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Corson-like compacta and related function spaces

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Supervisor's statement

The dissertation is ready to be reviewed.

April 2nd, 2025 ....

Author's statement

I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Contents

Chapter 1

Introduction

Compactness has always played an important role in analysis. One of the most well known classes of compact spaces with roots in functional analysis is the class of Eberlein compact spaces, i.e. weakly compact subsets of Banach spaces. An important extension of this class is the class of Corson compact spaces, i.e. compact spaces that can be embedded into Σ-products of Euclidean lines (see [Ne] and [AMN]). Both of these classes for decades have been the object of research by numerous mathematicians from many research centers around the world (see [HNV]).

The thesis is mainly devoted to the study of the classes of κ-Corson compact spaces, which are a generalization of the notion of Corson compact space (the class of Corson compacta coincides with the class of ω_1-Corson compacta), and a certain subclass of Eberlein compact spaces - the class of NY compact spaces.

For arbitrary infinite cardinal number κ, κ-Corson compact spaces were first considered by Kalenda [Ka2] and in [BM] (where the terminology was slightly different). Trying to examine them systematically one discovers quickly that the case $\kappa = \omega$ is quite special. In turn, for regular, uncountable cardinal numbers κ, there is a number of properties of κ-Corson compacta that may be seen as natural analogues of known features of the usual Corson compact spaces. In Sections 2.2 – 2.4 we investigate ω-Corson compacta and a wider, somewhat more natural, class of NY compacta, first considered by Nakhmanson and Yakovlev [NY].

In Chapter 2, extending results from [NY], we give internal characterizations of ω-Corson compacta (Theorem 2.3.6) and of NY compacta (Theorem 2.3.1). We discuss the stability of these classes of compacta under countable products (Corollary 2.4.2), continuous images (Corollary 2.4.3), and the functor P which assigns to a compact space K the compact space $P(K)$ of regular probability Borel measures equipped with the *weak** topology (Remark 2.4.11). In [NY] it was proved that all NY compacta are hereditarily metacompact, in Section 2.4 we investigate this covering property for Eberlein compacta not belonging to the class of NY compact spaces (Example 2.4.5, Theorem 2.4.9).

In Section 2.5 we show that the question, for which infinite cardinal numbers κ, the class of

κ-Corson compact spaces is closed under the functor P is naturally related to the notion of a caliber of measures (so is, typically, undecidable in the usual set theory). Results presented in Chapter 2 come from the article [MPZ].

In Chapter 3 we consider spaces of real continuous functions $C_p(K)$ on some compact spaces K. In the theory of function spaces there are many theorems answering questions of the following type: Assume we are given spaces X and Y and a topological property $\mathcal{P}$. Assume further that the function spaces $C_p(X)$ and $C_p(Y)$ are (linearly) homeomorphic. Is it true that if a space X has the property $\mathcal{P}$, then the space Y also must have this property ? There is a variety of topological properties for which the answer to such question is in the positive. Somewhat better type of such theorems characterise given topological property $\mathcal{P}$ of a space X in terms of properties of the function space $C_p(X)$. In this chapter we will present results of both types.

Section 3.2 contains a proof of Theorem 3.2.11 stating that, for σ-compact spaces X and Y, where X is strongly countable dimensional, if function spaces $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic, then Y is strongly countable dimensional as well. We also investigate the case when X is finite dimensional (Theorem 3.2.12).

In Section 3.3 we show that the class of NY compact spaces is invariant under linear homeomorphisms of function spaces. In fact we prove several slightly more general theorems (Theorems 3.3.9, 3.3.11, 3.3.12). Recently these results were refined by Krupski and Avilés. They proved that the class of NY compact spaces is preserved by homeomorphisms of function spaces [AK]. We also give an example of a compact space K and a ω-Corson compact space L such that $C_p(K)$ is linearly homeomorphic to a linear subspace of $C_p(L)$ but K is not Eberlein compact (Example 3.3.13).

Section 3.4 is devoted to the proof of a characterisation (Theorem 3.4.8) of κ-Corson compact spaces K in terms of topological properties of function spaces $C_p(K)$ for regular uncountable cardinal numbers κ, which extends results of Pol [P] and Bell and Marciszewski [BM]. Additionally, using results from Chapters 2 and 3 we show that the class of ω-Corson compact spaces K is invariant under linear homeomorphisms of function spaces $C_p(K)$ (Theorem 3.4.14).

In the final section of Chapter 3 we discuss the stability of κ-Corson compacta K with respect to isomorphisms of Banach spaces $C(K)$. Theorems discussed in this section come from [MPZ]. Results presented in the rest of Chapter 3 come from the article [Za], and were partially announced and discussed in [MPZ].

In the first part of the last chapter we discuss two results concerning continuous linear operators between general locally convex, linear topological spaces. We show that every open operator is always weakly open (Corollary 4.1.5). We also present a theorem generalising the well known Banach theorem about closed transformations to a general setting of locally convex spaces (Theorem 4.1.7). In the second part we show that c_0-product of function spaces defined on a pseudocompact space is always homeomorphic to its countable power (Theorem 4.2.3). This theorem generalises theorem of Gul'ko and Khmyleva stating that the space c_0 of sequences converging to 0, endowed with the pointwise convergence topology, is homeomorphic to its countable power.

1.1. Σ_κ-products

We often write I for the closed unit interval $[0,1]$. In the sequel, κ always stands for an infinite cardinal number.

Given x in the product space $\mathbb{R}^\Gamma$, we write $\operatorname{supp}(x) = \{\gamma \in \Gamma : x(\gamma) \neq 0\}$. The Σ_κ-product of real lines, denoted by $\Sigma_\kappa(\mathbb{R}^\Gamma)$, is the space of all $x \in \mathbb{R}^\Gamma$ satisfying $|\operatorname{supp}(x)| < \kappa$ and $\Sigma_\kappa([0,1]^\Gamma)$ is defined analogously. Traditionally, $\Sigma_{\omega_1}(\mathbb{R}^\Gamma)$ is written as $\Sigma(\mathbb{R}^\Gamma)$ and $\Sigma_\omega(\mathbb{R}^\Gamma)$ is written as $\sigma(\mathbb{R}^\Gamma)$. However, we need to introduce more general notation for the case $\kappa = \omega$.

Let $\{X_\gamma : \gamma \in \Gamma\}$ be a family of nonempty topological spaces, and let a_γ be a fixed point in X_γ. The σ-product of the family $\{(X_\gamma, a_\gamma) : \gamma \in \Gamma\}$ is defined as

$$\sigma(X_\gamma, a_\gamma, \Gamma) = \left\{(x_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} X_\gamma : |\{\gamma \in \Gamma : x_\gamma \neq a_\gamma\}| < \omega\right\}.$$

If $X_\gamma = I$ and $a_\gamma = 0$, for all $\gamma \in \Gamma$, then $\sigma(X_\gamma, a_\gamma, \Gamma)$ is simply denoted by $\sigma(I, \Gamma)$. Likewise, if $X_\gamma = I^\omega$ and $a_\gamma = (0, 0, \dots)$, for all $\gamma \in \Gamma$, then $\sigma(X_\gamma, a_\gamma, \Gamma)$ is denoted by $\sigma(I^\omega, \Gamma)$.

1.2. Locally convex linear topological spaces

By a linear topological space X we will always mean a real linear space endowed with a Hausdorff topology such that addition $+ : X \times X \to \mathbb{R}$ and scalar multiplication $\cdot : \mathbb{R} \times X \to X$ are continuous functions. A subset A of X is said to be convex if $tx + (1-t)y \in A$ for any $x, y \in A$ and $t \in (0,1)$. A linear topological space X is called locally convex if its topology has a basis consisting of convex sets. For a subset A of a linear topological space, let conv A denote the convex hull of A. Let X^* be the dual space of X consisting of continuous linear functionals $\phi : X \to \mathbb{R}$. We will always consider the dual space endowed with the weak* topology, generated by sets of the form

$$W_{x,U} = \{\phi \in X^* : \phi(x) \in U\},$$

for $x \in X$ and an open subset U of $\mathbb{R}$. For a continuous linear operator $T : X \to Y$ between two linear topological spaces, the dual operator $T^* : Y^* \to X^*$ is given by the formula $T^*(\phi) = \phi \circ T$. The dual operator is linear and continuous.

For a Banach space X, let B_X denote the closed unit ball of X.

1.3. Function spaces

For a Tychonoff space X, let $C_p(X)$ denote the space of real continuous functions on the space X, endowed with the topology of pointwise convergence. This is the topology generated by sets of the form

$$N(x_1, \dots, x_n, U_1, \dots, U_n) = \{f \in C_p(X) : f(x_1) \in U_i \text{ for } i \in \{1, \dots, n\}\},$$

for $x_1, \dots, x_n \in X$ and open $U_1, \dots, U_n \subseteq \mathbb{R}$. For a topological space X, let $C_c(X)$ denote the space of real continuous functions on X, endowed with the compact-open topology. Subbasic open sets in this topology have the form

$$[K, U] = \{f \in C_c(X) : f(K) \subseteq U\},$$

where K is a compact subset of X, and U is an open subset of $\mathbb{R}$. For a topological space X, function spaces $C_p(X)$ and $C_c(X)$ are locally convex linear topological spaces. For a compact space K, let $C(K)$ denote the Banach space of real, continuous functions on K endowed with the *sup* norm generating the topology, which is, in this case, equal to the compact-open topology. As usual, we identify the dual space $C(K)^*$ with the space $M(K)$ of signed Radon measures on K having finite variation and write $\mu(g)$ for $\int_K g \, \mathrm{d}\mu$. The symbol $M_1(K)$ stands for the unit ball of $M(K)$, equipped with the *weak** topology inherited from $C(K)^*$. Finally, $P(K)$ is the subset of regular probability measures on K.

1.4. Other topological notions and notation

In the whole thesis apart from Section 4.1 all topological spaces are assumed to be completely regular. As usual, the topological weight of a space X is denoted by $w(X)$. For an infinite cardinal number κ, $D(\kappa)$ denotes the discrete space of cardinality κ, and $A(\kappa)$ is the one-point compactification of $D(\kappa)$. In the general case, for a locally compact space X, $A(X)$ denotes the one-point compactification of the space X.

Recall that for any compact space K, its Aleksandrov duplicate $AD(K)$ is defined as the space $AD(K) = K \times 2$ in which all the points $(x, 1)$ for $x \in K$ are isolated and basic neighborhoods of a point $(x, 0)$ have the form $(U \times 2) \setminus \{(x, 1)\}$, where U is an open neighborhood of x in K.

A space X is said to be *scattered* if every nonempty subset $A \subseteq X$ contains a point isolated in A. For a scattered space K, its *Cantor-Bendixson height* $ht(X)$ is the minimal ordinal number α such that the Cantor-Bendixson derivative $K^{(\alpha)}$ of the space K is empty. The Cantor-Bendixson height of a compact scattered space is always a nonlimit ordinal.

For a topological space X, sets of the form $f^{-1}((0, 1])$, where $f : X \to [0, 1]$ is a continuous function, are called the *cozero sets*.

A family $\mathcal{U}$ of subsets of a topological space X is called T_0*-separating if*, for every pair of distinct points x, y of X, there is $U \in \mathcal{U}$ containing exactly one of the points x, y.

Given a family $\mathcal{U}$ of subsets of a topological space X we define $ord(x,\mathcal{U}) = |\{U \in \mathcal{U} : x \in U\}|$ for $x \in X$ and $ord(\mathcal{U}) = sup\{ord(x,\mathcal{U}) : x \in X\}$. We say that $\mathcal{U}$ is *point-finite* if $ord(x,\mathcal{U}) < \omega$ for all $x \in X$.

A Hausdorff space X is called *metacompact* if every open cover $\mathcal{U}$ of X has a point-finite, open refinement $\mathcal{V}$. It is σ*-metacompact* if every open cover $\mathcal{U}$ of X has an open refinement which is a union of countably many point-finite families. A Hausdorff space is said to be hereditarily metacompact (σ-metacompact) if its every subspace is metacompact (σ-metacompact).

By (finite) dimension of a normal space X we will always mean the covering dimension $dim(X)$. Recall that a normal space X is called *strongly countable dimensional* if X is a countable union of closed subspaces of finite covering dimension, see [En2].

By s we will denote the countable product of real lines.

Chapter 2

The classes of Corson-like compact spaces

For an infinite cardinal number κ, a compact space is called κ-Corson compact if it is homeomorphic to a subset of $\Sigma_\kappa(\mathbb{R}^\Gamma)$ for some set Γ. Equivalently, instead of $\mathbb{R}^\Gamma$ one can consider I^Γ. Every compact space K is κ-Corson compact for $\kappa > w(K)$, see Remark 2.1.1. Traditionally, instead of ω_1-Corson compact space we will write Corson compact space.

A compact space is said to be NY compact if it is homeomorphic to a subset of $\sigma(X_\gamma, a_\gamma, \Gamma)$ for some compact, metrizable X_γ's, $a_\gamma \in X_\gamma$ and some set Γ. It is easy to observe that a compact space is NY compact if and only if it is homeomorphic to a subspace of $\sigma(I^\omega, \Gamma)$ for some set Γ (see Proposition 2.2.1). The class of NY compact spaces is invariant under taking continuous images [NY].

Recall that a space is called Eberlein compact if it is homeomorphic to a compact subset of a Banach space endowed with the weak topology. Equivalently, by the celebrated Amir-Lindenstrauss theorem, a compact space is Eberlein compact if and only if it is homeomorphic to a subset of

$$c_0(\Gamma) = \{x \in \mathbb{R}^\Gamma : |\{\gamma \in \Gamma : |x(\gamma)| < \epsilon\}| < \omega \text{ for every } \epsilon > 0\}.$$

Every Eberlein compact space is Corson compact, since for every $x \in c_0(\Gamma)$, $|supp(x)| \leq \omega$. Since $\sigma(I^\omega, \Gamma) \sim \sigma(\prod_{n\in\mathbb{N}}[0, \frac{1}{n}], 0, \Gamma) \subseteq c_0(\Gamma \times \mathbb{N})$, every NY compact space is Eberlein compact. Finally, every ω-Corson compact space is NY compact because I is compact and metrizable.

2.1. Definitions and introductory observations

Remark 2.1.1 ([MPZ, Remark 2.1]). *We record here the following preliminary observations.*

(a) While defining κ-Corson compacta, one can clearly replace $\Sigma_\kappa(\mathbb{R}^\Gamma)$ by $\Sigma_\kappa([0,1]^\Gamma)$.

(b) *If K is κ-Corson compact, then K embeds into $\Sigma_\kappa([0,1]^\Gamma)$ where Γ is of size $w(K)$.*

(c) Every *compact space K is κ-Corson compact for $\kappa > w(K)$.*

(d) *A compact space K is κ-Corson compact if and only if there is a family $\mathcal{F}$ of continuous functions $K \to [0,1]$ that separates the points of K and $|\{f \in \mathcal{F} : f(x) \neq 0\}| < \kappa$ for every $x \in K$.*

Proof. For the proof of (a), let K be a compact subspace of $\Sigma_\kappa(\mathbb{R}^\Gamma)$. Since K is a compact space it is bounded on every coordinate in the product $\mathbb{R}^\Gamma$. Without loss of generality we can assume that K is a subspace of $\Sigma_\kappa([-1,1]^\Gamma)$. Now, identify the segment $[-1,1]$ with $(\{0\}\times[0,1])\cup([0,1]\times\{0\})$ by bending it at point 0. In this way we make $\Sigma_\kappa([-1,1]^\Gamma)$ a subspace of $\Sigma_\kappa([0,1]^{\Gamma\times 2})$.

To prove (b), assume that K is a compact subset of $\Sigma_\kappa([0,1]^\lambda) \subseteq [0,1]^\lambda$. Let $i : K \to [0,1]^{w(K)}$ be a homeomorphic embedding. By Tietze-Urysohm theorem, there exists a continuous extension of i, $\widetilde{i} : [0,1]^\lambda \to [0,1]^{w(K)}$. For $\alpha < w(K)$ let $\pi_\alpha : [0,1]^{w(K)} \to [0,1]$ be the projection onto the α-th coordinate. By [En1, Exercise 2.7.12 (d)], for every $\alpha < w(K)$, there exists a countable set $A_\alpha \subseteq w(K)$ and a continuous function $f_\alpha : [0,1]^{A_\alpha} \to [0,1]$ such that $\pi_\alpha \circ \widetilde{i} = f_\alpha \circ \pi_{A_\alpha}$. For $\alpha < w(K)$, let $\pi_{A_\alpha} : [0,1]^\lambda \to [0,1]^{A_\alpha}$ and $\pi : [0,1]^{w(K)} \to [0,1]^{\bigcup_{\alpha\in w(K)} A_\alpha}$ be the projections. Notice that $|\bigcup_{\alpha<w(K)} A_\alpha| \leq w(K)$. We will show now that $\pi\restriction_K$ is injective. Indeed, if $x, y \in K$ and $\pi(x) = \pi(y)$, then $\pi_\alpha(\widetilde{i}(x)) = f_\alpha(\pi_{A_\alpha}(x)) = f_\alpha(\pi_{A_\alpha}(y)) = \pi_\alpha(\widetilde{i}(y))$ for every α. This means that $\widetilde{i}(x) = \widetilde{i}(y)$, but $\widetilde{i}\restriction_K = i$ is an injection, therefore $x = y$. The restriction $\pi\restriction_K$ is a continuous injection from a compact space to a Hausdorff space, and therefore it is a homeomorphic embedding.

Condition (c) follows from the universality of the Tychonoff cubes.

To prove (d), assume that K is κ-Corson compact. By (a), we can assume that K is a subspace of $\Sigma_\kappa([0,1]^\Gamma)$. For $\gamma \in \Gamma$, let $\pi_\gamma : K \to \mathbb{R}$ be the projection onto the γ-th coordinate. Clearly, the family $\{\pi_\gamma : \gamma \in \Gamma\}$ separates points in K since every two distinct points must have at least one different coordinate. Lastly, $|\{\gamma \in \Gamma : \pi_\gamma(x) \neq 0\}| < \kappa$ for every $x \in K$ since $K \subseteq \Sigma_\kappa([0,1]^\Gamma)$.

Assume now that K is a compact space and there exists a family $\mathcal{F}$ as in (d). We want to prove that K is κ-Corson compact. Cosider the diagonal transformation

$$F = \mathop{\triangle}_{f\in\mathcal{F}} f : K \to [0,1]^{|\mathcal{F}|}.$$

Since $\mathcal{F}$ separates points, the function F is a continuous injection and therefore a homeomorphic embedding because K is compact. Clearly, we have $F(K) \subseteq \Sigma_\kappa([0,1]^{|\mathcal{F}|})$ because

$$|\{f \in \mathcal{F} : f(x) \neq 0\}| < \kappa$$

for every $x \in K$. □

The following result is a straightforward generalisation of the well known Rosenthal-type characterisation of Corson compacta (see e.g. [BKT]).

Proposition 2.1.2 ([MPZ, Proposition 2.2]). *Let κ be an uncountable cardinal number. For a compact space K, the following conditions are equivalent*

(a) K is κ-Corson;

(b) There exists a family $\mathcal{U}$ consisting of cozero subsets of K which is T_0-separating, and $\mathrm{ord}(x,\mathcal{U}) < \kappa$ *for all $x \in K$.*

If a space K is homeomorphic to a weakly compact subset of a Hilbert space, then K is said to be *uniform Eberlein compact*. Note that the class of uniform Eberlein compacta contains all metrizable compact spaces.

2.1.1. Boolean algebras

For a Boolean algebra $\mathfrak{B}$ we denote by $\mathrm{ult}(\mathfrak{B})$ its Stone space (of all ultrafilters on $\mathfrak{B}$); the Stone isomorphism

$$\mathfrak{B} \ni b \mapsto \widehat{b} = \{p \in \mathrm{ult}(\mathfrak{B}) : b \in p\}$$

identifies $\mathfrak{B}$ with the algebra of clopen subsets of $\mathrm{ult}(\mathfrak{B})$.

Following [BKT], we say that a Boolean algebra $\mathfrak{B}$ is *κ-Corson* if there is $G \subseteq \mathfrak{B}$ that generates $\mathfrak{B}$ and every centered subfamily $G_0 \subseteq G$ has size $< \kappa$.

For uncountable κ this property can be characterized as follows.

Lemma 2.1.3 ([MPZ, Lemma 2.3]). *Given $\kappa > \omega$, a Boolean algebra $\mathfrak{B}$ is κ-Corson if and only if its Stone space* $\mathrm{ult}(\mathfrak{B})$ *is κ-Corson compact.*

The proof of the lemma above can be found in [BKT]; it is an obvious analogue of the case $\kappa = \omega_1$ which was already noted in [MP, Lemma 2.2]. Note that Lemma 2.1.3 fails for $\kappa = \omega$, see Remark 2.2.4.

2.2. Basic properties of ω-Corson and NY compact spaces

We start by the following easy observations.

Proposition 2.2.1 ([MPZ, Proposition 3.2]). *Let K be a compact space K.*

(a) K is ω-Corson if and only if it can be embedded into some σ-product of metrizable finitely dimensional compacta.

(b) K is NY compact if and only if it can be embedded into the σ-product $\sigma(I^\omega, \Gamma)$ for some set Γ.

Proof. The forward implication in (a) follows from Remark 2.1.1(a). To check the reverse implication, suppose that Ξ is an embedding of K into $\sigma(X_\gamma, a_\gamma, \Gamma)$, where X_γ's are compact metrizable spaces of finite dimension.

For every $\gamma \in \Gamma$ there exist a natural number k_γ and a homeomorphic embedding $\Phi_\gamma : X_\gamma \hookrightarrow [-1,1]^{k_\gamma}$ such that $\Phi_\gamma(a_\gamma) = \mathbf{0}^{k_\gamma}$; hence we can treat K as a subspace of $\sigma([-1,1], 0, \Gamma')$ for a suitable set Γ'. We identify $[-1,1]$ with $(\{0\} \times [0,1]) \cup ([0,1] \times \{0\})$ by rotating $[-1,1]$ over 90 degrees. In this way we turn a subspace of $\sigma([-1,1], 0, \Gamma')$ into a subspace of $\Sigma_\omega([0,1]^{\Gamma' \times 2})$.

Part (b) follows immediately from the fact that that any metrizable compactum can be embedded into the Hilbert cube I^ω and I^ω is homogeneous. □

Let us note that Proposition 2.2.1(b) gives another equivalent definition of the class $\mathcal{NY}$ which will be often used below.

Proposition 2.2.2 ([MPZ, Proposition 3.3])**.** *Every ω-Corson compact space is Eberlein compact and strongly countable dimensional.*

Proof. If K is a ω-Corson compact space, then it is also Eberlein compact by the very definition. To verify the second statement we can assume that $K \subseteq \sigma(I, \Gamma)$.

Note that $\sigma(I, \Gamma) = \bigcup_n A_n$ where

$$A_n = \{x \in \sigma(I, \Gamma) : |\operatorname{supp}(x)| \leq n\}.$$

Moreover, every A_n is a compact space of dimension n, see [EP]. It follows that $\dim(A_n \cap K) \leq \dim A_n = n$ and hence K is strongly countable dimensional. □

Proposition 2.2.3 ([MPZ, Proposition 3.4])**.** *A metrizable compact space K is ω-Corson if and only if it is strongly countable dimensional.*

Proof. This follows immediately from Proposition 2.2.2 and the result from [En2] stating that the space $\sigma(I, \omega)$ is universal for the class of separable metrizable spaces that are strongly countable dimensional. □

It becomes clear that the Hilbert cube is NY compact but not ω-Corson compact. We shall see later that there are zero-dimensional Eberlein compacta which are not ω-Corson. The assertion of Proposition 2.2.3 in fact holds for NY compact spaces K, see Corollary 2.4.1.

Remark 2.2.4 ([MPZ, Remark 3.5])**.** *Given a Boolean algebra $\mathfrak{B}$ and its Stone space $K = \operatorname{ult}(\mathfrak{B})$, the following conditions are equivalent:*

(i) K is ω-Corson compact;

(ii) K has a T_0-separating, point-finite family $\mathcal{U}$ consisting of clopen subsets;

(iii) K is scattered Eberlein compact;

(iv) K is scattered Corson compact.

Indeed, (i)⇔(ii) follows immediately from the very definition and (ii) implies that K can be embedded into some $\sigma(\{0,1\},\Gamma)$; it is easy to check that every compact subset of such a σ-product is scattered. Then (iv)⇒(iii) is a result of Alster [Al].

We shall indicate that the Rosenthal-type characterization of κ-Corson compacta given by Proposition 2.1.2 fails for $\kappa = \omega$.

Proposition 2.2.5 ([MPZ, Proposition 3.6])**.** *For a compact space K, the following conditions are equivalent:*

(a) There exists a T_0-separating, point-finite family $\mathcal{U}$ consisting of cozero subsets of K;

(b) K is a scattered Eberlein compact space.

In view of Remark 2.2.4, Proposition 2.2.5 follows immediately from the following lemma.

Lemma 2.2.6 ([MPZ, Lemma 3.7])**.** *If a compact space K has a point-finite, T_0 - separating family $\mathcal{U}$ of open sets, then K is scattered.*

Proof. Suppose that K is not scattered and let $A \subseteq K$ be a nonempty subset without isolated points. To arrive at contradiction, we inductively construct nonempty open sets V_n such that $\overline{V_{n+1}} \subseteq V_n$ and $|\{U \in \mathcal{U} : V_n \subseteq U\}| \geq n$ for every n.

To start, pick $x, y \in A$, $x \neq y$. There exists $V_1 \in \mathcal{U}$ such that $|V_1 \cap \{x, y\}| = 1$; say $x \in V_1$. As x is not isolated in A, there is $y_1 \in A \cap V_1 \setminus \{x\}$. In turn, there exists $U_1 \in \mathcal{U}$ separating x and y_1. Assume, for instance, that $y_1 \in U_1$ and let V_2 be an open set such that $y_1 \in V_2 \subseteq \overline{V_2} \subseteq U_1 \cap V_1$. Continuing in this fashion, we get the required sequences of V_n's.

Now the set $F = \bigcap_n V_n = \bigcap_n \overline{V_n}$ is nonempty by compactness. If $z \in F$ then z belongs to infinitely many elements of the family $\mathcal{U}$, a contradiction. □

We close this section by a few observations on the stability of our classes under some operations. It was proved in [Ma, Proposition 2.8] that the Aleksandrov duplicate $AD(K)$ of an (uniform) Eberlein compact space K is (uniform) Eberlein compact. Using the same argument one can show a counterpart of this observation for κ-Corson and NY compacta.

Proposition 2.2.7 ([MPZ, Proposition 3.8])**.** *For any infinite cardinal number κ, the Aleksandrov duplicate $AD(K)$ of a κ-Corson compact (NY compact) space K is κ-Corson compact (NY compact).*

Proposition 2.2.8 ([MPZ, Proposition 3.9])**.** *Let $\kappa > \omega$ and let K_t, $t \in T$, be a family of κ-Corson compact spaces (NY compact spaces). Then the one point compactification $\alpha(\bigoplus_{t \in T} K_t)$ of the discrete union $\bigoplus_{t \in T} K_t$ is κ-Corson compact (NY compact, respectively).*

Proof. We will prove this fact only for κ-Corson compacta (for NY compacta the proof is very similar).

Denote the point at infinity of $\alpha(\bigoplus_{t\in T} K_t)$ by ∞. We can assume that $K_t \subseteq \Sigma_\kappa(\mathbb{R}^{\Gamma_t})$ for some set Γ_t, and the sets Γ_t are pairwise disjoint and disjoint from T.

Consider $\Gamma = T \cup \bigcup_{t\in T} \Gamma_t$. Let $\psi : \alpha(\bigoplus_{t\in T} K_t) \to \Sigma_\kappa(\mathbb{R}^\Gamma)$ be defined by

$$\psi(x)(\gamma) = \begin{cases} x(\gamma) & \text{if } x \in K_t\,,\ \gamma \in \Gamma_t\,,\ t \in T \\ 1 & \text{if } x \in K_t\,,\ \gamma = t\,,\ t \in T \\ 0 & \text{if } x = \infty\,, \gamma \in \Gamma \end{cases}$$

for $x \in \alpha(\bigoplus_{t\in T} K_t)$. A routine verification shows that ψ is an embedding. □

We conclude the section with an observation related to the wider class of κ-Valdivia compact spaces considered by Kalenda [Ka2]. The following provides a positive answer to a question posed by Ondřej Kalenda in a conversation.

Proposition 2.2.9 ([MPZ, Proposition 6.5]). *For every metrizable compactum K there is an embedding $f : K \to I^\omega$ such that $f(K) \cap \sigma(I, \omega)$ is dense in $f(K)$.*

Proof. Let K be a metrizable compactum. We can assume that $K \subseteq I^\omega$. Let D be a countable, dense subset of K, and E be a countable, dense subset of I^ω, contained in $\sigma(I, \omega)$. Put $F = D \cup E$. The Hilbert cube I^ω is *countable dense homogeneous*, i.e., given countable dense subsets $A, A' \subseteq I^\omega$, there is a homeomorphism $h : I^\omega \to I^\omega$ such that $h(A) = A'$. Let h be such homeomorphism applied for $A = F$ and $A' = E$. Then the restriction $h|_K$ is an embedding such that $h|_K(K) \cap \sigma(I, \omega)$ contains $h(D)$ hence is dense in $h|_K(K)$. □

2.3. Characterizations of ω-Corson compacta and NY compacta

The main result of this section presents a characterization of ω-Corson compact spaces given in Theorem 2.3.6. Our theorem builds on a characterization of the class $\mathcal{NY}$ obtained by Nakhmanson and Yakovlev [NY], which we expand by adding another condition, see Theorem 2.3.1(iv).

The following theorem offers several internal characterizations of compacta from the class $\mathcal{NY}$. Recall that a family $\mathcal{A}$ of subsets of a topological space X is closure preserving if

$$\overline{\bigcup \mathcal{A}'} = \bigcup\{\overline{A} : A \in \mathcal{A}'\}$$

for any subfamily $\mathcal{A}' \subseteq \mathcal{A}$.

Theorem 2.3.1 ([MPZ, Theorem 4.1]). *For a compact space K, the following conditions are equivalent:*

(i) K is NY compact;

(ii) there exists a T_0-separating family $\mathcal{U} = \bigcup\{\mathcal{U}_\gamma : \gamma \in \Gamma\}$ consisting of cozero subsets of K, where each $\mathcal{U}_\gamma$ is countable and the family $\{\bigcup\mathcal{U}_\gamma : \gamma \in \Gamma\}$ is point-finite;

(iii) K has a closure preserving cover consisting of metrizable compacta;

(iv) K is hereditarily metacompact and each nonempty subspace A of K contains a nonempty relatively open subspace U of countable weight.

The equivalences (i)⇔(ii)⇔(iii) here were already proved in [NY]. To incorporate condition (iv) we need some preparations. The proposition given below was also proved in [Ya, Theorem 11] using the equivalent assumption that the compact space K in question has a closure preserving cover consisting of metrizable compacta. Our argument starting from the assumption that $K \in \mathcal{NY}$ seems to be shorter.

Proposition 2.3.2 ([MPZ, Proposition 4.2])**.** *If $K \in \mathcal{NY}$, then each nonempty subspace A of K contains a nonempty relatively open subspace U of countable weight.*

Proof. By Proposition 2.2.1(b), we can assume that $K \subseteq \sigma(I^\omega, \Gamma)$ for some Γ. Obviously, it is enough to prove the assertion for the closure $\overline{A}$ of a nonempty set $A \subseteq K$. Since $\overline{A}$ is a compact subset of $\sigma(I^\omega, \Gamma)$, for simplicity we can assume that $\overline{A} = K$.

Suppose that there exists a nonempty open set $U \subseteq K$ such that its every nonempty open subset has uncountable weight. We can assume that U is a basic open set, that is

$$U = \{x \in K : x(\gamma) \in V_\gamma \text{ for } \gamma \in \Gamma_0\},$$

for some finite $\Gamma_0 \subseteq \Gamma$ and some family of open sets $\{V_\gamma : \gamma \in \Gamma_0\}$ in I^ω.

Note that for every countable $\Gamma' \subseteq \Gamma$ there must be $x \in U$ and $\gamma \in \Gamma \setminus \Gamma'$ such that $x(\gamma) \neq 0$. Indeed, otherwise U would be contained in a countable product of Hilbert's cubes so it would have a countable base.

Take $x \in U$ and $\gamma_1 \in \Gamma \setminus \Gamma_0$ such that $x(\gamma_1) \neq 0$. Take open G with $x \in G \subseteq K$ and such that for every $y \in G$ we have $y(\gamma_1) \neq 0$. Then find open $W \subseteq K$ such that $x \in W \subseteq \overline{W} \subseteq U$ and put $U_1 = W \cap G$. Finally, set $\Gamma_1 = \Gamma_0 \cup \{\gamma_1\}$. Then for every $y \in U_1$ we have $y(\gamma_1) \neq 0$ and $x \in U_1 \subseteq \overline{U_1} \subseteq U$.

Repeating the above argument, we can inductively construct nonempty open subsets U_n of K and distinct coordinates $\gamma_n \in \Gamma$ such that $U_{n+1} \subseteq \overline{U_{n+1}} \subseteq U_n$ and if $z \in U_n$, then $z(\gamma_n) \neq 0$. Clearly, by compactness, $F = \bigcap_n U_n = \bigcap_n \overline{U_n} \neq \emptyset$ and any $w \in F$ has infinitely many nonzero coordinates, a contradiction. □

For the next lemma we slightly modify the argument used in [NY, Theorem 2].

Lemma 2.3.3 ([MPZ, Lemma 4.3]). *If $K \in \mathcal{NY}$ then for any family $\mathcal{V}$ of open subsets of K, there exists a point-finite open refinement $\mathcal{U}$ such that $\bigcup\mathcal{U} = \bigcup\mathcal{V}$ and every $U \in \mathcal{U}$ is σ-compact.*

Proof. Consider $K \subseteq \sigma(X_\gamma, \xi_\gamma, \Gamma) = Z$ where $\{X_\gamma : \gamma \in \Gamma\}$ is a family of compact metrizable spaces and $\xi_\gamma \in X_\gamma$ for every $\gamma \in \Gamma$.

Since, for a σ-compact $V \subseteq Z$, the intersection $V \cap K$ is also σ-compact it is enough to prove that for any family $\mathcal{V}$ of open subsets of Z, there exists a point-finite open refinement $\mathcal{U}$ such that $\bigcup\mathcal{U} = \bigcup\mathcal{V}$ and every $U \in \mathcal{U}$ is σ-compact. Moreover, without loss of generality, we can assume that the family $\mathcal{V}$ consists of basic open sets in Z.

For $B \in [\Gamma]^{<\omega}$, let $\pi_B : Z \to \prod_{\gamma\in B} X_\gamma$ be the projection restricted to Z. Observe that, for each open set $U \subseteq \prod_{\gamma\in B} X_\gamma$, the set

$$\pi_B^{-1}(U) = U \times \sigma(X_\gamma, \xi_\gamma, \Gamma \setminus B)$$

is σ-compact being a product of σ-compact spaces: a σ-product and an open subset of a metrizable compact space. In particular, each basic open set in Z is σ-compact, hence all $V \in \mathcal{V}$ are σ-compact.

Given $B \in [\Gamma]^{<\omega}$, let

$$U(B) = \{x \in Z : \forall_{\gamma\in B}\ x(\gamma) \neq \xi_\gamma\} = \pi_B^{-1}\Big(\prod_{\gamma\in B}(X_\gamma \setminus \{\xi_\gamma\})\Big).$$

Every $U(B)$ is open in Z and σ-compact, and the family $\{U(B) : B \in [\Gamma]^{<\omega}\}$ is point-finite.

For $B \in [\Gamma]^{<\omega}$, let

$$F(B) = \{x \in Z : x(\gamma) = \xi_\gamma \iff \gamma \in \Gamma \setminus B\}.$$

Observe that $\pi_B|F(B)$ maps $F(B)$ homeomorphically onto $\prod_{\gamma\in B}(X_\gamma \setminus \{\xi_\gamma\})$. By the definition of a σ-product, $\bigcup\{F(B) : B \in [\Gamma]^{<\omega}\} = Z$.

Let $\mathcal{K}(B) = \{V \cap F(B) : V \in \mathcal{V}\}$ for $B \in [\Gamma]^{<\omega}$. Since $F(B)$ is metrizable, there exists a point-finite family $\mathcal{M}(B)$ of open subsets of $F(B)$, such that $\mathcal{M}(B) \prec \mathcal{K}(B)$ and $\bigcup\mathcal{M}(B) = \bigcup\mathcal{K}(B)$. Each set $W \in \mathcal{M}(B)$ is σ-compact because $F(B)$ is σ-compact and metrizable.

Let

$$\mathcal{M}'(B) = \{\pi_B(W) : W \in \mathcal{M}(B)\}.$$

Since $\pi_B|F(B)$ is a homeomorphism, $\mathcal{M}'(B)$ is a point-finite family of open sets in the product $\prod_{\gamma\in B}(X_\gamma \setminus \{\xi_\gamma\})$, hence the sets from $\mathcal{M}'(B)$ are also open in $\prod_{\gamma\in B} X_\gamma$.

Let W be an element of $M(B)$. Since $\mathcal{M}(B) \prec \mathcal{K}(B)$, there exists $V \in \mathcal{V}$ such that $W \subseteq V \cap F(B)$. Let

$$\widetilde{W} = \pi_B^{-1}(\pi_B(W)) \cap V,$$

then $\widetilde{W} \cap F(B) = W$, and $\pi_B(\widetilde{W}) = \pi_B(W)$.

The set $\pi_B(W)$ is σ-compact as a continuous image of a σ-compact set. The set

$$\pi_B^{-1}(\pi_B(W)) = \pi_B(W) \times \sigma(X_\gamma, \xi_\gamma, \Gamma \setminus B)$$

is σ-compact being a product of σ-compact sets. Finally, the set $\widetilde{W}$ is σ-compact as an intersection of two σ-compact sets.

Let

$$\begin{aligned}\mathcal{P}(B) &= \{\widetilde{W} : W \in \mathcal{M}(B)\};\\ \mathcal{R}(B) &= \{\pi_B^{-1}(\pi_B(W)) : W \in \mathcal{M}(B)\}.\end{aligned}$$

The family $\mathcal{R}(B)$ is point-finite, because $\mathcal{M}'(B)$ is point-finite, so $\mathcal{P}(B)$ is point-finite as a shrinking of $\mathcal{R}(B)$. For each $W \in \mathcal{M}(B)$ we have $W \subseteq F(B)$ and $\widetilde{W} \subseteq U(B)$, therefore $\bigcup \mathcal{P}(B) \subseteq U(B)$.

Put $\mathcal{U} = \bigcup\{\mathcal{P}(B) : B \in [\Gamma]^{<\omega}\}$; this family is point-finite because the family $\{U(B) : B \in [\Gamma]^{<\omega}\}$ and all families $\mathcal{P}(B), B \in [\Gamma]^{<\omega}$, are point-finite. Each set $\widetilde{W} \in \mathcal{U}$ is open in Z and σ-compact.

From the definition of sets $\widetilde{W}$, it immediately follows that $\mathcal{U} \prec \mathcal{V}$.

Finally, we shall check that $\bigcup \mathcal{U} = \bigcup \mathcal{V}$. Let $x \in V \in \mathcal{V}$. Then $x \in F(B)$, for some B, so $x \in V \cap F(B) \in K(B)$. Because $\mathcal{M}(B) \prec \mathcal{K}(B)$ and $\bigcup \mathcal{M}(B) = \bigcup \mathcal{K}(B)$, there exists $W \in \mathcal{M}(B)$ such that $x \in W \subseteq V \cap F(B)$. As we observed before $\widetilde{W} \cap F(B) = W$, so $x \in \widetilde{W}$, that is $x \in \bigcup \mathcal{P}(B)$, therefore $x \in \bigcup \mathcal{U}$. □

Let us recall that one can adapt the Cantor-Bendixson rank of scattered spaces to treat any topological property $\mathcal{P}$. Namely, for any ordinal number α we define the α-th derivative $X^{(\alpha)}$ of a given topological space X with respect to $\mathcal{P}$ as follows:

1. $X' = X^{(1)} = X \setminus \bigcup\{U \subseteq X : U$ is open and has property $\mathcal{P}\}$;

2. $X^{(\alpha+1)} = (X^{(\alpha)})'$;

3. $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$ for a limit ordinal α.

If $X^{(\alpha)} = \emptyset$ for some α's then the height $ht(X)$ (with respect to $\mathcal{P}$) is defined as $ht(X) = \min\{\alpha : X^{(\alpha)} = \emptyset\}$. Note that if X is compact, then $ht(X)$ is a successor ordinal number.

Lemma 2.3.4 ([MPZ, Lemma 4.4]). *Let $\mathcal{P}$ be a hereditary topological property. For any topological spaces $Y \subseteq X$ we have $Y^{(\alpha)} \subseteq X^{(\alpha)}$ for every ordinal number α.*

Proof. Note first that $Y' \subseteq X'$. Indeed, if $x \in Y \setminus X'$ then there exists an open neighbourhood of x, $U \subseteq X$ with property $\mathcal{P}$. Then $U \cap Y$ also has the property $\mathcal{P}$ and $x \in U \cap Y$ which implies $x \notin Y'$.

The general follows by induction on α: for the successor step we apply the above observation while the limit case is obvious. □

Theorem 2.3.5 ([MPZ, Theorem 4.5])**.** *Let K be a compact, hereditarily metacompact space and suppose that every nonempty subspace A of K contains a nonempty, relatively open subspace U of countable weight. Then $K \in \mathcal{NY}$.*

Proof. Let $\mathcal{P}$ be property of having countable weight. Let $X^{(\alpha)}$ be α-th derivative of space X with respect to property $\mathcal{P}$. Note first that if $ht(K) = 1$ then

$$K = \bigcup \{U \subseteq K : U \text{ is open and } w(U) \leq \omega\}.$$

By compactness, K is then a finite union of open sets of countable weight, and therefore K has countable weight itself. Consequently, K can be embedded into I^ω, so $K \in \mathcal{NY}$.

Observe that, by our assumption on K, $ht(K)$ is well-defined so we can check the assertion by induction on $ht(X)$. Let $ht(X) = \alpha = \beta + 1$ and assume that the theorem holds for all compact spaces of smaller height.

The space $K^{(\beta)}$ is of height 1; by the introductory remark, there is an embedding $i : K^{(\beta)} \to I^\omega$. Using the Tietze-Urysohn theorem, i can be extended to a continuous function $f : K \to I^\omega$.

For every $x \in K \setminus K^{(\beta)}$ there is an open set $U_x \subseteq K$ such that $\overline{U_x} \cap K^{(\beta)} = \emptyset$. The space $\overline{U_x}$ is hereditarily metacompact and, by Lemma 2.3.4, $(\overline{U_x})^{(\beta)} \subseteq K^{(\beta)}$. Moreover, $\overline{U_x} \cap K^{(\beta)} = \emptyset$, so $(\overline{U_x})^{(\beta)} = \emptyset$. It follows, that $ht(\overline{U_x}) < \alpha$ and, by the inductive assumption, $\overline{U_x}$ can be embedded in $\sigma(I^\omega, \Gamma_x)$, for some Γ_x.

Now the family $\mathcal{U} = \{U_x : x \in K \setminus K^\beta\}$ forms an open cover of $K \setminus K^{(\beta)}$. Hence, by hereditary metacompactness, there exists a point-finite open cover $\mathcal{V} = \{V_\xi : \xi \in \kappa\}$ of $K \setminus K^\beta$ that is inscribed in $\mathcal{U}$. By virtue of Lemma 2.3.3, each set V_ξ has a point-finite cover $\mathcal{W}_\xi$ consisting of open σ-compact sets. The family $\mathcal{W} = \bigcup \{\mathcal{W}_\xi : \xi \in \kappa\}$ is then a point-finite cover of $K \setminus K^{(\beta)}$ consisting of open σ-compact sets and $\mathcal{W}$ is inscribed in $\mathcal{V}$.

It follows that for every $W \in \mathcal{W}$ there is an embedding

$$g_W : \overline{W} \to \sigma(I^\omega, \Gamma_W) \subseteq (I^\omega)^{\Gamma_W},$$

for some Γ_W. Again, by the Tietze-Urysohn theorem, g_W can be extended to a continuous function $G_W : K \to (I^\omega)^{\Gamma_W}$.

Note that for any open, σ-compact set $P \subseteq K$ there exists a continuous function $h_P : K \to [0, 1]$ such that $P = h_P^{-1}((0, 1])$. For every $W \in \mathcal{W}$ we define $H_W : K \to I^\omega$ by

$H_W = h_W \cdot G_W$. Consider now the function

$$T = \Big(\mathop{\triangle}_{W \in \mathcal{W}} H_W \Big) \triangle \Big(\mathop{\triangle}_{W \in \mathcal{W}} h_W \Big) \triangle f : K \to \sigma(I^\omega, \Gamma).$$

Each $x \in K$ belongs to only finitely many elements of $\mathcal{W}$, hence $H_W(x) = 0 = h_W(x)$ except for a finite number of W's. Consequently, the range of T is a subset of $\sigma(I^\omega, \Gamma)$.

To complete the proof, it remains to prove that T is injective. Consider any $x, y \in K$, $x \neq y$.

Case (1) If $x, y \in K^\beta$ then $T(x) \neq T(y)$ since $f(x) \neq f(y)$.

Case (2) If $x \in K^{(\beta)}$ and $y \notin K^{(\beta)}$ then there is $W \in \mathcal{W}$ such that $y \in W$ and $x \notin W$; then $h_W(x) = 0 \neq h_W(y)$.

Case (3) If $x, y \notin K^{(\beta)}$ and $x, y \in W$ for some $W \in \mathcal{W}$ then either $h_W(x) \neq h_W(y)$ (and so $T(x) \neq T(y)$) or $h_W(x) = h_W(y)$ but then then $H_W(x) \neq H_W(y)$ because $G_W(x) = g_W(x) \neq g_W(y) = G_W(y)$.

Case (4) The remaining case is $x \in W$ and $y \notin W$ for some $W \in \mathcal{W}$ (or vice versa); then $h_W(x) \neq 0 = h_W(y)$ so again $T(x) \neq T(y)$.

□

Proof of Theorem 2.3.1. The equivalences (i)⇔(ii)⇔(iii) were proved in [NY, Theorem 1] and [NY, Lemma 1]. The implication (i)⇒(iv) follows from Proposition 2.3.2 and Lemma 2.3.3. The reverse implication is given by Theorem 2.3.5. □

We can now present the following counterpart of Theorem 2.3.1.

Theorem 2.3.6 ([MPZ, Theorem 4.6]). *For a compact space K, the following conditions are equivalent:*

(i) K is ω-Corson;

(ii) K has a closure preserving cover consisting of finite dimensional metrizable compacta;

(iii) K is hereditarily metacompact and each nonempty subspace A of K contains a nonempty relatively open, finite dimensional subspace U of countable weight.

Here we again build on the results from [NY] and [Ya]; in particular, in the proof below we use a slight modification of Yakovlev's reasoning from [Ya, Theorem 3(a)].

Proposition 2.3.7 ([MPZ, Proposition 4.7]). *Every ω-Corson compact space K has a closure-preserving cover consisting of metrizable, compact, finitely dimensional sets.*

Proof. By Proposition 2.2.1, we can assume that K is a compact subspace of $\sigma(I, \Gamma)$, for some Γ. For any finite $\Gamma_0 \subseteq \Gamma$ we put

$$A_{\Gamma_0} = \{x \in K : \forall_{\gamma \in \Gamma \setminus \Gamma_0}\ x(\gamma) = 0\},$$

and consider $\mathcal{A} = \{A_{\Gamma_0} : \Gamma_0 \in [\Gamma]^{<\omega)}\}$. Then the family $\mathcal{A}$ covers K and consists of finitely dimensional metrizable compacta so it remains to prove that it is closure-preserving.

Since K is Eberlein compact, it is Frechet-Urysohn so, in particular, the closure in K coincides with the sequential closure. Therefore it suffices to check that the union of any subfamily of $\mathcal{A}$ is sequentially closed. For that purpose consider $x_n \in A_{\Gamma_n} \in \mathcal{A}$ and assume that the sequence of x_n converges to $x \in K$.

Suppose that $x \notin \bigcup_n A_{\Gamma_n}$; then for every n there is $\gamma_n \notin \Gamma_n$ such that $x(\gamma_n) \neq 0$. Since $x \in K$, it has only finitely many nonzero coordinates and therefore there exists $\gamma \in \Gamma$ such that the set $N = \{n : \gamma_n = \gamma\}$ is infinite. Since $\gamma_n = \gamma \notin \Gamma_n$ for $n \in N$, we get

$$0 \neq x(\gamma) = \lim_{n \in N} x_n(\gamma) = \lim_{n \in N} x_n(\gamma_n) = 0,$$

a contradiction. □

Lemma 2.3.8 ([MPZ, Lemma 4.8])**.** *Let X be a separable metrizable topological space and let $\mathcal{A}$ be a closure-preserving cover of the space X consisting of closed subsets. Then there exists a countable subfamily $\mathcal{A}' \subseteq \mathcal{A}$ such that $\bigcup \mathcal{A}' = X$.*

Proof. Suppose that every countable family $\mathcal{A}' \subseteq \mathcal{A}$ is not a cover of X. Then we can construct by induction on $\alpha < \omega_1$ countable subfamilies $\mathcal{A}_\alpha \subseteq \mathcal{A}$ such that for $\alpha < \beta < \omega_1$ we have $\bigcup \mathcal{A}_\alpha \subsetneq \bigcup \mathcal{A}_\beta$. However, the space $Y = \bigcup_{\alpha<\omega_1}(\bigcup \mathcal{A}_\alpha)$ is separable and metrizable; therefore it does not allow strictly ascending uncountable chains of closed sets, a contradiction. □

Proof of Theorem 2.3.6. The implication (i)⇒(ii) is given by Proposition 2.3.7.

To prove (ii)⇒(iii) note first that by Theorem 2.3.1 and Lemma 2.3.3, the space K is hereditarily metacompact.

It remains to show the second part of (iii). If $\overline{A}$ meets the condition (iii), then so does A — we can assume that A is closed. We can also assume that $A = K$, because condition (ii) is inherited by closed subspaces. By Proposition 2.3.2, there exists an open, nonempty set $V \subseteq K$ such that $w(V) \leq \omega$. The set V is a Baire space as it is an open subset of a compact space. Let $\mathcal{A}$ be a cover as in (ii). By virtue of Lemma 2.3.8, there exists a countable $\mathcal{A}' \subseteq \mathcal{A}$ such that $V \subseteq \bigcup \mathcal{A}'$. Hence, there exists $A' \in \mathcal{A}'$ such that $\mathrm{int}(A' \cap V) \neq \emptyset$ and the set $U = \mathrm{int}(A \cap V)$ is as required.

For the proof of (iii)⇒(i) we closely follow the argument from the proof of Theorem 2.3.5. This time we consider property $\mathcal{P}$ of having countable weight and finite dimension.

We verify (iii) by transfinite induction on $ht(K)$ with respect to $\mathcal{P}$. If $ht(K) = 1$, then

$$K = \bigcup \{U \subseteq K : U \text{ is open}, w(U) \leq \omega, \dim(U) < \infty\}.$$

Since K is compact, it is a finite union of open sets of countable weight and finite dimension; consequently, K is metrizable and finitely dimensional. The space K is ω-Corson, because it can be embedded in the cube I^k, for a natural number k, and of course I^l can be embedded in $\sigma(I, \omega)$.

For the inductive step one can repeat the inductive step of the proof of Theorem 2.3.5, replacing the σ-products of Hilbert cubes by the σ-products of unit intervals. □

2.4. More properties of ω-Corson compacta and NY compacta

From the characterizations of ω-Corson and NY compact space given in Section 2.3 we can easily derive the following strengthening of Proposition 2.2.3.

Corollary 2.4.1 ([MPZ, Corollary 5.1]). *An NY compact space K is ω-Corson if and only if it is strongly countable dimensional.*

Proof. The "only if" part follows immediately from Proposition 2.2.2.

Let K be an NY compact space which is a countable union of finite dimensional compacta A_n. We will show that K satisfies condition (iii) of Theorem 2.3.6. By condition (iv) of Theorem 2.3.1, K is hereditarily metacompact. Similarly as in the proof of the implication (ii)⇒(iii) of Theorem 2.3.1, we only need to verify the second part of (iii) for a nonempty closed subset A of K.

By Theorem 2.3.1(iv), there exists a relatively open, nonempty set $V \subseteq A$ such that $w(V) \leq \omega$. The set V is a Baire space since it is an open subset of a compact space A. It follows, that for some n, the intersection $V \cap A_n$ has a nonempty interior in V. The set $U = \mathrm{int}_V(V \cap A_n)$ being a Lindelöf subspace of a finite dimensional compact space A_n, is finite dimensional (cf. [En2, Theorem 3.1.23]), so it has all required properties. □

The classes of ω-Corson and NY compacta are clearly closed under finite products; they are, however, not closed under taking nontrivial countable products. Namely, from condition (iv) of Theorem 2.3.1 we can easily deduce the following fact which also allows one to produce simple examples of Eberlein compacta which are not NY compact.

Corollary 2.4.2 ([MPZ, Corollary 5.2]). *For any sequence $(K_n)_{n\in\omega}$ of nonmetrizable Eberlein compacta, the product $\prod_{n\in\omega} K_n$ does not belong to $\mathcal{NY}$.*

Nakhmanson and Yakovlev noted that one can use condition (iii) of Theorem 2.3.1 to conclude the following.

Corollary 2.4.3 ([MPZ, Corollary 5.3]). *The class $\mathcal{NY}$ is closed under taking continuous images.*

Note that the class of ω-Corson compact spaces is clearly not stable under taking Hausdorff continuous images, as the Hilbert cube is a continuous image of the Cantor set 2^ω.

In the context of hereditary metacompactness of NY compacta, the following result of Gruenhage [G, Theorem 2.2] is worth recalling (here $\Delta = \{(x,x) : x \in K\}$ is the diagonal).

Theorem 2.4.4. *For a compact space K, the following conditions are equivalent:*

(i) K is Eberlein compact;

(ii) K^2 is hereditarily σ-metacompact;

(iii) $K^2 \setminus \Delta$ is σ-metacompact.

We shall now mention two examples showing that one cannot omit any of the two properties named by Theorem 2.3.1(iv) and Theorem 2.3.6(iii).

The first example is essentially due to Gruenhage, see his remarks at the end of Section 2 in [G]. He described it as an inverse limit $X = \varprojlim X_n$ of spaces X_n, where $X_0 = A(\omega_1)$, X_{n+1} is obtained from X_n, by replacing each isolated point of X_n, by a copy of $A(\omega_1)$, and the bonding maps are the obvious quotient maps. It is stated in [G], without a proof, that the space $X^2 \setminus \Delta$ is metacompact. We present below a different description of that space and prove that its every finite power is hereditarily metacompact (Theorem 2.4.9 shows that we cannot improve this for infinite products). Recall here that, in general, a square L^2 of a hereditarily metacompact (even hereditarily Lindelöf) compact space L need not to be hereditarily metacompact, which is witnessed by the *double arrow* space L (cf. [En1, 53.B(a)]).

Recall also that *polyadic* spaces, introduced by Mrówka, are the continuous images of spaces of the form $A(\kappa)^\lambda$.

Example 2.4.5 ([MPZ, Example 5.5]). *There exists a zero-dimensional uniform Eberlein and polyadic compact space K such that $K \notin \mathcal{NY}$ but K^n is hereditarily metacompact for every n.*

Proof. Consider the tree $T = \omega_1^\omega \cup \bigcup_{n\in\omega} \omega_1^n$ with the standard order $\preceq$ given by inclusion. For $s \in \omega_1^n$, $n \in \omega$, let $V_s = \{t \in T : s \preceq t\}$ (a wedge in T).

Our example K is the tree T equipped with the coarse wedge topology, i.e., the topology generated by the sets V_s, $T \setminus V_s$ for $s \in \omega_1^n$, $n \in \omega$. Since T has only one minimal element, and every branch in T has the greatest element, the space K is compact (cf. [Ny2, Theorem 3.4]). From the definition of the topology on T it immediately follows that K is zero-dimensional.

To show that K is uniform Eberlein compact and polyadic it is enough to check that it is a continuous image of the product $A(\omega_1)^\omega$. Indeed, one can easily verify that the map

$\varphi : A(\omega_1)^\omega \to K$ defined by

$$\varphi((x_n)_{n=0}^\infty) = \begin{cases} (x_i)_{i=0}^\infty & \text{if } x_i \in \omega_1 \text{ for all } i \in \omega \\ (x_i)_{i=0}^{n-1} & \text{if } x_n = \infty \text{ and } x_i \in \omega_1 \text{ for all } i < n\,, \end{cases}$$

for $(x_i)_{i=0}^\infty \in A(\omega_1)^\omega$, is a continuous surjection.

Let

$$\mathcal{B} = \{V_s \setminus \bigcup_{i=0}^{k} V_{t_i} : s \in \omega_1^n,\ t_0, \ldots, t_k \in \omega_1^{n+1},\ n, k \in \omega\}\,.$$

The family $\mathcal{B}$ is a base of K having the following properties (observe that (1)⇒(2) and (3)⇒(4)):

1. $u \in V_s \setminus \bigcup_{i=0}^{k} V_{t_i}$ if and only if $s \preceq u$ and $t_i \npreceq u$ for $i \leq k$;

2. for a fixed $u \in T$, the collection $\mathcal{B}_u = \{U \in \mathcal{B} : u \in U\}$ is closed under finite unions;

3. $V_s \setminus \bigcup_{i=0}^{k} V_{t_i} \subseteq V_{s'} \setminus \bigcup_{i=0}^{k'} V_{t'_i}$ implies $s' \prec s$ or $s' = s$ and $\{t'_0, \ldots, t'_{k'}\} \subset \{t_0, \ldots, t_k\}$;

4. for a fixed $u \in T$, the collection $\mathcal{B}_u$ does not contain infinite strictly increasing chains.

Properties (2) and (4) imply that the base $\mathcal{B}$ is *point-additively Noetherian* (cf. [Ny1, Definition 1]). Therefore, Theorems 1 and 5 from [Ny1] yield that every finite product K^n is hereditarily metacompact.

Finally, the space K does not belong to $\mathcal{NY}$, since each nonempty open subset of K contains a copy of $A(\omega_1)$, which violates the conclusion of Proposition 2.3.2. □

Example 2.4.6 ([MPZ, Example 5.6]). *For the second example consider any scattered compact space K which is not Eberlein compact. Then K has the property that each nonempty subspace A of K contains a nonempty relatively open separable, metrizable, finite dimensional subspace U.*

Another relevant example comes from [Ya]:

Example 2.4.7 ([MPZ, Example 5.7]). *The product $A(\omega_1)^\omega$ is Eberlein compact, but not hereditarily metacompact.*

As we show below, such a property is shared by all infinite product of nonmetrizable Eberlein compacta, see Theorem 2.4.9.

Lemma 2.4.8 ([MPZ, Lemma 5.8]). *Let K be a compact, hereditarily metacompact topological space. If $\varphi : K \to L$ is a continuous surjection then L is hereditarily metacompact as well.*

Proof. For any $B \subseteq L$, by our assumption, the space $A = \varphi^{-1}[B]$ is metacompact. By compactness of K, the mapping φ is closed and so is the restriction of φ to A (if F is closed in A then $F = A \cap H$ for some closed subset H of K; then $\varphi[F] = \varphi[H] \cap B$)). By Worrell's theorem, see [En1, Theorem 5.3.7], metacompactness is preserved by closed mappings so B is metacompact. □

Theorem 2.4.9 ([MPZ, Theorem 5.9]). *If $(K_n)_{n\in\omega}$ is a sequence of nonmetrizable Eberlein compact spaces then the product $\prod_{n\in\omega} K_n$ is not hereditarily metacompact.*

Proof. It is well-known that a nonmetrizable Eberlein compactum is not *ccc*, so for each n, there exists a family $\{U_n^\alpha : \alpha \in \omega_1\}$ of pairwise disjoint, nonempty, open sets of K_n. Pick any $x_n^\alpha \in U_n^\alpha$. Consider

$$L_n = (K_n \setminus \bigcup_{\alpha\in\omega_1} U_n^\alpha) \cup \{x_n^\alpha : \alpha \in \omega_1\}.$$

One can easily verify that the quotient space $L_n/(K_n \setminus \bigcup_{\alpha<\omega_1} U_n^\alpha)$ obtained by identifying all points in $K_n \setminus \bigcup_{\alpha<\omega_1} U_n^\alpha$ is homeomorphic to $A(\omega_1)$. Pick a homeomorphism h_n witnessing this. Let $q_n : L_n \to L_n/(K_n \setminus \bigcup_{\alpha<\omega_1} U_n^\alpha)$ be the quotient map, and let $f_n = h_n \circ q_n$. The map

$$\prod_{n\in\omega} f_n : \prod_{n\in\omega} L_n \to A(\omega_1)^\omega$$

is a continuous surjection, and therefore, by Example 2.4.7 and Lemma 2.4.8, the product $\prod_{n\in\omega} K_n$ is not hereditarily metacompact. □

Recall that the preorder $\leq^*$ on ω^ω is defined by $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. A subset A of ω^ω is called *unbounded* if it is unbounded with respect to this preorder. We refer below to the classical cardinal number:

$$\mathfrak{b} = \min\{|A| : A \text{ is an unbounded subset of } \omega^\omega\}.$$

It is well known that, for every $n \geq 1$, the statement $\mathfrak{b} = \omega_n$ is consistent with ZFC,
(cf. [vD, Theorem 5.1]).

The next result was proved in [Ma] for a proper subclass $\mathcal{EC}_{wc}$ of the class of NY compacta. We do not know if it can be proved for all Eberlein compact spaces.

Theorem 2.4.10 ([MPZ, Theorem 5.10]). *Assuming that $\mathfrak{b} > \omega_1$, each nonmetrizable compact space $K \in \mathcal{NY}$ contains a closed nonmetrizable zero-dimensional subspace L.*

Proof. Our argument follows closely the proof of Theorem 4.15 in [Ma].

Let $K \in \mathcal{NY}$; we can assume that $K \subseteq \sigma(I^\omega, \Gamma)$ for some set Γ. Since K is nonmetrizable, obviously the set Γ must be uncountable. We can also assume that, for each $\gamma \in \Gamma$, there is $x_\gamma \in K$ such that $x_\gamma(\gamma) \neq (0, 0, \dots)$. Given $\gamma \in \Gamma$, the set

$$F_\gamma = \{\delta \in \Gamma : x_\gamma(\delta) \neq (0, 0, \dots)\}$$

is finite and nonempty. Using the Δ-system lemma we can find a finite set $A \subseteq \Gamma$ and a set $S \subseteq \Gamma$ of size ω_1 such that, for any distinct $\alpha, \beta \in S$, $F_\alpha \cap F_\beta = A$. Now, we can identify the product $(I^\omega)^\Gamma$ with the product $I^{\omega \times \Gamma}$, and apply Lemma 4.10 from [Ma] for the sets $X = \{x_\gamma : \gamma \in S\}$ and $\Gamma_0 = \omega \times A$. $\square$

Remark 2.4.11 ([MPZ, Remark 5.11])**.** *It follows from Proposition 2.3.2 that no nonmetrizable, compact convex subset K of a topological vector space E is NY compact because every nonempty open set $V \subseteq K$ contains a copy of the space K. Indeed, for a fixed $x_0 \in V$, by the continuity of the linear operations in E, and the compactness of K, the set $\{(1-\varepsilon)x_0 + \varepsilon y : y \in K\}$ - an affine copy of K, is contained in V, for suitably small $\varepsilon > 0$.*

In particular, the space $P(K)$ of all probability Radon measures on a compact space K is NY compact if and only if K is metrizable (recall that $P(K)$ is Eberlein compact if and only if K is Eberlein compact).

2.5. Measures on κ-Corson compacta

For a point x in K $t(x,K)$ denotes the tightness of K at x (recall that $t(x,K) \le \tau$ means that whenever $X \subseteq K$ and $x \in \overline{X}$, then there is $A \subseteq X$ such that $|A| \le \tau$ and $x \in \overline{A}$). The tightness of a space is $\le \tau$ if $t(x,K) \le \tau$ for every $x \in K$. The strong tightness of a space K is $\le \tau$ if for every $X \subseteq K$ and $x \in \overline{X}$ there is $A \subseteq X$ such that $|A| < \tau$ and $x \in \overline{A}$.

The theorem below was first proved in [BKT, Corollary 2.17], another proof was given in [MPZ, Theorem 9.2], see Theorem 3.4.10.

Theorem 2.5.1. *For a regular, uncountable cardinal number κ the class of κ-Corson compact spaces is invariant under continuous images.*

Remark 2.5.2 ([MPZ, Remark 12.1])**.** *Recall that the space $P(K)$ shares many topological properties with $M_1(K)$, the space of all signed measures from the unit ball in $C(K)^*$. In particular, $M_1(K)$ is a continuous image of $T \times P(K) \times P(K)$, where $T = \{(t,s) \in \mathbb{R}^2 : |t| + |s| \le 1\}$.*

To discuss the tightness we first recall [Ka2, Lemma 1.19].

Lemma 2.5.3. *For every $\kappa > \omega$ and any Γ the space $\Sigma_\kappa([0,1]^\Gamma)$ has strong tightness κ if and only if κ is regular.*

Definition 2.5.4. *A regular cardinal κ is a* caliber of Radon measures *if, for every compact space K, a measure $\mu \in P(K)$, and a family $\{A_\alpha : \alpha < \kappa\}$ in $Bor(K)$ of μ-positive sets, there is $x \in K$ such that $\{\alpha < \kappa : x \in A_\alpha\}$ has cardinality κ.*

Theorem 2.5.5 ([MPZ, Theorem 12.5])**.** *For a regular uncountable cardinal number κ the following are equivalent* *(i)* $P(K)$ *is* κ*-Corson compact for every* κ*-Corson compact space* K*;*

(ii) $M_1(K)$ *is* κ*-Corson compact for every* κ*-Corson compact space* K*;*

(iii) For every $K \subseteq \Sigma_\kappa([0,1]^\Gamma)$ *and* $\mu \in P(K)$ *there is* $S \subseteq \Gamma$ *such that* $|S| < \kappa$ *and*

$$\mu\left(\{x \in K; \operatorname{supp}(x) \subseteq S\}\right) = 1;$$

(iv) κ *is a caliber of Radon measures.*

Proof. The equivalence (i)⇔(ii) follows from $P(K) \subseteq M_1(K)$, Theorem 2.5.1 and Remark 2.5.2.

For (i)⇒(iii) consider $K \subseteq \Sigma_\kappa([0,1]^\lambda)$ and $\mu \in P(K)$. It is not difficult to check that

$$\mu \in \overline{\operatorname{conv}\{\delta_x : x \in K\}}.$$

By Lemma 2.5.3 applied to $P(K)$, there is a set $D \subseteq K$ such that $|D| < \kappa$ and

$$\mu \in \overline{\operatorname{conv}\{\delta_x : x \in D\}}.$$

We shall check that the following set satisfies (iii):

$$S = \bigcup_{x \in D} \operatorname{supp}(x).$$

Write $C(\gamma) = \{x \in K : x(\gamma) = 0\}$; if $\gamma \notin S$ then $\delta_x(C(\gamma)) = 1$ for every $x \in D$ and hence $\nu(C(\gamma)) = 1$ for every ν from the set $\operatorname{conv}\{\delta_x : x \in D\}$. Since μ is in its closure, we have $\mu(C(\gamma)) = 1$ as well. Finally, the set

$$\{x \in K : \operatorname{supp}(x) \subseteq S\} = \bigcap_{\gamma \in \Gamma \setminus S} C(\gamma)$$

is an intersection of closed sets of full measure μ so it has measure 1, by regularity of μ.

To check (iii)⇒(i) take $K \subseteq \Sigma_\kappa([0,1]^\Gamma)$ and consider the family $\mathcal{F} \subseteq C(K)$ of functions $f_{(I,\varphi)}$ where

$$f_{(I,\varphi)}(x) = \prod_{\gamma \in I} x(\gamma)^{\varphi(\gamma)}, \quad I \subseteq [\Gamma]^{<\omega}, \varphi : I \to \mathbb{N}.$$

By the Stone-Weierstrass theorem, the linear span of $\mathcal{F}$ is norm dense in $C(K)$, and therefore the family $\mathcal{F}$ distinguishes elements of $P(K)$. Given $\mu \in P(K)$, take a set S as in *(iii)*; then $\mu(f_{(I,\varphi)}) > 0$ implies $I \subseteq S$ so, treating $\mathcal{F}$ as a family of continuous functions on $P(K)$ and using Remark 2.1.1(d), we conclude that $P(K)$ is κ-Corson.

To argue for (iii)⇒(iv), suppose that κ is not a caliber of a measure $\mu \in P(L)$ for some compact space L. This is witnessed, in view of regularity of μ, by a family $\mathcal{A} = \{A_\xi : \xi < \kappa\}$ of closed subsets of L such that $\mu(A_\xi) > 0$ for every ξ and

$$\Big|\{\xi < \kappa : y \in A_\xi\}\Big| < \kappa,$$

for every $y \in L$. Consider the Boolean algebra $\mathfrak{B}$ of subsets of L generated by $\mathcal{A}$. Then $\mathfrak{B}$ is a κ-Corson algebra by Lemma 2.1.3. Moreover,

$$\theta : \mathrm{ult}(\mathfrak{B}) \to 2^{\mathcal{A}}, \theta(p) = \chi_{\widehat{A}}(p) \text{ for } p \in \mathrm{ult}(\mathfrak{B}),$$

is an embedding onto $K \subseteq \Sigma_\kappa(2^{\mathcal{A}})$. The measure μ defines a measure $\widehat{\mu} \in P(\mathrm{ult}(\mathfrak{B}))$ via the Stone isomorphism, and if we take the image measure $\nu = \theta[\widehat{\mu}] \in P(K)$, then

$$\nu(\{x \in K : x(A) = 1\} = \mu(A) > 0,$$

for every $A \in \mathcal{A}$, so we have proved that (iii) does not hold.

Finally, we verify (iv)⇒(iii). If $K \subseteq \Sigma_\kappa([0,1]^\Gamma)$ and $\mu \in P(K)$, then the family of sets of the form $C_\gamma = \{x \in K : x(\gamma) > 0\}$, $\gamma \in \Gamma$, does not contain κ many sets with nonempty intersection so, by *(iv)*, the set $S = \{\gamma \in \Gamma : \mu(C_\gamma) > 0\}$ must be of size $< \kappa$ and hence μ is supported by $\{x \in K : \mathrm{supp}(x) \subseteq S\}$. □

Corollary 2.5.6 ([MPZ, Corollary 12.6])**.** *For every $\mathfrak{c}^+$-Corson compact space K the spaces $P(K)$ and $M_1(K)$ are $\mathfrak{c}^+$-Corson compact.*

Chapter 3

Function spaces Corson-like compacta

3.1. Linear topological spaces and dual operators

We note here two observations about dual operators.

For two locally convex linear topological spaces X and Y, a continuous operator $T : X \to Y$ has a dense image if and only if its dual operator is injective ([S, Corollary 21/Chapter 26]).

From [Sc, Theorem 7.3] and [S, Corollary 21/Chapter 26] it follows that, for two locally convex linear topological spaces X and Y, a continuous linear operator $T : X \to Y$ is a weakly open (onto its image) injection if and only if the dual operator is a surjection.

3.2. Behaviour of dimension under transformations of function spaces

In 1980 Pavlovski proved that the covering dimension of metrizable, compact spaces is preserved under linear homeomorphisms of function spaces endowed with the pointwise convergence topology ([Pv]). Later in 1982 Pestov showed that this holds for arbitrary Tychonoff spaces (see [Pe]). In 1990 Arhangel'skii asked if it is true that $dim\, Y \leq dim\, X$ when there exists a continuous, linear surjection of $C_p(X)$ onto $C_p(Y)$. In 1997 Leiderman, Morris and Pestov showed that the answer to this question is in the negative. Namely, they proved that for every metrizable, finite dimensional compact space X, there exists a continuous, linear surjection from $C_p(I)$ to $C_p(X)$ (see [LMP]).

In 1997 Leiderman, Levin and Pestov proved that, for every finite dimensional, metrizable, compact space Y, there exists a 2-dimensional, metrizable, compact space X such that there is a continuous, open, linear surjection of $C_p(X)$ onto $C_p(Y)$. They also proved that, for every $n \in \mathbb{N}$, there exist an n-dimensional, compact, metrizable space Y and 1-dimensional, compact, metrizable space X such that there is a continuous, linear, open surjection from $C_p(X)$ to $C_p(Y)$. Lastly, they gave an example of a continuous, linear surjection from $C_p([0,1])$ to $C_p([0,1])$ which is not open. For details see [LLP].

In the same year Levin and Sternfeld noticed that one can construct a continuous surjection of $C_p(I)$ onto $C_p(X)$ where X is an infinite dimensional space (see [LMP, Remark 4.6]).

In 2011 Levin proved a strengthening of results from [LMP] and [LLP], namely he showed that for every metrizable, finite dimensional, compact space X, the space $C_p(X)$ is a continuous, linear, open image of $C_p([0,1])$ (see [L]).

For a compact space K, define $I(K) = \bigcup\{U \subseteq K : \text{U is open and finite dimensional}\}$. Let $K^{(0)} = K$, $K^{(\alpha+1)} = K^{(\alpha)} \setminus I(K)$ for every ordinal number α and $K^{(\lambda)} = \bigcap_{\alpha<\lambda} K^{(\alpha)}$ for a limit ordinal λ. The fd-height of a compact space K is the minimal ordinal number α such that $K^{(\alpha)} = \emptyset$. A compact space is strongly countable dimensional if and only if its fd-height is a countable, ordinal number.

In 2017 Gartside and Feng proved that if K is a compact, metrizable space and has finite fd-height, then there exists a continuous, linear surjection from $C_p(I)$ to $C_p(K)$. They also showed that if X is submetrizable and has k_ω sequence of compact sets of finite fd-height, then there exists a continuous, linear surjection from $C_p(\mathbb{R})$ to $C_p(X)$. Finally, they showed that if there is a continuous, linear surjection of $C_p(\mathbb{R})$ onto $C_p(X)$, then X is submetrizable, has k_ω sequence and is strongly countable dimensional (for details and definitions see [GF]).

Also in 2017 Kawamura and Leiderman proved that for compact spaces K and L if there exists a continuous linear surjection $T : C_p(K) \to C_p(L)$, then if K is zero-dimensional, then L is zero-dimensional as well (see [KL]).

In 2019 Górak, Krupski and Marciszewski proved that a space X is compact, metrizable and strongly countable dimensional if and only if for every $\epsilon > 0$ there exists a $(1+\epsilon)$-good, uniformly continuous surjection $u_\epsilon : C_p(I) \to C_p(X)$ such that $\|u_\epsilon(f)\|_\infty \leq \|f\|_\infty$ for every $f \in C_p(I)$. They also proved that if X is compact, metrizable, and there exists a uniformly continuous surjection from $C_p(X)$ to $C_p(Y)$ which is c-good for some $c > 0$, then if X is zero-dimensional, then Y is zero-dimensional, and if X is strongly countable dimensional, then so is Y (for details and definitions see [GKM]).

Recently Ali Emre Eysen and Vesko Valov extended resluts of Górak, Krupski and Marciszewski and proved that if X and Y are σ-compact, separable metric spaces and there exists a c-good, uniformly continuous surjection $T : C_p(X) \to C_p(Y)$ for some $c > 0$, then if X is strongly countable dimensional, then Y is strongly countable dimensional as well, and if X is zero-dimensional, then Y is zero-dimensional as well. Finally, they showed that, if X and Y are Tychonoff spaces and there exists a continuous, linear surjection $T : C_p(X) \to C_p(Y)$, then if X is zero-dimensional, then Y is zero-dimensional as well, see [EV].

The following two lemmas are [M, Proposition 6.7.2] and [M, Proposition 6.7.4].

For a space X, let $L_p(X)$ denote the dual space of $C_p(X)$, endowed with the $weak^*$ topology.

Lemma 3.2.1. *Every space X embeds as a closed subset in $L_p(X)$.*

Lemma 3.2.2. *For a space X, every $\phi \in L_p(X)$ is of the form*

$$\phi(f) = \sum_{k=1}^{n} a_k f(x_k)$$

where $a_k \in \mathbb{R}$, $x_k \in X$ and $n \in \mathbb{N}$.

Definition 3.2.3. *For a space X and $\phi \in L_p(X)$, let $supp(\phi) = \{x_1, \ldots, x_n\} \subseteq X$ if $\phi(f) = \sum_{k=1}^{n} a_k f(x_k)$ for every $f \in C_p(X)$ and some nonzero $a_1, \ldots, a_n \in \mathbb{R}$.*

Lemma 3.2.4 ([Za, Lemma 4.4])**.** *Let X be a space, and let $n \in \mathbb{N}$. The set*

$$A_n = \{\phi \in L_p(X) : |supp(\phi)| \leq n\}$$

is closed in $L_p(X)$.

Proof. Pick any $\phi \in L_p(X) \setminus A$, and let $supp(\phi) = \{x_1, \ldots, x_m\}$ where $m \geq n+1$. There exists pairwise disjoint open sets $U_i \subseteq X$ such that $U_i \cap supp(\phi) = \{x_i\}$ for every $i \in \{1, \ldots, m\}$. Since the space X is Tychonoff, there exists $f_i \in C_p(X)$ such that $f_i \restriction_{X \setminus U_i} \equiv 0$ and $f_i(x_i) = 1$ for every $i \in \{1, \ldots, m\}$. The set

$$V = \{\phi \in L_p(X) : \phi(f_i) \neq 0, \text{ for } i \in \{1, \ldots, m\}\}$$

is an open neighbourhood of ϕ disjoint from A. □

The following theorem and lemma are [En1, Theorem 3.3.7] and [En1, Lemma 3.1.6].

Theorem 3.2.5 (Theorem on dimension rising mappings)**.** *If $f : X \to Y$ is a closed mapping from a normal space X onto a normal space Y and there exists $k \in \mathbb{N}$ such that $|f^{-1}(y)| \leq k$ for every $y \in Y$, then $dim\, Y \leq dim\, X + (k-1)$.*

Lemma 3.2.6. *If a normal space X can be represented as a union of a sequence $K_1, K_2, \ldots$ of subspaces such that for every $i \in \mathbb{N}$ and every $F \subseteq K_i$ closed in X we have $dim\, F \leq n$ and the union $\bigcup_{j \leq i} K_j$ is closed for every $i \in \mathbb{N}$, then $dim\, X \leq n$.*

Lemma 3.2.7 ([Za, Lemma 4.7])**.** *Let X be a finite dimensional, normal topological space, and let $A \subseteq X$ be a closed subset, then $dim\, X/A \leq dim\, X$.*

Proof. Let $\pi : X \to X/A$ be the quotient map. Let $\mathcal{U}$ be a finite open cover of X/A. Consider

$$\mathcal{V} = \{\pi^{-1}(U) : U \in \mathcal{U}\},$$

it is a open cover of X. There exists $V_0 \in \mathcal{V}$ such that $A \subseteq V_0$. Let

$$\mathcal{V}' = \{V_0\} \cup \{V \setminus A : V \in \mathcal{V} \setminus \{V_0\}\},$$

it is an open cover of X. By [En1, Theorem 3.2.1] there exists a shrinking $\mathcal{W}$ of $\mathcal{V}'$ such that $ord(\mathcal{W}) \leq dim\, X + 1$. Notice that there is precisely one $W_0 \in \mathcal{W}$ such that $W_0 \cap A \neq \emptyset$, and so for this W_0, we have $A \subseteq W_0$. The family

$$\mathcal{U}' = \{\pi(W) : W \in \mathcal{W}\}$$

is an open cover of X/A inscribed in U and $ord(\mathcal{U}') = ord(\mathcal{W}) \leq dimX + 1$. Consequently, we have $dim\, X/A \leq dim\, X$. □

Definition 3.2.8. *A topological space X is called strongly countable dimensional if it can be represented as a union of countably many closed, finite dimensional subspaces.*

Definition 3.2.9 ([Za, Definition 4.9])**.** *For $\kappa \geq \omega$, we say that a topological space X is strongly κ-dimensional if it can be represented as a union of κ many closed, finite dimensional subspaces.*

Definition 3.2.10 ([Za, Definition 4.10])**.** *For $\kappa \geq \omega$, we say that a topological space X is κ-compact if it is a union of its κ many compact subspaces.*

A theorem of Gartside and Feng shows the thesis of the theorem below for the case when T is a surjection, $X = \mathbb{R}$ and $\kappa = \omega$. Recall that by a result of Górak, Krupski and Marciszewski for a metrizable, compact X and $\kappa = \omega$, it is enough to assume that T is a uniformly continuous surjection which is c-good for some $c > 0$.

Theorem 3.2.11 ([Za, Theorem 4.11])**.** *Let $\kappa \in Card$, $\kappa \geq \omega$. Assume that X and Y are κ-compact Tychonoff spaces. Assume further there exists a continuous, linear transformation $T : C_p(X) \to C_p(Y)$ such that the image of T is dense in $C_p(Y)$. If X is strongly κ-dimensional, then Y is strongly κ-dimensional as well.*

Proof. It is easy to notice that $X = \bigcup_{\alpha\in\kappa} X_\alpha$ where X_α's are compact, finite dimensional, and the family $\{X_\alpha : \alpha \in \kappa\}$ is closed under finite unions. Let

$$s^\alpha_{n,m} : X^n_\alpha \times ([-m, -1/m] \cup [1/m, m])^n \to L_p(X)$$

be given by the formula

$$s^\alpha_{n,m}(x_1, \ldots , x_n, a_1, \ldots , a_n)(f) = a_1 f(x_1) + \cdots + a_n f(x_n).$$

Since $\{X_\alpha : \alpha \in \kappa\}$ is closed under finite unions, we have

$$\{\mathbf{0}\} \cup \bigcup\{im(s^\alpha_{n,m}) : \alpha \in \kappa, n, m \in \mathbb{N}\} = L_p(X).$$

The function $s^\alpha_{n,m}$ is closed as a continuous function from a compact space. Define

$$S_n = \{\phi \in L_p(X) : |supp(\phi)| < n\}$$

and

$$Y_{\alpha,n} = \{x \in X^n_\alpha : x_k = x_l \text{ for some } k \neq l\},$$

then

$$Y_{\alpha,n} \times ([-m, -1/m] \cup [1/m, m])^n = (s^\alpha_{n,m})^{-1}(S_n).$$

Clearly, the space $Y_{\alpha,n}$ is a closed subspace of X^n_α, and by Lemma 3.2.4, the space S_n is closed in $L_p(X)$. Put

$$Z_{\alpha,n,m} = X^n_\alpha \times ([-m, -1/m] \cup [1/m, m])^n/(Y_{\alpha,n} \times ([-m, -1/m] \cup [1/m, m])^n).$$

By Lemma 3.2.7, we have $dim\,(Z_{\alpha,n,m}) \leq n \cdot dim\, X_\alpha + n$, moreover $Z_{\alpha,n,m}$ is compact and Hausdorff. Let $p^\alpha_{n,m} : Z_{\alpha,n,m} \to L_p(X)/S_n$ be given by the formula

$$p^\alpha_{n,m}([x]) = \begin{cases} [s^\alpha_{n,m}(x)] & \text{if } x \notin Y_{\alpha,n} \times ([-m, -1/m] \cup [1/m, m])^n \\ [\mathbf{0}] & \text{if } x \in Y_{\alpha,n} \times ([-m, -1/m] \cup [1/m, m])^n \end{cases}.$$

It is a continuous function from a compact, Hausdorff space into a Hausdorff space, so it is closed. For every $\phi \in L_p(X)$, we have $|(p^\alpha_{n,m})^{-1}([\phi])| \leq n!$, since $|(p^\alpha_{n,m})^{-1}([\mathbf{0}])| = 1$ and $|(p^\alpha_{n,m})^{-1}([\phi])| = n!$ for $\phi \notin S_n$. By the theorem on dimension rising mappings, we have

$$dim\, im(p^\alpha_{n,m}) \leq dim\, Z_{\alpha,n,m} + n! - 1 \leq n \cdot dim\, X_\alpha + n + n! - 1,$$

that is

$$dim\, im(s^\alpha_{n,m})/(im(s^\alpha_{n,m}) \cap S_n) \leq n \cdot dim\, X_\alpha + n + n! - 1.$$

We will prove by induction on n that, for every $n, m \in \mathbb{N}$ and $\alpha \in \kappa$, we have

$$dim\, im(s^\alpha_{n,m}) \leq n \cdot dim\, X_\alpha + n + n! - 1.$$

Firstly, as $s^\alpha_{1,m}$ is a homeomorphic embedding for every $m \in \mathbb{N}$ and $\alpha \in \kappa$, and $S_1 = \{\mathbf{0}\}$, we get that

$$dim\, im(s^\alpha_{1,m})/(im(s^\alpha_{1,m}) \cap S_1) = dim\, im(s^\alpha_{1,m}) = dim\,(X_\alpha \times [-m, m]) \leq dim\, X_\alpha + 1$$

for every $m \in \mathbb{N}$ and $\alpha \in \kappa$. This shows the thesis for $n = 1$. Assume that for every $k < n, m \in \mathbb{N}$ and $\alpha \in \kappa$

$$dim\, im(s^\alpha_{k,m}) \leq k \cdot dim\, X_\alpha + k + k! - 1 = D(\alpha, k).$$

By the inductive assumption, the space $\bigcup_{m\in\mathbb{N}}\bigcup_{k\leq n-1} im(s^{\alpha}_{k,m})$ is a union of countably many compact spaces of dimension at most $D(\alpha, n-1)$, so by Lemma 3.2.6,

$$dim \bigcup\{im(s^{\alpha}_{k,m}) : m \in \mathbb{N}, k \leq n-1\} \leq D(\alpha, n-1).$$

Obviously,

$$im(s^{\alpha}_{n,m}) = (im(s^{\alpha}_{n,m}) \cap S_n) \cup (im(s^{\alpha}_{n,m}) \setminus S_n).$$

Firstly, notice that $im(s^{\alpha}_{n,m}) \cap S_n$is a closed subset of $\bigcup_{m\in\mathbb{N}}\bigcup_{k\leq n-1} im(s^{\alpha}_{k,m})$, and therefore $im(s^{\alpha}_{n,m}) \cap S_n$ is a normal space of dimension at most $D(\alpha, n-1)$.

Secondly, let $F \subseteq im(s^{\alpha}_{n,m}) \setminus S_n$ be closed in $im(s^{\alpha}_{n,m})$, and let

$$\pi^{\alpha}_{n,m} : im(s^{\alpha}_{n,m}) \to im(s^{\alpha}_{n,m})/(im(s^{\alpha}_{n,m}) \cap S_n)$$

be the quotient map. The set $\pi^{\alpha}_{n,m}(F) \sim F$ is closed in $im(s^{\alpha}_{n,m})/(im(s^{\alpha}_{n,m}) \cap S_n)$, and therefore

$$dim\, F = dim\, \pi^{\alpha}_{n,m}(F) \leq dim\, im(s^{\alpha}_{n,m})/(im(s^{\alpha}_{n,m}) \cap S_n) \leq n \cdot dim\, X_{\alpha} + n + n! - 1 = D(\alpha, n)$$

By Lemma 3.2.6, we have

$$dim\, im(s^{\alpha}_{n,m}) \leq max(D(\alpha, n-1), D(\alpha, n)) = D(\alpha, n).$$

As

$$L_p(X) = \{\mathbf{0}\} \cup \bigcup\{im(s^{\alpha}_{n,m}) : \alpha \in \kappa, n, m \in \mathbb{N}\},$$

the space $L_p(X)$ is strongly κ-dimensional. By [S, Corollary 21/Chapter 26], there exists a continuous, linear injection $T^* : L_p(Y) \to L_p(X)$. By Lemma 3.2.1, the space Y is homeomorphic to a closed subspace Y' of $L_p(Y)$. Let $Y' = \bigcup_{\alpha\in\kappa} Y_{\alpha}$ where Y_{α}'s are compact spaces. For $\alpha \in \kappa$, the restriction $T^* \restriction_{Y_{\alpha}}$ is a homeomorphic embedding, so Y_{α} is strongly κ-dimensional. Since $Y \sim Y' = \bigcup_{\alpha\in\kappa} Y_{\alpha}$, the space Y is strongly κ-dimensional as well. □

In the theorem below, to have a bound on dimension of Y, the assumption that support is bounded is necessary because by a theorem of Levin, for every metrizable, finite dimensional, compact space K, the space $C_p(K)$ is a continuous, linear, open image of $C_p([0, 1])$. For the standard definition of $supp_T(y)$ see [M]. Equivalently, $supp_T(y) = supp(T^*(e(y)))$ where $e(y) : C_p(Y) \to \mathbb{R}$ is the evaluation at point $y \in Y$.

Theorem 3.2.12 ([Za, Theorem 4.12]). *Let X and Y be σ-compact spaces such that X is finite dimensional. If there exists a continuous, linear transformation $T : C_p(X) \to C_p(Y)$ such that $T(C_p(X))$ is dense in $C_p(Y)$, and $|supp_T(y)| \leq p$ for some $p \in \mathbb{N}$ and every $y \in Y$, then*

$$dim(Y) \leq p \cdot dim\, X + p + p! - 1.$$

Proof. Using definitions from the proof of Theorem 3.2.11 for $\kappa = \omega$, we have

$$X = \bigcup_{k \in \mathbb{N}} X_k$$

where X_k's are compact, and

$$L_p(X) = \bigcup \{im(s^k_{n,m}) : k, n, m \in \mathbb{N}\} \cup \{\mathbf{0}\}.$$

Let $Y' \subseteq L_p(Y)$, $Y' \sim Y$ be the subspace of evaluations (Lemma 3.2.1). Since $|supp_T(y)| \le p$ for every $y \in Y$, there exists a continuous injection

$$T^* \restriction_{Y'} : Y' \to \{\phi \in L_p(X) : |supp(\phi)| \le p\} = \bigcup \{im(s^k_{n,m}) : k, m \in \mathbb{N}, n \le p\} \cup \{\mathbf{0}\}.$$

Let $Y' = \bigcup_{l \in \mathbb{N}} Y_l$ for some compact spaces Y_l. For every $l \in \mathbb{N}$, the restriction $T^* \restriction_{Y_l}$ is a homeomorphic embedding. We have

$$dim\, im(s^k_{n,m}) \le n \cdot dim\, X + n + n! - 1$$

for every $k, n, m \in \mathbb{N}$, so

$$dim\, Y_l \le dim \bigcup_{k,m \in \mathbb{N}, n \le p} im(s^k_{n,m}) \cup \{\mathbf{0}\} \le p \cdot dim\, X + p + p! - 1$$

for every $l \in \mathbb{N}$ by Lemma 3.2.6. Again by Lemma 3.2.6, we can conclude that

$$dim\, Y \le p \cdot dim\, X + p + p! - 1.$$

□

3.3. Function spaces on NY compact spaces

The main result of this section is the invariance of the class of NY compact spaces K under linear homeomorphisms of function spaces $C_p(K)$. In fact we prove several more general theorems (Theorems 3.3.9, 3.3.11, 3.3.12). Our reasoning relies on the condition (iv) in Theorem 2.3.1.

Lemma 3.3.1 ([Za, Lemma 5.2]). *Let K be a compact space such that $K = \bigcup_{n \in \mathbb{N}} K_n$ where $\{K_n : n \in \mathbb{N}\}$ is a sequence of compact spaces with the property that every nonempty subspace A of K_n contains a nonempty, relatively open subspace U of countable weight. Then the space K has the same property.*

Proof. Assume there exists $A \subseteq K$ such that every relatively open $U \subseteq A$ has uncountable weight, then $\overline{A}$ has the same property. Indeed, assume there is a nonempty, relatively open $U \subseteq \overline{A}$ of countable weight, then $U \cap A \neq \emptyset$, and therefore $U \cap A \subseteq A$ is a relatively open, nonempty subset of A with $w(U \cap A) \le \omega$, contradiction.

Without loss of generality, we can assume that A is closed and therefore compact. Then it has the same property as K, so we can assume that $A = K$. As $K = \bigcup_{n\in\mathbb{N}} K_n$, by the Baire category theorem, there is K_n with nonempty interior. By the assumption, there exists a nonempty, open $V \subseteq int_K(K_n)$ with $w(V) \le \omega$. Then V is a nonempty, open subset of K of countable weight. $\square$

Lemma 3.3.2 ([Za, Lemma 5.3])**.** *Let K be an NY compact space, and let L be a compact space. If there exists a continuous, linear transformation $T : C_p(K) \to C_p(L)$ such that $T(C_p(K))$ is dense in $C_p(L)$, then L is a countable union of NY compact spaces.*

Proof. For $n, m \in \mathbb{N}$, let

$$s_{n,m} : K^n \times [-m, m]^n \to L_p(K)$$

be given by the formula

$$s_{n,m}((x_1, \dots, x_n), (t_1, \dots, t_n)) = \sum_{k=1}^{n} t_k e(x_k)$$

where $e(x)(f) = f(x)$ for every $f \in C_p(K)$ and $x \in K$. Functions $s_{n,m}$ are continuous. By Lemma 3.2.2, we have

$$\bigcup\{im(s_{n,m}) : n, m \in \mathbb{N}\} = L_p(K).$$

A finite product of NY compact spaces is again NY compact, so $K^n \times [-m, m]^n$ is NY compact for $n, m \in \mathbb{N}$. As NY compact spaces are preserved under continuous images, the space $L_p(K)$ is a union of countably many NY compact spaces. By [S, Corollary 21/Chapter 26], there exists a continuous, linear injection $T^* : L_p(L) \to L_p(K)$, and by Lemma 3.2.1, the space L is homeomorphic to a subspace L' of $L_p(L)$. Since L is a compact space, the transformation $T^* \restriction_{L'}$ is a homeomorphic embedding. We obtain that L is a union of countably many NY compact spaces. $\square$

Using Lemmas 3.3.1 and 3.3.2 we obtain

Proposition 3.3.3 ([Za, Proposition 5.4])**.** *Let K be an NY compact space, and let L be a compact space. Assume there exists a continuous, linear transformation $T : C_p(K) \to C_p(L)$ such that $T(C_p(K))$ is dense in $C_p(L)$. Then every nonempty subspace A of L contains a nonempty, relatively open subspace of countable weight.*

The above proposition shows that the space L follows the second part of condition *iii*) from Theorem 2.3.1. Now it suffices to show that L is hereditarily metacompact.

The following lemma is [En1, Lemma 5.3.5].

Lemma 3.3.4. *For every open cover $\{U_s : s \in S\}$ of a metacompact space, there exists a point-finite open cover $\{V_s : s \in S\}$ such that $V_s \subseteq U_s$ for every $s \in S$.*

The following lemma is a generalisation of [En1, Lemma 5.3.6] for closed transformations f, and is proved in an analogous way.

Lemma 3.3.5 ([Za, Lemma 5.6]). *Let $f : X \to Y$ be a continuous map from a metacompact space X onto space Y such that the image of every closed subset F of X is an F_σ set in Y. Then for every open cover $\mathcal{U}$ of Y which is a union of countably many point-finite families, there exists a point-finite open refinement $\mathcal{V}$.*

Proof. Let $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ where $\mathcal{U}_n$'s are point-finite families. By Lemma 3.3.4, there exists a point-finite open cover $\{G_n : n \in \mathbb{N}\}$ of the space X such that $G_n \subseteq f^{-1}(U_n)$ where $U_n = \bigcup \mathcal{U}_n$. The set $E_n = X \setminus \bigcup_{k \geq n} G_k$ is closed for $n \in \mathbb{N}$. One can easily see that $E_1 \subseteq E_2 \subseteq \ldots$. Since the family $\{G_n : n \in \mathbb{N}\}$ is point finite, the family $\{E_n : n \in \mathbb{N}\}$ is a cover of X. Moreover, we have

$$f(E_n) \subseteq f\Big(\bigcup_{k<n} G_k\Big) = \bigcup_{k<n} f(G_k) \subseteq \bigcup_{k<n} U_k, \text{ for } n \in \mathbb{N}.$$

For $n \in \mathbb{N}$, let $f(E_n) = \bigcup_{k \in \mathbb{N}} E_n^k$ where E_n^k are closed. Without loss of generality we can assume that $E_n^k \subseteq E_n^{k+1}$ and $E_n^k \subseteq E_{n+1}^k$ for $k, n \in \mathbb{N}$. For $n \in \mathbb{N}$, let

$$\mathcal{V}_n = \{U \setminus E_n^n : U \in \mathcal{U}_n\}.$$

It suffices to show that $\mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ is a point-finite cover of Y. We shall prove that $\mathcal{V}$ is a cover of Y.

Pick $y \in Y$, and let $n(y)$ be the smallest number such that $y \in U_{n(y)}$. Since we have $f(E_{n(y)}) \subseteq \bigcup_{n<n(y)} U_n$, there exists $U \in \mathcal{U}_{n(y)}$ such that $y \in U \setminus f(E_{n(y)})$, then $y \in U \setminus E_{n(y)}^k$ for every $k \in \mathbb{N}$, in particular $y \in U \setminus E_{n(y)}^{n(y)} \in \mathcal{V}_{n(y)}$.

It remains to show that $\mathcal{V}$ is point-finite. Choose any $y \in Y$. As $Y = \bigcup_{n \in \mathbb{N}} f(E_n)$, the point y belongs to a set $f(E_n)$ for some $n \in \mathbb{N}$. Consequently, $y \in E_n^k$ for some $k \in \mathbb{N}$, and if $m = max(n, k)$, then $y \in E_m^m$. By the definition of $\mathcal{V}_n$, we have $y \notin \bigcup_{n>m} \mathcal{V}_n$. Since the families $\mathcal{U}_n$ are point-finite, the point y belongs to only finitely many elements of $\mathcal{V}$. □

Lemma 3.3.6 ([Za, Lemma 5.7]). *Every σ-metacompact space which is a union of its countably many closed metacompact subspaces is metacompact.*

Proof. Assume X is σ-metacompact, and $X = \bigcup_{n \in \mathbb{N}} X_n$ where $X_n \subseteq X$ are closed and metacompact. Let $\mathcal{U}$ be an open cover of X. By σ-metacompactness, there exists a σ-point finite, open cover $\mathcal{V} \prec \mathcal{U}$.

Consider $Y = \dot{\bigcup}_{n \in \mathbb{N}} X_n$ endowed with the topology of disjoint union, and let $f : Y \to X$ be such that $f \restriction_{X_n}$ is the identity embedding of X_n into X. Clearly, the space Y is metacompact, and the transformation f is a continuous surjection with the property that image of every closed set in Y is an F_σ set in X. By Lemma 3.3.5, there exists a point-finite open cover $\mathcal{W} \prec \mathcal{V} \prec \mathcal{U}$. The space X is metacompact. □

The following theorem is [A, Theorem 7.1].

Theorem 3.3.7. *For a compact space K, the following conditions are equivalent:*

(i) K is Eberlein compact;

(ii) $C_p(K)$ has a σ-compact dense subspace.

Lemma 3.3.8 ([Za, Lemma 5.10])**.** *An Eberlein compact space which is a union of its countably many closed, hereditarily metacompact subspaces is hereditarily metacompact.*

Proof. Let K be an Eberelein compact space, and let $K = \bigcup_{n\in\mathbb{N}} K_n$ where K_n's are hereditarily metacompact, compact spaces. Let $A \subseteq K$. By Theorem 2.4.4, the space A is σ-metacompact. The spaces $A \cap K_n$ are closed in A and metacompact. By Lemma 3.3.6, the space A is metacompact. This shows that K is hereditarily metacompact. □

Theorem 3.3.9 ([Za, Theorem 5.11])**.** *Let K be an NY compact space, and let L be a compact space. If there exists a continuous, linear transformation $T : C_p(K) \to C_p(L)$ such that $T(C_p(K))$ is dense in $C_p(L)$, then L is NY compact as well.*

Proof. By Theorem 2.3.1 and Proposition 3.3.3, it suffices to show that L is hereditarily metacompact. By Lemma 3.3.2, we have $L = \bigcup_n L_n$ where L_n is *NY* compact for every $n \in \mathbb{N}$. By Theorem 3.3.7, the space $C_p(K)$ has a σ-compact dense subspace D, then $T(D)$ is a σ-compact dense subspace of $C_p(L)$, and again by Theorem 3.3.7, the space L is Eberlein compact. By Lemma 3.3.8, the space L is hereditarily metacompact. □

The following Lemma is a direct consequence of known characterizations of Eberlein compact spaces, WCG spaces and stability of Eberlein compact spaces under taking continuous images, see [F, Theorem 1.2.4].

Lemma 3.3.10 ([Za, Lemma 5.12])**.** *Let K and L be compact spaces. Assume there exists a closed linear homeomorphic embedding $T : C_p(K) \to C_p(L)$. If L is Eberlein compact, then K is Eberlein compact as well.*

Proof. Let $C(K), C(L)$ denote the Banach spaces of real function spaces on K and L respectively. Since the topology on $C(L)$ is stronger than the topology on $C_p(L)$, the image $T(C(K))$ is closed in $C(L)$ and therefore is a Banach space. By the closed graph theorem, transformation $T : C(K) \to C(L)$ is also continuous. By the open mapping theorem, it is a homeomorphic embedding. Consequently, the dual transformation $T^* : C(L)^* \to C(K)^*$ is a continuous, linear surjection. By the open mapping theorem, the function T^* is open, so there exists $c > 0$ such that $c \cdot B_{C(K)^*} \subseteq T^*(B_{C(L)^*})$.

Since L is Eberlein compact, the space $(B_{C(L)^*}, \omega^*)$ is also Eberlein compact. Obviously $T^*(B_{C(L)^*})$ is an Eberlein compactum as a continuous image of an Eberlein compact space, then

$B_{C(K)^*}$ is also Eberlein compact. Finally, the space K is homeomorphic to a subspace of $B_{C(K)^*}$, so it is Eberlein compact as well. $\square$

Theorem 3.3.11 ([Za, Theorem 5.13]). *Let K be an Eberlein compact space, and let L be an NY compact space. If there exists a linear homeomorphic embedding $T : C_p(K) \to C_p(L)$, then K is NY compact as well.*

Proof. From the proof of Lemma 3.3.2, we know that if L is NY compact, then $L_p(L)$ is a union of countably many NY compact spaces. By [Sc, Theorem 7.3] and [S, Corollary 21/Chapter 26], there exists a continuous surjection $T^* : L_p(L) \to L_p(K)$, so $L_p(K)$ is a union of countably many NY compact spaces as well. By Lemma 3.2.1, the space K is a countable union of NY compact spaces and therefore closed hereditarily metacompact spaces. By Lemma 3.3.8, the space K is hereditarily metacompact. By Lemma 3.3.1, every subspace of K has an open set of countable weight. The space K is NY compact by Theorem 2.3.1. $\square$

By Lemma 3.3.10 and Theorem 3.3.11, we obtain

Theorem 3.3.12 ([Za, Theorem 5.14]). *Let K and L be compact spaces. Assume there exists a closed, linear, homeomorphic embedding $T : C_p(K) \to C_p(L)$. If L is NY compact, then K is NY compact as well.*

The following example shows that assumptions in Theorems 3.3.11, 3.3.12 are necessary.

Example 3.3.13 ([Za, Example 5.15]). *There exist a compact space K and a ω-Corson compact space L such that there is a linear, homeomorphic embedding $T : C_p(K) \to C_p(L)$, but K is not Eberlein compact.*

Proof. Let $Z = 2^\omega \cup 2^{<\omega}$, where points from $2^{<\omega}$ are isolated in Z, and basic neighbourhoods of a point $x \in 2^\omega$ are of the form

$$\{x\} \cup \{x \restriction_n : n \geq k\}, \text{ for some } k \in \omega.$$

Let $K = 2^\omega \cup 2^{<\omega} \cup \{\infty\}$ be the one-point compactification of Z. Notice that $2^{<\omega}$ is a countable, dense subset of K. Clearly, the space K is not metrizable as it has an uncountable discrete set 2^ω. This shows that K is not Eberlein compact because every separable Eberlein compactum is metrizable.

Let $L = (2^\omega \cup \{\infty\}) \dot\cup (\omega + 1)$, where $2^\omega \cup \{\infty\}$ is the one-point compactification of 2^ω endowed with the discrete topology and $\omega + 1$ is endowed with the order topology. Clearly, we have $L \sim A(\mathfrak{c}) \,\dot\cup\, A(\omega)$, and therefore L is ω-Corson compact, since it is a disjoint union of two ω-Corson compact spaces.

Let $\sigma : 2^{<\omega} \to \omega$ be a bijection, and let $T : C_p(K) \to C_p(L)$ be given by the formulas:

$$T(f) \restriction_{2^\omega \cup \{\infty\}} = f \restriction_{2^\omega \cup \{\infty\}},$$

and

$$T(f)(n) = \frac{f(\sigma^{-1}(n))}{n+1}$$

for $n \in \omega$, and $T(f)(\omega) = 0$. Since $f \in C_p(K)$ where K is compact, the function f is bounded, and therefore $T(f)(n) \to 0$ when $n \to \infty$, so $T(f) \in C_p(L)$ for every $f \in C_p(K)$. It is easy to see that T is continuous, linear, injective and open onto its image, and hence the transformation T is a homeomorphic embedding.

□

3.4. A characterisation of κ-Corson compact spaces in terms of topological properties of function spaces $C_p(K)$

In this section we present a generalisation of results due to Pol [P] and Bell and Marciszewski [BM]. Pol's theorem characterises Corson compacta K in terms of function spaces $C_p(K)$. Bell and Marciszewski extended this result to κ^+-Corson compact spaces. We also conclude, using results from [MPZ] and Sections 3.2, 3.3 that the class of ω-Corson compact spaces K is invariant under linear homeomorphisms of function spaces $C_p(K)$.

Given an infinite cardinal numbers κ and λ, let $L_\kappa(\lambda)$ denote the set $\lambda \cup \{\infty\}$ topologized as follows: all points $\alpha \in \lambda$ are isolated, and open neighbourhoods of ∞ are of the form $\{\infty\} \cup A$ where $A \subseteq \lambda$ and $|\lambda \setminus A| < \kappa$. For an infinite cardinal number κ, let $\mathcal{L}_\kappa$ denote the class of all spaces which are continuous images of closed subsets of the countable product $L_\kappa(\lambda)^\omega$. Pol's result states that a compact space K is Corson compact if and only if $C_p(K) \in \mathcal{L}_{\omega_1}$. Bell and Marciszewski extended this theorem to κ^+-Corson compact spaces.

We generalise the results from [BM] and [P] to κ-Corson compact spaces for any regular, uncountable cardinal number κ. Our reasoning is a modification of the proof from [BM] which in turn follows the arguments from [P]. Let κ be a regular, uncountable cardinal number, and let $\lambda \in Card$, $\lambda > \kappa$. For $A \subseteq L_\kappa(\lambda)$ such that $\infty \in A$, define the retraction $r_A : L_\kappa(\lambda) \to A$

$$r_A(x) = \begin{cases} x & \text{if } x \in A \\ \infty & \text{if } x \notin A \end{cases}.$$

Let $R_A = \prod_{n\in\mathbb{N}} r_A : L_\kappa(\lambda)^\omega \to A^\omega$. For $n \in \mathbb{N}$, let $P_n : L_\kappa(\lambda)^\omega \to L_\kappa(\lambda)^\omega$ be given by the formula $P_n(x_1, x_2, \dots) = (x_1, \dots x_n, \infty, \infty, \dots)$.

Lemma 3.4.1 ([Za, Lemma 6.1]). *Let $F \subseteq L_\kappa(\lambda)^\omega$ be a closed subset, and let A be a subset of $L_\kappa(\lambda)$, such that $\infty \in A$ and $|A| \geq \kappa$. Then there exists $B \subseteq L_\kappa(\lambda)$ such that $A \subseteq B$, $|B| = |A|$ and $R_B(F) \subseteq F$.*

Proof. For every $z \in L_\kappa(\lambda)^\omega \setminus F$, fix a basic open neighbourhood

$$V^z = V_1^z \times V_2^z \times \dots V_k^z \times L_\kappa(\lambda) \times \dots$$

of z, disjoint from F such that $k = k(z)$ is minimal for all such neighbourhoods. Define

$$C_z = \bigcup \{L_\kappa(\lambda) \setminus V_i^z : \infty \in V_i^z, i \leq k\},$$

and observe that $|C_z| < \kappa$. We construct the set B by induction. Start with $B_1 = A$, and put

$$B_{n+1} = \bigcup \{C_z : z \in P_n(B_n^\omega) \setminus F\} \cup B_n,\ n \in \mathbb{N}.$$

Finally, define $B = \bigcup_{n\in\mathbb{N}} B_n$. By induction, if $|B_n| = |A|$, then $|P_n(B_n^\omega)| = |A|$, so $|C_z| < \kappa$ implies $|B_{n+1}| = |A|$. Then, we have $|B| = |A|$. We will show that $R_B(F) \subseteq F$.

Let $x = (x_i) \in F$, and suppose that $y = (y_i) = R_B(x) \notin F$. Notice that $y_i \in B$ for every $i \in \mathbb{N}$. Take $z = (z_i) = P_k(y)$, where $k = k(y)$. Then $z_i = y_i$ for $i \leq k$, hence $z \notin F$ and $k(z) = k$. Choose $n \geq k$ such that $z_i \in B_n$, for $i \leq k$. Observe that $C_z \subseteq B$. For each $i \leq k$, we have $z_i = y_i = r_B(x_i)$, and we can consider two possibilities: Firstly, if $z_i \neq \infty$, then $x_i = r_B(x_i)$ by the definition of r_B, so $x_i = z_i \in V_i^z$. Secondly, if $z_i = \infty$, then $x_i \notin C_z$. Indeed, if $x_i \in C_z \subseteq B$, then $z_i = r_B(x_i) = x_i \neq \infty$, since $\infty \notin C_z$. Therefore again $x_i \in V_i^z$. This shows that x belongs to V^z which is disjoint from F, contradiction. □

Lemma 3.4.2 ([Za, Lemma 6.2]). *Let $F \subseteq L_\kappa(\lambda)^\omega$ be a closed subset, and let A be a subset of $L_\kappa(\lambda)$ such that $\infty \in A$ and $|A| < \kappa$. Then there exists $B \subseteq L_\kappa(\lambda)$ such that $A \subseteq B$, $|B| < \kappa$ and $R_B(F) \subseteq F$.*

Proof. Let us define the set B the same way as in the proof of Lemma 3.4.1. We will show that $|B| < \kappa$. Firstly, we have $|B_2| < \kappa$ as it is a union of less than κ sets of cardinality less than κ. It is easy to see that, by induction, $|B_n| < \kappa$ for every $n \geq 2$. Finally, we have $|B| = |\bigcup_{n\in\mathbb{N}} B_n| < \kappa$ because κ is regular. The inclusion $R_B(F) \subseteq F$ can be shown in the same way as in Lemma 3.4.1. □

Lemma 3.4.3 ([Za, Lemma 6.3]). *Let $F \subseteq L_\kappa(\lambda)^\omega$ be a closed subset, where $\lambda \geq \kappa$. There exists a family $\{A_\alpha : \alpha \leq \lambda\}$ of subsets of $L_\kappa(\lambda)$ such that:*

(1) $\infty \in A_0$,

(2) $|A_\alpha| < \kappa$, *for* $\alpha < \kappa$,

(3) $|A_\alpha| = |\alpha|$, *for* $\kappa \leq \alpha \leq \lambda$,

(4) $A_\alpha \subseteq A_\beta$, *for* $\alpha \leq \beta \leq \lambda$,

(5) $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$, *for* $\alpha \in \mathrm{Lim} \cap (\lambda + 1)$,

(6) $A_\lambda = L_\kappa(\lambda)$,

(7) $R_{A_\alpha}(F) \subseteq F$ *for* $\alpha \leq \lambda$.

Proof. By Lemma 3.4.2 for F and $A = \{\infty\}$, there exists $A_0 \subseteq L_\kappa(\lambda)$ such that $\infty \in A_0$, $|A_0| < \kappa$ and $R_{A_0}(F) \subseteq F$. We will construct sets A_α for $\alpha \geq 1$ by induction. Assume we have constructed A_α for $\alpha \leq \beta < \kappa$. By Lemma 3.4.2, there exists $B \subseteq L_\kappa(\lambda)$ such that

$$A_\beta \cup (\beta + 1) \subseteq B, R_B(F) \subseteq F \text{ and } |B| < \kappa.$$

Let $A_{\beta+1} = B$. For $\beta \in Lim \cap (\kappa + 1)$, put $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$. As $\beta + 1 \subseteq A_{\beta+1}$ for $\beta < \kappa$, we have $|A_\kappa| = \kappa$.

Assume that we constructed A_α for $\alpha \leq \beta < \lambda$ where $\beta \geq \kappa$. By Lemma 3.4.1, there exists $B \subseteq L_\kappa(\lambda)$, such that $R_B(F) \subseteq F$, $|B| = |A_\beta|$ and $A_\beta \cup (\beta + 1) \subseteq B$. For a limit ordinal $\kappa < \beta \leq \lambda$, let $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$.

It remains, to check that if $R_{A_\alpha}(F) \subseteq F$ for $\alpha < \beta$ where $\beta \in Lim$, then $R_{A_\beta}(F) \subseteq F$. Assume that there exists $x \in F$ such that $R_{A_\beta}(x) \notin F$. Let $V = \prod_{i\in\mathbb{N}} V_i$ a basic neighbourhood of $R_{A_\beta}(x)$ such that $V_i = L_\kappa(\lambda)$ for $i > n$ and $V \cap F = \emptyset$. For every $k \in \{1, 2, \ldots, n\}$, there exists $\alpha_k \leq \beta$ such that for every $\alpha_k < \alpha < \beta$, we have $r_{A_{\alpha_k}}(x_k) = r_{A_\alpha}(x_k)$. Consequently, the point $R_{A_\alpha}(x)$ does not belong to F for α greater than $max(\alpha_1, \ldots, \alpha_n)$, contradiction. □

Let α be a limit ordinal number. Observe that for every $x \in A_\alpha$ there exists a minimal $\alpha_0 < \alpha$ for which $x \in A_{\alpha_0}$, so for $\alpha_0 \leq \beta \leq \alpha$, we have $r_{\alpha_0}(x) = r_\beta(x)$. This shows that, for a limit ordinal α and $y \in L_\kappa(\lambda)^\omega$,

$$R_{A_\alpha}(y) = \lim_{\beta \longrightarrow \alpha} R_{A_\beta}(y). \tag{3.1}$$

Let κ be an infinite cardinal number. Recall that a space is called κ-Lindelöf if every open cover has a subcover of cardinality less than κ. The following proposition follows from [No, Corollary 4.2].

Proposition 3.4.4 ([Za, Proposition 6.4])**.** *Let κ be an infinite regular cardinal number. Every space from the class $\mathcal{L}_\kappa$ is κ-Lindelöf.*

Proposition 3.4.5 ([Za, Proposition 6.5])**.** *For a closed set $F \subseteq L_\kappa(\lambda)^\omega$, there exists a continuous, linear injection $T : C_p(F) \to \Sigma_\kappa(\mathbb{R}^\lambda)$.*

Proof. If $\lambda < \kappa$, then $L_\kappa(\lambda)$ is discrete, and so F is metrizable, and $w(F) \leq \lambda$. There exists a dense $D \subseteq F$ such that $|D| = \lambda$. Let $T : C_p(F) \to \mathbb{R}^D$ be the restriction map $T(f) = f \restriction_D$. Clearly T is linear, injective and continuous, and $\mathbb{R}^D$ embeds in $\Sigma_\kappa(\mathbb{R}^\lambda)$ as $|D| < \kappa$.

Assume $\lambda \geq \kappa$, and assume that the thesis holds for every $\mu < \lambda$. Let F be a closed subset of $L_\kappa(\lambda)^\omega$, and let the family $\{A_\alpha : \alpha \leq \lambda\}$ be as in Lemma 3.4.3. Let $R_\alpha = R_{A_\alpha} \restriction_F$ and $F_\alpha = R_\alpha(F)$. Function $G : C_p(F_\alpha) \to C_\alpha$ given by the formula $G(f) = f \circ R_\alpha$ where

$$C_\alpha = \{f \circ R_\alpha : f \in C_p(F_\alpha)\} \subseteq C_p(F)$$

is a linear homeomorphism. Since $A_\alpha \subseteq A_\beta$ for $\alpha < \beta$, we have $R_\alpha = R_\alpha \circ R_\beta$. This implies $C_\alpha \subseteq C_\beta$ for $\alpha < \beta$.

We will check that $F_\alpha = F \cap A_\alpha^\omega$. Firstly, $F_\alpha = R_\alpha(F) \subseteq A_\alpha^\omega$ as $im\,(r_{A_\alpha}) \subseteq A_\alpha$ because $\infty \in A_\alpha$. Secondly, we have $R_{A_\alpha}(F) \subseteq F$ (condition (7) from Lemma 3.4.3). Finally, if $x \in F \cap A_\alpha^\omega$, then $R_{A_\alpha}(x) = x$, so $R_{A_\alpha}(x) = x \in F$, therefore $x \in R_{A_\alpha}(F) = F_\alpha$.

Since $|A_\alpha| < \lambda$, $F_\alpha \subseteq L_\kappa(A_\alpha)^\omega$ and $C_p(F_\alpha) \sim C_\alpha$ for $\alpha < \lambda$, by inductive hypothesis, there exists a continuous linear injection $T_\alpha : C_\alpha \to \Sigma_\kappa(\mathbb{R}^{|A_\alpha|})$. For the special case $\lambda = \kappa$, define $T_\kappa : C_p(F_\kappa) \to \prod_{\alpha\in\kappa} \Sigma_\kappa(\mathbb{R}^{|A_\alpha|})$ as

$$T_\kappa(f) = (T_0(f \circ R_0), T_{\alpha+1}(f \circ R_{\alpha+1} - f \circ R_\alpha)_{\alpha\in[0,\kappa)}),$$

and $T_\lambda : C_p(F_\lambda) \to \prod_{\alpha\in\lambda} \Sigma_\kappa(\mathbb{R}^{|A_\alpha|})$ for $\lambda > \kappa$ as

$$T_\lambda(f) = (T_\kappa(f \circ R_\kappa), (T_{\alpha+1}(f \circ R_{\alpha+1} - f \circ R_\alpha))_{\alpha\in[\kappa,\lambda)}).$$

Note that we always have $F_\lambda = F$. It is clear that T_κ is continuous and linear. Let us check that it is injective. Assume that $T_\kappa(f) = \mathbf{0}$ for some nonzero $f \in C_p(F_\kappa)$. Let $\alpha = min\{\beta \leq \kappa : f \circ R_\beta \neq \mathbf{0}\}$ (such β exists since $R_\kappa = id_F$). If $\alpha = 0$, then $T_0(f \circ R_0) \neq \mathbf{0}$ because T_0 is injective, so we have a contradiction. If α is a successor ordinal, then let $\alpha = \beta + 1$. In this case

$$T_{\beta+1}(f \circ R_{\beta+1} - f \circ R_\beta) = T_{\beta+1}(f \circ R_{\beta+1}) - T_{\beta+1}(f \circ R_\beta) \neq \mathbf{0}$$

because $T_{\beta+1}(f \circ R_\beta) = \mathbf{0}$ by the definition of α, and $T_{\beta+1}(f \circ R_{\beta+1}) \neq \mathbf{0}$ by injectivity of $T_{\beta+1}$. If α is a limit ordinal number, then by Equation 3.1, we arrive at contradiction.

We can identify $\prod_{\alpha\in\kappa} \Sigma_\kappa(\mathbb{R}^{|A_\alpha|})$ with a subspace of $\mathbb{R}^S$ where $S = \dot{\bigcup}_{\alpha\in\kappa} A_\alpha$, $|S| = \kappa$. Now we will check that for every $f \in C_p(F)$, $|supp(T(f))| < \kappa$. By regularity of κ, it is enough to show that

$$T_{\alpha+1}(f \circ R_{\alpha+1} - f \circ R_\alpha) \neq \mathbf{0}$$

for less than κ ordinal numbers α because $|A_\alpha| < \kappa$ for $\alpha < \kappa$. Let $f \in C_p(F)$ and suppose to the contrary that we can find a set $T \subseteq S$ of cardinality κ such that, for every $\alpha \in T$, there exists $x^\alpha = (x_i^\alpha) \in F$ for which $f(R_{\alpha+1}(x^\alpha)) - f(R_\alpha(x^\alpha)) \neq \mathbf{0}$. Since $cf(\kappa) = \kappa > \omega$, we can assume that, for every $\alpha \in T$, we have

$$|f(R_{\alpha+1}(x^\alpha)) - f(R_\alpha(x^\alpha))| > \epsilon$$

for some $\epsilon > 0$. Using Proposition 3.4.4 for F, we can find a point $x = (x_i) \in F$ such that for every neighbourhood U of x we have $|\alpha \in T : R_\alpha(x^\alpha) \in U| = \kappa$. Let

$$V = V_1 \times \cdots \times V_k \times L_\kappa(\lambda) \times \ldots$$

be a basic neighbourhood of x in $L_\kappa(\lambda)^\omega$ such that $diam(f(V \cap F)) < \epsilon$ and $V_i = \{x_i\}$ if $x_i \neq \infty$.

Let

$$C = \bigcup \{L_\kappa(\lambda) \setminus V_i : \infty \in V_i, i \leq k\}.$$

Since $|C| < \kappa$, and we have κ many points $R_\alpha(x^\alpha)$ in V with $\alpha \in T$, we can find $\alpha \in T$ such that $R_\alpha(x^\alpha) \in V$ and the set $A_{\alpha+1} \setminus A_\alpha$ is disjoint from C. Then we also have $R_{\alpha+1}(x^\alpha) \in V$. Indeed, we can show that $R_{\alpha+1}(x^\alpha)_i \in V_i$ for all $i \leq k$.

If $\infty \notin V_i$, then $R_\alpha(x^\alpha)_i = r_{A_\alpha}(x_i^\alpha) \in V_i$ and $\infty \notin V_i$, so $r_{A_\alpha}(x_i^\alpha) = x_i = x_i^\alpha \in A_\alpha \subseteq A_{\alpha+1}$. Hence $R_{\alpha+1}(x^\alpha)_i = r_{A_{\alpha+1}}(x_i^\alpha) = x_i^\alpha = x_i \in V_i$.

If $\infty \in V_i$, then $(A_{\alpha+1} \setminus A_\alpha) \cap C = \emptyset$ implies that $A_{\alpha+1} \setminus A_\alpha \subseteq V_i$. There are two cases. Firstly, if $x_i^\alpha \in A_{\alpha+1} \setminus A_\alpha$, then $R_{\alpha+1}(x^\alpha)_i = r_{A_{\alpha+1}}(x_i^\alpha) = x_i^\alpha \in V_i$. Secondly, if $x_i^\alpha \notin A_{\alpha+1} \setminus A_\alpha$, then $R_{\alpha+1}(x^\alpha)_i = R_\alpha(x^\alpha)_i = r_{A_\alpha}(x_i^\alpha) \in V_i$.

By the definition of V, we have a contradiction with $|f(R_{\alpha+1}(x^\alpha)) - f(R_\alpha(x^\alpha))| > \epsilon$. Analogous reasoning yields that T_λ is injective and $im\, T_\lambda \subseteq \Sigma_\kappa(\mathbb{R}^\lambda)$ for $\lambda > \kappa$. □

Proposition 3.4.6 ([Za, Proposition 6.6]). *If $C_p(K) \in \mathcal{L}_\kappa$, then K is a κ-Corson compact space.*

Proof. Let $F \subseteq L_\kappa(\lambda)^\omega$ be a closed subset. Assume that there is a continuous surjection $\phi : F \to C_p(K)$. Define $\phi' : C_p(C_p(K)) \to C_p(F)$ as $\phi'(g) = g \circ \phi$. It is a well known fact, that in this situation ϕ' is a continuous injection. Let $\psi : K \to C_p(C_p(K))$ be the standard embedding given by evaluations. By Proposition 3.4.5, there exists a continuous linear injection $\theta : C_p(F) \to \Sigma_\kappa(\mathbb{R}^\Gamma)$ for some Γ. The function $\theta \circ \phi' \circ \psi : K \to \Sigma_\kappa(\mathbb{R}^\Gamma)$ is a continuous injection from a compact, Hasudorff space into a Hausdorff space, therefore it is a homeomorphic embedding. This shows that K is a κ-Corson compact space. □

The theorem below can be proved using the same arguments as in the proof of Theorem 6.1 in [BM].

Theorem 3.4.7 ([Za, Theorem 6.7]). *Let κ be an uncountable cardinal number. If space K is κ-Corson compact, then $C_p(K) \in \mathcal{L}_\kappa$.*

By Proposition 3.4.6 and Theorem 3.4.7, we get

Theorem 3.4.8 ([Za, Theorem 6.8]). *For a compact space K and a regular $\kappa > \omega$, the space K is κ-Corson compact if and only if $C_p(K) \in \mathcal{L}_\kappa$.*

Theorem 3.4.9 ([Za, Theorem 6.9]). *Let κ be a regular, uncountable cardinal number. Assume that K is a κ-Corson compact space and L is compact. If there exists a continuous surjective transformation $T : C_p(K) \to C_p(L)$, then L is κ-Corson compact as well.*

Proof. By Theorem 3.4.8, we have $C_p(K) \in \mathcal{L}_\kappa$. There exists $\lambda \in Card$, a closed subset F of $L_\kappa(\lambda)^\omega$ and a continuous surjection $f : F \to C_p(K)$. The composition $T \circ f : F \to C_p(L)$

is a continuous surjection as well, and therefore by Theorem 3.4.8, the space L is κ-Corson compact. ☐

The theorem below was originally proved in [BKT, Corollary 2.17]

Theorem 3.4.10 ([Za, Theorem 6.10])**.** *For a regular, uncountable cardinal number κ the class of κ-Corson compact spaces is invariant under continuous images.*

Proof. Let K be a κ-Corson compact space for some regular, uncountable κ. By Theorem 3.4.8, we have $C_p(K) \in \mathcal{L}_\kappa$. If L is a continuous image of K, then there exists a closed homeomorphic embedding of $C_p(L)$ into $C_p(K)$, consequently $C_p(L) \in \mathcal{L}_\kappa$. By Theorem 3.4.8, the space L is κ-Corson compact. ☐

Corollary 4.2 from Noble [No] implies that, for every uncountable cardinal λ, the space $L_\kappa(\lambda)^\omega$ is κ-Lindelöf, for every uncountable regular cardinal number κ, and it is κ^+-Lindelöf, for every singular κ. Hence we obtain

Proposition 3.4.11 ([MPZ, Proposition 11.2])**.** *Let κ be an uncountable cardinal number and X be an element of $\mathcal{L}_\kappa$. Then*

(A) if κ is regular, then X is κ-Lindelöf;

(B) if κ is singular, then X is κ^+-Lindelöf.

Corollary 3.4.12 ([MPZ, Corollary 11.4])**.** *Let κ be an uncountable cardinal and K be a κ-Corson compact space. Then*

(A) if κ is regular, then $C_p(K)$ is κ-Lindelöf;

(B) if κ is singular, then $C_p(K)$ is κ^+-Lindelöf.

The following fact belongs to the folklore

Proposition 3.4.13 ([MPZ, Proposition 11.5])**.** *Let κ be an infinite cardinal and $[0,\kappa]$ be the ordinal space equipped with the order topology. Then the function space $C_p([0,\kappa])$ contains a closed discrete subspace of cardinality κ.*

Proof. One can easily verify that the subspace

$$\{f \in C_p([0,\kappa]) \cap 2^{[0,\kappa]} : f(\kappa) = 0 \text{ and } f \text{ is nonincreasing}\} = \{\chi_{[0,\alpha]} : \alpha < \kappa\} \cup \{\chi_\emptyset\}$$

has the required properties. ☐

Since the space $[0,\kappa]$ is κ-Corson for singular κ, and it is κ^+-Corson for regular κ, the above proposition shows that the results of Corollary 3.4.12 cannot be improved. In particular, this gives a negative answer to a part of Question 5.5 from [BKT].

By Theorems 3.2.11, 3.3.9 and Corollary 2.4.1, we obtain

Theorem 3.4.14 ([Za, Theorem 6.12]). *Let K and L be compact spaces. Assume there exists a continuous, linear transformation $T : C_p(K) \to C_p(L)$ such that $T(C_p(K))$ is dense in $C_p(L)$. If K is ω-Corson compact, then L is ω-Corson compact as well.*

3.5. On Banach spaces $C(K)$

Let us say that a class of compact spaces $\mathcal{C}$ is B-stable if for any $L \in \mathcal{C}$ and any compact space K, the fact that the Banach spaces $C(K)$ and $C(L)$ are isomorphic implies $K \in \mathcal{C}$. Clearly, the class of metrizable compacta is B-stable, since K is metrizable if and only if $C(K)$ is separable. The class of Eberlein compacta is B-stable, as L is Eberlein compact if and only if $C(L)$ is weakly compactly generated, which is an isomorphic property, see e.g. [Ne].

The class of ω-Corson compact spaces K is not B-stable: $C(I)$ and $C(I^\omega)$ are isomorphic by Miljutin's theorem (cf. Semadeni [Se]), while I^ω is not ω-Corson. We do not know if the class $\mathcal{NY}$ is B-stable. The following open question has been around for several years, see e.g. [AMN, 3.9] and [Ne, 6.45].

Problem 3.5.1. *Is the class of Corson compacta B-stable?*

The problem has a relatively easy answer 'yes' if ω_1 is a caliber of Radon measures. However, it is still unclear if Problem 3.5.1 can be resolved within the usual axioms of set theory; there are partial positive solutions in [Pl1]. It seems natural to formulate a more general problem.

Problem 3.5.2. *Given an uncountable cardinal number κ, is the class of κ-Corson compacta B-stable?*

The positive answer to 3.5.2 follows for every regular number κ for which the class of κ-Corson compacta is stable under $P(\cdot)$. In fact, we have the following more general result.

Theorem 3.5.3 ([MPZ, Theorem 13.3]). *Suppose that a regular cardinal number κ is a caliber of Radon measures. If K is κ-Corson compact and $T : C(L) \to C(K)$ is an isomorphic embedding, then L is κ-Corson compact.*

Proof. We can assume that $c \cdot \|g\| \leq \|Tg\| \leq \|g\|$ for some $c > 0$ and every $g \in C(L)$. We consider the dual operator

$$T^* : M(K) \to M(L), \quad T^*\mu(g) = \mu(Tg) \text{ for } \mu \in M(K) \text{ and } g \in C(L).$$

Then T^* is surjective and $weak^* - weak^*$ continuous; moreover, $T^*[M_1(K)] \supseteq c \cdot M_1[L]$. By Theorem 2.5.5, $M_1[K]$ is κ-Corson compact, so $T^*[M_1(K)]$ is also κ-Corson compact. The space L embeds into $T^*[M_1(K)]$ via the mapping $L \ni y \mapsto c \cdot \delta_y$, and we are done. □

Corollary 2.5.6 and Theorem 3.5.3 yield the following.

Corollary 3.5.4 ([MPZ, Corollary 13.4]). *The class of $\mathfrak{c}^+$-Corson compacta is B-stable.*

Chapter 4

Miscellaneous results

In this chapter we present observations which are byproduct of our investigation in the theory function spaces. It's possible that these facts are known to specialists, but we couldn't find it anywhere in the literature.

All linear topological spaces are assumed to be Hausdorff.

4.1. Linear operators

The following lemma is [S, Theorem 11/Chapter 14]. This Lemma inspired the author to investigate if, under some assumptions, every continuous, open linear operator is weakly open.

Lemma 4.1.1. *A continuous linear operator between two linear topological spaces is weakly continuous.*

The following lemma is [R, Theorem 3.6]

Lemma 4.1.2. *Every continuous linear functional on a linear subspace M of a locally convex linear topological space X can be extended to a continuous linear functional on X.*

Let us note here that from the above lemma it follows that the weak topology of a linear subspace M of a locally convex linear space X is equal to the subspace topology of the space X endowed with the weak topology.

The following lemma is [Sc, Corollary 2/Section 4.1].

Lemma 4.1.3. *Let Y be a closed subspace of a topological vector space X. The weak topology on the quotient space X/Y is equal to the quotient topology of X/Y where X is endowed with weak topology, and Y is endowed with the subspace topology of the weak topology on X.*

Theorem 4.1.4. *Let X be a linear topological space, and let Y be a locally convex linear topological space. Let $T : X \to Y$ be a continuous linear transformation with the property that image*

$T(U)$ of every weakly open set U in X is open in $T(X)$, then the image of every weakly open set in X is weakly open in $T(X)$.

Proof. Let $\pi : X \to X/ker(T)$ be the quotient map. Consider $T' : X/ker(T) \to Y$ given by the formula $T'([x]) = T(x)$. It is a well defined continuous, linear transformation which is additionally injective.

Let $U \subseteq X/ker(T)$ be a weakly open subbasic set. There exists $\phi \in (X/ker(T))^*$ and open $V \subseteq \mathbb{R}$ such that $U = \phi^{-1}(V)$. Define $\widetilde{\phi} : X \to \mathbb{R}$ as $\widetilde{\phi}(x) = \phi([x]) = (\phi \circ \pi)(x)$. Clearly, it is a continuous, linear functional on X. We have

$$W = T'(U) = T'(\phi^{-1}(V)) = T(\pi^{-1}(\phi^{-1}(V))) = T((\phi \circ \pi)^{-1}(V)) = T(\widetilde{\phi}^{-1}(V)).$$

By assumption, the set W is open in $T(X)$. Consequently, transformation T' is a continuous linear injection between two locally convex linear topological spaces with the property that image of every weakly open set is open. This means that the inverse transformation $(T')^{-1} : T(X) \to (X, \omega)$ is continuous. By Lemma 4.1.1, used for the space $T(X)$ endowed with the subspace topology and X endowed with the weak topology, we obtain that
$(T')^{-1} : (T(X), \omega) \to (X, \omega)$ is continuous. This means that T' is weakly open. Let $U \subseteq X$ be a weakly open set, then

$$U' = \bigcup \{U + v : v \in ker(T)\} \subseteq X$$

is also a weakly open set as a union of weakly open sets, and $T(U) = T(U')$. Clearly $\pi(U')$ is open in the quotient topology of X endowed with the weak topology and $T(X)$ endowed with the weak topology. By Lemma 4.1.3, set $\pi(U')$ is weakly open in X/Y. Then

$$T(U) = T(U') = T'(\pi(U'))$$

is weakly open. □

Corollary 4.1.5. *Let X be a linear topological space, and let Y be a locally convex linear topological space. If $T : X \to Y$ is a continuous, open, linear transformation, then T is weakly open.*

Lemma 4.1.6. *Let X be a topological vector space. Sets of the form*

$$V = \{\phi \in X^* : \phi(x_1) \in U_1, \ldots, \phi(x_n) \in U_n\}$$

for open subsets $U_k \subseteq \mathbb{R}$ and linearly independent vectors $x_1, \ldots, x_n \in X$ form a basis of X^*.

Proof. Let ψ be a continuous functional on X and let

$$W = \{\phi \in X^* : \phi(x_1) \in U_1, \ldots, \phi(x_n) \in U_n\}$$

be its open basic neighbourhood. If vectors $x_1, \dots, x_n$ are linearly independent, we are done. Without loss of generality assume that

$$x_n = \sum_{i=1}^{n-1} a_i x_i$$

for some $a_1, \dots, a_{n-1} \in \mathbb{R}$. The function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ given by the formula

$$f(x_1, \dots, x_{n-1}) = \sum_{i=1}^{n-1} a_i x_i$$

is continuous, so there exist open neighbourhoods $V_1, \dots, V_{n-1} \subseteq \mathbb{R}$ of points $\psi(x_1), \dots \psi(x_{n-1})$ such that

$$f(\phi(x_1), \dots \phi(x_{n-1})) = \sum_{i=1}^{n-1} a_i \phi(x_i) = \phi(x_n) \in U_n,$$

whenever $\phi(x_i) \in V_i$ for every $i \in \{1, \dots, n-1\}$. Then

$$\psi \in \{\phi \in X^* : \phi(x_1) \in V_1, \dots, \phi(x_{n-1}) \in V_{n-1}\} \subseteq W.$$

We repeat the process described above until we are left with linearly independent vectors $\{y_1, \dots y_k\} \subseteq \{x_1, \dots, x_n\}$. □

Theorem 4.1.7. *Let X be a linear topological space, and let Y be a convex linear topological space. Let $T : X \to Y$ be a continuous linear transformation. The following conditions are equivalent:*

(i) $T(X)$ is closed in Y;

(ii) T^* is open in weak topology onto its image.*

Proof. For $a) \implies b)$, let $y \in Y$, and let $U \subseteq \mathbb{R}$ be an open set. Consider an open subbasic set

$$V = \{\phi : \phi \in Y^*, \phi(y) \in U\}.$$

Then

$$T^*(V) = \{\phi \circ T : \phi \in Y^*, \phi(y) \in U\}.$$

If $y \in T(X)$, then $y = T(x)$ for some $x \in X$, so

$$T^*(V) = \{\phi \circ T : \phi \in Y^*, \phi(T(x)) \in U\}$$

is clearly an open set in $T^*(Y^*)$.

If $y \notin T(X)$, take arbitrary $\psi \in Y^*$ and consider $\psi \restriction_{T(X)}$. Since $T(X) \subseteq Y$ is closed, by [R, Theorem 3.5], there exists $\overline{\psi} \in Y^*$ such that $\overline{\psi} \restriction_{T(X)} = \psi$ and $\overline{\psi}(y) \in U$. So $\psi \circ T = \overline{\psi} \circ T$

and $\overline{\psi}(y) \in U$. Consequently, the set $T^*(V) = T^*(Y^*)$ is open.
For $b) \implies a)$, assume there exists $y \in \overline{T(X)} \setminus T(X)$. Consider an open subbasic set

$$W = \{\phi \in Y^* : \phi(y) \in U\}$$

for some open $\emptyset \neq U \subsetneq \mathbb{R}$. By $b)$, the set

$$T^*(W) = \{\phi \circ T : \phi \in Y^*, \phi(y) \in U\}$$

is a nonempty open set in $T^*(Y^*)$. There exists an open subbasic set

$$\emptyset \neq V = \{\phi \circ T : \phi \in Y^*, \phi(T(x_1)) \in U_1, ..., \phi(T(x_n)) \in U_n\} \subseteq T^*(W).$$

By Lemma 4.1.6, we can assume that $T(x_1), ..., T(x_n)$ are linearly independent. As $y \notin T(X)$, vectors $y, T(x_1), ..., T(x_n)$ are linearly independent as well. By the Hahn-Banach theorem, there exists $\phi \in Y^*$ such that $\phi(y) \notin U$, but $\phi(T(x_i)) \in U_i$ for $i \in \{1, 2, ..., n\}$. Since $V \subseteq T^*(W)$, there must exist $\psi \in Y^*$ such that $\psi(y) \in U$ and $\psi \circ T = \phi \circ T$. We have that $\psi \restriction_{T(X)} = \phi \restriction_{T(X)}$, and therefore $\psi \restriction_{\overline{T(X)}} = \phi \restriction_{\overline{T(X)}}$, since ψ and ϕ are continuous, which contradicts that $\psi(y) \in U$ while $\phi(y) \notin U$. □

By Theorem 4.1.7 and [S, Corollary 21/Chapter 26] we obtain.

Theorem 4.1.8. *Let X be a linear topological space, and let Y be a convex linear topological space. Let $T : X \to Y$ be a continuous linear transformation. The following are equivalent:*

(i) T is a surjection;

(ii) T^* is injective and weak open onto its image.*

The above theorem is known for Banach spaces, see [S, Theorem 3/Chapter26]. Recall that every weakly continuous linear operator between two Banach spaces is continuous, therefore it is a natural question to ask whether it is true that Corollary 4.1.5 can be reversed for Banach spaces. Note that here we need to drop the assumption that the image of T is a Banach space, since every continuous linear surjective transformation between Banach spaces is open by the open mapping theorem. The answer to this question is given by the following proposition.

Proposition 4.1.9. *Let X be a Banach space, and let Y be a normed linear space. Assume that $T : X \to Y$ is a continuous, linear, weakly open surjection. Then Y is a Banach space, and so T is open.*

Proof. Let $\widetilde{Y}$ be the completion of the space Y, and let $i : Y \hookrightarrow \widetilde{Y}$ be a linear homeomorphic embedding. Let $\widetilde{T} = i \circ T$. By [Sc, Theorem 7.3], the dual transformation $\widetilde{T}^*$ has a closed range. By [R, Theorem 4.14], the transformation $\widetilde{T}$ has a closed range $\widetilde{T}(X) = i(Y) \sim_{lin} Y$ and therefore it is a Banach space. Consequently, by the open mapping theorem, the transformation T is open. □

Note that in the proof of Proposition 4.1.9 we used the Banach theorem [R, Theorem 4.14]. This theorem can be concluded from Theorem 4.1.7 and a theorem of Hausdorff [En1, Problem 5.5.8 (d)].

4.2. Some remarks on homeomorphisms and linear homeomorphisms of function spaces

For normed spaces $(E_n, \|\cdot\|_n)$, their c_0-product $\left(\prod_{n\in\mathbb{N}} E_n\right)_0$ is the set of all sequences $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} E_n$ such that $\|x_n\|_n$ converges to zero. For a pseudocompact space X we consider the supremum norm on the space $C_p(X)$. Now we will show that every c_0-product of function spaces $C_p(X)$ for pseudocompact X is homeomorphic to its countable power. This generalises a theorem proved in [GK] stating that $c_0 \sim (c_0)^\omega$ where c_0 is endowed with the pointwise convergence topology. The Lemma below is [GK, Corollary]. Let us point out that the original proof of Lemma 4.2.1 seems to be incomplete. One needs deeper analysis of transformations used in the proof. A more detailed proof can be found in [T, Problem 019].

Lemma 4.2.1. *Let X be a pseudocompact space. If $C_p(X) \sim \left(\prod_{n\in\mathbb{N}} C_p(X)\right)_0$, then $C_p(X) \sim \left(C_p(X)\right)^\omega$.*

In the following lemma and theorem we consider the following norms

$$\|(f_n)_{n\in\mathbb{N}}\| = \sup_{n\in\mathbb{N}}(\sup_{x\in X}(|f_n(x)|)), \text{ for } (f_n)_{n\in\mathbb{N}} \in \Big(\prod_{n\in\mathbb{N}} C_p(X)\Big)_0$$

and

$$\|(r,(f_n)_{n\in\mathbb{N}})\| = max\{|r|, \|(f_n)_{n\in\mathbb{N}}\|\}, \text{ for } (r,(f_n)_{n\in\mathbb{N}}) \in \mathbb{R}\times\Big(\prod_{n\in\mathbb{N}} C_p(X)\Big)_0.$$

Lemma 4.2.2. *For a pseudocompact space X, there exists a linear homeomorphism*

$$h : \mathbb{R}\times\Big(\prod_{n\in\mathbb{N}} C_p(X)\Big)_0 \to \Big(\prod_{n\in\mathbb{N}} C_p(X)\Big)_0$$

such that

$$\frac{1}{3}\|z\| \le \|h(z)\| \le 3\|z\|.$$

Let h be given by the formula

$$h(r,(f_k)_{k\in\mathbb{N}}) = (f_1 - f_1(x_0) + r, (f_k - f_k(x_0) + f_{k-1}(x_0))_{k\ge 2}).$$

Clearly, $h(r,(f_k)_{k\in\mathbb{N}}) \in \left(\prod_{n\in\mathbb{N}} C_p(X)\right)_0$ for every $r\in\mathbb{R}$ and $(f_k)_{k\in\mathbb{N}} \in \left(\prod_{n\in\mathbb{N}} C_p(X)\right)_0$. It is easy to see that h is continuous. We will show that the inverse function is given by the formula

$$h^{-1}((g_k)_{k\in\mathbb{N}}) = (g_1(x_0), (g_k - g_k(x_0) + g_{k+1}(x_0))_{k\ge 1}).$$

Indeed

$$h(g_1(x_0),(g_k - g_k(x_0) + g_{k+1}(x_0))_{k\geq 1})_1 =$$

$$g_1 - g_1(x_0) + g_2(x_0) - (g_1(x_0) - g_1(x_0) + g_2(x_0)) + g_1(x_0) = g_1,$$

and, for $n \geq 2$,

$$h(g_1(x_0),(g_k - g_k(x_0) + g_{k+1}(x_0))_{k\geq 1})_n =$$

$$g_n - g_n(x_0) + g_{n+1}(x_0) - (g_n(x_0) - g_n(x_0) + g_{n+1}(x_0)) + (g_{n-1}(x_0) - g_{n-1}(x_0) + g_n(x_0)) = g_n,$$

and

$$h^{-1}(f_1 - f_1(x_0) + r,(f_k - f_k(x_0) + f_{k-1}(x_0))_{k\geq 2})_1 = f_1(x_0) - f_1(x_0) + r = r,$$

$$h^{-1}(f_1 - f_1(x_0) + r,(f_k - f_k(x_0) + f_{k-1}(x_0))_{k\geq 2})_2 =$$

$$f_1 - f_1(x_0) + r - (f_1(x_0) - f_1(x_0) + r) + (f_2(x_0) - f_2(x_0) + f_1(x_0)) = f_1$$

and, for $n \geq 3$,

$$h^{-1}(f_1 - f_1(x_0) + r,(f_k - f_k(x_0) + f_{k-1}(x_0))_{k\geq 2})_n =$$

$$f_{n-1}-f_{n-1}(x_0)+f_{n-2}(x_0)-(f_{n-1}(x_0)-f_{n-1}(x_0)+f_{n-2}(x_0))+(f_n(x_0)-f_n(x_0)+f_{n-1}(x_0)) = f_{n-1}.$$

This shows that the above formulas for h and h^{-1} do define mutually inverse functions. Clearly, the inverse transformation h^{-1} is continuous, and therefore h is a homeomorphism. Its easy to see that, for every $z \in \mathbb{R} \times \left(\prod_{n\in\mathbb{N}} C_p(X)\right)_0$, we have $\|h(z)\| \leq 3\|z\|$, and for every $w \in \left(\prod_{n\in\mathbb{N}} C_p(X)\right)_0$ we have $\left\|h^{-1}(w)\right\| \leq 3\|w\|$, consequently

$$\frac{1}{3}\|z\| \leq \|h(z)\| \leq 3\|z\|$$

for every $z \in \mathbb{R} \times \left(\prod_{n\in\mathbb{N}} C_p(X)\right)_0$.

Theorem 4.2.3. *For a pseudocompact space X, we have*

$$\left(\left(\prod_{n\in\mathbb{N}} C_p(X)\right)_0\right)^\omega \sim \left(\prod_{n\in\mathbb{N}} C_p(X)\right)_0.$$

Proof. Consider the space $Y = (\dot{\bigcup}_{n\in\mathbb{N}} X_n)\dot{\cup}\{a\}$, where $X_n \sim X$ for every $n \in \mathbb{N}$. The topology on $\dot{\bigcup}_{n\in\mathbb{N}} X_n$ is the topology of a disjoint union, and all open neighbourhoods of a consist of $\{a\}$ and cofinitely many copies of X. Notice that Y is pseudocompact as well.

Let $g : C_p(Y) \to \mathbb{R} \times \left(\prod_{n\in\mathbb{N}} C_p(X_n)\right)_0$ be given by the formula

$$g(f) = (f(a),(f\restriction_{X_n} - f(a))_{n\in\mathbb{N}}).$$

It is easy to see that g is a linear homeomorphism. Additionally $\|g(f)\| \leq 2\|f\|$ for every $f \in C_p(Y)$. Its inverse function is given by the formulas

$$g^{-1}(r,(f_n)_{n\in\mathbb{N}}) \restriction_{X_n} = f_n + r, \qquad g^{-1}(r,(f_n)_{n\in\mathbb{N}})(a) = r.$$

Clearly, $\left\|g^{-1}(r,(f_n)_{n\in\mathbb{N}}))\right\| \leq 2\|(r,(f_n)_{n\in\mathbb{N}})\|$ for every $(r,(f_n)_{n\in\mathbb{N}})) \in \mathbb{R} \times \left(\prod_{n\in\mathbb{N}} C_p(X_n)\right)_0$, and therefore $\frac{1}{2}\|f\| \leq \|g(f)\| \leq 2\|f\|$ for every $f \in C_p(Y)$. By Lemma 4.2.2, there exists a linear homeomorphism

$$h : \mathbb{R} \times \Big(\prod_{n\in\mathbb{N}} C_p(X)\Big)_0 \to \Big(\prod_{n\in\mathbb{N}} C_p(X)\Big)_0$$

such that $\frac{1}{3}\|z\| \leq \|h(z)\| \leq 3\|z\|$. Consequently, $p = h \circ g : C_p(Y) \to \left(\prod_{n\in\mathbb{N}} C_p(X)\right)_0$ is a linear homeomorphism such that $\frac{1}{6}\|f\| \leq \|p(f)\| \leq 6\|f\|$ for every $f \in C_p(Y)$. The function

$$F : \Big(\prod_{n\in\mathbb{N}} C_p(Y)\Big)_0 \to \Big(\prod_{n\in\mathbb{N}} \Big(\prod_{n\in\mathbb{N}} C_p(X)\Big)_0\Big)_0$$

given by $F((f_n)_{n\in\mathbb{N}}) = (p(f_n))_{n\in\mathbb{N}}$ is a linear homeomorphism. It is easy to see that F is continuous, injective and open onto its image. The inequality $\|p(f)\| \leq 6\|f\|$ implies that $(p(f_n))_{n\in\mathbb{N}}$ converges to 0, and the inequality $\frac{1}{6}\|f\| \leq \|p(f)\|$ implies that F is onto. Consequently,

$$\Big(\prod_{n\in\mathbb{N}} C_p(Y)\Big)_0 \sim \Big(\prod_{n\in\mathbb{N}} \Big(\prod_{n\in\mathbb{N}} C_p(X)\Big)_0\Big)_0 \sim \Big(\prod_{n\in\mathbb{N}} C_p(X)\Big)_0 \sim C_p(Y).$$

By Lemma 4.2.1, we have $C_p(Y) \sim (C_p(Y))^\omega$, that is

$$\Big(\prod_{n\in\mathbb{N}} C_p(X)\Big)_0 \sim \Big(\Big(\prod_{n\in\mathbb{N}} C_p(X)\Big)_0\Big)^\omega$$

□

We enclose this chapter with the following theorem which is already known for the case where the function space is endowed with the pointwise convergence topology.

Theorem 4.2.4. *If a topological space X is not pseudocompact, then $C_c(X) \sim_{lin} s \times C_c(X)$, and so $C_\omega(X) \sim_{lin} s \times C_\omega(X)$ as well.*

Proof. By assumption, there exists an unbounded function $f \in C(X)$. The function $f^2 \in C(X)$ is also unbounded, so without the loss of generality, we can assume that $f \geq 0$. There exists a sequence of positive numbers $(z_n)_{n\in\mathbb{N}} \subseteq im(f)$ such that $z_{n+1} \geq z_n + 2$. Pick $x_n \in X$ satysfying $f(x_n) = z_n$. The set $A = \{x_n : n \in \mathbb{N}\}$ is discrete and closed (it is discrete as $f^{-1}(z_n - 1, z_n + 1) \cap A = \{x_n\}$). Define $C = \{z_n : n \in \mathbb{N}\}$, and let $s : C_c(C) \to C_c(\mathbb{R})$ be an extension operator such that $s(f)$ is constant on $(-\infty, z_1]$ and linear on every interval $[z_n, z_{n+1}]$ for $n \in \mathbb{N}$. Let $T : C_c(A) \to C_c(X)$ be given by the formula

$$T(h) = s(h \circ f^{-1} \restriction_C) \circ f.$$

Clearly, it is a linear extension operator. Now we will show that the operator T is continuous. By [S, Proposition 1/Chapter 5], it is enough to check continuity at the point equal to the constant function $\mathbf{0} \in C_c(A)$.

For a compact $K \subseteq X$ and $\epsilon > 0$, consider an open basic neighbourhood of the zero function in $C_c(X)$

$$[K, (-\epsilon, \epsilon)] = \{f \in C_c(X) : f(K) \subseteq (-\epsilon, \epsilon))\}.$$

For $h \in C_c(A) \sim s$, we have

$$T(h) \in [K, (-\epsilon, \epsilon)] \iff (s(h \circ f^{-1} \restriction_C) \circ f)(K) \subseteq (-\epsilon, \epsilon).$$

Let $K' = f(K)$, then K' is a compact set as well, and

$$T(h) \in [K, (-\epsilon, \epsilon))] \iff s(h \circ f^{-1} \restriction_C)(K') \subseteq (-\epsilon, \epsilon).$$

For a function $p \in C_c(C)$, to have the property that $(s(p))(K') \subseteq (-\epsilon, \epsilon)$, it is enough to assume that $|p \restriction_S| < \epsilon$, where $S = \{z_n : n \in \{1, .., m\}\}$ is the minimal set consisting of initial elements of C, containing a number larger than $max(K')$. If $|(h \circ f^{-1} \restriction_C) \restriction_S| < \epsilon$ (that is $|h \restriction_{f^{-1}(S) \cap A}| < \epsilon$), then $(s(h \circ f^{-1} \restriction_C) \circ f)(K) \subseteq (-\epsilon, \epsilon)$. Consequently,

$$\mathbf{0} \in \{h \in C_c(A) : |h(x_n)| < \epsilon \text{ for } n \in \{1, .., m\}\} \subseteq T^{-1}([K, (-\epsilon, \epsilon)]).$$

The operator T is a continuous, linear extension operator. In the same way as in [M, Proposition 6.6.6.] one can show that in this case $C_c(X) \sim_{lin} C_c(A) \times C_c(X|A)$ and therefore

$$C_c(X) \sim_{lin} C_c(A) \times C_c(X|A) \sim_{lin} s \times C_c(X|A) \sim_{lin} s \times s \times C_c(X|A) \sim_{lin} s \times C_c(X).$$

By Lemma 4.1.1, we obtain $C_\omega(X) \sim_{lin} s \times C_\omega(X)$. □

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