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Statistics of return times and Hausdorff dimension

PhD dissertation

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Author's declaration:
aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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the dissertation is ready to be reviewed.

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Abstract

The thesis contains several results concerning the quantitive aspects of Poincaré recurrence.

In particular, bounds on the limit $\liminf_{n\to+\infty} n^\beta d(T^n(x), x)$ (and similar expressions) are obtained in different settings, for dynamical systems both deterministic and random.

The exponential distrubution of return/entry times is proved, again in the deterministic and random situation.

The recurrence within a space is linked to the Hausdorff dimension of the said space and we show how this may be used to estimate the dimension.

Additionally, using a more specific method (though similar in gist), the dimension of certain indecomposable continua occuring naturally in the dynamics of $\lambda \exp(z)$ is calculated.

Streszczenie

Rozprawa zawiera kilka wyników dotyczących jakościowych aspektów lematu Poincaré o powracaniu.

W szczególności, wskazujemy szacowania granicy $\liminf_{n\to+\infty} n^\beta d(T^n(x), x)$ (i podobnych wyrażeń) przy różnych założeniach, dla układów dynamicznych zarówno deterministycznych jak i losowych.

Udowodniony jest także rozkład wykładniczy czasów powrotu/wejścia, ponownie w sytuacjach deterministycznej i losowej.

Tempo powracania w danej przestrzeni zostaje powiązane z wymiarem Hausdorffa tejże przestrzeni oraz pokazujemy jak można użyć tej obserwacji do szacowania wymiaru.

Dodatkowo, używając bardziej precyzyjnej metody (choć podobnej co do idei), wyznaczamy wymiar pewnych kontinuów nierokładalnych pojawiających się naturalnie w dynamice holomorficznej dla funkcji $\lambda \exp(z)$.

Keywords

All Chapters:	Poincaré recurrence, dimension theory, Hausdorff dimension
Chapters 3–6:	measure-preserving transformations, ergodicity, decay of correlations
Chapters 5, 6:	random maps, statistical properties
Chapter 7:	holomorphic dynamics, indecomposable continua

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Chapters 5, 6:	37H99
Chapter 7:	37F35, 37B45

Contents

Chapter 1

Introduction

1.1 Basic definitions and facts

One of the basic results in the theory of dynamical systems is the Poincaré recurrence theorem:

Theorem 1.1. *Let $(X, \mathcal{F}, \mu, T)$ be a measure preserving dynamical system. Then for any measurable set A*

$$\mu(A) = \mu \{x \in A : T^n(x) \in A \text{ for infinitely many } n\}, \tag{1.1.1}$$

which is perhaps better stated in plain English: almost any point returns to its starting set infinitely many times.

This result may be restated and/or improved in many ways. We may look at how often this *returning* occurs, leading to the ergodic theorem. If, instead of one set A, one takes a decreasing, nested family of sets A_n, then we can try to find out about the behaviour of those *returns* in the limit $n \to +\infty$.

Most of this thesis is devoted to metrical spaces, let us define:

Definition 1.2. A *metrical measure preserving dynamical system* is a quintuplet $(X, \mathcal{F}, \mu, d, T)$, where $(X, \mathcal{F}, \mu)$ is a measurable space, T preserves the measure μ, d is a metric on X and the σ-field $\mathcal{F}$ contains all Borel sets.

Remark. We shall call this simply a *metric measure system.*

In such spaces one can easily prove a restatement of Poincaré's result [Fur81, p. 61]:

Theorem 1.3. *Let $(X, \mathcal{F}, \mu, d, T)$ be a metric measure system and assume that (X, d) is separable. Then μ–almost every point $x \in X$ is recurrent, which means that*

$$\liminf_{n\to\infty} d(x, T^n(x)) = 0. \tag{1.1.2}$$

One of the pioneering papers concerning this topic [Bos93] improves this result, e.g. showing that

$$\liminf_{n\to\infty} n^{\beta} d(T^n(x), x) < +\infty, \tag{1.1.3}$$

for $\beta = \frac{1}{\alpha}$, whenever the Hausdorff α–measure is σ–finite on X.

To state other results we need to define *the entrance time* (also *hitting time*) into a measurable set U:

$$\tau(U, x) = \tau_U(x) = \inf\{k \geq 1 \colon T^k(x) \in U\}.$$

Whenever $x \in U$ this is usually called *the return time* (often *first return time*).

Note. If $x \in U$ and $\mu(U) > 0$ then $\tau_U(x) < \infty$ almost everywhere because of Poincaré recurrence theorem. For general x the same is true, if the system is ergodic.

Let us observe some straightforward properties of $\tau_U(x)$:

A) For any fixed x this function is monotone non–decreasing as U decreases, i.e. $\tau_U \geq \tau_V$ if $U \subset V$.

B) It takes only integer values and so has jump–type discontinuities as U decreases.

C) If the measure is preserved by T, then there is a clear correspondence between *entry* and *return* times:

$$\mu(\{x \colon \tau_U(x) = n\}) = \mu(\{x \in U \colon \tau_U(x) \geq n\}). \tag{1.1.4}$$

Proof. $\{\tau_U(x) > n\} = \{T(x) \notin U \wedge \tau_U(T(x)) > n-1\} = T^{-1}(U' \cap \{\tau_U(x) > n-1\}$. Invariance of μ gives $\mu(\{\tau_U > n\}) = \mu(\{\tau_U > n-1\}) - \mu(U \cap \{\tau_U > n-1\})$ thus obtaining the result. $\square$

D) If the measure is ergodic we have a result usually known as the Kac's lemma [Kac47]:

Theorem 1.4. *If the dynamical system (X, T, μ) is measure preserving and ergodic, then for any measurable set U*

$$\int_U \tau_U(x)\, d\mu(x) = 1.$$

Using the probability theory notions this could be stated as

$$\mathbb{E}(\mu(U)\tau_U) = \sum_{k=1}^{+\infty} k\mu(U \cap \{\tau_U = k\}) = 1,$$

where the expectation is computed with respect to the induced (conditional) probability measure on U, i.e. $\mu_U(\cdot) = \dfrac{\mu(\cdot \cap U)}{\mu(U)}$.

It is natural to ask what statistical properties this normalized variable $\mu(U)\tau_U$ has. For example in [Cha03] the author finds conditions for the existence of moments of higher order, depending on the mixing properties of the system.

A different approach, currently rapidly developing is to describe asymptotics for hitting (or return) times. If instead of one set U, we take a family of sets U_n (usually $\mu(U_n) \to 0$), we may ask questions about the resulting sequence of variables and their limit.

Weak limits have been shown to exist for suitably chosen U_n's (decreasing cylinders or balls). In many classes of mixing systems the limiting distribution has been shown to be exponential (normalization leads to parameter equal 1), check [You99], [HSV99] and [CG93]. Nonexponential asymptotics may be found e.g. in [CdF90]. For a survey of results one may read [Aba04], for instance.

Other interesting results are in the study of all possible asymptotics for return [Lac02] and hitting [LK05] times. Additionally, if one of those limiting (as $\mu(U_n) \to 0$) distributions (return or hitting time) is exponential, then so is the other [HSV99].

Again we may look at how the family of variables $\tau_{U_n}\mu(U_n)$ behave/deviate from the limiting distribution by looking at $\liminf \tau_{U_n}\mu(U_n)$. This is strictly related to (1.1.3) — cf. section 3.2.

Note that this time is makes sense to also ask about $\limsup \tau_{U_n}\mu(U_n)$. However, the author cannot prove any substantial results about *limsup* at this time; so this will not be discussed further.

In a metric space, taking as the sets U_n a family of concentric balls $B(x,r)$, we may study yet another expression:

$$\frac{\log \tau_{B(x,r)}(x)}{-\log(r)}. \tag{1.1.5}$$

Obviously, it is closely connected with (1.1.3). The authors of [BS01] prove that

$$\liminf_{r\to 0} \frac{\log \tau_{B(x,r)}(x)}{-\log(r)} \leq \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log(r)}, \tag{1.1.6}$$

and that the same is true taking $\limsup$ instead of $\liminf$.
Additionally, they show that — under additional assumptions — both lower (resp. upper) limits are equal.

Note. Expression on the right hand side is the (lower / upper) pointwise dimension of measure μ (cf. Def. 2.3).

It is easy to see that the examining the limit $\tau_{B(x,r)}(x) \cdot \mu(B(x,r))$ is more precise than the expression (1.1.5).

1.2 Thesis overview

The first chapter contains some historical information and motivations for pursuing the subject as well as basic problems and areas of interest within the field of study. It also introduces some basic notations and definitions.

The next chapter is dedicated to measures and their properties. It does not concern the recurrence as such, but introduces some features of measures that will be needed throughout the thesis. All results concerning measures have been separated into this part.

The third chapter contains several smaller results. It starts with a basic theorem, which can then be applied to many systems using the inducing map technique. Afterwards, we learn how some of the results may be rewritten to get other expressions (appearing widely in the literature). Then we get a result about recurrence with almost no assumptions (apart from ergodicity obviously). We finish by showing how those theorems may be applied to estimate the Hausdorff dimension and measure of a set. This idea (but not straightforward application) will return in the last chapter.

Chapter four shows that assuming the exponential decay of correlations in a dynamical system gets us some strong estimations on the rate of recurrence — it is as fast as possible. We introduce some systems, for which the theorems are applicable; and the proofs fill the rest of this chapter.

We introduce random dynamical systems in chapter five. Firstly, we lay out some theory and using results from ch. 3 arrive at a simple, but interesting result. Then we show two model settings and in these situations state and prove the equivalents of results from chapter 4.

In chapter six we turn our attention to the distribution of return times. We start by proving that this distribution is in fact exponential for a group of deterministic systems using a technique introduced in [HSV99]. Afterwards, we proceed to significantly alter this method in order to prove the same results for the random mappings.

The last chapter is somewhat different than the others. We introduce a family of indecomposable continua in a complex plane. Then, instead of observing recurrence (or proving results about the phenomena), we use our good knowledge of how points return to themselves to gets some bounds on the Hausdorff dimension of those continua. Even though we do not state results about recurrence directly, the recurring is a keystone of the proof (at least in the non-trivial cases).

1.3 Often used notions

This section gives some basic definitions and notations, that will be used throughout the entire thesis.

Unless explicitly stated otherwise, the space X is always assumed to be a metric space and the measure μ is a Borel, probability measure. All the maps T considered preserve the measure μ and are ergodic.

We will use all the following notations as a ball of radius r around x:

$$B(x,r) = B_r(x) = B_x(r).$$

Mostly we will write the second one, but sometimes to emphasize which argument is fixed and which is changing, we will need the others. Sometimes, if we work with a fixed point x, we will write simply B_r.

We shall be mostly working with entrance times to balls and then for brevity denote

$$\tau_r^y(x) = \tau_{B(y,r)}(x).$$

If we work with return times and it should not cause confusion we will write $\tau_r(x) = \tau_r^x(x)$.

Another notation we will use (though rarely) is for *the earliest return of a set*, i.e.

$$\tau(U) = \inf_{x \in U} \{\tau_U(x)\}.$$

The α–Hausdorff measure of a set A will be written as $H_\alpha(A)$ and similarly the α–packing measure as $\Pi_\alpha(A)$.

The Hausdorff dimension of A will be denoted as $\dim_H(A)$ and accordingly $\dim_P(A)$ will be the packing dimension.

Chapter 2

Measures

In this chapter we introduce a few behavioural traits that we will expect our measures to observe. Then we will find some relations between those notions. All properties of measure needed and proved within the thesis have been separated into this chapter.
Recall that X is a metric space and μ is a Borel, probability measure.

2.1 Definitions

This property of measure will be needed in many theorems so let us define it here.

Definition 2.1. We shall say that a measure has *doubling property at x* if there exist $\sigma(x) < +\infty$ and $\rho(x) > 0$ such that $\mu(B(x, 2r)) \leq \sigma(x)\mu(B(x, r))$ for all $r \leq \rho(x)$. We shall say that the measure has *doubling property almost everywhere* if it has doubling property on a set of full measure and the functions $\sigma(x)$ and $\rho(x)$ are measurable.

This is a weaker version of the well–known property: the measure ν is called *doubling* if there exists $C > 0$ such that

$$\nu(B(x, 2r)) \leq C\nu(B(x, r)) \qquad \text{for all } x \in X \text{ and } r > 0. \tag{2.1.1}$$

Remark. In this case we have an interesting result [LS98]:
A complete metric space X carries a nontrivial doubling measure iff X is a doubling space, i.e there exist M such that any ball $B(x, r)$ may be covered by at most M balls of radii $\frac{r}{2}$.

If we had weakened this property slightly more, then it would be satisfied for all Borel measures. Precisely speaking, this result comes from [BS01]:

Proposition 2.2. *Any Borel probability measure on $\mathbb{R}^n$ is* weakly diametrically regular*, i.e. for μ–almost every $x \in \mathbb{R}^n$ and every $\varepsilon > 0$ there exists δ such that for all $r < \delta$*

$$\mu(B(x, 2r)) \leq \mu(B(x, r))r^{-\varepsilon}. \tag{2.1.2}$$

Another quantity that will be used often:

Definition 2.3. For a measure μ define the *lower* $\underline{d}_\mu$ *and upper* $\overline{d}_\mu$ *pointwise dimension* of the measure μ

$$\underline{d}_\mu(z) = \liminf_{r\to 0} \frac{\ln\big(\mu(B_r(z))\big)}{\ln r}, \qquad \overline{d}_\mu(z) = \limsup_{r\to 0} \frac{\ln\big(\mu(B_r(z))\big)}{\ln r}.$$

These functions are closely related to other measure–dimensions. Define *the Hausdorff dimension* of a probability measure μ as

$$\dim_H \mu = \inf\{\dim_H Z \colon \mu(Z) = 1\}, \tag{2.1.3}$$

where $\dim_H Z$ denotes the Hausdorff dimension of the set Z.

In a similar way we may also define $\dim_P \mu$ — a *packing dimension* of measure. The relation between these definitions is given by the following result (e.g. [PU10, Thm. 8.6.5]).

Proposition 2.4. *If μ is a probability measure in the Euclidean space, then*

$$\dim_H \mu = \operatorname{ess\,sup} \underline{d}_\mu$$
$$\dim_P \mu = \operatorname{ess\,sup} \overline{d}_\mu.$$

Definition 2.5. The measure μ is called *exact–dimensional* if

$$\underline{d}_\mu = \overline{d}_\mu \qquad \mu\text{–a.e.} \tag{2.1.4}$$

Remark. A lot of measures have this property, e.g. hyperbolic Bowen–Ruelle–Sinai measures [Led87] and hyperbolic measures invariant under a $\mathcal{C}^{1+\alpha}$ diffeomorphism of a smooth compact surface [You82]. The Eckmann–Ruelle conjecture states that a general hyperbolic measure is exact dimensional — this has been confirmed in [BPS99].

Another interesting property of such measures is:

Proposition 2.6. *If μ is exact–dimensional in $\mathbb{R}^n$, then*

$$\dim_H \mu = \dim_P \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu, \tag{2.1.5}$$

where $\dim_B \mu$ are the so-called box dimensions *(upper and lower) of measure μ.*

To introduce the next property we need one technical definition.

Definition 2.7. A function $l \colon \mathbb{R}_+ \to \mathbb{R}_+$ will be called *subpoly* if

$$\lim_{r\to 0} l(r) r^\varepsilon = 0 \tag{2.1.6}$$

for every $\varepsilon > 0$.

Note. Basic example of *subpoly* function is $l(r) = -\ln(r)$.

And our property is defined as follows:

Definition 2.8. A measure is said to have a *thin annuli property* if for μ–almost every x there exists a *subpoly* function $\kappa_x(r) > 0$ such that

$$\lim_{r\to 0} \frac{\mu\left(B(x, r + r^{\kappa_x(r)}) \setminus B(x,r)\right)}{\mu\left(B(x,r)\right)} = 0. \qquad \mu\text{–a.e.}$$

Example 2.9. Any geometric measure (cf. sec. 2.2) trivially has *thin annuli property* for any fixed $\kappa > 1$.

2.2 Measure vs. packing dimension

Definition 2.10. We shall say that μ has the upper β–property, if there exist measurable functions $D(x)$ and $R(x)$ both positive μ–a.e. such that for μ–almost every $x \in X$

$$\forall_{r<R(x)}\ \mu\left(B(x,r)\right) \geq D(x) \cdot r^{\beta}. \tag{2.2.1}$$

Note. This is slightly stronger than saying that the upper pointwise dimension $\overline{d}_{\mu}(z) \leq \beta$ almost everywhere.

And the reverse inequality has a corresponding name:

Definition 2.11. We shall say that μ has the lower β–property, if there exist measurable functions $D(x)$ and $R(x)$ both positive μ–a.e. such that for μ–almost every $x \in X$

$$\forall_{r<R(x)}\ \mu\left(B(x,r)\right) \leq C(x) \cdot r^{\beta}. \tag{2.2.2}$$

Note. This is stronger than $\underline{d}_{\mu}(z) \geq \beta$ almost everywhere.

If the measure satisfies both those conditions with the same β, then it is usually called a *geometric measure.*

In the Euclidean space $\mathbb{R}^n$ the first property has a nice characterization.

Lemma 2.12. *If X is a Borel bounded subset of $\mathbb{R}^n$, then μ (Borel, probability measure on X) has the upper β–property iff there exists a set A of full measure such that the packing measure Π_{β} is σ–finite on A.*
In particular, if Π_{β} is σ–finite on X, then the upper β–property holds and if the measure μ has the upper β–property then the packing dimension $\mathrm{PD}(\mu) \leq \beta$.

Proof. We shall use a volume lemma (in literature often called Frostman–type lemma), namely Theorem 8.6.2. from [PU10], which states that if A is a bounded subset of $\mathbb{R}^n$ and $0 < D < +\infty$ then:

a) If for all $x \in A$
$$\liminf_{r\to 0} \frac{\mu\left(B(x,r)\right)}{r^{\beta}} \leq D$$
then $\mu(E) \leq b(n) D \Pi_{\beta}(E)$ for every Borel subset $E \subset A$ ($b(n)$ is a constant depending only on the dimension).

b) If for all $x \in A$
$$\liminf_{r\to 0} \frac{\mu\left(B(x,r)\right)}{r^{\beta}} \geq D$$
then $\mu(E) \geq D \Pi_{\beta}(E)$ for every Borel subset $E \subset A$.

First, assume the β–property holds on the set A_β (of full measure) and let us construct the set A of σ–finite Π_β measure.

Fix $\lambda > 0$, there exists a set $A_\lambda \subset A_\beta$ of measure $\mu(A_\lambda) \geq 1-\lambda$ and a constant $D_\lambda > 0$ such that $D(x) \geq D_\lambda$ for all $x \in A_\lambda$. Now take a Borel subset $E_\lambda \subset A_\lambda$ of measure $\mu(E_\lambda) = \mu(A_\lambda)$ (μ is a Borel regular measure). Part (b) of the above theorem tells us that $\Pi_\beta(E_\lambda) \leq D^{-1}\mu(E_\lambda) < +\infty$. Take $A = \bigcup_{k=1}^{+\infty} E_{1/k}$. Observing that $\Pi_\beta(E_{1/k}) < +\infty$ for all k and $\mu(A) = 1$ ends this part of the proof.

Assume that the β–property does not hold. This means that there is a set H, $\mu(H) > 0$ such that for every $x \in H$

$$\liminf_{r\to 0} \frac{\mu\left(B(x,r)\right)}{r^\beta} = 0.$$

A is σ–finite with respect to Π_β, so $A = \bigcup_{n=1}^{+\infty} A_n$, where $\Pi_\beta(A_n) < +\infty$. Look at sets $H_n = H \cap A_n$. The measure $\mu(H) > 0$, so there has to exist n such that $\mu(H \cap A_n) > 0$. Take and fix a Borel subset $E \subset H_n$ of positive μ measure; for any $D > 0$ part (a) of the volume lemma shows that $\Pi_\beta(E) \geq \big(Db(n)\big)^{-1}\mu(E)$ and so $\Pi_\beta(E) = +\infty$, which is a contradiction. $\square$

2.3 Three properties

A natural question is: are there any relations between the three notions: doubling property, thin annuli property and pointwise dimensions.

A lot of measures appearing in dynamical systems have positive and finite pointwise dimensions and also the other two properties. Unfortunately for general measures we may give only two interesting results.

Theorem 2.13. *If a Borel measure μ on a metric space X has doubling property at $x \in X$, then $\overline{d}_\mu(x) < \infty$.*

The proof is given later in this section.

It is easy to construct a measure that shows the reverse implication is not true. Unfortunately, a short form of such a measure is not pretty.

Example 2.14. Fix a point x and put $B_r = B_r(x)$. We may set $\mu(B_r) = r^{1+p(r)}$, where $p(r) = \dfrac{v(r)}{\sqrt{-\ln(r)}}$ and for v we need a quickly changing function; e.g. $v(r) = \sin(\pi \log_2 r)$ or

$$v(r) = \begin{cases} 1 & \text{for} \quad 2^{-2n} \leq r < 2^{-2n+1} \\ -1 & \text{for} \quad 2^{-2n-1} \leq r < 2^{-2n} \end{cases}$$

(the function is just taken in such a way that $v(r)$ and $v(r/2)$ have different signs).

Also $p(r)$ goes to 0 (as $r \to 0$) — this means that $\underline{d}_\mu(x) = \overline{d}_\mu(x) = 1$. On the other hand,

$$\frac{\mu(B_r)}{\mu(B_{r/2})} = r^{p(r)-p(r/2)} 2^{1+p(r/2)} = \exp\left(\left(\frac{v(r)}{\sqrt{-\ln(r)}} - \frac{v(r/2)}{\sqrt{-\ln(r/2)}}\right)\ln(r)\right) 2^{1+p(r/2)}$$

and if we take r such that $v(r) < 0$ then the exponent $\approx \sqrt{-\ln(r/2)}$ and we get $\limsup_{r\to 0} \dfrac{\mu(B_r)}{\mu(B_{r/2})} = +\infty$, which means that we do not have the doubling property.

A measure may have positive and finite pointwise dimensions at x, but not the thin annuli property — e.g. if there is a countable family of concentric circles $S(x, r_n)$ of positive measure; the dimensions may behave 'correctly', but the thin annuli fails.

Example 2.15. Let us say that the measure around a fixed point x is concentrated only on circles $S(x, 2^{-n})$ and $\mu\left(S(x, 2^{-n})\right) = 2^{-n}$. Then the limit in thin annuli property equals 1 for $r = 2^{-n}$ for any $\kappa > 0$. (On other radii, however, it is equal to 0 with accordance to Thm. 2.17). The dimensions satisfy $\underline{d}_\mu(x) = \overline{d}_\mu(x) = 1$.

Example 2.16. If $\mu(B_r(x)) = \frac{1}{-\ln(r)}$, then the measure has thin annuli property, but $\underline{d}_\mu(x) = \overline{d}_\mu(x) = 0$.

Our second positive result shows that a measure has thin annuli property for most r's.

Theorem 2.17. *If μ is a Borel measure on $X = \mathbb{R}^d$, then for μ–a.e. $x \in X$, any $\kappa > 1$ and any $A > 0$ the set of r for which*

$$\frac{\mu\left(B(x, r + r^\kappa) \setminus B(x, r)\right)}{\mu\left(B(x, r)\right)} > A \tag{2.3.1}$$

has zero density at point $r = 0$.
In other words, if we denote the set of points that do not satisfy the thin annuli property with exponent κ and constant A (the condition above) as Z, then

$$\lim_{u\to 0} \frac{l(Z \cap [0, u])}{l([0, u])} = 0, \text{ where } l \text{ is the Lebesgue measure.} \tag{2.3.2}$$

Moreover, $l(Z \cap [0, u]) = O(-u^\kappa \ln(u))$.

The proof is given later in this section. Putting a *subpoly* function $\kappa(r) = -\ln(r)$ we get a more legible result.

Corollary 2.18. *If μ is a Borel measure on $X = \mathbb{R}^d$, then for μ–a.e. $x \in X$ and most sequences of radii (r_n) decreasing to 0:*

$$\lim_{n\to+\infty} \frac{\mu\left(B(x, r_n + r_n^{-\ln(r_n)}) \setminus B(x, r_n)\right)}{\mu\left(B(x, r_n)\right)} = 0,$$

where "most sequences" mean that $r_n \notin Z$ and $l(Z \cap [0, u]) = O(-u^{-ln(u)} \ln(u))$.
This trivially implies that the lower limit *(in the definition of* thin annuli*) equals 0.*

Proof of Thm. 2.13. Fix a point x. Let us denote $f(r) = \mu(B(x,r))$.
Observe that f is non-decreasing. Now the doubling property at x states that

$$f(r) \leq C \cdot f\left(\frac{r}{2}\right),$$

for a certain $1 < C < \infty$ (dependent on x). Combining this k times and setting $r = 1$ we arrive at

$$f\left(\frac{1}{2^k}\right) \geq \frac{f(1)}{C^k}. \tag{2.3.3}$$

Let us now take any radius $s > 0$. There is a unique n for which $\frac{1}{2^{n+1}} \leq s < \frac{1}{2^n}$. Using this, monotonicity of f and (2.3.3) we get the following estimates

$$f(s) \geq f\left(\frac{1}{2^{n+1}}\right) \geq \frac{f(1)}{C^{n+1}} = \frac{f(1)}{C \cdot (2^n)^{\log_2 C}} > \frac{f(1)}{C} s^{\log_2 C}. \tag{2.3.4}$$

This means that the measure has upper β–property for $\beta = \log_2 C$ ($C > 1$ so $\beta > 0$) and $\overline{d}_\mu(x) \leq \log_2 C$. $\square$

Proof of Thm. 2.17. By Prop. 2.2 for any $\varepsilon > 0$ and all sufficiently small r we have

$$\mu\big(B(x,2r)\big) \leq \mu\big(B(x,r)\big) r^{-\varepsilon}. \tag{2.3.5}$$

Fix $r_0 > 0$ satisfying the above and let us say we have a sequence of n points r_n satisfying (2.3.1), i.e. radii not satisfying the thin annuli, such that

$$r_0 \leq r_1 \leq r_1 + r_1^\kappa < r_2 \leq r_2 + r_2^\kappa < r_3 \leq \cdots \leq r_n + r_n^\kappa \leq 2r_0. \tag{2.3.6}$$

This means that the annuli defined by radii r_n do not intersect. By (2.3.1), for any $1 \leq p \leq n$

$$\frac{\mu\left(B(x, r_p + r_p^\kappa)\right)}{\mu\left(B(x, r_p)\right)} > 1 + A. \tag{2.3.7}$$

Using this estimate n times we arrive at

$$\begin{aligned} \mu\left(B(x,r_0)\right) \leq \mu\left(B(x,r_1)\right) \leq \frac{\mu\left(B(x, r_1 + r_1^\kappa)\right)}{1+A} \leq \frac{\mu\left(B(x,r_2)\right)}{1+A} \leq \cdots \\ \cdots \leq \frac{\mu\left(B(x,r_n)\right)}{(1+A)^n} \leq \frac{\mu\left(B(x,2r_0)\right)}{(1+A)^n} \end{aligned} \tag{2.3.8}$$

and applying (2.3.5) yields

$$\leq \frac{\mu\left(B(x,r_0)\right) r_0^{-\varepsilon}}{(1+A)^n}.$$

This shows that

$$r_0^\varepsilon \leq (1+A)^{-n}, \text{ giving the estimate:} \qquad n \leq \frac{-\varepsilon \ln(r_0)}{\ln(1+A)}. \tag{2.3.9}$$

Now let us divide the set of radii $[r_0, 2r_0]$ into intervals of length $(2r_0)^\kappa$, i.e. define:

$$I_1 = [r_0, r_0 + (2r_0)^\kappa) \ \dots \ I_n = [r_0 + (n-1)(2r_0)^\kappa, r_0 + n(2r_0)^\kappa) \ \dots$$

for n until $(n_{\max}+1)(2r_0)^\kappa \geq r_0$.

Observe that if we take a point $s \in I_p$, then $s + s^\kappa \in I_p \cup I_{p+1}$. This means that the annulus defined by any radius r_l intersects at most 2 intervals. So at most $\frac{-2\varepsilon \ln(r_0)}{\ln(1+A)}$ intervals (of radii) may contain a point (radius) that satisfies (2.3.1).

The total length of those intervals is bounded by (Z denotes the set of those points)

$$l(Z \cap [r_0, 2r_0]) \leq (2r_0)^\kappa \cdot \frac{-2\varepsilon \ln(r_0)}{\ln(1+A)} = C\left(-r_0^\kappa \ln(r_0)\right). \tag{2.3.10}$$

We end our proof by summing estimates:

$$l(Z \cap [0, r_0]) \leq \sum_{n=1}^{\infty} l\left(Z \cap \left[\frac{r_0}{2^n}, \frac{r_0}{2^{n-1}}\right]\right) \leq \sum_{n=1}^{\infty} C\left(-\left(\frac{r_0}{2^n}\right)^\kappa \ln\left(\frac{r_0}{2^n}\right)\right) = O\left(-r_0^\kappa \ln(r_0)\right). \tag{2.3.11}$$

Obviously, this shows that

$$\lim_{u\to 0} \frac{l(Z \cap [0,u])}{l([0,u])} = \lim_{u \to 0} \frac{O\left(-u^\kappa \ln(u)\right)}{u} = 0 \qquad \text{for } \kappa > 1. \tag{2.3.12}$$

$\square$

Chapter 3

Lower limit

This chapter is devoted to preliminary results concerning the expression (and its variants):

$$\liminf_{n\to\infty} n^{\beta} d(T^n(x), x).$$

Firstly, we show that the limit is bounded, and also see what happens, if we change T into a first return mapping $\widehat{T}$. Secondly, we rewrite the limit into other forms. Later, we prove some recurrence results for sets more general than balls. Finally, we show how the recurrence rates may be used to estimate the Hausdorff dimension.

Some of the results are interesting in themselves (like the first subsection); some will be applied in later chapters (e.g. section 3.3) and some hint on a technique that is either useful as such or will be applied (in a different form) later (as in subsection 3.1.2).

3.1 Estimating by density

3.1.1 First result

In [Bos93] the following theorem is stated and proved in the case of $\mu = H_\alpha$, i.e. for $g \equiv 1$. The proof is easily adaptable and has been done in my Master's thesis. For completeness it is given at the end of this chapter in sec. 3.5.

Theorem 3.1. *Let $(X, \mathcal{F}, \mu, d, T)$ be a metrical measure preserving dynamical system. In addition suppose that $\mu \approx H_\alpha$ for some $\alpha > 0$ and that $g := \frac{dH_\alpha}{d\mu}$ is bounded from above. Then for μ – almost every $x \in X$ we have*

$$\liminf_{n\to\infty} n^{1/\alpha} d(T^n(x), x) \leq (\operatorname{ess\,sup} g)^{1/\alpha}. \tag{3.1.1}$$

Remark. Since the measures μ and H_α are equivalent, then the inequality takes place also for H_α– almost every $x \in X$. Note also that g is the inverse of the usually taken density.

Now we shall 'localize' this theorem obtaining:

Corollary 3.2. *With the assumptions as above, additionally assuming μ is ergodic with respect to T we have for μ – almost every $x \in X$*

$$\liminf_{n\to\infty} n^{1/\alpha} d(T^n(x), x) \leq g(x)^{1/\alpha}. \tag{3.1.2}$$

Remark 1. The density g is defined only almost everywhere, so $g(x)$ really means

$$g(x) = \lim_{r\to 0} (\operatorname*{ess\,sup}_{B(x,r)} g).$$

Remark 2. This result for $X = [0,1]$ ($\alpha = 1$), has been proved in [Cho02]. The proof, however, resembles the technique used in subsection 3.2.1 and works only in a 1-dimensional space.

Actually, we may drop some assumptions and arrive at

Proposition 3.3. *Let $(X, \mathcal{F}, \mu, d, T)$ be a metrical measure preserving dynamical system and let the measure be ergodic. Then for μ – almost every $x \in X$ we have*

$$\liminf_{n\to\infty} n^{1/\alpha} d(T^n(x), x) \leq g(x)^{1/\alpha}, \textit{ where } g(x) = \limsup_{r\to 0} \tfrac{H_\alpha(B_x(r))}{\mu(B_x(r))}. \tag{3.1.3}$$

Proof. If $\mu \perp H_\alpha$, then the limit is zero by a result from [Bos93, Thm 1.2] (which states that the recurrence limit vanishes if $H_\alpha(X) = 0$). If $g(x) = \infty$, then the limit is trivially true. Finally the remaining case is dealt with by Cor. 3.2. □

Many systems satisfy the assumptions of Cor. 3.2 as basically all that is needed is the equivalence of the invariant measure to some Hausdorff measure H_β. For example:

1. The Gauss transformation on the unit interval $f(x) = \dfrac{1}{x} - \left[\dfrac{1}{x}\right]$ with the invariant measure $d\mu(x) = \dfrac{1}{\ln(2)} \dfrac{1}{1+x}$.
2. The logistic transformation on the unit interval $g(x) = 4x(1-x)$ with the invariant measure $d\nu(x) = \dfrac{1}{\pi\sqrt{x(1-x)}}$. Observe that the density is not bounded, but is separated from 0 and this is exactly what is needed in the assumptions.

Note. Those systems are poor examples for use of Cor. 3.2, because Prop 4.6 shows that the observed limit is actually equal to 0. Better application is in section 3.4.

Proof of Corollary 3.2. Fix x and $r > 0$ and consider $S(y)$ — the first return function to the set $B(x,r)$. It is easy to see that S preserves $\mu|_{B(x,r)}$. Put

$$\nu = \mu|_{B(x,r)} \cdot \frac{1}{\mu(B(x,r))}. \tag{3.1.4}$$

That means that ν is a probabilistic measure on $B(x,r)$ preserved by S. Also the new density fulfills the equation:

$$h = \frac{dH_\alpha}{d\nu} = g|_{B(x,r)} \cdot \mu(B(x,r)). \tag{3.1.5}$$

Using Theorem 3.1 for a system $\big(B(x,r), \mathcal{F}|_{B(x,r)}, \nu, d|_{B(x,r)}, S\big)$ we get

$$\liminf_{k\to\infty}\ k^{1/\alpha} d(S^k(y), y) \le (\operatorname{ess\,sup} h)^{1/\alpha}. \tag{3.1.6}$$

Denote by $n_k(y)$ the time of k-th return of y to $B(x,r)$. Then $S^k(y) = T^{n_k(y)}(y)$ and also

$$\lim_{k\to\infty} \frac{k}{n_k(y)} = \mu(B(x,r)) \tag{3.1.7}$$

for μ – almost every y because of the ergodic theorem. Thus, the limit in (3.1.6) transforms to

$$\liminf_{k\to\infty} \left(\frac{k}{n_k(y)}\right)^{1/\alpha} n_k(y)^{1/\alpha} \cdot d(T^{n_k(y)}(y), y) \ge \mu(B(x,r))^{1/\alpha} \cdot \liminf_{n\to\infty}\ n^{1/\alpha} d(T^n(y), y). \tag{3.1.8}$$

It remains to compile (3.1.5), (3.1.6) and (3.1.8) obtaining

$$\mu(B(x,r))^{1/\alpha} \cdot \liminf_{n\to\infty}\ n^{1/\alpha} d(T^n(y), y) \le (\operatorname{ess\,sup} g|_{B(x,r)})^{1/\alpha} \cdot \mu(B(x,r))^{1/\alpha}. \tag{3.1.9}$$

Letting $r \to 0$ we finish the proof. □

3.1.2 Inducing

The above proof may be summarized in this way: first, prove the needed result on a subset of the entire space; second, using an induced transformation get the result everywhere.

The same approach has been used e.g. by [BSTV03] to prove exponential return statistics for many systems. We may use such a technique in a general situation:

Assumptions:

1) $(X, \mathcal{F}, \mu, d, T)$ — a metrical measure preserving dynamical system, μ-ergodic;

2) $\widehat{X} \subset X$, open and $\mu(\widehat{X}) < \infty$; we set $\widehat{\mu} = \mu|_{\widehat{X}}$ meaning $\widehat{\mu}(A) = \frac{\mu(A\cap\widehat{X})}{\mu(\widehat{X})}$;

3) $\widehat{T}\colon \widehat{X} \to \widehat{X}$ induced transformation (not only *first return*), i.e. we have $k\colon \widehat{X} \to \mathbb{N}$ such that $\widehat{T}(x) := T^{k(x)}(x) \in \widehat{X}$; $\widehat{T}$ is $\widehat{\mu}$-preserving; (it follows that $\widehat{\mu}$ is ergodic)

4) μ is finite, which happens iff $\int_{\widehat{X}} k\, d\widehat{\mu} < \infty$.

Remark. We can also start with a finite, ergodic measure $\widehat{\mu}$ on $\widehat{X}$ preserving $\widehat{T}$ — with which we may define a measure μ on $X = \bigcup_{n=0}^{+\infty} T^{-n}(\widehat{X})$ by an equation:

$$\mu(A) = \sum_{n=0}^{+\infty} \widehat{\mu}\left(\widehat{X} \cap T^{-n}(A) \setminus \bigcup_{j=1}^{n} T^{-j}(\widehat{X})\right) = \int_{\widehat{X}} \sum_{n=0}^{k-1} \mathbb{1}_A \circ T^n \, d\widehat{\mu}$$

(k satisfies assumptions above). Such defined measure μ is preserves T and is ergodic.

Theorem 3.4. *Keeping the above assumptions (1) – (4) we have* $\liminf_{n\to\infty} n^\alpha d(T^n(x), x) = 0$ *iff* $\liminf_{n\to\infty} n^\alpha d(\widehat{T}^n(x), x) = 0$ *for any* $x \in \widehat{X}$ *.*
The same is true for the limit being finite instead of zero.

Proof. The left to right implication is quite obvious. $\widehat{X}$ is open, so if we take any sequence n_k fulfilling the lower limit, then it satisfies $T^{n_k}(x) \in \widehat{X}$ for all k sufficiently large. That means there exists a sequence c_k such that $\widehat{T}^{c_k}(x) = T^{n_k}(x)$, also $c_k \le n_k$ and implications follows.

The right to left implication needs some work. Let us start by defining a sum $A_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} k\left(\widehat{T}^i(x)\right)$. From the ergodic theorem it follows that

$$\lim_{n\to\infty} A_n(x) = \int_{\widehat{X}} k \, d\widehat{\mu} \qquad \text{for } \widehat{\mu}\text{–a.e. } x \in \widehat{X}. \tag{3.1.10}$$

We shall denote the RHS (right–hand side) by letter c. Let us also observe that

$$\widehat{T}^n(x) = T^{\sum_{i=0}^{n-1} k\left(\widehat{T}^i x\right)}(x) = T^{nA_n(x)}(x),$$

which gives us

$$\begin{aligned}
\liminf_{n\to\infty} n^\alpha d\left(\widehat{T}^n(x), x\right) &= \lim_{k\to\infty} n_k^\alpha d\left(\widehat{T}^{n_k}(x), x\right) = \\
&= \lim_{k\to\infty} \left(\frac{n_k}{n_k A_{n_k}(x)}\right)^\alpha \left(n_k A_{n_k}(x)\right)^\alpha \cdot d\left(T^{n_k A_{n_k}}(x), x\right) \ge \\
&\ge \left(\frac{1}{c}\right)^\alpha \cdot \liminf_{n\to\infty} \; n^\alpha d\left(T^n(x), x\right),
\end{aligned}$$

which finishes the proof. □

3.2 Reformulating the limit

In this section we will see how Theorem 3.1 can be rephrased. We will observe different limits and see what they have in common with the one from previous section.

3.2.1 Creating a suitable metric

Instead of seeking systems with appropriate metric (so that the density is bounded), we could try to start with a measurable dynamical system and simply construct a metric satisfying the assumptions.

Let us assume that X is contained in the real line. Then we will define a function $\rho(x,y) = \mu\big([x,y]\big)$. It is easy to check that ρ is a metric iff μ is non-atomic and $\operatorname{supp}\mu = X$ (no open interval intersecting X can have zero measure).

Observe that $B_\rho(x,r) = (s,t)$, where $\mu([s,x)]) = \mu([x,t]) = r$, if possible, as it may happen that e.g. $\mu\big([x,+\infty)\big) < r$. In general $s = \inf\{z \in X\colon \mu([z,x)] \leq r)\}$ and accordingly for t. Note that $r \leq \mu(X)$.

This gives $\mu\left(B_\rho(x,r)\right) = 2r$, if x is *in the interior* of X and $r \leq \mu(B_\rho(r)) \leq 2r$ in the general case (e.g. the rightmost point of X) for r small enough. Obviously, the Hausdorff measure built using this metric satisfies assumptions of Thm. 3.1. In addition, for every point (except the left/rightmost, if they exist) $\frac{dH_1}{d\mu} = 1$.

Corollary 3.5. *For a system (X,T,μ), where $X \subset \mathbb{R}$, $\operatorname{supp}\mu = X$ and μ is non-atomic we have*

$$\liminf_{n\to+\infty} n \cdot \mu\big([T^n(x), x]\big) \leq 1. \tag{3.2.1}$$

3.2.2 Changing limits

If X is not a subset of the real line, then the metric cannot be defined so easily. (It is still possible, although we may only construct a quasi-metric, i.e. the triangle inequality has to be weakened.)

Instead of building a new metric, we will find a better (more general) expression (for which we want to find the limits).

In the first chapter we introduced a variable $\mu(U)\tau_U$, what we would like to do now is investigate how this is related to the limits in the previous section. To do that we will use notions from section 2.2.

Let us observe the limit

$$\liminf_{n\to+\infty} n^{1/\beta} d\left(T^n(x), y\right),$$

which is almost the same as in previous section, but here y may not be equal to x. This shows the speed in which x approaches target point y. Now let us observe how this limit is achieved. Define two sequences inductively:

$$n_1 = 1 \longrightarrow d_1 = d(T^{n_1}x, y)$$
$$n_{k+1} = \inf\{n\colon d(T^n x, y) < d_k\} \longrightarrow d_{k+1} = d(T^{n_{k+1}}x, y).$$

So n_k's are subsequent *closest* approaches to y and d_k's are the distances of those approaches. It is trivial to see that the lower limit is realised on sequence n_k.

Let us set $U_k = \overline{B(d_k, y)}$. Definition of n_k gives that $\tau_{U_k} = n_k$. Assuming the upper β–property (cf. section 2.2) gives:

$$\tau_{U_k}\mu(U_k) = n_k\mu(U_k) \geq n_k D(y) d_k^\beta = D(y)\left(n_k^{1/\beta} d(T^{n_k}x, y)\right)^\beta . \tag{3.2.2}$$

And assuming the lower β–property reverses the inequality. This implies the following statement.

Corollary 3.6. *If the system satisfies the upper β–property at point y (we may take y equal to x), then*

$$\liminf_{r\to 0} \tau_r^y(x)\mu(B_r(y)) = 0 \Longrightarrow \liminf_{n\to+\infty} n^{1/\beta} d\left(T^n(x), y\right) = 0,$$
$$\liminf_{r\to 0} \tau_r^y(x)\mu(B_r(y)) < +\infty \Longrightarrow \liminf_{n\to+\infty} n^{1/\beta} d\left(T^n(x), y\right) < +\infty.$$

And if the system satisfies the lower β–property, then we get the reverse implications. Finally if the measure is geometric, then one lower limit is finite (resp. 0) iff the other is finite (resp. 0).

Now the next question is: what is the difference between taking arbitrary y and taking $y = x$, i.e. between observing the *return times* and *entry times*. Or perhaps there is none?

The paper [BC13] defines and names three different limits

$$\begin{aligned}
\phi_\alpha(x,y) &= \liminf_{n\to+\infty} n^\alpha d(T^n(x), y) && \textit{connectivity gauge},\\
\psi_\alpha(x,y) &= \liminf_{n\to+\infty} n^\alpha d(T^n(x), T^n(y)) && \textit{proximality gauge},\\
\rho_\alpha(x) &= \phi_\alpha(x,x) = \liminf_{n\to+\infty} n^\alpha d(T^n(x), x) && \textit{recurrence gauge}.
\end{aligned}$$

We are mostly interested in the third and somewhat in first. This is partly because of two results from the mentioned paper.

Proposition 3.7 ([BC13] Thm. 1). *In a metric measure preserving system (X, T, μ, d) for any $\alpha > 0$ both $\phi_\alpha(x,y)$ and $\psi_\alpha(x,y) \in \{0, \infty\}$ for almost all (x,y).*

Remark. The $\rho_\alpha(x)$ may behave quite differently, e.g. the golden mean rotation on an interval $T(x) = (x + \varphi) \mod 1$ has $\rho_1(x) = \frac{1}{\sqrt{5}}$ for all x; see e.g. [Khi64].

As for proximality gauge, the situation is even clearer.

Proposition 3.8 ([BC13] Prop. 2). *If the metric meas. preser. system is weakly mixing, then either $\psi_\alpha(x,y) = 0$ $\mu \times \mu$–a.e. or $\psi_\alpha(x,y) = +\infty$ $\mu \times \mu$–a.e.*

Note. Weakly mixing gives that $S = T \times T$ is ergodic. ψ is sub-invariant w.r.t. S and the result follows from the previous one.

3.3 Lower limit for non-ball sets

In this section we will see what can be proved for $\tau_U \mu(U)$ for some sets U, more general than balls. We will work with minimal assumptions on the system.

We pursue those results, because it is interesting to see what can be proved for sets different than balls, but most of all we will be able to apply this section in the setting of random dynamical system, cf. chapter 5 and Thm. 5.16.

Definition 3.9. A two–parameter family of sets $D_r(x)$ will be called *ball–like* if

a) $x \in D_r(x)$ for all r;

b) $D_r(x) \subset D_s(x)$ for all $r < s$;

c) $D_r(x) \subset D_{2r}(y)$, if $y \in D_r(x)$;

d) $\mu(D_r(x)) \to 0$ as $r \to 0$.

The families of sets satisfying assumptions (a)–(c) include (apart from balls, obviously) e.g. cylinders with height somehow dependent on radius (or constant). The (d) on the other hand depend mostly on the measure.

3.3.1 Lower limit bounded

Definition 3.10. We shall say that the measure is *C–doubling* or *strongly doubling* on family D_r if $\mu(D_{2r}(x)) \leq C\mu(D_r(x))$ for all $r \leq \rho(x)$ and $\rho(x)$ is a measurable a.e. positive function.

Definition 3.11. We shall say that the family D_r is *Lebesgue–compatible with* μ if for any set $A \subset X$ of measure $\mu(A) > 0$, there exists $x_0 \in X$ such that

$$\lim_{r \to 0} \frac{\mu\left(D_r(x_0) \cap A\right)}{\mu(D_r(x_0))} = 1. \tag{3.3.1}$$

Remark 1. If D_r is a family of balls in $\mathbb{R}^n$, then (3.3.1) holds for any Radon measure μ and almost every x. This is a generalisation of the Lebesgue density theorem, may be found e.g. in [Mat95] Cor. 2.14. As we only need one point, this is a weaker assumption.

Remark 2. In $\mathbb{R}^n$ any locally finite, Borel regular measure is a Radon measure, again check [Mat95] Cor. 1.11.

Theorem 3.12. *Let (X, d) be a separable metric space and T be a transformation preserving a Borel, probability measure μ. We also assume that the system (X, T, μ) is ergodic. If the family D_r is* Lebesgue–compatible with μ *and the measure μ is C–doubling on D_r, then for μ–a.e. x*

$$\liminf_{r \to 0} \tau_{D_r(x)}(x) \cdot \mu\left(D_r(x)\right) \leq C^2. \tag{3.3.2}$$

First we need a simple observation:

Lemma 3.13. *For any measurable set $V \subset X$ we define*

$$V^t = \{x \in V : \tau_V(x) > t\} = V \cap \bigcap_{l=1}^{t} T^{-l}(X \setminus V).$$

Then $\mu(V^t) \leq \frac{1}{t+1}$.

Proof of Lemma. The sets V^t, $T^{-1}(V^t)$, ..., $T^{-t}(V^t)$ are disjoint and all have the same measure, so $(t+1)\mu(V^t) \leq 1$. □

Proof of Thm. 3.12. Now, let us denote $\mu_r(x) = \mu(D_r(x))$, $\tau_r(x) = \tau(D_r(x), x)$ and $\tau_r^y(x) = \tau(D_r(y), x)$.
Note that this is a generalisation of the notation used previously.
Let us proceed by contradiction, for some fixed $\gamma > 0$ and $\rho > 0$ the set

$$W = \{x \in X : \forall_{s<\rho} \tau_s(x) \cdot \mu_s(x) \geq C^2 + \gamma\},$$

has positive measure. Now from the assumption we have that there exists $x_0 \in W$ such that

$$\lim_{r \to 0} \frac{\mu(D_r(x_0) \cap W)}{\mu_r(x_0)} = 1.$$

Put $\varepsilon = \frac{1}{2}\frac{\gamma}{C^2+\gamma}$; there exists $\rho_1 > 0$ such that for all $r < \rho_1$

$$\frac{\mu(D_r(x_0) \cap W)}{\mu_r(x_0)} \geq 1 - \varepsilon = 1 - \frac{1}{2}\frac{\gamma}{C^2 + \gamma}. \tag{3.3.3}$$

Take any $r < \min(\rho/2, \rho_1)$ and fix $s = 2r$. Then definition of W and (3.3.3) give

$$\frac{\mu(\{y \in D_r(x_0) : \tau_s(y)\mu_s(y) \geq C^2 + \gamma\})}{\mu_r(x_0)} \geq 1 - \varepsilon \tag{3.3.4}$$

and on the other hand, lemma 3.13 used for $V = D_r(x_0)$ states that

$$\mu(\{y \in D_r(x_0) : \tau_r^{x_0}(y) > t\}) \leq \frac{1}{t+1} \tag{3.3.5}$$

or taking $t - 1$ instead of t and dividing by $\mu(D_r(x_0))$

$$\frac{\mu(\{y \in D_r(x_0) : \tau_r^{x_0}(y) \geq t\})}{\mu_r(x_0)} \leq \frac{1}{t\mu_r(x_0)} \tag{3.3.6}$$

or taking the complement

$$\frac{\mu(\{y \in D_r(x_0) : \tau_r^{x_0}(y) < t\})}{\mu_r(x_0)} \geq 1 - \frac{1}{t\mu_r(x_0)}. \tag{3.3.7}$$

Put $\left[t = \dfrac{1}{(1-2\varepsilon)\mu_r(x_0)}\right]$. The right-hand side becomes 2ε. As the sum of the (conditional) measures of sets in (3.3.4) and (3.3.7) exceeds 1 we get that those sets must intersect. This gives a point $y \in D_r(x_0)$, for which

$$\tau_r^{x_0}(y) < \frac{\frac{1}{1-2\varepsilon}}{\mu_r(x_0)}$$
$$\tau_{2r}^{y}(y) > \frac{C^2+\gamma}{\mu_{2r}(y)}.$$

Now $D_r(x_0) \subset D_{2r}(y) \implies \tau_r^{x_0}(y) \geq \tau_{2r}^{y}(y)$ and $D_{2r}(y) \subset D_{4r}(x_0) \implies \mu_{2r}(y) \leq \mu_{4r}(x_0)$ so

$$\frac{C^2+\gamma}{\mu_{4r}(x_0)} \leq \frac{C^2+\gamma}{\mu_{2r}(y)} < \tau_{2r}^{y}(y) \leq \tau_r^{x_0}(y) < \frac{\frac{1}{1-2\varepsilon}}{\mu_r(x_0)}. \tag{3.3.8}$$

This means that

$$\frac{1}{C^2} \leq \frac{\mu_r(x_0)}{\mu_{4r}(x_0)} < \frac{1}{1-2\varepsilon}\frac{1}{C^2+\gamma}, \tag{3.3.9}$$

which contradicts the choice of ε (3.3.3). □

3.3.2 Lower limit finite

Now we will weaken the assumptions and alter the theorem slightly to arrive at a more useful result.

Definition 3.14. We shall say that the measure has the *doubling property* on D_r if $\mu(D_{2r}(x)) \leq C(x)\mu(D_r(x))$ for all $r \leq \rho(x)$ and $\rho(x)$, $C(x)$ are measurable (positive, finite, but not necessarily bounded from above or separated from 0) functions.

Definition 3.15. We shall say that the family D_r is *weakly Lebesgue–compatible with* μ if for any set $A \subset X$ of measure $\mu(A) > 0$, there exists $\alpha > 0$ such that

$$\exists_{\delta>0}\forall_{r<\delta}\exists_{x_0\in X}\frac{\mu\left(D_r(x_0)\cap A\right)}{\mu(D_r(x_0))} \geq \alpha. \tag{3.3.10}$$

Note. This is trivially a weaker notion than the previous one (Def. 3.11).

Remark. To see the difference between 'strong' and weak *Lebesgue–compatibility* take a family $D_r(\omega, y) = \Omega \times B_r(y)$ (for a certain space Ω) of subsets of $\Omega \times Y$. Then the set $A \subset \Omega \times Y$ and there is no possibility of having the *Lebesgue–compatibility* (Def. 3.11), but in many 'normal' situations (e.g. $\Omega = Y = \mathbb{R}$) the family D_r will be *weakly Lebesgue–compatible with* μ. Check Figure 3.1.

This is precisely how the Thm. 3.16 will be applied later for random systems in sec. 5.2.3.

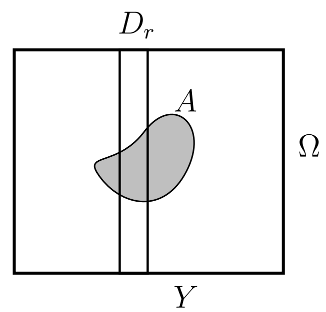

Figure 3.1: *Weak Lebesgue–compatibility* for cylinders.

Theorem 3.16. *Let (X, d) be a separable metric space and T be a transformation preserving a Borel, probability measure μ. We also assume that the system (X, T, μ) is ergodic. If the family D_r is* weakly Lebesgue–compatible with *μ and the measure μ has* doubling property *on D_r, then*

$$\liminf_{r\to 0} \tau_{D_r(x)}(x) \cdot \mu\left(D_r(x)\right) < +\infty \qquad \mu\text{–}a.e. \tag{3.3.11}$$

Proof. We will use notations as in the previous proof and, as before, define a family of sets

$$W_M = \left\{x \in X \colon \exists_{\rho>0} \forall_{s<\rho} \tau_s(x) \cdot \mu_s(x) \geq M\right\}.$$

The proof will be by contradiction, so we assume that $\liminf = +\infty$ on a set of positive measure, i.e. $\mu(\bigcap_M W_M) > 0$. Put $W = \bigcap_M W_M$.

By Def. 3.14 the function $C(x)$ is measurable and finite, so there exists C such that the set $A_C = \{x \in X \colon C(x) \leq C\}$ have measure $\mu(A_C) > 1 - \mu(W)$. This means that $\mu(\widehat{W}) > 0$, where $\widehat{W} = W \cap A_C$.

Using the assumption on *Leb–compatibility* for the set $\widehat{W}$ gives constants δ and α. The same constants can be used for any $\widehat{W_M} = W_M \cap A_C$, because $W_M \supset W$.

Take $M = \frac{4C^2}{\alpha}$, any $r < \min(\delta, \rho/2)$ and put $s = 2r$. This gives:

$$\frac{\mu\left(\{y \in D_r(x_0) \colon \tau_{2r}(y)\mu_{2r}(y) \geq M\}\right)}{\mu_r(x_0)} \geq \alpha \tag{3.3.12}$$

and, as before, lemma 3.13 gives (check equations (3.3.5) to (3.3.7))

$$\frac{\mu\left(\{y \in D_r(x_0) \colon \tau_r^{x_0}(y) < t\}\right)}{\mu_r(x_0)} \geq 1 - \frac{1}{t\mu_r(x_0)}. \tag{3.3.13}$$

Put $t = \dfrac{2}{\alpha\mu_r(x_0)}$. The right-hand side becomes $1 - \alpha/2$, so the sets must intersect. This gives a point $y \in D_r(x_0)$, for which

$$\tau_r^{x_0}(y) < \frac{\frac{2}{\alpha}}{\mu_r(x_0)}$$
$$\tau_{2r}^{y}(y) \geq \frac{M}{\mu_{2r}(y)}$$

and we arrive at

$$\frac{M}{\mu_{4r}(x_0)} \leq \frac{M}{\mu_{2r}(y)} \leq \tau_{2r}^{y}(y) \leq \tau_r^{x_0}(y) < \frac{\frac{2}{\alpha}}{\mu_r(x_0)}. \tag{3.3.14}$$

Using doubling property and the value of M, this transforms to

$$\frac{1}{C^2} \leq \frac{\mu_r(x_0)}{\mu_{4r}(x_0)} < \frac{2}{M\alpha} = \frac{1}{2C^2}, \tag{3.3.15}$$

and this contradiction ends the proof. □

3.4 Hausdorff dimension by recurrence

The results in the first sections of this chapter show that the behaviour of the recurrence is governed by the Hausdorff (or packing) dimension of the space. We may try to use this backward: if one can compute the lower limit, then this gives as some information on the Hausdorff dimension.

First of all, let us recall a result from [Bos93] (Theorem 1.2, part 2):

Proposition 3.17. *If $(X, d, T, \mu, \mathcal{F})$ is a metric, Borel–measure preserving dynamical system and $H_\beta(X) = 0$, then for μ–almost every $x \in X$*

$$\liminf_{n\to\infty} n^{1/\beta} \cdot d(T^n(x), x) = 0. \tag{3.4.1}$$

In the following example we shall calculate (or estimate from below) the above limit proving that some β–Hausdorff measure of the underlying set is positive. Moreover, we may use Prop. 3.3 (or Cor. 3.2 if its assumptions are satisfied); this will give us an estimation of $g(x)$ (from below), leading to an estimation of $H_\beta(X)$ from below.

As an example let us take the canonical Cantor set C. Every point $x \in C$ has a unique coding (x_n) in triadic representation using only 0 and 2, that is $x = \sum_{k=1}^{+\infty} \frac{x_k}{3^k}$. It follows that the (Euclidean) distance between points x and y is given by a formula $|x - y| = \left|\sum_{k=1}^{+\infty} \frac{x_k - y_k}{3^k}\right|$. Let us define a transformation T on the coding space (triadic representation), by an inductive scheme:

A) Start with $n = 1$.

B) If the symbol x_n equals 0, then add 2 to it (new $T(x)_n = 2$) and finish.

C) If the symbol $x_n = 2$, then make it equal 0 (new $T(x)_n = 0$), increase n by 1, and return to (B).

This 'program' will run indefinitely, if our point x has code $[222\ldots]$ (i.e. if $x = 1$), but mathematically this is not an issue ($T(1) = 0$).

In other words — we scan the code for the first code (x_n) equal 0, set this first code equal 2 and all the previous codes (i.e. (x_k) for $k < n$) equal 0.

Usually this is called an ‘adding machine’ — if we were using digits 0 and 1 the transformation would be equivalent to adding 1 to the first digit of a binary number (written in reverse).

This transformation is a piecewise isometry and it preserves the Cantor measure μ (defined to be equally distributed on the cylinders). Let us start calculations by taking the point $z = 0 = [0000\ldots]$ and looking at the forward iterates:

$$T(z) = \frac{2}{3} = [2000\ldots],\ T^2(z) = \frac{2}{9} = [0200\ldots],\ T^3(z) = \frac{8}{9} = [2200\ldots].$$

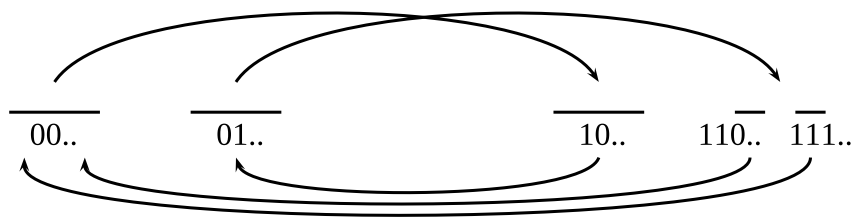

Figure 3.2: *Adding machine* transformation on a Cantor set.

To calculate the limit inferior we only need to look at the subsequent “closest returns” (as in the previous section), i.e. we can omit all n for which there exists $k < n$ such that $|T^k(z) - z| < |T^n(z) - z|$. For the point $z = 0$ it is obvious that those returns will occur for the powers of 2. Writing this more precisely we get:

$$\begin{aligned} \left|T^{2^n}(z) - z\right| &= \frac{2}{3^{n+1}} && \text{for all } n, \\ \left|T^k(z) - z\right| &\geq \frac{2}{3^{n+1}} && \text{for all } 2^n < k < 2^{n+1}. \end{aligned}$$

This leads to calculating the lower limit for any $\beta > 0$.

$$\liminf_{n\to+\infty} n^{1/\beta}|T^n(z) - z| = \lim_{n\to+\infty} (2^n)^{1/\beta}\frac{2}{3^{n+1}} = \lim_{n\to+\infty} \frac{2}{3}\frac{(2^{1/\beta})^n}{3^n}.$$

Now let us look at the limit for any point $x \in C$. Look at the coding of x — $[x_1x_2x_3\ldots]$. After 2^n iterates the first n symbols will be the same and the $(n+1)$–st symbol will be different. What we do not control are the later symbols, which can lower the distance slightly. (e.g. the distance between $[2000\ldots]$ and $[0222\ldots]$ is equal to $1/3$). However, we can estimate the distance between x and $T^n(x)$ from below by $\frac{2}{3^{n+1}} - \sum_{k=n+2}^{+\infty} \frac{2}{3^k}$, which leads to the following estimates:

$$\left|T^{2^n}(x) - x\right| \geq \frac{1}{3^{n+1}} \qquad \text{for all } n,$$
$$\left|T^k(x) - x\right| \geq \frac{1}{3^{n+1}} \qquad \text{for all } 2^n < k < 2^{n+1}.$$

This clearly shows that for all $x \in C$

$$\liminf_{n\to+\infty} n^{1/\beta}|T^n(x) - x| \geq \frac{1}{3}\frac{(2^{1/\beta})^n}{3^n}.$$

The limit is positive for $\beta = \frac{\ln(2)}{\ln(3)}$ and so by using ('backwards') Prop. 3.17 we know that the Hausdorff dimension $\mathrm{HD}(C) \geq \frac{\ln(2)}{\ln(3)}$, moreover $H_\beta(C) > 0$.

Note. A slightly more rigorous calculation shows that $\liminf \geq \frac{4}{9}$. Such estimation will be shown in the next (more interesting) example.

Observe now that Prop. 3.3 gives that $g(x)^{1/\beta} \geq \frac{1}{3}$ for all x (where $g(x) = \frac{dH_\beta}{d\mu}$), or using the more careful estimate: $g(x)^{1/\beta} \geq \frac{4}{9}$. This leads to an inequality.

$$H_{\log_3 2}(C) = \int_C g(x)d\mu(x) \geq \mu(C)\left(\frac{4}{9}\right)^{\log_3 2} \approx 0.6. \tag{3.4.2}$$

This is not a very strong result — in reality $H_{\log_3 2}(C) = 1$, but on the other hand, the estimate has been acquired with little effort.

Observe that the calculation also shows that $H_{\log_3 2} \gg \mu$ (which is trivial in this case, but leads to an interesting result).

Take any subset $A \subset C$ of positive measure μ. We want to show that $H_{\log_3 2}(A) > 0$. Take a new measure $\nu = \frac{1}{\mu(A)}\mu\Big|_A$ and a new transformation S — the first return mapping for T into set A. S preserves the probability measure ν and by the lower limit for S is equal to the lower limit for T divided by $\mu(A)$. (Check proof of corollary 3.2 for details or use Thm. 3.4). This means that

$$\liminf_{n\to+\infty} n^{1/\beta}|S^n(x) - x| > 0 \qquad \text{for } \nu\text{–a.e. } x \in A.$$

And Prop. 3.17 tells us that $H_{\log_3 2}(A) > 0$.

Note that we proved the following result.

Proposition 3.18. *Take a metric measure system (X, T, μ, d) and a measurable subset Z. If for almost every $x \in Z \subset X$ $\liminf_{n\to\infty} n^{1/\beta} \cdot d(T^n(x), x) > 0$, then $\mu \ll H_\beta$ on Z.*

A similar calculation may be done on a Sierpiński gasket (triangle).

This time every point has a code in $\{0, 1, 2\}^{\mathbb{N}}$, cf. Figure 3.3. The transformation T on the symbolic space is defined as before — adding 1 to a code (treated as a ternary number) and this gives a transformation on our fractal S.

We have to be slightly more careful this time as the coding is not unique: e.g. the centre of the left leg of the main triangle can be written as $011\ldots = 100\ldots$ However, there are only countably many such points (those are the points connecting the triangles) and so we may simply ignore them.

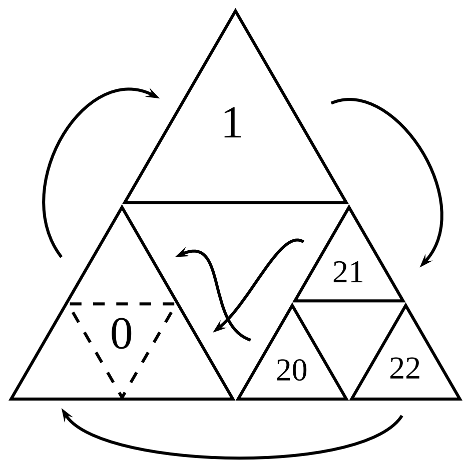

Figure 3.3: *Adding machine* transformation on a Sierpiński gasket.

The closest returns occur at times equal 3^n. Let us take one sample point and look at those returns. Fix $x_0 = 01220122\ldots$ and denote $x_n = T^n(x_0)$. The returns are as follows

$$\begin{aligned} x_0 &= 01220122\ldots \\ x_1 &= 112201\ldots \Longrightarrow ||x_1 - x_0|| = \frac{1}{2}, \\ x_3 &= 022201\ldots \Longrightarrow ||x_3 - x_0|| = \frac{1}{4}, \\ x_9 &= 010011\ldots \Longrightarrow ||x_9 - x_0|| = \frac{\sqrt{31}}{32} > \frac{1}{8}, \\ x_{27} &= 012011\ldots \Longrightarrow ||x_{27} - x_0|| = \frac{\sqrt{3}}{32} < \frac{1}{16}, \\ x_{81} &= 012211\ldots \Longrightarrow ||x_{81} - x_0|| = \frac{1}{32}. \end{aligned}$$

It is easy to verify the following statement:
For any point x_0 the n–th closest return is in distance $\frac{1}{2^{n+1}}$, unless the n–th symbol of x_0 is the digit 2. In that case the distance is even bigger, if the next symbol is also 2; or else

(worst case scenario) it is equal to $\frac{\sqrt{3}}{2^{n+2}}$.

$$\liminf_{n\to+\infty} n^{1/\beta}||T^n(x)-x|| \geq \frac{\sqrt{3}}{4}\frac{(3^{1/\beta})^n}{2^n}.$$

This limit is finite for $\beta = \frac{\ln(3)}{\ln(2)}$, which give the Hausdorff dimension and also we may estimate the β–Hausdorff measure of the set $H_{\log_2 3}(S) \geq \left(\frac{\sqrt{3}}{4}\right)^{\log_2 3} \geq 0.26$.

Again, this is not the best estimate. Although the precise measure is not known, it has been recently proved in [Mór09] that $H_{\log_2 3}(S) \geq 0.77$. Nevertheless, our estimate was almost effortless.

3.5 Proof of Theorem 3.1

For completeness we include the proof of the theorem mentioned in the beginning of this chapter.

Theorem. *Let $(X, \mathcal{F}, \mu, d, T)$ be a metrical measure preserving dynamical system. In addition suppose that $\mu \approx H_\alpha$ for some $\alpha > 0$ and that $g := \frac{dH_\alpha}{d\mu}$ is bounded from above. Then for μ – almost every $x \in X$ we have*

$$\liminf_{n\to\infty}\ n^{1/\alpha} d(T^n(x), x) \leq (\operatorname{ess\,sup} g)^{1/\alpha}. \tag{3.5.1}$$

Proof. Denote $\beta := \frac{1}{\alpha}$, $c := \operatorname{ess\,sup}\, g$ and consider the set

$$X(u) := \{x \in X :\ \ n^\beta d(T^n(x), x) > c^\beta + u,\ \ \forall_{n\geq 1} \text{ such that } d(T^n(x), x) < u\}. \tag{3.5.2}$$

From its definition it is obvious that $X(u_1) \subseteq X(u_2)$ for $u_1 \geq u_2$. It is easily seen that the sets $X(u)$ are μ-measurable.

In order to prove this theorem we need to show that $\mu(X') = 0$, where

$$X' = \{x \in X :\ \liminf_{n\to\infty}\ n^\beta d(T^n(x), x) > c^\beta\}. \tag{3.5.3}$$

Notice now that

$$X' = \bigcup_{k\geq 1} X\left(\frac{1}{k}\right).$$

Indeed, if $x \in X'$, then $\liminf\ n^\beta d(T^n(x), x) > c^\beta$, hence

$$\exists_N \exists_a \forall_{n\geq N}\ \ n^\beta d(T^n(x), x) > c^\beta + a.$$

Taking $u = \min\{a, \min_{n\leq N} d(T^n(x), x)\}$ we get that $x \in X(u)$. Proof of the opposite inclusion is trivial.

Thus, is suffices to show that $\mu(X(u)) = 0$ for any $u > 0$.

Suppose the opposite: $\mu(X(u_0)) > 0$ for some u_0 and denote $Y := X(u_0)$. Since $H_\alpha \gg \mu$, then $\mu(Y) > 0 \Rightarrow H_\alpha(Y) > 0$. Put $p := H_\alpha(Y)$. Next take $\varepsilon > 0$ such that

$$\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^\beta < 1 + \frac{u_0}{c^\beta}. \tag{3.5.4}$$

Lemma 3.19. *If the measure ν is nonatomic and finite on a set Z, then*

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \ \textit{for any } \nu\textit{-measurable } U \subset Z \quad \operatorname{diam}(U) < \delta \implies \nu(U) < \varepsilon. \tag{3.5.5}$$

Proof. Assume the opposite, that for an ε there does not exists a suitable δ. This would mean that there is a sequence of sets $U_i \subset Z$, such that $\nu(U_i) \geq \varepsilon$ for all i and

$$\lim_{i\to\infty} \operatorname{diam} U_i = 0.$$

This implies that $\nu(U) \geq \varepsilon$, where

$$U = \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} U_i = \{z \in Z : z \in U_i \text{ for infinitely many } i \geq 1\}.$$

Now take any $u \in U$ and consider instead of sequence U_i its infinite subsequence U_{i_k} of sets containing u. Then $\{x : x \text{ belongs to infinitely many sets } U_{i_k}\} = \{u\}$, that is $\nu\{u\} \geq \varepsilon$ which contradicts nonatomicity of ν on Z. □

Observe now that if the measure μ has an atom in $\{y_0\}$, then y_0 is a periodic point (because T preserves the measure μ). Hence $y_0 \notin Y$, so μ is nonatomic on the set Y. This lets us take a suitable δ (using the lemma above) for an ε; if needed we decrease δ to be smaller than u_0. From the definition of measure $H_{\alpha,\delta}$ there exists a division of Y into separate sets U_i of diameters less than δ, such that

$$p - \frac{\varepsilon p}{2} < \sum_i (\operatorname{diam} U_i)^\alpha < p + \frac{\varepsilon p}{2}. \tag{3.5.6}$$

Recall now that $p = H_\alpha(Y)$. Denote $\operatorname{diam} U_i =: r_i$, taking suitable δ now ensures that

$$\mu(U_i) < \varepsilon. \tag{3.5.7}$$

Afterwards define the set

$$J := \{i \geq 1 : (1-\varepsilon) r_i^\alpha > H_\alpha(U_i)\}. \tag{3.5.8}$$

We get that for every $i \in J$ we can divide the set U_i into separate subsets U_{ij}, such that

$$\sum_{j\geq 1} (r_{ij})^\alpha < (1-\varepsilon) r_i^\alpha, \quad \text{where} \quad r_{ij} = \operatorname{diam} U_{ij}. \tag{3.5.9}$$

This is a straight conclusion from the definition of the Hausdorff measure. Considering the covering $\{U_i, U_{ij}\}$ and from the definition of the measure $H_{\alpha,\delta}$ we get

$$\begin{aligned} p - \frac{\varepsilon p}{2} \le H_{\alpha,\delta}(Y) \le \sum_{i \notin J}(r_i^\alpha) + \sum_{\substack{i\in J\\ j\ge 1}}(r_{ij})^\alpha < \sum_{i\notin J}(r_i^\alpha) + (1-\varepsilon)\sum_{i\in J}(r_i^\alpha) = \\ = \sum_{i\ge 1}(r_i^\alpha) - \varepsilon\sum_{i\in J}(r_i^\alpha) \le p + \frac{\varepsilon p}{2} - \varepsilon \sum_{i\in J}(r_i^\alpha). \end{aligned} \tag{3.5.10}$$

Hence

$$p - \frac{\varepsilon p}{2} < p + \frac{\varepsilon p}{2} - \varepsilon \sum_{i\in J}(r_i^\alpha).$$

Which means that

$$\sum_{i\in J}(r_i^\alpha) < p. \tag{3.5.11}$$

So by using only the sets $\{U_i\}_{i\in J}$ we cannot cover the entire set Y, thus there exists an $i \notin J$. In other words we have a nonempty set U_k, such that $(1-\varepsilon)r_k^\alpha \le H_\alpha(U_k)$.

From the definition of density g

$$H_\alpha(U_k) = \int_X \mathbb{1}_{U_k} dH_\alpha = \int_X g\mathbb{1}_{U_k} d\mu \le \operatorname{ess\,sup} g \cdot \mu(U_k) = c\mu(U_k).$$

Hence

$$\mu(U_k) \ge \frac{1-\varepsilon}{c} \cdot r_k^\alpha.$$

Denote $\mu(U_k) =: u_k$, take both sides of the inequality to the power of β and divide by the product of both sides, obtaining

$$\left(\frac{1}{u_k}\right)^\beta \le \left(\frac{c}{1-\varepsilon}\right)^\beta \cdot \frac{1}{r_k}. \tag{3.5.12}$$

Since T preserves μ, then

$$T^{-n}U_k \cap U_k \ne \emptyset \text{ for some } n \le 1 + \frac{1}{u_k}. \tag{3.5.13}$$

Indeed, $\mu(T^{-i}U_k) = u_k$, thus if U_k, $T^{-1}(U_k) \dots T^{-n}(U_k)$ were separate, then the measure of the entire space would be greater than $\mu\left(\bigcup_{i=1}^n T^{-i}(U_k)\right) = \left(1 + \frac{1}{u_k}\right) \cdot u_k = 1 + u_k > 1$, which is not possible.

To finish the proof take any $x \in T^{-n}U_k \cap U_k$. This point fulfills both $x \in U_k$ and $T^n(x) \in U_k$, so

$$d(T^n(x), x) \le \operatorname{diam} U_k = r_k. \tag{3.5.14}$$

This implies

$$n^\beta d(T^n(x), x) \leq n^\beta \cdot r_k \leq \left(1 + \frac{1}{u_k}\right)^\beta \cdot r_k = \left(\frac{1}{u_k}\right)^\beta (1 + u_k)^\beta \cdot r_k \leq$$
$$\leq \left(\frac{c}{1-\varepsilon}\right)^\beta \cdot \frac{1}{r_k} \cdot (1+\varepsilon)^\beta \cdot r_k = \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^\beta \cdot c^\beta < c^\beta + u_0,$$

where the inequalities are implied in order by (3.5.14); (3.5.13); (3.5.12) & (3.5.7) and from the choice of ε — (3.5.4). But $x \in U_k \subset Y$, and we arrive at a contradiction. □

Chapter 4

Vanishing limit

This chapter gives a stronger result (than in the previous chapter) on the rate of recurrence and the rate on approach to a fixed point, i.e. we are interested in the limits:

$$\liminf_{r\to 0} \tau_r^x(x)\cdot\mu\big(B(x,r)\big) \qquad \text{and} \qquad \liminf_{r\to 0} \tau_r^y(x)\cdot\mu\big(B(y,r)\big).$$

We show that (under additional assumptions) these rates are as fast as possible, i.e. the limits are equal 0. We introduce some systems, for which the theorems are applicable. All proofs are at the end of this chapter.

4.1 Setting and results

Let (X,d) be a separable metric space and T a transformation preserving a Borel, probability measure μ. We also assume that the system (X,T,μ) is ergodic.

Recall that for brevity we shall often write B_r^y as as ball $B(y,r)$, also whenever the centre is fixed and obvious from the context we shall simply write B_r. Also $\tau_r^y(x) = \tau_{B(y,r)}(x)$ and we shall omit the superscript y whenever it does not cause confusion.

Definition 4.1. If T is non-singular with respect to a measure m, i.e. $m(A) = 0 \implies m(T(A)) = 0$, then (by Radon-Nikodym Thm.) there exists a m–integrable function ϕ (the inverse Jacobian) satisfying

$$m(T^{-1}(A)) = \int_A \phi dm \qquad \text{for every measurable } A \subset X.$$

If the function T is finite-to-one (or even countable-to-one), then for every measure $\mu \ll m$ we may define:

Definition 4.2. *The Perron-Frobenius* (transfer) *operator* $\mathcal{L}_\mu$ defined on $L^1(X,\mu)$ is given by the equation

$$\mathcal{L}_\mu(g)(x) = \sum_{y\in T^{-1}(x)} (\phi\circ T)(y)g(y).$$

Recall that $\mathcal{L}_\mu(\mathbb{1}) = \mathbb{1}$, because μ is T–invariant. Another useful property is that $\mu(T^{-1}A \cap B) = \displaystyle\int_A \mathcal{L}(\mathbb{1}_B)\, d\mu$.

For a precise introduction check e.g. [PU10, Ch.5].

Definition 4.3. We say that a dynamical system has an *exponential decay of correlations* in Lipschitz–continuous functions, if there exists $0 < \gamma < 1$ and a integrable function $C\colon X \to (0, +\infty)$ such that for all $g \in \mathrm{Lip}$, $n \geq 0$ and $x \in X$

$$\left|\mathcal{L}_\mu^n(g)(x) - \mu(g)\right| \leq C(x)\gamma^n ||g||_L, \tag{4.1.1}$$

where $||\cdot||_L$ is the usual norm in the space of Lipschitz–continuous functions.

Remark 1. We get analogous definitions to Def. 4.3, if g is a Hölder–continuous function (we take Hölder norm) or if g has bounded variation (denoted by BV–functions). (Norms are reminded in section 5.2, check 5.2.1 and 5.2.4.)

Remark 2. Usually this property is defined and proved with respect to Hölder–continuous functions, in which case a classic approximation argument gives the above for Lipschitz–continuous. Also any Lipschitz function f on a bounded interval has bounded variation and satisfies $||f||_{BV} \leq D||f||_L$ (where D is the length of the interval). So exponential decay for BV–functions gives exponential decay for Lipschitz–continuous functions as well.

Main results of this chapter follow.

Theorem 4.4 (Quick entrance). *If (X, T, μ) has exponential decay of correlations (for Lipschitz or Hölder or BV functions), the measure has doubling property at y and $\underline{d}_\mu(y) > 0$ then*

$$\liminf_{r\to 0} \tau_r^y(x) \cdot \mu\big(B(y,r)\big) = 0 \quad \textit{for } \mu\textit{–a.e. } x \in X. \tag{4.1.2}$$

Remark. We use the convention that $0 \cdot \infty = 0$, i.e. we ignore the (uninteresting) case when $y \notin \operatorname{supp}\mu$ and $\tau_r^y(x)$ may be $= +\infty$.

Theorem 4.5 (Quick return). *(X, T, μ) has exponential decay of correlations (as above), the measure has doubling property a.e. and $0 < \underline{d}_\mu(x) \leq \overline{d}_\mu(x) < +\infty$ a.e. Then*

$$\liminf_{r\to 0} \tau_r^x(x) \cdot \mu\big(B(x,r)\big) = 0 \quad \mu\textit{–a.e.} \tag{4.1.3}$$

The proofs of these theorems are at the end of this chapter.

Note. Actually, for functions of bounded variation the proofs are slightly easier, because the indicator function of a ball has bounded variation and does not need to be approximated by a Lipschitz continuous functions (check 4.3.17). The proof, however, is written in the general case.

4.2 Applications

Combining the above results with the notion of upper β–property (Def. 2.10) and Cor. 3.6 we arrive at a nice result:

Proposition 4.6. *Suppose that (X, T, μ) has exponential decay of correlations, the measure has upper β–property, doubling property at y and $0 < \underline{d}_\mu(y)$. Then*

$$\liminf_{n\to+\infty} n^{1/\beta} d\left(T^n(x), y\right) = 0 \quad \mu\text{–}a.e.$$

Moreover, if the measure has doubling property a.e. and $0 < \underline{d}_\mu(x) \le \overline{d}_\mu(x) < +\infty$ a.e., then

$$\liminf_{n\to+\infty} n^{1/\beta} d\left(T^n(x), x\right) = 0 \quad \mu\text{–}a.e. \tag{4.2.1}$$

That is a considerable generalisation of Theorem 4.2 from [Bos93], which proved the limit (4.2.1) only for $T(x) = m \cdot x \ (\mathrm{mod} 1)$ on $X = [0, 1)$.

Recall that if X is a subset of the Euclidean space, then using results from subsection 2.2 gives a clearer result.

Corollary 4.7. *Let X be a Borel subset of $\mathbb{R}^k$ such that the packing measure Π_β is σ–finite on X. T is a mapping on X preserving a Borel, probability measure μ with an exponential decay of correlations. The measure μ has doubling property and positive dimension $0 < \underline{d}_\mu(x)$ a.e. Then*

$$\liminf_{n\to\infty} n^{1/\beta} \cdot d(T^n(x), x) = 0 \tag{4.2.2}$$

for μ–almost every $x \in X$.

Proof. By Lemma 2.12, if Π_β is σ–finite, then the measure has upper β–property, which in turn gives that $\overline{d}_\mu(x) \le \beta$ a.e. This gives all the assumptions of Proposition 4.6. □

Remark. Actually, Cor. 4.7 is still true without assuming *doubling property*. However, the proof needs to be rewritten slightly: proving Lemma 4.14 using maximal ε–separated sets in $\mathbb{R}^k$ and using β–property instead of doubling in estimates just after that Lemma.

4.2.1 Examples

Classes of systems for which Corollary 4.7 may be applied include the following:

Example 4.8. Rational functions:

A hyperbolic rational function of degree ≥ 2 on the Riemann sphere has an invariant probability measure which is equivalent (up to a constant factor) to some β–Hausdorff measure, where $\beta = \mathrm{HD}(J_f)$. (see e.g. [Sul83]).

This gives both the doubling and β–property, hence we may apply Prop. 4.6.

Actually, the measure is also equivalent to the appropriate packing measure, so $\Pi_\beta(J(f)) < +\infty$, satisfying assumptions of Corollary 4.7.

Example 4.9. Maps on the interval:

A piecewise expanding transformation $f\colon [a,b] \to [a,b]$ with the function $\frac{1}{|f'(x)|}$ of bounded variation admits an acim (w.r.t. the Lebesgue measure) with a density of bounded variation. (see e.g. [BG97], Thm. 5.2.1)

The density can be redefined on a countable set to become lower semicontinuous and positive on an open set ([BG97], Thm. 8.1.2) and also bounded away from zero on the support.

If in addition the system is weakly mixing, then we have an exponential decay of correlations in the functions of bounded variation ([BG97], Thm 8.3.1), which certainly includes those Lipschitz functions that are needed in the proof.

This class of examples includes the Gauss transformation and the tent map, which is conjugate to the logistic transformation.

Example 4.10. Conformal IFS:

One can also apply the result to some conformal graph directed Markov systems (conformal iterated function systems). Definitions and necessary results may be found in [MU03b], conformal systems are introduced in chapter 4.

The systems have an appropriate conformal measure ([MU03b, Thm. 3.2.3], check also Lemma 4.2.2) and exhibit the decay of correlations (Thm. 2.4.6).

We only need to check two assumptions:

- First is the β–property. All finite conformal systems have it (Thm. 4.2.11) and also some infinite systems — those with finite packing measure of the limit set.

- The second is the doubling property. Again it is satisfied for all finite systems. (And also for some infinite ones [MU03a]).

However, by the Remark after Cor. 4.7 we do not need the doubling property. We could also extend the result to some finite parabolic IFS [MU03a, ch. 8], if the packing measure if finite (for the β–property). And Thm. 1.6 in [MU98] shows that if the limit set has dimension less or equal 1, then the packing measure is finite.

It should be added that if the dimension is strictly less than 1, then the appropriate Hausdorff measure of the limit set is equal to 0 and the statement can also be deduced from Boshernitzan's result.

4.3 Proofs of Theorems 4.4 and 4.5

First of all, we shall need two lemmas: the first proves that a series diverges, the second provides an estimation on partial sums of that series.

Lemma 4.11. *For any $\alpha > 0$, $A > 0$, $B \in \mathbb{R}$ we define a sequence:*

$$\begin{aligned} b_1 &= \alpha \\ b_{n+1} &= b_n + A\log_2(b_n) + B. \end{aligned}$$

If α is large enough, then the sequence goes to $+\infty$ and the series $\sum_{n=1}^{\infty} \frac{1}{b_n}$ diverges. Obviously, we could have taken any other base (greater than 1) of the logarithm.

Lemma 4.12. *Take a sequence defined as in Lemma 4.11 and fix $C > 0$ and $D > 1$. The sequence is terminated when $b_{N+1} \geq \omega = C \cdot D^{\alpha}$. Then we have*

$$\lim_{\alpha \to +\infty} \sum_{n=1}^{N} \frac{1}{b_n} = +\infty.$$

Proof of Lemma 4.11. The sequence b_n is strictly increasing for $\alpha > 2^{-B/A}$. We divide the set $[\alpha, +\infty)$ into subsets: $I_1 = [\alpha, 2\alpha)$, $I_2 = [2\alpha, 4\alpha) \ldots I_p = [2^{p-1}\alpha, 2^p\alpha)$. Observe that when $b_n \in I_p$, we have

$$b_{n+1} = b_n + A\log_2(b_n) + B \leq b_n + A\log_2\left(2^p\alpha\right) + B = b_n + pA + A\log_2\alpha + B. \quad (4.3.1)$$

This shows that the distance between the consecutive elements in I_p is at most $pA + A\log_2\alpha + B$. So the amount of the elements in I_p is at least

$$\#\{n\colon b_n \in I_p\} \geq \left\lfloor \frac{\text{length of } I_p}{\text{max. dist. between elements}} \right\rfloor \geq \frac{2^{p-1}\alpha}{pA + A\log_2\alpha + B} - 1. \quad (4.3.2)$$

The lemma is proved by the following inequalities:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{b_n} &\geq \sum_{p=1}^{\infty} \sum_{b_n \in I_p} \frac{1}{b_n} \geq \sum_{p=1}^{\infty} \#\{n\colon b_n \in I_p\} \cdot \min\{b_n\colon b_n \in I_p\} \geq \\ &\geq \sum_{p=1}^{\infty} \left(\frac{2^{p-1}\alpha}{pA + A\log_2\alpha + B} - 1 \right) \frac{1}{2^p\alpha} \geq \\ &\geq \sum_{p=1}^{\infty} \frac{1}{2\left(pA + A\log_2\alpha + B\right)} - \sum_{p=1}^{\infty} \frac{1}{2^p\alpha} = +\infty. \end{aligned}$$

□

Proof of Lemma 4.12. Proof follows as above. We have the family I_p of intervals, but in this case the family is finite, with the last set I_M given by the inequalities $2^M\alpha \leq \omega < 2^{M+1}\alpha$. This gives $M = \left\lfloor \log_2\left(\frac{\omega}{\alpha}\right) \right\rfloor$. Recall that $N = \#\{n\colon b_n < \omega\}$. Repeating the estimation we arrive at

$$\sum_{p=1}^{M} \frac{1}{2\left(pA + A\log_2\alpha + B\right)} - \sum_{p=1}^{M} \frac{1}{2^p\alpha}. \quad (4.3.3)$$

The second series is estimated easily:

$$-\sum_{p=1}^{M}\frac{1}{2^p\alpha} \geq -\sum_{p=1}^{\infty}\frac{1}{2^p\alpha} \geq -\frac{1}{\alpha}. \tag{4.3.4}$$

Before estimating the first part let us calculate M

$$\begin{aligned} M = \left[\log_2\left(\frac{\omega}{\alpha}\right)\right] &\geq \log_2\omega - \log_2\alpha - 1 = \log_2\left(C\cdot D^{\alpha}\right) - \log_2\alpha - 1 = \\ &= \alpha\log_2 D - \log_2\alpha + \log_2 C - 1. \end{aligned}$$

The second estimation uses integrals and the estimate on M:

$$\begin{aligned} \sum_{p=1}^{M}\frac{1}{2\left(pA + A\log_2\alpha + B\right)} &\geq \frac{1}{2A}\int_1^M \frac{1}{p+\log_2\alpha+\frac{B}{A}}\,dp \geq \\ &\geq \frac{1}{2A}\ln\left(M+\log_2\alpha+\frac{B}{A}\right) - \frac{1}{2A}\ln\left(1+\log_2\alpha+\frac{B}{A}\right) \geq \\ &\geq \frac{1}{2A}\ln\left(\alpha\log_2 D+\log_2 C - 1+\frac{B}{A}\right) - \frac{1}{2A}\ln\left(1+\log_2\alpha+\frac{B}{A}\right). \end{aligned}$$

In the last inequality the first term is of order $\ln(\alpha)$ and the second $\ln\ln(\alpha)$. So for certain $U, V, W > 0$ we may write

$$\sum_{p=1}^{M}\frac{1}{2\left(pA + A\log_2\alpha + B\right)} \geq U\log\alpha - V\log\log\alpha - W. \tag{4.3.5}$$

Combining estimates (4.3.4) and (4.3.4) and returning to the sequence b_n we get

$$\sum_{n=1}^{N}\frac{1}{b_n} \geq U\log\alpha - V\log\log\alpha - W - \frac{1}{\alpha} \overset{\alpha\to+\infty}{\longrightarrow} +\infty, \tag{4.3.6}$$

thus finishing the proof. $\square$

Proof of Theorem 4.4. First, let us fix $y \in X$ and for brevity set $f_r(x) = \tau_r^y(x)\cdot\mu\big(B(y,r)\big)$. All the balls will be centered at y so we may write B_r instead of $B(y,r)$. Proving 4.1.2 is equivalent to proving that

$$\mu\left(\{x \in X\colon \forall_{\varepsilon>0}\,\forall_{\rho>0}\,\exists_{r<\rho}\; f_r(x) \leq \varepsilon\}\right) = 1.$$

Taking the complement we need

$$\mu\left(\{x \in X\colon \exists_{\varepsilon>0}\,\exists_{\rho>0}\,\forall_{r<\rho}\; f_r(x) \geq \varepsilon\}\right) = 0. \tag{4.3.7}$$

Let us define sets

$$A_r^{\varepsilon} = \big\{f_r(x) \geq \varepsilon\big\} = \left\{\tau_r(x) \geq \frac{\varepsilon}{\mu(B_r)}\right\}. \tag{4.3.8}$$

Observe that is suffices to prove that

$$\mu\left(\bigcap_{r<\rho} A_r^\varepsilon\right) = 0 \quad \text{for any fixed } \varepsilon \text{ and } \rho. \tag{4.3.9}$$

Indeed, if (4.3.7) were false, i.e. the set would have positive measure, then there would exist global constants $\varepsilon > 0$ and $\delta > 0$ such that $\mu\left(\{x \in X \colon \forall_{r<\rho}\, f_r(x) \geq \varepsilon\}\right) > 0$. That would make (4.3.9) false. From now on let us fix ε and ρ.

A point belongs to A_r^ε, if its trajectory omits B_r for some time. Precisely speaking:

$$x \in A_r^\varepsilon \iff Tx, T^2x, \dots T^l x \notin B_r, \quad \text{where } l = \left[\frac{\varepsilon}{\mu(B_r)}\right]. \tag{4.3.10}$$

This gives

$$A_r^\varepsilon = \bigcap_{k=1}^{\left[\frac{\varepsilon}{\mu(B_r)}\right]} T^{-k} B'_r, \quad \text{where } B'_r = X \setminus B_r. \tag{4.3.11}$$

Thus the set in (4.3.9) may be written as

$$\bigcap_{r<\rho} \bigcap_{k=1}^{\left[\frac{\varepsilon}{\mu(B_r)}\right]} T^{-k} B'_r. \tag{4.3.12}$$

By further decreasing ρ if needed, we may guarantee that $\left[\frac{\varepsilon}{\mu(B_r)}\right] \geq 1$ so the second intersection (indexed by k) is over a non-empty set. Changing the order of intersecting gives

$$\bigcap_{k=1}^{+\infty} \bigcap_{r \leq r_k} T^{-k} B'_r, \tag{4.3.13}$$

where $r_k = \sup\left\{r \colon \frac{\varepsilon}{\mu(B_r)} \geq k\right\}$.

The sets B'_r and likewise $T^{-k}B'_r$ form an increasing family of sets as $r \searrow 0$ (for k fixed), so $\bigcap_{r \leq r_k} T^{-k} B'_r = T^{-k} B'_{r_k}$; leaving us to prove that

$$\mu\left(\bigcap_{k=1}^{+\infty} T^{-k} B'_{r_k}\right) = 0. \tag{4.3.14}$$

Firstly, we need an estimate on $\mu(B_{r_k})$. Definition of r_k shows that

$$\frac{\varepsilon}{\mu(B_{2r_k})} < k, \quad \text{because } \ 2r_k > r_k \tag{4.3.15}$$

and combining this with doubling property, we get (writing $\sigma = \sigma(y)$)

$$\frac{\varepsilon}{k} < \mu(B_{2r_k}) \leq \sigma \mu(B_{r_k}). \tag{4.3.16}$$

So $\mu(B_{r_k}) > \frac{\varepsilon}{k\sigma}$ and we know that the intersecting sets in equation (4.3.14) have the measure most $1 - \frac{\varepsilon}{k\sigma}$.

If the events $T^{-k}B'_{r_k}$ were independent, the result would follow from the Borel-Cantelli lemma. In this setting instead of independence we will use the exponential decay of correlations.

We will need to approximate the characteristic function of the complement of a ball. Let us set

$$\phi_{r,\kappa}(t) = \begin{cases} 0 & \text{for } 0 \le t \le \frac{r}{2} \\ \frac{2}{r}t - 1 & \text{for } \frac{r}{2} \le t \le r \\ 1 & \text{for } t \ge r \end{cases}, \tag{4.3.17}$$

and the approximating function is $g_r(z) = \phi_r(d(y,z))$. It has Lipschitz constant equal to $\frac{2}{r}$ and $||g_r||_L \le \frac{3}{r}$ (for $r \le 1$).

Recall that by the definition of the transfer operator we have

$$\mu\left(E \cap T^{-n}F\right) = \int_F \mathcal{L}_\mu^n(\mathbb{1}_E)\, d\mu. \tag{4.3.18}$$

Now we shall estimate the measure in (4.3.14) by taking just a subset of k_n with gaps between them big enough, so that their intersections have a small measure by the decay of correlations. The subset (k_n) will be defined later, we start with the estimations.

$$\begin{aligned} \mu\left(\bigcap_{n=1}^{+\infty} T^{-k_n}B'_{r_{k_n}}\right) &= \mu\left(T^{-k_1}B'_{r_{k_1}} \cap \bigcap_{n=2}^{+\infty} T^{-k_n}B'_{r_{k_n}}\right) = \\ &= \mu\left(B'_{r_{k_1}} \cap \bigcap_{n=2}^{+\infty} T^{-k_n+k_1}B'_{r_{k_n}}\right) = \\ &= \mu\left(B'_{r_{k_1}} \cap T^{-k_2+k_1}\bigcap_{n=2}^{+\infty} T^{-k_n+k_2}B'_{r_{k_n}}\right) = \end{aligned} \tag{4.3.19}$$

we used preserving of μ by T. Now set $E = B'_{r_{k_1}}$, $F = \bigcap_{n=2}^{+\infty} T^{-k_n+k_2}B'_{r_{k_n}}$ and use the Perron-Frobenius operator

$$\begin{aligned} &= \int_F \mathcal{L}_\mu^{k_2-k_1}(\mathbb{1}_E)\, d\mu \le \int_F \mathcal{L}_\mu^{k_2-k_1}(g_{r_{k_1}})\, d\mu \le \\ &\le \mu(F)\cdot\left(\mu(g_{r_{k_1}}) + \int_F C(x)d\mu(x)\cdot\gamma^{k_2-k_1}\cdot\frac{3}{r_{k_1}}\right) \le \\ &\le \mu\left(B'_{r_{k_2}} \cap \bigcap_{n=3}^{+\infty} T^{-k_n+k_2}B'_{r_{k_n}}\right)\left(\mu(g_{r_{k_1}}) + C\gamma^{k_2-k_1}\frac{3}{r_{k_1}}\right) \end{aligned}$$

for $C = \int_X C(x)d\mu(x)$. Following inductively we get an estimation:

$$\mu\left(\bigcap_{n=1}^{+\infty} T^{-k_n}B'_{r_{k_n}}\right) \le \prod_{n=1}^{+\infty}\left(\mu(g_{r_{k_n}}) + C\gamma^{k_{n+1}-k_n}\frac{3}{r_{k_n}}\right). \tag{4.3.20}$$

From the definition of g_r and the doubling property (recall that $\sigma = \sigma(y)$ is the doubling constant) we get another estimation.

$$\mu(g_{r_k}) \leq \mu(B'_{\frac{1}{2}r_k}) \leq 1 - \mu(B_{\frac{1}{2}r_k}) \leq 1 - \frac{1}{\sigma}\mu(B_{r_k}) \leq 1 - \frac{\varepsilon}{k\sigma^2}. \tag{4.3.21}$$

Now we define k_n (inductively) in such a way that it satisfies the following condition:

$$C\gamma^{k_{n+1}-k_n}\frac{3}{r_{k_n}} \leq \frac{\varepsilon}{2\sigma^2 k_n}. \tag{4.3.22}$$

Leaving only $\gamma^{k_{n+1}}$ on the left side and taking a logarithm gives a recurrence relation

$$k_{n+1} \geq k_n + \log_\gamma\left(\frac{\varepsilon}{6C\sigma^2}\right) - \log_\gamma(k_n) + \log_\gamma(r_{k_n}), \tag{4.3.23}$$

we take the smallest k_{n+1} satisfying that inequality, and $k_1 = 1$.

Finally, using the definition of the sequence k_n we arrive at an estimation:

$$\begin{aligned} \mu\left(\bigcap_{k=1}^{+\infty} T^{-k}B'_{r_k}\right) &\leq \mu\left(\bigcap_{n=1}^{+\infty} T^{-k_n}B'_{r_{k_n}}\right) \leq \\ &\leq \prod_{n=1}^{+\infty}\left(\mu(g_{r_{k_n}}) + C\gamma^{k_{n+1}-k_n}\frac{3}{r_{k_n}}\right) \leq \\ &\leq \prod_{n=1}^{+\infty}\left(1 - \frac{\varepsilon}{\sigma^2 k_n} + \frac{\varepsilon}{2\sigma^2 k_n}\right) \leq \\ &\leq \prod_{n=1}^{+\infty}\left(1 - \frac{\varepsilon}{2\sigma^2 k_n}\right). \end{aligned} \tag{4.3.24}$$

The last product is equal to 0, if the sum $\sum_{n=1}^{+\infty}\frac{1}{k_n} = +\infty$, which we shall state and prove as a separate lemma. So the proof of Thm. 4.4 ends by using Lemma 4.13. ∎

Lemma 4.13. *Under assumptions of Thm. 4.4 let (k_n) be a sequence defined by the recurrence relation (4.3.23). Then the sum $\sum_{n=1}^{+\infty}\frac{1}{k_n}$ diverges.*

Proof of Lemma 4.13. We want to show that we may use Lemma 4.11 for sequence k_n. To do that, let us clean up the formula:

$$\begin{aligned} k_1 &= 1 \\ k_{n+1} &= k_n + P - \log_\gamma(k_n) + \log_\gamma(r_{k_n}), \qquad P \text{ is large but fixed.} \end{aligned}$$

We assumed that $\underline{d}(y) > 0$, so there exists $W > 0$ such that for all sufficiently small r we have

$$\frac{\ln \mu(B_r)}{\ln r} \geq W$$
$$\ln \mu(B_r) \leq W \ln r.$$

On the other hand the definition of r_k and doubling property (4.3.16) give

$$\frac{\varepsilon}{\sigma k} \leq \mu(B_{r_k}) < \frac{\varepsilon}{k}.$$

Combining the above, we get:

$$\begin{aligned} \ln \frac{\varepsilon}{\sigma k} &\leq \ln \mu(B_{r_k}) \leq W \ln r_k \\ \log_\gamma \frac{\varepsilon}{\sigma k} &\geq W \log_\gamma r_k \qquad \text{for any } \gamma < 1 \\ \log_\gamma r_k &\leq -\frac{1}{W} \log_\gamma(k) + \frac{1}{W} \log_\gamma \frac{\varepsilon}{\sigma}. \end{aligned}$$

We use this estimate in the definition of k_{n+1} obtaining

$$k_{n+1} \leq k_n + \widetilde{P} - \left(1 + \frac{\kappa}{W}\right) \log_\gamma(k_n). \tag{4.3.25}$$

This is precisely the type of sequence introduced in Lemma 4.11 for $A = -\left(\frac{W+\kappa}{W \log_2(\gamma)}\right) > 0$ and $B = \widetilde{P}$. It remains to use Lemma 4.11. $\square$

Proof of Theorem 4.5. First of all, for any $\lambda > 0$ we may find a set G_λ and constants d_λ, D_λ, σ_λ and $\widehat{\rho}_\lambda$ such that

$$\begin{aligned} &\mu(G_\lambda) \geq 1 - \lambda \\ &\text{and for all } x \in G_\lambda \\ &\underline{d}_\mu(x) \geq 2d_\lambda \\ &\overline{d}_\mu(x) \leq \frac{1}{2} D_\lambda \\ &\left.\begin{array}{rcl} \sigma(x) & \leq & \sigma_\lambda \\ \rho(x) & \geq & \widehat{\rho}_\lambda \end{array}\right\} \text{from the doubling property.} \end{aligned}$$

We fix $\lambda > 0$ and we shall omit the subscripts from now on: writing $\widehat{\rho}$ instead of $\widehat{\rho}_\lambda$ etc. Observe that the estimate on the lower and upper pointwise dimension gives

$$r^d \geq \mu\big(B(x,r)\big) \geq r^D, \tag{4.3.26}$$

for $x \in G$ and all sufficiently small r. Shrinking $\widehat{\rho}$ if necessary, we may assume this for all $r \leq \widehat{\rho}$ independently of x.

We will prove that for x in the subset of G of measure only slightly smaller than $\mu(G)$ the limit (4.1.3) is as small as needed. Precisely speaking, we will prove that

$$\mu\left(\left\{x \in G\colon \forall_{\varepsilon>0}\forall_{\rho>0}\exists_{r\leq\rho}\tau_r^x(x)\cdot\mu\big(B(x,r)\big)\leq\varepsilon\right\}\right)\geq 1-2\lambda. \tag{4.3.27}$$

As λ can be taken arbitrarily close to zero this will end the proof. Also it suffices to prove this only for $\rho \leq \widehat{\rho}$ and from now on we will assume that. As in the previous proof we fix $\varepsilon > 0$ and $\rho > 0$ and it will suffice to prove that

$$\mu\left(\left\{x \in G\colon \forall_{r\leq\rho}\tau_r^x(x)\cdot\mu\big(B(x,r)\big)\geq\varepsilon\right\}\right)\leq\lambda. \tag{4.3.28}$$

Fix $R = R(\varepsilon,\rho,\lambda) > 0$, such that $\mu(B_R(x)) \leq \varepsilon$ for $x \in G$; this is possible because of (4.3.26). Let us cover the set G with balls $B(x_i, R)$ centered at points of G. Also, we need the covering to have the property that $\sum_i \mu\left(B(x_i, 2R)\right) \leq \beta$ where β does not depend on R.

Lemma 4.14. *Under assumptions of Thm. 4.5 and for any R such a covering exists.*

Proof of Lemma. We take a covering $\mathcal{A}_{\frac{2}{5}R}$ centered at G consisting of balls of diameter $\frac{4}{5}R$. Using the Vitali $5r$–lemma we get a disjoint subset $\mathcal{A}'$ such that $\mathcal{A}'_{2R}$ is a covering. Because $\sigma(x)$ is bounded on G we can easily estimate

$$\begin{aligned}\sum_{i\in\mathcal{A}'}\mu(B(x_i,2R)) &\leq \sigma\sum\mu(B(x_i,R)\leq\sigma^2\sum\mu(B(x_i,\frac{1}{2}R))\leq\\ &\leq\sigma^3\sum\mu(B(x_i,\frac{1}{4}R))\leq\sigma^3\sum\mu(B(x_i,\frac{2}{5}R))\leq\\ &\leq\sigma^3\mu(X)=\sigma^3=\beta.\end{aligned}$$

Notice that β depends on σ and in turn on λ, but this does not pose a problem. □

Take any point $x \in G$ and assume $r > R$. There exists an x_i such that $d(x, x_i) < R$. Moreover, we have $B\big(x_i, r - d(x_i, x)\big) \subset B(x, r) \subset B\big(x_i, r + d(x_i, x)\big)$ for any $r > R$. From the condition on x_i we get $B(x_i, r - R) \subset B(x, r) \subset B(x_i, r + R)$.

Using monotonicity of the measure and the return time (observe that if $C \subset D$, then $\tau_C(x) \geq \tau_D(x)$ and $\mu(C) \leq \mu(D)$) we may write

$$\begin{aligned}\tau_r^x(x)\mu\big(B(x,r)\big) &\leq \tau_{r-R}^{x_i}(x)\mu\big(B(x_i,r+R)\big) =\\ &= \tau_{r-R}^{x_i}(x)\mu\big(B(x_i,r-R)\big)\cdot\frac{\mu\big(B(x_i,r+R)\big)}{\mu\big(B(x_i,r-R)\big)}.\end{aligned}$$

Now let us take $r > 2R$ and denote $\delta := r - R$. Observe that $R < \delta$. Then the above takes the form

$$\begin{aligned}\tau_r^x(x)\mu\big(B(x,r)\big) &\leq \tau_\delta^{x_i}(x)\mu\big(B(x_i,\delta)\big)\cdot\frac{\mu\big(B(x_i,\delta+2R)\big)}{\mu\big(B(x_i,\delta)\big)}\leq\\ &\leq \tau_\delta^{x_i}(x)\mu\big(B(x_i,\delta)\big)\cdot\frac{\mu\big(B(x_i,3\delta)\big)}{\mu\big(B(x_i,\delta)\big)}\leq\\ &\leq \tau_\delta^{x_i}(x)\mu\big(B(x_i,\delta)\big)\sigma^2.\end{aligned}$$

For brevity denote $B_i = B(x_i, R)$. We will use the above inequality in estimating the set from (4.3.28):

$$
\begin{aligned}
&\mu\left(\left\{x \in G \colon \forall_{r\leq\rho}\tau_r^x(x)\mu\big(B(x,r)\big) \geq \varepsilon\right\}\right) \leq \\
&\quad\leq \mu\left(\left\{x \in G \colon \forall_{2R\leq r\leq\rho}\tau_r^x(x)\mu\big(B(x,r)\big) \geq \varepsilon\right\}\right) \leq \\
&\quad\leq \sum_i \mu\left(B_i \cap \left\{x \in G \colon \forall_{2R\leq r\leq\rho}\tau_r^x(x)\mu\big(B(x,r)\big) \geq \varepsilon\right\}\right) \leq \\
&\quad\leq \sum_i \mu\left(B_i \cap \left\{x \in B_i \colon \forall_{2R\leq r\leq\rho}\tau_r^x(x)\mu\big(B(x,r)\big) \geq \varepsilon\right\}\right) \leq \\
&\quad\leq \sum_i \mu\left(B_i \cap \left\{x \in B_i \colon \forall_{R\leq\delta\leq\rho-R}\tau_\delta^{x_i}(x)\mu\big(B(x_i,\delta)\big) \geq \frac{\varepsilon}{\sigma^2}\right\}\right) =: Z.
\end{aligned}
$$

And now we recall the proof of the previous theorem and observe that the expression above be bounded exactly as in (4.3.12)

$$
Z \leq \sum_i \mu\left(B_i \cap \bigcap_{R\leq\delta\leq\rho-R} \bigcap_{k=1}^{\left[\frac{\varepsilon}{\mu(B_\delta^i)}\right]} T^{-k}(B_\delta^i)'\right), \tag{4.3.29}
$$

where $B_\delta^i = B(x_i, \delta)$ and as before $B' = X \setminus B$. (Now we may use two notations for one set: $B_i = B_R^i$.) Recall that R was defined so that $\mu(B_R^i) \leq \varepsilon$. This makes the second set of indices (over k) non-empty. Making the transformations exactly as in the proof of Thm. 4.4 (equations (4.3.12) to (4.3.14)), we arrive at needing to estimate

$$
\sum_i \mu\left(B_i \cap \bigcap_{k=1}^{\omega} T^{-k}(B_{r_k}^i)'\right) \leq \lambda, \qquad \text{where } \omega = \left[\tfrac{\varepsilon}{\mu(B_R^i)}\right]. \tag{4.3.30}
$$

Now we will need another approximating Lipschitz-continuous function:

$$
\phi_{r,\kappa}(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq r \\ 2 - \frac{t}{r} & \text{for } r \leq t \leq 2r \\ 0 & \text{for } t \geq 2r \end{cases}, \tag{4.3.31}
$$

and the function we will use is $h_r^i(z) = \phi(d(z, x_i))$. It has similar properties as the previously used function $g(z)$, $||h_r||_L \leq \frac{3}{r}$ for $r \leq 1$.

Fix $\alpha \in \mathbb{N}$. We will need another assumption on R (equivalently on ω). On one hand it will have to be small enough for the measure of the intersection to be small. And on the other the ball cannot be to small for the estimate below to hold (γ is the constant from the decay of correlations).

$$
C\gamma^\alpha \frac{3}{R} \leq \mu\left(B_{2R}^i\right) \text{ for all } i. \tag{4.3.32}
$$

Using the ball estimates (4.3.26) we may write this more precisely

$$
R\mu\left(B_{2R}^i\right) \geq R(2R)^D \geq C\gamma^\alpha. \tag{4.3.33}
$$

From now on we will assume that (the second line defines $\widetilde{\gamma} = \gamma^{\frac{1}{D+1}}$ and $\widetilde{C}$ as the rest)

$$R \geq (3C)^{\frac{1}{D+1}} 2^{\frac{-D}{D+1}} \gamma^{\frac{\alpha}{D+1}} =: \widetilde{C} \cdot \widetilde{\gamma}^{\alpha}. \tag{4.3.34}$$

Using the definition of ω (4.3.30) and the ball estimate again (4.3.26)

$$\omega \leq \frac{\varepsilon}{\mu(B_R^i)} \leq \frac{\varepsilon}{R^D} \leq \widehat{C} \cdot \widehat{\gamma}^{-\alpha}, \tag{4.3.35}$$

where $\widehat{\gamma} = \widetilde{\gamma}^D$ and $\widehat{C} = \dfrac{\varepsilon}{\widetilde{C}^D}$.

As before we will use the Perron-Frobenius operator to estimate the measure. This time, however, we need an initial *jump* to get some independence between B_i and $T^{-k}(B_{r_1}^i)$. This is where we will use α.

$$\begin{aligned}\mu\left(B_i \cap \bigcap_{k=1}^{\omega} T^{-k}(B_{r_k}^i)'\right) &\leq \mu\left(B_i \cap \bigcap_{k=\alpha}^{\omega} T^{-k}(B_{r_k}^i)'\right) = \\ &= \mu\left(B_i \cap T^{-\alpha} \bigcap_{k=\alpha}^{\omega} T^{-k+\alpha}(B_{r_k}^i)'\right) =\end{aligned}$$

Set $F = \bigcap_{k=\alpha}^{\omega} T^{-k+\alpha}(B_{r_k}^i)'$ and use the transfer operator (h_R^i was defined under (4.3.31)):

$$\begin{aligned}&= \int_F \mathcal{L}^{\alpha}(\mathbb{1}_{B_i})(x)d\mu \leq \int_F \mathcal{L}(h_R^i)(x)d\mu \leq \\ &\leq \mu\left(\bigcap_{k=\alpha}^{\omega} T^{-k+\alpha}(B_{r_k}^i)'\right) \cdot \left(\mu(h_R^i) + C\frac{3}{R}\gamma\right) \leq \\ &\leq \mu\left(\bigcap_{k=\alpha}^{\omega} T^{-k+\alpha}(B_{r_k}^i)'\right)\left(\mu(B_{2R}^i) + C\frac{3}{R}\gamma\right) \leq \\ &\leq \mu\left(\bigcap_{k=\alpha}^{\omega} T^{-k+\alpha}(B_{r_k}^i)'\right)\Big(2\mu(B(x_i, 2R)\Big),\end{aligned}$$

where the last inequality used (4.3.32).

Summing these estimates over i we get

$$\sum_i \mu\left(B_i \cap \bigcap_{k=1}^{\omega} T^{-k}(B_{r_k}^i)'\right) \leq \mu\left(\bigcap_{k=\alpha}^{\omega} T^{-k+\alpha}(B_{r_k})'\right) \cdot 2\beta. \tag{4.3.36}$$

It remains to prove that the set on the right–hand side has a measure as small as needed. We proceed as in the previous proof, defining the sequence k_n, but this time stopping when $k_{N+1} > \omega$ (so this is a finite sequence):

$$\begin{aligned}k_1 &= \alpha \\ k_{n+1} &= k_n + P - \log_{\gamma}(k_n + 1) + \log_{\gamma}(r_{k_n}).\end{aligned}$$

Estimations of the measure of the intersection are exactly as in the proof of Thm. 4.4: starting with (4.3.19), defining the sequence k_n identically and estimating up to (4.3.24). It remains to use lemma 4.13 to clear up the definition k_n, i.e. say that $\log_\gamma(r_{k_n}) \approx \log_\gamma(k_n)$.

Finally, lemma 4.12 shows that the sum $\displaystyle\sum_{n=1}^{N} \frac{1}{k_n} \to +\infty$, so the measure of the intersection is as small as needed. This ends the proof. $\square$

Chapter 5

Random systems

This chapter is devoted to the *random* dynamical systems. In the first section we lay some theoretical and historical groundwork. We recall some known results and provide a brief introduction to the topic.

Then we gives the precise settings in which the recurrence is examined. The definitions and examples come from works of J. Buzzi and a paper by V. Mayer, B. Skorulski and M. Urbański.

Finally we give the results about recurrence in such system and their proofs end this chapter.

5.1 Introduction

Random dynamical systems is field of study attracting more and more interest in the recent years.

If one applies the theory of dynamical systems to a physical phenomena, then in most cases there surfaces a problem of small errors — from approximations, observations or simply some unaccounted-for noise. In many cases those problems are dismissed by saying that those small perturbations cannot dominate on the predicted behaviour of the system. The field of random systems verifies such statements formally.

There are several different ways of introducing randomness into dynamical systems. We show a few of them. As mentioned, this section is just an introduction and contains only already known facts and results; by no means is this extensive or thorough.

5.1.1 Dispersion

Let us take a dynamical system (T, X) and at every step $x \mapsto T(x)$ the target point is dispersed around $T(x)$ given by some probability distribution $\mathbb{Q}$. This is a Markov process and under some assumptions it has an invariant measure μ_Q. A natural question is: what can we tell about μ_Q? And what happens if this perturbation become smaller and smaller?

For example, this is the precise setting introduced in [BG97]:
Let $\mathbb{Q}_\varepsilon(x, A)$ be a family such that:

- $\mathbb{Q}_\varepsilon(x, \cdot)$ is a probability measure for any x,
- $\mathbb{Q}_\varepsilon(\cdot, A)$ is a measurable function for any A.

Set $P_\varepsilon(x, A) = \mathbb{Q}_\varepsilon(T(x), A)$ and let F_ε be a Markov process given by the transfer function P_ε. If we assume that P_ε tends to a single point distribution P_0, we call this *small stochastic perturbation*. Such defined F_ε is a stochastic perturbation of the transformation T. Under mild conditions (e.g. the Doeblin condition /or a variant of it/, though it usually gives also ergodicity) F_ε has an invariant measure, where invariance means that

$$\mu_\varepsilon(A) = \int P_\varepsilon(x, A)\, d\mu_\varepsilon(x).$$

Then the authors prove existence of the limit measure $\mu_\varepsilon \rightharpoonup \mu_0$ and some good spectral properties of the Perron–Frobenius operator.

In a similar setting Baladi and Young [BY93] prove the following.
An expanding $\mathcal{C}^r$ map T on a $\mathcal{C}^\infty$ compact, connected Riemannian manifold without boundary, when given a small stochastic perturbation (of a certain type, check the cited paper for specifics), has these properties in the space of $\mathcal{C}^{r-1}$ functions:
($||\cdot||$ is the typical $\mathcal{C}^{r-1}$ norm, ϕ and ψ are in this space)

- μ_ε tends to μ_0 — this is often called *stochastic stability*,
- τ_ε tends to τ_0, where τ_0 is *the rate of decay of correlations* for unperturbed map T, i.e. the smallest number for which the following holds:

  $$\left| \int (\phi \circ T^n) \cdot \psi\, d\mu_0 - \int \phi\, d\mu_0 \int \psi\, d\mu_0 \right| \le C\tau_0^n,$$

  where $C = C(||\phi||, ||\psi||)$ for all ϕ and ψ and all n;

  and the rate τ_ε for the random map is defined (again for all ϕ and ψ and C as above)

$$\left| \int \left(\int \phi(y) P_\varepsilon^n(x, dy) \right) \cdot \psi(x)\, d\mu_\varepsilon(x) - \int \phi\, d\mu_\varepsilon \int \psi\, d\mu_\varepsilon \right| \le C\tau_\varepsilon^n.$$

5.1.2 Random mappings

A different (though, in many cases equivalent to the previous one) way of introducing randomness to a system is by changing the map at every step.

As a trivial example take two maps $T_1 = x + 1$ and $T_2 = x - 1$ and choose T_1 or T_2 independently. This obviously leads to a random walk on a real line.

The interesting theory will surface, if the consecutive choices are not entirely random (independent of each other). Let us now formalize the concept.

Definition 5.1. Let us take a probability space $(\Omega, \mathbb{P})$, its ergodic automorphism S and a family of maps $T_\omega \colon X \to X$ (for $\omega \in \Omega$).

A *random dynamical system* is the skew-product:

$$\begin{aligned} F \colon \Omega \times X &\to \Omega \times X \\ (\omega, x) &\mapsto (S\omega, T_\omega(x)). \end{aligned}$$

Let us denote the trajectory of the point as $T^n_\omega(x) = T_{S^{n-1}\omega} \circ \ldots \circ T_{S\omega} \circ T_\omega(x)$.

We will also need some measures; there are (again) different approaches to this:

a) Simply assume that the skew-product $(F, \Omega \times X)$ has an invariant probability measure η; the projection of this measure is also invariant, i.e. define $\pi_\Omega(\omega, x) = \omega$ and denote the pushforward measure $\mathbb{P} = (\pi_\Omega)_*\eta$. Now the invariance $F_*(\eta) = \eta$ gives $S_*(\mathbb{P}) = \mathbb{P}$.

b) We are given a certain measure $\mathbb{P}$ and also an invariant measure on the skew-product. Then we should check whether the invariant measure on the entire system η satisfies $\mathbb{P} = (\pi_\Omega)_*\eta$.

c) If S is not an automorphism, but a one–side shift, then quite often the invariant measure η is a product measure: $\eta = \mathbb{P} \otimes \mu$; in this case μ is usually called stationary measure. This includes the situation when the maps are chosen independently with identical distribution. And this justifies the name, because (for i.i.d. random maps) this measure μ is the stationary measure for the Markov process equivalent to this random system (check also previous subsection).

d) There is also a more general approach: the random part is given a priori as an ergodic map S and measure $\mathbb{P}$ and we find a family of measures μ_ω, which is T_ω invariant, which means $(T_\omega)_*\mu_\omega = \mu_{S\omega}$. The measures can then be recovered as $\mu(A) = \int \mu_\omega(A)\, d\mathbb{P}(\omega)$ and $\eta(Z) = \int \mu_\omega(Z \cap \{\omega\} \times X)\, d\mathbb{P}(\omega)$. (There may be some problems with measurability, check e.g. [MSU08] for details). Note that this is usually not a product measure. We will investigate this setting in a bit more detail.

Definition 5.2. We will call a random dynamical system *ergodic* and *invariant*, if in the deterministic skew-product system $(\Omega \times X, F, \eta)$ the measure η is ergodic and invariant w.r.t. F.

Note. It is also possible to change the spaces X on every step, so $T_\omega \colon X_\omega \to X_{S\omega}$, check [MSU08] for details. However, it is difficult to obtain a meaningful definition of recurrence in such a situation. Sometimes, those sets X_ω have a natural correspondence between them; then we may treat recurrence to a point as approaching the set of those points (related to our starting point). We will not pursue this here.

Remark. Later, in the proofs, we will sometimes omit the subscript (or superscript) ω where it should not cause confusion. Also the placement of ω symbol may vary.

5.1.3 Operators

Let us assume that we have a random dynamical system (Def. 5.1) and that X is a metric space. We also have a family of measures μ_ω on X and those measures are all Borel ($\mathcal{F}_\omega = \mathcal{F}$).

- In many cases there exists a common reference measure m such that $\mu_\omega \ll m$; denote the density $d\mu_\omega = h_\omega dm$.
- In others, there exists another family ν_ω of conformal measures and $d\mu_\omega = h_\omega d\nu_\omega$.

Remark. The main problem with the *presentation* of our results regarding the random systems will be combining the two cases. The ideas and methods of the proofs are exactly the same, but the different measures force us to write substantially different expressions. We will try to avoid this whenever possible and write the equations in such a way that will work in both situations.

In such situations (similarly to the deterministic case — sec. 4.1) we may introduce the transfer operator (or rather a family of operators).

We omit some technical details here (assuming non-singularity of the reference measure leading to the definition of the Jacobian J_ω; assuming the maps T_ω to be countable-to-one). As before, for a precise introduction of the Perron-Frobenius operator we suggest e.g. [PU10, Ch.5].

The precise settings, assumptions and known results are in the next sections.

Definition 5.3. *The Perron-Frobenius* (transfer) *operator* is given by

$$\mathcal{L}_\omega(g)(x) = \sum_{y \in T_\omega^{-1}(x)} g(y) J_\omega^{-1}(y).$$

Those operators fix the family of densities (with respect to either the measure m or the measures ν_ω) and acts as a shift on them, i.e.

$$\mathcal{L}_\omega^n(h_\omega) = h_{S^n\omega}.$$

We want to define (as in the deterministic case) the decay of correlations. This time, however, we will use the correlation function instead of the *P–F operator*, simply because the definition is clearer.

Definition 5.4. The random correlation function is *(note the different measures)*

$$\mathrm{Corr}_\omega(\phi, \psi, n) := \int_X \phi \cdot \psi \circ T_\omega^n \, d\mu_\omega - \int_X \phi \, d\mu_\omega \int_X \psi \, d\mu_{S^n\omega}, \tag{5.1.1}$$

where ϕ and ψ are maps $X \to \mathbb{R}$.

Definition 5.5. We say that a random, measure preserving dynamical system has *ω-exponential decay of correlations* in Lipschitz–continuous functions, if for a fixed $\omega \in \Omega$ there exists $0 < \gamma < 1$ and $C_\omega < +\infty$ such that for all $g \in \mathrm{Lip}$, $n \geq 0$, $x \in X$ and all $\xi = S^k\omega$ $(k \in \mathbb{N})$

$$|\mathrm{Corr}_\xi(\phi, \psi, n)| \leq C_\omega ||\phi||_L ||\psi||_\infty \gamma^n, \tag{5.1.2}$$

where $|| \cdot ||_L$ is the usual norm in the space of Lipschitz–continuous functions.

Remark 1. Observe that C_ω may behave 'wildly' when we change ω. In particular we do not require or expect integrability w.r.t $\mathbb{P}$. Compare this to the next definition.

Remark 2. Recall that as before (def. 4.3 and following remarks): if the system has exponential decay in either Hölder–continuous functions or those with bounded variation, then it has for Lipschitz–continuous.

Definition 5.6. We say that a random dynamical system has *pathwise exponential decay of correlations* in Lipschitz–continuous functions, if the above inequality holds for almost all $\omega \in \Omega$, i.e.

$$|\mathrm{Corr}_\omega(\phi, \psi, n)| \leq C(\omega) ||\phi||_L ||\psi||_\infty \gamma^n, \tag{5.1.3}$$

where γ is independent of ω and $C(\omega)\colon \Omega \to \mathbb{R}_+$ is $\mathbb{P}$–integrable.

Note. This is sometimes called *fiber decay of correlations.*

This setting will be more useful to observe recurrence, which we introduce in the next subsection.

5.1.4 Random recurrence

Let (X, d) be a separable metric space on which we have a random dynamical system. As above there are several ways of defining the rate of recurrence:

- in the first setting, we may ask about distribution (or density) of return times, i.e. define $p_n = \mathbb{P}\big(F_\varepsilon^n(x) = x\big)$,
- in the second one, fixing ω gives us fixed sequence of maps and we can define recurrence in the typical way,
- we may also integrate this fixed–trajectory return time over whole probability space losing dependence on ω,
- or we can treat the entire $\Omega \times X$ as our (standard) system, but look at return time to a cylinder $B_r(x) \times \Omega$.

Let us now formalize these notions. We will skip the first one as it is not really suitable for our purposes.

For a measurable set U and a fixed 'random trajectory' ω we define the entrance time (in other words this is fiber-wise recurrence)

$$\tau_U^\omega(x) = \inf\{k \geq 1 \colon T_\omega^k(x) \in U\}.$$

If $x \in U$, then this is sometimes called *quenched return time*, check [MR11].

As before we shall be working with entrance times to balls so let us define

$${}^\omega\tau_r^y(x) = \tau_{B(y,r)}^\omega(x).$$

Recall that ${}^\omega\tau_r^x(x)$ is called simply return time. As mentioned before (because the symbol is almost illegible) we shall omit the superscript y (or x) whenever it does not cause confusion.

As mentioned we may integrate this with respect to $\mathbb{P}$ arriving at *annealed return time* (again after [MR11]):

$$\mathbb{T}_U(x) = \int_\Omega \tau_U^\omega(x)\, d\mathbb{P}(\omega).$$

And the last notion is defined as

$$\tau_C(\omega, x) = \inf\{k \colon F^k(\omega, x) \in \Omega \times C\},$$

which is the same as $\tau_U^\omega(x)$, but treats the random system as a standard dynamical system (skew-product).

5.2 Precise Setting

Now we will introduce two main classes of examples, the first comes from works of J. Buzzi: [Buz98] and [Buz99]. All quoted results in the subsection come from these papers.

The other examples come from a paper by Mayer et al [MSU08]. The next subsection gives the definitions and results from their work.

5.2.1 Bounded variation setting

Let $(\Omega, \mathbb{P}, \mathcal{B})$ be a probability space with an ergodic automorphism S and let us take a family of maps $T_\omega \colon [0,1] \to [0,1]$ (for $\omega \in \Omega$). Random dynamical system is realized by the skew-product:

$$\begin{aligned} F \colon \Omega \times [0,1] &\to \Omega \times [0,1] \\ (\omega, x) &\mapsto (S\omega, T_\omega(x)). \end{aligned}$$

Assume that for every ω the maps T_ω are piecewise monotone and Lebesgue non-singular, i.e. $l(T_\omega(E)) = \int_E |T_\omega'|\, dl$; where l denotes the Lebesgue measure.

The author uses the following definition of variation (after Keller, see [Kel85] or [HK82]).

$$\operatorname{var}(g) = \inf_{h = g (\mathrm{mod}\ l)} \sup_{n \in \mathbb{N}} \sup_{0 = s_0 < \cdots < s_n = 1} \sum_{k=1}^{n} |h(s_k) - h(s_{k-1})|.$$

Recall that the space of functions with bounded variation is equipped with a norm

$$||g||_{\mathrm{BV}} = ||g||_\infty + \mathrm{var}(g). \tag{5.2.1}$$

Additionally we assume that

i) the following functions are measurable on Ω: $\mathrm{var}(1/T'_\omega)$, $\operatorname{ess\,inf}(|T'_\omega|)$, number and the boundaries of monotonicity intervals (i.e. if T_ω is monotonic on $(a^\omega_{i-1}, a^\omega_i)$ for $1 \le i \le N^\omega$, then a^ω_k is measurable for every k and so is N^ω),

ii) bounded variation of expanding maps: there exist W and a such that $\mathrm{var}(1/T'_\omega) \le W$ and $\operatorname{ess\,inf}(|T'_\omega|) \ge a > 1$,

iii) F is *covering*, i.e. for each subinterval $I \subset [0,1]$ and for $\mathbb{P}$–a.e. ω, there exists $M(\omega, I)$ such that
$$\operatorname*{ess\,inf}_{x\in[0,1]}(\mathcal{L}^n_\omega \mathbb{1}_I) > 0 \qquad \forall n \ge M.$$

Definition 5.7. A random system of functions as above satisfying the assumptions (i) and (ii) is called *an admissible random Lasota-Yorke map.*

Remark 1. The assumptions in Buzzi's paper (and so in the definition) are weaker: maps are expanding in average, logarithm of variation in $\mathbb{P}$–integrable; but for clarity we have written it as above. For precise setting check [Buz99].

Remark 2. J. Buzzi also shows that all the results (see below) may be proved for the so-called *multi-dimensional β–transformations*, i.e.

$$\begin{aligned} T_B\colon [0,1)^d &\longrightarrow [0,1)^d \\ x &\longmapsto Bx \mod \mathbb{Z}^d, \end{aligned}$$

where B is a $d \times d$ matrix. The definition of variation needs to be reworked (check [Kel85]), but all the other assumptions are easily adaptable.

In this situation we get some good properties of the transfer operator. First of all, we get measures:

Proposition 5.8 ([Buz98] Thm. 0.3). *Suppose that we have an admissible random Lasota-Yorke map. Then there exist finitely many ergodic F–invariant measures absolutely continuous w.r.t. $\mathbb{P} \times l$.*

Moreover, if we add the *covering* assumption we get exponential decay of correlations:

Proposition 5.9 ([Buz99] Main Thm.). *Under the assumptions (i), (ii) and (iii), we have:*

1. *There exists a normalized density h on $\Omega \times X$, which is invariant; i.e.*
$$\mathcal{L}_\omega(h_\omega) = h_{S\omega},$$
where $h_\omega = h(\omega, \cdot)$. Also this density h is unique modulo l and $\mathrm{var}(h_\omega) < \infty$ *for almost all ω.*

2. *Set $\eta = h \cdot (\mathbb{P} \times l)$ and $\mu_\omega = h_\omega \cdot l$. Then we have a nice bound on the correlations (5.1.1), i.e. there exists $\gamma < 1$ and $K(\omega)$, such that for ϕ of bounded variation and ψ bounded and $n \geq 0$:*
$$|\mathrm{Corr}_\omega(\phi, \psi, n)| \leq K(\omega)||\phi||_{\mathrm{BV}}||\psi||_\infty \gamma^n. \tag{5.2.2}$$

Definition 5.10. All systems satisfying assumptions of Prop. 5.9 and *additionally* assuming that the measures $\mu_{S^n\omega}$ are equivalent with densities bounded from above and below will be called *type A* of random systems.

Remark. We may check the equivalence of measures along one fiber by looking at how the P-F operator works on h_ω. For example, we will have bounds on the respecitve densities if the Jacobians $J^{-1}_{S^n\omega}$ are uniformly bounded.

5.2.2 Hölder setting

The second class of examples and all the quoted results in this subsection come from [MSU08]. As mentioned before we will apply the authors' results for maps T_ω all working on one space X (instead of a family of spaces X_ω).

We start with $(\Omega, \mathcal{B}, \mathbb{P}, S)$ — a measure preserving dynamical system with ergodic and invertible map S. (X, ρ) will be a compact metric space and (as before) F is the associated skew-product.

Definition 5.11. The map F is called *expanding random map* if the mappings T_ω are continuous, open and surjective and there exists $\xi > 0$ and measurable functions $A_\omega > 1$ and $\eta_\omega > 0$, for which the following hold:

A) *Uniform openness.* $T_\omega\big(B(x, \eta_\omega)\big) \supset B(T_\omega(x), \xi)$ for every (ω, x);

B) *Measurably expanding.* $\rho(T_\omega(y), T_\omega(x)) \geq A_\omega \rho(y, x)$ whenever $\rho(y, x) < \eta_\omega$;

C) *Measurability of the degree.* The map $\omega \mapsto \deg(T_\omega) = \sup_{x \in X} \# T_\omega^{-1}(\{x\})$ is measurable;

D) *Topological exactness.* There exists a measurable function $n(\omega)$ such that the image $T_\omega^{n(\omega)}\big(B(x, \xi)\big) = X$ for a.e. ω and every $x \in X$.

To get the measures we need to introduce the notion of the random potential.

Denote by $\mathcal{H}^\xi(X)$ the space of all Hölder continuous functions on a compact, metric space X with exponent ξ, i.e. $f \in \mathcal{H}^\xi(X)$ if f is continuous and $v_\xi(f) < \infty$, where

$$v_\xi(f) = \inf \left\{ H_f \colon |f(x) - f(y)| \leq H_f(\rho(x,y))^\xi \right\}. \tag{5.2.3}$$

This space has a norm:

$$||f||_\xi = ||f||_\infty + v_\xi(f). \tag{5.2.4}$$

Definition 5.12. We will call a function $\phi \colon \Omega \times X \to \mathbb{R}$ *α–Hölder continuous*, if

i) $\phi_\omega := \phi(\omega, \cdot) \in \mathcal{C}(X, \mathbb{R})$;

ii) $\omega \longmapsto ||\phi_\omega||_\infty$ is measurable;

iii) $||\phi||_1 := \int_\Omega ||\phi_\omega||_\infty \, d\mathbb{P}(\omega) < \infty$;

iv) $\phi_\omega \in \mathcal{H}^\xi(X)$ and $\omega \longmapsto H_\omega := v_\xi(\phi_\omega)$ is a measurable function $H \colon \Omega \to [1, +\infty)$;

v) $\int_\Omega \ln(H_\omega) \, d\mathbb{P} < \infty$.

Recall that in a case of a general potential ϕ the *transfer* operators are given by

$$\mathcal{L}_\omega(g)(x) = \sum_{y \in T_\omega^{-1}(x)} g(y) e^{\phi_\omega(x)}. \tag{5.2.5}$$

We may define three 'groups' of random systems in this setting:

1. The first one is just as Def. 5.11 — the general setting;

2. The second is a further generalisation — the first two assumptions are relaxed to saying that all inverse branches exist and that functions are expanding in the mean: $A_\omega > 0$ and $\int_\Omega \ln(A_\omega) \, d\mathbb{P} > 0$;

Remark. It is proved in the cited paper that the second 'group' of random systems may be reduced to the first one by applying the induction procedure.

3. The third one is smaller and consists of the systems, called *uniformly expanding random maps*, where T_ω are somewhat similar to each other:

Definition 5.13. A system (X, F, μ) is called a *uniformly expanding random map*, if:
F is an expanding random map, the measure μ is constructed with an *α–Hölder continuous* potential and

- $A := \inf_{\omega \in \Omega} A_\omega > 1$;
- $\deg(T) := \sup_{\omega \in \Omega} \deg(T_\omega) < \infty$;
- $\widehat{n} := \sup_{\omega \in \Omega} n(\omega) < \infty$;
- the potential satisfies $H_\omega \leq H$. (cf. def 5.12)

Remark. Most of the results (Sec. 5.3 and sec. 6.2) in chapters 5 and 6 will be proved for *uniformly expanding random maps* — this is still a very large class of examples. However, some theorems (6.9, 6.10) may be proved in the general setting (the first subgroup) after adding additional assumptions (such as integrability of the degree and of A_ω).

Proposition 5.14 ([MSU08] Thm. 3.1, 3.2). *Under the assumptions of Def. 5.11 and Def. 5.12:*

1. *For the P-F operator defined in (5.2.5) there exists a unique Borel probability measure ν on $\Omega \times X$ such that for $\mathbb{P}$–almost every ω and $\nu_\omega = (\pi_\Omega)_*\nu$*

$$\mathcal{L}_\omega^* \nu_\omega = \lambda_\omega \nu_\omega \qquad \text{for } \lambda_\omega = \nu_{S\omega}(\mathcal{L}_\omega(\mathbb{1})). \tag{5.2.6}$$

2. *There exists a unique density h such that for $\mathbb{P}$–almost every ω and $h_\omega = h(\omega, \cdot)$*

$$\mathcal{L}_\omega h_\omega = \lambda_\omega h_{S\omega} \qquad \text{and } \nu_\omega(h_\omega) = 1. \tag{5.2.7}$$

3. *Measures $\mu_\omega = h_\omega \nu_\omega$ are invariant, which means $(T_\omega)_*\mu_\omega = \mu_{S\omega}$.*

4. *There exists $\gamma < 1$ and measurable functions $A, Q \colon \Omega \to (0, +\infty)$ such that for every $f \in L^1(\mu_\omega)$ and every $g \in \mathcal{H}^\xi(X)$*

$$\begin{aligned}\Big|\mu_\omega(f \circ T_\omega^n \cdot g) - \mu_{S^n\omega}(f)\mu_\omega(g)\Big| &\le \\ &\le \mu_{S^n\omega}(|f|)A(\omega)\left(\mu_\omega(|g|) + \frac{4}{Q(\omega)}\nu_\xi(gh_\omega)\right)\gamma^n.\end{aligned} \tag{5.2.8}$$

Moreover, ([MSU08] sec 3.11) for uniformly expanding random maps, whenever the potentials have bounded Hölder constants $H_\omega \le H$, the functions A and Q are bounded; and so are the densities $1/D \le h \le D$ and their Hölder constants. Which means that we may rewrite the inequality as:

$$|\mathrm{Corr}_\omega(g, f, n)| \le C \cdot \mu_{S^n\omega}(|f|) \cdot ||g||_\xi \cdot \gamma^n. \tag{5.2.9}$$

Remark **([MSU08] sec. 4.1 and 4.2).** Additionally, assume that $\ln(\deg(T_\omega)) \in L^1(\mathbb{P})$ and that the transfer operator is *measurable*, i.e. if g is measurable and $g_\omega \in \mathcal{C}(X)$, then $\mathcal{L}(g)$ is measurable and $(\mathcal{L}(g))_\omega \in \mathcal{C}(X)$.

Then λ_ω, h_ω, μ_ω, ν_ω are measurable and we may define μ as

$$\mu(A) = \int_\Omega \int_X h_\omega A_{|\omega \times X}\, d\nu_\omega\, d\mathbb{P}. \tag{5.2.10}$$

Definition 5.15. All uniformly expanding random maps, i.e. systems with properties as in Prop. 5.14 with the 'moreover' part, will be called *type B* of random systems.

5.2.3 First random recurrence result

We will now apply results from section 3.3 to prove a theorem about recurrence in random systems. Firstly, we do it in a general situation. Next, we apply the techinque in a precise situation: the bounded variation setting. We wish to emphasize, however, that the method itself is quite general.

Take a random system F (Def. 5.1) on $Z = \Omega \times X$, where the map S on Ω is a one-sided shift, with an invariant probability product measure $\eta = \mathbb{P} \otimes \mu$ preserved by F. Define $D_r(\omega, x) = \Omega \times B_r(x)$.
We will apply Thm. 3.16 to this system and arrive at the following:

Theorem 5.16. *Suppose that $Z = \Omega \times X$ is a separable metric space and the system (Z, F, η) is ergodic. Assume additionally that the family $D_r(\omega, x)$ is* weakly Lebesgue–compatible with *η; the measure μ is non–atomic and has* doubling property a.e. *Then for $\mathbb{P}$–a.e. ω*

$$\liminf_{r\to 0} {}^{\omega}\tau_r^x(x) \cdot \mu\left(B_r(x)\right) < +\infty \qquad \mu\text{–a.e.} \tag{5.2.11}$$

Proof. First of all, $D_r(\omega, x) = \Omega \times B_r(x)$ is clearly a *ball–like* family (cf. def 3.9); non–atomicity of μ is needed for property (d). Also $\tau_{D_r(\omega,x)}(x) = {}^{\omega}\tau_r^x(x)$. The measures satisfy $\eta(D_r(\omega, x)) = \mu(B_r(x))$, so η has *doubling property* on D_r (subsec. 3.3.2) iff μ has it on B_r, which is exactly the same as say that μ has *doubling property a.e.*

Thus we get all the assumptions of Thm. 3.16; applying it ends the proof. □

We may apply this technique e.g. to the bounded variation setting to get a simpler statement. Precise assumptions are in the subsection 5.2.1.

Theorem 5.17. *Take an admissible random Lasota–Yorke map (def. 5.7) and an ergodic invariant measure for the skew product $\eta \ll \mathbb{P} \times l$ with the density $h(\omega, x)$ (cf. Thm. 5.8). Define a probability measure μ on $X = [0, 1]$ by $\mu(I) = \eta(\Omega \times I)$. Then*

$$\liminf_{r\to 0} {}^{\omega}\tau_r^x(x) \cdot \mu\left(B_r(x)\right) < +\infty \qquad \text{for } \mathbb{P}\text{–a.e. } \omega \text{ and } \mu\text{–a.e. } x. \tag{5.2.12}$$

Remark. Observe that we may have no decay of correlations in the situation above.

Proof. This time our measure η is not a product measure as in the previous theorem, so we may not apply it directly. However, the method of proof is identical.

Take our density $h(\omega, x)$ and define $H(x) = \int_\Omega h(\omega, x) d\mathbb{P}$. Then $\mu = H \cdot l$ and μ is absolutely continuous w.r.t. the Lebesgue measure l on the interval $[0, 1]$.

As before, define $D_r(\omega, z) = \Omega \times B_r(z)$. As this does not depend on ω, we will write $D_r(z)$. Again, it is a *ball–like* family and $\eta(D_r(z)) = \mu(B_r(z))$. We need to check two assumptions of Thm. 3.16: the doubling property of η and that the family D_r is *weakly Lebesgue–compatible with* η (cf. Def. 3.15).

Let us start with the latter. Take any set A of positive measure $\eta(A) > 0$. Cut it into strips $A = \bigcup_x A_x \times \{x\}$.

$$\eta(D_r(z) \cap A) = \int_\Omega \int_X \mathbb{1}_{D_r \cap A} h(\omega, x) d\mathbb{P} dl = \int_{B_r(z)} \left(\int_{A_x} h(\omega, x) d\mathbb{P}(\omega) \right) dl(x) =: \int_{B_r(z)} f(x) dl,$$

where the last equality defines $f(x)$. f is clearly a positive, measurable function and $\int_X f(x)dl = \eta(A) > 0$. To get weak Lebesgue–compatibility (Def. 3.15) we need to show that there exists $z \in [0,1]$ and $\alpha > 0$ such that for small enough r

$$\alpha \leq \frac{\eta\left(D_r(z) \cap A\right)}{\eta(D_r(z))} = \frac{\int_{B_r(z)} f(x)dl}{\int_{B_r(z)} H(x)dl} = \frac{\int_{B_r(z)} \int_{A_x} h(\omega, x)d\mathbb{P}dl}{\int_{B_r(z)} \int_{\Omega} h(\omega, x)d\mathbb{P}dl},$$

which is straightforward from e.g. the Fubini theorem.

Checking the doubling property of η is equivalent to checking it on μ:

$$\frac{\eta(D_{2r}(\omega, z))}{\eta(D_r(\omega, z))} = \frac{\mu(B_{2r}(z))}{\mu(B_r(z))} = \frac{\int_{z-2r}^{z+2r} H(x)dx}{\int_{z-r}^{z+r} H(x)dx}.$$

If H is continuous at z and $H(x) > 0$, then the fraction goes to 2 as $r \to 0$ and we have the doubling property ($C(z) = 2$). If $H(x) = 0$ in an open interval $I \ni z$, then $\mu(B_r(z)) = 0$ for all r small enough and the doubling property is trivially satisfied. Finally, H has bounded variation, so it is continuous almost everywhere and $H = 0$ on a set of positive measure only if the set contains an interval.

So all the assumptions are satisfied and Thm. 3.16 gives us the required result. □

5.3 Quick random recurrence

5.3.1 Results

We will state and prove the random counterparts of the results from Chapter 3. Those results may be applied to maps of both *type A* and *type B*, but we need additional assumptions.

Theorem 5.18 (Quick random entrance). *Take either*

- *a random system of* Type A *(Def. 5.10)*

or

- *a random system of* Type B *(Def. 5.15).*

Assume additionally that μ_ω has a doubling property at y and $\underline{d}_{\mu_\omega}(y) > 0$. Then

$$\liminf_{r \to 0} {}^{\omega}\tau_r^y(x) \cdot \mu_\omega\big(B(y, r)\big) = 0 \quad \mu_\omega\text{–a.e.} \tag{5.3.1}$$

Remark. Observe that in the proof of Thm. 5.17 we have shown that systems of Type A have the doubling property a.e. The measures μ_ω are absolutely continuous w.r.t. the Lebsgue measure, so $d_{\mu_\omega} > 0$ a.e.

Theorem 5.18 follows from a more general result:
(Systems of *Type A* and *Type B* fulfill all the required assumptions below: exponential decay, appropriate measures, etc.)

Theorem 5.19. *If the random system has ω-exponential decay of correlations — either in Hölder continuous functions (Type B) or those of bounded variation (Type A), all measures μ_ξ (for $\xi = S^n\omega$, $n \geq 0$) are equivalent to μ_ω with densities uniformly bounded from above and below, μ_ω has a doubling property at y and $\underline{d}_{\mu_\omega}(y) > 0$ then*

$$\liminf_{r\to 0} {}^\omega\tau_r^y(x) \cdot \mu_\omega\big(B(y,r)\big) = 0 \quad \textit{for } \mu_\omega\textit{–a.e. } x \in X \tag{5.3.2}$$

Note. As before we could rewrite this limit to different form, cf. Section 4.2 or parts of Chapter 3.

We may write a similar result for the return times.

Theorem 5.20 (Quick random return). *Take (as previously)*

- *a random system of* Type A*,*

or

- *a random system of* Type B.

Assume additionally that μ_ω has a doubling property a.e. and $0 < \underline{d}_{\mu_\omega}(x) \leq \overline{d}_{\mu_\omega}(x) < +\infty$ a.e. Then

$$\liminf_{r\to 0} {}^\omega\tau_r^x(x) \cdot \mu_\omega\big(B(x,r)\big) = 0 \quad \mu_\omega\textit{–a.e.} \tag{5.3.3}$$

And this again follows trivially from a more general theorem below. Read also the Remark after Thm. 5.18

Theorem 5.21. *If the random system has ω-exponential decay of correlations — again in Hölder cont. functions (Type B) or bounded variation (Type A), all measures μ_ξ (for $\xi = S^n\omega$, $n \geq 0$) are equivalent to μ_ω with densities uniformly bounded from above and below, μ_ω has a doubling property a.e. and $0 < \underline{d}_{\mu_\omega}(x) \leq \overline{d}_{\mu_\omega}(x) < +\infty$ a.e. then*

$$\liminf_{r\to 0} {}^\omega\tau_r^x(x) \cdot \mu_\omega\big(B(x,r)\big) = 0 \quad \textit{for } \mu_\omega\textit{–a.e. } x \in X \tag{5.3.4}$$

Remark. We assume that the measures along a path are equivalent (with bounded densities), so saying that μ_ω has a property (e.g. doubling) is equivalent to saying that all measures μ_ξ have that property (for $\xi = S^n\omega$, $n \geq 0$).

5.3.2 Proofs

The proofs are very similar to those for the previous chapter, cf. Section 4.3, but with some additional problems originating in the added randomness.

We will write the full proof of Thm. 5.19 to show that this generalisation is doable and then sketch the proof of Thm. 5.21 to omit unnecessary (and unneeded) repetition.

As before we will need two technical convergence results, which we have already proved: Lemma 4.11 and Lemma 4.12.

Proof of Theorem 5.19. First, let us fix $y \in X$; ω is fixed by the statement.

For brevity define $\omega^k = S^k\omega$ (for $k \geq 0$) and as before set $f_r^\omega(x) = {}^\omega\tau_r^y(x) \cdot \mu_\omega\big(B(y,r)\big)$. We will write B_r instead of $B(y,r)$. Proving 5.3.2 is equivalent to proving that

$$\mu_\omega\left(\{x \in X \colon \forall_{\varepsilon>0}\, \forall_{\rho>0}\, \exists_{r<\rho}\, f_r^\omega(x) \leq \varepsilon\}\right) = 1.$$

Let us define sets

$${}^\omega A_r^\varepsilon = \left\{f_r^\omega(x) \geq \varepsilon\right\} = \left\{{}^\omega\tau_r^y(x) \geq \frac{\varepsilon}{\mu_\omega(B_r)}\right\}. \tag{5.3.5}$$

Observe that is suffices to prove that

$$\mu_\omega\left(\bigcap_{r<\rho} {}^\omega A_r^\varepsilon\right) = 0 \quad \text{for any fixed } \varepsilon \text{ and } \rho. \tag{5.3.6}$$

A point belongs to ${}^\omega A_r^\varepsilon$, iff its trajectory defined using a sequence of T_{ω^n} omits B_r for some time. Precisely speaking (cf. random trajectory in Def. 5.1):

$$x \in {}^\omega A_r^\varepsilon \iff T_{\omega^0}(x), T_{\omega^1} \circ T_{\omega^0}(x), \ldots, T_\omega^l(x) \notin B_r, \quad \text{where } l = \left[\frac{\varepsilon}{\mu_\omega(B_r)}\right]. \tag{5.3.7}$$

We can define T_ω^{-k} and write

$${}^\omega A_r^\varepsilon = \bigcap_{k=1}^{\left[\frac{\varepsilon}{\mu_\omega(B_r)}\right]} T_\omega^{-k} B_r', \quad \text{where } B_r' = X \setminus B_r. \tag{5.3.8}$$

Using this we rewrite the set from (5.3.6) as

$$\bigcap_{r<\rho} \bigcap_{k=1}^{\left[\frac{\varepsilon}{\mu_\omega(B_r)}\right]} T_\omega^{-k} B_r'. \tag{5.3.9}$$

Changing the order of intersection gives

$$\bigcap_{k=1}^{+\infty} \bigcap_{r \leq r_k} T_\omega^{-k} B_r', \tag{5.3.10}$$

where $r_k = \sup\left\{r \colon \frac{\varepsilon}{\mu_\omega(B_r)} \geq k\right\}$.

The sets B'_r and likewise $T_\omega^{-k}B'_r$ form an increasing family of sets as $r \searrow 0$ (for k fixed), so $\bigcap_{r \le r_k} T_\omega^{-k}B'_r = T_\omega^{-k}B'_{r_k}$. It leaves us to prove that

$$\mu_\omega\left(\bigcap_{k=1}^{+\infty} T_\omega^{-k}B'_{r_k}\right) = 0. \tag{5.3.11}$$

Definition of r_k shows that

$$\frac{\varepsilon}{\mu_\omega(B_{2r_k})} < k, \quad \text{because} \;\; 2r_k > r_k \tag{5.3.12}$$

and combining this with doubling property, we get (writing $\sigma = \sigma(y)$)

$$\frac{\varepsilon}{k} < \mu_\omega(B_{2r_k}) \le \sigma\mu_\omega(B_{r_k}). \tag{5.3.13}$$

So $\mu_\omega(B_{r_k}) > \frac{\varepsilon}{k\sigma}$ and we know that in the equation (5.3.11) the measures of sets intersected are at most $1 - \frac{\varepsilon}{k\sigma}$. Observe that because the measures μ_ω and μ_{ω^k} are equivalent (with uniformly bounded densities), we could write the same inequality for all μ_{ω^k} — at most by increasing σ a little.

As before we need a family of approximation functions. Note that in the bounded variation situation this is unnecessary as $\mathbb{1}_B \in \mathrm{BV}$. Let us set

$$\phi_r(t) = \begin{cases} 0 & \text{for } 0 \le t \le \frac{r}{2} \\ \frac{2}{r}t - 1 & \text{for } \frac{r}{2} \le t \le r \\ 1 & \text{for } t \ge r \end{cases},$$

and define an approximating function $g_r(z) := \phi_r(d(y,z))$. It has Lipschitz constant equal to $\frac{2}{r}$ and so we have a bound on the Hölder norm for any ξ: $||g_r||_\xi \le \frac{3}{r}$ (for $r \le 1$).

We will use the functions g_r to get a bound on the measure of the intersection

$$\mu_\omega\left(B'_r \cap T_\omega^{-n}F\right) = \mu_\omega\left(\mathbb{1}_{B'_r} \cdot \mathbb{1}_F \circ T_\omega^n\right) \le \mu_\omega\left(g_r \cdot \mathbb{1}_F \circ T_\omega^n\right). \tag{5.3.14}$$

The last expression may be rewritten using the correlation function (5.1.1):

$$\mu_\omega\left(g_r \cdot \mathbb{1}_F \circ T_\omega^n\right) = \mathrm{Corr}_\omega(g_r, \mathbb{1}_F, n) + \mu_\omega(g_r)\mu_{\omega^n}(F). \tag{5.3.15}$$

Also observe (and recall) that

$$\mu_\omega\left(T_\omega^{-n}B\right) = \mu_{\omega^n}\left(B\right). \tag{5.3.16}$$

Now we shall estimate the measure in (5.3.11) by taking just a subset of k_n, which will be defined later.

$$\mu_\omega\left(\bigcap_{n=1}^{+\infty} T_\omega^{-k_n}B'_{r_{k_n}}\right) =$$

$$= \mu_\omega\left(T_\omega^{-k_1}B'_{r_{k_1}} \cap \bigcap_{n=2}^{+\infty} T_\omega^{-k_n}B'_{r_{k_n}}\right) =$$

and using (5.3.16)

$$= \mu_{\omega^{k_1}}\left(B'_{r_{k_1}} \cap \bigcap_{n=2}^{+\infty} T_{\omega^{k_1}}^{-k_n+k_1} B'_{r_{k_n}}\right) =$$
$$= \mu_{\omega^{k_1}}\left(B'_{r_{k_1}} \cap T_{\omega^{k_1}}^{-k_2+k_1} \bigcap_{n=2}^{+\infty} T_{\omega^{k_2}}^{-k_n+k_2} B'_{r_{k_n}}\right) =$$

Now set $F = \bigcap_{n=2}^{+\infty} T_{\omega^{k_2}}^{-k_n+k_2} B'_{r_{k_n}}$, use (5.3.14) and (5.3.15); and afterwards the decay of correlations (5.2.9) and the Hölder norm of $g_{r_{k_1}}$ (BV-norm of $||g_r||_{\mathrm{BV}} \leq 3$).

$$= \mu_{\omega^{k_1}}\left(\mathbb{1}_{B'_{r_{k_1}}} \cdot \mathbb{1}_F \circ T_{\omega^{k_1}}^{k_2-k_1}\right) \leq \mu_{\omega^{k_1}}\left(g_{r_{k_1}} \cdot \mathbb{1}_F \circ T_{\omega^{k_1}}^{k_2-k_1}\right) \leq$$
$$\leq \left|\mathrm{Corr}_{\omega^{k_1}}(g_{r_{k_1}}, \mathbb{1}_F, k_2-k_1)\right| + \mu_{\omega^{k_1}}(g_{r_{k_1}})\mu^{\omega^{k_2}}(F) \leq$$
$$\leq \mu_{\omega^{k_2}}(F) \cdot \left(\mu_{\omega^{k_1}}(g_{r_{k_1}}) + C \cdot \gamma^{k_2-k_1} \cdot ||g_{r_{k_1}}||_\xi\right) \leq$$
$$\leq \mu_{\omega^{k_2}}\left(B'_{r_{k_2}} \cap \bigcap_{n=3}^{+\infty} T_{\omega^{k_2}}^{-k_n+k_2} B'_{r_{k_n}}\right)\left(\mu_{\omega^{k_1}}(g_{r_{k_1}}) + C\gamma^{k_2-k_1}\frac{3}{r_{k_1}}\right).$$

Following inductively we get an estimation

$$\mu_\omega\left(\bigcap_{n=1}^{+\infty} T_\omega^{-k_n} B'_{r_{k_n}}\right) \leq \prod_{n=1}^{+\infty}\left(\mu_{\omega^{k_n}}(g_{r_{k_n}}) + C\gamma^{k_{n+1}-k_n}\frac{3}{r_{k_n}}\right). \tag{5.3.17}$$

From equivalence of μ_ω i μ_{ω^k}, the definition of g_r and the doubling property we get another estimation (as before — increasing σ, if necessary, we get this inequality for all k)

$$\mu_{\omega^k}(g_{r_k}) \leq \mu_{\omega^k}(B'_{\frac{1}{2}r_k}) \leq 1 - \mu_{\omega^k}(B_{\frac{1}{2}r_k}) \leq 1 - \frac{1}{\sigma}\mu_{\omega^k}(B_{r_k}) \leq 1 - \frac{\varepsilon}{k\sigma^2}. \tag{5.3.18}$$

In the equation (5.3.17) we want the factors to be strictly smaller than one, so we need to set a condition on k_n:

$$C\gamma^{k_{n+1}-k_n}\frac{3}{r_{k_n}} \leq \frac{\varepsilon}{2\sigma^2 k_n}. \tag{5.3.19}$$

This in turn gives a recurrence relation

$$k_{n+1} \geq k_n + \log_\gamma\left(\frac{\varepsilon}{6C\sigma^2}\right) - \log_\gamma(k_n) + \log_\gamma(r_{k_n}), \tag{5.3.20}$$

and we take the smallest k_{n+1} satisfying that inequality, $k_1 = 1$.

Finally, we arrive at an estimation:

$$\begin{aligned}
\mu_\omega\left(\bigcap_{k=1}^{+\infty} T_\omega^{-k} B'_{r_k}\right) &\leq \mu_\omega\left(\bigcap_{n=1}^{+\infty} T_\omega^{-k_n} B'_{r_{k_n}}\right) \\
&\leq \prod_{n=1}^{+\infty}\left(\mu_{\omega^{k_n}}(g_{r_{k_n}}) + C\gamma^{k_{n+1}-k_n}\frac{3}{r_{k_n}}\right) \\
&\leq \prod_{n=1}^{+\infty}\left(1 - \frac{\varepsilon}{\sigma^2 k_n} + \frac{\varepsilon}{2\sigma^2 k_n}\right) \\
&\leq \prod_{n=1}^{+\infty}\left(1 - \frac{\varepsilon}{2\sigma^2 k_n}\right).
\end{aligned}$$

The last product is equal to 0 (ending the proof), if the sum $\sum_{n=1}^{+\infty}\frac{1}{k_n} = +\infty$, which follows from applying first Lemma 4.13 and then Lemma 4.11. □

Notes on the proof of Theorem 5.21. We simply need to rewrite the proof of Thm. 4.5 but "adding ω, wherever necessary":

- The measures are equivalent, so all properties of μ_ω are held (uniformly) for all measures $\mu_{S^n\omega}$ along the path.
- This allows to find a suitable: set G_λ to uniformly bound all parameters and later a covering, correct for all measures.
- The estimations are written exactly as in the proof above — using the random transfer operator as on page 65.
- The function h_r approximating the characteristic function of a ball has $||h_r||_\xi \leq \frac{3}{r}$.
- Finally, we use Lemma 4.12 to prove that our partial sums tend to $+\infty$, ending the proof.

□

Chapter 6

Exponential statistic

The aim of this chapter is to prove the exponential law for statistic of return (and entry) times. First section is devoted to proving this result in the deterministic situation. The second section proves this in the random setting. Its worth noting that the randomized situation is much harder and more toilsome than the non-random one, unlike the results from previous chapters, which were not hard to adapt to random maps.

The idea of the proof in the deterministic situation has been inspired by an unpublished notes [UZ]. It should be noted, however, that in this work the assumptions are weaker and the proof itself is written in a significantly different way.

(The authors of the note had problems with the *thin annuli assumption*, which was partially resolved here. Moreover, it is probable that we may drop this assumption entirely.)

6.1 Non-random result

We will start with some definitions. We will need some definitions from chapter 2; recall that a *subpoly* function (Def. 2.7) is basically $-\ln(r)$ (perhaps in some power).
Also in sec. 1.3 we introduced $\tau(U) = \inf_{x\in U} \tau_U(x)$ as the first return of a set into itself.

Definition 6.1. We will call a metric measure preserving dynamical system $(T, X, \mu, \mathcal{F}, \rho)$ *weakly Markov*, if it satisfies the assumptions (i) to (iii):

i) *Decay of correlations.* There exists $\gamma \in (0, 1)$ such that for all $g \in \mathcal{H}_\xi$, all $f \in L_1(\mu)$ and every $n \in \mathbb{N}$ we have

$$|\mu\left(f \circ T^n \cdot g\right) - \mu(g) \cdot \mu(f)| \leq C\gamma^n ||g||_\xi \mu(|f|); \tag{6.1.1}$$

ii) *Pointwise sensible.* $0 < \underline{d}_\mu(x) \leq \overline{d}_\mu(x) < +\infty$ for μ–a.e. $x \in X$;

iii) *No small return.* $\liminf\limits_{r\to 0} \dfrac{\tau\big(B_r(x)\big)}{-\ln(r)} > 0$ for μ–a.e. $x \in X$.

And if it also satisfies (iv), then we will call the system *weakly Markov with thin annuli.*

iv) *Thin Annuli.* There exists a *subpoly* (def. 2.7) function $\kappa_x(r)$ positive a.e. such that

$$\lim_{r\to 0} \frac{\mu\left(B(x, r + r^{\kappa_x(r)}) \setminus B(x,r)\right)}{\mu\left(B(x,r)\right)} = 0. \tag{6.1.2}$$

Remark 1. The *no small return* property has been proved for many system e.g. in [STV02]. This also is easily proved for expanding maps, check the proof in the next section (Lemma 6.13).

Remark 2. M. Urbański introduced the so-called *Loosely Markov* systems in [Urb07]. These systems assume (ii), a slightly stronger version of (i) and a weak partition–existence condition, which implies (iii).

Example 6.2. Weakly Markov systems include all the examples from subsection 4.2.1.

Theorem 6.3. *For a* weakly Markov system with thin annuli *the entry time tends to the exponential one law*

$$\sup_{t\geq 0} \left| \mu\left(\left\{z \colon \tau_{B_r(x)}(z) > \frac{t}{\mu(B_r(x))}\right\}\right) - e^{-t} \right| \xrightarrow{r\to 0} 0 \tag{6.1.3}$$

and also the normalized return time tends to the exponential one law

$$\sup_{t\geq 0} \left| \mu_{B_r(x)}\left(\left\{z \colon \tau_{B_r(x)}(z) > \frac{t}{\mu(B_r(x))}\right\}\right) - e^{-t} \right| \xrightarrow{r\to 0} 0. \tag{6.1.4}$$

Remark. Recall that earlier (Thm. 2.17 and Cor. 2.18) we have proved that the thin annuli property is satisfied for most radii. More precisely, that in an interval $[0, s]$ the Lebesgue measure of radii with *thick* annuli is of order $O(-s^{\kappa}\ln(s))$. For example, this gives that the lower limit in (6.1.2) is equal to 0 for any Borel measure.

This may be written as a separate result:

Theorem 6.4. *For a* weakly Markov system *the entry time tends has the exponential distribution*

$$\lim_{n\to\infty} \sup_{t\geq 0} \left| \mu\left(\left\{z \colon \tau_{B_{r_n}(x)}(z) > \frac{t}{\mu(B_{r_n}(x))}\right\}\right) - e^{-t} \right| = 0 \tag{6.1.5}$$

for most *choices of the sequence r_n;*
where most choices *mean that the subset of radii within an interval* $[0, s]$ *must omit a set of order* $O(-s^{-\ln(s)}\ln(s))$.
The same is true for the return time.

Remark. If the system has positive Lyapunov exponents and the pointwise dimensions are bounded from above and separated from zero, then as the function κ we may take $\kappa(r) = r^{-\varepsilon}$ for a sufficiently small ε.

This follows from slightly more subtle estimates in the proof and knowing that in this case the limit in *no small return* is separated from zero by a result from [STV02].

The basic idea for proof of Thm. 6.3 is to apply the results obtained in [HSV99]. More precisely, we will need to use two theorems from there.
First, that the distribution of the first return time into a fixed set is close to the exponential law if and only if the distributions of the first return time and first entry are close.
Second, that we can bound this, mentioned in the previous sentence, *closeness* by quite easy to control expressions. We will finish the proof by estimating those expressions.

Proof of Theorems 6.3 and 6.4. The proof is divided into 3 separate steps and so it follows straightforward from connecting three theorems.
Thm. 6.5 gives the limiting exponential distribution, if we prove that a certain function $d(B_r)$ goes to 0.
Thm. 6.6 bounds $d(B_r)$ from above by 3 quantities $a_N(B_r)$, $b_N(B_r)$ and $N \cdot \mu(B_r)$.
Finally, Thm. 6.7 shows that for a certain $N(r)$ those 3 quantities all tend to 0 simultaneously, ending the proof of Thm 6.3.

The proof of Thm. 6.4 goes exactly the same, but we use Thm. 6.7 only for these radii for which we have the thin annuli property (and we know most radii have it by Cor. 2.18). □

Let us start with some notation; it has been introduced in the aforementioned paper. For a fixed set U let us define

$$\begin{aligned} c(k,U) &= \mu_U\left(\tau > k\right) - \mu\left(\tau > k\right), \\ c(U) &= \sup_{k\in\mathbb{N}} \left|c(k,U)\right|. \end{aligned}$$

And this is the first result from [HSV99]:

Theorem 6.5. *For a measure preserving transformation the distributions of both the first return time and first entry time differs from the exponential law by an expression which tends to 0, i.e. for entry time*

$$\sup_{t\geq 0}\left|\mu\left(\left\{z\colon \tau_U(z) > \frac{t}{\mu(U)}\right\}\right) - e^{-t}\right| \leq d(U) \tag{6.1.6}$$

and also for return time

$$\sup_{t\geq 0}\left|\mu_U\left(\left\{z\colon \tau_U(z) > \frac{t}{\mu(U)}\right\}\right) - e^{-t}\right| \leq d(U), \tag{6.1.7}$$

where $d(U) = 4\mu(U) + c(U)\big(1 - \ln c(U)\big)$.

The second theorem (also from [HSV99]) gives an estimate on the value of $c(U)$.

Theorem 6.6. *With the transformation as above:*

$$c(U) \leq \inf_{N\in\mathbb{N}} \left\{a_N(U) + b_N(U) + N\mu(U)\right\},$$

where

$$\begin{aligned} a_N(U) &= \mu_U\left(\{\tau_U \leq N\}\right), \\ b_N(U) &= \sup_{V\in\mathcal{B}} \left|\mu_U\big(T^{-N}V\big) - \mu\big(V\big)\right| = \\ &= \sup_{V\in\mathcal{B}} \left|\frac{\mu\big(U \cap T^{-N}V\big) - \mu(U)\mu\big(V\big)}{\mu(U)}\right| \end{aligned}$$

and $\mathcal{B}$ is the σ-algebra of Borel sets.

Note. that for a fixed set U: $a_N(U)$ grows to 1 as $N \to +\infty$, whereas $b_N(U)$ tends to 0 (provided the system has some mixing properties). The tricky part is to find a number N such that b_N has become small, but a_N and $N \cdot \mu(U)$ haven't grown too big.

The proof of Thm. 6.3 follows easily from those 2 theorems and the result below.

Theorem 6.7. *If the system is weakly Markov, then there exists $n_r(x)$ such that all three values $a_{n_r}(B_r(x))$ and $b_{n_r}(B_r(x))$ and $n_r \cdot \mu(B_r(x))$ tend to 0 as $r \to 0$ for almost all x.*

Proof. We will write B_r instead of $B_r(x)$, when dependence on x is not important. Put $n_r = \mu(B_r)^{-\theta}$. Obviously if $\theta < 1$ we get $n_r \cdot \mu(B_r) \to 0$ instantly. So it remains to find θ such that both a_{n_r} and b_{n_r} will tend to 0.

Firstly, let us rewrite the *no small return* assumption.

There exists a Borel set V of full μ measure and measurable functions $\chi(x), \rho_1(x)$ positive a.e. such that

$$B_r(x) \cap T^{-k}\big(B_r(x)\big) = \emptyset \tag{6.1.8}$$

for all $x \in V$, all $r < \rho_1(x)$ and all integers $k \leq \chi(x)\ln(1/r)$.

Secondly, the assumptions on pointwise dimension give that there exists a set W, again of full measure such that for all $x \in W$

$$r^{2\overline{d}_\mu(x)} \leq \mu\left(B_r(x)\right) \leq r^{\underline{d}_\mu(x)/2}, \tag{6.1.9}$$

for all $r < \rho_2(x)$, for a certain measurable, positive a.e. function $\rho_2(x)$.

Now let us define a family of Lipschitz continuous functions approximating a characteristic function on a ball; depending on a parameter radius $r > 0$, $\kappa(r) > 0$, and $x \in X$. First — auxiliary:

$$\phi_r^\kappa(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq r \\ r^{-\kappa(r)}(r + r^{\kappa(r)} - t) & \text{for } r \leq t \leq r + r^{\kappa(r)} \\ 0 & \text{for } t \geq r + r^{\kappa(r)} \end{cases},$$

The functions we are looking for are $g^\kappa_{r,x}(z) = \phi^\kappa_r(\rho(z,x))$. We will denote them simply as g_r. Their Lipschitz constant equals $r^{-\kappa(r)}$ (as metric ρ is 1–Lipschitz). In particular their Hölder norm (needed in the definition of exponential decay of correlations) is bounded $||g_r||_\xi \leq 1 + r^{-\kappa(r)} \approx r^{-\kappa(r)}$ (for small r).

When estimating a_N the function $\kappa(r)$ may be taken constant $= \kappa > 0$. Fix $x \in V \cap W$ and sufficiently small r. For brevity we will write g_r for $g^\kappa_{r,x}$ and set $f_r = \mathbb{1}_{B_r}$. Note that $f_r \leq g_r$.

Recall that

$$a_N(B_r) = \mu_{B_r}(\tau_{B_r} \leq N) = \mu_{B_r}\left(\bigcup_{n=1}^{N} T^{-n}(B_r)\right) \leq \sum_{n=1}^{N} \frac{\mu\left(B_r \cap T^{-n}(B_r)\right)}{\mu(B_r)}.$$

And as $x \in V$ we know that a few first intersections are empty (put $\chi = \chi(x)$):

$$a_N(B_r) \leq \sum_{n=-\chi\ln(r)}^{N} \frac{\mu\left(B_r \cap T^{-n}(B_r)\right)}{\mu(B_r)}.$$

The assumption on decay of correlations (6.1.1) gives

$$\begin{aligned}\mu\left(B_r \cap T^{-k}(B_r)\right) &= \mu\left(f_r \circ T^n \cdot f_r\right) \leq \mu\left(f_r \circ T^n \cdot g_r\right) \leq \\ &\leq \mu(g_r)\cdot\mu(f_r) + C\gamma^n ||g_r||_\xi \mu(f_r) \leq \mu(f_r)\left(\mu(g_r) + C\gamma^n r^{-\kappa}\right).\end{aligned}$$

Which allows us to rewrite the estimate on a_N and later bound the sum's elements as simply as possible.

$$\begin{aligned}a_N(B_r) &\leq \sum_{n=-\chi\ln(r)}^{N}\left(\mu(g_r) + C\gamma^n r^{-\kappa}\right) \leq N\mu(g_r) + Cr^{-\kappa}\sum_{n=-\chi\ln(r)}^{+\infty}\gamma^n = \\ &= N\mu(g_r) + \frac{C}{1-\gamma} r^{-\kappa}\gamma^{-\chi\ln(r)} = N\mu(g_r) + Dr^{-\kappa-\chi\ln(\gamma)}.\end{aligned}$$

If $\kappa \leq 1$ we get an estimate (using 6.1.9):

$$\mu(g_r) \leq \mu\left(B(x, r + r^\kappa)\right) \leq \mu\left(B(x, 2r^\kappa)\right) \leq 2^{\underline{d}_\mu(x)/2} r^{\kappa\underline{d}_\mu(x)/2}. \tag{6.1.10}$$

Finally insert $N = n_r = \mu(B_r)^{-\theta}$ and let us rewrite the estimate (again using 6.1.9).

$$\begin{aligned}a_{n_r}(B_r) &\leq \mu(B_r)^{-\theta}\cdot 2^{\underline{d}_\mu(x)/2} r^{\kappa\underline{d}_\mu(x)/2} + Dr^{-\kappa-\chi\ln(\gamma)} \leq \\ &\leq Er^{-2\theta\overline{d}_\mu(x)} r^{\kappa\underline{d}_\mu(x)/2} + Dr^{-\kappa-\chi\ln(\gamma)}.\end{aligned}$$

If we take $\kappa < -\chi\ln(\gamma)$ and then any θ so small that $2\theta\overline{d}_\mu(x) < \kappa\underline{d}_\mu(x)/2$, we arrive at the conclusion that

$$\lim_{r\to 0} a_{n_r}(B_r) = 0. \tag{6.1.11}$$

Now we turn to estimating $b_{n_r}(B_r)$. The point x remains fixed, fix a Borel set V, but take function $\kappa(r)$ satisfying the *thin annuli* assumption (6.1.2).

$$\begin{aligned}\big|\mu\big(B_r \cap T^{-N}V\big) - \mu(B_r)\mu\big(V\big)\big| &= \big|\mu\big(\mathbb{1}_V \circ T^N \cdot f_r\big) - \mu(\mathbb{1}_V)\mu(f_r)\big| \leq \\ &\leq \big|\mu\big(\mathbb{1}_V \circ T^N \cdot f_r\big) - \mu\big(\mathbb{1}_V \circ T^N \cdot g_r\big)\big| + \\ &+ \big|\mu\big(\mathbb{1}_V \circ T^N \cdot g_r\big) - \mu(\mathbb{1}_V)\mu(g_r)\big| + \\ &+ \big|\mu(\mathbb{1}_V)\mu(g_r) - \mu(\mathbb{1}_V)\mu(f_r)\big|.\end{aligned}$$

So $\mu(B_r)b_{n_r}(B_r)$ is bounded by the supremum (over all sets Borel V) of the above three elements.

The third expression bounding b_{n_r} is controlled easily:

$$\begin{aligned}\mu(B_r)^{-1}\big|\mu(\mathbb{1}_V)\mu(g_r) - \mu(\mathbb{1}_V)\mu(f_r)\big| &\leq \mu(B_r)^{-1}\big(\mu(g_r) - \mu(f_r)\big) \leq \\ &\leq \mu(B_r)^{-1}\big(\mu(B(x, r + r^{\kappa(r)}) - \mu(B(x,r)\big) = \\ &= \frac{\mu\left(B(x, r + r^{\kappa(r)}) \setminus B(x,r)\right)}{\mu(B_r)}.\end{aligned}$$

And this tends to 0 because of the thin annuli assumption. The first element is bounded identically as before.

$$\big|\mu\big(\mathbb{1}_V \circ T^N \cdot f_r\big) - \mu\big(\mathbb{1}_V \circ T^N \cdot g_r\big)\big| \leq \big(\mu(g_r) - \mu(f_r)\big).$$

And for the second we may use the decay of correlations:

$$\big|\mu\big(\mathbb{1}_V \circ T^N \cdot g_r\big) - \mu(\mathbb{1}_V)\mu(g_r)\big| \leq C\gamma^N r^{-\kappa(r)}\mu(\mathbb{1}_V) \leq C\gamma^N r^{-\kappa(r)}.$$

Using the pointwise dimensions (6.1.9) we get $n_r = \mu(B_r)^{-\theta} \geq r^{-\theta\underline{d}_\mu(x)/2}$ and using again we arrive at

$$\begin{aligned}\mu(B_r)^{-1}\left|\mu\big(\mathbb{1}_V \circ T^N \cdot g_r\big) - \mu(\mathbb{1}_V)\mu(g_r)\right| &\leq C r^{-\kappa(r) - 2\overline{d}_\mu(x)}\gamma^{r^{-\theta\underline{d}_\mu(x)/2}} = \\ &= Ce^{-\kappa(r)\ln(r) - 2\overline{d}_\mu(x)\ln(r) + r^{-\theta\underline{d}_\mu(x)/2}\ln(\gamma)},\end{aligned}$$

and the last estimate converges to zero as $r \to 0$ if

$$\lim_{r\to 0} \kappa(r)\ln(r) r^{\theta\underline{d}_\mu(x)/2} = 0, \tag{6.1.12}$$

which follows from the *subpoly* assumption on $\kappa(r)$ (Def. 2.7). We conclude that

$$\lim_{r\to 0} b_{n_r}(B_r) = 0. \tag{6.1.13}$$

and this ends the proof. □

6.2 Random setting

The section deals with the random setting. The method is similar to that in the deterministic situation, but there are significant differences and problems related mostly to the number of measures (i.e. the fact there is more than one). We will begin by proving equivalents of two theorems from [HSV99]. (Their statements are quite different to the deterministic versions.)

The first one will give the exponential distribution, if certain expressions tend to 0. The second will give easier-to-use bounds on those expressions. Finally, we will give some estimates on those bounds.

We will obtain our result only for *Type B* maps (i.e. the setting from [MSU08]), check subsection 5.2.2). Densities for *Type A* maps may not be continuous, thus we cannot prove Lemma 6.12. (The result is still probable, but the proof will have to use different estimates.)

6.2.1 Theorem

The main result is as follows:

Theorem 6.8. *Suppose we have a uniformly expanding random map (recall that we assume a bounded potential) for which the 'invariant' measure μ_ω has the* thin annuli *property and has finite, positive pointwise dimensions. Then this system has the exponential law distribution of the return times into balls.*

The proof is divided into 2 parts. The first part (theorems 6.9 and 6.10) is true in a more general setting. Stated precisely, these are the assumptions of those two theorems:

A) *Skew-product.* Take an ergodic automorphism S, preserving a probability measure $\mathbb{P}$ on a set Ω. The random system $F\colon \Omega \times X \to \Omega \times X$ is defined by $F(\omega, x) = (S\omega, T_\omega(x))$.

B) *Pathwise measures.* There exists a family of measures μ^ω satisfying a property $\mu^\omega\left(T_\omega^{-n}(A)\right) = \mu^{S^n\omega}(A)$.

C) *Reference measure.* The maps T_ω are non–singular with respect to a reference measure m and all measures $\mu^\omega \ll m$.

D) *Path-equivalent measures.* There is a normalized density h on $\Omega \times X$ which satisfies $\mathrm{d}\mu^\omega = h_\omega \mathrm{d}m$, $(0 < h_\omega(x) < +\infty)$; and also μ^ξ are equivalent (for $\xi = S^n\omega$) with a bounded and separated from zero constant D.

Note. As of this moment we will write μ^ω instead of μ_ω, because we need the subscript for the conditional measure.

Remark. Recall that a uniformly expanding random map has all of the above properties and also the exponential decay of correlations (5.2.9).

The assumptions for the final theorem are:

E) The system is a *uniformly expanding random map.* (So we have (A)–(D))

F) *Pointwise dimensions.* $0 < \underline{d}_{\mu^\xi}(x) \leq \overline{d}_{\mu^\xi}(x) < +\infty$ for m–almost all $x \in X$ and all $\xi = S^n\omega$ (i.e. for all n and a fixed ω).

G) *Thin Annuli.* There exists a *subpoly* function $\kappa_x(r) > 0$ such that for all $\xi = S^n\omega$ (equivalently any ξ):

$$\lim_{r\to 0} \frac{\mu^\xi\left(B(x, r + r^{\kappa_x(r)}) \setminus B(x,r)\right)}{\mu^\xi\left(B(x,r)\right)} = 0 \qquad \text{for } \mu^\xi\text{–a.e. } x \in X. \tag{6.2.1}$$

And now the precise statement of Thm. 6.8 follows.

Theorem 6.8. *If a uniformly expanding random map also satisfies the assumptions (F) and (G), then the normalized entry time tends to the exponential one law*

$$\forall_{t\geq 0} \left|\mu^\omega\left(\left\{z\colon \tau^\omega_{B_r(x)}(z) > \frac{t}{\mu(B_r(x))}\right\}\right) - e^{-t}\right| \xrightarrow{r\to 0} 0 \tag{6.2.2}$$

and also the normalized return time tends to the exponential one law

$$\forall_{t\geq 0} \left|\mu^\omega_{B_r(x)}\left(\left\{z\colon \tau^\omega_{B_r(x)}(z) > \frac{t}{\mu(B_r(x))}\right\}\right) - e^{-t}\right| \xrightarrow{r\to 0} 0. \tag{6.2.3}$$

And *if a uniformly expanding random map satisfies only the assumptions (E) and (F), then then limits are still true for* most *sequences of radii (cf. Thm. 6.4).*

Note. This means that the cumulative distribution function of return (entry) times limits pointwise on e^{-t}. By Dini's theorem this convergence is uniform on every compact subset and since both functions tend to 0 at $+\infty$, the convergence is uniform. So one may write the above limits with $\sup_t$ instead of $\forall_t$ as in the previous section or [HSV99].

Remark. This result (exponential distribution) may be proved with a considerably weaker assumptions, but this proof is not written here.
Instead of taking a uniformly expanding random map we may take a general expanding random map, additionally assuming: integrability of the degree, the expanding constants and exponent of the Hölder constants: $\int \deg(T_\omega)\, d\mathbb{P}$, $\int A_\omega\, d\mathbb{P}$, $\int e^{H_\omega}\, d\mathbb{P}$ all finite. However, we still need to assume (F) and (G).

6.2.2 Intermediate results

The proof of Thm. 6.8 is divided into three results stated below.
First, we need some notation. For a fixed set U let us define:

$$\begin{aligned} c^\omega(k,U) &= \mu^\omega_U\left(\tau^\omega_U > k\right) - \mu^\omega\left(\tau^\omega_U > k\right) \\ \hat{c}^\omega(U) &= \sup_{k\in\mathbb{N}} \left|c^\omega(k,U)\right| \\ c^\omega(U) &= \sup_{l\in\mathbb{Z}} \left|\hat{c}^{S^l\omega}(U)\right|. \end{aligned} \tag{6.2.4}$$

Whenever ω is fixed and omitting it should not cause difficulties, we will do so (e.g. we will write $c(U)$ instead of $c^\omega(U)$). For brevity (again, only when not leading to confusion) we will denote

$$\mu_l := \mu^{S^l\omega}; \qquad h_l := h_{S^l\omega}; \qquad \tau_U^l = \tau^l(U) := \tau^{S^l\omega}(U). \tag{6.2.5}$$

Theorem 6.9. *Under conditions (A)–(D): the distributions of both the first return time and first entry time differs from the exponential law by an expression which tends to 0, i.e. for entry time*

$$\left|\mu^\omega\left(\left\{z\colon \tau_U^\omega(z) > \frac{t}{\mu(U)}\right\}\right) - e^{-t}\right| \le d^\omega(U,t) \tag{6.2.6}$$

and also for return time

$$\left|\mu_U^\omega\left(\left\{z\colon \tau_U^\omega(z) > \frac{t}{\mu(U)}\right\}\right) - e^{-t}\right| \le d^\omega(U,t), \tag{6.2.7}$$

where for every fixed t we have $d^\omega(U,t) \to 0$, if $\mu(U) \to 0$ and $c^\omega(U) \to 0$.

Remark. We will only use this theorem for $U = B(x,r)$ and $r \to 0$. This will make the proof slightly simpler — cf. equation 6.3.9 and explanation beneath it.

The second auxiliary theorem gives a useful (and usable) estimate on the value of $c(U)$.

Theorem 6.10. *Under conditions (A)–(D):*

$$c^\omega(U) \le \inf_{N\in\mathbb{N}} \left\{ a_N^\omega(U) + b_N^\omega(U) + \sum_{l=1}^{N} \mu_l(U) \right\},$$

where

$$\begin{aligned} a_N^\omega(U) &= \mu_U^\omega\left(\{\tau_U^\omega \le N\}\right), \\ b_N^\omega(U) &= \sup_{V\in\mathcal{B}} \left|\mu_U^\omega\big(T_\omega^{-N}V\big) - \mu^{S^N\omega}\big(V\big)\right| = \\ &= \sup_{V\in\mathcal{B}} \left|\frac{\mu^\omega\big(U\cap T_\omega^{-N}V\big) - \mu^\omega(U)\mu^{S^N\omega}\big(V\big)}{\mu^\omega(U)}\right| \end{aligned}$$

and $\mathcal{B}$ is the σ-algebra of Borel sets.

Note. For a fixed set U: $a_N^\omega(U)$ grows to 1 as $N \to +\infty$, whereas $b_N^\omega(U)$ tends to 0 (provided the system has some mixing properties). The tricky part is to find a number N such that b_N^ω has become small, but a_N^ω and $\sum \mu_l$ haven't grown too big.

The third theorem estimates $a_n^\omega(B_r(x))$ and $b_n^\omega(B_r(x))$ and allows to find a well fitting N. Taken together these results prove Theorem 6.8.

Theorem 6.11. *Under assumptions of Thm. 6.8 there exists $n = n_\omega(r)$ such that all three values $a_n^\omega(B_r)$ and $b_n^\omega(B_r)$ and $\sum_{l=1}^n \mu_l(B_r)$ simultaneously tend to 0 as $r \to 0$.*

6.3 Proofs

The proof of the main result of this chapter is divided (as in the non–random case Thm. 6.3) into three separate results, which we will prove in this section.

Proof of Theorem 6.8. The proof is divided into 3 separate steps and so it follows straightforward from connecting three theorems.
Thm. 6.9 proves the limiting exponential distribution, if we know that $c^\omega(B_r)$ goes to 0.
Thm. 6.10 bounds $c^\omega(B_r)$ from above by a sum of $a_N^\omega(B_r)$, $b_N^\omega(B_r)$ and $N \cdot \mu^\omega(B_r)$.
Finally Thm. 6.11 shows that there exists $N^\omega(r)$ for which those 3 quantities all tend to 0 simultaneously, ending the proof. □

6.3.1 Random sum lemma

In proving the theorems we will need an estimate on the average (random pathwise) measure of a set.

Lemma 6.12. *Under the assumptions of Thm. 6.8: if the densities h_ω are equicontinuous and a set $\overline{V}$ is compact, then*

$$\forall_{\varepsilon>0}\exists_Q\forall_{k>Q} \qquad \left|\sum_{l=1}^{k}\mu_l(U) - k\mu(U)\right| \le \varepsilon k m(U) \tag{6.3.1}$$

for all subsets $U \subset V$ (with the same constant Q).

Remark 1. We will use this theorem taking shrinking concentric balls as U, so the compactness assumption will be automatically satisfied and Q will be fixed for all of balls.

Remark 2. We have a uniformly expanding random map with a Hölder potential; recall that we assume that all the potentials ϕ_ω have the same exponent and a uniformly bounded Hölder constant (def. 5.12). Then the densities are Hölder–continuous with the same exponent and again the same constant — thus equicontinuous.

Proof. Let us start by fixing x and recalling that $S(\omega)$ is an ergodic transformation. We will write $h(x,\omega) = h_x(\omega) = h_\omega(x)$ depending on which variable will be changing. This means that for $\mathbb{P}$-almost every ω

$$\frac{\sum_{l=1}^{M} h_x(S^l\omega)}{M} \longrightarrow \int_\Omega h_x(\omega)\mathrm{d}\mathbb{P}(\omega) = h(x) \tag{6.3.2}$$

as $M \to +\infty$. Denote the set (of full $\mathbb{P}$–measure) of those ω's for which the above limit holds as A_x.

We choose a countable subset $Z \subset \overline{U}$ dense in $\overline{U}$ and define $A = \bigcap_{x\in Z} A_x$. Obviously $\mathbb{P}(A) = 1$ and for all $x \in Z$ and $\omega \in A$ the limit in the equation (6.3.2) holds.

Next, we fix $\omega \in A$. We have a family of equicontinuous functions $h_l(x)$ (recall notations 6.2.5). Define

$$f_k(x) = \frac{1}{k}\sum_{l=1}^{k} h_l(x) \text{ for } x \text{ in } \overline{U}.$$

The functions $f_k(x)$ are equicontinuous as well. Moreover, we know that $f_k(x) \to h(x)$ for all $x \in Z$.

Using the Arzela-Ascoli theorem we get a subsequence f_{k_n} that tends uniformly to a limit function which we will call $f(x)$.

f is continuous as a uniform limit of continuous functions and equation (6.3.2) shows that $h(x) = f(x)$ for all x in a dense set Z. Since, h is also continuous, we know that $f(x) = h(x)$ for all $x \in \overline{U}$. Note that this would be also true for all $x \in X$, but we will not need it.

We now have a family of functions f_k defined on a compact set converging pointwise on a dense set Z to h and having a subsequence converging uniformly. This means that $f_k(x)$ converges uniformly to $h(x)$ on the set Z and density implies that the convergence is uniform everywhere.

We get that for every $\varepsilon > 0$ and all sufficiently large k

$$\begin{aligned}
\sup_{x\in U} |f_k(x) - h(x)| &\leq \varepsilon \\
\sup_{x\in U} \left|\sum_{l=1}^{k} h_l(x) - kh(x)\right| &\leq \varepsilon k \\
\left|\int_U \sum_{l=1}^{k} h_l(x)\mathrm{d}m - k\int_U h(x)\mathrm{d}m\right| &\leq \varepsilon k m(U) \\
\left|\sum_{l=1}^{k} \mu_l(U) - k\mu(U)\right| &\leq \varepsilon k m(U),
\end{aligned}$$

thus ending the proof. □

6.3.2 Proof of the first Theorem — 6.9

Using the previous lemma we obtain the first result. Recall that as the set U we will take concentric balls $B_r(x)$ of shrinking radius. This simplifies the proof slightly.

Proof of Theorem 6.9. Let us start by observing that

$$\begin{aligned}
\{x\colon \tau_U^{\omega} > k\} &= \{x\colon T_\omega x \notin U \wedge \tau_U^{S\omega}(T_\omega x) > k-1\} = \\
&= T_\omega^{-1}\left(U' \cap \left\{y\colon \tau_U^{S\omega}(y) > k-1\right\}\right).
\end{aligned}$$

So we get

$$\mu^{\omega}\left(\{\tau_U^{\omega} > k\}\right) = \mu^{S\omega}\left(U' \cap \left\{\tau_U^{S\omega} > k-1\right\}\right), \tag{6.3.3}$$

which leads to equations ($c^\omega(k,U)$ was defined in (6.2.4))

$$\begin{aligned}\mu^\omega\left(\{\tau_U^\omega > k\}\right) &= \mu^{S\omega}\left(\left\{\tau_U^{S\omega} > k-1\right\}\right) - \mu^{S\omega}\left(U \cap \left\{\tau_U^{S\omega} > k-1\right\}\right) = \\ &= \mu^{S\omega}\left(\left\{\tau_U^{S\omega} > k-1\right\}\right) - \mu^{S\omega}(U)\cdot \mu_U^{S\omega}\left(\left\{\tau_U^{S\omega} > k-1\right\}\right) = \\ &= \mu^{S\omega}\left(\left\{\tau_U^{S\omega} > k-1\right\}\right)\left(1-\mu^{S\omega}(U)\right) + \\ &\qquad\qquad - \mu^{S\omega}(U)\cdot c^{S\omega}(k-1,U).\end{aligned}$$

We may repeat this for an element on the RHS:

$$\mu^{S\omega}\left(\left\{\tau_U^{S\omega} > k-1\right\}\right) \le \mu^{S^2\omega}\left(\left\{\tau_U^{S^2\omega} > k-2\right\}\right)\left(1-\mu^{S^2\omega}(U)\right) - \mu^{S^2\omega}(U)\cdot c^{S^2\omega}(k-2,U). \tag{6.3.4}$$

Continuing this calculation and combining the inequalities gives the following

$$\mu^\omega\left(\{\tau_U^\omega > k\}\right) = \prod_{l=1}^{k}\left(1-\mu_l(U)\right) - \sum_{l=1}^{k}\mu_l(U)\cdot c^{S^l\omega}(k-l,U)\cdot\prod_{j=1}^{l-1}\left(1-\mu_l(U)\right). \tag{6.3.5}$$

Using the definition of $c^\omega(U)$ (6.2.4) (as ω is fixed we will write $c(U) = c^\omega(U)$) and bounding the second product by 1 we arrive at the estimation

$$\left|\mu^\omega\left(\{\tau_U^\omega > k\}\right) - \prod_{l=1}^{k}\left(1-\mu_l(U)\right)\right| \le c(U)\sum_{l=1}^{k}\mu_l(U). \tag{6.3.6}$$

Now let us fix $t \ge 0$ and take $k = k_t = \left[\frac{t}{\mu(U)}\right]$. As τ_U takes only discrete values we have

$$\left\{z\colon \tau_U^\omega(z) > \frac{t}{\mu(U)}\right\} = \left\{z\colon \tau_U^\omega(z) > k_t\right\}. \tag{6.3.7}$$

We get a trivial estimate for every $t \ge 0$

$$\begin{aligned}&\left|\mu^\omega\left(\left\{z\colon \tau_U^\omega(z) > \frac{t}{\mu(U)}\right\}\right) - e^{-t}\right| \le \\ &\le \underbrace{\left|\mu^\omega\left\{z\colon \tau_U^\omega(z) > k_t\right\} - \prod_{l=1}^{k_t}\left(1-\mu_l(U)\right)\right|}_{I} + \underbrace{\left|\prod_{l=1}^{k_t}\left(1-\mu_l(U)\right) - e^{-t}\right|}_{II}.\end{aligned} \tag{6.3.8}$$

Recall that we need to prove that for any fixed t the RHS of (6.3.8) goes to 0 as $c(U) \to 0$ and $\mu(U) \to 0$ (i.e. a $c(U_n), \mu(U_n) \to 0$ for a fixed sequence U_n). We will show this separately for I and II.

The first step is to use Lemma 6.12 from the previous subsection to estimate $\sum \mu_l(U)$. We want to show that for any small set V and all measurable subsets $U \subset V$ there exists $\varepsilon > 0$ such that

$$\varepsilon m(U) \le \frac{1}{2}\mu(U). \tag{6.3.9}$$

This is easy, since the density is separated from zero — say $h(x) \geq D$ for almost every x. We may take $\varepsilon = \frac{D}{2}$.

Thus for all $k > Q$ Lemma 6.12 gives a good estimate:

$$\frac{1}{2}k\mu(U) \leq \sum_{l=1}^{k} \mu_l(U) \leq \frac{3}{2}k\mu(U). \tag{6.3.10}$$

Let us fix $q = -\ln(c(U))$ and $k_q = \left[\frac{q}{\mu(U)}\right]$. Since we take $c(U) \to 0$ we have that q is large enough: $q > Q$.

We will start by estimating I (from (6.3.8)) for $t \leq q$. Using the above inequality (6.3.10) we get

$$\begin{aligned} I &\leq c(U)\sum_{l=1}^{k_t} \mu_l(U) \leq c(U)\sum_{l=1}^{k_q} \mu_l(U) \leq c(U)\frac{3}{2}k_q\mu(U) \leq \\ &\leq c(U)\frac{3}{2}q \leq -\frac{3}{2}c(U)\ln(c(U)). \end{aligned} \tag{6.3.11}$$

Now for $t > q$ we will estimate both parts of I separately.

$$\prod_{l=1}^{k_t}(1-\mu_l(U)) = e^{\sum_{l=1}^{k_t}\ln(1-\mu_l(U))} \leq e^{-\sum_{l=1}^{k_t}\mu_l(U)}, \tag{6.3.12}$$

where in the inequality we used the estimate: $\ln(1+x) \leq x$.

We will also use the following (basic) inequalities:

$$-a - a^2 \leq \ln(1-a) \leq -a \qquad \text{for } a \in \left(0, \tfrac{1}{2}\right) \tag{6.3.13}$$

$$1 - 2|x| \leq e^x \leq 1 + 2|x| \qquad \text{for } -1 \leq x \leq 1. \tag{6.3.14}$$

Continuing the estimate (6.3.12) (only the exponent) and using (6.3.10):

$$\sum_{l=1}^{k_t}\mu_l(U) \geq \sum_{l=1}^{k_q}\mu_l(U) \geq \frac{1}{2}k_q\mu(U) \geq \frac{1}{2}\big(q - \mu(U)\big) \geq -\frac{1}{2}\big(\ln(c(U)) + \mu(U)\big). \tag{6.3.15}$$

Returning to the product this means that

$$\prod_{l=1}^{k_t}(1-\mu_l(U)) \leq c(U)^{1/2} \cdot e^{1/2\cdot\mu(U)} \leq \sqrt{c(U)}(1+\mu(U)), \tag{6.3.16}$$

where we used the estimate (6.3.14). The second item in I is estimated accordingly, using again (6.3.6):

$$\begin{aligned} \mu^\omega\left(\{\tau_U^\omega > k_t\}\right) \leq \mu^\omega\left(\{\tau_U^\omega > k_q\}\right) &\leq \prod_{l=1}^{k_q}(1-\mu_l(U)) + c(U)\sum_{l=1}^{k_q}\mu_l(U) \leq \\ &\leq \sqrt{c(U)}(1+\mu(U)) - \frac{3}{2}c(U)\ln(c(U)), \end{aligned}$$

where the second line comes from estimates above (6.3.16) and (6.3.11). Finally, this estimate with (6.3.16) is used to estimate I.

$$I \leq \sqrt{c(U)}(1+\mu(U)) + -\frac{3}{2}c(U)\ln(c(U)) + \sqrt{c(U)}(1+\mu(U)). \tag{6.3.17}$$

So $I \to 0$ if $\mu(U) \to 0$ and $c(U) \to 0$ for all $t \geq 0$.

To estimate II (6.3.8) we need to be slightly more careful. As before fix t and fix ε. This gives the constant Q (recall that Q depends only on ε and not on U). Then take (shrink if necessary) U small enough so that $k_t = \left[\frac{t}{\mu(U)}\right] > Q$.

Start by using estimate (6.3.14):

$$\begin{aligned} II = \left|\prod_{l=1}^{k_t}(1-\mu_l(U)) - e^{-t}\right| &= \left|e^{\sum_{l=1}^{k_t}\ln(1-\mu_l(U))} - e^{-t}\right| = \\ &= e^{-t}\left|e^{t+\sum_{l=1}^{k_t}\ln(1-\mu_l(U))} - 1\right| \leq \\ &\leq 2e^{-t}\left|t + \sum_{l=1}^{k_t}\ln\left(1-\mu_l(U)\right)\right|. \end{aligned} \tag{6.3.18}$$

Now using estimate on logarithm (6.3.13) and the definition of k_t gives

$$II \leq 2e^{-t}\left|k_t\mu(U) + \mu(U) - \sum_{l=1}^{k_t}\big(\mu_l(U) + \mu_l^2(U)\big)\right|. \tag{6.3.19}$$

We may use Lemma 6.12 obtaining:

$$II \leq 2e^{-t}\left(\varepsilon k_t m(U) + \mu(U) - \sum_{l=1}^{k_t}\mu_l^2(U)\right). \tag{6.3.20}$$

Let us say that the densities $h_l(x) \leq W$. Then we may write:

$$\sum_{l=1}^{k_t}\mu_l^2(U) \leq \sum_{l=1}^{k_t} W\mu(U)\mu_l(U) \leq W\mu(U)\big(k_t\mu(U) + \varepsilon k_t m(U)\big). \tag{6.3.21}$$

It remains to put back the definition of k_t, recall that $h(x) > 0$ a.e. so $m(U)/\mu(U)$ is finite, say $\leq Z$ and observe that function $t \mapsto te^{-t}$ is bounded:

$$\begin{aligned} II &\leq 2e^{-t}\big(\varepsilon tZ + \mu(U) + W\mu(U)t + W\mu(U)\varepsilon tZ\big) \leq \\ &\leq 2te^{-t}\big(\varepsilon Z + W\mu(U) + W\mu(U)\varepsilon Z\big) + 2\mu(U) \leq 2\varepsilon Z + O(\mu(U)). \end{aligned}$$

And the last expression can be as small as needed thus ending the first part of the theorem.

Since the difference $|\mu^\omega(\{\tau_U^\omega > k\}) - \mu_U^\omega(\{\tau_U^\omega > k\})|$ between the entry and return time is bounded by $c(U)$ — we get directly the second inequality (for the return time) as well. $\square$

6.3.3 Proof of the second Theorem — 6.10

Proof of Theorem 6.10. Recall definitions of a_N^ω and b_N^ω from the statement of the theorem. We want to prove that for every k and every N

$$|c^\omega(k,U)| \le a_N^\omega(U) + b_N^\omega(U) + \sum_{l=1}^{N} \mu_l(U). \tag{6.3.22}$$

Firstly, for $k \le N$ the estimate is simple.

$$\begin{aligned}
|c_\omega(k,U)| &= |\mu_U^\omega(\tau_U^\omega > k) - \mu^\omega(\tau_U^\omega > k)| = |\mu_U^\omega(\tau_U^\omega \le k) - \mu^\omega(\tau_U^\omega \le k)| \le \\
&\le |\mu_U^\omega(\tau_U^\omega \le k)| + |\mu^\omega(\tau_U^\omega \le k)| = \\
&= a_k^\omega(U) + \mu^\omega\left(\left\{U \cup T_\omega^{-1}(U) \cup \ldots \cup T_\omega^{-k}(U)\right\}\right) \le \\
&\le a_k^\omega(U) + \sum_{l=1}^{k} \mu^\omega\left(T_\omega^{-l}(U)\right) \le a_N^\omega(U) + \sum_{l=1}^{N} \mu_l(U).
\end{aligned}$$

The estimates are a bit harder for $k > N$. We will split the inequality into 3 parts each bounded by a different part of the right-hand side.

Start by observing that $\{\tau_U^\omega > k\} = \{\tau_U^\omega > N \wedge \tau_U^{S^N\omega} \circ T_\omega^N > k-N\}$ so the sets $\{\tau_U^\omega > k\}$ and $\{\tau_U^{S^N\omega} \circ T_\omega^N > k-N\}$ differ only on set $\{\tau_U^\omega \le N\}$ and

$$\left|\mu_U^\omega\big(\{\tau_U^\omega > k\}\big) - \mu_U^\omega\big(\{\tau_U^{S^N\omega} \circ T_\omega^N > k-N\}\big)\right| \le \mu_U^\omega\big(\{\tau_U^\omega \le N\}\big) = a_N^\omega(U). \tag{6.3.23}$$

Also we have (straight from the definition of $b_N^\omega(U)$)

$$\left|\mu_U^\omega\left(T_\omega^{-N}\left(\{\tau_U^{S^N\omega} > k-N\}\right)\right) - \mu^{S^N\omega}\left(\{\tau_U^{S^N\omega} > k-N\}\right)\right| \le b_N^\omega(U). \tag{6.3.24}$$

We will use (6.3.3) to get a series of equations

$$\begin{aligned}
\mu^\omega(\{\tau_U^\omega > k\}) &= \mu^{S\omega}\left(\left\{\tau_U^{S\omega} > k-1\right\}\right) - \mu^{S\omega}\left(U \cap \left\{\tau_U^{S\omega} > k-1\right\}\right) = \\
&= \mu^{S^2\omega}\left(\left\{\tau_U^{S^2\omega} > k-2\right\}\right) - \mu^{S\omega}\left(U \cap \left\{\tau_U^{S\omega} > k-1\right\}\right) + \\
&- \mu^{S^2\omega}\left(U \cap \left\{\tau_U^{S^2\omega} > k-2\right\}\right) = \\
&\ldots \\
&= \mu^{S^N\omega}\left(\left\{\tau_U^{S^N\omega} > k-N\right\}\right) - \mu^{S\omega}\left(U \cap \left\{\tau_U^{S\omega} > k-1\right\}\right) + \ldots \\
&\ldots - \mu^{S^N\omega}\left(U \cap \left\{\tau_U^{S^N\omega} > k-N\right\}\right).
\end{aligned}$$

Set $\Delta_l = \left\{\tau_U^{S_l\omega} > k-l\right\}$. Then the last equation gives an estimate

$$\begin{aligned}
&\left|\mu^{S^N\omega}\left(\{\tau_U^{S^N\omega} > k-N\}\right) - \mu^\omega(\{\tau_U^\omega > k\})\right| \le \\
&\le \mu^{S\omega}(U \cap \Delta_1) + \ldots + \mu^{S^N\omega}(U \cap \Delta_N) \le \sum_{l=1}^{N} \mu_l(U).
\end{aligned}$$

Summing the above inequality, (6.3.23) and (6.3.24) ends the proof. ∎

6.3.4 No small return lemma

Before we prove Thm. 6.11 we will need to know that no sets (sufficiently small) return too quickly to themselves (compare to *no small return* assumption: (iv) from def. 6.1).

Lemma 6.13. *If maps T_ω are uniformly expanding and measures μ^ω have positive pointwise dimension (assumptions* E *and* F*) then there exists a Borel set V_ω of full μ_ω measure and a measurable function $\chi(x)$ such that*

$$B_r(x) \cap T_\omega^{-k}\big(B_r(x)\big) = \emptyset \tag{6.3.25}$$

for all $x \in V$, all sufficiently small $r > 0$ and all integers $k \leq \chi(x)\ln(1/r)$. Or in other words

$$\liminf_{r \to 0} \frac{\tau^\omega\big(B_r(x)\big)}{-\ln(r)} > 0 \qquad \text{for } \mu_\omega\text{–a.e. } x \in X. \tag{6.3.26}$$

Moreover, one set V may be taken for all ω on one fixed random trajectory.

Proof. Fix ω and a sequence of transformations T_ω, $T_{S\omega}\ldots$ We want to find two functions $\sigma(x)$ and $\chi(x)$ (both > 0 a.e.) so that

$$\forall_{x \in S} \forall_{r < \sigma(x)} \forall_{k = 1..\chi(x)\ln(1/r)} \quad B_r(x) \cap T_\omega^{-k}\big(B_r(x)\big) = \emptyset.$$

We will need a bound on the number of periodic points. Note that here they are meant with respect to the finite sequence T_ω, $T_{S\omega}\ldots$ We call that ω–pathwise periodic. So a point y has period 3 if $T_\omega^3(y) = T_{S^2\omega} \circ T_{S\omega} \circ T_\omega(y) = y$. In other words this is a periodic point for a sequence of type $\omega_0, \omega_1, \omega_2, \omega_0, \omega_1 \ldots$

Observe that if we have a countable collection of maps $T_i \colon X \to X$ uniformly expanding of finite degree (recall that uniformly expanding random maps have bounded degree — def. 5.13) we have the following inequality

$$\#\mathrm{Fix}(T_1 \circ \ldots \circ T_n) \leq e^{nW} \tag{6.3.27}$$

for any subset $T_1 \ldots T_n$ and a certain W. The proof of that fact follows.

Let us say that x_0 is a fixed point of the composition $T^n \overset{df}{=} T_1 \circ \ldots \circ T_n$. Take a ball $B(x_0, \xi)$, where ξ is the radius from 'openness' condition (Def. 5.11). There exists a well defined (cf. [PU10, Def. 4.1.3]) inverse map $T^{-n} \colon B(x_0, \xi) \to B(x_0, \xi)$ taking $x_0 \mapsto x_0$. Also, for any $x \in B(x_0, \xi)$ the map satisfies [PU10, Lemma 4.1.4]

$$\rho\left(T^{-n}(x), T^{-n}(x_0)\right) \leq A^{-n}\rho(x, x_0),$$

where A is the uniform expansion constant. This shows that there cannot be any fixed point (apart from x_0) of the chosen inverse map T^{-n} in $B(x_0, \xi)$.

On the other hand, the degrees of maps T_k are uniformly bounded by a number, which we denote by D. So degree of T^n is bounded by D^n, and there are at most D^n inverse branches of T^{-n}. As shown above, every branch may have at most one fixed point, so there may not be more than D^n fixed points of T^n on $B(x_0, \xi)$.

The space is compact, so a covering with balls $B(x_i, \xi)$ will consist of a finite number of balls, say C. Combining the results we get that

$$\#\mathrm{Fix}(T_1 \circ \ldots \circ T_n) \leq C \cdot D^n. \tag{6.3.28}$$

We return to the proof of our lemma. Observe that if Z is *connected component of* $T_\omega^{-k}\big(B_r(x)\big)$ and $Z \subset B_r(x)$, then there is a fixed point y of T_ω^k in $B_r(x)$.The proof of the lemma will end by using the bound on the number of periodic points to estimate the measure of sets returning too fast and then using the Borel-Cantelli lemma. This is the idea — precise proof follows.

Set $K = \log_A(4)$ (where A is the uniform expanding constant) and define $\phi(x) = \frac{1}{2}\min\left\{\rho\left(x, T_\omega^{-l}(x)\right) : 1 \leq l \leq K\right\}$. This minimum is positive if x is not a periodic point of period $\leq K$ and there are finitely many such points. Also the set of those points has measure zero, which follows from finiteness of the upper pointwise dimension of measure. This together with definition of lower pointwise dimension of measure gives:

For any $\varepsilon > 0$ there exists a measurable set F of measure $\mu^\omega(F) > 1 - \varepsilon$ and positive constants ϕ, d and δ such that for all $x \in F$ and all $r < \delta$:

$$\phi(x) \geq \phi \tag{6.3.29}$$

$$\mu_\omega(B_r(x)) \leq r^d. \tag{6.3.30}$$

Observe that for $x \in F$ and $r < \phi$ we have (because of choice of $\phi(x)$)

$$B_r(x) \cap T_\omega^{-k}\big(B_r(x)\big) = \emptyset$$

for $k \leq K$. Additionally, because of maps being uniformly expanding, for $k \geq K$ and for $r \leq \eta/4$ (recall that η is the radius of balls on which we have the expanding property)

$$\mathrm{diam}\big(\text{connected component of } T_\omega^{-k}(B_{2r})\big) \leq \frac{1}{4}\mathrm{diam}(B_r).$$

Now this implies that if $B_r(x)$ intersects a component of $T_\omega^{-k}\big(B_r(x)\big)$ then $B_{2r}(x)$ contains a component of $T_\omega^{-k}\big(B_{2r}(x)\big)$. This means that there is a periodic point y (ω–pathwise) of period at most k in $B_{2r}(x)$.

This gives a covering of our set of points returning too quickly:

$$C_r^N := \big\{x \colon \exists_{l \leq N} B_r(x) \cap T_\omega^{-l}(B_r(x)) \neq \emptyset\big\} \subset \bigcup_{p_i \in \mathrm{Per}(\leq N)} B_{2r}(p_i). \tag{6.3.31}$$

The periodic points need not lie in the set F (where we have control over the measure of balls), but every periodic point *used to cover points* x is close to at least one of those points. Simply put: for every p_i there exists $x_i \in F$

$$B_{2r}(p_i) \subset B_{4r}(x_i).$$

Using 6.3.30 we get an estimate on the measure.

$$\mu^\omega(C_r^N) \leq \sum_{p_i \in \mathrm{Per}(\leq N)} \mu^\omega(B_{2r}(p_i)) \leq \sum_{x_i} \mu^\omega(B_{4r}(x_i)) \leq$$
$$\leq \#\mathrm{Per}(\leq N) \cdot (4r)^d \leq e^{NW + d\ln(4r)}.$$

Fix χ and define a set of *bad* points:

$$A_R = \left\{x \colon \exists_{R/2 \leq r \leq R}\ \exists_{l \leq -\chi \ln(r)} \quad B_r(x) \cap T_\omega^{-l}(B_r(x)) \neq \emptyset\right\} \tag{6.3.32}$$

Its measure may be estimated by (increasing radius does not remove intersections)

$$\mu^\omega(A_R) \leq \mu^\omega\left(\left\{x \colon \exists_{l \leq -\chi \ln(R/2)} \quad B_R(x) \cap T_\omega^{-l}(B_R(x)) \neq \emptyset\right\}\right) \leq$$
$$\leq e^{-\chi \ln(R/2) + d \ln(4R)} \leq R^P, \tag{6.3.33}$$

for a certain $P > 0$ if χ was small enough.

This means that $\sum \mu^\omega(A_{2^{-n}}) < +\infty$, so almost every $x \in F$ does not belong to any set A_R for sufficiently small R.

Finally, as ε goes to 0 we reach our result, i.e. we define $V_\omega = \bigcup_{\varepsilon > 0} F_\varepsilon$.

Last part of the lemma is achieved by taking $V = \bigcap_{n=0}^{+\infty} V_{S^n\omega}$, which has needed properties as the measures are equivalent. $\square$

6.3.5 Proof of the third Theorem — 6.11

Proof of Theorem 6.11. The proof is similar to the non-random setting, with some changes due to the randomness and more importantly to the difference of the form of the previous theorems (compared to the deterministic setting).

We want to show that three quantities (definitions are in the statement of Thm. 6.10): $a_N^\omega(B_r)$, $b_N^\omega(B_r)$ and $\sum_{l=1}^N \mu_l(B_r)$ are small for a certain $N(r)$.

First of all, the assumptions on pointwise dimension give that there exists a set W of full measure μ^ω such that for all $x \in W$

$$r^{2\overline{d}_{\mu^\omega}(x)} \leq \mu^\omega\left(B_r(x)\right) \leq r^{\underline{d}_{\mu^\omega}(x)/2} \qquad \text{for all sufficiently small } r. \tag{6.3.34}$$

Observe that this set has full measure in all measures $\mu^{S^n\omega}$ along the path.

Again we will need to approximate the ball. As first function take:

$$\phi_r^\kappa(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq r \\ r^{-\kappa(r)}(r + r^{\kappa(r)} - t) & \text{for } r \leq t \leq r + r^{\kappa(r)} \\ 0 & \text{for } t \geq r + r^{\kappa(r)} \end{cases}.$$

And as before we take $g_{r,x}^\kappa(z) = \phi_r^\kappa(\rho(z,x))$. Their Hölder norm is bounded $||g||_\xi \leq r^{-\kappa(r)}$ (for small r).

First, $\kappa(r)$ can be taken constant $=\kappa>0$. Take the set V as in Lemma 6.13 and fix $x\in V\cap W$ and sufficiently (for sets V and W) small r. We will write g_r for $g^\kappa_{r,x}$ and set $f_r=\mathbb{1}_{B_r}$.

The first expression is bounded as before

$$a^\omega_N(B_r)=\mu^\omega_{B_r}\left(\tau^\omega_{B_r}\leq N\right)=\mu^\omega_{B_r}\left(\bigcup_{n=1}^{N}T^{-n}_\omega(B_r)\right)\leq\sum_{n=1}^{N}\frac{\mu^\omega\left(B_r\cap T^{-n}_\omega(B_r)\right)}{\mu^\omega(B_r)}.$$

The point $x\in V$ so that sum is zero at first terms:

$$a^\omega_N(B_r)\leq\sum_{n=-\chi\ln(r)}^{N}\frac{\mu^\omega\left(B_r\cap T^{-n}_\omega(B_r)\right)}{\mu^\omega(B_r)}.$$

The assumption on decay of correlations gives

$$\begin{aligned}\mu^\omega\left(B_r\cap T^{-k}_\omega(B_r)\right)&=\mu^\omega\left(f_r\circ T^n_\omega\cdot f_r\right)\leq\mu^\omega\left(f_r\circ T^n\cdot g_r\right)\leq\\&\leq\mu^{S^n\omega}(g_r)\cdot\mu^\omega(f_r)+C\gamma^n||g_r||_\xi\mu^\omega(f_r)\leq\\&\leq\mu^\omega(f_r)\left(\mu^{S^n\omega}(g_r)+C\gamma^nr^{-\kappa}\right).\end{aligned}$$

So using the equivalence of measures we may rewrite it and bound a^ω_N simply as

$$\begin{aligned}a^\omega_N(B_r)&\leq\sum_{n=-\chi\ln(r)}^{N}\left(\mu^{S^n\omega}(g_r)+C\gamma^nr^{-\kappa}\right)\leq ND\mu^\omega(g_r)+Cr^{-\kappa}\sum_{n=-\chi\ln(r)}^{+\infty}\gamma^n=\\&=ND\mu^\omega(g_r)+\frac{C}{1-\gamma}r^{-\kappa}\gamma^{-\chi\ln(r)}=ND\mu^\omega(g_r)+Dr^{-\kappa-\chi\ln(\gamma)}.\end{aligned}$$

For $\kappa\leq1$ and using (6.3.34) we get

$$\mu^\omega(g_r)\leq\mu^\omega\left(B(x,r+r^\kappa)\right)\leq\mu^\omega\left(B(x,2r^\kappa)\right)\leq2^{\underline{d}_{\mu^\omega}(x)/2}r^{\kappa\underline{d}_{\mu^\omega}(x)/2}.\tag{6.3.35}$$

Put $N=n_r=\mu^\omega(B_r)^{-\theta}$ and again using (6.3.34):

$$\begin{aligned}a^\omega_{n_r}(B_r)&\leq\mu^\omega(B_r)^{-\theta}\cdot2^{\underline{d}_{\mu^\omega}(x)/2}r^{\kappa\underline{d}_{\mu^\omega}(x)/2}+Dr^{-\kappa-\chi\ln(\gamma)}\leq\\&\leq Er^{-2\theta\overline{d}_{\mu^\omega}(x)}r^{\kappa\underline{d}_{\mu^\omega}(x)/2}+Dr^{-\kappa-\chi\ln(\gamma)}.\end{aligned}$$

Take $\kappa<-\chi\ln(\gamma)$ and afterwards a θ so small that $2\theta\overline{d}_{\mu^\omega}(x)<\kappa\underline{d}_{\mu^\omega}(x)/2$. This shows that

$$\lim_{r\to0}a^\omega_{n_r}(B_r)=0.\tag{6.3.36}$$

Using the boundedness of densities we see that

$$\sum_{l=1}^{n_r}\mu_l(B_r)\leq n_rD\mu^\omega(B_r)\leq D\mu^\omega(B_r)^{1-\theta},\tag{6.3.37}$$

so the third expression, we want to estimate, will limit at 0 for $\theta < 1$.

The second expression $b^{\omega}_{n_r}(B_r)$ remains to be estimated. Point x is fixed, as is a Borel set V, but the function $\kappa(r)$ is taken satisfying the *thin annuli* assumption.

$$\begin{aligned}\left|\mu^{\omega}\big(B_r \cap T^{-N}V\big) - \mu^{\omega}(B_r)\mu^{S^n\omega}\big(V\big)\right| &= \left|\mu^{\omega}\big(\mathbb{1}_V \circ T^N \cdot f_r\big) - \mu^{\omega}(\mathbb{1}_V)\mu^{S^n\omega}(f_r)\right| \leq \\ &\leq \left|\mu^{\omega}\big(\mathbb{1}_V \circ T^N \cdot f_r\big) - \mu^{\omega}\big(\mathbb{1}_V \circ T^N \cdot g_r\big)\right| + \\ &+ \left|\mu^{\omega}\big(\mathbb{1}_V \circ T^N \cdot g_r\big) - \mu^{\omega}(\mathbb{1}_V)\mu^{S^n\omega}(g_r)\right| + \\ &+ \left|\mu^{\omega}(\mathbb{1}_V)\mu^{S^n\omega}(g_r) - \mu^{\omega}(\mathbb{1}_V)\mu^{S^n\omega}(f_r)\right|.\end{aligned}$$

So as before the estimation of $\mu^{\omega}(B_r)b^{\omega}_{n_r}(B_r)$ is done by three parts.

For the third part we use the thin annuli assumption.

$$\begin{aligned}\mu^{\omega}(B_r)^{-1}\left|\mu^{\omega}(\mathbb{1}_V)\mu^{S^n\omega}(g_r) - \mu^{\omega}(\mathbb{1}_V)\mu^{S^n\omega}(f_r)\right| &\leq \mu^{\omega}(B_r)^{-1}\big(\mu^{S^n\omega}(g_r) - \mu^{S^n\omega}(f_r)\big) \leq \\ &\leq \mu^{\omega}(B_r)^{-1}\Big(\mu^{S^n\omega}\big(B(x, r + r^{\kappa(r)})\big) - \mu^{S^n\omega}\big(B(x,r)\big)\Big) = \\ &= \frac{\mu^{S^n\omega}\left(B(x, r + r^{\kappa(r)}) \setminus B(x,r)\right)}{\mu^{\omega}(B_r)}.\end{aligned}$$

And this tends to 0. The first part is bounded this way as well:

$$\begin{aligned}\mu^{\omega}(B_r)^{-1}\left|\mu^{\omega}\big(\mathbb{1}_V \circ T^N \cdot f_r\big) - \mu^{\omega}\big(\mathbb{1}_V \circ T^N \cdot g_r\big)\right| &\leq \mu^{\omega}(B_r)^{-1}\big(\mu^{\omega}(g_r) - \mu^{\omega}(f_r)\big) \\ &\leq \frac{\mu^{\omega}\left(B(x, r + r^{\kappa(r)}) \setminus B(x,r)\right)}{\mu^{\omega}(B_r)}\end{aligned}$$

The second needs the decay of correlations:

$$\left|\mu^{\omega}\big(\mathbb{1}_V \circ T^N \cdot g_r\big) - \mu^{\omega}(\mathbb{1}_V)\mu^{S^n\omega}(g_r)\right| \leq C\gamma^N r^{-\kappa(r)}\mu^{\omega}(\mathbb{1}_V) \leq C\gamma^N r^{-\kappa(r)}$$

Bound on the pointwise dimensions (6.3.34) gives us $n_r = \mu^{\omega}(B_r)^{-\theta} \geq r^{-\theta\underline{d}_{\mu^{\omega}}(x)/2}$ and (using it again) we get

$$\begin{aligned}\mu^{\omega}(B_r)^{-1}\left|\mu^{\omega}\big(\mathbb{1}_V \circ T^N \cdot g_r\big) - \mu^{\omega}(\mathbb{1}_V)\mu^{S^n\omega}(g_r)\right| &\leq C r^{-\kappa(r) - 2\overline{d}_{\mu^{\omega}}(x)}\gamma^{r^{-\theta\underline{d}_{\mu^{\omega}}(x)/2}} = \\ &= Ce^{-\kappa(r)\ln(r) - 2\overline{d}_{\mu^{\omega}}(x)\ln(r) + r^{-\theta\underline{d}_{\mu^{\omega}}(x)/2}\ln(\gamma)},\end{aligned}$$

and the second expression converges to zero as $r \to 0$ if

$$\lim_{r\to 0} \kappa(r)\ln(r) r^{\theta\underline{d}_{\mu^{\omega}}(x)/2} = 0, \tag{6.3.38}$$

which follows from the definition of *subpoly* functions.

The three estimates go to 0 and we reach a conclusion

$$\lim_{r\to 0} b^{\omega}_{n_r}(B_r) = 0, \tag{6.3.39}$$

which ends our proof. □

Chapter 7

Indecomposable continua

This chapter deals with a rather different problem. Instead of stating and proving some results about recurrence, we will use this behaviour of points in a proof of a different problem.

Here we calculate the Hausdorff dimensions of some special sets appearing in complex dynamics. As it happens, one of the key steps in the proof is observing the recurrence of points.

However, this is not as easy as the idea in section 3.4 and we shall devote the entire chapter to this result. To see an outline of the role of recurrence in this proof go to sec. 7.2.1.

The contents of this chapter is being published as [PZ].

The chapter is organised as follows. Firstly, there is a small introduction to complex dynamics. Then in section 7.2 we formulate some conditions on the map ("super-growing parameters", see Definition 7.1). Then we introduce and describe some forward invariant sets for the dynamics of the map f_λ, where λ is a super-growing parameter.

Then we formulate the main result of this chapter (Theorem 7.4) on the upper bound of the Hausdorff dimension of these forward invariant sets, namely, that the Hausdorff dimension of such set is not larger than 1.

The proof of Theorem 7.4 is divided into several steps, and it is presented in section 7.3 and its subsections.

Afterwards, in section 7.4 we show in detail how Theorem 7.4 can be used to estimate the dimension of the particular, dynamically defined indecomposable continua. In each case when the application of Theorem 7.4 is possible, we get the strongest possible upper bound on the dimension, proving that the Hausdorff dimension of the continuum is equal to one. See Theorems 7.17 and 7.19 for the precise statement.

Finally, in the subsection 7.4.3 we show that the one–dimensional Hausdorff measure of an indecomposable continuum in the plane is not σ-finite. As mentioned earlier, one could try to prove this using the technique from section 3.4, but the proof is simply too easy, to look for a different one.

7.1 Complex dynamics

The dynamics of transcendental meromorphic functions has been developed as a counterpart to the theory of rational iterations. A model family $\lambda \mapsto \lambda \exp(z)$ has been a subject of a particular interest, playing a similar role as the family of quadratic polynomials $z^2 + c$ in the iterations of rational maps, see e.g. [Dev99] for a review of results on this family.

There are several properties of the iterations of entire (and, more general, transcendental) maps which have no counterpart in the dynamics of rational maps.

Below, we recall basic definitions and facts.

7.1.1 Introduction

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a transcendental meromorphic function. As usually, we shall denote by f^n the n-th iterate of f: $f^n = f \circ f \circ \cdots \circ f$. The sequence $f^n(z)_{n=0}^{\infty}$ is called the trajectory of z. The Fatou set $F(f)$ consists of all points $z \in \mathbb{C}$ for which there exists a neighbourhood $U \ni z$ such that the family of iterates $f^n_{|U}$ is defined in U and forms a normal family. The complement of $F(f)$ is called the Julia set of f and it is denoted by $J(f)$. An intuitive characterisation of the Julia set says that it carries the chaotic part of the dynamics. See e.g. [Ber96] and [JU08] for a detailed presentation of the theory.

For the particular family of maps $f_\lambda(z) = \lambda e^z$ the structure of the Julia set and the dynamics have been studied intensively. In 1981 M. Misiurewicz answered the old question (formulated by Fatou) and proved that the Julia set of the map $f_1(z) = e^z$ is the whole plane ([Mis81]). A striking result, proved independently by M. Lyubich [Lyu87] and M. Rees [Ree86] says that, nevertheless, the map is not ergodic with respect to the two-dimensional Lebesgue measure, and for Lebesgue almost every point z the ω-limit set of z (i.e. the set of accumulation points of the trajectory) is just the trajectory of the singular value 0 plus the point at infinity.

On the other hand, there is an open set of parameters λ for which there exists a periodic attracting orbit (i.e. a point p_λ such that $f_\lambda^k(p_\lambda) = p_\lambda$ and $|(f_\lambda^k)'(p_\lambda)| < 1$ for some $k \in \mathbb{N}$). In this case, the Fatou set is nonempty and it contains a neighbourhood of this periodic attracting orbit. Actually, it turns out that in this case the Fatou set is just the basin of attraction of this periodic orbit, i.e.

$$F(f_\lambda) = \left\{ z \in \mathbb{C} : \lim_{n\to\infty} f_\lambda^{nk}(z) = f^i(p_\lambda) \text{ for some } i \in \{0, 1, \dots k-1\} \right\}.$$

Moreover, this basin of attraction must contain the whole trajectory of the singular value 0. For instance, if $\lambda \in (0, \frac{1}{e})$, then f_λ has a real attracting fixed point p_λ. For these values of λ the Fatou set of f_λ is connected and simply connected and its complement $J(f_\lambda)$ is a "Cantor bouquet" of curves. See [Dev99] or [Sch07] for a survey of results in this direction.

A general classification theorem for the components of the Fatou set leads to the following corollary: if the parameter λ is chosen so that $f_\lambda^n(0) \to \infty$ then the Julia set of f_λ is the whole plane: $J(f_\lambda) = \mathbb{C}$ (see [EL92] or [GK86]).

In this work, we shall always assume that the parameter λ is chosen so that $f_\lambda^n(0) \to \infty$ sufficiently fast. Thus, in particular, for all maps f_λ considered here we have $J(f_\lambda) = \mathbb{C}$ (see section 7.2 for setting of precise assumptions).

7.1.2 Motivation for the problem

It is known that the set of escaping points

$$I(f_\lambda) = \{z : f_\lambda^n(z) \to \infty\}$$

can be described in terms of so-called "dynamic rays" (see [SZ03] and [Rem06]). Each dynamic ray is a curve $g_{\underline{s}} : (t_{\underline{s}}, \infty) \to \mathbb{C}$. Moreover, $\mathrm{Re}(g_{\underline{s}}(t)) \to +\infty$ as $t \to +\infty$. Each such curve is characterised by an infinite sequence $\underline{s} = (s_0, s_1, \dots,)$ of integers. The sequence $\underline{s}$ is frequently called the "external address" of the curve $g_{\underline{s}}$. The dynamical meaning of the sequence $\underline{s}$ is the following: Let us divide the plane $\mathbb{C}$ into horizontal strips

$$P_k = \{z \in \mathbb{C} : (2k-1)\pi - \mathrm{Arg}(\lambda) < \mathrm{Im}(z) \leq (2k+1)\pi - \mathrm{Arg}(\lambda)\}. \tag{7.1.1}$$

If z is a point on the curve $g_{\underline{s}}$, $z = g_{\underline{s}}(t)$ with t sufficiently large, then, for every $n \geq 0$, $f_\lambda^n(z) \in P_{s_n}$.

The classification of escaping points (see [SZ03], Corollary 6.9) says that this family of curves almost exhausts the set $I(f_\lambda)$. Namely, if $z \in I(f_\lambda)$ then either z belongs to some curve $g_{\underline{s}}$, or z is a landing point of some curve $g_{\underline{s}}$, or else the singular value 0 escapes, $0 = g_{\underline{s}}(t_0)$ for some $t_0 > t_{\underline{s}}$, and z is eventually mapped to the initial piece of the curve $g_{\underline{s}}$, cut off by the point 0, i.e.

$$f_\lambda^n(z) = g_{\underline{s}}(t')$$

for some $t_{\underline{s}} < t' < t_0$.

As an example, let us consider $\lambda = 1$ and the dynamical ray corresponding to the sequence $\underline{s} = (0, 0, 0, \dots)$. This set is just the real line $\mathbb{R}$. The set

$$\bigcup_{n \geq 0} f_\lambda^{-n}(\mathbb{R}) \cap \{\mathrm{Im}(z) \in [0, \pi]\}$$

is a union of infinitely many arcs extending to infinity. It was observed first by R. Devaney (see [Dev99]) that the closure of this set, after a natural compactification at ∞, becomes an indecomposable continuum. This continuum can be equivalently described as the set of points whose trajectories remain in the strip $\mathrm{Im}(z) \in [0, \pi]$.

Next, again for $\lambda = 1$, R. Devaney and X. Jarque discovered the existence of indecomposable continua defined as an accumulation set, in the Riemann sphere, of some dynamic rays. More precisely, they considered in [DJ02] the rays of the form $g_{\underline{s}}$, where $\underline{s} = (t_{m_1}, 0_{n_1}, t_{m_2}, 0_{n_2} \dots)$, where t_{m_j} are blocks of integers, of length m_j, with all digits $\leq M$. If the blocks of zeros 0_{n_j} are sufficiently long, then the set of accumulation points of the ray $g_{\underline{s}}$ becomes an indecomposable continuum containing $g_{\underline{s}}$.

A more general result appeared in [Rem07]. The author shows the following: assume that the singular value 0 of the map f_λ is on a dynamic ray, or it is a landing point of such a ray. Then there are uncountably many dynamic rays g whose accumulation set (in the Riemann sphere) is an indecomposable continuum containing g.

The considered problem was motivated by the above-mentioned examples. Our goal was to give a bound for the Hausdorff dimension of dynamically defined indecomposable continua appearing in the exponential dynamics. It was already observed in [Dev93] that M.Lyubich's result [Lyu87] implies that two-dimensional Lebesgue measure of the indecomposable continuum described in [Dev99], is equal to 0. Actually, it is easy to see that the same remark applies to the examples considered in [DJ02].

Our goal in this chapter is to prove, for a class of dynamically defined indecomposable continua for the exponential family, that their Hausdorff dimension is equal to one, so it takes on the smallest possible value. This class of continua contains, in particular, the examples described in [Dev99] and [DJ02]. So, our result is an essential strengthening of the previously known estimates.

Our results apply also to a subclass of the class of continua described in [Rem07]. It is now natural to ask about the possible values of the Hausdorff dimension of other dynamically defined indecomposable continua (appearing in the dynamics of exponential maps), not covered by our considerations (in particular — all the continua described in[Rem07]).

7.2 Result

After the introduction we are ready to formulate the results. As usual, we start with some definitions.

Throughout this chapter we will work with the function $f_\lambda(z) = \lambda e^z$ for which the trajectory of the singular value 0 tends to infinity exponentially fast. We keep the following definition and notation, introduced in [UZ07]. Let

$$\beta_n = f_\lambda^n(0), \quad \alpha_n = \mathrm{Re}(\beta_n). \tag{7.2.1}$$

Definition 7.1. We say that the parameter λ is *super-growing* if $\alpha_n \to \infty$ and there exists a constant $c > 0$ such that, for all n large enough,

$$\alpha_{n+1} \geq c e^{\alpha_n}. \tag{7.2.2}$$

Note that the above condition is equivalent to

$$|\beta_{n+1}| \geq |\lambda| \exp\left(\frac{c}{|\lambda|}|\beta_n|\right) \tag{7.2.3}$$

and to

$$\alpha_n \geq \frac{c}{|\lambda|}|\beta_n|. \tag{7.2.4}$$

From now on, unless stated otherwise, we assume that the parameter λ satisfies the super-growing condition (7.2.2), with the constant c. We shall keep the symbols α and β (7.2.1). To simplify the notation, we will write $f(z)$ instead of $f_\lambda(z)$, if it does not lead to a confusion.

Remark. A natural group of examples, for which the condition (7.2.2) is satisfied, is if $|\mathrm{Im}(\beta_n)|$ is bounded and $f_\lambda^n(0) \to \infty$ as $n \to +\infty$. Moreover, it was proved in [Wei94] that the Hausdorff dimension of parameters λ satisfying the super-growing condition is equal to two. (See also [FS09] for a detailed description of the structure of parameters λ for which the trajectory of the singular value escapes to ∞.)

We will be able to prove our results for sets that do not grow too much at infinity.

Definition 7.2. Let $W \subset \mathbb{C}$ be a closed set. Denote by $W(R)_+$ the intersection $W(R)_+ = W \cap \{\mathrm{Re} z = R\}$ and, analogously, $W(R)_- = W \cap \{\mathrm{Re} z = -R\}$. Let

$$w(R) = \max(\mathrm{diam} W(R)_+, \mathrm{diam} W(R)_-).$$

We call the set W *thin* if there is a constant $K > 0$ such that

$$\frac{|z|}{|\mathrm{Re}(z)| + 1} < K \tag{7.2.5}$$

for all $z \in W$, and also

$$\lim_{R\to\infty} \frac{\log_+ w(R)}{\log R} = 0. \tag{7.2.6}$$

Note. Obviously, every horizontal strip

$$\{z \in \mathbb{C} : |\mathrm{Im}(z)| \le P\}$$

is a thin set.

Note also that the condition (7.2.5) implies that there is a cone symmetric with respect to the real axis, with the opening angle smaller that π and $R_0 > 0$ such that the intersection $W \cap |\{\mathrm{Re}(z)| \ge R_0\}$ is contained in this cone.

The following definition introduces the set which will be an object of our estimates.

Definition 7.3. Let $\lambda \in \mathbb{C} \setminus \{0\}$ and let W be a thin set. We define the set Λ_W as

$$\Lambda_W = \{z \in \mathbb{C} : f^n(z) \in W \text{ for every } n \ge 0\}.$$

Observe that the set Λ_W is a forward invariant, i.e. $f(\Lambda_W) \subset \Lambda_W$, closed subset of $\mathbb{C}$.

Note. We may think of this as a set of very strongly recurrent points, i.e. points that return to W at every single step.

Remark. Since we do not assume any dynamical condition on W, it may even happen that the set Λ_W is empty. However, in all our applications (see section 7.4) the set Λ_W contains a non-trivial continuum, so it has Hausdorff dimension at least one.

We shall prove the following.

Theorem 7.4. *Let λ be a super-growing parameter, let W be a thin set and let Λ_W be the set defined in Definition 7.3.*

Put

$$Y_M = \{z \in \mathrm{C} : |\mathrm{Re}(z)| \geq M\}. \tag{7.2.7}$$

Then

$$\lim_{M\to\infty} \dim_H(\Lambda_W \cap Y_M) \leq 1. \tag{7.2.8}$$

Note. The limit exists because the function $M \mapsto \dim_H(\Lambda_W \cap Y_M)$ is non-increasing.

So, writing Λ_W as a union $\Lambda_W = \Lambda_{W,bd} \cup \Lambda_{W,ubd}$, where $\Lambda_{W,bd}$, $\Lambda_{W,ubd}$ denote the subset of points in Λ_W with bounded (resp.: unbounded) trajectory, we have

Corollary 7.5.

$$\dim_H(\Lambda_{W,ubd}) \leq 1.$$

Proof. Obviously, for every $M > 0$,

$$\Lambda_{W,ubd} \subset \bigcup_{n=0}^{\infty} f^{-n}(\Lambda_W \cap Y_M).$$

Since f is an analytic non-constant map, every Borel set A has the same Hausdorff dimension as its preimage $f^{-1}(A)$. Taking a countable union of sets of the same dimension does not increase the dimension. So, Corollary follows immediately from Theorem 7.4. $\square$

7.2.1 Recurrence in the proof

The estimate on the Hausdorff dimension is obtained in a typical way — by building a sequence of coverings and estimating their measure.

The set $\Lambda_W \cap Y_M$ is naturally divided into two sets: W_M^+ on $\{\mathrm{Re}(z) > 0\}$ and W_M^- other on $\{\mathrm{Re}(z) < 0\}$.

For the first set the covering is built easily. We use the fact that if $\mathrm{Re}(z)$ is big enough, then $|f'(z)|$ is as big as we need it to be.

The second set requires more work and it is here, where we utilise the recurrence.
We show that almost any point returns arbitrarily close to 0 (which is a *special* point for e^z). The only way a point can get close to 0 is through the set $\{\mathrm{Re}(z) < -M\}$ for large M. Then we show that a point close to 0 must not leave the neighbourhood of the trajectory of zero for a certain number of iterations (check subsec. 7.3.3 and particularily fig. 7.4). Such recurrent behaviour of points in Λ_W allows us the define the covering on the set W_M^-.

Additionally, we show that — in the many interesting cases, cf. sec. 7.4 — all points of Λ_W return to $\Lambda_W \cap Y_M$ infinitely often. This proves that $\dim_H(\Lambda_W) = 1$.

7.3 Proof

The proof of Theorem 7.4 is split into several steps. In subsection 7.3.1 we claim the existence of the special induced map, and we formulate, in Proposition 7.6, some numerical estimates for this map. Next, the precise definition of the map and the proof of the required estimates is presented in subsections 7.3.2 and 7.3.3. Finally, in subsection 7.3.4 we use these estimates to conclude the proof of Theorem 7.4.

7.3.1 The induced map

Obviously, to prove Theorem 7.4 it is enough to consider integer values of M. So, from now one we shall assume that $M \in \mathbb{N}$.

In this subsection, we claim the existence of an auxiliary induced map, with certain properties, see Proposition 7.6 below.

First, we cut each horizontal strip P_k (see (7.1.1)) into rectangles

$$R_r^k = \{z \colon r \leq \mathrm{Re}(z) < r+1\} \cap P_k. \tag{7.3.1}$$

For an arbitrary $M \in \mathbb{N}$ denote by Z_M the family of all rectangles R_r^k intersecting $W \cap Y_M$.

Note that our assumption (7.2.6) on W implies that the number $n(r)$ of rectangles in Z_M intersecting the lines $\mathrm{Re}(z) = \pm r$, satisfies

$$\log n(r) = o(\log(r)).$$

The map F will be defined in the union W_M of all rectangles intersecting the set $W \cap Y_M$:

$$W_M = \bigcup_{R_r^k \in Z_M} R_r^k.$$

The set W_M splits naturally into two subsets:

$$W_M^+ = W_M \cap \{z : \mathrm{Re}(z) > 0\},$$

$$W_M^- = W_M \cap \{z : \mathrm{Re}(z) < 0\}.$$

Or, writing explicitely:

$$W_M = W_M^+ \cup W_M^- = \left(W_M \cap Y_M^+\right) \cup \left(W_M \cap Y_M^-\right),$$

where $Y_M^+ = \{z \in \mathbb{C} \colon \mathrm{Re}(z) \geq M\}$ and $Y_M^- = \{z \in \mathbb{C} \colon \mathrm{Re}(z) \leq -M\}$.

The next Proposition summarizes the required properties of the induced map F.

Proposition 7.6. *Let W be a thin set. Let λ be a super-growing parameter. For every $\delta > 0$ there exist $M \in \mathbb{N}$ and a map F defined in W_M, with the following properties:*

- *F is constructed with appropriate iterates of the map f and for every rectangle $R_r^k \in Z_M$ we have $F_{|R_r^k} = f^n$ for some $n \in \mathbb{N}$ (dependent on r and k).*
- $$F(\Lambda_W \cap Y_M) \subset \Lambda_W \cap Y_M. \tag{7.3.2}$$
- *For every rectangle $R_r^k \in Z_M$ the set $R_r^k \cap F^{-1}(W_M)$ can be covered by a family $\mathcal{F}_r^k$ of disjoint subsets of R_r^k such that each set $Q \in \mathcal{F}_r^k$ is mapped by F bijectively onto its image $F(Q)$, which is contained in some rectangle $\widehat{R}_s^m \in Z_M$. Moreover, the holomorphic branch of F^{-1} mapping $F(Q)$ back onto Q is well defined in a neighbourhood of the whole rectangle $\widehat{R}_s^m$ and*

$$\sum_{Q \in \mathcal{F}_r^k} \sup_{w \in Q} |F'(w)|^{-(1+\delta)} < \frac{1}{2}. \tag{7.3.3}$$

7.3.2 Definition and the estimates for the map F in W_M^+

Let λ be a super-growing parameter. Recall that we use the simplified notation $f = f_\lambda$ and that W is a thin set.

We start with a simple, but very useful lemma.

Lemma 7.7. *Fix some $\delta \in (0, 1)$, and put $\delta' = \frac{\delta}{2}$. There exist $\widehat{C}$ (independent of δ) and $M > 0$ (depending on δ) such that, for $r \geq M$, and every rectangle R_r^k*

$$\sum_{\widehat{R}_s^l \cap f(R_r^k) \neq \emptyset} \sup_{w \in f^{-1}\left(\widehat{R}_s^m\right) \cap R_r^k} |f'(w)|^{-(1+\delta)} \leq \widehat{C} e^{-r\delta'}, \tag{7.3.4}$$

where the summation runs over all rectangles $\widehat{R}_s^m \in Z_M$ intersecting $f(R_r^k)$.

Proof. We assume now that M is so large that $w(s) < s^{\delta'}$ for all $s \geq M$.

If $z \in R_r^k$, we have

$$|f'(z)| = |f(z)| = |\lambda| e^{\mathrm{Re}(z)} \geq |\lambda| e^r \tag{7.3.5}$$

The number of rectangles $\widehat{R}_s^m \in Z_M$ for which $\widehat{R}_s^m \cap f(R_r^k) \neq \emptyset$ can be estimated (very roughly) by

$$\sum_{M \leq s \leq |\lambda| e^{r+1}} w(s) \leq \sum_{M \leq s \leq |\lambda| e^{r+1}} s^{\delta'} \leq C e^{(1+\delta')(r+1)}.$$

Thus,

$$\sum_{\widehat{R}_s^m \cap f(R_r^k) \neq \emptyset} \sup_{w \in f^{-1}\left(\widehat{R}_s^m\right) \cap R_r^k} |f'(w)|^{-(1+\delta)} \leq C e^{(1+\delta')(r+1)} |\lambda|^{-(1+\delta)} e^{-r(1+\delta)} = \widehat{C} e^{-r\delta'},$$

where C and $\widehat{C}$ are constants independent of M and δ (see Figure 7.1). $\square$

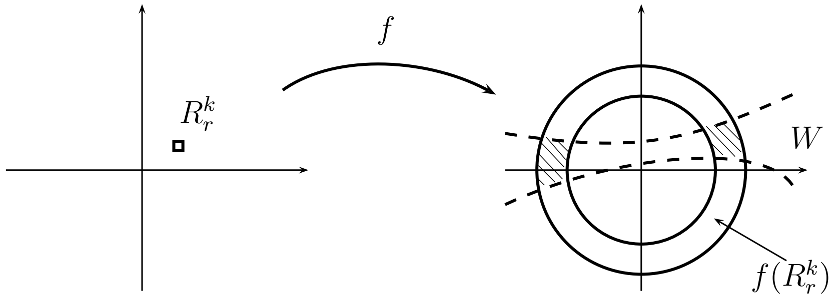

Figure 7.1: f in W_M^+.

Now, we turn to the the definition of the family $\mathcal{F}_r^k$ and prove an estimate similar to (7.3.3).

Denote by $\mathcal{F}_r^k$ the family of connected components Q of sets of the form $f^{-1}(\widehat{R}_s^m) \cap R_r^k$. Note that there are at most two components, if $\widehat{R}_s^m$ intersects the negative real line; and at most one, if it does not. (In Figure 7.2 the 'middle' set $\widehat{R}_s^m$ has two disjoint preimages in R_r^k.) The branch of the logarithm is well defined in a neighbourhood of any $\widehat{R}_s^m \in Z_M$, and so is our holomorphic branch of the inverse map. We thus proved the inequality:

$$\sum_{Q \in \mathcal{F}_r^k} \sup_{w \in Q} |f'(w)|^{-(1+\delta)} \leq 2\widehat{C} e^{-r\delta'}. \tag{7.3.6}$$

Let us return to the proof of Proposition 7.6. The map F is defined in W_M^+ simply as $F(z) = f(z)$. Thus, F is one-to-one, if restricted to any rectangle R_r^k.

Assume that M is so large that $\widehat{C} e^{-M\delta'} < \frac{1}{4}$. Then the inequality (7.3.6) proves the estimate (7.3.3):

$$\sum_{Q \in \mathcal{F}_r^k} \sup_{w \in Q} |F'(w)|^{-(1+\delta)} \leq 2\widehat{C} e^{-r\delta'} < \frac{1}{2}. \tag{7.3.7}$$

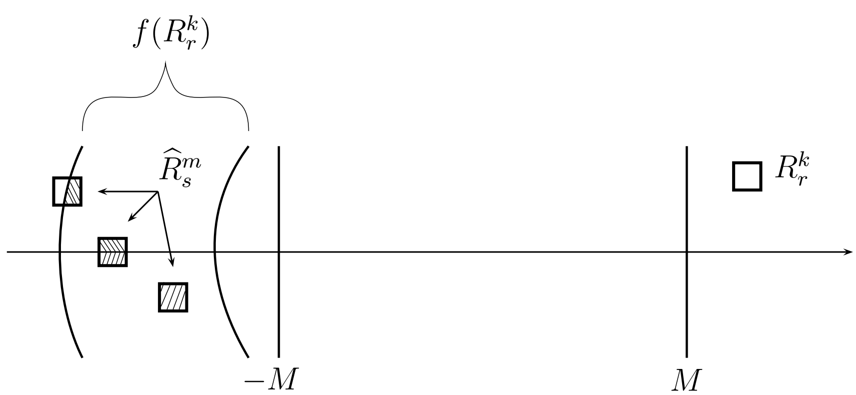

Figure 7.2: Intersections $\widehat{R}_s^m \cap f(R_r^k)$.

Lemma 7.8. *For M large enough the following inclusion holds:*

$$F(\Lambda_W \cap Y_M^+) \subset \Lambda_W \cap Y_M,$$

where $Y_M^+ = \{z \in \mathbb{C} \colon \operatorname{Re}(z) \geq M\}$.

Proof. It is obvious that $f(\Lambda_W) \subset \Lambda_W$ (cf. Def. 7.3). So, $F(\Lambda_W \cap Y_M^+) \subset \Lambda_W$.

Let $z \in \Lambda_W \cap Y_M^+$. Since $f(z) \in W$, we have

$$K(|\operatorname{Re}(f(z))| + 1) > |f(z)| = |\lambda| e^{\operatorname{Re}(z)} \geq |\lambda| e^M.$$

where K is the constant coming from the definition of a thin set.

Thus,

$$|\operatorname{Re}(F(z))| = |\operatorname{Re}(f(z))| \geq \frac{1}{K}|\lambda| e^M - \frac{1}{K}.$$

Thus, if M is large enough, we conclude that $|\operatorname{Re}(F(z))| \geq M$. □

Remark. Note that, in this part of the proof we did not use the *super-growing* property. We used only the fact that the domain of the map is contained in $\operatorname{Re}(z) \geq M$ and that the set W is thin. It is worth to observe that above calculation is close in spirit to the argument contained in [Kar99].

7.3.3 Definition and the estimates for the map F in W_M^-.

This part is more difficult. Also here we will be interested in some recurrence, but let us start with the following easy observation.

Proposition 7.9. *If the singular value* 0 *does not belong to the set* Λ_W*, then there exists* $M > 0$ *such that the set* Λ_W *is contained in the right half-plane* $\operatorname{Re}(z) \geq -M$.

Proof. If $0 \notin \Lambda_W$, then there exists $k \geq 0$ such that $f^k(0) \notin W$, so there exists a ball $B(0, \eta)$ such that $f^k(z) \notin W$ for every $z \in B(0, \eta)$. Consequently, $B(0, \eta) \cap \Lambda_W = \emptyset$. Thus, if $\operatorname{Re}(w) < \log \eta - \log |\lambda|$, then $w \notin \Lambda_W$ (since $f(w) \in B(0, \eta)$). □

Thus, the part of the proof contained in this subsection is void in the case when $0 \notin \Lambda_W$. Or in other words no point that is far to the left of the plane may return to W at every step.

The definition of the map F and the proof of (7.3.3) for r negative is more involved and it uses the *super-growing* condition.

The proof of the following technical lemma is straightforward and left to the reader.

Lemma 7.10. *Let* λ *be a super-growing parameter,* $\alpha_n = \operatorname{Re} f_\lambda^n(0)$. *Then for every* $\varepsilon > 0$ *there exists* N *such that for all* $n > N$

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n < \varepsilon \alpha_{n+1}. \tag{7.3.8}$$

Below, we keep the assumption that λ is a super-growing parameter and, as before, we abbreviate the notation writing $f := f_\lambda$. Let us fix some constants.

The Koebe distortion theorem implies that for every univalent map f defined on some ball $B(z,r)$, the distortion of f', restricted to the ball $B(z,\frac{r}{2})$ is bounded by some constant, independent of the map. This constant will be denoted by L. Recall the *super-growing* equivalent conditions (7.2.2), (7.2.3) and (7.2.4). Put

$$D = \frac{1}{4}\frac{c}{|\lambda|} \leq \frac{1}{4},$$

where the constant c comes from the super-growing conditions.

It follows from the condition (7.2.3) that there exists l_0 such that, for all $l \geq l_0$, the ball $B(\beta_l, 2D|\beta_l|)$ contains only one point of the trajectory of the point 0. Put

$$B_l := B(\beta_l, D|\beta_l|).$$

Thus, the inverse branches of f^l are well defined in B_l, with distortion bounded by L.

Let f_0^{-l} denote the branch of f^{-l} following the backward trajectory $\beta_l, \beta_{l-1}, \ldots \beta_0 = 0$; we put $\widetilde{B}_l = f_0^{-l}(B_l)$. Then $\widetilde{B}_l$ is a topological disc containing the point 0. Since $|(f^l)'(0)| = |\beta_1| \cdot \cdots \cdot |\beta_l|$ and since the branch f_0^{-l}, sending β_l to 0, is also well defined on the ball twice larger $2 \cdot B_l = B(\beta_l, 2D|\beta_l|)$, the set $\widetilde{B}_l$ is contained in the ball

$$B\left(0, \frac{LD|\beta_l|}{|\beta_1|\cdots|\beta_{l-1}||\beta_l|}\right) = B\left(\frac{LD}{|\beta_1|\cdots|\beta_{l-1}|}\right) \tag{7.3.9}$$

and contains the ball

$$B\left(0, \frac{\frac{1}{4}D|\beta_l|}{|\beta_1|\cdots|\beta_{l-1}||\beta_l|}\right) = B\left(0, \frac{\frac{1}{4}D}{|\beta_1|\cdots|\beta_{l-1}|}\right). \tag{7.3.10}$$

Next, put $\widetilde{G}_l = f^{-1}(\frac{1}{e}\widetilde{B}_l)$. where $\frac{1}{e}\widetilde{B}_l$ is the image of $\widetilde{B}_l$ under the rescaling $z \mapsto \frac{1}{e}z$. So, $\widetilde{G}_l$ is an unbounded set containing some left half-plane. Finally, put $G_l = \widetilde{G}_l \cap W$ (see Figure 7.3).

Lemma 7.11. *For all l large enough,* $\mathrm{cl}\widetilde{G}_{l+1} \subset \widetilde{G}_l$. *Consequently,* $\mathrm{cl}G_{l+1} \subset G_l$.

Proof. Indeed, it follows from (7.3.9) and (7.3.10) that, for l large, $\mathrm{cl}\widetilde{B}_{l+1} \subset \widetilde{B}_l$. □

Finally, set V_l to be the union of all rectangles $R_r^k \in Z_M$ which intersect the set $G_l \setminus G_{l+1}$. Thus the sets V_l are not disjoint, but, for large l, one rectangle R_r^k may intersect two sets V_l at most (see Figure 7.3).

Given $l_0 \in \mathbb{N}$, we put $M = [\alpha_{l_0}] + 1$. Note that, by the above estimates, we have:

Lemma 7.12. *If l_0 is large enough then*

$$W_M^- \subset \bigcup_{l=l_0+1}^{\infty} V_l.$$

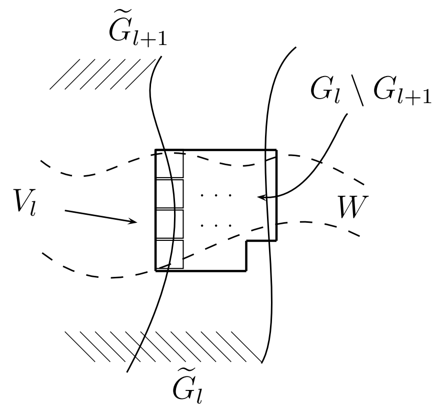

Figure 7.3: Definitions of sets V_l, $\widetilde{G}_l$, etc.

Proof. Using the fact that $\widetilde{B}_l$ contains the ball $B(0, \frac{\frac{1}{4}D}{|\beta_1|\cdots|\beta_{l-1}|})$, we conclude that the set $\widetilde{G}_l$ contains the left half-plane

$$\{\mathrm{Re}(z) < -\log 4 - 1 + \log D - (\alpha_{l-2} + \cdots + \alpha_0) - l \log |\lambda|\}. \tag{7.3.11}$$

It follows from (7.3.11) and Lemma 7.10 that, for large l, $\widetilde{G}_l$ contains the left half-plane

$$\{\mathrm{Re} z \leq -\alpha_{l-1}\}.$$

Thus,

$$W_M^- = W \cap \{\mathrm{Re} z \leq -([\alpha_{l_0}] + 1)\} \subset W \cap \widetilde{G}_{l_0+1} = G_{l_0+1} \subset \bigcup_{l=l_0+1}^{\infty} V_l.$$

□

Below, we define the map F, separately on each set V_l. On each of those sets we know how long a point follows the trajectory of 0. This allows the following.

For $z \in V_l$ define $F(z) = f \circ f^l \circ f = f^{l+2}$. Note that the set $\widetilde{G}_l$ is mapped by f onto $\frac{1}{e}\widetilde{B}_l$. Thus, f maps V_l into $\widetilde{B}_l$ and we have

$$V_l \xrightarrow{f} \widetilde{B}_l \xrightarrow{f^l} B_l \xrightarrow{f} f(B_l).$$

Note that the map is neither onto nor injective. Obviously, the map $F|_{V_l}$ has a holomorphic extension to $\mathbb{C}$.

The following lemma can be proved similarly as Lemma 7.8.

Lemma 7.13. *Put $M = [\alpha_{l_0}] + 1$. If l_0 is sufficiently large then, for all $l \geq l_0$,*

$$F(\Lambda_W \cap V_l) \subset \Lambda_W \cap Y_M.$$

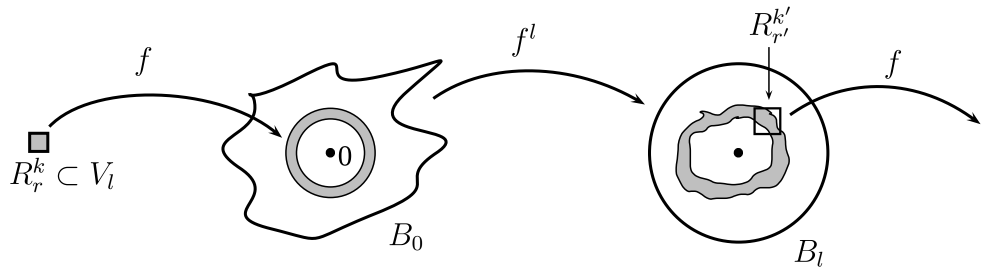

Figure 7.4: Defining F in W^-_M.

Proof. It is obvious that $F(\Lambda_W \cap V_l) \subset \Lambda_W$. Next, $f^{l+1}(V_l) \subset B_l$, so taking $w \in \Lambda_W \cap V_l$, we have for $f^{l+1}(w) = z \in f^{l+1}(V_l)$:

$$\mathrm{Re}(z) \geq \mathrm{Re}(\beta_l) - D|\beta_l| \geq \alpha_l - \frac{1}{4}\alpha_l = \frac{3}{4}\alpha_l.$$

Thus,

$$|F(w)| = |f(f^{l+1})(w)| \geq e^{\frac{1}{2}|\alpha_l|},$$

if l is large enough. As in the proof of Lemma 7.8 we conclude that

$$|\mathrm{Re}(F(w))| \geq \frac{1}{K}|F(w)| - \frac{1}{K} \geq \frac{1}{K}|\lambda|e^{\frac{1}{2}\alpha_l} - \frac{1}{K} > [\alpha_l] + 1 \geq [\alpha_{l_0}] + 1$$

for all $l > l_0$, if l_0 is large enough. Therefore, $F(w) \in Y_M$.

□

Now, we turn to the proof of the estimate (7.3.3) (for negative r). Let us fix some $\delta \in (0, 1)$. Let, as in subsection 7.3.2, $\delta' = \delta/2$.

Let R^k_r be a rectangle in V_l. Then f^{l+1} maps R^k_r bijectively onto its image, contained in B_l, and for $z \in R^k_r$ we have:

$$|(f^{l+1})'(z)| = |f'(z)||(f^l)'(f(z))| = |f(z)||(f^l)'(f(z))| \geq \frac{\frac{1}{4e}D}{|\beta_1\beta_2 \dots \beta_l|} \cdot L^{-1}|\beta_1\beta_2 \dots \beta_l| = \frac{D}{4eL}. \tag{7.3.12}$$

The whole set B_l is covered by $O(|\beta_l|)^2)$ rectangles $R^{k'}_{r'}$ of the initial partition (7.3.1).

On each such rectangle $R^{k'}_{r'}$ the map f is a bijection onto its image. Thus, as in subsection 7.3.2, there a family $\mathcal{F}^{k'}_{r'}$ of disjoint sets $Q^{k'}_{r'}$, such that the set $R^{k'}_{r'} \cap f^{-1}(W_M)$ can be written as a union of the sets $Q^{k'}_{r'} \in \mathcal{F}^{k'}_{r'}$. Each set $Q^{k'}_{r'}$ is defined as the intersection $f^{-1}(\widehat{R}^m_s) \cap R^{k'}_{r'}$ where $\widehat{R}^m_s$ is some rectangle from the family Z_M. Obviously, the inverse map f^{-1} is well defined and holomorphic in a neighbourhood of every rectangle $\widehat{R}^m_s$.

Moreover, since the ball B_l is contained in the set $\{\mathrm{Re}(z) > \frac{1}{2}\alpha_l\}$, for each such rectangle $R^{k'}_{r'}$ Lemma 7.7 tells that

$$\sum_{Q^{k'}_{r'} \in \mathcal{F}^{k'}_{r'}} \sup_{z \in Q^{k'}_{r'}} |f'(z)|^{-(1+\delta)} < \widehat{C}e^{-\delta' \cdot \frac{1}{2}\alpha_l}. \tag{7.3.13}$$

Now, recall that f^{l+1} maps the rectangle R_r^k bijectively onto its image, contained in B_l, see Figure 7.4. Let $g = (f^{l+1}|_{R_r^k})^{-1}$ be the inverse map.

Taking all the preimages $g(Q)$, $Q \in \mathcal{F}_{r'}^{k'}$ over all rectangles $R_{r'}^{k'}$ intersecting B_l, we obtain the family $\mathcal{F}_r^k$ of disjoint subsets of the rectangle R_r^k, such that the union $\bigcup_{Q\in\mathcal{F}_r^k} Q$ covers the whole set $R_r^k \cap F^{-1}(W_M)$ and each set $Q \in \mathcal{F}_r^k$ is mapped by $F = f^{l+2}$ bijectively onto its image, contained in some rectangle $\widehat{R}_s^m \in Z_M$.

Denote by $\mathcal{G}_l$ the family of all rectangles $R_{r'}^{k'}$ intersecting the ball B_l. Using (7.3.12) and (7.3.13), we get the following estimate.

$$\begin{aligned}
&\sum_{Q\in\mathcal{F}_r^k} \sup_{w\in Q} |F'(w)|^{-(1+\delta)} \leq \\
&\leq \sup_{w\in R_r^k} |(f^{l+1})'(w)|^{-(1+\delta)} \cdot \sum_{R_{r'}^{k'}\in\mathcal{G}_l} \left(\sum_{Q_{r'}^{k'}\in\mathcal{F}_{r'}^{k'}} \sup_{z\in Q_{r'}^{k'}} |(f')(z)|^{-(1+\delta)} \right) \\
&\leq (4eLD^{-1})^{(1+\delta)} \cdot \#(\mathcal{G}_l) \cdot \widehat{C} e^{-\delta' \frac{1}{2}\alpha_l}.
\end{aligned} \tag{7.3.14}$$

Since the cardinality $\#(\mathcal{G}_l)$ can be estimated by $\mathcal{O}(|\beta_l|^2)$, we can write

$$\sum_{Q\in\mathcal{F}_r^k} \sup_{w\in Q} |F'(w)|^{-(1+\delta)} \leq C_1 |\beta_l|^2 e^{-\frac{1}{2}\delta'\alpha_l} = C_1 |\beta_l|^2 |\lambda|^{\frac{1}{2}\delta'} |\beta_{l+1}|^{-\frac{1}{2}\delta'},$$

where C_1 is another constant. Using the *super-growing* condition we easily conclude that the right-hand side of the above inequality can be estimated from above by

$$|\beta_{l+1}|^{-\frac{\delta'}{4}},$$

for all $l \geq l_0$, if l_0 is large enough. So, finally we get, for every rectangle R_r^k intersecting V_l,

$$\sum_{Q\in\mathcal{F}_r^k} \sup_{w\in Q} |F'(w)|^{-(1+\delta)} \leq |\beta_{l+1}|^{-\frac{\delta'}{4}} < \frac{1}{2}. \tag{7.3.15}$$

This ends the proof of Proposition 7.6, with $M = [\alpha_{l_0}] + 1$, and F given by

$$F(z) = \begin{cases} f(z) & \text{for} \quad z \in W_M^+ \\ f^{l+2}(z) & \text{for} \quad z \in V_l, \quad l \geq l_0 + 1 \end{cases}. \tag{7.3.16}$$

Remark. Recall that $W_M^- \subset \bigcup_{l=l_0}^{\infty} V_l$, so the map F is defined everywhere in W_M. However, since the sets V_l are not disjoint, there are rectangles R_r^k that are included in both V_l and V_{l+1}. In such case we define the map F on the rectangle R_r^k, choosing arbitrarily one of two possible ways.

7.3.4 Estimates of the dimension

In this subsection, we finish the proof of Theorem 7.4. Fix an arbitrary $\delta > 0$ and let M be so large that the statement of Proposition 7.6 holds true.

We shall define inductively, for every rectangle $R_r^k \in Z_M$, and for every n, a cover $(\mathcal{F}_r^k)^n$ of the set $\Lambda_W \cap R_r^k$, such that

$$\sum_{K \in (\mathcal{F}_r^k)^n} (\mathrm{diam} K)^{1+\delta} < (2\pi + 1) \cdot \frac{1}{2^n}. \tag{7.3.17}$$

The cover $(\mathcal{F}_r^k)^1$ is defined simply as the family of sets $\mathcal{F}_r^k$ described in Proposition 7.6. The condition (7.3.17) is satisfied since

$$\sum_{Q \in \mathcal{F}_r^k} (\mathrm{diamQ})^{1+\delta} \leq \sum_{Q \in \mathcal{F}_r^k} \sup_{w \in Q} |F'(w)|^{-(1+\delta)} \cdot (2\pi + 1) \leq \frac{1}{2} \cdot (2\pi + 1).$$

Assume that the covers $(\mathcal{F}_r^k)^{n-1}$ have been already defined for every rectangle $R_r^k \in Z_M$. So, fix some rectangle $R_r^k \in Z_M$, and, next, some set $Q_r^k \in \mathcal{F}_r^k$. It then follows from the construction that the set Q_r^k is mapped by F bijectively onto its image $F(Q_r^k)$, which is contained in some rectangle $\widehat{R}_s^m \in Z_M$.

By the inductive assumption, for the rectangle $\widehat{R}_s^m \in Z_M$ there is a cover $(\widehat{\mathcal{F}}_s^m)^{n-1}$ of the set $\widehat{R}_s^m \cap \Lambda_W$, such that

$$\sum_{\widehat{K} \in (\widehat{\mathcal{F}}_s^m)^{n-1}} (\mathrm{diam} \widehat{K})^{1+\delta} < (2\pi + 1) \frac{1}{2^{n-1}}.$$

This allows us to define a cover $\mathcal{Q}_r^k$ of the set $Q_r^k \cap \Lambda_W$ as the family of all sets of the form $F_*^{-1}(\widehat{K})$ where $\widehat{K} \in (\widehat{\mathcal{F}}_s^m)^{n-1}$ and F_*^{-1} is the inverse of the bijective map $F : Q_r^k \to F(Q_r^k)$.

Obviously,

$$\sum_{\widehat{K} \in (\widehat{\mathcal{F}}_s^m)^{n-1}} (\mathrm{diam} F_*^{-1}(\widehat{K}))^{1+\delta} \leq (2\pi + 1) \frac{1}{2^{n-1}} \cdot \sup_{w \in Q_r^k} |F'(w)|^{-(1+\delta)}.$$

Finally, we define the family $(\mathcal{F}_r^k)^n$ as the union (over all sets $Q_r^k \in \mathcal{F}_r^k$) of all the covers $\mathcal{Q}_r^k$, described above. Since $\mathcal{Q}_r^k$ is a cover of $Q_r^k \cap \Lambda_W$, we obtain, by taking the union, a cover of $R_r^k \cap \Lambda_W$.

Moreover, by (7.3.3):

$$\sum_{K \in (\mathcal{F}_r^k)^n} (\mathrm{diam} K)^{1+\delta} \leq \sum_{Q_r^k \in \mathcal{F}_r^k} \sup_{w \in Q_r^k} |F'(w)|^{-(1+\delta)} \cdot (2\pi + 1) \frac{1}{2^{n-1}} \leq \frac{1}{2} \cdot (2\pi + 1) \frac{1}{2^{n-1}} = (2\pi + 1) \frac{1}{2^n}. \tag{7.3.18}$$

This ends the proof of the inequality $\dim_H(\Lambda_W \cap R_r^k) \leq 1 + \delta$.
The conclusion $\dim_H(\Lambda_W \cap Y_M) \leq 1 + \delta$ is immediate and thus Theorem 7.4 is proved.

7.4 Application — Hausdorff dimension of indecomposable continua

In this section we apply Theorem 7.4 to show that the Hausdorff dimension of several indecomposable continua, described in section 7.1, is equal to one.

The strategy is the following. Since every non-trivial continuum in the plane has Hausdorff dimension at least one, it is enough to prove the upper bound on the dimension. We shall use Theorem 7.4, and, more precisely, Corollary 7.5. In order to make use of Corollary 7.5, we shall check that our continuum, minus at most one point, is contained in the set $\Lambda_{W,ubd}$ for some thin set W. This has already observed in several particular cases (see [DJ02], [Dev93], and [FRS08]) for the most general statement).

7.4.1 Dynamic rays

For the completeness, we outline the proof of Lemma 7.15 below. This is a particular case, needed for the considered here calculations. We start with a standard fact. See e.g. [UZ07], Lemma 2.2 for its proof.

Lemma 7.14. *Fix some parameter λ for which the singular trajectory $f_\lambda^n(0)$ escapes (i.e. $f_\lambda^n(z) \to \infty$). Fix some $R > 0$, and denote by F_R the set of points $z \in \mathbb{C}$ for which the whole trajectory $f_\lambda^n(z)$ remains in the closed ball $\overline{B}(0, R)$. Then the map $f_{\lambda|F_R}$ is expanding: there exist $c > 0$ and $\gamma > 1$ such that for every $z \in F_R$*

$$|(f_\lambda^n)'(z)| \geq c\gamma^n.$$

We now assume that the parameter λ is chosen so that the singular value 0 escapes sufficiently fast. More precisely, it follows from the classification of the escaping points ([SZ03] Corollary 6.9) that the fact that $f_\lambda^n(0)$ escapes to ∞ implies that there is a dynamic ray $g_{\underline{r}}$ such that $0 = g_{\underline{r}}(t_0)$ for some $t_0 \geq t_{\underline{r}}$. In Lemma 7.15 below, we shall assume additionally that $t_0 > t_{\underline{r}}$, i.e. that the point 0 is located on an (open) ray.

We need the following.

Lemma 7.15. *Fix some parameter λ, such that the singular trajectory $f_\lambda^n(0)$ escapes, and $0 = g_{\underline{r}}(t_0)$ for some $t_0 > t_{\underline{r}}$. Let $g_{\underline{s}} : (t_{\underline{s}}, \infty)$ be a dynamic ray. Denote by G_s the closure of the set $\{g_{\underline{s}}(t), t > t_{\underline{s}}\}$ in $\mathbb{C}$. Then at most one point in $G_{\underline{s}}$ has bounded trajectory.*

Proof. In the outline below we use notation from [SZ03] and [Rem07]. To simplify the notation, we write below $g_{\underline{r}}$ to denote the arc $\{g_{\underline{r}}(t), t \geq t_0\}$. Recall that we write $f = f_\lambda$. Now, it is convenient to use a "dynamical coding", as proposed in [Rem07], section 2. The preimage $f^{-1}(g_{\underline{r}})$ is a union of countably many curves $g_{k\underline{r}}$, which cut the plane, defining countably many open strips S_k, where S_k is bounded by $g_{k\underline{r}}$ and $g_{(k+1)\underline{r}}$. Denote also by $\overline{S}_k$ the closure of S_k.

Each strip S_k is mapped by f univalently onto $\mathbb{C} \setminus g_{\underline{r}}$. Denote by f_k^{-1} the inverse map: $f_k^{-1} : \mathbb{C} \setminus g_{\underline{r}} \to S_k$.

Let $g_{\underline{s}}$ be an arbitrary dynamic ray. It is easy to check that the ray $g_{\underline{s}}$ does not intersect the ray $g_{k\underline{r}}$ unless $g_{\underline{s}} = g_{k\underline{r}}$. Thus, all the points on the ray $g_{\underline{s}}$ share the same "dynamic address": there is a sequence $\underline{\tilde{s}} = \tilde{s}_0, \tilde{s}_1 \dots$ such that for every point $z \in g_{\underline{s}}$ and every n we have $f^n(z) \in S_{\tilde{s}_n}$. Obviously, for every $z \in G_{\underline{s}}$ we have $f^n(z) \in \overline{S}_{\tilde{s}_n}$.

Now, let us fix some $R > 0$, and consider the set F_R defined in Lemma 7.14. Since all points in the rays $g_{k\underline{r}}$ escape to ∞, and the trajectory of points $z \in F_R$ remains bounded, there exists $\delta > 0$ such that the trajectory of every point $z \in F_R$ is δ-separated from the boundary curves of all strips S_k, and, by the same reason, from the curves $f^k(g_{\underline{r}})$. (Here we use the fact that $f^k(z) \to \infty$ uniformly on the curve $g_{\underline{r}} := g_{\underline{r}}(t_0, \infty)$, so, actually, only finitely many of curves $f^k(g_{\underline{r}})$ intersect the ball $\overline{B}(0, R)$.)

This easily implies the following: there exists a constant $L > 0$ such that, for arbitrary two points $z, w \in F_R$, belonging to the same strip, there exist a topological disc $U = U_{z,w}$ on which each composition

$$f_{k_0}^{-1} \circ f_{k_1}^{-1} \circ \dots \circ f_{k_{n-1}}^{-1}$$

is well defined, with distortion bounded by L.

Let $x, y \in F_R$. If $x, y \in G_{\underline{s}}$, then they have the same dynamic code $\underline{\tilde{s}}(x) = \underline{\tilde{s}}(y) = \tilde{s}_0, \tilde{s}_1, \dots, \tilde{s}_{n-1}, \tilde{s}_n \dots$. Put $z = f^n(x)$, $w = f^n(y)$ and take the inverse branch following our coding

$$f_*^{-n} = f_{\tilde{s}_0}^{-1} \circ f_{\tilde{s}_1}^{-1} \circ \dots \circ f_{\tilde{s}_{n-1}}^{-1}.$$

Then $f_*^{-n}(z) = x$ and $f_*^{-n}(w) = y$ (because $\underline{\tilde{s}}(x) = \underline{\tilde{s}}(y)$).
Expanding property (Lemma 7.14) together with the bounded distortion property give

$$|x - y| = |f_*^{-n}(z) - f_*^{-n}(w)| \leq \frac{L \cdot 2R}{c} \gamma^{-n}.$$

Since n can be taken arbitrarily large, we get $x = y$. □

Note that, actually, we proved in Lemma 7.15 the following fact:

Lemma 7.16. *Under the assumption and notation of Lemma 7.15, let $\underline{\tilde{s}} = (\tilde{s}_0, \tilde{s}_1, \tilde{s}_2 \dots)$ be a dynamic address and let $H_{\underline{\tilde{s}}}$ be the set of points $z \in \mathbb{C}$ such that, for every $n \in \mathbb{N}$, $f_\lambda^n(z) \in \overline{S}_{\tilde{s}_n}$. Then at most one point in $H_{\underline{\tilde{s}}}$ has bounded trajectory.*

7.4.2 Dimensions

We are ready to formulate the following corollaries.

Theorem 7.17. *Let $f(z) = \exp(z)$ and let Λ be the indecomposable continuum described in [Dev93]:*

$$\Lambda = \{z \in \mathbb{C} : \forall n \geq 0 \;\; \mathrm{Im}(f^n z) \in [0, \pi]\}.$$

Then $\dim_H(\Lambda) = 1$.

Proof. This is an immediate consequence of Theorem 7.4 and Lemma 7.16. Putting

$$W = \{z : \mathrm{Im}(z) \in [0, \pi]\}$$

we see immediately that $\Lambda = \Lambda_W$ is contained in the set of points whose trajectory remains in the closed dynamic strip $\overline{S}_0 = \mathbb{R} \times [0, 2\pi]$. (Note, besides, that points from $S_0 \setminus W$ will leave S_0 immediately.) Thus, using using Lemma 7.16 we see that $\mathrm{card}(\Lambda \setminus \Lambda_{W,ubd}) \leq 1$. Therefore, by Theorem 7.4, $\dim_H(\Lambda) \leq 1$. Since Λ is a non-trivial continuum, $\dim_H(\Lambda) \geq 1$. This gives the required equality. □

Remark. The inclusion $\mathrm{card}(\Lambda \setminus \Lambda_{W,ubd}) \leq 1$ can be also deduced from the detailed description of ω–limit sets of the points in Λ, provided in [Dev93].

Now, let us consider a more general situation: Assume that the parameter λ satisfies $f_\lambda^n(0) \to \infty$. In particular, this implies that the singular value 0 is on a dynamic ray or is the landing point of such a ray. Denote this ray, as in the proof of Lemma 7.15, by $g_{\underline{r}} : (t_{\underline{r}}, \infty) \to \mathbb{C}$. For such maps, L. Rempe provides the construction of uncountably many dynamically defined indecomposable continua. Each such a continuum is defined as the accumulation set of some dynamic ray (see [Rem07], Theorem 1.2). Namely, one considers the set R_1 of external addresses of the following form:

$$\underline{s} = s(n_1, n_2, n_3, \dots) := T_1 r_0 r_1 ... r_{n_1-1} T_{n_1} r_0 r_1 ... r_{n_2-1} T_{n_2} r_0 r_1 ... r_{n_3-1} T_{n_3} \dots$$

where $T_n := 2 + \max_{k \leq n} r_k$. A suitably chosen, uncountable subset $R \subset R_1$ has the required property: for every ray $\underline{s} \in R$ the accumulation set of the ray $g_{\underline{s}}$ is an indecomposable continuum.

Before formulating the result about the continua described in [Rem07], we note the following auxiliary fact.

Lemma 7.18. *Let $\lambda \in \mathbb{C} \setminus \{0\}$ is chosen so that the trajectory $f_\lambda^n(0)$ escapes to ∞.*

Let $g_{\underline{u}} : (t_{\underline{u}}, \infty) \to \mathbb{C}$ be a dynamic ray such that the sequence u_n is bounded. If the ray $g_{\underline{u}}$ "lands", i.e. if there exists a finite limit

$$\lim_{t \searrow t_{\underline{u}}} g_{\underline{u}}(t) = a \in \mathbb{C},$$

then the trajectory $f_\lambda^n(a)$ does not escape to ∞.

Proof. Let $K = \sup_n\{|u_n|\}$. As mentioned above, since the point 0 escapes, there is an external ray $g_{\underline{r}} : (t_{\underline{r}}, \infty) \to \mathbb{C}$ such that 0 is either its landing point or it is contained in $g_{\underline{r}}(t_{\underline{r}}, \infty)$ ray. Then, as in the proof of Lemma 7.15, and using its notation, we consider the dynamic coding defined with use of the curves $g_{k\underline{r}}$. Since the rays do not intersect, every image of the ray $f_\lambda^n(g_{\underline{u}}([t_{\underline{u}}, \infty)))$, is contained in some strip $S_k(n)$, and it is easy to see, using the behaviour of the rays at infinity, that $|k(n)| < K'$ for some $K' \geq K$, i.e. the dynamic address is also bounded.

Denote by $\Sigma_{K'}$ the set of all infinite sequences $\underline{w} = (w_n)_{n=0}^\infty$, with integer entries, and such that $|w_n| < K'$ for all n. Every point $z \in \mathbb{C}$ has its dynamic "geometric code"

$\underline{s}(z)$ defined by $s_i(z) = k$ if $f_\lambda^n(z) \in P_k$. According to [DK84], section 3 (see also [SZ03], Prop. 3.4), there exists $M > 0$ depending on K' and λ such that the set of "directly escaping points":

$$E_{M,K'} = \{z \in \mathbb{C} : \mathrm{Re} f_\lambda^n(z) \geq M \quad \text{for all} \quad n \geq 0 \text{ and } \quad \underline{s}(z) \in \Sigma_{K'}\}$$

is a union of "tails of rays" $g_{\underline{s}}(t)$. Each tail is a curve to ∞, and, actually, a graph of a function defined in $\{x \geq M\}$.

Now, let a be the landing point of the tail $g_{\underline{u}}$, and assume that $f_\lambda^n(a) \to \infty$. Then there exists n_0 such that for $n \geq n_0$ $\mathrm{Re} f_\lambda^n(a) > M + 1$. Consequently, $f_\lambda^{n_0}(a) \in E_{M,K'}$ and, since $\mathrm{Re} f^{n_0}(a) > M + 1$, $f^{n_0}(a)$ is in a tail of some ray (thus, it is located in an open ray). This is a contradiction since f_λ maps ends of rays to ends of rays. □

Theorem 7.19. *Assume that the parameter λ satisfies $f_\lambda^n(0) \to \infty$. Assume additionally that $\mathrm{Im}(f_\lambda^n(0))$ remains bounded. Let Λ be the indecomposable continuum constructed in [Rem07]. Then $\dim_H(\Lambda) = 1$.*

Proof of Theorem 7.19. As mentioned in section 7.2, it is easy to see that the assumptions of the Theorem imply that the parameter λ satisfies the super-growing condition (7.2.2). Since $0 \in I(f_\lambda)$, it follows from the classification of escaping points ([SZ03]) that 0 belongs to some dynamic ray $g_{\underline{r}}(t_{\underline{r}}, \infty) \to \mathbb{C}$, or it is a landing point of a ray. Lemma 7.18 allows us to exclude this second possibility. Therefore, $0 = g_{\underline{r}}(t)$ for some $t > t_{\underline{r}}$. Open dynamic rays are smooth curves [VdS88], see also [FS09] for the estimates of the second derivative of the parametrization $t \mapsto g_{\underline{r}}(t)$, $t \in (t_{\underline{r}}, \infty)$. In particular, the ray $g_{\underline{r}}(t_{\underline{r}}, \infty)$ has a tangent vector at 0. Let us consider, as in the proof of Lemma 7.15, the rays $g_{k\underline{r}}$. Then the existence of a tangent vector of $g_{\underline{r}}$ at 0 guarantees that for every $k \in \mathbb{Z}$ there exists a finite limit

$$\lim_{\mathrm{Re} z \to -\infty,\, z \in g_{k\underline{r}}} \mathrm{Im}(z) = A_k.$$

Moreover, it follows from the general bounds for the parametrization of hairs (see [SZ03], Prop 3.4) that there exists a finite limit

$$\lim_{\mathrm{Re}(z) \to +\infty,\, z \in g_{k\underline{r}}} \mathrm{Im}(z) = B_k = 2\pi k - \mathrm{Arg}(\lambda).$$

Obviously, $A_{k+1} = A_k + 2\pi$, $B_{k+1} = B_k + 2\pi$.

Now, let $\underline{s}$ be an arbitrary external address with bounded entries, and let $g_{\underline{s}} : (t_{\underline{s}}, \infty) \to \mathbb{C}$ be the corresponding external ray. Put $G_{\underline{s}} = \overline{g_{\underline{s}}}$. As in the proof of Lemma 7.15 we notice that $G_{\underline{s}}$ is contained in the closure of some dynamic strip S_k bounded by the curves $g_{k\underline{r}}$ and $g_{(k+1)\underline{r}}$.

Moreover, again by the above-mentioned Prop. 3.4 in [SZ03] we know that for every dynamic ray $g_{\underline{s}}$

$$\lim_{t \to +\infty} \mathrm{Im}(g_s(t)) = 2\pi s_1 - \mathrm{Arg}(\lambda).$$

It now easily follows from two above observations that for every bounded sequence $g_{\underline{s}}$ there exists $K > 0$ such that

$$\bigcup_{n\geq 0} f_\lambda^n(G_{\underline{s}}) \subset W = \{z \in \mathbb{C} : |\mathrm{Im} z| \leq K\}\,.$$

Thus,

$$G_{\underline{s}} \subset \Lambda_W.$$

By Lemma 7.15 we have $\mathrm{card}(G_{\underline{s}} \setminus \Lambda_{W,ubd}) \leq 1$. Therefore, using Theorem 7.4, we conclude that $\dim_H(\Lambda) \leq 1$, and, again, since Λ is a non-trivial continuum, $\dim_H(\Lambda) = 1$. □

As a corollary we have

Corollary 7.20. *If Λ is an indecomposable continuum constructed — for the map $f_1(z) = \exp(z)$ — in [DJ02], then* $\dim_H(\Lambda) = 1$.

7.4.3 Non-rectifiability of continua

As a complement of our result let us note the following general remark:

Proposition 7.21. *One-dimensional Hausdorff measure of an indecomposable continuum in the plane is not σ-finite.*

To prove Proposition 7.21 we will need to state some facts about indecomposable continua. See [Kur68] for definitions and properties. Let us denote the continuum by $\mathcal{C}$.

Definition 7.22. For a point p define the set K_p as a union of all proper subcontinua containing p, in other words $K_p = \{x \colon \exists$ a proper subcontinuum S of $\mathcal{C}$ containing p and $x\}$. The set K_p is called the *composant* of the point p.

The following facts are easy to verify, cf. [Kur68], section 48.

Fact 1. *Every composant of an indecomposable metric continuum is an F_σ set of the first category.*

Fact 2. *In an indecomposable metric continuum every composant is a dense subset of $\mathcal{C}$.*

Fact 3. *An indecomposable metric continuum is a union of uncountably many disjoint composants.*

The next fact follows immediately from the definition of Hausdorff measure.

Fact 4. *A connected measurable set $A \subset \mathbb{R}^2$ with* $\mathrm{diam}(A) > 0$ *has positive 1-dimensional Hausdorff measure $\mathcal{H}_1(A) > 0$.*

The above facts show that an indecomposable continuum consists of uncountably many disjoint sets of positive measure $\mathcal{H}_1$. This proves that the whole set cannot be σ-finite with respect to $\mathcal{H}_1$. The formal proof follows.

Proof of Proposition 7.21. Assume the opposite: the continuum $\mathcal{C}$ can be represented as a union of countably many disjoint measurable sets of finite measure:

$$\mathcal{C} = \bigcup_{i=1}^{\infty} C_i, \;\; C_i \cap C_j = \emptyset \text{ (for } i \neq j\text{) and } \mathcal{H}_1(C_i) < +\infty.$$

The set $\mathcal{C}$ is also a union of disjoint composants of some points: $\mathcal{C} = \bigcup_{p\in P} K_p$, where $\mathrm{card}(P) > \aleph_0$. Define the sets $K_p^i = K_p \cap C_i$.

The set $\mathcal{C}$ has a positive diameter a. Since every K_p is dense in $\mathcal{C}$, the diameter of K_p is bigger than $\frac{1}{2}a$. Using Fact 4 we see that $\mathcal{H}_1(K_p) > 0$.

As $K_p = \bigcup_i K_p^i$ this means that for every p at least one K_p^i has positive measure $\mathcal{H}_1(K_p^i) > 0$. Let us denote these chosen sets by $\widehat{K}_p$.

Define $A_n = \{p \in P \colon \mathcal{H}_1(\widehat{K}_p) > \frac{1}{n}\}$. The set P is not countable so there exists $A_n = A_{n_0}$ containing uncountably many p's. We denote them by $\widehat{p}$.

Finally, since there are countably many sets C_i, one of them contains infinitely many of $\widehat{K}_{\widehat{p}}$'s. Choosing a countable subset $(\widehat{p}_k)_{k=1}^{\infty}$, such that $\widehat{K}_{\widehat{p}_k} \subset C_i$, we can write

$$\mathcal{H}_1(C_i) \geq \mathcal{H}_1\left(\bigcup_{k=1}^{\infty} \widehat{K}_{\widehat{p}_k}\right) = \sum_{k=1}^{\infty} \mathcal{H}_1\left(\widehat{K}_{\widehat{p}_k}\right) = \infty$$

thus giving a contradiction. ☐

Combining the result from this section (Prop. 7.21) with the previous estimates on the continua (Thm. 7.17 and Cor. 7.20) we arrive at the following:

Theorem 7.23. *Let $f(z) = \exp(z)$ and let Λ be an indecomposable continuum described either in [Dev93] or in [DJ02].*
Then H_1 is not σ–finite on Λ, but $H_{1+\varepsilon}(\Lambda) = 0$ (for all $\varepsilon > 0$).

Bibliography

[Aba04] M. Abadi, *Sharp error terms and necessary conditions for exponential hitting times in mixing processes*, Ann. Probab. **32** (2004), 243–264.

[BC13] M. D. Boshernitzan and J. Chaika, *Diophantine properties of IETs and general systems: quantitative proximality and connectivity*, Invent. Math. **192** (2013), 375–412.

[Ber96] W. Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. **29** (1996), 151–188.

[BG97] A. Boyarski and P. Góra, *Laws of chaos: Invariant measures and dynamical systems in one dimension*, Birkhäuser Boston, 1997.

[Bos93] M. D. Boshernitzan, *Quantitative recurrence results*, Invent. Math. **113** (1993), 617–631.

[BPS99] L. Barreira, Ya. Pesin, and J. Schmeling, *Dimension and product structure of hyperbolic measures*, Annals of Math. **149** (1999), no. 3, 755–783.

[BS01] L. Barreira and B. Saussol, *Hausdorff dimension of measures via Poincaré recurrence*, Commun. Math. Phys. **219** (2001), 443–463.

[BSTV03] H. Bruin, B. Saussol, S. Troubetzkoy, and S. Vaienti, *Return time statistics via inducing*, Ergodic Theory Dynam. Systems **23** (2003), 991–1013.

[Buz98] J. Buzzi, *Absolutely continuous S.R.B. measures for random Lasota-Yorke maps*, Trans. Amer. Math. Soc **352** (1998), 3289–3303.

[Buz99] ———, *Exponential decay of correlations for random Lasota–Yorke maps*, Commun. Math. Phys. **208** (1999), 25–54.

[BY93] V. Baladi and L. S. Young, *On the spectra of randomly perturbed expanding maps*, Commun. Math. Phys. **156** (1993), 355–385.

[CdF90] Z. Coelho and E. de Faria, *Limit laws for entrance times for homeomorphisms of the circle*, Israel J. Math. **69** (1990), 235–249.

[CG93] P. Collet and A. Galves, *Statistics of close visits to the indifferent fixed point of an interval map*, J. Statist. Phys. **72** (1993), 459–478.

[Cha03] J.-R. Chazottes, *Hitting and returning to non-rare events in mixing dynamical systems*, Nonlinearity **16** (2003), 1017–1034.

[Cho02] Geon Ho Choe, *Recurrence of transformations with absolutely continuous invariant measures*, App. Math. Comput. **129** (2002), 501–516.

[Dev93] R. Devaney, *Knaster-like continua and complex dynamics*, Ergodic Theory Dynam. Systems **13** (1993), 627–634.

[Dev99] ______, *Cantor bouquets, explosions, and knaster continua: Dynamics of complex exponentials*, Publ. Mat. **43** (1999), 27–54.

[DJ02] R. Devaney and X. Jarque, *Indecomposable continua in exponential dynamics*, Conform. Geom. Dyn. **6** (2002), 1–12.

[DK84] R. Devaney and M. Krych, *Dynamics of exp(z)*, Ergodic Theory Dynam. Systems **4** (1984), 35–52.

[EL92] A. Eremenko and M. Lyubich, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier **42** (1992), no. 4, 1–32.

[FRS08] M. Förster, L. Rempe, and D. Schleicher, *Classification of escaping exponential maps*, Proc. Amer. Math. Soc. **136** (2008), 651–663.

[FS09] M. Förster and D. Schleicher, *Parameter rays for exponential family*, Ergodic Theory Dynam. Systems **29** (2009), 515–544.

[Fur81] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton University Press, 1981.

[GK86] L. Goldberg and L. Keen, *A finiteness theorem for a dynamical class of entire functions*, Ergodic Theory Dynam. Systems **6** (1986), 183–192.

[HK82] F. Hofbauer and G. Keller, *Ergodic properties of invariant measures for piecewise monotonic transformations*, Math. Z. **180** (1982), 119–140.

[HSV99] M. Hirata, B. Saussol, and S. Vaienti, *Statistics of return times: a general framework and new applications*, Commun. Math. Phys. **206** (1999), 33–55.

[JU08] Kotus J. and M. Urbański, *Fractal measures and ergodic theory of transcendental meromorphic functions*, London Math. Soc. Lect. Notes. **348** (2008).

[Kac47] M. Kac, *On the notion of recurrence in discrete stochastic processes*, Bull. Amer. Math. Soc. **53** (1947), 1002–1010.

[Kar99] B. Karpińska, *Hausdorff dimension of the hairs without endpoints for* $\lambda \exp(z)$, C. R. Acad. Sci. Paris **328** (1999), 1039–1044.

[Kel85] G. Keller, *Generalized bounded variation and applications to piecewise monotonic transformations*, Z. Wahr. verw. Geb. **69** (1985), 461–478.

[Khi64] A. Ya. Khinchin, *Continued fractions*, The University of Chicago Press, Chicago and London, 1964.

[Kur68] K. Kuratowski, *Topology, vol. ii*, Academic Press, New York, 1968.

[Lac02] Y. Lacroix, *Possible limit laws for entrance times of an ergodic aperiodic dynamical system*, Israel J. Math. **132** (2002), 253–264.

[Led87] F. Ledrappier, *Dimension of invariant measures*, Proceedings of the conference on ergodic theory and related topics II; Teubner-Texte Math., **94** (1987), 116–124.

[LK05] Y. Lacroix and M. Kupsa, *Asymptotics for hitting times*, Ann. Probab. **33** (2005), 610–619.

[LS98] J. Luukkainen and E. Saksman, *Every complete doubling metric space carries a doubling measure*, Proc. Amer. Math. Soc. **126** (1998), no. 2, 531–534.

[Lyu87] M. Lyubich, *Measurable dynamics of an exponential.(English translation)*, Soviet Math. Dokl. **35** (1987), no. 1, 223–226.

[Mat95] P. Mattila, *Geometry of sets and measures in euclidean spaces*, Cambridge University Press, 1995.

[Mis81] M. Misiurewicz, *On iterates of* e^z, Ergodic Theory Dynam. Systems **1** (1981), 103–106.

[Mór09] P. Móra, *Estimate of the Hausdorff measure of the sierpinski triangle*, Fractals **17** (2009), no. 02, 137–148.

[MR11] P. Marie and J. Rousseau, *Recurrence for random dynamical systems*, Discr. Contin. Dyn. Syst. **30** (2011), no. 1, 1–16.

[MSU08] V. Mayer, B. Skorulski, and M. Urbański, *Random distance expanding mappings, thermodynamic formalism, Gibbs measures, and fractal geometry*, to appear Springer's Lecture Notes in Math (2008).

[MU98] R. D. Mauldin and M. Urbański, *Fractal measures for parabolic IFS*, Adv. Math. **136** (1998), 225–253.

[MU03a] ———, *The doubling property of conformal measures for infinite iterated function systems*, J. Number Th. **102** (2003), 23–40.

[MU03b] ______, *Graph directed Markov systems: Geometry and dynamics of limit sets*, Cambridge Univ. Press, 2003.

[PU10] F. Przytycki and M. Urbański, *Conformal fractals: Ergodic theory methods*, Cambridge University Press, 2010.

[PZ] Ł. Pawelec and A. Zdunik, *Indecomposable continua in exponential dynamics–Hausdorff dimension*, Submitted., pre-print: http://arxiv.org/abs/1405.7784.

[Ree86] M. Rees, *The exponential map is not recurrent*, Math. Z. **191** (1986), no. 4, 593–598.

[Rem06] L. Rempe, *Topological dynamics of exponential maps on their escaping sets*, Ergodic Theory Dynam. Systems **26** (2006), 1939–1975.

[Rem07] ______, *On nonlanding dynamic rays of exponential maps*, Ann. Acad. Sci. Fenn. Math. **32** (2007), 353–369.

[Sch07] D. Schleicher, *The dynamical fine structure of iterated cosine maps and a dimension paradox*, Duke Math. J. **136** (2007), 343–356.

[STV02] B. Saussol, S. Troubetzkoy, and S. Vaienti, *Recurrence, dimensions, and Lyapunov exponents*, J. Statist. Phys. **106** (2002), 623–634.

[Sul83] D. Sullivan, *Conformal dynamical systems*, Springer, 1983.

[SZ03] D. Schleicher and J. Zimmer, *Periodic points and dynamic rays of exponential maps*, Ann. Acad. Sci. Fenn. Math. **28** (2003), no. 2, 327–354.

[Urb07] M. Urbański, *Recurrence rates for Loosely Markov dynamical systems*, J. Australian Math. Soc. **82** (2007), 39–57.

[UZ] M. Urbański and A. Zdunik, *Exponential distribution for Loosely Markov dynamical systems*, unpublished.

[UZ07] M. Urbański and A. Zdunik, *Geometry and ergodic theory of non-hyperbolic exponential maps*, Trans. Amer. Math. Soc. **359** (2007), 3973–3997.

[VdS88] M. Viana da Silva, *The differentiability of the hairs of* $\exp(z)$, Proc. Amer. Math. Soc. **103** (1988), 1179–1184.

[Wei94] Qiu Weiyuan, *Hausdorff dimension of the M-set of* $\lambda \exp(z)$, Acta Math. Sin. **10** (1994), no. 4, 362–368.

[You82] L. S. Young, *Dimension, entropy and Lyapunov exponents,*, Ergodic Theory Dynam. Systems **2** (1982), 109–124.

[You99] ______, *Recurrence times and rates of mixing*, Israel J. Math. **110** (1999), 153–188.