the twisting homomorphism, this happens if a linking number between L_1 and some L_i is nonzero, and otherwise $H_0(\mathring{D}^2 \times S^1; \mathbb{C}^\omega) \cong \mathbb{C}$, since the space $\mathring{D}^2 \times S^1$ is connected.
For each torus T_e, one of the S^1 factors is glued to a meridian of some $\hat{F}_i, i \neq 1$. The corresponding integral homology class gets mapped to $\omega_i \neq 0$ and so we have that $H_*(\amalg T_e) = 0$ by Corollary 2.7.

Finally, consider $H_1(\mathring{D}^2 \times S^1; \mathbb{C}^\omega)$. The space $\mathring{D}^2$ is homotopy equivalent to a bouquet of circles $\bigvee_k S^1$, where $k = \sum_{i=2}^{\mu} |\operatorname{lk}(L_1, L_i)|$. By the Künneth formula (Proposition 2.6) we obtain

$$H_1(\mathring{D}^2 \times S^1; \mathbb{C}^\omega) = (H_1(\mathring{D}^2; \mathbb{C}^\omega) \otimes H_0(S^1; \mathbb{C}^\omega)) \oplus (H_0(\mathring{D}^2; \mathbb{C}^\omega) \otimes H_1(S^1; \mathbb{C}^\omega)).$$

Since the class of $m_1 = \mathrm{pt} \times S^1$ is mapped to 1, we have that $H_*(S^1; \mathbb{C}^\omega) \cong H_*(S^1; \mathbb{C})$ and so

$$H_1(\mathring{D}^2 \times S^1; \mathbb{C}^\omega) \cong H_1(\mathring{D}^2; \mathbb{C}^\omega) \oplus H_0(\mathring{D}^2; \mathbb{C}^\omega).$$

The twisted homology groups of a bouquet of circles have already been computed in the proof of Lemma 4.21, and so we get that $H_1(\mathring{D}^2 \times S^1; \mathbb{C}^\omega) \cong \mathbb{C}^{k-1}$ if there is some nonzero linking number between components of L, and $H_1(\mathring{D}^2 \times S^1; \mathbb{C}^\omega) \cong \mathbb{C}[m_1]$ otherwise.

Next, we will consider the space X_L:

Lemma 4.27. *Let $L = L_1 \cup L'$ be a μ-components link together with $\omega = (1, \omega')$ for $\omega' \in \mathbb{T}_*^{\mu-1}$ defining a twisted coefficient system on the link exterior X_L. Denote by λ the product $\Pi_{i=2}^{\mu} \omega_i^{\operatorname{lk}(L_1, L_i)}$ and let $[\tilde{m}_1]$ be the homology class of the meridian of L_1 in $H_1(X_{L'}; \mathbb{C}^\omega)$. Then,*

$$H_1(X_L; \mathbb{C}^\omega) \cong \begin{cases} H_1(X_{L'}; \mathbb{C}^\omega), & \text{if } \lambda \neq 1 \\ H_1(X_{L'}; \mathbb{C}^\omega), & \text{if } \lambda = 1 \text{ and } [\tilde{m}_1] = 0 \text{ in } H_1(X_L; \mathbb{C}^\omega) \\ H_1(X_{L'}; \mathbb{C}^\omega) \oplus \mathbb{C}[\tilde{m}_1], & \text{otherwise,} \end{cases}.$$

Proof. We have the decomposition of $X_{L'}$ into $X_L \cup \nu(L_1)$, glued along $\partial \nu(L_1)$. This gives a Mayer-Vietoris sequence

$$H_1(\partial \nu(L_1); \mathbb{C}^\omega) \xrightarrow{(\iota, \kappa)} H_1(X_L; \mathbb{C}^\omega) \oplus H_1(\nu(L_1); \mathbb{C}^\omega) \to H_1(X_{L'}; \mathbb{C}^\omega) \to$$
$$\to H_0(\partial \nu(L_1); \mathbb{C}^\omega) \xrightarrow{\cong} H_0(\nu(L_1); \mathbb{C}^\omega) \to 0,$$

as the H_0 groups of X_L and $X_{L'}$ vanish.

Now, consider first the case when $\lambda \neq 1$. As the longitude of L_1 is mapped to λ in the twisted coefficients system, all the groups $H_*(\partial \nu(L_1); \mathbb{C}^\omega), H_*(\nu(L_1); \mathbb{C}^\omega)$ vanish by Corollary 2.7. The isomorphism between $H_1(X_L; \mathbb{C}^\omega)$ and $H_1(X_{L'}; \mathbb{C}^\omega)$ then follows.
If $\lambda = 1$, the exact sequence, after removing the final isomorphism, looks as follows:

$$\mathbb{C}[m_1] \oplus \mathbb{C}[\ell_1] \to H_1(X_L; \mathbb{C}^\omega) \oplus H_1(\nu(L_1); \mathbb{C}^\omega) \to H_1(X_{L'}; \mathbb{C}^\omega) \to 0.$$

In the leftmost morphism the map from $\mathbb{C}[\ell_1]$ to $H_1(\nu(L_1); \mathbb{C}^\omega)$ is an isomorphism and the map from $\mathbb{C}[m_1]$ to $H_1(\nu(L_1); \mathbb{C}^\omega)$ is trivial. Therefore, after applying Lemma 2.2 the Mayer-Vietoris sequence reduces to

$$\mathbb{C}[m_1] \xrightarrow{\iota} H_1(X_L; \mathbb{C}^\omega) \to H_1(X_{L'}; \mathbb{C}^\omega) \to 0,$$

and the lemma follows.

∎

Finally, can consider the whole space M_L:

Proof of Theorem 4.25. To obtain M_L we glue X_L and $-\operatorname{Pb}(L)$ along $\partial\nu(L)$. This gives the Mayer-Vietoris sequence

$$H_1(\partial\nu(L_1);\mathbb{C}^\omega) \to H_1(X_L;\mathbb{C}^\omega) \oplus H_1(-\operatorname{Pb}(L);\mathbb{C}^\omega) \to H_1(M_L;\mathbb{C}^\omega) \to H_0(\partial\nu(L_1);\mathbb{C}^\omega) \to 0$$

as the H_0 groups of X_L and $-\operatorname{Pb}(L)$ vanish and the boundary of neighborhoods of other link components are acyclic.

The corresponding Mayer-Vietoris sequence for $\partial W_{L'}$ shows that $H_1(\partial W_{L'}) = H_1(X_{L'})$, as in that case we glue an acyclic space along an acyclic space.

Consider the case when $\lambda \neq 1$. In that case, since the longitude of L_1 is mapped to λ, we have

$$0 \to H_1(X_L;\mathbb{C}^\omega) \oplus H_1(-\operatorname{Pb}(L);\mathbb{C}^\omega) \to H_1(M_L;\mathbb{C}^\omega) \to 0,$$

as $\partial\nu(L)$ is acyclic as a product of a space with a properly twisted circle. Therefore we get that

$$\eta_L = \dim(H_1(M_L);\mathbb{C}^\omega) = \dim(H_1(X_{L'});\mathbb{C}^\omega) + \dim(H_1(-\operatorname{Pb}(L));\mathbb{C}^\omega),$$

and so by applying Lemma 4.26 and Proposition 4.3 to the terms on the right we obtain

$$\eta_L(1,\omega_2,...,\omega_\mu) = \eta_{L\setminus L_1}(\omega_2,...,\omega_\mu) + \sum |\operatorname{lk}(L_1,L_i)| - 1$$

in this case.

Now, let us consider the case where $\operatorname{lk}(L_1,L_i) = 0$ for any i. Then, $H_*(\partial\nu(L_1);\mathbb{C}^\omega) \cong H_*(\partial\nu(L_1);\mathbb{C})$ and the Mayer-Vietoris sequence takes the following form:

$$\mathbb{C}[m_1] \oplus \mathbb{C}[\ell_1] \to H_1(X_{L'};\mathbb{C}^\omega) \oplus \mathbb{C}[\tilde{m}_1] \oplus H_1(-\operatorname{Pb}(L);\mathbb{C}^\omega) \to H_1(M_L;\mathbb{C}^\omega) \to$$
$$\to H_0(\partial\nu(L)) \xrightarrow{\simeq} H_0(-\operatorname{Pb}(L);\mathbb{C}^\omega) \to 0$$

and in this case $H_1(-\operatorname{Pb}(L);\mathbb{C}^\omega)$ is one dimensional, generated by the image of the meridian of L_1 as a consequence of Proposition 4.23, since in the case of zero linking numbers there are no punctures in disk bounding L_1 in $\operatorname{Pb}(L)$.
We have now that $H_1(M_L;\mathbb{C}^\omega)$ is isomorphic to

$$[H_1(X_{L'};\mathbb{C}^\omega) \oplus \mathbb{C}[\tilde{m}_1] \oplus H_1(-\operatorname{Pb}(L);\mathbb{C}^\omega)]/\operatorname{Im}(H_1(\partial\nu(L_1);\mathbb{C}^\omega))$$

and so by substituting in the dimensions of homology groups of $X_{L'}$ and $-\operatorname{Pb}(L)$ we get

$$\dim(H_1(M_L;\mathbb{C}^\omega)) = \eta_{L\setminus L_1}(\omega_2,\dots,\omega_\mu) + 1 + \dim(\mathbb{C}[\tilde{m}_1]) - \dim(\operatorname{Im}(H_1(\partial\nu(L_1);\mathbb{C}^\omega))$$

We know from Lemma 2.10 that the kernels of homomorphisms induced by the inclusions of $H_1(\partial X_L;\mathbb{C}^\omega) \simeq H_1(\partial\nu(L_1);\mathbb{C}^\omega)$ into $H_1(X_L;\mathbb{C}^\omega)$ and into $H_1(-\operatorname{Pb}(L);\mathbb{C}^\omega)$ are 1-dimensional, and so are their images. The image of the direct sum of these morphisms will therefore be 2-dimensional unless the kernels of these two homomorphisms coincide.
The kernel of $H_1(\partial\nu(L_1);\mathbb{C}^\omega) \to H_1(-\operatorname{Pb}(L);\mathbb{C}^\omega)$ is just the 1-dimensional space generated by ℓ_1 in this case by Proposition 4.24, as the vertex corresponding to L_1 in Γ_L is isolated when all the linking numbers $\operatorname{lk}(L_1,L_i)$ vanish.
For the second homomorphism we need to consider the slope of L_1 in L. The two kernels coincide if and only if the slope is zero. On the other hand $[\tilde{m}_1] \neq 0$ if and only if the slope is finite. Finally, we obtain

$$\dim\,(H_1(M_L;\mathbb{C}^\omega)) = \eta_L(1,\omega_2,...,\omega_\mu) = \begin{cases} \eta_{L\setminus L_1}(\omega_2,...,\omega_\mu)+1, & \text{if the slope is zero} \\ \eta_{L\setminus L_1}(\omega_2,...,\omega_\mu)-1, & \text{if the slope is infinite} \\ \eta_{L\setminus L_1}(\omega_2,...,\omega_\mu), & \text{otherwise.} \end{cases}$$

Since the dimension of a space must be nonnegative it is worth checking if the value of η_L is actually nonnegative in the case of infinite slope. Indeed, in that case we have by Lemma 4.27 that $\dim H_1(X_L;\mathbb{C}^\omega) = \dim H_1(X_{L'};\mathbb{C}^\omega) = \eta_{L\setminus L_1}(\omega_2,\dots,\omega_\mu)$. The dimension of $H_1(X_L;\mathbb{C}^\omega)$ has to be strictly positive, as otherwise the homomorphism induced on first homology groups by the inclusion $\partial X_L \hookrightarrow X_L$ would be trivial and its kernel would be of dimension 2, contradicting Lemma 2.10. Thus, the nullity of $L\setminus L_1$ is indeed positive in this case, as desired.

Finally, consider the case where $\lambda = 1$ and there exists a nonzero linking number between L_1 and some L_i. We then have the following Mayer-Vietoris sequence

$$\mathbb{C}[m_1]\oplus\mathbb{C}[\ell_1] \to H_1(X_{L'};\mathbb{C}^\omega)\oplus\mathbb{C}[\tilde{m}_1]\oplus H_1(-\operatorname{Pb}(L);\mathbb{C}^\omega) \to H_1(M_L;\mathbb{C}^\omega) \to \mathbb{C} \to 0.$$

We can compare the dimensions of spaces in the exact sequence to obtain

$$\dim(H_1(M_L;\mathbb{C}^\omega)) = \dim[H_1(X_{L'};\mathbb{C}^\omega)\oplus\mathbb{C}[\tilde{m}_1]\oplus H_1(-\operatorname{Pb}(L);\mathbb{C}^\omega)/\operatorname{Im}(\mathbb{C}[m_1]\oplus\mathbb{C}[\ell_1])]+1.$$

In this case we have that the kernel of $H_1(\partial\nu(L_1);\mathbb{C}^\omega)\to H_1(-\operatorname{Pb}(L);\mathbb{C}^\omega)$ is generated by m_1 by Proposition 4.24 and so the image ℓ_1 is nonzero in $H_1(-\operatorname{Pb}(L);\mathbb{C}^\omega)$. We get then that the image of m_1 under the connecting homomorphism is nonzero if and only if $\tilde{m}_1\neq 0$, that is the space $\mathbb{C}[\tilde{m}_1]$ does not vanish. Combining these facts we see that the dimension of

$$H_1(X_{L'};\mathbb{C}^\omega)\oplus\mathbb{C}[\tilde{m}_1]\oplus H_1(-\operatorname{Pb}(L);\mathbb{C}^\omega)/\operatorname{Im}(\mathbb{C}[m_1]\oplus\mathbb{C}[\ell_1])$$

is always equal to

$$\dim(H_1(X_L;\mathbb{C}^\omega))+\dim(H_1(-\operatorname{Pb}(L);\mathbb{C}^\omega))-1,$$

since the dimension of the image of $(\mathbb{C}[m_1]\oplus\mathbb{C}[\ell_1])$ is equal to 2 if and only if the $\mathbb{C}[\tilde{m}_1]$ summand does not vanish.

By substituting the dimensions of the spaces by Lemma 4.26 and Proposition 4.3 we arrive at the following expression,

$$\eta_{L\setminus L_1}(\omega_2,...,\omega_\mu)+\sum|\operatorname{lk}(L_1,L_i)|-1,$$

valid for any slope of L_1 in L.

These three cases together yield the desired formula of the theorem. □

4.5. **Limits of signature and nullity from the 4-dimensional point of view.** We can now use Lemma 2.1 together with the two formulas for the signature and nullity at 1 to obtain

Theorem 4.28. *Let L be a μ-component link with $\mu\geq 2$. Then, for each (ω_1,ω'), $\omega'\in\mathbb{T}_*^{\mu-1}$*

(1) *If $\operatorname{lk}(L_1,L_i)\neq 0$ for some L_i then*

$$\Big|\lim_{\omega_1\to 1^\pm}\sigma_L(\omega_1,\omega')-\sigma_{L\setminus L_1}(\omega')\Big| \leq \eta_{L\setminus L_1}(\omega')+\sum_{2\leq i\leq\mu}|\operatorname{lk}(L_1,L_i)|-1-\lim_{\omega_1\to 1^\pm}\eta_L(\omega_1,\omega').$$

(2) *If $\operatorname{lk}(L_1,L_i)=0$ for all $2\leq i\leq\mu$, then*

$$\Big|\lim_{\omega_1\to 1^\pm}\sigma_L(\omega_1,\omega')-\sigma_{L\setminus L_1}(\omega')-\operatorname{sgn}((L_1/L)(\omega'))\Big| \leq \eta_{L\setminus L_1}(\omega')+s(\omega')-\lim_{\omega_1\to 1^\pm}\eta_L(\omega_1,\omega'),$$

where

$$s(\omega') = \begin{cases} +1 & \text{if } (L_1/L)(\omega')=0 \\ -1 & \text{if } (L_1/L)(\omega')=\infty \\ 0 & \text{otherwise.} \end{cases}$$

In order to prove this theorem, we will need the following lemma, proven by David Cimasoni [6, Lemma 5.1].

Lemma 4.29. *Let Λ_μ denote the group ring $\mathbb{C}[\mathbb{Z}^\mu]$, and let $Q(\Lambda_\mu)$ be its fraction field. Suppose that (W, ψ) is a compact connected oriented 4-manifold over $\mathbb{Z}^\mu$ with connected boundary, such that the composition*

$$H_1(\partial W) \to H_1(W) \xrightarrow{\psi} \mathbb{Z}^\mu$$

is surjective and $H_1(W;\Lambda_\mu) = 0$. Then, for any $j = 1, \dots, \mu$, there exists a Hermitian matrix H_j over $Q(\Lambda_\mu)$ such that for any $\omega \in U_j \coloneqq \{\omega \in \mathbb{T}^\mu \colon \omega_j \neq 1\}$, the intersection form

$$Q_\omega \colon H_2(W;\mathbb{C}^\omega) \times H_2(W;\mathbb{C}^\omega) \to \mathbb{C}$$

is represented by $H_j(\omega)$.

Proof of Theorem 4.28. From Lemma 4.15 we know that the meridional homomorphism $\pi_1(W_F) \to \mathbb{Z}^\mu$ is an isomorphism. This implies that the conditions of Lemma 4.29 are satisfied and taking any $j \neq 1$ we have a one-parameter family of matrices $H_j(e^{it}, \omega')$ the signature and nullity of which coincide with the signature and nullity of W_F at $(e^{\pm it}, \omega')$. Thus, Lemma 2.1 implies that

$$\left| \lim_{\omega_1 \to 1^\pm} \sigma_L(\omega_1, \omega') - \sigma_L(1, \omega') \right| \leq \eta_L(1, \omega') - \lim_{\omega_1 \to 1^\pm} \eta_L(\omega_1, \omega').$$

Now, substituting in the formulas for $\sigma_L(1, \omega'), \eta_L(1, \omega')$ given by Theorems 4.20 and 4.25 proves the desired result. □

This gives us different bounds on the limit of the limit of signature of L then the ones provided by Theorem 3.5. Interestingly enough, either one of these can give stronger information about the limit of $\sigma_L(\omega_1, \omega')$ depending on the link in question. In general, Theorem 4.28 will be better in case of links where L_1 is algebraically split or otherwise the linking numbers $\mathrm{lk}(L_1, L_i)$ are small in absolute value, and Theorem 3.5 - when the linking numbers are large. To illustrate this, we now return to the family of twist links of Example 3.18.
There, Theorem 3.5 was unable to determine the limit of the signature, since the right-hand side was greater than 0. This will not be the case for Theorem 4.28. Since all the Seifert matrices of Θ_k are equal to $[k]$ we get a formula for its multivariable Conway function by the main result of [4]:

$$\nabla_{\Theta_k}(t_1, t_2) = k(t_1 - t_1^{-1})(t_2 - t_2^{-1}).$$

Now, we wish to apply Theorem 2.37 to obtain the slope $(L_1/\Theta_k)(\omega_2)$ for any ω_2. We do not need to specify which component of Θ_k is denoted by L_1, as there there exists an isotopy of Θ_k exchanging the components. We have

$$2\nabla_{\Theta_k \setminus L_1}(\sqrt{\omega_2}) = \frac{1}{\sqrt{\omega_2} - \sqrt{\omega_2}^{-1}},$$
$$\frac{\partial \nabla_{\Theta_k}}{\partial t_1}(1, \sqrt{\omega_2}) = 2k(\sqrt{\omega_2} - \sqrt{\omega_2}^{-1}).$$

Therefore,

$$(L_1/\Theta_k)(\omega_2) = -2k(\sqrt{\omega_2} - \sqrt{\omega_2}^{-1})^2 = 2k(1-\omega_2)(1-\omega_2^{-1}) = 2k|1-\omega_2|^2,$$

and the sign of the slope is equal to the sign of k. Thus, theorem 4.28 yields, for $k \neq 0$,

$$\left| \lim_{\omega_1 \to 1^\pm} \sigma_{\Theta_k}(\omega_1, \omega') - 0 - \operatorname{sgn}(k) \right| \leq 0 + 0 - 0.$$

This gives a precise value the limit of the signature, unlike the 3-dimensional inequality.

4.6. **Concordance invariance of the twisted signature.** In the case of knots, Levine–Tristram signature has been successfully used to investigate the properties of the knot concordance group. In particular, it can be shown that the signature provides a lower bound for a knot's slicing number, that is the minimal number of crossing changes required to make a knot K concordant to an unknot [7, Section 1.4].
We would therefore wish that our definition of link signature and nullity also behaves well with regards to this relation. First, we want to generalize the notion of concordance root from Definition 2.28:

Definition 4.30. *We call a tuple $\omega \in \mathbb{T}^\mu$ a* concordance root *if there exists a Laurent polynomial $p \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}]$ such that $p(1, \dots, 1) = \pm 1$ and $p(\omega) = 0$. We denote the set of $\omega \in \mathbb{T}^\mu$ that are* not *concordance roots by $\mathbb{T}^\mu_!$.*

This generalization of the definition turns out to satisfy desired properties. In particular, the following generalization of Lemma 2.29 holds:

Lemma 4.31. *Let k be a non-negative integer and let ω lie in $\mathbb{T}^\mu_!$. If (X, Y) is a pair of CW-complexes over $B\mathbb{Z}^\mu$ such that $H_i(X, Y; \mathbb{Z}) = 0$ for $0 \leq i \leq k$ the the groups $H_i(X, Y; \mathbb{C}^\omega)$ also vanish for $0 \leq i \leq k$.*

To prove this, we wish to make use of the following fact, quoted here after [10] and proved in more generality in [25, Section 3]. Here, $U \subset \Lambda = \mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}]$ is the multiplicative subset consisting of polynomials p such that $p(1, \dots, 1) \neq \pm 1$:

Proposition 4.32. Let $g\colon \mathbb{Z}[\mathbb{Z}^m]^k \to \mathbb{Z}[\mathbb{Z}^m]^k$ be a $\mathbb{Z}[\mathbb{Z}^m]$–module homomorphism with the property that $\mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}^m]} g$ is an isomorphism. Then

$$U^{-1}\Lambda \otimes_{\mathbb{Z}[\mathbb{Z}^m]} g\colon (U^{-1}\Lambda)^k \to (U^{-1}\Lambda)^k$$

is also an isomorphism. Consequently, so is $\mathbb{C}^\omega \otimes_{\mathbb{Z}[\mathbb{Z}^m]} g$.

Proof of Lemma 4.31. First, we proceed by reproducing the proof of [10, Lemma 2.16]. We make the following abbreviations $C^\mathbb{Z} := C(X, Y; \mathbb{Z})$ and $C^\Lambda := C(X, Y; \Lambda)$ for the cellular chain complexes of the pairs (X, Y). For the remainder of the proof, i will be an arbitrary integer $0 \leq i \leq k$. The chain complex $C^\mathbb{Z}$ consists of finitely generated free $\mathbb{Z}$-modules, and as $H_i(C^\mathbb{Z}) = 0$, it admits a partial contraction, i.e. homomorphisms $s_i\colon C_i^\mathbb{Z} \to C_{i+1}^\mathbb{Z}$ with

$$\mathrm{id}_{C_i^\mathbb{Z}} = s_{i-1} \circ d_i + d_{i+1} \circ s_i.$$

Consider the chain map $\varepsilon\colon C^\Lambda \to C^\mathbb{Z}$ of chain complexes over Λ, which is induced by tensoring with the augmentation map. Pick a lift s_i^Λ of s_i under ε, which is a homomorphism $s_i^\Lambda\colon C_i^\Lambda \to C_{i+1}^\Lambda$ of Λ-modules such that the following diagram commutes:

$$\begin{tikzcd} C_i^\Lambda \arrow[r, "s_i^\Lambda"] \arrow[d, "\varepsilon"] & C_{i+1}^\Lambda \arrow[d, "\varepsilon"] \\ C_i^\mathbb{Z} \arrow[r, "s_i"] & C_{i+1}^\mathbb{Z}. \end{tikzcd}$$

Such a lift exists because C_i^Λ consists of free modules and the map ε is surjective. Consider the partial chain map

$$f_i = s_{i-1}^\Lambda \circ d_i + d_{i+1} \circ s_i^\Lambda.$$

By construction, $\mathbb{Z}\otimes_\Lambda f_i = s_{i-1}\circ d_i + d_{i+1}\circ s_i = id_{C_i^{\mathbb{Z}}}$ and so $U^{-1}\Lambda\otimes_\Lambda f_i$ is also an isomorphism; see Proposition 4.32 We obtain that $U^{-1}\Lambda\otimes_\Lambda s_i^\Lambda$ is a partial chain contraction for $U^{-1}\Lambda\otimes_\Lambda C^\Lambda$ and

$$H_i(X,Y;U^{-1}\Lambda) = H_i(U^{-1}\Lambda\otimes_\Lambda C^\Lambda) = 0.$$

Now, we make use of the fact that $\mathbb{C}^\omega$ is a $U^{-1}\Lambda$ module for $\omega\in\mathbb{T}_!^\mu$. This means that we can consider the chain complex

$$C^\Lambda\otimes_{U^{-1}\Lambda} U^{-1}\Lambda\otimes_\Lambda \mathbb{C}^\omega,$$

for which $\mathbb{C}^\omega\otimes s_i^\Lambda$ is a partial contraction, and therefore $H_i(X,Y;\mathbb{C}^\omega)=0$ for $i\leq k$. □

Using this, we want to make the following observation about homology cobordisms (see Proposition 2.27) and their signatures:

Proposition 4.33. Let W with $\partial W = M\cup M'$ be a $\mathbb{Z}$-homology cobordism over $\mathbb{Z}^\mu$ between 3-manifolds M,M'. Then, for each $\omega\in\mathbb{T}_!^\mu$, we have:

$$\operatorname{sign} W = \operatorname{sign}_\omega W = 0,$$

Proof. First, since $H_i(W,M;\mathbb{Z})$ vanish for $i=1,2$ we get that the map $H_2(M;\mathbb{Z})\to H_2(W;\mathbb{Z})$ induced by the inclusion of M is onto, and since the image of the boundary annihilates the intersection form, it follows that the intersection form is zero, and so the signature of W also is zero.

As $\omega\in\mathbb{T}_!^\mu$, Lemma 4.31 that the relative homology groups $H_i(W,M;\mathbb{C}^\omega)$ also vanish and therefore by the same argument we have $\operatorname{sign}_\omega W = 0$. □

In order to obtain results about concordance invariance of the nullity and signature we will proceed by modifying the arguments of [10] and [14] to accommodate the case of $\omega_1 = 1$. In theory, this is not necessary to do in the case of ordered link considered in this thesis.This is because the Torres formulae obtained previously depend only on the signature and nullity of a sublink as well as the linking numbers of components of L and certain slopes. All of these are invariant under concordance as proved in the aforementioned papers. However, we present a modification of these proofs here, since these arguments can be easily turned into corresponding statements about more general definitions of signature and nullity of colored links, where we might not have easily tractable Torres formulae.
First, we consider the invariance of the slope of L_1 relative to L under concordance. The first half of the proof here is essentially the same as the proof of [14, Theorem 3.2], however we make use of the generalization of Lemma 4.31 to simplify the argument:

Lemma 4.34. *Let L^0,L^1 be two concordant μ-component links. Since $\operatorname{lk}(L_i^0,L_j^0)=\operatorname{lk}(L_i^1,L_j^1)$ for any i,j, we know that the set of admissible $\omega'\in\mathbb{T}^{\mu-1}$ for both of these links coincide. Then, for each $\omega'\in\mathbb{T}_{*!}^{\mu-1}$ the following equality holds:*

$$(L_1^0/L^0)(\omega) = (L_1^1/L^1)(\omega).$$

Proof. Let $A=A'\cup A''$ be the concordance between L^0,L^1, where A' is the concordance restricted to $L^s\setminus L_1^s =: L'^s$. Consider an open tubular neighborhood $T_{A'\cup A''}$ of A. Denote

$$U = S^3\times[0,1]\setminus T_{A''}, U_{L'} = S^3\times[0,1]\setminus T_A$$

and let

$$X^s := U\cap(S^3\times s), X_{L'}^s = U_{L'}\cap(S^3\times s),$$

for $s=0,1$. The inclusions $X_{L'}^s\hookrightarrow U_{L'}$ send the meridians of $L'^s\cup {L_1}^s$ to those of $A'\cup A''$, see for example the proof of [10, Lemma 2.5].
The relative Mayer-Vietoris sequence (for homology with integer coefficients) applied to

$$(S^3\times I, S^3\times s) = (U_{L'},X_{L'}^s)\cup(\bar{T}_A,\bar{T}_{L^s}) = (U,X^s)\cup(\bar{T}_{A''},\bar{T}_{{L_1}^s})$$

where $\bar{T}_*$ stands for closure of a tubular neighborhood T_*, gives us

$$H_*(U_{L'}, X^s_{L'}) = H_*(U, X^s) = 0$$

for $s = 0, 1$. In particular the inclusions $X^s_L \hookrightarrow U_{L'}$ induce isomorphisms

$$H_1(X^0) \xrightarrow{\cong} H_1(U_{L'}) \xleftarrow{\cong} H_1(X^1),$$

preserving the meridians. Since the Seifert-framed longitude of link L_i component can be characterized as the unique longitude for which it's image in $H_1(X_L)$ expressed as a sum of meridians has a zero coefficient at m_i, this identification also has to take Seifert-framed longitudes of L^0 to Seifert-framed longitudes of L^1. This now means that these isomorphisms agree with the identification of $H_1(\partial\nu(L^s))$ with $\mathbb{Z}^{2\mu}$ by the basis of meridians and Seifert-framed longitudes. These isomorphisms give us that the zeroth and the first relative homology groups with integer coefficients must vanish and since the inclusions are maps of manifolds over $\mathbb{Z}^\mu$, Lemma 4.31 gives us isomorphisms

$$H_1(X^0; \mathbb{C}^\omega) \xrightarrow{\cong} H_1(U_{L'}; \mathbb{C}^\omega) \xleftarrow{\cong} H_1(X^1; \mathbb{C}^\omega),$$

where $\omega = (1, \omega')$.

Since the concordance takes the meridian to the meridian, and the Seifert-framed longitude to the Seifert-framed longitude, that means that $\alpha m^0_1 + \beta\ell^0_1$ vanishes in $H_1(X^0; \mathbb{C}^\omega)$ if and only if $\alpha m^1_1 + \beta\ell^1_1$ does in $H_1(X^1; \mathbb{C}^\omega)$, which is precisely what we wanted to prove. $\square$

Now, we can finally consider the behavior of the signature and the nullity under concordance. For signature, we have the following theorem, the proof of which is a generalization of the proof of [10, Theorem 5.13]:

Theorem 4.35. *If two μ-component links L^0, L^1 are concordant, then for each $\omega = (1, \omega')$ with $\omega' \in \mathbb{T}^{\mu-1}_{*!}$, we have*

$$\sigma_{L^0}(\omega) = \sigma_{L^1}(\omega).$$

Proof. We start by considering the exterior of a link concordance between L^0 and L^1. This has as its boundary three pieces, glued along boundaries of link neighborhoods: $-X_{L^0}, X_{L^1}$ and $\partial\nu(D)$, the boundary of a tubular neighborhood of the concordance D. We denote this manifold by W_X. Note that the boundary ∂W_X is homeomorphic to the gluing $-X_{L^0} \cup X_{L^1}$ with the identification of boundaries of neighborhoods of links identifies meridians and preferred longitudes to each other. Also, note that since W_X is homotopy equivalent to the concordance exterior, we have $H_*(W_X, X_{L^i}; \mathbb{Z}) = 0$ by Proposition 2.27.

Now, choose bounding surfaces F^0, F^1 for L^0, L^1 respectively, both obtained by pushing a C-complex into B^4. Then, we can consider the manifolds W_{F^0} and W_{F^1} associated to these bounding surfaces as in Definition 4.19. Recall that the boundary of each W_{F^s} is the unions $X_{L^s} \cup -\mathrm{Pb}(L^s)$, glued along $\partial\nu(L^s)$.

We then form a manifold Υ by gluing W_{F^0} and $-W_{F^1}$ to W_X along the common parts of their boundaries, that is the link exteriors X_{L^s}. Now, we wish to compute the signature defect of Υ in two different ways. To that end, first we need the following proposition from [25]:

Proposition 4.36. Let L be a μ-component link and let X_L denote the link exterior. Then the kernel

$$\ker(H_1(\partial\nu(L); \mathbb{Z}) \to H_1(X_L; \mathbb{Z}))$$

is generated by expressions of the form

$$\ell_i - \sum \mathrm{lk}(L_i, L_j) m_j,$$

where m_i, ℓ_i denote respectively the meridian and zero-framed longitude of the i-th component.

First, we claim that the signature defect is additive with respect to the gluings defining Υ:

$$\operatorname{dsign}_\omega \Upsilon = \operatorname{dsign}_\omega W_{F^0} + \operatorname{dsign}_\omega(-W_{F^1}) + \operatorname{dsign}_\omega W_X.$$

To prove that, we simply prove the additivity of both twisted and untwisted signature for this gluing.
In the untwisted case, we need to apply the Novikov-Wall theorem twice: First, we have

$$\operatorname{sign}(W_{F^0} \cup W_X) = \operatorname{sign} W_{F^0} + \operatorname{sign} W_X + \mathcal{M}_0,$$

where $\mathcal{M}_0$ is the Maslov index associated to inclusions of $\partial\nu(L)$ into $\operatorname{Pb}(L^0), X_{L^0}, X_{L^1}$. Since by Lemma 4.36 the kernels of the second two inclusions depend only on the linking numbers, which are preserved by concordance, the Maslov index has to vanish by Corollary 2.18.
For the second gluing, we have

$$\operatorname{sign} \Upsilon = \operatorname{sign}(W_{F^0} \cup W_X) + \operatorname{sign} W_{F^1} + \mathcal{M}_1,$$

where $\mathcal{M}_1$ is the Maslov index associated to the inclusions of $\partial\nu(L)$ into $\operatorname{Pb}(L^0), X_{L^1}, -\operatorname{Pb}(L^1)$. However, since the spaces $\operatorname{Pb}(L^s)$ depend only on the linking numbers of L^s, the first and third space also only depend on linking numbers. and so again these spaces coincide and the Maslov index vanishes. Therefore, we get the additivity of untwisted signature.
For the twisted signature, we have that $H_1(\partial\nu(L^s;\mathbb{C}^\omega)) = H_1(\partial\nu(L_1^s;\mathbb{C}^\omega))$ which might not be a trivial space. It is non-trivial if and only if $\prod \omega_i^{\operatorname{lk}(L_1^s,L_i^s)} = 1$, in which case it is isomorphic to $\mathbb{C}^2$. Then, the kernel of the inclusion into X_{L^s} is given by the slope $(L_1^s/L^s)(\omega)$. We therefore get that for the first gluing the twisted signature is additive, since for $\omega \in \mathbb{T}_!^\mu$ the slope is preserved by concordance and two of the kernels coincide, forcing the Maslov index to vanish.
Similarly, for the second gluing, Proposition 4.24 gives us the kernels of the homomorphisms induced by inclusions of $\partial\nu(L)$ into $\operatorname{Pb}(L^0)$ and $-\operatorname{Pb}(L^1)$. These depend only on whether the linking numbers are zero or non-zero. Since concordances preserve linking numbers, these two kernels coincide and the Maslov index vanishes.
Now that we know that the signature defect is additive, we wish to identify $\operatorname{dsign}_\omega W_{F^s}$ with the signature of L^s. Since $\operatorname{dsign}_\omega(-W) = -\operatorname{dsign}_\omega(W)$ we get

$$\operatorname{dsign}_\omega \Upsilon = \sigma_{L^0}(\omega) - \sigma_{L^1}(\omega) + \operatorname{dsign}_\omega W_X.$$

Finally, we want to show that the signature defects of both Υ and W_X vanish:
Since we took W_X to be a homology cobordism, Proposition 4.33 yields $\operatorname{dsign}_\omega W_X = 0$.
The boundary of $\Upsilon = \operatorname{Pb}(L^0) \cup -\operatorname{Pb}(L^1)$ can be described as a plumbed manifold in its own right. The vertices of the graph defining it correspond to components of L^s and the set of edges is the union of the sets of edges in the graphs associated to L^0, L^1 (under the identification of link components induced by the concordance.). To each vertex we assign $S^2 = D^2 \cup D^2$.
Now, for any i, j the set of edges between v_i, v_j consists of $|\operatorname{lk}(L_i, L_j)|$ edges with positive sign and the same number of edges with negative sign, coming from the other plumbed manifold. Therefore, $\partial\Upsilon$ has the structure of a *balanced* plumbed manifold. We get by Lemma 4.11 that $\partial\Upsilon$ bounds a 4-manifold with a vanishing signature defect, as each vertex is assigned a sphere and thus the collection of curves L_{F_i} is empty. Now, by corollary 2.34 this implies that every manifold bounding it has signature defect equal to zero, and in particular $\operatorname{dsign}_\omega \Upsilon = 0$.
Finally, putting this all together gives us

$$0 = \sigma_{L^0}(\omega) - \sigma_{L^1}(\omega) + 0,$$

which was precisely what we wanted to show. □

Theorem 4.37. *If two μ-component links L^0, L^1 are concordant, then for each $\omega = (1, \omega')$ with $\omega' \in \mathbb{T}_{*!}^{\mu-1}$, we have*

$$\eta_{L^0}(\omega) = \eta_{L^1}(\omega).$$

Proof. Since $\eta_{L^s} = \dim H_1(\partial W_{F^s}; \mathbb{C}^\omega)$ by Lemma 4.17, we wish to look at the decompositions

$$\partial W_{F^s} = X_{L^s} \cup -\operatorname{Pb}(L^s)$$

and the associated Mayer-Vietoris sequences. We obtain then an exact sequence:

$$0 \to \ker(\iota_*) \to H_1(\partial\nu(L); \mathbb{C}^\omega) \xrightarrow{\iota_*} H_1(X_{L^s}; \mathbb{C}^\omega) \oplus H_1(-\operatorname{Pb}(L^s); \mathbb{C}^\omega) \to H_1(\partial W_{F^s}; \mathbb{C}^\omega) \to$$
$$\to H_0(\partial\nu(L); \mathbb{C}^\omega) \to 0 \oplus H_0(-\operatorname{Pb}(L^s); \mathbb{C}^\omega) \to 0,$$

where the last two zeroes are a consequence of Proposition 2.5. From Proposition 4.23 we get that the dimension of $H_*(-\operatorname{Pb}(L^s); \mathbb{C}^\omega)$ depend only on the linking numbers of L^s, and so coincide for $s = 0, 1$. By Proposition 4.24 we get that the kernel of $H_1(\partial\nu(L); \mathbb{C}^\omega) \to H_1(-\operatorname{Pb}(L^s); \mathbb{C}^\omega)$ also depends only on the linking numbers. The kernel of of the homomorphism to $H_1(X_{L^s}; \mathbb{C}^\omega)$ is determined by the slope of L_1^s in L^1 and so by Lemma 4.34 these kernels also coincide for $s = 0, 1$. Therefore, the kernel of ι_* also has the same dimension for both links. Finally, the dimension of $H_1(X_{L^s}; \mathbb{C}^\omega)$ has been calculated in the proof of Theorem 4.25 and depends only on the nullity of $L^s \setminus L_1^s$, which is invariant for $\omega' \in \mathbb{T}_{*!}^{\mu-1}$, and the linking numbers. Since we have that the dimensions of all the other spaces in this exact sequence coincide for $s = 0, 1$ then by taking the Euler characteristics we see that the dimensions of $H_1(\partial W_{F^s}; \mathbb{C}^\omega)$ also have to be the same, and so the nullities coincide. □

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