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Julia sets in random holomorphic dynamics

PhD dissertation

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Author's declaration:
I hereby declare that this dissertation is my own work.

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This dissertation is ready to be reviewed.

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Abstract

This dissertation studies the properties of Julia sets arising in random iteration of holomorphic functions. It consists of two parts, each one concerning a natural generalization of a classical family of functions from autonomous complex dynamics.

The first part deals with random iterations of the quadratic family, that is we consider a sequence of compositions of $z^2 + c_n$ where the values of c_n change along iteration. In this setting the Julia set depends on the sequence $(c_n)_{n\geqslant 0}$ and not just on one parameter. Descriptions of the set of sequences for which the Julia set is totally disconnected have been of some interest. The main result of this section is an answer to a question posed by Brück, Büger and Reitz in 1999. They asked whether the Julia set is almost always (with respect to the product of Lebesgue measures) totally disconnected, when the sequences are chosen from the product space of disks of large enough radius $R > \frac{1}{4}$. We give a positive answer to this question by showing that any $R > \frac{1}{4}$ is sufficient, and in fact prove the statement for much more general spaces than products of disks. In particular, a result which may be surprising from the point of view of autonomous dynamics, one can also take sequences from the infinite product of copies of the main cardioid of the Mandelbrot set and the same statement holds. Lastly this part also contains some results on the maximal measure of the above quadratic Julia sets. In particular we prove a formula for the Hausdorf dimension of said measure, as well as several results on its dependence on the parameter space.

In the second part of the dissertation we study random iterations of the exponential family $\lambda_n e^z$. Once again the Julia set depends on a sequence (λ_n), but this time we are concerned with characterizing sequences of real numbers for which the Julia set is the whole complex plane. In the autonomous setting the Julia set of λe^z is the whole plane for $\lambda > \frac{1}{e}$, thus unsurprisingly various sequences converging to $\frac{1}{e}$ are considered. Among other results we prove that the Julia set for a sequence (λ_n) where $\lambda_n = \frac{1}{e} + \frac{1}{n^p}$ is the whole plane for $p < \frac{1}{2}$. On the other hand we provide an example of a sequence for which the Fatou set is non-empty, despite all the trajectories of the singular value 0 escaping to infinity.

Keywords: Random dynamical system, complex dynamics, Julia set, quadratic system, exponential family

AMS Subject Classification: 37F10, 37H12

Streszczenie

Poniższa rozprawa bada własności zbiorów Julii pojawiających się w losowej iteracji funkcji holomorficznych. Składa się ona z dwóch części, w każdej badamy uogólnienie pewnej klasycznej rodziny funkcji znanej z autonomicznej dynamiki holomorficznej.

W pierwszej części zajmujemy się losowymi iteracjami rodziny kwadratowej, to znaczy rozważamy złożenia funkcji postaci $z^2 + c_n$ gdzie wartości c_n zmieniają się wraz z iteracją. W tej sytuacji zbiór Julii zależy od całego ciągu $(c_n)_{n\geqslant 0}$, a nie tylko od jednego parametru. Jednym z interesujących problemów w tej dziedzinie był opis zbioru ciągów dla których zbiór Julii jest całkowicie niespójny. Główny wynik tej części rozprawy odpowiada na pytanie zadane przez Brücka, Bügera oraz Reitza w 1999. W swojej pracy spytali czy zbiór Julii jest prawie zawsze (względem produktu miar Lebesgue'a) całkowicie niespójny, o ile wybieramy ciągi z produktu dysków o dostatecznie dużym promieniu $R > \frac{1}{4}$. Odpowiadamy na powyższe pytanie twierdząco, dowodząc tezy dla dowolnego $R > \frac{1}{4}$. Ponadto dowodzimy mocniejszego twierdzenia które daje powyższy wniosek nie tylko dla produktów dysków ale i ogólniejszych przestrzeni. W szczególności, co może być zaskakujące z punktu widzenia autonomicznej dynamiki holomorficznej, ta sama teza jest prawdziwa jeśli wybieramy ciąg z produktu kopii głównej kardioidy zbioru Mandelbrota. Wreszcie w ostatniej sekcji tej części rozprawy zajmujemy się miarą maksymalną powyższych zbiorów Julii. Między innymi dowodzimy wzoru na jej wymiar Hausdorfa oraz pokazujemy pewną zależność wymiaru od przestrzeni z której losujemy ciągi.

W drugiej części rozprawy badamy iteracje rodziny eksponencjalnej $\lambda_n e^z$. Ponownie zbiór Julii zależy od ciągu (λ_n), ale tym razem zajmujemy się opisem ciągów liczb rzeczywistych dla których zbiór Julii jest całą płaszczyzną. W autonomicznej sytuacji zbiór Julii dla λe^z jest płaszczyzną zespoloną dla $\lambda > \frac{1}{e}$, dlatego naturalnie w rozprawie szczególną uwagę poświęcamy ciągom zbieżnym do $\frac{1}{e}$. Dowodzimy między innymi tego że zbiór Julii dla (λ_n) gdzie $\lambda_n = \frac{1}{e} + \frac{1}{n^p}$ jest całą płaszczyzną gdy $p < \frac{1}{2}$. Z drugiej strony wskazujemy przykład ciągu dla którego zbiór Fatou jest niepusty, pomimo tego że wszystkie trajektorie wartości singularnej 0 uciekają do nieskończoności.

Słowa kluczowe: Losowe układy dynamiczne, dynamika holomorficzna, zbiór Julii, rodzina kwadratowa, rodzina eksponencjalna

Klasyfikacja tematyczna AMS: 37F10, 37H12

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Contents

Chapter 1

Introduction and main results

This dissertation consists of two parts, both studying Julia sets occuring in random iteration of holomorphic functions. We begin by giving a brief overview of each of the two parts, as well as random holomorphic dynamics in general. The subsequent sections will present a more detailed description of the contents of each part, and give an introduction of relevant results from the field.

The general setting we operate in throughout the dissertation is as follows: take a sequence of holomorphic functions $(f_n)_{n\geqslant 0}^{\infty}$ defined on the whole plane (or Riemann sphere) and consider the compositions

$$F_n(z) = f_n \circ f_{n-1} \circ ... \circ f_1(z).$$

For the sequence of compositions $(F_n)_{n=1}^{\infty}$ one can naturally generalize the notions of the Fatou and Julia sets.

Definition 1.1 (Non-autonomous Fatou set) Let $\{F_n\}_{n\in\mathbb{N}}$ be a family of holomorphic functions $F_n : \mathbb{C} \to \mathbb{C}$ (or $F_n : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$). The Fatou set for the family $\{F_n\}_{n\in\mathbb{N}}$ is the set of points $z \in \mathbb{C}$ (or $z \in \hat{\mathbb{C}}$) for which there exists an open set $U \ni z$ such that the family $\{F_n|_U\}_{n\in\mathbb{N}}$ is normal in the sense of Montel.

Definition 1.2 (Non-autonomous Julia set) Let $\{F_n\}_{n\in\mathbb{N}}$ be a family of holomorphic function $F_n : \mathbb{C} \to \mathbb{C}$ (or $F_n : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$). The Julia set for this family is the complement of the Fatou set.

Despite the above sets being defined in the standard way, some of their properties known from autonomous complex dynamics, where the family considered are the iterates of a single function, do not hold in our setting. It is for example easy to construct a family for which the Julia set is empty. In general, in a non-autonomous setting we do not have a reasonable notion of a periodic point, as the function changes along iteration. Moreover we also do not have such a strong notion of invariance of the Fatou and Julia sets as in the autonomous case. This immediately means that a lot of standard theory, such as classification of Fatou components, either does not apply at all or has to be heavily reformulated and proven using different techniques.

In view of the above, it is reasonable to make some assumptions on the family $\{F_n\}_{n\in\mathbb{N}}$ rather than consider compositions of arbitrary holomorphic functions. What is commonly done in this field, is that one can take f_n to be some function dependent on a parameter $a(n)$, which is chosen, possibly with some distribution, from some Borel set $V \subset \mathbb{C}$. Then the Julia set for the compositions (F_n) depends on the sequence of parameters $a(n)$, and one can prove results that hold for a "typical" Julia set - with respect to some product measure on the space of sequences of parameters $a(n)$. In this dissertation we deal with two families of functions dependent on a parameter: the quadratic family, and the exponential family.

The first part of the dissertation, contained in Chapter 2, studies typical connectivity of the Julia set for a sequence of compositions of functions of the form $z^2 + c_n$. The setting in which c_n are all chosen from a disk of radius R was considered by R. Brück, M. Büger and S. Reitz in [8], and other related papers, such as [6], [9]. In [8], the authors ask whether for a large enough radius R the Julia set is totally disconnected for almost every sequence (with respect to the product of uniform distributions on the disk). A positive answer to that question is the main result of this section, as we shall see any $R > \frac{1}{4}$ is sufficient. In fact we prove typical total disconnectedness for a much more general space of sequences than the product of disks. Additionally we study the maximal measure of the resulting quadratic Julia sets. Among other results we prove a formula for the Hausdorff dimension of the maximal measure, and some theorems on the regularity of the dimension as a function of the space from which the sequences of parameters are chosen. This part of the dissertation is mostly based on a joint work with Anna Zdunik [21].

The second part of the dissertation, contained in Chapter 3, studies the sequence of compositions of exponential functions $\lambda_n e^z$, where λ_n are positive real numbers. In [30] Mariusz Urbański and Anna Zdunik prove, among other results, that the Julia set for this family is the whole plane, provided the numbers λ_n are greater than some constant greater than $\frac{1}{e}$. The number $\frac{1}{e}$ plays a special role for the exponential family, as in the autonomous case e^{z-1} has a Fatou set consisting of the basin of attraction of the parabolic point 1. Thus e^{z-1} has a non-empty Fatou set, unlike λe^z for $\lambda > \frac{1}{e}$. In this part of the dissertation we expand on the aforementioned result from [30], weakening the assumption of the sequence being separated from $\frac{1}{e}$. One of the main results is that for $\lambda_n = \frac{1}{e} + \frac{1}{n^p}$ where $p < \frac{1}{2}$ the Julia set is the whole plane. We also give an example of a sequence $(\lambda_n)_{n=1}^{\infty}$ for which the Fatou set is non-empty but the trajectory of the singular value 0 escapes to infinity for the sequence $(\lambda_n)_{n=N}^{\infty}$, for all $N \in \mathbb{N}$. This section is based mostly on [20].

1.1 Random quadratic maps

We consider non–autonomous compositions of quadratic polynomials $f_c = z^2 + c$ where at each step c is chosen randomly from some bounded Borel $V \subset \mathbb{C}$ (e.g. the disc $\mathbb{D}(0, R)$). Let us introduce the parameter space $\Omega = V^{\mathbb{N}}$. The space Ω is equipped with a natural left shift map σ. Namely, for every $\omega \in \Omega$, $\omega = (c_0, c_1, c_2, \dots)$ put

$$\sigma(\omega) = (c_1, c_2, \dots).$$

Next, for every $\omega \in \Omega$, $\omega = (c_0, c_1, \dots)$ denote by f_ω the map f_{c_0}.
Then the non-autonomous composition f_ω^n is given by the formula

$$f_\omega^n := f_{c_{n-1}} \circ f_{c_{n-2}} \circ \dots \circ f_{c_0}.$$

The global dynamics can be described as a skew product $F : \Omega \times \mathbb{C} \to \Omega \times \mathbb{C}$,

$$F(\omega, z) = (\sigma(\omega), f_\omega(z)).$$

Then for all $n \in \mathbb{N}$ we have that

$$F^n(\omega, z) = (\sigma^n(\omega), f_\omega^n(z)).$$

So, every sequence $\omega \in \Omega$ determines a sequence of non-autonomous iterates: $(f_\omega^n)_{n \in \mathbb{N}}$.

Let μ be a Borel probability measure on V. We denote by $\mathbb{P}$ the product distribution on Ω generated by μ. Then $(\Omega, \mathbb{P})$ becomes a measurable space, and $\sigma : \Omega \to \Omega$ is an ergodic measure preserving endomorphism.

Analogously to the autonomous case, it is natural to consider the following objects:

- (*escaping set*, or *basin of infinity*)

$$\mathcal{A}_\omega = \{z \in \mathbb{C} : f_\omega^n(z) \xrightarrow[n\to\infty]{} \infty\}$$

- (non-autonomous Julia set)

$$J_\omega = \{z \in \mathbb{C} : \text{for every open set } U \ni z \text{ the family } f^n_{\omega|U} \text{ is not normal.}\} \tag{1.1}$$

- (non-autonomous Fatou set)

$$F_\omega = \mathbb{C} \setminus J_\omega \tag{1.2}$$

- (non- autonomous filled-in Julia set)

$$K_\omega = \mathbb{C} \setminus \mathcal{A}_\omega. \tag{1.3}$$

The following proposition, which can be found in [9] (Theorem 1) is analogous to the autonomous case.

Proposition 1.3. *Let $\omega \in \mathbb{D}(0, R)^{\mathbb{N}}$. Then*

$$J_\omega = \partial \mathcal{A}_\omega,$$

Let us also note the following straightforward observations:

Proposition 1.4. *For every $\omega \in \Omega$*

- $J_{\sigma\omega} = f_\omega(J_\omega)$,
- $\mathcal{A}_{\sigma\omega} = f_\omega(\mathcal{A}_\omega)$.

The study of iterates of non-autonomous and random rational maps, and, in particular, non- autonomous and random polynomials, originated from the seminal paper [17] by J. Fornæss and N. Sibony. This field was since developed by many authors.

A systematic study of non- autonomous dynamics of quadratic polynomials was done by R. Brück, M. Büger, S.Reitz, see [6; 8; 9]. Some other interesting results related to random polynomial dynamics in general have also been achieved by M. Comerford, in [12], [13]. Finally in [22] V. Mayer, M. Skorulski and M. Urbański among other results confirm a conjecture by R. Brück and M. Büger, concerning the typical Hausdorff dimension of a certain random quadratic Julia set.

In [8] the authors focus on the question of connectedness of the Julia set, giving, among other results, a transparent sufficient and necessary condition for the Julia set to be connected:

Theorem 1.5 (Theorem 1.1. in [8])**.** *Let $\omega \in \mathbb{D}(0, R)^{\mathbb{N}}$, $R > 0$. The Julia set J_ω is disconnected if and only if there exists $k \in \mathbb{N}$ such that*

$$f^n_{\sigma^k\omega}(0) \xrightarrow[n\to\infty]{} \infty.$$

Note that the point 0 plays a special role, since it is a common critical point of all maps f_c. Recall that in the autonomous case, i.e. the iterates of a single map f_c, the Julia set is disconnected if and only if $f_c^n(0) \xrightarrow[n\to\infty]{} \infty$. Moreover, if the Julia set $J(f_c)$ is disconnected, then it is totally disconnected. The last statement is no longer true in the non-autonomous case considered here; in particular, one can easily construct sequences ω for which J_ω is disconnected but has finitely many components.

Looking at the above characterization of connected Julia sets J_ω, it might seem reasonable to conjecture that the condition

$$f^n_{\sigma^k\omega}(0) \xrightarrow[n\to\infty]{} \infty \quad \text{for every} \quad k \in \mathbb{N}$$

is the right characterization of totally disconnected Julia sets J_ω.

However, this condition is neither necessary nor sufficient. Indeed, in [8] the authors construct an example of a sequence $\omega \in \mathbb{D}(0,R)^{\mathbb{N}}$ such that for every $k \in \mathbb{N}$ $f^n_{\sigma^k\omega}(0) \to \infty$ as $n \to \infty$, but the Julia set J_ω is only disconnected, not totally disconnected (see Example 4.4 in [8]). On the other hand, Example 4.5 in the same paper shows that the Julia set may be totally disconnected even if for infinitely many $k \in \mathbb{N}$ $f^n_{\sigma^k\omega}(0)$ does not tend to infinity as $n \to \infty$.

Clearly, the behaviour of the (typical) dynamics depends on the domain from which the parameters c_n are chosen. In particular in case of a disk $\mathbb{D}(0,R)$, the dynamics depend on R.

If $R \leqslant 1/4$ then for every $\omega \in \mathbb{D}^{\mathbb{N}}$, $\omega = (c_i)_{i=0}^{\infty}$, the Julia set J_ω is connected (see Remark 1.2 in [8]). Note that in this case all parameters c_i are chosen from the main cardioid in the Mandelbrot set.

For $R > 1/4$ the situation changes drastically. Indeed, the disc $\mathbb{D}(0,R)$ now contains parameters from the complement of the Mandelbrot set $\mathcal{M}$. So, it is evident that putting, for instance, $\omega = (c,c,c,\dots)$, where $c \in \mathbb{D}(0,R) \setminus \mathcal{M}$, one obtains a totally disconnected Julia set J_ω.

This motivates the following question, which was raised in [8] and [6]: what is a typical behaviour of the Julia set J_ω, in terms of connectedness? More formally, in [8] and [6] the authors introduce subsets of $\Omega = \mathbb{D}(0,R)^{\mathbb{N}}$, denoted by $\mathcal{D}$, $\mathcal{D}_N$, $\mathcal{D}_\infty$, $\mathcal{T}$, and described in terms of connectedness:

$$\mathcal{D} = \{\omega \in \Omega : J_\omega \quad \text{is disconnected}\}$$

$$\mathcal{D}_N = \{\omega \in \Omega : J_\omega \quad \text{has at least } N \text{ connected components}\}$$

$$\mathcal{D}_\infty = \{\omega \in \Omega : J_\omega \quad \text{has infinitely many connected components}\}$$

$$\mathcal{T} = \{\omega \in \Omega : J_\omega \quad \text{is totally disconnected}\}$$

$$\mathcal{F} = \{\omega \in \Omega : \forall k \in \mathbb{N}\ f^n_{\sigma^k\omega}(0) \xrightarrow[n\to\infty]{} \infty\}$$

Clearly, for $N > 1$, $\mathcal{D} \supset \mathcal{D}_N \supset \mathcal{D}_\infty \supset \mathcal{T}$. But, as mentioned above, the set $\mathcal{F}$ is neither contained in nor it contains $\mathcal{T}$.

Here, typicality may be understood in topological or metric sense. The space $\Omega = \mathbb{D}(0,R)^{\mathbb{N}}$ carries the natural product topology induced by the standard topology on $\mathbb{D}(0,R)$. Note that this topology is completely metrizable.

The space Ω also carries the natural product measure $\mathbb{P} := \otimes_{n=0}^{\infty}\lambda_R$ where each λ_R is the normalized Lebesgue measure on $\mathbb{D}(0,R)$. In [8] the authors prove that $\mathbb{P}(\mathcal{D}) = 1$, provided $R > 1/4$ (Theorem 2.3 in [8]). It can be deduced from the proof, in a rather straightforward way, that $\mathbb{P}(\mathcal{F}) = 1$ and also (although it is not explicitly stated in the paper) that $\mathbb{P}(\mathcal{D}_\infty) = 1$.

The work [6] deals with topological aspects of typicality of the above sets. In particular, the author proves (assuming $R > 1/4$) that

- the set $\mathcal{T}$ is dense in Ω (Theorem 1.1)
- the set $\mathcal{D}_\infty$ has empty interior in Ω (Theorem 1.2)
- for every $N \in \mathbb{N}$ the set $\mathcal{D}_N$ is an open dense subset of Ω, which immediately implies that
- the set $\mathcal{D}_\infty$ is of the second Baire category.

In [6] the author asked if the set $\mathcal{T}$ is also of the second Baire category. This question was positively answered by Z. Gong, W. Qiu and Y. Li in [18].

However, the question about metric typicality of $\mathcal{T}$, formulated in [8], remained open until the publication of [21]:

Question [BBR] Is it true that $\mathbb{P}(\mathcal{T}) = 1$ provided that $R > 1/4$ is large enough?

The answer to the above question, along with several more general statements, is one of the main results of this dissertation. Precisely, we prove the following.

Theorem 1.6. *Let $R > 1/4$. Consider $\Omega = \mathbb{D}(0,R)^{\mathbb{N}}$ equipped with the product distribution $\mathbb{P} := \otimes_{n=0}^{\infty} \lambda_R$. Let*

$$\mathcal{T} = \{\omega \in \Omega : J_\omega \quad \text{is totally disconnected}\}$$

Then $\mathbb{P}(\mathcal{T}) = 1$.

In other words, a typical (metrically) Julia set J_ω is totally disconnected.

One might expect that the statement of Theorem 1.6 is a consequence of how for $R > 1/4$ the disc $\mathbb{D}(0,R)$ intersects the complement of the Mandelbrot set $\mathcal{M}$. However, the following generalization shows that the analogous statement holds true also for domains which are completely contained in the Mandelbrot set.

Theorem 1.7. *Let V be an open and bounded set such that $\mathbb{D}(0,\frac{1}{4}) \subset V$ and $V \neq \mathbb{D}(0,\frac{1}{4})$. Consider the space $\Omega = V^{\mathbb{N}}$ equipped with the product $\mathbb{P}$ of uniform distributions on V. Then for $\mathbb{P}$–almost every sequence $\omega \in \Omega$ the Julia set J_ω is totally disconnected.*

Theorem 1.7 leads immediately to the following corollary, which might be surprising from the point of view of autonomous holomorphic dynamics.

Corollary 1.8. *Let $\Omega = B^{\mathbb{N}}$ where B is the main cardioid of the Mandelbrot set, and let Ω be equipped with the product of uniform distributions on B. Then for almost every sequence $\omega \in \Omega$ the Julia set J_ω is totally disconnected.*

Finally the last part of the chapter on quadratic dynamics deals with the properties of the maximal measure on the random quadratic Julia sets, that is, the equilibrium measure for potential theory. We shall denote this measure by μ_ω, as it depends on the sequence $\omega \in \Omega$. Let us introduce the global Green function defined by

$$\mathbf{g}(0) := \int g_\omega(0) d\mathbb{P}(\omega),$$

where g_ω is the standard Green function associated with the non-autonomous Julia set J_ω. Under some assumptions on the dynamical system, which will be later explained in greater detail, we prove a formula for the Hausdorf dimension of the maximal measure.

Theorem 1.9. *Let $\mathcal{S}$ be a bounded random system of independent quadratic maps. Assume additionally that $\mathcal{S}$ is typically fast escaping.*

Then for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\dim_H(\mu_\omega) = \frac{\log 2}{\chi} = \frac{\log 2}{\log 2 + \mathbf{g}(0)} < 1 \tag{1.4}$$

A number of other results on the dependence of the dimension on the parameter space are also established. Below is one such theorem for the product of disks.

Theorem 1.10. *Let $\Omega = D(0, R)$, and let us denote $\mathbf{g}(0) = \mathbf{g}_R(0)$ to underline the dependence on R. Then the function*

$$R \to \frac{\log 2}{\log 2 + \mathbf{g}_R(0)}$$

is continuous for $R \in (0, \infty)$.

Actually, for $R < \frac{1}{4}$ the function in the theorem above is not just continuous but constant, since as we will later show for $R < \frac{1}{4}$ we have $\mathbf{g}_R(0) = 0$. Finally one somewhat surprising result establishes that when the parameter space is the product of small disks around some value c outside of the Mandelbrot set, the dimension does not depend on the size of this disk, i.e. it is constant. For more details see Section 2.3.8 and in particular Proposition 2.79.

1.2 Random exponential family

We consider a sequence ω of positive bounded real numbers, i.e. $\omega = (\lambda_n)_{n=1}^{\infty}$. Let us denote $f_{\lambda_n}(z) = \lambda_n e^z$ and let F_ω^n be the composition $F_\omega^n := f_{\lambda_n} \circ f_{\lambda_{n-1}} \circ ... \circ f_{\lambda_1}$. We denote the Fatou and Julia sets by $\mathcal{F}_\omega$ and $\mathcal{J}_\omega$ respectively. Throughout the dissertation we will sometimes consider autonomous sequences, i.e. such $(\lambda_n)_{n=1}^{\infty}$ where $\lambda_n = \lambda$ for all n. In those cases we drop the above notation for the more standard one, and write f_λ^n instead of F_ω^n as well as $\mathcal{J}_\lambda$ in place of $\mathcal{J}_\omega$.

For a sequence of iterates of λe^z, where λ is fixed, it is well known that the Julia set is the whole plane for $\lambda > \frac{1}{e}$. In particular the case of $\lambda = 1$ is a famous result of Misiurewicz [25]. For $\lambda = \frac{1}{e}$ the picture is different, as there is a nonempty Fatou set, namely the parabolic basin of the fixed point 1. It should not be surprising then that $\frac{1}{e}$ plays a special role in the non-autonomous case as well.

We open our considerations by stating a theorem which was an auxiliary result in [30].

Theorem 1.11. *Let $M, \bar{\lambda}$ be positive real numbers satisfying $M > \bar{\lambda} > \frac{1}{e}$. Let $\omega = (\lambda_n)_{n=1}^{\infty}$ be a sequence of real numbers satisfying $M > \lambda_n > \bar{\lambda}$ for all n. Then $\mathcal{J}_\omega = \mathbb{C}$.*

We shall present some key corollaries arising from the proof of the above. Since those were not explicitly stated by the authors of [30], we will also provide their proof (or rather a modification of it). These generalizations of Theorem 1.11 give us sufficient conditions for the Julia set being the whole plane and will be crucial later in the dissertation even when dealing with sequences more general than those satisfying the assumptions of Theorem 1.11. Most of these results come from [20]. Some of them were also part of the author's master thesis, and thus are not to be considered original results for the purpose of this dissertation. Nevertheless they are included in Chapter 3 for a full overview. The author was careful to specify which results are fully part of the doctoral thesis.

The main theorems of this part of the dissertation give examples of sequences converging to $\frac{1}{e}$, but for which the Julia sets are the whole plane. They also provide some quantitative results in this area. We prove the following two theorems, note that Theorem 1.13 is a corollary from the proof of Theorem 1.12.

Theorem 1.12. *There exists a sequence $\omega = (\lambda_n)_{n=1}^{\infty}$ satisfying $\lambda_n \searrow \frac{1}{e}$ such that the Julia set is the whole plane. The sequence can be chosen to satisfy $\limsup_{n\to\infty}(\lambda_n - e^{-1})n^{\frac{1}{2}} < \infty$.*

Theorem 1.13. *There exists a constant $C > 0$ such that if $\omega = (\lambda_n)_{n=1}^{\infty}$ is a sequence satisfying $\lambda_n = \frac{1}{e} + \frac{C}{n^p}$ for $p < \frac{1}{2}$, then $\mathcal{J}_\omega = \mathbb{C}$.*

Finally, in the autonomous setting the trajectory of 0 plays a key role, as this is the unique singular value. For $\lambda > \frac{1}{e}$ the iterates of 0 converge to infinity, and the Julia set is the whole plane. But for $0 < \lambda \leqslant \frac{1}{e}$ the iterates of the point 0 are bounded and there is a non-empty

Fatou set. As we shall see, one can also easily prove in the non-autonomous setting that if the iterates of 0 are bounded, the Fatou set is non-empty. The converse however is not true. Denote by $\sigma\omega$ the sequence ω shifted to the left. We prove the following result.

Theorem 1.14. *There exists a sequence $\omega = (\lambda_n)_{n=1}^{\infty}$ satisfying $\lambda_n > 0$ such that*

$$\forall_{k\in\mathbb{N}} \lim_{n\to\infty} F^n_{\sigma^k\omega}(0) = \infty,$$

but $\mathcal{J}_\omega \neq \mathbb{C}$.

1.3 Organization of the dissertation

The dissertation is split into two main parts.

Chapter 2 contains results relating to the random quadratic family. Section 2.1 provides necessary preliminaries, mainly concerning a non-autonomous generalization of the Green's function and related theory. Then Section 2.2 provides the proof of the main result. Finally Section 2.3 concerns the dimension of the maximal measure on the non-autonomous quadratic Julia sets.

Chapter 3 contains results relating to the random exponential family. Section 3.1 serves as a short introduction to the autonomous case. Section 3.2 deals with sequences which are bounded away from $\frac{1}{e}$, while the next section drops that assumption and contains some of the main results of the chapter. Finally, Section A contains a modified proof from [30], which is a key component for other results in this chapter.

Chapter 2

Random quadratic maps

In this chapter we prove several results on the non-autonomous quadratic family, our primary focus being connectedness of the Julia set for random quadratic iterations. The main result is Theorem 1.7, which proves that the Julia set is almost always totally disconnected for a wide class of spaces. In particular this answers an open question from [8], whether the Julia set is totally disconnected if we choose the values c_n independently and with uniform distribution from disks of radius $R > \frac{1}{4}$ centered at 0. Actually, the result we prove is stronger than the above statement and can be applied to many other sequence spaces, not just products of disks. The above theorem and more come from a joint work with Anna Zdunik in [21].

Section 2.3 concerns the subject of the equilibrium measure (from potential theory) defined on the above Julia sets. Among other results, we prove a formula for the Hausdorff dimension of said measure.

2.1 Motivation and preliminaries

We will open with a short introduction to the autonomous theory of quadratic iteration. In this section we consider composition of the function z^2+c where c is fixed. In a sense this is the only important class of quadratic functions, as every other is conjugated to one of this form. Moreover for different values of c_1, c_2 the functions z^2+c_1 and z^2+c_2 are not holomorphically conjugated, thus the parameter space of c represents the different holomorphic conjugacy classes of quadratic functions on the Riemann sphere. Every polynomial (and quadratic maps in particular) has a superattracting fixed point at infinity, and thus a basin of infinity $A(\infty)$ in its Fatou set. The following classic result can be found for instance in [11].

Theorem 2.1. *The Julia set of any polynomial of degree $d \geqslant 2$ coincides with the set $\partial A(\infty)$.*

This along with the following definition can give a very general idea of what the Julia set may look like in the polynomial case.

Definition 2.2 The filled in Julia set $\mathcal{K}$ of a polynomial P is the union of the Julia set and all bounded components of the Fatou set.

Remark 2.3 If $\mathcal{K}$ is the filled in Julia set for a polynomial P then $z \in \mathcal{K}$ if and only if the iterates $P^n(z)$ are bounded.

The above remark gives a simple method for computing pictures of the (filled in) Julia set. On the following pages the reader will find examples of Julia sets of $z^2 + c$, for various values of c, all computer generated by the method above. Finally, recall the following classic theorem.

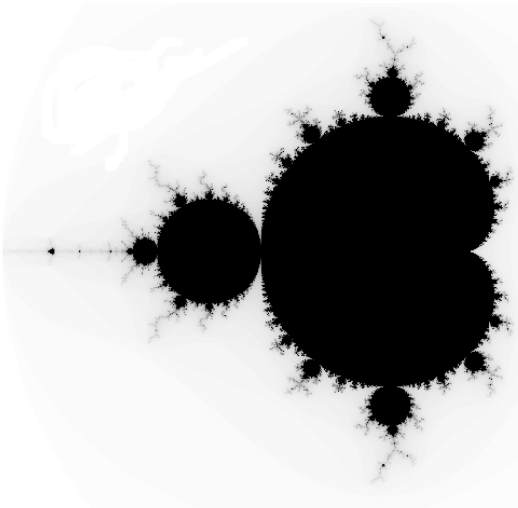

Figure 2.1: The Mandelbrot set

Theorem 2.4. *Let $f_c(z) = z^2 + c$. Then the Julia set of the function f_c is connected if and only if the iterates of the critical point 0 under f_c are bounded. In the other case the Julia set is totally disconnected.*

The above dichotomy makes it natural to divide the parameter plane into two sets: one of parameters c for which the Julia set is totally disconnected, and the other for which it is connected. The set of c for which $z^2 + c$ has a connected Julia set is the famous Mandelbrot set.

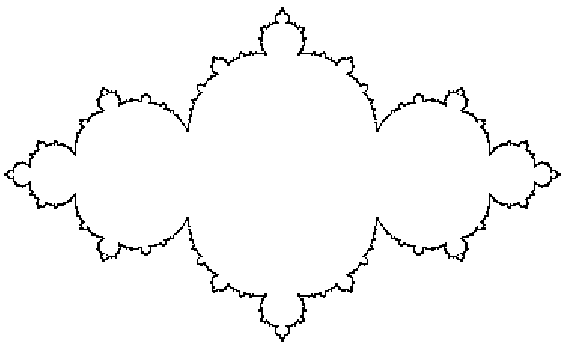

Figure 2.2: Connected Julia set for $c = -0.70$

Let us justify briefly the pictures provided in this section. The value $c = -0.7$ is inside the main cardioid of the Mandelbrot set, and thus the Julia set is connected. Its interior is the bounded Fatou component on which the iterates of $z^2 - 0.7$ converge to an attracting fixed point. Similarly for $c = -0.15 + 0.6i$. The value $c = 0.4$ is outside of the Mandelbrot set, and thus the Julia set is totally disconnected.

The following proposition from [11] gives a basic description of the Mandelbrot set.

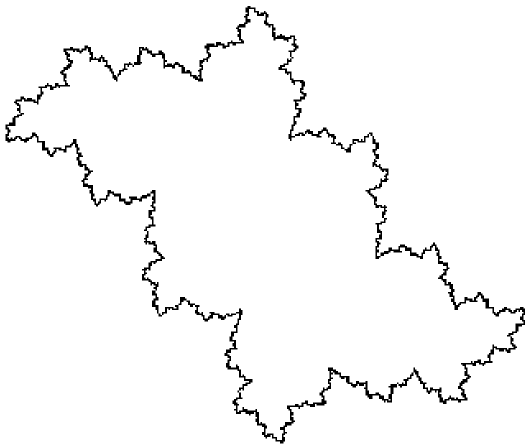

Figure 2.3: Connected Julia set for $c = -0.15 + 0.6i$

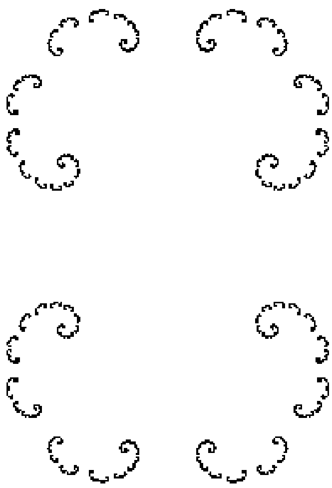

Figure 2.4: Totally disconnected Julia set for $c = 0.40$

Proposition 2.5. *The Mandelbrot set is a closed simply connected subset of the disk* $D(0, 2)$ *which meets the real axis in the interval* $[-2, \frac{1}{4}]$.

The Mandelbrot set is a complex fractal whose properties are not within the scope of this dissertation. We end this topic stating the following famous conjecture, by many considered one of the central open problems of holomorphic dynamics. For more on this the reader can refer to [16].

Conjecture 2.6. *The Mandelbrot set is locally connected.*

2.1.1 Green's function on $\mathcal{A}_\omega$.

Let us introduce the notion of the Green's function, along with some of its applications to complex dynamics. Its non autonomous generalization will also be an important tool later in the dissertation.

Definition 2.7 Let D be a proper subdomain of $\hat{\mathbb{C}}$. A Green's function for D is a map $g_D : D \times D \to (-\infty, \infty]$, such that for each $w \in D$:

1. $g_D(\cdot, w)$ is harmonic on $D \setminus \{w\}$ and bounded outside of every neighbourhood of w
2. $g_D(w, w) = \infty$ and as $z \to w$ we have: $g_D(z, w) = \begin{cases} \ln|z| + O(1), & w = \infty \\ -\ln|z - w| + O(1), & w \neq \infty \end{cases}$
3. $g_D(z, w) \to 0$ as $z \to \zeta$ for nearly every $\zeta \in \partial D$

The statement "nearly every" in the above definition means everywhere outside of some polar set, which are negligible sets from the point of view of potential theory. The formal definition will be supplied later, see Definition 2.35.
In the context of this dissertation we will always consider the Green's function of the basin of infinity (for a sequence of quadratic polynomials) with a pole at infinity. That is, keeping with the notation from the above definition, in our case $D = \mathcal{A}(\infty)$ and $g_D(z, w) = g_{\mathcal{A}(\infty)}(z, \infty)$. For brevity, let P be some quadratic polynomial with basin of infinity $\mathcal{A}_P(\infty)$, we denote $g_P(z) = g_{\mathcal{A}(\infty)}(z, \infty)$. Recall the following result, which ties in Green's function with complex dynamics.

Theorem 2.8. *Let* $\phi(z)$ *be the holomorphic function conjugating* P *to* z^2 *on a neighbourhood of infinity. Then* $\ln|\phi(z)|$ *coincides with* $g_P(z)$ *on said neighbourhood. Moreover the following functional equation holds:*

$$g_P(P(z)) = 2g_P(z)$$

for $z \in \mathcal{A}(\infty)$.

The above in particular means that level curves of the Green's function are mapped onto level curves by P, which makes the Green function a convenient tool for studying escape rates to infinity (it has a wide variety of deeper applications, but that will be its main use in this dissertation). Although the statement has to be slightly modified, we shall see that a similar theorem will hold in non-autonomous dynamics, which we will put to use later in the dissertation.

Notation. We now define the non-autonomous Green's function and prove some of its properties, which are mostly generalizations of their autonomous counterparts. We shall use the following notation. For every $r > 0$ denote $\mathbb{D}_r := \mathbb{D}(0, r)$ and $\mathbb{D}_r^* := \mathbb{C} \setminus \overline{\mathbb{D}}_r$.
We write $\omega = \{(c_n)\}$ to denote an infinite sequence of parameters, even if no probability distribution is specified. For most of our applications we will have $n \in \mathbb{N}$ although there will be cases where we consider sequences such that $n \in \mathbb{Z}$. For brevity we keep the same notation in both cases. For such a sequence we use both notations:

$$f_\omega^n = f_{c_{n-1},c_{n-2},\dots,c_1,\dots c_0} = f_{c_{n-1}} \circ f_{c_{n-2}} \circ \cdots \circ f_{c_1} \circ f_{c_0}.$$

For a full overview of classic Green function theory we refer to [27]. Let us recall the well-known formula for the autonomous case: for the map $f_c(z) := z^2 + c$ and its basin of infinity $\mathcal{A}_c$, the Green's function with a pole at infinity is given by:

$$g_c(z) = \lim_{n\to\infty} \frac{1}{2^n} \log |f_c^n(z)|.$$

The following proposition, which is a non-autonomous generalization of the above, comes from [17]. Actually the authors of [17] worked in a more general setting, but the reformulation below is enough for our purposes. We provide a sketch of the proof, slightly modified compared to the original one. For more details and an overview of Green's function in non-autonomous polynomial dynamics we refer to the original authors' work in [17].

Proposition 2.9. *Let V be a bounded Borel subset of $\mathbb{C}$, put $\Omega = V^{\mathbb{N}}$. Let μ be a Borel probability measure on V, and $\mathbb{P}$ - the product distribution on Ω generated by μ. For every $\omega \in \Omega$ the following limit exists: $g_\omega : \mathcal{A}_\omega \to \mathbb{R}$*

$$g_\omega(z) = \lim_{n\to\infty} \frac{1}{2^n} \log |f_\omega^n(z)|. \tag{2.1}$$

The function $z \mapsto g_\omega(z)$ is the Green's function on $\mathcal{A}_\omega$ with pole at infinity. Putting $g_\omega \equiv 0$ on the complement of $\mathcal{A}_\infty$, g_ω extends continuously to the whole plane. With z fixed, the function $\omega \mapsto g_\omega(z)$ is $\mathbb{P}-$ measurable.

Sketch of a proof. We start with proving existence of the limit $g_\omega(z)$ and its harmonicity. Let R be large enough so that for all $c \in V$ we have $|f_c(z)| > |z|$ for $|z| > R$. Also, let R be large enough so that for all $|z| > R$ and all $c \in V$ we have

$$\frac{|z|^2}{R} \leqslant |z^2 + c|, \tag{2.2}$$

this can clearly be done because V is a bounded set. The last condition may seem somewhat arbitrary, but it will become clear shortly. Let us denote $U_k = (f_\omega^k)^{-1}(\mathbb{D}_R^*)$. Then we have $U_k \subset U_{k+1} \subset \mathcal{A}_\omega(\infty)$ as well as $\mathcal{A}_\omega(\infty) = \bigcup_{k=1}^{\infty} U_k$. Let us denote $g_n(z) = \frac{1}{2^n} \ln \frac{|f_\omega^n(z)|}{R}$ for $z \in U_n$. Note that $g_n(z) \leqslant g_{n+1}(z)$. Indeed, this inequality is equivalent to

$$2\ln\Big(\frac{|f_\omega^n(z)|}{R}\Big) \leqslant \ln\Big(\frac{|f_\omega^{n+1}(z)|}{R}\Big)$$

which can be rewritten as

$$\ln\Big(\frac{|f_\omega^n(z)^2|}{R}\Big) \leqslant \ln\Big(|f_\omega^n(z)^2 + c_{n+1}|\Big).$$

Since $|f_\omega^n(z)| > R$, then the last inequality is true for all $z \in U_n$ by (2.2). Thus $(g_n(z))_{n=1}^\infty$ is an increasing (and also locally bounded) sequence of harmonic functions (each $g_n(z)$ defined on U_n), convergent to a harmonic function $G(z)$ which can be defined on $\mathcal{A}_\omega(\infty)$. Since $g_n(z)$ has a limit we have

$$G(z) = \lim_{n\to\infty} \frac{1}{2^n} \ln\left|\frac{f_\omega^n(z)}{R}\right| = \lim_{n\to\infty} \frac{1}{2^n} \ln |f_\omega^n(z)| = g_\omega(z).$$

Thus we conclude that $g_\omega(z)$ is a well defined harmonic function in $\mathcal{A}_\omega$. We now move on to proving that this is the Green's function for $\mathcal{A}_\omega$.

Since the set V is bounded we know that for every $\epsilon > 0$, there exists $R_\epsilon > 0$ such that for all $|z| > R_\epsilon$ and $c \in V$ we have

$$(1-\epsilon)|z|^2 \leqslant |f_c(z)| \leqslant (1+\epsilon)|z|^2.$$

Without loss of generality assume also that R_ϵ is large enough so that for all $|z| > R_\epsilon$ and all $c \in V$ we also have $|f_c(z)| > R_\epsilon$. This means that for a sequence $\omega \in \Omega$ we have the following inequalities for the iterates

$$(1-\epsilon)^{2^n-1}|z|^{2^n} \leqslant |f_\omega^n(z)| \leqslant (1+\epsilon)^{2^n-1}|z|^{2^n}.$$

Finally taking the logarithm yields

$$\ln(1-\epsilon) + \ln(z) \leqslant \frac{1}{2^n} \ln|f_\omega^n(z)| \leqslant \ln(1+\epsilon) + \ln(z).$$

It is clear that at ∞ we have

$$g_\omega(z) = \ln(z) + o(1).$$

We now prove that $\lim\limits_{z \to \partial\mathcal{A}_\omega} g_\omega(z) = 0$. Let us denote $\omega_m = \sigma^m(\omega)$, that is the sequence ω starting from index m. We have

$$g_\omega(z) = \lim_{k\to\infty} \frac{1}{2^k} \ln|f_\omega^k(z)| = \frac{1}{2^m} \lim_{k\to\infty} \frac{1}{2^{k-m}} \ln|f_{\omega_m}^{k-m}(f_\omega^m(z))|.$$

Pick a large R so that $\{|z| > R\} \subset \mathcal{A}_\omega$ for all $\omega \in \Omega$. It is elementary to see that for any second degree polynomial g there exists a constant M such that for all $|z| < R$ we have

$$|g^n(z)| < |M|^{2^n}.$$

Let $c > 0$ be such that $c > |c'|$ for all $c' \in V$, and pick $M > 0$ as above for the function $g(z) = z^2 + c$. In that case it is easy to see that we have

$$|f_\omega^n(z)| < M^{2^n},$$

for any $\omega \in \Omega$. From this we deduce that there exists a constant C_R independent of k, m, ω such that for $|z| \leqslant R$ the following holds

$$\frac{1}{2^{k-m}} \ln|f_{\omega_m}^{k-m}(z)| < C_R.$$

This implies $g_\omega(z) < \frac{C_R}{2^m}$ on the set $V_m = \{|f_\omega^m(z)| \leqslant R\}$. Note that V_m is a descending (i.e. $V_{m+1} \subset V_m$) sequence of sets containing the Julia set. This concludes the proof that $\lim\limits_{z \to \partial\mathcal{A}_\omega} g_\omega(z) = 0$. We now have shown that $g_\omega(z)$ is the Green's function for $\mathcal{A}_\omega$. Finally denote by $\log^+(z)$ the function $\max(\log(|z|), 0)$, then it is clear that one can define each $g_\omega(z)$ on the whole plane as the limit

$$g_\omega(z) = \lim_{n\to\infty} \frac{1}{2^n} \ln^+(|f_\omega^n(z)|).$$

Thus $\omega \mapsto g_\omega$ is measurable as a limit of measurable functions. This concludes the proof.

□

The authors of [17] also note the following corollary of the above.

Corollary 2.10. *For every $\omega \in \Omega$ the basin of infinity $\mathcal{A}_\omega$ is a regular set for logarithmic potential theory.*

Finally the following is another simple corollary, which will be useful for finding estimates for the Green function. As with the previous remarks, it is also analogous to a property of classic autonomous Green functions.

Corollary 2.11. *We have*

$$g_{\sigma\omega}(f_\omega(z)) = 2g_\omega(z). \tag{2.3}$$

Proof. This follows directly from the formula (2.1), defining the Green's function g_ω. □

2.1.2 Estimates for Green's function.

Proposition 2.12. *For every $\epsilon > 0$, $R > 0$ there exists $R_1 > 0$ such that for every $R_0 > R_1$ and every $\omega \in \mathbb{D}(0,R)^{\mathbb{N}}$ we have*

$$f_\omega(\mathbb{D}^*_{R_0}) \subset \mathbb{D}^*_{2R_0}, \tag{2.4}$$

$$\mathbb{D}^*_{R_0} \subset \mathcal{A}_\omega, \tag{2.5}$$

$$|g_\omega(z) - \log|z|| < \epsilon \quad \textit{in} \quad \mathbb{D}^*_{R_0}. \tag{2.6}$$

Proof. First, since $|c_n| < R$ for all n, one can choose $R_1 > 0$ to ensure

$$f_\omega(\mathbb{D}^*_{R_0}) \subset \mathbb{D}^*_{2R_0} \tag{2.7}$$

for every $R_0 \geqslant R_1$. This guarantees (2.4) and(2.5).

Let $a_0(z) = \log|z|$, $a_n(z) = \frac{1}{2^n}(\log|f^n_\omega(z)|)$ for $n \geqslant 1$, and note that we have for all $n \geqslant 0$:

$$\begin{aligned} a_{n+1}(z) &= \frac{1}{2^{n+1}}(\log|f^{n+1}_\omega(z)|) \\ &= \frac{1}{2}(\frac{1}{2^n}\log|(f^n_\omega(z))^2 + c_n|) \\ &= \frac{1}{2^n}(\log|f^n_\omega(z)| + \frac{1}{2}\log|1 + \frac{c_n}{(f^n_\omega(z))^2}|) \\ &= a_n(z) + \frac{1}{2^{n+1}}(\log|1 + \frac{c_n}{(f^n_\omega(z))^2}|). \end{aligned}$$

In particular this means that $g_\omega(z) = a_0 + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}(\log|1 + \frac{c_n}{(f^n_\omega(z))^2}|) = \log|z| + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}(\log|1 + \frac{c_n}{(f^n_\omega(z))^2}|)$.

Choosing $R_0 \geqslant R_1$ large enough we can have $\sum_{n=0}^{\infty} \frac{1}{2^{n+1}}(\log|1 + \frac{c_n}{(f^n_\omega(z))^2}|) < \epsilon$ on $\mathbb{D}^*_{R_0}$. This yields

$$|g_\omega(z) - \log|z|| < \epsilon$$

which concludes the proof. □

The following is an immediate consequence of item (2.4) of Proposition 2.12.

Corollary 2.13. *For every $\omega \in \mathbb{D}(0,R)^{\mathbb{N}}$ we have $K_\omega \subset \mathbb{D}_{R_0}$.*

Determining constants. Now, for every R we fix some $R_0 > R$ satisfying the conditions formulated in Proposition 2.12 with $\epsilon := 1$, in particular,

$$|g_\omega(z) - \log|z|| < 1 \quad \text{in} \quad \mathbb{D}^*_{R_0} \tag{2.8}$$

Next, for every $R > 0$ let us fix also some $\tilde{R}_0 \in (R_0, R_0^2 - R)$, say,
$\tilde{R}_0 = \frac{1}{2}(R_0 + R_0^2 - R)$. Then for every $\omega \in \mathbb{D}(0,R)^{\mathbb{N}}$ $f_\omega^{-1}(\mathbb{D}_{\tilde{R}_0}) \subset \mathbb{D}_{R_0}$. By Proposition 2.12,

$$G = G(R) := \sup_{|R_0| \leqslant |z| \leqslant \tilde{R}_0, \omega \in \mathbb{D}(0,R)^{\mathbb{N}}} (g_\omega(z)) < \infty \tag{2.9}$$

Proposition 2.14. *For every $R > 0$, for every $\omega \in \mathbb{D}(0,R)^{\mathbb{N}}$,*

$$\sup_{z \in \mathbb{D}_{R_0}} g_\omega(z) \leqslant \log R_0 + 1.$$

In particular, the function $\mathbb{D}(0,R)^{\mathbb{N}} \ni \omega \mapsto g_\omega(0)$ is bounded above, and

$$\sup_{\omega \in \mathbb{D}(0,R)^{\mathbb{N}}} g_\omega(0) \leqslant \log R_0 + 1.$$

Proof. Since $|g_\omega(z)| \leqslant \log|z| + 1$ in $\mathbb{D}^*_{R_0}$, we have, in particular, $g_{\omega|\partial\mathbb{D}_{R_0}} \leqslant \log R_0 + 1$. By the Maximum Principle, the same estimate holds in the whole disc $\mathbb{D}_{R_0}$, in particular, for $z = 0$. So, $\sup_{\omega \in \mathbb{D}(0,R)^{\mathbb{N}}} g_\omega(0) \leqslant \log R_0 + 1$. □

2.1.3 Escape rate of the critical point.

We introduce the following definition.

Definition 2.15 Let $\omega \in \mathbb{D}(0,R)^{\mathbb{N}}$. For every $z \in \mathbb{D}(0,R_0)$ we denote by $k(z,\omega)$ the escape time of z from $\mathbb{D}_{R_0}$:

$$k(z,\omega) = \begin{cases} \min\{j : |f_\omega^j(z)| \geqslant R_0\} & \text{if} \quad z \in \mathcal{A}_\omega \\ \infty \quad \text{if} \quad z \in K_\omega. \end{cases} \tag{2.10}$$

Proposition 2.16. *For every $z \in \mathcal{A}_\omega \cap \mathbb{D}_{R_0}$*

$$(\log R_0 - 1)2^{-k(z,\omega)} \leqslant g_\omega(z) \leqslant 2(\log R_0 + 1)2^{-k(z,\omega)} \tag{2.11}$$

Proof. Recall that, by (2.3),

$$g_\omega(z) = g_{\sigma^{k(z,\omega)}\omega}(f_\omega^{k(z,\omega)}(z)) \cdot 2^{-k(z,\omega)},$$

and

$$g_{\sigma^{k(z,\omega)}\omega}(f_\omega^{k(z,\omega)}(z)) = 2g_{\sigma^{k(z,\omega)-1}\omega}(f_\omega^{k(z,\omega)-1}(z)) \leqslant 2(\log R_0 + 1),$$

since $f_\omega^{k(z,\omega)-1}(z)) \in \mathbb{D}_{R_0}$.

On the other hand,

$$g_{\sigma^{k(z,\omega)}\omega}(f_\omega^{k(z,\omega)}(z)) \geqslant \log|f_\omega^{k(z,\omega)}(z)| - 1 \geqslant \log R_0 - 1.$$

This implies that (2.11) holds. □

Our estimates show that the distribution of the random variable $\log^- g_\omega(0)$ is roughly the same as that of $k(0,\omega)$.

Definition 2.17 Let V be a bounded Borel subset of $\mathbb{C}$, $V \subset \mathbb{D}(0,R)$. Let μ be a probability Borel measure on V, and let $\mathbb{P}$ be the product distribution on $V^{\mathbb{N}}$ generated by μ. Fix the values $R_0 = R_0(R)$ and $G = G(R)$ according to (2.9). We say that the critical point 0 is *typically fast escaping* if there exists $\gamma > 0$ such that

$$\mathbb{P}\left(\left\{\omega \in \Omega : g_\omega(0) < \frac{G}{2^k}\right\}\right) < e^{-\gamma k} \tag{2.12}$$

2.2 Total disconnectedness; the main result

In this section we shall provide a sufficient condition for a sequence ω to have a totally disconnected Julia set J_ω. Thus to finish the proof of Theorem 1.7 it will remain to show that the condition from this section is satisfied for almost every sequence ω.

Recall that in Section 2.1.2 we assigned, for every $R > 0$ the values R_0 and $\tilde{R}_0$.

Lemma 2.18. *Choose an arbitrary radius $\rho \in [R_0, \tilde{R}_0]$ and let $D := \mathbb{D}_\rho$. Then the filled -in Julia set K_ω, i.e. the set of points z whose trajectories $f^n_\omega(z)$ do not escape to ∞ can be written as*

$$K_\omega := \bigcap_{k\in\mathbb{N}} (f^k_\omega)^{-1}(D).$$

Proof. Since the trajectory of every point $z \in \bigcap_{k\in\mathbb{N}} (f^k_\omega)^{-1}(D)$ is bounded, it is clear that

$$\bigcap_{k\in\mathbb{N}} (f^k_\omega)^{-1}(D) \subset K_\omega.$$

On the other hand, if $z \notin \bigcap_{k\in\mathbb{N}} (f^k_\omega)^{-1}(D)$ then, for some $k \in \mathbb{N}$, $|f^k_\omega(z)| \geqslant \rho \geqslant R_0$, and it follows from the choice of R_0 that

$$f^n_\omega(z) = f^{n-k}_{\sigma^k\omega}(f^k_\omega(z)) \xrightarrow[n\to\infty]{} \infty,$$

so $z \notin K_\omega$. □

Observe that $\bigcap_{k\in\mathbb{N}} (f^k_\omega)^{-1}(D)$ is an intersection of a descending sequence of sets. At each level k the set

$$D^k(\omega) := (f^k_\omega)^{-1}(D)$$

is a union of pairwise disjoint topological discs $D^k_j(\omega)$, each of them being mapped by f^k_ω onto D with some degree $d^k_j \leqslant 2^k$.

Now put $\rho = \tilde{R}_0$, i.e., put $D := \mathbb{D}_{\tilde{R}_0}$. The following proposition formulates, in terms of degree of the maps $f^k_\omega : D^k_j(\omega) \to D$, a sufficient condition for total disconnectedness of the Julia set J_ω.

Proposition 2.19. *Let $\omega \in \mathbb{D}(0,R)^{\mathbb{N}}$. If there exists $N \in \mathbb{N}$ such that for infinitely many integers $k \in \mathbb{N}$, for each component $D^k_j(\omega)$ of the set $D^k(\omega) = (f^k_\omega)^{-1}(D)$ the degree of the map*

$$f^k_\omega : D^k_j(\omega) \to D$$

is at most N, then the Julia set J_ω is totally disconnected.

Proof. In what follows, to simplify the notation we write D_j^k and D^k in place of $D_j^k(\omega)$ and $D^k(\omega)$, respectively. Recall that R_0 and $\tilde{R}_0$ were chosen in Section 2.1.2 in such a way that

$$\forall_{\nu\in\mathbb{D}(0,R)^{\mathbb{N}}} \quad \mathbb{D}^*_{R_0} \subset A_\omega \quad \text{and} \quad f_\nu^{-1}(\mathbb{D}_{\tilde{R}_0}) \subset \mathbb{D}_{R_0}. \tag{2.13}$$

Denote by P the annulus

$$P = \{z : R_0 < |z| < \tilde{R}_0\}.$$

For every $k \in \mathbb{N}$ and for every component D_j^k of D^k the map $f_\omega^k : D_j^k \to D$ is a proper holomorphic map onto D.

By the assumption there exists an increasing sequence of positive integers $\{k_n\}$ such that the maps

$$f_\omega^{k_n} : D_j^{k_n} \to D$$

have degree at most N for all j.

Now, let us divide the annulus P into N nested geometric annuli with the same modulus M. Just as P, these N annuli all lie in the intersection of D and all basins of infinity $\mathcal{A}_\nu$, $\nu \in \mathbb{D}(0,R)^{\mathbb{N}}$, by (2.13).

Let us pick a point z in the Julia set J_ω, and let $D_{j_n}^{k_n}$ be the component of D^{k_n} such that $z \in D_{j_n}^{k_n}$.
Since the degree of $f_\omega^{k_n}$ on $D_{j_n}^{k_n}$ is at most N, one of the N annuli contains no critical values of $f_\omega^{k_n}$; let us choose such an annulus and denote it by P_n. Consider now the (possibly smaller) disc $D' \subset D$, bounded by the outer boundary circle of the annulus P_n, and let ${D'}_{j_n}^{k_n}$ be the connected component of $(f_\omega^{k_n})^{-1}(D')$, containing the point z. The map $f_\omega^{k_n} : {D'}_{j_n}^{k_n} \to D'$ is also proper, and the preimage of the annulus P_n under this map, denoted here by P'_n is again a (topological) annulus. The map $f_\omega^{k_n}$ restricted to P'_n is a covering map of degree at most N, so, the modulus of P'_n is at least M/N.

The point z lies in some connected component of $(f_\omega^{k_n})^{-1}(\mathbb{D}_{R_0})$ contained in $D_{j_n}^{k_n}$, so, in particular, it lies in the bounded component of the complement of the annulus P'_n.

Observation. Now, let us recall that, according to the choice of R_0 and $\tilde{R}_0$, for every $\nu \in \mathbb{D}(0,R)^{\mathbb{N}}$ we have that

$$f_\nu^{-1}(\mathbb{D}_{\tilde{R}_0}) \subset \mathbb{D}_{R_0}.$$

So, in particular, for every $k \geqslant 1$, $f^{-1}_{\sigma^{k-1}\omega}(\mathbb{D}_{\tilde{R}_0}) \subset \mathbb{D}_{R_0}$. This also implies that for any k and any $\omega \in \mathbb{D}(0,R)^{\mathbb{N}}$, each component of $(f_\omega^{(k+1)})^{-1}(\mathbb{D}_{\tilde{R}_0})$ is contained in some component of $(f_\omega^k)^{-1}(\mathbb{D}_{R_0})$ (since each such component is mapped by f_ω^k onto some component of $f^{-1}_{\sigma^{k-1}\omega}(\mathbb{D}_{\tilde{R}_0})$). Clearly, the same is true with $k+1$ being replaced by any arbitrary integer $m > k$.

We shall apply this observation for $k := k_n$ and $m := k_{n+1}$. As before, for k_{n+1} we find a topological annulus P'_{n+1} of modulus at least M/N, in the connected component of $(f_\omega^{k_{n+1}})^{-1}(\mathbb{D}_{\tilde{R}_0})$ containing the point z, and such that z lies in the bounded component of the complement of the annulus P'_{n+1}.

Using the above observation we conclude that the annulus P'_{n+1} is contained in the component of $(f_\omega^{k_n})^{-1}(\mathbb{D}_{R_0})$ containing the point z; in particular, it is contained in the bounded component of the complement of P'_n.

In this way, we obtain a nested infinite sequence of disjoint annuli P'_n, each of modulus at least M/N, all contained in $\mathcal{A}_\omega$. The point z is in the bounded component of the complement of each of the annuli.

Now let us fix n and consider the topological annulus $\mathcal{P}_n$ that is bounded by the boundaries of D and $D_{j_n}^{k_n}$. Since it contains the nested sequence of annuli $P'_1, P'_2, ..., P'_n$, each of

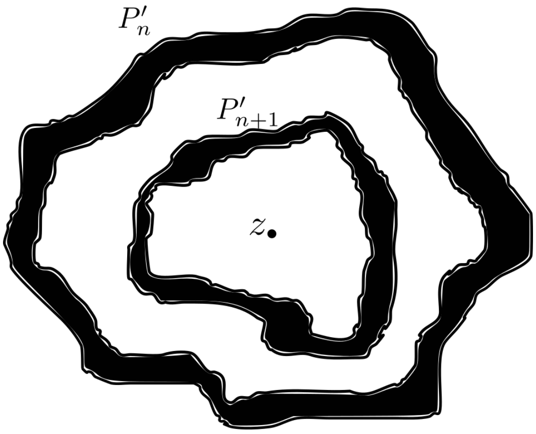

Figure 2.5: Nesting sequence of annuli P'_n

modulus at least $\frac{M}{N}$, then, by Grötzsch inequality, it must have modulus at least $n\frac{M}{N}$ (see, e.g., [4], Proposition 5.4 or [24], Theorem B5). This in turn means it contains an actual geometric annulus of modulus at least $n\frac{d}{N} - C$ (where C is some constant), which separates the components of the boundary of $\mathcal{P}_n$. (see, e.g., Theorem 2.1 in[24]). Since for every n the connected component of K_ω containing the point z, is contained in the bounded component of the complement of $\mathcal{P}_n$, this implies that the component of K_ω containing z must have arbitrarily small diameter, i.e. it is the single point z.

Since the choice of the point z was arbitrary, finally this means the Julia set is totally disconnected, which concludes the proof of Proposition 2.19. □

2.2.1 Typically fast escaping critical point and total disconnectedness

In this section, we check that the condition formulated in Definiton 2.17 is sufficient to prove that the assumptions of Propositon 2.19 are satisfied for $\mathbb{P}$–a.e. ω. More precisely, we prove the following.

Theorem 2.20. *Let V be a bounded Borel subset of $\mathbb{C}$, $V \subset \mathbb{D}(0, R)$. Let R_0, G be the values assigned to R as in Section 2.1.2. Let μ be a Borel probability measure on V and let $\mathbb{P}$ be the product distribution on $\Omega = V^{\mathbb{N}}$, generated by μ.*

If the critical point 0 *is* typically fast escaping, *i.e., if* (2.12) *holds, then the assumptions of Propositon 2.19 are satisfied for $\mathbb{P}$–almost every $\omega \in \Omega$. Thus, for $\mathbb{P}$-a.e. $\omega \in \Omega$ the Julia set J_ω is totally disconnected.*

Actually, the property from (2.12) is stronger than necessary, since to apply our proof all that is needed is for the series of probabilities to be convergent. In all our applications the bounds are indeed exponential, nevertheless the reader will soon see that the following remark is also true.

Remark 2.21 The statement of Theorem 2.20 is still true if one replaces (2.12) with

$$\sum_{k=0}^{\infty} \mathbb{P}\left(\{\omega \in \Omega : g_\omega(0) < \frac{G}{2^k}\}\right) < \infty.$$

Define the sets

$$A_k = \{\omega \in \Omega : g_\omega(0) < \frac{1}{2^k} G\}.$$

Before proving Theorem 2.20 we explain in the next proposition the role of the sets A_k in possible application of Proposition 2.19. We apply the setting and the notation of Theorem 2.20.

Proposition 2.22. *(a): If*

$$\sigma^i\omega \notin A_{k-i} \quad \textit{for all} \quad i = 0, \dots, k-1$$

then for every connected component D^k_j of the preimage $(f^k_\omega)^{-1}(D)$ the degree of the map

$$f^k_\omega : D^k_j \to D$$

is equal to 1.

(b): If the above holds for all but l indices then for every connected component of the set $f^{-k}_\omega(D)$ the degree of

$$f^k_\omega : D^k_j \to D$$

is bounded above by $N = 2^l$.

Proof. It follows from (2.9) that for every $\nu \in \Omega$ and for every $z \in D$ $g_\nu(z) \leqslant G$. Let D^k_* be some component of $(f^k_\omega)^{-1}(D)$. Consider the sequence of maps

$$D^k_* \xrightarrow[f_\omega]{} D^{k-1}_* \xrightarrow[f_{\sigma\omega}]{} D^{k-2}_* \dots \xrightarrow[f_{\sigma^{k-2}\omega}]{} D^1_* \xrightarrow[f_{\sigma^{k-1}\omega}]{} D,$$

where we denoted by D^{k-i}_* the consecutive images of D^k_* under the maps $f_\omega, f_{\sigma\omega} \dots f_{\sigma^{k-1}\omega}$. Note that $f^k_\omega : D^k_* \to D$ is just the composition of the above sequence of maps.

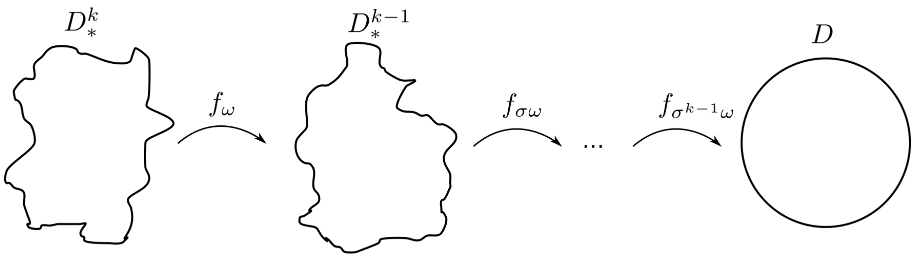

Figure 2.6: The functions $f_{\sigma^i\omega}$ are either univalent or degree two

If D^{k-i}_* contains the critical point 0 then $f_{\sigma^i\omega} : D^{k-i}_* \to D^{k-i-1}_*$ is a degree two map; otherwise it is univalent.

Now, if

$$\sigma^i\omega \notin A_{k-i} \tag{2.14}$$

then $g_{\sigma^i\omega}(0) \geqslant \frac{1}{2^{k-i}}G$, while for every $z \in D^{k-i}_*$ we have that

$$g_{\sigma^i\omega}(z) = \frac{1}{2^{k-i}} g(f^{k-i}_{\sigma^i\omega}(z)) < \frac{1}{2^{k-i}} \cdot G$$

This implies that $0 \notin D^{k-i}_*$ and, consequently, the map $f_{\sigma^i\omega} : D^{k-i}_* \to D^{k-i-1}_*$ is univalent. So, if (2.14) happens for all $i = 0, \dots k-1$ then the map

$$f^k_\omega : D^k_* \to D$$

is univalent, so of degree one.

If (2.14) fails to hold for l indices i, then for these indices the degree of the map $f_{\sigma^i\omega} : D^{k-i}_* \to D^{k-i-1}_*$ is equal to one or two, while for all other indices it is equal to one, so that the degree of the composition $f^k_\omega : D^k_* \to D$ is at most $N = 2^l$. Propositon 2.22 is proved. □

Proof of Theorem 2.20. We consider now the extended probability space

$$\tilde{\Omega} := V^{\mathbb{Z}},$$

with product probability, which we denote by $\tilde{\mathbb{P}}$. The left shift σ, considered in $\tilde{\Omega}$ is now a measurable automorphism of the space $\tilde{\Omega}$. There is a natural measurable projection

$$\pi : (\tilde{\Omega}, \tilde{\mathbb{P}}) \to (\Omega, \mathbb{P})$$

$$\tilde{\Omega} \ni (\dots c_{-2}, c_{-1}, c_0, c_1, c_2, \dots) \xrightarrow{\pi} (c_0, c_1, c_2 \dots) \in \Omega$$

This projection transforms the measure $\tilde{\mathbb{P}}$ onto the measure $\mathbb{P}$, i.e., $\tilde{\mathbb{P}} \circ \pi^{-1} = \mathbb{P}$.
For each $\tilde{\omega} \in \tilde{\Omega}$ the iterates $f^n_{\tilde{\omega}}$ are defined as previously, i.e., for $\tilde{\omega} = (\dots c_{-2}, c_{-1}, c_0, c_1, c_2 \dots)$

$$f^n_{\tilde{\omega}}(z) = f_{c_{n-1}} \circ \dots \circ f_{c_1} \circ f_{c_0}(z).$$

The Julia set is defined analogously to (1.1) and denoted by $J_{\tilde{\omega}}$. Similarly, the Green function $g_{\tilde{\omega}}$ is defined as in (2.1).

Considering the extended space $\tilde{\Omega}$ in this context may seem artificial, since the iterates $f^n_{\tilde{\omega}}$ depend only on the "future", i.e. only non-negative items $(c_j)_{j \geqslant 0}$ are used to define $f^n_{\tilde{\omega}}$ or its Julia set. Nevertheless, the proof is based on the construction of appropriate backward trajectories, which we shall describe below. Let

$$E_k = \{\tilde{\omega} \in \tilde{\Omega} : g_{\sigma^{-k}\tilde{\omega}}(0) < \frac{1}{2^k} G\}, \quad k = 0, 1, 2, \dots$$

Let us note that the following estimate holds.

Proposition 2.23. *If the critical point is typically fast escaping, i.e., if* (2.12) *holds, then*

$$\tilde{\mathbb{P}}(E_k) < e^{-\gamma k},$$

where γ comes from the estimate formulated in (2.12).

Proof. We have the estimates for the measure $\mathbb{P}$ of the set $A_k \subset \Omega$, given by (2.12).

Now, let

$$\tilde{A}_k := \pi^{-1}(A_k) = V^{\mathbb{N}} \times A_k$$

Then,

$$\tilde{\mathbb{P}}(\tilde{A}_k) = \mathbb{P}(A_k).$$

Now, note that $E_k = \sigma^k(\tilde{A}_k)$, which implies that

$$\tilde{\mathbb{P}}(E_k) = \tilde{\mathbb{P}}(\tilde{A}_k) = \mathbb{P}(A_k) < e^{-\gamma k}.$$

□

It follows from Proposition 2.23 and Borel–Cantelli Lemma that almost every $\tilde{\omega} \in \tilde{\Omega}$ belongs to at most finitely many sets E_k. This implies that there exists $K \in \mathbb{N}$ and a set $E \subset \tilde{\Omega}$ such that

$$\tilde{\mathbb{P}}(E) > 0 \tag{2.15}$$

and

$$E \cap (\bigcup_{k=K}^{\infty} E_k) = \emptyset.$$

Thus, for every $\tilde{\omega} \in E$ and every $k \geqslant K$ we have that

$$g_{\sigma^{-k}\tilde{\omega}}(0) \geqslant \frac{1}{2^k}G$$

Now, using ergodicity of the left shift σ on $\tilde{\Omega}$, we conclude that $\tilde{\mathbb{P}}$— almost surely a sequence $\tilde{\nu} \in \tilde{\Omega}$ visits E infinitely many times under the iterates of σ.

Let $k \in \mathbb{N}$. For $\nu \in \Omega$ we introduce the following.

Property (K,k): $\sigma^i\nu \in A_{k-i}$ for more than K indices $i \in \{0, \dots k-1\}$

Lemma 2.24. *If Property (K,k) holds for $\nu \in \Omega$ and if $\tilde{\nu} \in \pi^{-1}(\nu)$, then $\sigma^k\tilde{\nu} \notin E$*

Proof. Indeed, let $\tilde{\nu} \in \pi^{-1}(\nu)$. Now, $\sigma^i\nu \in A_{k-i}$ means that

$$g_{\sigma^i\tilde{\nu}}(0) = g_{\sigma^i\nu}(0) < \frac{1}{2^{k-i}}G.$$

Putting $m := k - i$, this can be rewritten as

$$g_{\sigma^{-m}(\sigma^k\tilde{\nu})}(0) < \frac{1}{2^m}G.$$

i.e.,

$$\sigma^k\tilde{\nu} \in E_m \tag{2.16}$$

Since (2.16) happens for more than K indices m, the definition of the set E implies that $\sigma^k\tilde{\nu} \notin E$. $\square$

Let B be the set of elements $\nu \in \Omega$, for which **Property (K,k)** happens for all but finitely many integers k. Put $\tilde{B} := \pi^{-1}(B)$. It follows from Lemma 2.24 that every point $\tilde{\nu} \in \pi^{-1}(B)$ visits E at most finitely many times under the iterates of σ. It thus follows that $\tilde{\mathbb{P}}(\tilde{B}) = 0$, and, consequently $\mathbb{P}(B) = 0$.

Now, let $\nu \notin B$. Then for infinitely many positive integers k **Property** (K, k) does not hold. Pick such k. Then $\sigma^i\nu \notin A_{k-i}$ for all but at most K indices $i \in \{0, \dots, k-1\}$. Thus, the assumption of Proposition 2.22, (b) is satisfied for all such integers k. Applying this Proposition we see that the assumption of Proposition 2.19 is satisfied for ν. This allows to conclude that the Julia set J_ν is totally disconnected for all $\nu \notin B$. This concludes the proof of Theorem 2.20. $\square$

2.2.2 Proof of the main result

In this section we complete the proofs of Theorem 1.6 and Theorem 1.7. As shown in Theorem 2.20, it is enough to check that the estimate (2.12) holds i.e. the critical point is *typically fast escaping* under the assumptions of both theorems. First let us note that under the assumptions of Theorem 1.6 the estimate (2.12) was actually proved in [8] (see Theorem 2.2 in that paper). Obviously Theorem 1.7 implies Theorem 1.6, thus let us focus on the more general setting presented in Theorem 1.7. We shall conclude the proof of Theorem 1.7 with the following Proposition.

Proposition 2.25. *Let V be a bounded open set such that $D(0, \frac{1}{4}) \subset V$ and $V \neq D(0, \frac{1}{4})$. Take $\Omega = V^{\mathbb{N}}$ to be the product space equipped with the product of uniform distributions on V, denoted by $\mathbb{P}$. There exists a constant $\gamma > 0$ such that*

$$\mathbb{P}\left(\{\omega \in \Omega : g_\omega(0) < \frac{G}{2^k}\}\right) < e^{-\gamma k}$$

where G is set as in (2.9).

Proof. To prove Proposition 2.25 we shall use the estimates (2.11). We also need a lemma, which follows the general scheme of the proof of Theorem 2.2 in [8]:

Lemma 2.26. *Let V be an open and bounded set, such that $\mathbb{D}(0, \frac{1}{4}) \subset V \subset \mathbb{D}(0, R)$ and $V \neq \mathbb{D}(0, \frac{1}{4})$. Consider the space $\Omega = V^{\mathbb{N}}$ with the product of uniform distributions on V. Then there exists $\gamma > 0$ such that for every $z \in \mathbb{C}$*

$$\mathbb{P}(k(z, \omega) > k) \leqslant e^{-\gamma k},$$

where $k(z, \omega)$ is the escape time of z from the disc $\mathbb{D}_{R_0}$, defined in Definition 2.15.

Proof. Let $c \in V$ be a point such that $|c| > \frac{1}{4}$, say $|c| > \frac{1}{4} + \epsilon$ for some small $\epsilon > 0$.

Let us pick a point $c' \in \mathbb{D}(0, \frac{1}{2})$ (not neccesarily in V), such that $|c'| = \frac{1}{2} - \frac{\epsilon}{2}$ and $\arg(c') = \frac{\arg(c)}{2}$, where arg takes values from the interval $[0, 2\pi)$. In particular, pick ϵ small enough so that $\frac{1}{2} - \frac{\epsilon}{2} > 0$. Observe that for the parabolic map $f(w) = w^2 + \frac{1}{4}$ we have

$$f^n(w) \xrightarrow[n\to\infty]{} \frac{1}{2} \tag{2.17}$$

for every real w satisfying $|w| < \frac{1}{2}$.

Consider first z such that $|z| < \frac{1}{2}$. We claim that one can choose $N \in \mathbb{N}$ and the parameters $c_1, c_2, ..., c_N \in \mathbb{D}(0, \frac{1}{4})$ in a way that $f^N_\omega(z) = c'$. Indeed, first note that since $|z| < \frac{1}{2}$ the set

$$\left\{z^2 + c : c \in \mathbb{D}\left(0, \frac{1}{4}\right)\right\}$$

contains the disc $\{w : |w| < \rho\}$, where $\rho = \frac{1}{4} - |z|^2 > 0$. So, we can choose c_0 such that, putting $w = z^2 + c_0$ we have $|w| < \rho$, and, adjusting c_0, we can additionally achieve that the argument of w is as we wish.

Using (2.17) we find $N > 0$ and real parameters $\tilde{c}_1, \ldots, \tilde{c}_{N-1} \in (0, \frac{1}{4})$ such that

$$f^{N-1}_{\tilde{c}_{N-1},\ldots,\tilde{c}_1}(|w|) = |c'|.$$

Now, choosing appopriate c_0, we adjust the argument of w in such a way that

$$\left[2^{N-1}\arg(w) = 2^{N-1}\arg(f_{c_0}(z)\right]_{\mathrm{mod}2\pi} = \arg(c') \tag{2.18}$$

Next, for $n = 1, \ldots, N-1$, we choose c_n in such a way that $|c_n| = \tilde{c}_n$ and

$$\arg(c_n) = \arg\left((f^n_{c_{n-1},\ldots c_1,c_0}(z))^2\right)$$

so that

$$\begin{aligned}|f^{n+1}_{c_n,c_{n-1},\ldots,c_1,c_0}(z)| &= |(f^n_{c_{n-1},\ldots c_1,c_0}(z))^2 + c_n| = |f^n_{c_{n-1},\ldots c_1,c_0}(z))|^2 + |c_n| \\ &= |f^n_{c_{n-1},\ldots c_1,c_0}(z))|^2 + \tilde{c}_n,\end{aligned}$$

and, in consequence, $|f^N_{c_{N-1},\ldots c_0}(z)| = |c'|$ and $\arg(f^N_{c_{N-1},\ldots c_0}(z)) = \arg(c')$, thus $f^N_{c_{N-1},\ldots c_0}(z) = c'$. Now since $f^N_{c_{N-1},\ldots c_0}(z) = c'$ and $\arg[(c')^2] = \arg(c)$, then, putting $c_N := c$ we obtain

$$|f^{N+1}_{c_{N+1},\ldots c_0}(z)| = |c'|^2 + c > \left(\frac{1}{2} - \frac{\epsilon}{2}\right)^2 + \frac{1}{4} + \epsilon > \frac{1}{2}.$$

Recall that for v real, $v > \frac{1}{2}$ we have $f^n(v) \xrightarrow[n\to\infty]{} \infty$. This means we can pick parameters $c_{N+2}, c_{N+3}, \ldots, c_{N+N_1-1} \in \mathbb{D}(0, \frac{1}{4})$ for some N_1 (again, adjusting the argument appropriately) in such a way that $|f^{N+N_1}_{c_{N+N_1-1},\ldots,c_0}(z)| > R_0 + 1$. For $|z| > \frac{1}{2}$ we obtain the same statement even in a easier way; one only has to repeat the second part of the reasoning above. The case of z with $|z| = \frac{1}{2}$ needs a small modification: choosing an appropriate c_0 in $\mathbb{D}(0, \frac{1}{4})$, we obtain $|z^2 + c_0| < \frac{1}{2}$ and the previously described procedure applies.

So, finally, we checked the following: For every $z \in \mathbb{C}$, there exists $M = M_z$ and a sequence $c_0, c_1, \ldots, c_M$, $c_i \in V$, such that

$$|f^M_{c_{M-1},\ldots,c_0}(z)| > R_0 + 1.$$

Clearly, the same is true with c_i slightly perturbed, so, if we take $\delta > 0$ sufficiently small and put

$$A_z = \mathbb{D}(c_0, \delta) \times \cdots \times \mathbb{D}(c_{M-1}, \delta) \times \mathbb{D}(0, R)^{\mathbb{N}}$$

then $\mathbb{P}(A_z) > 0$ and, for all $\omega \in A_z$,

$$|f^M_\omega(z)| > R_0 + \frac{1}{2}.$$

Since the family

$$\{f^M_{\omega\ |\mathbb{D}_{R_0}}, \omega \in \Omega\},$$

with fixed M, is equicontinuous, we conclude that there exists $U_z \ni z$, an open neighbourhood of z such that for all $v \in U_z$, $\omega \in A_z$ we have $|f^M_\omega(v)| > R_0$. Actually because of (3.13), for all $N \geqslant M$ there holds

$$|f^N_\omega(v)| > R_0.$$

By compactness of $\overline{\mathbb{D}}_{R_0}$, there exists a finite cover of $\overline{\mathbb{D}}_{R_0}$ by a finite collection of the sets U_{z_i}. Taking $\alpha := \min_{z_i} \mathbb{P}(A_{z_i})$ and $M = \max_{z_i} M_{z_i}$, we can write

$$\exists_{M\in\mathbb{N}} \exists_{\alpha>0} \forall_{z\in\mathbb{C}}\ \mathbb{P}(\{|f^M_\omega(z)| > R_0\}) > \alpha.$$

Consequently, putting $S^k(z) = \{\omega \in \Omega : |f^k_\omega(z)| < R_0\}$, we know that for any z we have $\mathbb{P}(S^M(z)) < 1 - \alpha$.

We proceed to estimate $\mathbb{P}(S^k(z))$ exactly like in [8], using the fact that $\mathbb{P}$ is the product measure :

$$\mathbb{P}(S^{k+M}(z)) = \int_{S^k(z)} \mathbb{P}(S^M(f^k_\omega(z))) d\mathbb{P}(\omega) \leqslant (1-\alpha)\mathbb{P}(S^k(z))$$

which applied repeatedly yields the existence of a constant $\gamma > 0$ such that

$$\mathbb{P}(k(z,\omega) > k) = \mathbb{P}(S^k(z)) \leqslant e^{-k\gamma}.$$

□

Applying the above result for $z = 0$, together with the previously established (2.11), yields the claim, with possibly modified constant γ. This ends the proof of Proposition 2.25. □

It is important to point out that Lemma 2.26 is the only part of the proof of the main result that uses the assumption on the parameter space, i.e. that it contains points from outside of the disk $D(0, \frac{1}{4})$. As mentioned before, if $R \leqslant \frac{1}{4}$ then the resulting Julia set is always connected, thus the proof above illustrates exactly the role this assumption fulfills.

Taking V to be the main cardioid yields the following interesting corollary of Theorem 1.7.

Theorem 2.27. *Let $\Omega = B^{\mathbb{N}}$ where B is the main cardioid of the Mandelbrot set, and let Ω be equipped with the product of uniform distributions on V. Then for almost every sequence $\omega \in \Omega$ the Julia set J_ω is totally disconnected.*

2.2.3 Generalizations of the proof of Theorem 1.7

A number of generalizations can be made by simple adaptations of the proof from the previous sections. For instance it can be seen by inspecting the proof of Lemma 2.26 that the uniform distribution does not play any important role.

Theorem 2.28. *Let $R > \frac{1}{4}$, and let μ be a Borel probability distribution on $\mathbb{D}(0, R)$ such that $\operatorname{supp}(\mu) \supset \mathbb{D}\left(0, \frac{1}{4}\right)$ and $\mu\left(\mathbb{D}(0, R) \setminus (\overline{\mathbb{D}(0, \frac{1}{4})}\right) > 0$. Now consider the product measure of μ on $\mathbb{D}(0, R)^{\mathbb{N}}$. The Julia set for a sequence $\{c_n\} \subset \mathbb{D}(0, R)^{\mathbb{N}}$ is almost always totally disconnected, with respect to this product measure.*

The following result comes from [18], Theorem 2.2 but can also be inferred easily from our proof.

Remark 2.29 For every $c \notin \mathcal{M}$ there exists a neighbourhood $U(c)$ such that $J(c_n)$ is totally disconnected if all $c_n \in U(c)$.

Indeed, in this case it is easy to see that

$$\inf_{\omega=(c_n), c_n \in U} g_\omega(0) > a > 0$$

for some constant a, depending on U. So, with K sufficiently large, the set E defined in Section 2.2.1 is just the whole space $\tilde{\Omega}$. By Lemma 2.24 we conclude that for every $\nu \in \Omega = U(c)^{\mathbb{N}}$ and for all k

$$\sigma^i \nu \notin A_{k-i}$$

happens for all but at most K indices $i \in \{0, \ldots, k-1\}$, which, by Proposition 2.22 and Proposition 2.19 immediately implies that every Julia set J_ω is totally disconnected.

Another easy adaptation of the proof yields an answer to a question from [8] (see Remark 2.5 in [8]), about connectivity of the Julia set when parameters are chosen from a circle of radius $\delta > \frac{1}{4}$.

Actually, the authors ask in Remark 2.5 in [8] if the set Julia set is almost surely disconnected. Our approach gives much more:

Proposition 2.30. *Let $\Omega = \partial\mathbb{D}(0, R)^{\mathbb{N}}$ where $R > \frac{1}{4}$ be equipped with the product of uniform distributions on the circle $\partial\mathbb{D}(0, R)$. Then for almost every $\omega \in \Omega$ the Julia set J_ω is totally disconnected.*

To repeat our proof in the above case we need the following version of Lemma 2.26.

Lemma 2.31. *Let $K = \partial\mathbb{D}(0, R)$ where $R > \frac{1}{4}$. Consider the space $\Omega = K^{\mathbb{N}}$ with the product of uniform distributions on K. Then there exists $\gamma > 0$ such that for all $z \in \mathbb{C}$,*

$$\mathbb{P}(k(z, \omega) > k) \leqslant e^{-\gamma k},$$

where $k(z, \omega)$ is the value defined in (2.10).

Proof. Take an arbitrary point $z \in \mathbb{C}$, let $c_1, c_2, c_3, ..., c_N \in K$ be a sequence of N parameters such that for all $n \leqslant N$

$$|f_\omega^n(z)| = |f_\omega^{n-1}(z)^2 + c_n| = |f_\omega^{n-1}(z)|^2 + |c_n|.$$

Recall that for iterations on the real line, with $f(x) = x^2 + R$ and $R > \frac{1}{4}$, we have for all x

$$\lim_{n\to\infty} f^n(x) = \infty.$$

Since $|c_n| = R > \frac{1}{4}$ by our choice of the numbers $c_1, ..., c_N$, for a large enough N, we will have $|f_\omega^N(z)| > R_0$. By continuity and compactness arguments, used exactly as in the proof of Lemma 2.26, we see that one can show something more, that is

$$\exists_{N\in\mathbb{N}}\exists_{\delta>0}\forall_{z\in\mathbb{C}}\ \mathbb{P}(\{\omega \in \Omega : |f_\omega^N(z)| > R_0\}) > \delta.$$

We finish the proof in exactly the same way as the proof of Lemma 2.26.

□

2.3 Dimension of maximal measure

We end this chapter with a section on the properties of the maximal (harmonic) measures of the Julia sets J_ω. Results from this section are from a joint work with Anna Zdunik (in preparation), which is tied closely to the results in [21]. We begin by introducing some definitions related to harmonic measures. For a better overview of harmonic measures and potential theory one can refer to [5], [27].

2.3.1 Preliminaries on potential theory

Let $K \subset \mathbb{C}$ be a compact set, and let μ be a finite Borel measure on K. We then introduce the following definitions.

Definition 2.32 The logarithmic potential of μ is defined by

$$u(z) = \int_K \ln \frac{1}{|z-\zeta|} d\mu(\zeta).$$

Definition 2.33 The energy of μ is given by

$$I(\mu) = \int_K \int_K \ln \frac{1}{|z-\zeta|} d\mu(\zeta) d\mu(z).$$

Definition 2.34 Let $P(K)$ be the collection of all Borel probability measures on K. Denote

$$V = \inf_{\mu \in P(K)} I(\mu),$$

then any measure $\mu^* \in P(K)$ that satisfies

$$I(\mu^*) = V$$

is called the equilibrium measure of K.

One can prove that with some mild assumption on K the equilibrium measure is unique. In our case what is called the harmonic measure for $\mathcal{A}_\omega$ is the equilibrium measure supported on the Julia set J_ω (see for instance Theorem 4.3.14 in [27]), thus we shall use these terms interchangeably (along with maximal measure).

Potential theory also comes with its own concept of a negligible set, defined as follows.

Definition 2.35 A set $E \subset \mathbb{C}$ is called polar if $I(\mu) = \infty$ for every finite Borel measure $\mu \neq 0$ for for which $\operatorname{supp} \mu$ is a compact subset of E. We say a property holds nearly everywhere if it holds everywhere outside of a some polar set.

Finally the last definition we introduce is that of capacity.

Definition 2.36 If $V = \inf_{\mu \in P(K)} I(\mu)$ then the capacity of a set K is defined by

$$\operatorname{cap}(K) = e^{-V}.$$

The following theorem, which can be found in [27], gives the relation between capacity and Green's function. This gives a way to compute capacity and shows the ties between this notion and complex dynamics.

Theorem 2.37. *Let K be a compact non-polar set and let $c(K)$ be its capacity. Let D be the component of $\hat{\mathbb{C}} \setminus K$ which contains ∞. Then*

$$g_D(z, \infty) = \log |z| - \log c(K) + o(1)$$

as $z \longrightarrow \infty$.

2.3.2 Preliminaries on maximal measure in the non-autonomous setting

The measure $\mu_\omega = \frac{1}{2\pi} \Delta g_\omega$ (i.e., the harmonic measure for the domain $\mathcal{A}_\omega$, evaluated at infinity) is supported on the Julia set J_ω, and, since $g_{\sigma\omega} \circ f_\omega = 2 g_\omega$, it satisfies

$$\mu_{\sigma\omega}(f_\omega(A)) = 2\mu_\omega(A) \tag{2.19}$$

if A is a Borel set such that f_ω is one-to-one on A. For a more detailed argument one can refer to [31] (more specifically, the beginning of proof of Theorem B).

We call this measure also the maximal measure since, analogously to the autonomous case, μ_ω is a fixed point of a (non- autonomous) transfer operator. Denote by C_{R_0} the space of continuous functions in $\overline{\mathbb{D}}_{R_0}$, equipped with the supremum norm. Consider a family of operators $\mathcal{L}_\omega : C_{R_0} \to C_{R_0}$ defined as

$$\mathcal{L}_\omega(\varphi)(w) = \frac{1}{2} \sum_{z \in f_\omega^{-1}(w)} \varphi(z),$$

where $\varphi \in C_{R_0}$. Because of the choice of R_0 (see (2.8)), each operator $\mathcal{L}_\omega$ acts continuously on the space C_{R_0}.

Let $\mathcal{L}_\omega^*$ be the conjugate operator, that is $\mathcal{L}_\omega^*(\mu)(\phi) = \mu(\mathcal{L}_\omega(\phi))$. Thus, $\mathcal{L}^*$ acts on the conjugate space $C_{R_0}^*$, i.e., the space of signed finite Borel measures on $\overline{\mathbb{D}}_{R_0}$.

Notice that $\mathcal{L}_\omega^*(\mu_{\sigma\omega}) = \mu_\omega$. Indeed, let $\varphi \in C_{R_0}$. Then

$$\mathcal{L}_\omega^* \mu_{\sigma\omega}(\varphi) = \mu_{\sigma\omega}(\mathcal{L}_\omega(\varphi)) = \mu_\omega(\varphi)$$

by the equation (2.19).

The equation (2.19) implies also that the measure $\{\mu_\omega\}$ is "fiberwise invariant":

$$\mu_{\sigma\omega} = \mu_\omega \circ (f_\omega)^{-1} \tag{2.20}$$

The following proposition says that the maximal measure can be expressed in terms of iterates of the (non- autonomous) transfer operator. The autonomous version of it was first observed by Brolin [5]. The non-autonomous variant below comes from [7]. We give below the proof of the author of [7].

Proposition 2.38. *For any point $z_0 \in \mathbb{D}^*_{R_0}$ the following holds:*

$$\mu_\omega = \lim_{n\to\infty} \frac{1}{2^n} \sum_{y\in (f^n_\omega)^{-1}(z_0)} \delta_y.$$

weakly, i.e., for every continuous function φ defined on a neighbourhood of J_ω,

$$\mu_\omega(\varphi) = \lim_{n\to\infty} \frac{1}{2^n} \sum_{y\in (f^n_\omega)^{-1}(z_0)} \varphi(y).$$

In other words,

$$\mu_\omega = \lim_{n\to\infty} (\mathcal{L}^{*,n}_\omega)(\delta_{z_0}),$$

where we denoted by $(\mathcal{L}^{*,n}_\omega)$ the composition of the operators:

$$\mathcal{L}^{*,n}_\omega = \mathcal{L}^*_\omega \circ \mathcal{L}^*_{\sigma\omega} \circ \cdots \circ \mathcal{L}^*_{\sigma^{n-1}\omega}$$

The proof of the above requires the following two lemmas from [7].

Lemma 2.39. *Let $\omega \in \Omega$ and let μ^* be the equilibrium measure for the Julia set J_ω. Then* $\operatorname{supp}\mu^* = \mathcal{J}_\omega$.

The above result comes from the fact that the set $\mathcal{J}_\omega \setminus \operatorname{supp}\mu^*$ should have zero capacity, but one can show that any open (with respect to the subspace topology) subset of $\mathcal{J}_\omega$ has positive capacity. For a detailed argument we refer the reader to [7].

Lemma 2.40. *Let $E, H \subset \mathbb{C}$ be compact sets with $E \subset H$ and $cap(E) = e^{-V} > 0$. Furthermore, let (μ_n) be a sequence of probability measures on H which converges weakly to a probability measure μ on E. Let u_n denote the logarithmic potential with respect to μ_n and μ^* denote the equilibrium measure on E, and suppose $\liminf_{n\to\infty} u_n(z) \geqslant V$ for $z \in E$ and* $\operatorname{supp}\mu^* = E$. *Then there holds $\mu = \mu^*$.*

Proof of Proposition 2.38. For $k \in \mathbb{N}$ let $z_{1,k}, z_{2,k}, \dots, z_{2^k,k}$ be the solutions of the equation $f^k_\omega(z) = z_0$. Then we have $z_{j,k} \in \mathcal{A}_\omega(\infty)$ and $z_{j,k} \in \mathbb{D}(0, |z_0|)$. This implies also $\operatorname{supp}\mu_\omega \subset \mathbb{D}(0, |z_0|)$. Since $|f^k_\omega(z)| \leqslant R_0$ for $z \in J_\omega$ and

$$|f^k_\omega(z) - z_0| = \prod_{j=1}^{2^k} |z - z_{j,k}|$$

thus for $z \in J_\omega$

$$\sum_{j=1}^{2^k} \ln|z - z_{j,k}| = \ln|f^k_\omega(z) - z_0| \leqslant \ln(R_0 + |z_0|) = C$$

which finally yields

$$u_k(z) := \frac{1}{2^k} \sum_{j=1}^{2^k} \ln\Big(\frac{1}{|z - z_{j,k}|}\Big) \geqslant -\frac{C}{2^k}.$$

This can be written as

$$u_k(z) = \int_{D(0,|z_0|)} \ln\Big(\frac{1}{|z-\zeta|}\Big) d\mu^k_\omega(\zeta) \geqslant -\frac{C}{2^k}$$

where

$$\mu^k_\omega = \frac{1}{2^k} \sum_{y\in (f^k_\omega)^{-1}(z)} \delta_y.$$

Note that combining Theorem 2.37 and Proposition 2.12 yields $\text{cap} J_\omega = 1$, and thus we have

$$\liminf_{k\to\infty} u_k(z) \geqslant 0 = \ln \text{cap} J_\omega.$$

It is a well known fact (see for example [29]) that every sequence of probability measures on a compact set H contains a weakly convergent subsequence. Thus we only need to show that for every subsequence of μ_ω^k which converges weakly to some ν we have $\nu = \mu_\omega$. Since the predecessors $(f_\omega^k)^{-1}(z_0)$ do not accumulate in $\mathcal{A}_\omega(\infty)$, we obtain $\text{supp}\, \nu \subset J_\omega$, and now the assertion follows from the above lemmas. □

2.3.3 Bounded random systems of quadratic maps

The formalism we use in this section for describing random iterations of quadratic polynomials is in part as in [2], [14]. The randomness is modeled by a measure preserving dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \sigma)$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and $\sigma : \Omega \to \Omega$ – an invertible measure preserving ergodic transformation. To every $\omega \in \Omega$ we associate in a measurable way a parameter $c = c(\omega) \in \mathbb{C}$, and consequently, the map denoted in the sequel by f_ω is defined as

$$f_\omega := f_{c(\omega)}.$$

This gives a natural way of defining random iterates: for $\omega \in \Omega$ put

$$f_\omega^n = f_{\sigma^{n-1}\omega} \circ \cdots \circ f_\omega. \tag{2.21}$$

In the subsequent sections we shall always assume that the function $\omega \mapsto c(\omega)$ is bounded, so that there exists $R > 0$ such that $c(\omega) \in \mathbb{D}(0, R)$ for every $\omega \in \Omega$. Finally, for this R we set an appropriate constant R_0 just like in Proposition 2.12.

Such a random system will be referred to as a *bounded random system of quadratic maps*. So, the formal definition is the following.

Definition 2.41 A bounded random system of quadratic maps $\mathcal{S}$ is formed by a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with an ergodic $\mathbb{P}$-measurable measure preserving automorphism $\sigma : \Omega \to \Omega$ and a measurable map $\omega \mapsto c(\omega) \in \mathbb{D}(0, R)$.

We shall use a formalism of random sets and random measures. We follow the presentation elaborated in [14]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a (complete) probability space, let X be a Polish space. Recall the following definition from [14].

Definition 2.42 A function $\phi : \Omega \times X \to \mathbb{R}$, $\phi(\omega, z) = \phi_\omega(z)$ is called a random continuous function if it satisfies the following conditions:

1. for all $x \in X$ the function $\omega \mapsto f(\omega, x)$ is measurable
2. for all $\omega \in \Omega$ the function $x \mapsto f(\omega, x)$ is continuous and bounded
3. $\omega \mapsto \sup\{|f(x, \omega)| : x \in X\}$ is integrable with respect to $\mathbb{P}$

The vector space of all real–valued random continuous functions is denoted by $C_b(\Omega \times X)$. Equipped with the norm

$$\|\phi\| := \int_\Omega \|\phi_\omega\|_\infty d\mathbb{P}(\omega)$$

it becomes a Banach space.

Put

$$\mathcal{X} := \Omega \times X.$$

Denote by $\mathcal{M}(\mathcal{X})$ the space of probability measures ν defined on the σ-algebra $\mathcal{F} \otimes \mathcal{B}$, and let $\pi_1 : \mathcal{X} \to \Omega$ be the projection onto the first coordinate, i.e.,

$$\pi_1(\omega, z) = \omega.$$

Let $\mathcal{M}_{\mathbb{P}}(X) \subset \mathcal{M}(\mathcal{X})$ be the set of all probability measures on $\mathcal{X}$ that project onto $\mathbb{P}$ under the map $\pi_1 : X \to \Omega$, i.e.

$$\mathcal{M}_{\mathbb{P}} = \{\mu \in \mathcal{M}(\mathcal{X}) : \mu \circ \pi_1^{-1} = \mathbb{P}\}.$$

Definition 2.43 A map $\mu : \Omega \times \mathcal{B} \to [0,1]$, $(\omega, B) \longmapsto \mu_\omega(B)$, is called a random probability measure on X if

- For every set $B \in \mathcal{B}$ the function $\Omega \ni \omega \longmapsto \mu_\omega(B) \in [0,1]$ is measurable,
- For $\mathbb{P}$-almost every $\omega \in \Omega$ the map $\mathcal{B} \ni B \mapsto \mu_\omega(B) \in [0,1]$ is a Borel probability measure.

A random measure μ will be frequently denoted as $\{\mu_\omega\}_{\omega\in\Omega}$ or $\{\mu_\omega : \omega \in \Omega\}$.

The set $\mathcal{M}_{\mathbb{P}}$ can be canonically identified with the collection of all random probability measures on X as follows.

Proposition 2.44 (see Propositions 3.3 and 3.6 in[14])**.** *With the above notation, for every measure $\mu \in \mathcal{M}_{\mathbb{P}}(X)$ there exists a unique random measure $\{\mu_\omega\}_{\omega\in\Omega}$ on X such that*

$$\int_{\Omega\times X} h(\omega, z)\, d\mu(\omega, z) = \int_\Omega \left(\int_X h(\omega, z)\, d\mu_\omega(z) \right) d\mathbb{P}(\omega)$$

for every bounded measurable function $h : \Omega \times X \to \mathbb{R}$ and for every measurable non- negative function $h : \Omega \times X \to \mathbb{R}$.

Conversely, if $\{\mu_\omega\}_{\omega\in\Omega}$ is a random measure on X, then for every bounded measurable or non- negative measurable function $h : \mathcal{X} \to \mathbb{R}$ the function $\Omega \ni \omega \longmapsto \int_{\mathbb{C}} h(\omega, z) d\mu_\omega(z)$ is measurable, and the assignment

$$\mathcal{F} \otimes \mathcal{B} \ni A \longmapsto \int_\Omega \int_X \mathbb{1}_A(\omega, z) d\mu_\omega(z) d\mathbb{P}(\omega),$$

defines a probability measure (a "global measure") $\mu \in \mathcal{M}_{\mathbb{P}}(\mathcal{X})$.

Another useful characterization of random probability measure on X is the following (see Remark 3.20 in [14]).

Proposition 2.45. *A family of Borel probability measures on X, $\{\mu_\omega : \omega \in \Omega\}$ is a random probability measure if and only if for every bounded continuous function $\varphi : X \to \mathbb{R}$ the map*

$$\omega \mapsto \mu_\omega(\varphi)$$

is measurable.

Definition 2.46 Following Definition 2.1 in [14] we say that a function $\Omega \ni \omega \mapsto C_\omega$, ascribing to each point $\omega \in \Omega$ a closed subset $C \subset X$ is called a random closed set if for each $z \in \mathbb{C}$ the function

$$\Omega \ni \omega \mapsto \mathrm{dist}(z, C_\omega) \in \mathbb{R}$$

is measurable.

Since the probability measure m on Ω is assumed to be complete, we can also give another characterization (see Proposition 2.4 in [14]):

Proposition 2.47. *We call a function $\Omega \ni \omega \mapsto C_\omega$ a random closed set if C_ω are closed for all ω and moreover the union*

$$C := \bigcup_{\omega \in \Omega} \{\omega\} \times C_\omega$$

is a measurable subset of $\Omega \times X$.

We shall use yet another equivalent characterization of random closed sets as in the following:

Proposition 2.48. *[see Proposition 2.4 in [14]] A collection $(C_\omega)_{\omega\in\Omega}$ of closed subsets of X is a random closed set if and only if for every open set $U \subset X$ the set $\{\omega \in \Omega : U \cap C_\omega \neq \emptyset\}$ is measurable.*

2.3.4 Random equilibrium measures

Let us now state some basic results on non autonomous equilibrium measures in the language of the previous section. We begin with the following proposition.

Proposition 2.49. *Let $\mathcal{S}$ be a bounded random system of quadratic maps. Then*

1. *The filled-in random Julia sets K_ω are random closed sets, i.e., the collection $\{K_\omega\}_{\omega\in\Omega}$ satisfies the condition formulated in Definition 2.46.*
2. *The collection g_ω of random Green functions with poles at infinity is a random continuous function in $\Omega \times \mathbb{D}_{R_0}$, where R_0 is picked for R as in Proposition 2.12.*
3. *The collection $\{\mu_\omega\}_{\omega\in\Omega}$ of random equilibrium (harmonic) measures at infinity on J_ω, defined by*
$$\mu_\omega = \frac{1}{2\pi}\Delta g_\omega$$
is a random probability measure.

Proof. To prove item (1), let $U \subset \mathbb{C}$ be an arbitrary open set. We need to show that the set

$$\{\omega : K_\omega \cap U \neq \emptyset\} \tag{2.22}$$

is measurable. Equivalently, we need to show measurability of the set

$$\{\omega : K_\omega \cap U = \emptyset\} = \{\omega : U \subset \mathcal{A}_\omega\}.$$

Let $(V_n)_{n\in\mathbb{N}}$ be an ascending sequence of open sets, relatively compact in U, such that $\bigcup_n V_n = U$.

Then $U \subset \mathcal{A}_\omega$ if and only if for all $n \geqslant 1$ $\overline{V}_n \subset \mathcal{A}_\omega$ and we are left to show that all the sets

$$\{\omega \in \Omega : \overline{V}_n \subset \mathcal{A}_\omega\} \tag{2.23}$$

are measurable.

Since $\overline{V}_n$ is compact, the condition expressed in (2.23) can be written in terms of Green function:

$$\{\omega \in \Omega : \overline{V}_n \subset \mathcal{A}_\omega\} = \{\omega : \inf_{z\in\overline{V}_n} g_\omega(z) > 0\} \tag{2.24}$$

Let $\{z_k\}_{k\in\mathbb{N}}$ be a dense set in U. The condition expressed in (2.24) can be written equivalently as

$$\{\omega : \inf_{z_k\in\overline{V}_n} g_\omega(z_k) > 0\} \tag{2.25}$$

Since for each k the function $\omega \mapsto g_\omega(z_k)$ is measurable, the set defined in (2.25) is measurable, and, therefore the sets defined in (2.24), (2.23) and (2.22) are measurable as well.

Item (2) follows directly from Proposition 2.9 and from the definition of a random continuous function.

To prove item (3) we use the characterization of random probability measures formulated in Proposition 2.45. To continue the proof we need two lemmas.

Lemma 2.50. *Denote by $\underline{c} = (c_0, c_1, \ldots, c_{n-1})$ a sequence of parameters in $\mathbb{D}(0, R)$ i.e. $\underline{c} \in \mathbb{D}(0, R)^n$, and let $f^n_{\underline{c}} = f_{c_{n-1}} \circ f_{c_{n-2}} \circ \ldots \circ f_{c_0}$. Let R_0 be the value assigned to R in Proposition 2.12.*
Let φ be a continuous function in $\overline{\mathbb{D}}_{R_0}$ and let $z_0 \in \overline{\mathbb{D}}^*_{R_0}$. Then for large enough n the function

$$(\mathbb{D}(0, R))^n \ni \underline{c} \mapsto \sum_{v \in (f^n_{\underline{c}})^{-1}(z_0)} \varphi(v) \tag{2.26}$$

is well defined and continous.

Proof. Since φ is a function on $\overline{\mathbb{D}}_{R_0}$ then for $\sum_{v \in (f^n_{\underline{c}})^{-1}(z_0)} \varphi(v)$ to be well defined we need to have

$$v \in \overline{\mathbb{D}}_{R_0}$$

for all $v \in (f^n_{\underline{c}})^{-1}(z_0)$ for sufficiently large n. Note that by (2.4) it is easy to see that there exists N such that for all $n > N$ we have for a $v \in (f^n_{\underline{c}})^{-1}(z_0)$

$$|v| < R_0.$$

Thus we conclude that

$$(\mathbb{D}(0, R))^n \ni \underline{c} \mapsto \sum_{v \in (f^n_{\underline{c}})^{-1}(z_0)} \varphi(v)$$

is well defined. We move on to proving continuity. The points $v \in (f^n_{\underline{c}})^{-1}(z_0)$ are roots of the polynomial $z \mapsto f^n_{\underline{c}}(z) - z_0$. The coefficients of this polynomial depend also polynomially on the coefficients of $\underline{c} = (c_0, \ldots c_{k-1})$. Continuity of the function (2.26) thus follows immediately from the fact that roots of a complex monic polynomial (counted with multiplicity) depend continuously on the coefficients.

□

Lemma 2.51. *Let φ be a continuous function in $\overline{\mathbb{D}}_{R_0}$. Let $z_0 \in \overline{\mathbb{D}}^*_{R_0}$. The function*

$$\Omega \ni \omega \mapsto \sum_{v \in (f^n_\omega)^{-1}(z_0)} \varphi(v)$$

is measurable.

Proof. It is enough to observe that the function

$$\omega \mapsto \underline{c}(\omega) = (c_0(\omega), c_1(\omega) \ldots c_{n-1}(\omega))$$

is measurable, whereas the function

$$\underline{c} \mapsto \sum_{v \in (f^n_{\underline{c}})^{-1}(z_0)} \varphi(v)$$

is continuous, by Lemma 2.50. □

Having Lemmas 2.50 and 2.51 at our disposal, we can conclude the proof of Proposition 2.49. Indeed, defining by $\mu_{n,\omega}$ the measure equidistributed over the set $\{v \in (f_\omega^n)^{-1}(z_0)\}$ (counted with multiplicity), we conclude from Lemma 2.51 that the functions

$$\omega \mapsto \mu_{n,\omega}(\varphi)$$

are measurable. Since for every ω the sequence $\mu_{n,\omega}(\varphi)$ converges to $\mu_\omega(\varphi)$ (see Proposition 2.38), we conclude that the limit function $\omega \mapsto \mu_\omega(\varphi)$ is measurable.

□

Since, by Proposition 2.49 $\{\mu_\omega\}_{\omega\in\Omega}$ is a random probability measure, we can consider the global measure μ on the product space $\Omega \times \mathbb{C}$, as in Proposition 2.44, and the global map (skew product)

$$F : \Omega \times \mathbb{C} \to \Omega \times \mathbb{C}$$

determined by the formula

$$F(\omega, z) = (\sigma\omega, f_\omega(z)) . \tag{2.27}$$

Proposition 2.52. *The global map F is μ–measurable and the measure μ is invariant under the map F.*

Proof. First, we prove measurability of the map F. It is enough to prove that for every set of the form $C \times B$ where $C \in \mathcal{F}$ and B is a compact subset of $\mathbb{C}$, its preimage $F^{-1}(C \times B)$ is measurable. We have, by definition of the map F,

$$F^{-1}(C \times B) = \bigcup_{\omega\in\sigma^{-1}(C)} \{\omega\} \times f_\omega^{-1}(B). \tag{2.28}$$

By the characterization of a random closed set, to prove measurability of (2.28), we need to prove that the collection

$$B_\omega = \begin{cases} f_\omega^{-1}(B) & \text{for} \quad \omega \in \sigma^{-1}(C), \\ \emptyset \quad \text{for} \quad \omega \notin \sigma^{-1}(C) \end{cases}$$

is a random closed set. To prove measurability of this set we use the characterization given in Proposition 2.48.

Let $U \subset \mathbb{C}$ be an open set. We need to check that the set

$$\{\omega \in \sigma^{-1}(C) : U \cap f_\omega^{-1}(B) = \emptyset\} = \sigma^{-1}(C) \cap \{\omega \in \Omega : U \cap f_\omega^{-1}(B) = \emptyset\}$$

is measurable. Clearly, it is sufficient to check measurability of the set

$$\{\omega \in \Omega : U \cap f_\omega^{-1}(B) = \emptyset\}.$$

Since by definition the function $\omega \mapsto c(\omega) \in \mathbb{D}(0, R)$ is measurable, it is in fact sufficient to verify measurability of the set

$$\{c \in \mathbb{D}(0, R) : U \cap f_c^{-1}(B) = \emptyset\}$$

or equivalently the set

$$\{c \in \mathbb{D}(0, R) : U \cap f_c^{-1}(B) \neq \emptyset\}.$$

Since U is open and $f_\omega^{-1}(B)$ is closed, it is easy enough to see that the above set is in fact open, and thus measurable.

Next, we check that F preserves the measure μ. So, we need to show that for every μ–measurable set A we have $\mu(F^{-1}(A)) = \mu(A)$.

It is enough to check this equality for every set of the form $A := C \times B$, where $C \in \mathcal{F}$ and B is a compact subset of $\mathbb{C}$, since these sets generate the product σ–algebra $\mathcal{F} \otimes \mathcal{B}$. We have:

$$F^{-1}(C \times B) = \bigcup_{\omega \in \sigma^{-1}(C)} \{\omega\} \times f_\omega^{-1}(B).$$

Thus,

$$\mu(F^{-1}(C \times B)) = \int_{\omega \in \sigma^{-1}(C)} \mu_\omega(f_\omega^{-1}(B))\mathbb{P}(\omega) = \int_{\omega \in \sigma^{-1}(C)} \mu_{\sigma\omega}(B)\mathbb{P}(\omega) = \int_{\omega \in C} \mu_\omega(B)\mathbb{P}(\omega)$$
$$= \mu(C \times B)$$

□

2.3.5 Random systems of quadratic maps. Dimension of the maximal measure

In this section we answer the question about Hausdorff dimension of the equilibrium measure μ_ω for bounded random systems of quadratic maps. Recall the following standard definition.

Definition 2.53 Let $(X, \mathcal{B}, \mu)$ be a measure space and by $\dim_H(A)$ let us denote the Hausdorff dimension of a set $A \subset X$. Then the Hausdorff dimension of the measure μ is defined as

$$\dim_H \mu := \inf\{\dim_H A \,:\, A \in \mathcal{B},\, \mu(A) = 1\}.$$

The first part of our considerations works in the general setting of Definition 2.41 (bounded random quadratic system). Let us recall that, as noted in Proposition 2.9, if $\mathcal{S}$ is a bounded random quadratic system, then the function $\omega \mapsto g_\omega(0)$ is $\mathbb{P}$– measurable and bounded (actually Proposition 2.9 was formulated in less general terms, but the same proof works for bounded random quadratic systems). Therefore, the integral

$$\mathbf{g}(0) := \int g_\omega(0) d\mathbb{P}(\omega)$$

is well- defined. This value will be referred to as the global Green function.

We now define the random Lyapunov exponent:

$$\chi_\omega = \int_{J_\omega} \log|f'_\omega|(z) d\mu_\omega(z) = \int_{J_\omega} \log|2z| d\mu_\omega(z), \tag{2.29}$$

Note that the above integral is well defined as the function integrated is bounded from above (although a priori the integral might be equal $-\infty$). Naturally, the global Lyapunov exponent will be defined as:

$$\chi = \int_\Omega \left(\int_{J_\omega} \log|f'_\omega|(x) d\mu_\omega(x) \right) d\mathbb{P}(\omega) = \int_{\mathcal{X}} \log|f'_\omega|(z) d\mu(\omega, z) \tag{2.30}$$

Proposition 2.54. *Global Lyapunov exponent can be calculated in terms of the global Green function:*

$$\chi = \log 2 + \mathbf{g}(0).$$

Proof.

$$\chi_\omega = \int_{J_\omega} \log|f'_\omega|(z)d\mu_\omega(z) = \int_{J_\omega} \log|2z|d\mu_\omega(z) = \log 2 + \int_{J_\omega} \log|z|d\mu_\omega(z)$$

So, we need to calculate the integral

$$\int_{J_\omega} \log|z|d\mu_\omega(z) \tag{2.31}$$

Recall that for a Borel measure η in $\mathbb{C}$, the formula

$$p_\eta(w) = \int \log|z-w|d\eta(z)$$

defines a subharmonic function in $\mathbb{C}$, and $\Delta p_\eta = 2\pi\eta$.

In the case we consider, $\eta = \mu_\omega$ is the equilibrium measure (i.e. the measure maximizing the integral

$$I_\nu = \iint \log|x-y|d\nu(x)d\nu(y),$$

where ν runs over all Borel probability measures on J_ω).

Since each set J_ω is regular for Dirichlet problem, the function $x \mapsto \int \log|x-y|d\mu_\omega(y)$ is constant on J_ω (see Theorems 4.2.1, 4.2.4 and 4.4.9 from [27]).

So, for each J_ω we have:

1. p_{μ_ω} is constant on J_ω, say $p_{\mu_\omega} \equiv P$ on J_ω.
2. $g_\omega \equiv 0$ on J_ω.

Since $\Delta p_{\mu_\omega} = \Delta g_\omega = 2\pi\mu_\omega$, we conclude that $p_{\mu_\omega} = g_\omega + h$ where h is some harmonic function in $\mathbb{C}$.

Observe that $h(z) \to 0$ as $z \to \infty$ (because both g_ω and p_{μ_ω} are of the form $\log|z| + o(1)$ as $z \to \infty$), and $h(z) \equiv P$ on J_ω. Since h is harmonic, this implies that $P = 0$, and consequently $p_{\mu_\omega} = g_\omega$.

In particular, $p_{\mu_\omega}(0) = g_\omega(0)$, which gives the value of the integral (2.31), and shows that

$$\chi_\omega = \log 2 + g_\omega(0).$$

The formula for the global Lyapunov exponent follows by a direct integration. □

Remark 2.55 Since $\mathcal{S}$ is a bounded random quadratic system, there exists $R > 0$ such that $c(\omega) \in \mathbb{D}(0, R)$ for every $\omega \in \Omega$. Using Proposition 1.4, it is clear that for every $\omega \in \Omega$

$$\chi_\omega \leqslant \log 2 + \log R_0,$$

where R_0 is the value coming from Proposition 1.4. This follows from the fact that for every $\omega \in \Omega$, $J_\omega \subset \mathbb{D}_{R_0}$, so $\log|f'_\omega(z)| = \log 2 + \log|z| \leqslant \log 2 + \log R_0$ for every $z \in J_\omega$.

In order to give the exact formula for the dimension of the measure μ_ω we need to restrict our considerations to a natural subclass of bounded random system of quadratic maps.

Definition 2.56 A bounded random system of quadratic maps $\mathcal{S}$ is called a *system of independent random quadratic maps* if the probability space Ω is the infinite product $V^{\mathbb{Z}}$ for some Borel bounded set $V \subset \mathbb{C}$ and $\mathbb{P}$ is the (completed) product distribution generated by some Borel distribution ν on V.

For typically fast escaping bounded system of independent random quadratic maps, we can say more about the dynamics of the skew product F defined in the formula (2.27). In Proposition 2.52 we proved that the global map F preserves the global measure μ. Now we complete the picture, proving ergodicity.

Proposition 2.57. *Let $\mathcal{S}$ be a typically fast escaping (see Definition 2.17) bounded system of independent random quadratic maps. Then the global map F defined in the formula* (2.27) *is ergodic with respect to the global measure μ.*

Before the proof of the above, we will need the following proposition, which is a corollary from the proof of Theorem 2.20.

Proposition 2.58. *(Complement to Theorem 2.20) There exists $K \in \mathbb{N}$ such that for $\mathbb{P}-$ almost every $\omega \in \Omega$ there are infinitely many indices k_n such that the following holds:*

For each component $\tilde{D}_j^{k_n}(\omega)$ of the set $(f_\omega^{k_n})^{-1}(D)$ the degree of the map

$$f_\omega^{k_n-K} : \tilde{D}_j^{k_n}(\omega) \to f_\omega^{k_n-K}(\tilde{D}_j^{k_n}(\omega))$$

is equal to one.

Moreover, for $\mathbb{P}$- a.e. ω the sequence $k_n = k_n(\omega)$ may be chosen in such a way that

$$\lim_{n\to\infty} \frac{k_{n+1}-k_n}{k_n} = 0 \tag{2.32}$$

Sketch of proof of Proposition 2.58. Let us remark that the first part of the above Proposition is a rather straightforward adaptation of the proof of Theorem 2.20, whereas the last equation is a consequence of ergodicity of the left shift on $\tilde{\Omega}$. Indeed, let $E \subset \tilde{\Omega}$ be a set of two-sided sequences from the proof of Theorem 2.20 defined in (2.15). Recall that $\mathbb{P}(E) > 0$. Then, by ergodicity of σ, for almost any $\omega \in \tilde{\Omega}$ we have

$$\lim_{M\to\infty} \frac{|m \leqslant M : \sigma^m(\omega) \in E|}{M} = \mathbb{P}(E).$$

Thus clearly we can pick k_n to satisfy

$$\lim_{n\to\infty} \frac{k_n}{n} = \frac{1}{\mathbb{P}(E)}.$$

Finally this yields the desired equality

$$\lim_{n\to\infty} \frac{k_{n+1}-k_n}{k_n} = \lim_{n\to\infty} \frac{\frac{k_{n+1}}{n} - \frac{k_n}{n}}{\frac{k_n}{n}} = 0.$$

□

Proof of Proposition 2.57. Assume the contrary, that is that F is not ergodic with respect to μ. Then there exist two F- invariant disjoint sets A, B of positive measure μ. Let us denote by A_ω the preimage of the projection $\pi^{-1}(\omega)$ where $\pi : A \to \Omega$. We define the set B_ω analogously. Then for each $\omega \in \Omega$ the sets A_ω, B_ω are Borel measurable and the functions $\omega \mapsto \mu_\omega(A_\omega), \omega \mapsto \mu_\omega(B_\omega)$ are $\mathbb{P}$-measurable (refer for example to [14], Corollary 3.4).

Invariance of A, B under F implies that

$$f_\omega^{-1}(A_{\sigma\omega}) = A_\omega,\ f_\omega^{-1}(B_{\sigma\omega}) = B_\omega.$$

Moreover invariance of μ implies that for $\mathbb{P}$ almost every $\omega \in \Omega$ we have

$$\mu_\omega(f_\omega^{-1}(A_{\sigma\omega})) = \mu_{\sigma\omega}(A_{\sigma\omega})$$

and

$$\mu_\omega(f_\omega^{-1}(B_{\sigma\omega})) = \mu_{\sigma\omega}(B_{\sigma\omega}).$$

From the above we have that the functions

$$\varphi(\omega) = \mu_\omega(A_\omega)$$

and

$$\psi(\omega) = \mu_\omega(B_\omega)$$

are σ-invariant, and thus, by ergodicity of σ, constant $\mathbb{P}$-almost everywhere.

Consider two Borel probability measures μ_A and μ_B , defined by their fiber measures as follows:

$$\mu_{A,\omega}(F) = \frac{\mu_\omega(A_\omega \cap F)}{\mu_\omega(A_\omega)},$$

and, analogously

$$\mu_{B,\omega}(F) = \frac{\mu_\omega(B_\omega \cap F)}{\mu_\omega(B_\omega)},$$

where F is a Borel set. Both $(\mu_{A,\omega})_{\omega\in\Omega}, (\mu_{B,\omega})_{\omega\in\Omega}$ are random probability measures. Indeed, to confirm that for $(\mu_{A,\omega})_{\omega\in\Omega}$ we need to check that the function

$$\omega \mapsto \mu_\omega(A_\omega \cap F)$$

is $\mathbb{P}$-measurable. This follows again from [14], Corollary 3.4., applied to the set $(\Omega \times F) \cap A$.

Note that if $S \subset \mathbb{C}$ is a Borel measurable set on which f_ω is injective then from the fact that $\varphi(\omega)$ is constant almost everywhere and from (2.19) we get

$$\mu_{A,\sigma\omega}(f_\omega(S)) = 2 \cdot \frac{\mu_\omega(A_\omega)}{\mu_{\sigma\omega}(A_{\sigma\omega})}\mu_{A,\omega}(S) = 2\mu_{A,\omega}(S).$$

Similarly if ${}^k_\omega$ is injective on S then

$$\mu_{A,\sigma^k\omega}(f_\omega^k(S)) = 2^k\mu_{A,\omega}(S).$$

Clearly the above equalities also hold if we replace the set A with B. Finally if S, S' are topological disks and $f_\omega^k : S \mapsto S'$ is a proper map of some degree $d \geqslant 1$ then

$$\mu_{A,\sigma^k\omega}(S') \leqslant 2^k\mu_{A,\omega}(S).$$

To prove the above note first that the map $f_\omega^k : S \mapsto S'$ is a branched covering map of degree d. Since the measure μ_ω has no atoms, one can find an arc $L \subset S'$ with one endpoint inside S' and the other in $\partial S'$, such that L contains all critical values and $\mu_{\sigma^k\omega}(L) = 0$. The set $S' \setminus L$ is simply connected. Choose a branch of $(f_\omega^k)^{-1}$ defined on $S' \setminus L$ and mapping this set onto some set $S'' \subset S$. Then $f_\omega^k : S'' \mapsto S' \setminus L$ is injective, and thus indeed we get

$$\mu_{A,\sigma^k\omega}(S') = \mu_{A,\sigma^k\omega}(S' \setminus L) = 2^k\mu_{A,\omega}(S'') \leqslant 2^k\mu_{A,\omega}(S).$$

Now denote as before $D = D_{R_0}$. Let $z \in \mathcal{J}_\omega$ and let $S_k(z)$ be the connected component of $(f_\omega^k)^{-1}(D)$ containing z. Choose an arbitrary $\omega \in \Omega$ for which the statement of Proposition 2.58 holds and let k_n be the sequence from this Proposition which is assigned to ω. Let $z \in \mathcal{J}_\omega$. Then the map

$$f_\omega^{k_n-K} : S_{k_n}(z) \mapsto f_\omega^{k_n-K}(S_{k_n}(z))$$

is injective, and therefore we have

$$\mu_{A,\omega}(S_{k_n}(z)) = \frac{1}{2^{k_n-K}}\mu_{A,\sigma^{k_n-K}\omega}(f_\omega^{k_n-K}(S_{k_n}(z))) \leqslant \frac{1}{2^{k_n-K}}.$$

For the measure $\mu_{B,\omega}(S_{k_n}(z))$ we will use the following lower estimate

$$\mu_{B,\omega}(S_{k_n}(z)) \geqslant \frac{1}{2^{k_n}} \mu_{\sigma^{k_n}\omega,B}(D) = \frac{1}{2^{k_n}}.$$

Combining the above we obtain

$$\frac{\mu_{A,\omega}(S_{k_n}(z))}{\mu_{B,\omega}(S_{k_n}(z))} \leqslant 2^K. \tag{2.33}$$

We shall show that this implies absolute continuity of the measure $\mu_{A,\omega}$ with respect to $\mu_{B,\omega}$.

Consider a set $C \subset \mathcal{J}_\omega$, such that $\mu_{B,\omega}(C) = 0$. Set $S_n(C) = \bigcup_{z \in C} S_n(z)$ and now since for almost every ω the set $\mathcal{J}_\omega$ is totally disconnected, we have

$$C = \bigcap_n S_n(C).$$

This yields

$$\mu_{A,\omega}(C) = \lim_{n\to\infty} \mu_{A,\omega}(S_n(C))$$

and analogously

$$\mu_{B,\omega}(C) = \lim_{n\to\infty} \mu_{B,\omega}(S_n(C)).$$

Thus for every $\delta > 0$ there exists n such that

$$\mu_{B,\omega}(S_{k_n}(C)) < \delta.$$

A straightforward application of (2.33) implies that we also have

$$\mu_{A,\omega}(S_{k_n}(C)) < \delta 2^K$$

and thus also

$$\mu_{A,\omega}(C) < \delta 2^K.$$

Since δ is arbitrarily small, we have $\mu_{A,\omega}(C) = 0$ and thus indeed, $\mu_{A,\omega}$ is absolutely continous with respect to $\mu_{B,\omega}$, which is a contradiction, since for almost all ω we have $A_\omega \cap B_\omega = \emptyset$.

□

In this section we prove the following theorem.

Theorem 2.59. *Let $\mathcal{S}$ be a bounded random system of independent quadratic maps. Assume additionally that $\mathcal{S}$ is typically fast escaping (see* (2.12)*).*

Then for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$\dim_H(\mu_\omega) = \frac{\log 2}{\chi} = \frac{\log 2}{\log 2 + \mathbf{g}(0)} < 1 \tag{2.34}$$

Remark 2.60 It is worth to compare the formula given in Theorem 2.59 with the estimate obtained in the paper of Brück, [7]. Using our notation, it is not difficult to observe that for every $\omega \in \mathbb{D}(0,R)^{\mathbb{N}}$ the Green function g_ω is Hölder continuous with any Hölder exponent smaller than

$$\alpha_0 = \frac{\log 2}{\log 2 + \log R_0}$$

Using this estimate, the author of [7] then applies the following theorem from [10], which is interesting in its own right.

Theorem 2.61. *Let $E \subset \mathbb{R}^m$ be a compact set, and let $D \subset \mathbb{R}^m$ be a domain bounded by a smooth outer surface Γ and by the set E. Denote by H_α the class of harmonic functions on D which satisfy the Hölder condition of order $0 < \alpha < 1$. Then E is removable for the class H_α if and only if E has $(m-2+\alpha)$-dimensional measure zero.*

With the help of the above the author of [7] concludes that for every $\omega \in \Omega$

$$\dim_H(J_\omega) \geqslant \alpha_0 = \frac{\log 2}{\log 2 + \log R_0},$$

the estimate which, for $\mathbb{P}$– typical ω is immediately implied by our Theorem 2.59, since for $\mathbb{P}$– typical ω we can estimate:

$$\dim_H(J_\omega) \geqslant \dim_H(\mu_\omega) = \frac{\log 2}{\chi} \geqslant \frac{\log 2}{\log 2 + \log R_0} = \alpha_0.$$

Finally, it is worth remarking that the estimate for the dimension of the maximal measure can also be directly proven using α_0-Hölder continuity of the Green function. For details the reader can see the the work done in [28] (specifically Lemma 3.5).

Proof of Theorem 2.59. .
Choose $\omega \in \Omega$ for which satisfying the assertion of Proposition 2.58 holds. Let $z \in J_\omega$. Consider the sequence $D_j^{k_n}(z)$ of connected components of $(f_\omega^{k_n})^{-1}(D)$, containing the point z. Every such component $D_j^{k_n}(z)$ is mapped by

$$f_\omega^{k_n-K}$$

univalently onto its image B, which is some component in the collection $\{D_i^K\}$ of connected components of $f^{-K}(D)$.

The following lemma is then a straightforward consequence of the Koebe distortion theorem.

Lemma 2.62. *There exists $C > 0$ such that the following holds. Choose an arbitrary $\omega \in \Omega$ satisfying Proposition 2.58 and $z \in J_\omega$. Let $k_n = k_n(\omega)$ be the sequence described in the assertion of Proposition 2.58. Then*

$$\mathbb{D}\left(z, \frac{1}{C} \cdot \frac{1}{|(f_\omega^{k_n-K})'(z)|}\right) \subset D_j^{k_n}(z) \subset \mathbb{D}\left(z, C \cdot \frac{1}{|(f_\omega^{k_n-K})'(z)|}\right)$$

Now, note that the measure μ_ω of such component $D_j^{k_n}(z)$ satisfies

$$\frac{1}{2^{k_n}} \leqslant \mu_\omega(D_j^{k_n}(z)) = \frac{1}{2^{k_n-K}} \cdot \mu_{\sigma^{k_n-K}\omega}(B) \leqslant \frac{1}{2^{k_n-K}}.$$

So, denoting $r_n = r_n(z) = \frac{1}{C}\frac{1}{|(f_\omega^{k_n-K})'(z)|}$ we conclude, using Proposition 2.57 applied to the function $(\omega, z) \mapsto \log|f_\omega'(z)|$, that for $\mathbb{P}$-a.e. $\omega \in \Omega$, for μ_ω a.e. $z \in J_\omega$:

$$\frac{-\log \mu_\omega(\mathbb{D}(z, r_n))}{-\log r_n} \geqslant \frac{(k_n - K)\log 2}{\log|(f_\omega^{k_n-K})'(z)| + \log C} \xrightarrow{n\to\infty} \frac{\log 2}{\chi}.$$

So,

$$\liminf_{n\to\infty} \frac{-\log \mu_\omega \mathbb{D}(z, r_n)}{-\log r_n} \geqslant \frac{\log 2}{\chi}$$

and, similarly, denoting $R_n = R_n(z) = \frac{C}{|(f_\omega^{k_n-K})'(z)|}$, we conclude that

$$\limsup_{n\to\infty} \frac{-\log \mu_\omega \mathbb{D}(z, R_n)}{-\log R_n} \leqslant \frac{\log 2}{\chi}.$$

Now, let $r > 0$ be an arbitrary radius in $(0, r_1)$. Take the largest n such that $r < r_n$. Then

$$r_{n+1} \leqslant r < r_n$$

and

$$\frac{-\log(\mu_\omega(\mathbb{D}(z,r))}{-\log r} \geqslant \frac{-\log(\mu_\omega \mathbb{D}(z,r_n))}{-\log r} = \frac{-\log(\mu_\omega \mathbb{D}(z,r_n))}{-\log r_n} \cdot \frac{-\log r_n}{-\log r} \tag{2.35}$$

Next, notice that

$$\frac{-\log r_n}{-\log r} \geqslant \frac{-\log r_n}{-\log r_{n+1}} = \frac{\log|(f_\omega^{k_n-K})'(z)| + \log C}{\log|(f_\omega^{k_{n+1}-K})'(z)| + \log C} = \frac{k_n}{k_{n+1}} \cdot \frac{\frac{1}{k_n}(\log|(f_\omega^{k_n-K})'(z)| + \log C)}{\frac{1}{k_{n+1}}(\log|(f_\omega^{k_{n+1}-K})'(z)| + \log C)} \tag{2.36}$$

Since, by Proposition 2.58, $\frac{k_n}{k_{n+1}} \to 1$ as $n \to \infty$, combining (2.35) and (2.36), we conclude that (again for almost every ω and almost every z with respect to the measure μ_ω):

$$\liminf_{r \to 0} \frac{-\log \mu_\omega(\mathbb{D}(z,r))}{-\log r} \geqslant \frac{\log 2}{\chi}.$$

An analogous reasoning with r_n replaced by R_n gives that

$$\limsup_{r \to 0} \frac{-\log \mu_\omega(\mathbb{D}(z,r))}{-\log r} \leqslant \frac{\log 2}{\chi}.$$

We conclude that for $\mathbb{P}$ -a.e. $\omega \in \Omega$ and and for a.e. z with respect to μ_ω the limit

$$\lim_{r \to 0} \frac{\log \mu_\omega(B(z,r))}{\log r}$$

exists, and and it is equal to $\frac{\log 2}{\chi} = \frac{\log 2}{\log 2 + \mathbf{g}(\mathbf{0})}$.

It is now standard (see,e.g. [26] Theorem 8.6.3) to conclude that for $\mathbb{P}$ -a.e. $\omega \in \Omega$

$$\dim \mu_\omega = \frac{\log 2}{\log 2 + \mathbf{g}(0)}.$$

□

2.3.6 Dimension of the maximal measure-dependence on a parameter

In Section 2.3.5 we calculated Hausdorff dimension of the maximal measure on the Julia set of a randomly iterated polynomial. We would now like to examine the dependence of this dimension on the parameter space Ω. Clearly, the global Green function $\mathbf{g}$ depends on this space, in particular if $\Omega = \mathbb{D}(0,R)^{\mathbb{N}}$ (along with a product of uniform distributions), then $\mathbf{g}$ depends on R.

So, actually, the formula for the dimension of μ_ω should be written as:

$$\dim_H(\mu_\omega) = \frac{\log 2}{\log 2 + \mathbf{g}_R(0)}$$

to underline the dependence on R.

This motivates the question of investigating the regularity of the function $R \to \mathbf{g}_R(0)$.

Note that a regularity of the above function automatically implies the same regularity of the function

$$R \mapsto \dim_H(\mu_\omega) = \frac{\log 2}{\log 2 + \mathbf{g}_R(0)}$$

Let us recall what is known for the deterministic case, $z^2 + c$. We have:

$$\dim_H(\mu) = \frac{\log 2}{\log 2 + g_c(0)} = \frac{\log 2}{\log 2 + \frac{1}{2} g_c(c)},$$

where g_c is the Green function on $\mathcal{A}_c$, the basin of ∞ for f_c.

The value of $g_c(c)$ appears in the formula for the conformal representation of the complement of the Mandelbrot set:

Let φ_c be the conjugacy of f_c to z^2, tangent to identity at infinity. Then φ_c can be extended to a conjugacy on the set

$$\{z : g_c(z) > g_c(0)\},$$

and for such z,

$$g_c(z) = \log |\varphi_c(z)|.$$

The function

$$\mathbb{C} \setminus \mathcal{M} \ni c \mapsto \varphi_c(c)$$

is a conformal isomorphism of the complement of the Mandelbrot set onto $\mathbb{D}^*$.

So, we see that the function $c \mapsto g_c(0) = \frac{1}{2} g_c(c)$ is harmonic, thus– real- analytic. Therefore, the function $\mathbb{C} \setminus \mathcal{M} \ni c \mapsto \dim_H(J_c)$ is real analytic. Moreover, the function

$$\mathbb{C} \setminus \mathcal{M} \ni c \mapsto \dim_H(\mu_c)$$

is constant on "equipotential" lines (level sets of the Green function) of the set $\mathbb{C} \setminus \mathcal{M}$.

2.3.7 Dependence of the dimension of the random maximal measure on the radius R

In this section we give an initial answer to the question about regularity of the function $R \to \mathbf{g}_R(0)$.

From now on let us denote $\Omega_R = \mathbb{D}(0, R)^{\mathbb{N}}$, where $R \in (R_1, R_2)$, for some $R_2 > R_1 > 0$. Pick R_0 large enough as in Proposition 2.12 for Ω_{R_2} and for $\epsilon = 1$. Then clearly this R_0 is an appropriate choice for Proposition 2.12 for all Ω_R where $R \in (R_1, R_2)$.

Lemma 2.63. *There exist $M, h > 0$ such that for all z, z' such that $|z - z'| < h$ and $R \in (R_1, R_2)$ and all $\omega, \omega' \in \Omega_R$ we have*

$$|g_\omega(z) - g_{\omega'}(z')| < M.$$

Proof. Let us set h such that for $|z - z'| < h$ we have $|\ln(z) - \ln(z')| < 1$. For $|z| > R_0$ we have $g_\omega(z) = \ln|z| + \epsilon(\omega, z)$, where $|\epsilon(\omega, z)| \leqslant 1$. (see Proposition 2.12). So assume $|z| > R_0 + h$ (and thus of course $|z'| > R_0$), then we have

$$|g_\omega(z) - g_{\omega'}(z')| = |\ln(z) - \ln(z') + \epsilon(\omega, z) - \epsilon(\omega', z)| < 2 + |\ln(z) - \ln(z')| < 3.$$

It remains to prove the lemma for $z \in \mathbb{D}_{R_0+h}$.
If $z \in \mathbb{D}_{R_0+h}$ then $g_\omega(z) \leqslant \ln|R_0| + C(h)$ (where $C(h)$ is some constant dependent on h), see again Proposition 2.12. Denoting $G := \ln|R_0| + C(h)$, we can estimate:

$$|g_\omega(z) - g_{\omega'}(z')| < 2G,$$

and setting $M = \max(2G, 3)$ concludes the proof.

□

Since each sequence ω is a function $\omega : \mathbb{N} \mapsto \mathbb{D}_R$, in subsequent lemmas we shall use the notation $\omega(i)$, for $i \in \mathbb{N}$.

Lemma 2.64. *For every ϵ, there exist $N \in \mathbb{N}$ and $\tilde{h} > 0$ such that for all $R \in (R_1, R_2)$ and all $\omega, \omega' \in \Omega_R$ the following holds:*

If $|\omega'(i) - \omega(i)| < \tilde{h}$ for all $i \leqslant N$, then $|g_\omega(0) - g_{\omega'}(0)| < \epsilon$.

Proof. Set ϵ and let N be such that

$$\max(\frac{G}{2^N}, \frac{M}{2^N}) < \epsilon,$$

where M is chosen (along with a constant h) as in the previous lemma and G is the constant which bounds $g_\omega(z)$ in $\mathbb{D}_{R_0}$. Finally let $\tilde{h}$ be small enough so that

$$|\omega'(i) - \omega(i)| < \tilde{h} \quad \text{for all} \quad i \leqslant N \Rightarrow |f_\omega^N(0) - f_{\omega'}^N(0)| < h$$

where h is chosen with M as in Lemma 2.63.

We consider now three cases: when 0 is escaping for both ω and ω', when it is escaping for one of these sequences, and when it is escaping for neither.

Assume first that 0 is escaping for both ω and ω'. Let $z = f_\omega^N(0)$ and $z' = f_{\omega'}^N(0)$. Since 0 is escaping, the formula

$$g_\omega(0) = \lim_{n\to\infty} \frac{1}{2^n} \ln|f_\omega^n(0)|$$

holds. We have

$$\begin{aligned}
|g_\omega(0) - g_{\omega'}(0)| &= \lim_{n\to\infty} \frac{1}{2^n} |\ln(|f_\omega^n(0)|) - \ln(|f_{\omega'}^n(0)|)| \\
&= \lim_{n\to\infty} \frac{1}{2^N} |\frac{1}{2^n} \ln(|f_\omega^{n+N}(0)|) - \frac{1}{2^n} \ln(|f_{\omega'}^{n+N}(0)|)| \\
&= \lim_{n\to\infty} \frac{1}{2^N} (\frac{1}{2^n} \ln(|f_{\sigma^N\omega}^n(z)|) - \frac{1}{2^n} \ln(|f_{\sigma^N\omega'}^n(z')|)) \\
&= \frac{1}{2^N} (g_{\sigma^N\omega}(z) - g_{\sigma^N\omega'}(z')) \\
&< \frac{M}{2^N}.
\end{aligned}$$

Assume now that 0 is escaping for ω but not ω'. This means $g_{\omega'}(0) = 0$, whereas $g_\omega(0) = \lim_{n\to\infty} \frac{1}{2^n} | \ln(|f_\omega^n(0)|)$. Moreover $R_0 > |f_{\omega'}^N(0)|$ which implies $R_0 + h > |f_\omega^N(0)|$. Thus we have:

$$\begin{aligned}
|g_\omega(0) - g_{\omega'}(0)| &= \lim_{n\to\infty} \frac{1}{2^n} | \ln(|f_\omega^n(0)|) \\
&= \lim_{n\to\infty} \frac{1}{2^{n+N}} | \ln(|f_\omega^{n+N}(0)|) \\
&= \frac{1}{2^N} \lim_{n\to\infty} \frac{1}{2^n} | \ln(|f_{\sigma^N\omega}^n(f_\omega^N(0))|) \\
&= \frac{1}{2^N} g_{\sigma^N\omega}(f_\omega^N(0)) \\
&< \frac{\tilde{G}}{2^N}.
\end{aligned}$$

where $\tilde{G} = \ln|R_0 + h| + 1$ is the bound of $g_{\sigma^N\omega}(z)$ in $\mathbb{D}_{R_0+h}$.

Finally if 0 is not escaping for ω, ω', then the result holds trivially as $g_\omega(0) = g_{\omega'}(0) = 0$. This concludes the proof.

□

For a sequence $\omega = (c_1, c_2, \dots)$ let us denote by $(1+h)\omega$ the sequence $(1+h)\omega := ((1+h)c_1, (1+h)c_2 \dots)$, and by $R\omega$ the sequence $R\omega = (Rc_0, Rc_1, \dots)$. Then Lemma 2.64 yields the following as a corollary.

Lemma 2.65. *For every $R > 0, \epsilon > 0$ there exists $h > 0$ such that for every $\omega \in \Omega_R$ we have $|g_\omega(0) - g_{(1+h)\omega}(0)| < \epsilon$.*

We denote by $\mathbb{P}_R$ the product distribution of uniform distributions on $\mathbb{D}(0,R)$ in the product space $\mathbb{D}(0,R)^{\mathbb{N}}$.

Theorem 2.66. *The function $\mathbf{g}_R(0) : (0,\infty) \to \mathbb{R}$ is a continuous function of R.*

Proof. Let $R \in (0,\infty)$, choose some $R_1 < R_2$ such that $0 < R_1 < R < R_2$.

We have

$$\mathbf{g}_R(0) = \int_{\mathbb{D}(0,R)^{\mathbb{N}}} g_\omega(0) d\mathbb{P}_R(\omega) = \int_{\mathbb{D}(0,1)^{\mathbb{N}}} g_{R\omega}(0) d\mathbb{P}(\omega).$$

Now from Lemma 2.65 we conclude that for any $\epsilon > 0$ there exists $h_\epsilon > 0$ such that for all $h < h_\epsilon$ we obtain

$$|\mathbf{g}_R(0) - \mathbf{g}_{R+h}(0)| \leqslant \int_{\mathbb{D}(0,1)^{\mathbb{N}}} |g_{R\omega}(0) - g_{(R+h)\omega}(0)| d\mathbb{P}(\omega) < \epsilon.$$

□

Note that for $R < \frac{1}{4}$ the function in the above theorem is constant, and thus on $(0, \frac{1}{4})$ the statement is obvious. The next result gives the information about the asymptotics of the generalized Green function at infinity.

Theorem 2.67. *The generalized Green function $\mathbf{g}_R(0)$ satisfies $\lim_{R\to\infty} \frac{\mathbf{g}_R(0)}{\ln(R)} = 1$.*

Proof. First, note that for any $\omega \in \Omega_R$ we have $g_\omega(0) \leqslant g_{\omega_R}(0)$ where ω_R is the constant sequence defined by $\omega_R(i) = R$. This is a straightforward consequence of the formula for the Green function in Proposition 2.9 - the sequence ω_R gives the fastest growth rate of $|f^n_{\omega_R}(0)|$ out of all possible sequences.

Consequently, we have

$$\mathbf{g}_R(0) = \int_{\Omega_R} g_\omega(0) d\mathbb{P}_R(\omega) \leqslant \int_{\Omega_R} g_{\omega_R}(0) d\mathbb{P}_R(\omega) = g_{\omega_R}(0).$$

We will now show a similar bound for $g_R(0)$ from below. Let us denote

$$E_{R,\delta} = \{\omega \in \Omega_R : |\omega(0)| > \delta R\},$$

where $\delta \in (0,1)$. Then clearly the measure $\mathbb{P}_R(E_{R,\delta})$ is simply $1 - \delta^2$. Take any sequence $\omega' \in E_{R,\delta}$ for which 0 is escaping (so almost every ω' will do). Let us denote $h_R(z) = z^2 - R$. Let us fix δ for now, and note that for R large enough and some $C > 0$ (both chosen for this set δ) we have

$$\begin{aligned} g_{\omega'}(0) &= \lim_{n\to\infty} \frac{1}{2^n} \ln(|f^n_{\omega'}(0)|) \\ &\geqslant \lim_{n\to\infty} \frac{1}{2^n} \ln(|h^{n-1}_R(\delta R)|) \\ &\geqslant \lim_{n\to\infty} \frac{1}{2^n} \ln(C(\delta R)^{2^n}) \\ &\geqslant \ln(\delta R). \end{aligned}$$

The first equality is just a formula for Green's function which holds when 0 is escaping, the first inequality follows from a simple induction argument. Indeed, by assumption we have

$$|f_{\omega'}(0)| \geqslant \delta R$$

and it is easy to see that we have

$$|f^n_{\omega'}(0)| = |f^{n-1}_{\omega'}(0)^2 + c_n| \geqslant |f^{n-1}_{\omega'}(0)|^2 - R = |h_R(f^{n-1}_{\omega'}(0))|.$$

Thus we have

$$\mathbf{g}_R(0) = \int_{\Omega_R} g_\omega(0) d\mathbb{P}_R(\omega) \geqslant \int_{E_{R,\delta}} g_\omega(0) d\mathbb{P}_R(\omega) \geqslant \int_{E_{R,\delta}} \ln(\delta R) d\mathbb{P}_R(\omega) \geqslant (1-\delta^2)\ln(\delta R).$$

So we end up with

$$(1-\delta^2)\ln(\delta R) \leqslant \mathbf{g}_R(0) \leqslant g_{\omega_R}(0).$$

The left hand side yields

$$\liminf_{R\to\infty} \frac{\mathbf{g}_R(0)}{\ln(R)} \geqslant \lim_{R\to\infty} \frac{(1-\delta^2)\ln(\delta R)}{\ln(R)} = 1-\delta^2$$

and since δ can be chosen arbitrarily close to 0 this concludes the proof in one direction. Now for the other inequality. Note that there exists a constant M such that for all n the polynomial $\tilde{h}_R(z) = z^2 + R$ satisfies

$$\tilde{h}^n_R(0) < (MR)^{2^n}.$$

Thus we have

$$\limsup_{R\to\infty} \frac{\mathbf{g}_R(0)}{\ln(R)} \leqslant \lim_{R\to\infty} \frac{g_{\omega_R}(0)}{\ln(R)} \leqslant \lim_{R\to\infty} \frac{\ln(MR)}{\ln(R)} = 1$$

which concludes the proof. □

2.3.8 Randomly perturbed parameter $c_0 \notin \mathcal{M}$.

In this subsection, we consider a related problem. Namely, we study random iterates of the maps $z^2 + c$ where c is chosen independently at each step from the disc $\mathbb{D}(c_0, \delta_0)$ (with some fixed distribution), where $c_0 \notin \mathcal{M}$ and δ_0 is small enough, in particular $\mathrm{dist}(c_0, \mathcal{M}) > \delta_0$.

Again, the dimension of the equilibrium measure is constant almost everywhere. It is given by the formula analogous to (2.34) (see Proposition 2.69).

Next, we study the dependence of this value on the "range of randomness" δ_0. This dependence turns out to be very regular.

More formally, let us introduce the product space:

$$\Omega = \mathbb{D}(0,1)^{\mathbb{N}}.$$

Let ν be a probability distribution on $\mathbb{D}(0,1)$, and let $\mathbb{P}$ be the completed (see [14] for details on measure completeness) product distribution on Ω. So, the space Ω consists of infinite sequences of points in $\mathbb{D}(0,1)$, denoted by $\omega = (v_0, v_1, \dots)$. We now introduce the dependence on λ. For each $\lambda \in \Lambda := \mathbb{D}(0, \delta_0)$ denote

$$\Omega_\lambda = \lambda \cdot \Omega,$$

i.e, a point $\omega_\lambda \in \Omega_\lambda$, is represented as $\lambda \cdot \omega = (\lambda v_0, \lambda v_1, \dots)$, where $\omega \in \Omega$.

For each $\omega = (v_0, v_1, \dots)$ we write $f_{\lambda,\omega}$ to denote the map

$$f_{\lambda,\omega} = f_{c_0 + \lambda v_0}. \tag{2.37}$$

Analogously,

$$f^n_{\lambda,\omega} = f_{c_0+\lambda v_{n-1}} \circ \dots f_{c_0+\lambda v_2} \circ f_{c_0+\lambda v_1} \circ f_{c_0+\lambda v_0}. \tag{2.38}$$

The Green's function for the non-autonomous sequence above shall be denoted by $g_{\lambda,\omega}$. Now put $R = |c_0| + \mathrm{dist}(c_0, \mathcal{M})$. Then all parameters $c_0 + \lambda v_j$ are in the disc $\mathbb{D}(0, R)$. Let R_0 be the value assigned to R as in Proposition 2.12.

We denote by $J_{\lambda,\omega}$ the Julia set for the non- autonomous sequence $f^n_{\lambda,\omega}$, and by $\mathcal{A}_{\lambda,\omega}$ the basin of infinity for this sequence.

Let us note the following (stability property).

Proposition 2.68. *For every $\epsilon > 0$ there exists $\delta_0 > 0$ such that each Julia set $J_{\lambda,\omega}$ with $\lambda \in \mathbb{D}(0, \delta_0)$ is ϵ -close to the Julia set $J(f_{c_0})$, i.e.,*

$$J_{\lambda,\omega} \subset B(J(f_{c_0}), \epsilon).$$

Proof. Choose an arbitrary $\epsilon > 0$. Note that on the set $\mathbb{D}(0, R_0) \setminus B(J(f_{c_0}), \epsilon)$ the iterates $f^n_{c_0}$ converge to ∞ uniformly. Therefore, one can choose a common value n_0 such that for every $z \in \mathbb{D}(0, R_0) \setminus B(J(f_{c_0}), \epsilon)$ we have $|f^{n_0}_{c_0}(z)| > 2R_0$. Consequently, for δ_0 small enough, every $z \in \mathbb{D}(0, R_0) \setminus B(J(f_{c_0}), \epsilon)$, every $\lambda \in \mathbb{D}(0, \delta_0)$ and every $\omega \in \Omega$ the estimate $|f^{n_0}_{\lambda,\omega}(z)| > R_0$ holds. It now follows from the choice of R_0 that $z \in \mathcal{A}_{\lambda,\omega}$.

□

We will denote by $\mu_{\lambda,\omega}$ the equilibrium (harmonic) measure on the Julia set $J_{\lambda,\omega}$ for the non- autonomous sequence $f^n_{\lambda,\omega}$. The following proposition is a simpler version of Theorem 2.59, its proof simplified by the fact that in this case, as noticed in Remark 2.29, the sequence k_n can be taken to be the sequence of all sufficiently large integers.

Proposition 2.69. *Let $c_0 \notin \mathcal{M}$. There exists $\delta_0 > 0$ (depending on the choice of c_0) such that the following holds. Choose an arbitrary $\lambda \in \Lambda = \mathbb{D}(0, \delta_0)$. For $\mathbb{P}$-a.e. $\omega \in \Omega$*

$$\dim_H(\mu_{\lambda,\omega}) = \frac{\log 2}{\chi} = \frac{\log 2}{\log 2 + \mathbf{g}_\lambda(0)} < 1 \tag{2.39}$$

where $\mathbf{g}_\lambda = \int_\Omega g_{\lambda,\omega} d\mathbb{P}(\omega)$.

Recall that by Proposition 2.38 we know that the following holds:

There exists $R_0 > 0$ such that for an arbitrary point $z_0 \in \mathbb{C}$ with $|z_0| > R_0$

$$\mu_{\lambda,\omega} = \lim_{n\to\infty} \frac{1}{2^n} \sum_{y \in (f^n_{\lambda,\omega})^{-1}(z_0)} \delta_y$$

(the convergence in weak-* topology).

We call a point $z_0 \in \mathbb{D}^*_{R_0}$ non-exceptional for c_0 if $z_0 \notin \bigcup_{n\in\mathbb{N}} f^n_{c_0}(0)$. Since $f^n_{c_0}(0) \to \infty$ as $n \to \infty$, this automatically implies that for a non-exceptional point z_0 we have that

$$\inf_{n\in\mathbb{N}} |z_0 - f^n_{c_0}(0)| > 0.$$

Lemma 2.70. *Let z_0 be non- exceptional for $c_0 \notin \mathcal{M}$. There exists $\delta_0 > 0$ and $\tau > 0$ such that for every $\omega \in \Omega$, for every $\lambda \in \Lambda := \mathbb{D}(0, \delta_0)$,*

$$\bigcup_{n\in\mathbb{N}} f^n_{\lambda,\omega}(0) \cap \mathbb{D}(z_0, \tau) = \emptyset.$$

Proof. The proof is straightforward and similar to that of Propositon 2.68. Put

$$2\tau = \inf_{n\in\mathbb{N}} |z_0 - f^n_{c_0}(0)|.$$

Remember that the iterates $f^n_{c_0}(0)$ converge to infinity, so, actually, the above infimum is taken over a finite set of integers $n \leqslant M$ for some M depending on c_0.

Then for δ_0 sufficiently close to 0 we have

$$z_0 \notin \bigcup_{n=0}^{\infty} \mathbb{D}(f^n_{\lambda,\omega}(0), \tau)$$

for any $\omega \in \Omega$, $\lambda \in \Lambda$.

□

Lemma 2.71. *Let $c_0 \notin \mathcal{M}$, and let $f_{\lambda,\omega}$ be defined as in (2.37) and (2.38). For every δ_0 small enough there exists $\eta > 0$ such that for all $\omega \in \Omega$, $\lambda \in \Lambda$, $J_{\lambda,\omega} \cap \mathbb{D}(0,\eta) = \emptyset$.*

Proof. Since $f^n_{c_0}(0) \to \infty$ as $n \to \infty$, there exists $n_0 > 0$ such that $|f^{n_0}_{c_0}(0)| > 2R_0$. Thus, for δ_0 small enough, and every ω, $|f^{n_0}_{\lambda,\omega}(0)| > R_0$, and, by an equicontinuity argument, the same holds for every $z \in \mathbb{D}(0,\eta)$ with $\eta > 0$ sufficiently small. The conclusion now follows from the choice of R_0. □

Lemma 2.72. *Let $c_0 \notin \mathcal{M}$ and z_0 be non–exceptional for c_0. For every $\epsilon > 0$ there exists $\delta_0 > 0$ and $n_0 \in \mathbb{N}$ such that for every $\omega \in \Omega$, $\lambda \in \Lambda = \mathbb{D}(0,\delta_0)$ and $n > n_0$ all preimages of z_0 under $f^n_{\lambda,\omega}$ are ϵ–close to J_{c_0}, i.e.,*

$$f^{-n}_{\lambda,\omega}(\{z_0\}) \subset B(J_{c_0}, \epsilon)$$

Proof. The proof is similar to that of Proposition 2.68. Let $\epsilon > 0$. For sufficiently small $\delta_0 > 0$ the iterates $f^n_{\lambda,\omega}(z)$ converge to infinity uniformly with respect to $\omega \in \Omega$, $\lambda \in \Lambda = \mathbb{D}(0,\delta_0)$ and $z \in \mathbb{C} \setminus B(J_{c_0}, \epsilon)$. So, there exists $n_0 > 0$ such that every point in

$$\bigcup_{n>n_0} (f^n_{\lambda,\omega})^{-1}(z_0)$$

must be in $B(J_{c_0}, \epsilon)$. This immediately yields the claim.

□

Lemma 2.73. *Let $c_0 \notin \mathcal{M}$ and let z_0 be non- exceptional for c_0. For sufficiently small $\delta_0 > 0$ the following holds: there exists n_1 such that for all $n > n_1$, for each $\omega \in \Omega$ and $\lambda \in \Lambda = \mathbb{D}(0,\delta_0)$, and each $y_0 \in (f^n_{\lambda,\omega})^{-1}(z_0)$ we have that*

$$|(f^n_{\lambda,\omega})'(y_0)| > 2.$$

Proof. Since the map f_{c_0} is expanding on its Julia set, there exists $\epsilon > 0$, $N \in \mathbb{N}$, $\delta_0 > 0$ and $\gamma > 1$ such that for every $(\lambda,\omega) \in \mathbb{D}(0,\delta_0) \times \Omega$

$$\inf_{z\in B(J_{c_0},\epsilon)} (|f^N_{\lambda,\omega})'(z)| > \gamma.$$

Modifying δ_0 if necessary, we can assume that Lemma 2.71 holds with δ_0, and choose n_0 such that the claim of Lemma 2.72 holds. So, if $y_0 \in f^{-n}_{\lambda,\omega}(z_0)$, with $n = n_0 + kN + r$, $r < N$, then

$$|(f^n_{\lambda,\omega})'(y_0)| > \gamma^k \cdot \left(\inf_{w \in B(J_{c_0},\epsilon)} |f_{\lambda,\omega})'| \right)^r \cdot \inf_{w \in f^{-n_0}_{\lambda,\omega}(z_0)} |(f^{n_0}_{\lambda,\omega})'(w)|$$

The first infimum is positive by Lemma 2.71. Modifying δ_0 if necessary and applying Lemma 2.70, it is easily seen that the second infimum is also positive.

Choosing k sufficiently large completes the proof. □

Proposition 2.74 (analytic continuation of preimages). *Let $c_0 \notin \mathcal{M}$. Fix z_0 a non-exceptional point for c_0. For δ_0 sufficiently small and $n > n_1$ (where n_1 comes from Lemma 2.73) the following holds:*

For an arbitrary $\omega \in \Omega$, $\lambda_0 \in \Lambda = \mathbb{D}(0, \delta_0)$ and for any $y_0 \in (f^n_{\lambda_0,\omega})^{-1}(z_0)$ there exists a unique holomorphic function

$$\mathbb{D}(0, \delta_0) = \Lambda \ni \lambda \mapsto y(\lambda, y_0)$$

such that

$y(\lambda_0, y_0) = y_0$ and for all $\lambda \in \Lambda$

$$f^n_{\lambda,\omega}(y(\lambda, y_0)) = z_0.$$

Proof. This is a consequence of the Implicit Function Theorem. Indeed, define a function

$$\Phi(y, \lambda) := f^n_{\lambda,\omega}(y) - z_0.$$

Then the point y_0 solves the equation $\Phi(y, \lambda) = 0$, with $\lambda = \lambda_0$. Now, observe that

$$\frac{\partial \Phi}{\partial y}(y_0, \lambda_0) = (f^n_{\lambda_0,\omega})'(y_0)$$

Since $|(f^n_{\lambda_0,\omega})'(y_0)| > 0$ (in fact Lemma 2.73 gives an even stronger inequality), this implies that $\frac{\partial \Phi}{\partial y}(y_0, \lambda_0) \neq 0$ and the Implicit Function Theorem defines the required function $y(\lambda, y_0)$ in some neighbourhood of λ_0. It can be extended to the whole set Λ using analytic continuation and the fact that Λ is simply- connected.

□

Proposition 2.75. *Let $\epsilon > 0$. Then there exits $\delta_0 > 0$ such that for every bounded harmonic function φ defined in a neighbourhood $B(J_{c_0}, \epsilon)$ of J_{c_0} and for every $\omega \in \Omega$, $\lambda \in \Lambda = \mathbb{D}(0, \delta_0)$ the function*

$$\lambda \mapsto \mu_{\lambda,\omega}(\varphi)$$

is harmonic in $\Lambda = \mathbb{D}(0, \delta_0)$.

Proof. This follows directly from Proposition 2.38 and 2.74. Indeed, we can write

$$\mu_{\lambda,\omega}(\varphi) = \lim_{n\to\infty} \frac{1}{2^n} \sum_{y \in (f^n_{\lambda,\omega})^{-1}(z_0)} \varphi(y) = \lim_{n\to\infty} \frac{1}{2^n} \sum_{y_0 \in (f^{-n}_{c_0})(z_0)} \varphi(y(\lambda, y_0)),$$

where $z_0 \in \mathbb{D}^*_{R_0}$ and z_0 is non-exceptional for c_0. Since each function $\lambda \mapsto y(\lambda, y_0))$ is holomorphic in Λ, each function $\lambda \mapsto \varphi(y(\lambda, y_0))$ is harmonic in Λ. Since φ is bounded, we obtain a family of bounded harmonic functions

$$\mathbb{D}(0,\delta_0) \ni \lambda \mapsto \psi_n(\lambda) := \frac{1}{2^n} \sum_{y \in (f^n_{\lambda,\omega})^{-1}(z_0)} \varphi(y)$$

This family of functions is, therefore, equicontinuous on compact subset of $\mathbb{D}(0,\delta_0)$. On the other hand, we know that $\psi_n(\lambda)$ converges pointwise to $\mu_{\lambda,\omega}(\varphi)$. We conclude that this convergence is actually uniform on compact subsets and, therefore, the function

$$\lambda \mapsto \mu_{\lambda,\omega}(\varphi)$$

is harmonic in Λ.

□

Proposition 2.76. *Let $c_0 \notin \mathcal{M}$. There exists $\delta_0 > 0$ such that the following holds: For every $\omega \in \Omega$ the function*

$$\lambda \mapsto g_{\lambda,\omega}(0)$$

is harmonic in $\Lambda = \mathbb{D}(0,\delta_0)$. Moreover, the function

$$\lambda \mapsto \mathbf{g}_\lambda(0)$$

is harmonic in Λ.

Proof. The function $\varphi(z) = \log|z|$ is harmonic in $\mathbb{C} \setminus \{0\}$. By Lemma 2.71 we conclude that for sufficiently small δ_0 there is a common neighbourhood V of all Julia sets $J_{\lambda,\omega}$, $\lambda \in \Lambda$, $\omega \in \Omega$, on which the function φ is harmonic and bounded.

Applying Proposition 2.75, we conclude that for every $\omega \in \Omega$ the function

$$\lambda \mapsto g_{\lambda,\omega}(0) = \int \ln|z| d\mu_{\lambda,\omega}(z) \tag{2.40}$$

is harmonic in Λ.

Recall the formula for the global Green function:

$$\mathbf{g}_\lambda(0) = \int_\Omega g_{\lambda,\omega}(0) d\mathbb{P}(\omega).$$

Now we can conclude from harmonicity of all the functions in the formula (2.40) (see for instance Theorem 2.4.8 in [27]) that the following is also a harmonic function in Λ:

$$\lambda \mapsto \mathbf{g}_\lambda(0) = \int_{\omega\in\Omega} g_{\lambda,\omega}(0) d\mathbb{P}(\omega).$$

□

Remark 2.77 It is worth noting that Proposition 2.76 can also be shown directly from the definition of Green's function, by obtaining harmonicity of $g_{\lambda,\omega}(0)$ (as a function of λ) as a limit of a sequence of bounded harmonic functions (see formula (2.1)). In this case we would apply (2.1) for $z = 0$ and consider λ to be the variable.

Restricting the values of λ to real ones, we obtain the following result on the dependence of the dimension of $\mu_{\delta_0,\omega}$ (which is constant for $\mathbb{P}$-a.e. ω) on the size of the "randomness disc" δ_0.

Proposition 2.78. *Let ν be a Borel probability measure on $\mathbb{D}(0,1)$. Let $\mathbb{P}$ be the product distribution on Ω generated by ν. There exists $\delta_0 > 0$ such that the function $\delta \mapsto \mathbf{g}_\delta(0)$ is real analytic in the segment $[0, \delta_0)$. Consequently, the function*

$$[0, \delta_0) \ni \delta \mapsto \frac{\log 2}{\log 2 + \mathbf{g}_\delta(0)}$$

is real analytic in $[0, \delta_0)$. Thus by Theorem 2.59, the dimension $[0, \delta_0) \ni \delta \mapsto \dim \mu_{\delta,\omega}$ (the function which is constant for $\mathbb{P}$-a.e. $\omega \in \Omega$), is also real analytic.

By real analytic in the segment $[0, \delta_0)$ we understand that the function can be extended to an analytic function on some neighbourhood of 0 (which is clearly true, as again, this is just a restriction of a harmonic function on the entire disc of radius δ_0).

Assuming that the initial distribution ν is uniform, we obtain a stronger, and striking result:

Proposition 2.79. *Let ν be the uniform distribution on $\mathbb{D}(0,1)$. Let $\mathbb{P}$ be the product distribution on Ω generated by ν. For sufficiently small δ_0 the function $\delta \mapsto \mathbf{g}_\delta(0)$ is constant in the segment $[0, \delta_0)$. Consequently, for sufficiently small δ_0 the function*

$$[0, \delta_0) \ni \delta \mapsto \frac{\log 2}{\log 2 + \mathbf{g}_\delta(0)}$$

is constant in $[0, \delta_0)$, which implies, by Theorem 2.59, that the dimension
$[0, \delta_0) \ni \delta \mapsto \dim \mu_{\delta,\omega}$ (the function which is constant for $\mathbb{P}$-a.e. $\omega \in \Omega$) is constant in $[0, \delta_0)$, and equal to the dimension of μ_{c_0}, i.e.

$$\frac{\log 2}{\log 2 + g_{c_0}(0)}.$$

The proof will rely on the following simple lemma.

Lemma 2.80. *Let $h(z)$ be a harmonic function on a disk $D(0,r)$, and let h be rotation invariant, that is for any $\theta \in \mathbb{R}$ and any $z \in D(0,r)$ we have $h(z) = h(e^{2\pi i\theta} z)$. Then $h(z)$ is constant.*

Proof. Since $D(0,r)$ is simply connected we know there exists a holomorphic function f on $D(0,r)$ such that $\mathrm{Re}(f) = h$ (see for instance [27] Theorem 1.1.2). Moreover this function f is unique up to a constant. Now assume f is not constant (otherwise we are finished), then it cannot be rotation invariant, for instance by the open mapping theorem. Thus let us pick some number $\theta \in [0,1)$ and let $g(z) = f(e^{2\pi i\theta} z)$. Then g is a holomorphic function such that the function $f - g$ is not constant. Indeed, if $(f-g)(z) = C$ for some constant $C \in \mathbb{C} \setminus \{0\}$, then for all n we have

$$f(e^{2\pi i n\theta} z) = Cn + f(z).$$

This is impossible, since for rational θ it would yield $f(z) = f(z) + Cn_1$ for some n_1 such that $n_1\theta$ is a natural number. On the other hand if θ is irrational, then for any $z \in D(0,r)$ one can pick $n_1 \in \mathbb{N}$ such that $f(e^{2\pi i n_1\theta} z)$ is arbitrarily close to $f(z)$ by continuity of f, but at the same time $f(e^{2\pi i n_1\theta} z) = f(z) + Cn_1$, which is a contradiction.

Now note that since h is rotation invariant, and $\mathrm{Re}(f) = h$, then by the definition of g we also have $\mathrm{Re}(g) = h$. But the holomorphic function which has h as its real part is unique up to a constant, and $f - g$ is a non-constant function, which concludes the proof.

□

We can now move on to the proof of Proposition 2.79.

Proof of Proposition 2.79. We already know by Proposition 2.76 that the function

$$\lambda \mapsto \mathbf{g}_\lambda(0) = \int_{\omega\in\Omega} g_{\lambda,\omega}(0) d\mathbb{P}(\omega)$$

is harmonic on $\Lambda = \mathbb{D}(0, \delta_0)$.

Now, observe that, in the case of the product distribution $\mathbb{P}$ generated by the uniform distribution on $\mathbb{D}(0, \delta_0)$, this function is also rotation invariant. Indeed, let $\eta \in \mathbb{S}^1$ be an arbitrary rotation angle. Then

$$\mathbf{g}_{\eta\cdot\lambda}(0) = \int_{\omega\in\Omega} g_{\eta\cdot\lambda,\omega}(0) d\mathbb{P}(\omega) = \int_{\omega\in\Omega} g_{\lambda,\eta\cdot\omega}(0) d\mathbb{P}(\omega),$$

where we denoted by $\eta\cdot\omega$ the sequence with all entries multiplied by η, i.e., if $\omega = (v_0, v_1, \dots)$ the $\eta \cdot \omega = (\eta v_0, \eta v_1, \dots)$. Now notice that the map:

$$E : \Omega \to \Omega$$

defined by

$$E(\omega) = \eta \cdot \omega$$

preserves the probability distribution $\mathbb{P}$ (here we use the fact that $\mathbb{P}$ is generated by the uniform distribution on the disc $\mathbb{D}(0, 1)$). Therefore,

$$\mathbf{g}_{\eta\cdot\lambda}(0) = \int_{\omega\in\Omega} g_{\eta\cdot\lambda,\omega}(0) d\mathbb{P}(\omega) = \int_{\omega\in\Omega} g_{\lambda,E(\omega),}(0) d\mathbb{P}(\omega) = \int_{\omega\in\Omega} g_{\lambda,\omega}(0) d\mathbb{P}(\omega) = \mathbf{g}_\lambda(0)$$

So, $\Lambda \ni \lambda \mapsto \mathbf{g}_\lambda(0)$ is harmonic in $\Lambda = \mathbb{D}(0, \delta_0)$ and rotation-invariant, thus constant, by Lemma 2.80.

□

Chapter 3

Random exponential family

In this chapter we present a number of results on the non-autonomous exponential family, the main ones being Theorem 3.13 and Proposition 3.15. These, along with most results from this chapter, were published in [20]. It is worth mentioning that some parts of [20] were also included in the author's master thesis, and thus are not to be treated as new results for the purpose of this dissertation. The author has made sure to mention when a theorem is only presented to give a full picture and is not to be considered a part of the doctoral thesis.

For a sequence of positive real numbers $\omega = (\lambda_1, \lambda_2, ...)$ let $f_{\lambda_n}(z) = \lambda_n e^z$. In this chapter we study the Julia sets for the family of compositions

$$F_\omega^n = f_{\lambda_n} \circ f_{\lambda_{n-1}} \circ ... \circ f_{\lambda_1}.$$

In Chapter 2 the main property of the quadratic Julia sets that we were interested in was connectedness. In this chapter we focus on a seemingly simpler problem: whether for a given sequence ω the corresponding Julia set $\mathcal{J}_\omega$ is the whole plane. As in the quadratic case, this can be viewed as a natural question arising from the study of autonomous holomorphic dynamical systems. Let us remind the reader that for $\lambda > \frac{1}{e}$ the Julia set of λe^z is the whole plane, which was essentially proven by Misiurewicz in [25]. For $0 < \lambda \leqslant \frac{1}{e}$ it is easy to see that a left half-plane is invariant, and thus must lie in the Fatou set. In particular e^{z-1} has a parabolic basin for its fixed point at 1, and all z with $\mathrm{Re}(z) < 1$ lie in this basin. The value $\frac{1}{e}$ will play a key role in our non-autonomous considerations as well.

3.1 Introduction and motivation; the autonomous case

As was in the quadratic case, the random exponential dynamics are motivated by the theory in the autonomous setting. In this section we shall introduce some of these classical results on the exponential family. We start with the following theorem due to Misiurewicz [25].

Theorem 3.1. *The Julia set for the function e^z is the whole complex plane.*

Although in the original paper it is not explicitly mentioned, one can easily extend the proof of the above to all functions λe^z where $\lambda > \frac{1}{e}$. That is, the following theorem holds.

Theorem 3.2. *Let $\lambda > \frac{1}{e}$, then the Julia set for the function λe^z is the whole complex plane.*

Both of the above theorems can now be viewed as corollaries from much more general complex dynamics theory, which was developed after the elementary proof by Misiurewicz. For an overview of dynamics of an entire function the reader might want to refer to [1], [3].

The picture for $0 < \lambda \leqslant \frac{1}{e}$ is quite different, as λe^z has fixed points on the real line in that case. For $\lambda = \frac{1}{e}$ in particular the function has a parabolic fixed point at 1. It is easy to see that the Fatou set must be non-empty in this case, as the following proposition indicates.

Proposition 3.3. *The left half-plane $H = \{\mathrm{Re}(z) < 1\}$ lies in the Fatou set of the map $e^{z-1} = \frac{1}{e}e^z$.*

Proof. Note that $(e^{z-1})(H) \subset H$. Thus by Montel's theorem the iterates of e^{z-1} must be normal on H. □

In fact the Fatou set of e^{z-1} is the basin of 1, whereas the Julia set has the so called "topological hair" structure, which has been first described by Devaney and Krych in [15].

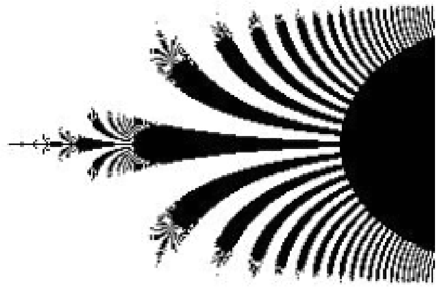

Figure 3.1: Julia set for e^{z-1}, the "Devaney hair".

The Julia set above is an uncountable union of pairwise disjoint simple curves, and it has many interesting properties. Apart from an extensive study done by the original authors in [15], McMullen also proved that the Hausdorff dimension of this set is 2 in [23]. Finally, Karpińska in [19] (see Theorem 1.1 in this work) proves the following striking result, that the hairs without their endpoints are in some sense a much smaller set. This is sometimes referred to as "Karpińska's paradox".

Theorem 3.4. *Let $0 < \lambda < \frac{1}{e}$ and let C_λ be the set of endpoints of Devaney hairs that form the Julia set J_λ. Then the Hausdorff dimension of $J_\lambda \setminus C_\lambda$ is 1.*

In the following sections we will tackle the question of whether the Julia set is the whole complex plane for various non-autonomous generalizations of this exponential family dynamical system.

3.2 Sequences bounded away from $\frac{1}{e}$ and related results

From now on we shall use the notation $F_\omega^{k,n} = f_{\lambda_{k+n}} \circ ... \circ f_{\lambda_{k+1}}$ for a composition of n functions beginning after the index k. This notation will be useful for formulating conditions independently of the first finitely many parameters in a sequence ω. We start with an overview of the case in which the sequence ω is separated from $\frac{1}{e}$, that is all λ_n satisfy $\lambda_n > \lambda$ for some fixed $\lambda > \frac{1}{e}$. In this situation the Julia set is the whole plane, as was shown by Urbański and Zdunik in [30]. The following theorem is from their work.

Theorem 3.5. *Let $M, \bar{\lambda}$ be positive real numbers satisfying $M > \bar{\lambda} > \frac{1}{e}$. Let $\omega = (\lambda_n)_{n=1}^{\infty}$ be a sequence of real numbers satisfying $M > \lambda_n > \bar{\lambda}$ for all n. Then $\mathcal{J}_\omega = \mathbb{C}$.*

The following propositions can both be seen as corollaries from the proof of the above theorem.

Proposition 3.6. *Let $M, \bar{\lambda}$ be positive real numbers satisfying $M > \bar{\lambda}$. Let $\omega = (\lambda_n)_{n=1}^{\infty}$ be a sequence of real numbers satisfying $M > \lambda_n > \bar{\lambda}$ for all n. Assume that for any $x \in \mathbb{R}$ and any $k \in \mathbb{N}$ we have $\lim_{n\to\infty} F_{\omega}^{k,n}(x) = \infty$. Finally assume there is no open set U, such that there exists a subsequence $(F_{\omega}^{n_k})_{k=1}^{\infty}$ of $(F_{\omega}^{n})_{n=1}^{\infty}$ that converges on U to a constant $a \in \mathbb{R}$ uniformly on compact sets. Then $\mathcal{J}_{\omega} = \mathbb{C}$.*

Proposition 3.7. *Let $M > \frac{1}{e}$ be a positive number and let $\omega = (\lambda_n)_{n=1}^{\infty}$ be a sequence of real numbers satisfying $M > \lambda_n \geqslant \frac{1}{e}$ for all n. Assume that for any $x \in \mathbb{R}$ and any $k \in \mathbb{N}$ we have $\lim_{n\to\infty} F_{\omega}^{k,n}(x) = \infty$. Suppose there is no open set U such that the sequence $(F_{\omega}^{n})_{n=1}^{\infty}$ converges on U to 1 uniformly on compact sets. Then $\mathcal{J}_{\omega} = \mathbb{C}$.*

Both of the above propositions follow from the work of authors of [30], but are not stated by them explicitly. Theorem 3.5 is Theorem 7 from [30], and the proof of both Propositions 3.6, 3.7 is nearly identical to that of Theorem 7 in [30]. The difference is that in the propositions we assume convergence to infinity on the real line, and that there is no Fatou component on which iterates converge to a real constant. These two properties are proved to be true for the sequences considered by Urbański and Zdunik as part of the proof of Theorem 3.5, but will not necessarily be true for more general sequences we shall consider in this document. This is why we reformulate the theorem in the form of these propositions, to have a useful criterion for when the Julia set is the whole plane. The proof of Proposition 3.6 (and, by extension, of Theorem 3.5 and Proposition 3.7) can be found in the appendix. It is a very minor modification of the proof of Theorem 3.5 as provided by the authors of [30].

In other words, the propositions state that if for a given sequence we are able to exclude the possibility of Fatou components on which the iterates converge to real constants, then one can repeat the proof in [30] for such a sequence, without the strong assumption of it being bounded from below by a constant strictly larger than $\frac{1}{e}$. Finding ways of applying these propositions is one of the goals of this chapter.

3.3 The Julia set for general sequences

In this section we prove various results that drop some of the assumptions of Theorem 3.5. The following subsection contains two theorems which deal with sequences that are in a sense "sometimes" bounded away from $\frac{1}{e}$. We demonstrate that in this case the picture is similar to that of autonomous iteration for $\lambda > \frac{1}{e}$ i.e. the Julia set is the whole plane. These results two (that is Theorem 3.8 and Theorem 3.9) come from the author's master thesis, and thus are not new results for the purpose of this dissertation. Nevertheless they give a more complete picture of non-autonomous exponential dynamics, and are included for that sake.

The more interesting situation arises from considering sequences converging to $\frac{1}{e}$ from above, and that is discussed in Subsection 3.3.2.

3.3.1 Sequences which do not converge to $\frac{1}{e}$

Theorem 3.8. *Let $M, \bar{\lambda}$ be positive real numbers satisfying $M > \bar{\lambda} > \frac{1}{e}$. Let $\omega = (\lambda_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that for all n we have $\lambda_n \in (\bar{\lambda}, M) \cup \{\frac{1}{e}\}$. Then the Julia set $\mathcal{J}_{\omega}$ is the whole plane if and only if $\lambda_n > \bar{\lambda}$ for infinitely many n.*

Proof. Let us assume that $\lambda_n > \bar{\lambda}$ only for finitely many n. In this case for sufficiently large N we have

$$\lambda_n = \frac{1}{e}$$

for all $n > N$. This means the Fatou set $\mathcal{F}_{\omega}$ is the preimage of the Fatou set of e^{z-1} under F_{ω}^{N}. Since the Fatou set of e^{z-1} is non-empty, this concludes the proof of the easier implication.

Now assume that $\lambda_n > \bar{\lambda}$ for infinitely many n. We shall show that

$$\forall_{x\in\mathbb{R}}\forall_{k\in\mathbb{N}} \lim_{n\to\infty} F_\omega^{k,n}(x) = \infty. \tag{3.1}$$

and that there is no open set U on which $F_\omega^n \to 1$ uniformly on compact sets. Then by Proposition 3.7 we will be done.

We begin by checking the first condition. For $x > 1$ the iterates of e^{x-1} converge to infinity, which means that so does $F_\omega^n(x)$. If $x \leqslant 1$, then iterates of e^{x-1} converge to the fixed point 1, thus for some n_0 we will have $F_\omega^n(x) > -\ln(\bar{\lambda})$ for all $n > n_0$. Now for $x > -\ln(\bar{\lambda})$ we have $\bar{\lambda}e^x > 1$. By our assumption, there exists $n_1 > n_0$ such that $\lambda_{n_1} > \bar{\lambda}$. Then we have $F_\omega^{n_1}(x) > 1$, and from this point the iterates converge to infinity. This reasoning can be applied starting from any index k, thus indeed (3.1) holds.

We now check the second condition from Proposition 3.7, that is that for no open set U does F_ω^n converge uniformly to 1 on compact sets. For fixed $\epsilon > 0$ and v_1, v_2 such that $v_2 > v_1$ let us denote the sets

$$S_\epsilon = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < \epsilon\}$$

$$V_\epsilon = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < \epsilon, v_1 < \mathrm{Re}(z) < v_2\}.$$

Now we set the constants as follows:

1. Fix $v_1 < 1$, $\epsilon \in (0, \frac{\pi}{2})$ and v_2 so that

$$\forall_{z\in S_\epsilon, \mathrm{Re}(z)>v_1} \mathrm{Re}(\bar{\lambda}e^z) > v_2 > 1.$$

2. If necessary decrease ϵ further so that

$$\forall_{z\in S_\epsilon, \mathrm{Re}(z)>v_1} \mathrm{Re}(e^{z-1}) > v_1.$$

3. Finally if necessary decrease ϵ so that

$$\forall_{z\in S_\epsilon, \mathrm{Re}(z)>v_2} \mathrm{Re}(e^{z-1}) > \mathrm{Re}(z) + \delta$$

   for some constant $\delta > 0$.

Let us assume that there is an open set U on which F_ω^n converge uniformly on compact sets to 1, then for a compact set $K \subset U$ after a finite number of iterations N_1 we should have

$$F_\omega^{N_1}(K) \subset V_\epsilon$$

and by the definition of V_ϵ this yields

$$\forall_{n>N_1}\forall_{z\in K} \mathrm{Re}(F_\omega^n(z)) > v_1.$$

Also by our assumptions we know that there is an index $N_2 > N_1$ such that $\lambda_{N_2} > \bar{\lambda}$, which by (1) gives us

$$\forall_{z\in K} \mathrm{Re}(F_\omega^{N_2}(z)) > v_2$$

and by (3)

$$\forall_{z\in K}\forall_{n>N_2} \mathrm{Re}(F_\omega^n(z)) > \mathrm{Re}(F_\omega^{n-1}(z)) + \delta.$$

Thus the real part of the iterations increase to infinity, and any convergence to a real constant would be a contradiction. □

The following theorem concerns randomly chosen sequences, but the proof is very similar to the one above.

Theorem 3.9. *Let $0 < \delta < \frac{1}{e}$. Let Ω be the product space $(\frac{1}{e} - \delta, \frac{1}{e} + \delta)^{\mathbb{N}}$ with the product of uniform distributions on $(\frac{1}{e} - \delta, \frac{1}{e} + \delta)$. Then for almost every sequence $\omega \in \Omega$ the Julia set $\mathcal{J}_\omega$ is the whole plane.*

Proof. Our goal is to apply Proposition 3.6. We shall show that the assumptions of the proposition hold for almost any sequence $\omega \in \Omega$.

Fix a sequence $\omega \in \Omega$. The function $(\frac{1}{e} - \delta)e^z$ has two fixed points on the real line, which we shall denote as p and q, with $p < q$. Similarly, for all $\lambda_n \in [\frac{1}{e} - \delta, \frac{1}{e})$ the function $\lambda_n e^z$ has two real fixed points, which lie in the interval $[p, q]$. Consider the sets

$$S_\epsilon = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < \epsilon\}.$$

$$V_\alpha = \{z \in \mathbb{C} : p - \alpha < \mathrm{Re}(z) < q + \alpha\}.$$

Let us set the constants $\alpha, \epsilon, \beta > 0$ so that

$$\forall_{z \in S_\epsilon \setminus V_\alpha} \forall_{\lambda \in (\frac{1}{e} - \delta, \frac{1}{e} + \delta)} \mathrm{Re}(f_\lambda(z)) > \mathrm{Re}(z) + \beta. \tag{3.2}$$

and

$$\forall_{z \in S_\epsilon \cap V_\alpha} \forall_{\lambda \in (\frac{1}{e} - \delta, \frac{1}{e} + \delta)} \mathrm{Re}(f_\lambda(z)) > p - \alpha. \tag{3.3}$$

Finally we may assume (by shrinking ϵ if necessary) that we also have

$$\forall_{z \in S_\epsilon \cap V_\alpha} \forall_{\lambda \in (\frac{1}{e} + \frac{\delta}{2}, \frac{1}{e} + \delta)} \mathrm{Re}(f_\lambda(z)) > \mathrm{Re}(z) + \beta. \tag{3.4}$$

Assume now that indeed there is a constant on the real line which is the limit of a subsequence of $(F_\omega^n)_{n=1}^\infty$, on some open set U. Moreover no subsequence of $(F_\omega^n)_{n=1}^\infty$ can have limits other than real constants (by proof of Proposition 3.6 in the appendix), which implies

$$\exists_{N \in \mathbb{N}} \forall_{n > N} F_\omega^n(U) \subset S_\epsilon.$$

Combined with (3.2) and (3.3) this yields

$$\exists_{N_1 > N} \forall_{n > N_1} F_\omega^n(U) \subset S_\epsilon \cap \{\mathrm{Re}(z) > p - \alpha\}.$$

Assume now that for a given sequence ω there exists $k \geqslant \frac{q - p + 2\alpha}{\beta}$ and an index n_2 such that $\lambda_n \in (\frac{1}{e} + \frac{\delta}{2}, \frac{1}{e} + \delta)$ for all $n \in [n_2, n_2 + k]$. Then (3.4) would imply that

$$\forall_{z \in U} \forall_{n > n_2 + k} \mathrm{Re}(F_\omega^n(z)) > q + \alpha.$$

This combined with (3.2) means that for such a sequence there can be no convergence to constants on the real line, as the real part is increased by β with all iterates.

Finally note that for a fixed k, the set of sequences ω which for inifinitely many indices n_2 satisfy $\lambda_n \in (\frac{1}{e} + \frac{\delta}{2}, \frac{1}{e} + \delta)$ for $n \in [n_2, n_2 + k]$, is a set of full measure, which concludes the proof.

□

Note that the uniform distribution does not play any key role in the reasoning above (and neither does the symmetry of the interval). In fact the proof of the previous theorem yields the following corollary:

Corollary 3.10. *Let $0 < \delta < \frac{1}{e}$. Let μ be a Borel probability measure on $(\frac{1}{e} - \delta, \frac{1}{e} + \delta)$ such that $\mu((\frac{1}{e}, \frac{1}{e} + \delta)) > 0$. Then for almost every (with respect to the product measure of μ on $(\frac{1}{e} - \delta, \frac{1}{e} + \delta)^{\mathbb{N}}$) sequence $\omega \in (\frac{1}{e} - \delta, \frac{1}{e} + \delta)^{\mathbb{N}}$ the Julia set $\mathcal{J}_\omega$ is the whole plane.*

3.3.2 Sequences converging to $\frac{1}{e}$

In this section we shall consider sequences ω which converge to $\frac{1}{e}$ from above. Lemma 3.11 and Proposition 3.12 are from the author's master thesis, and similarly to the results from the previous section, are included to give a more complete picture. The other results, in particular Theorem 3.13 and Proposition 3.15 are fully part of this dissertation. We start with the Lemma 3.11 which gives a simple sufficient condition for a non-empty Fatou set.

Lemma 3.11. *Let ω be a sequence of positive numbers for which there exists k such that for all $n > k$ we have $F_\omega^{k,n}(0) < 1$. Then $\mathcal{J}_\omega \neq \mathbb{C}$.*

Proof. Consider an arbitrary point z such that $\mathrm{Re}(z) < 0$. We have

$$\mathrm{Re}(f_{\lambda_n}(z)) = \cos(\mathrm{Im}(z))\lambda_n e^{\mathrm{Re}(z)} \leqslant \lambda_n e^{\mathrm{Re}(z)} = f_{\lambda_n}(\mathrm{Re}(z)).$$

Applying the above n times starting from index k yields the following:

$$\mathrm{Re}(F_\omega^{k,n}(z)) \leqslant F_\omega^{k,n}(\mathrm{Re}(z)) < F_\omega^{k,n}(0) < 1.$$

Thus after index k all iterations of the entire half-plane $P_- = \{z : \mathrm{Re}(z) < 0\}$ omit more than three points, so we can apply Montel's theorem to the family $(F_\omega^{k,n})_{n=1}^\infty$ on P_-. This implies that the preimage of P_- under F_ω^k lie in the Fatou set, which concludes the proof. □

Proposition 3.12. *There exists a constant C such that if a sequence $\omega = (\lambda_n)_{n=1}^\infty$ satisfies $\lambda_n \leqslant \frac{1}{e} + \frac{C}{n^2}$ for all n, then $\mathcal{J}_\omega \neq \mathbb{C}$.*

Proof. Let us first check by induction that for $\lambda_n = \frac{1}{e}e^{\frac{1}{n-1}}(1 - \frac{1}{n})$ we get

$$F_\omega^n(0) = 1 - \frac{1}{n}.$$

For $n = 1$ both sides of the equation are of course 0. Now, assuming the equality for n, we get

$$F_\omega^{n+1}(0) = f_{\lambda_{n+1}}(F_\omega^n(0)) = \lambda_{n+1}e^{F_\omega^n(0)} = \frac{1}{e}e^{\frac{1}{n}}(1 - \frac{1}{n+1})e^{1-\frac{1}{n}} = 1 - \frac{1}{n+1}.$$

This concludes the induction. Thus Lemma 3.11 implies the Fatou set is non-empty for $\lambda_n \leqslant \frac{1}{e}e^{\frac{1}{n-1}}(1 - \frac{1}{n})$. It suffices now to pick an appropriate constant C. Consider $C > 0$ such that

$$\begin{aligned}\frac{1}{e} + \frac{C}{n^2} &< \frac{1}{e} + \frac{e^{-1} - \frac{1}{ne}}{2(n-1)^2} \\ &= \frac{1}{e}(1 + \frac{1}{2(n-1)^2} - \frac{1}{2(n-1)^2 n}) \\ &= \frac{1}{e}(1 + \frac{1}{n-1} + \frac{1}{2(n-1)^2})(1 - \frac{1}{n}) \\ &< \frac{1}{e}e^{\frac{1}{n-1}}(1 - \frac{1}{n}).\end{aligned}$$

This concludes the proof.

□

One can also choose C such that for $\lambda_n = \frac{1}{e} + \frac{C}{n^2}$ all iterates of 0 starting from any index converge to infinity, and thus Lemma 3.11 cannot be applied. Same is of course true for $\lambda_n = \frac{1}{e} + \frac{1}{n^p}$ for $p < 2$. This is roughly speaking a consequence of the fact that the distance between the n-th iterate of 0 under e^{z-1} and the fixed parabolic point at 1 is of rate $\frac{1}{n}$.

It might seem reasonable to conjecture that the unboudedness of trajectories of 0 would imply that the Julia set is the whole plane. However, this is not true, as indicated by the next theorem,

Theorem 3.13. *There exists a bounded sequence of positive real numbers $\omega = (\lambda_n)_{n=1}^{\infty}$ such that*

$$\forall_{k\in\mathbb{N}} \lim_{n\to\infty} F_{\omega}^{k,n}(0) = \infty,$$

but $\mathcal{J}_\omega \neq \mathbb{C}$.

We shall require the following Lemma.

Lemma 3.14. *Let $f(z) = e^{z-1}$, and let V be an open set satisfying $\overline{V} \subset \{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < \frac{1}{2}, 0 < \mathrm{Re}(z) < 1\}$. Then we have*

$$\inf_{z\in V, n\in\mathbb{N}} \arg(f^n(z) - f^n(0)) > 0,$$

where arg *is the branch of the argument taking values from* $[0, 2\pi[$.

In other words, the Lemma says that we can fix an angle α, such that the angle between the real line and a line segment $[f^n(0), f^n(z)]$ is at least α, for any iterate n and any point z from V. The set V is chosen in such a way that it is compactly contained in the upper half-plane and in the basin of 1, the specific numbers in the theorem are somewhat arbitrary.

Proof. In order to arrive at the proof we shall show that there exists a neighbourhood of 1, such that for any two points z, a in it, where $a \in \mathbb{R}$ and $\mathrm{Im}(z) > 0$, we have

$$\arg(e^{z-1} - e^{a-1}) > \arg(z - a). \tag{3.5}$$

Indeed this is enough, since all iterates of both 0 and the set V will eventually land in the neighbourhood of 1 under iteration of e^{z-1}, and never leave that neighbourhood. This means that after a finite number of iterations, the expression $\arg(f^n(z) - f^n(0))$ will be increasing with respect to n for all z. Moreover by our choice of the set V the iterates $f^n(\overline{V})$ will stay in the upper half-plane, thus the expression $\arg(f^n(z) - f^n(0))$ can never be equal to 0. This is enough to conclude that the desired infimum is positive on $\overline{V}$ (as this is a compact set), which of course implies the same for V. Thus indeed proving (3.5) will be sufficient.

We will now show that (3.5) holds. Let $a, z \in B(1, \epsilon)$ with $\mathrm{Im}(z) > 0$ for some small $\epsilon > 0$, and moreover let $a \in \mathbb{R}$. Denote $x = \mathrm{Re}(z), y = \mathrm{Im}(z)$. Note that if $\mathrm{Re}(z-a) \leqslant 0$ then also $\mathrm{Re}(f^n(z) - f^n(a)) \leqslant 0$ for all iterations, since the iterates of a real point a will grow faster than the real part of iterates of any other point with real part a (a precise argument would look exactly like the proof of Lemma 3.11). This would imply $\arg(f^n(z) - f^n(a)) > \frac{\pi}{2}$ and yield the statement of Lemma 3.14. Finally this means without loss of generality we can assume that $\mathrm{Re}(z-a) > 0$ and $\mathrm{Re}(e^{z-1} - e^{a-1}) > 0$. We want to have (for a sufficiently small ϵ):

$$\frac{\mathrm{Im}(z)}{\mathrm{Re}(z-a)} < \frac{\mathrm{Im}(e^{z-1})}{\mathrm{Re}(e^{z-1} - e^{a-1})}$$

which can be rewritten as

$$\frac{y}{x-a} < \frac{e^{x-1}\sin(y)}{e^{x-1}\cos(y) - e^{a-1}}$$

where $z = x + iy$. Denote $x' = x - a > 0$, we need to prove

$$\frac{y}{x'} < \frac{\sin(y)}{\cos(y) - e^{-x'}}$$

which is equivalent to (the denominators on both sides of the inequality are positive)

$$y(\cos(y) - e^{-x'}) < \sin(y)x'.$$

Passing to Taylor series with respect to x' and y will now be enough to see that the last inequality is true:

$$y\left((1 - \frac{y^2}{2} + \frac{y^4}{4!} - ...) - (1 - x' + \frac{x'^2}{2} - ....)\right) < x'(y - \frac{y^3}{3!} + ...)$$

which can be rewritten as

$$x' - \left(\frac{y^2}{2} - ...\right) - \left(\frac{x'^2}{2} - ...\right) < x' - \frac{x'y^2}{3!} + ...$$

and finally this can be reorganized as the following (ignoring powers of order higher than 3):

$$\frac{x'y^2}{3!} + ... < \frac{y^2}{2} + \frac{x^2}{2} + ...$$

Last inequality is true for x', y sufficiently close to 0. Since $x' = \mathrm{Re}(z) - a$ and $y = \mathrm{Im}(z)$, then the inequality is true for z, a in some small neighbourhood of 1.

□

Proof of Theorem 3.13. Let us pick an open set V such that the assumptions of the previous Lemma are satisfied. We shall build the sequence $\omega = (\lambda_n)_{n=1}^{\infty}$ in a way that $V \subset \mathcal{F}_\omega$, thus the Fatou set will be non-empty. Let $(m_i)_{i=1}^{\infty}$ be a sequence of natural numbers and denote $M_n = \sum_{i=1}^{n} m_i$. The chosen sequence $\omega = (\lambda_n)_{n=1}^{\infty}$ will satisfy $\lambda_n = \frac{1}{e}$ for $n \in [M_k + 1, M_{k+1} - 1]$ and $\lambda_{M_k} > \frac{1}{e}$, for any natural k. What remains is to make the appropriate choice of the numbers m_i and λ_{M_k}.

Let again $f(z) = e^{z-1}$, then for any $\frac{\pi}{2} > \alpha_1 > 0$ there exists ϵ such that

$$\{z \in \mathbb{C} : \mathrm{Re}(z-1) > 0, \mathrm{Im}(z) < \epsilon, \arg(z-1) > \alpha_1\} \subset \mathcal{F}_{\frac{1}{e}}. \tag{3.6}$$

Here by $\mathcal{F}_{\frac{1}{e}}$ we denote the autonomous Fatou set for e^{z-1}. Indeed, it is well known that the Julia set of f is tangent to the real line at 1, thus for any angle α_1 we can choose an ϵ such that the above holds. Now for our set V let α_1 be a number satisfying

$$\inf_{z \in V, n \in \mathbb{N}} \arg(f^n(z) - f^n(0)) > \alpha_1 > 0,$$

the existence of which is given by Lemma 3.14. We pick ϵ for α_1 so that (3.6) is satisfied. Finally we can set m_1 large enough so that

$$f^{m_1-1}(V) \subset \{z : 0 < \mathrm{Im}(z) < \epsilon\}.$$

This can be done since V is chosen so that it lies in the basin of attraction of 1 and its iterates stay in the upper half-plane. Now we choose λ_{m_1} so that $F_\omega^{m_1}(0) = 1$. This means that $\lambda_{m_1} > \frac{1}{e}$, since otherwise we would have $F_\omega^{m_1}(0) < 1$. Moreover because of the choices of

α_1, ϵ, we know that $F_\omega^{m_1}(V)$ omits the autonomous Julia set $\mathcal{J}_{\frac{1}{e}}$, since $F_\omega^{m_1}(V)$ is contained in the set (3.6). Thus there is a natural number k_1 such that we have

$$f^{k_1}(F_\omega^{m_1})(V) = F_\omega^{m_1+k_1}(V) \subset \{\mathrm{Re}(z) < 1\}.$$

This is a consequence of the fact that in general compact subsets of the Fatou set of e^{z-1} are moved uniformly by e^{z-1} to $\{\mathrm{Re}(z) < 1\}$.
Now it is enough to repeat this construction for the set $F_\omega^{m_1+k_1}(V)$, by picking sufficiently large m_2 for the new angle α_2 (which maybe depends on the position of $F_\omega^{m_1+k_1}(V)$), and an appropriate $\lambda_{m_1+m_2} = \lambda_{M_2}$ so that $F_\omega^{m_1,m_2}(0) = 1$ and $F_\omega^{m_1+m_2}(V) \subset \mathcal{F}_{\frac{1}{e}}$. This way we build our sequence inductively by picking all m_n and λ_{M_n}.
It remains to be shown that this sequence satisfies our requirements. The set V is in the Fatou set by Montel's theorem, since by the construction all its iterates omit the entire real line.
The iterates of 0 starting from any index converge to infinity. Indeed, if we start from say $k \in [M_n, M_{n+1})$, then the monotonicity of the exponential yields

$$F_\omega^{k,(M_{n+1}-k)+m_{n+2}}(0) = F_\omega^{M_{n+1},m_{n+2}}(F_\omega^{k,(M_{n+1}-k)}(0)) > F_\omega^{M_{n+1},m_{n+2}}(0) = 1.$$

Since the iterates of $f^n(z)$ itself for any point $z > 1$ converge to infinity, then the same is true for the non-autonomous sequence we constructed (where we compose either f or f multiplied by a constant bigger than 1). This concludes the proof.

□

Note that the Fatou set produced in the above construction differs significantly from the autonomous Fatou set for e^{z-1}. For instance, in the example above the entire real line along with all its non-autonomous preimages lie in the Julia set. We now turn to another construction, which in some sense is "opposite" to the previous one. Namely we construct a sequence for which the Fatou set is empty, but the sequence λ_n converges to $\frac{1}{e}$.

Theorem 3.15. *There exists a sequence $\omega = (\lambda_n)_{n=1}^\infty$ satisfying $\lambda_n \searrow \frac{1}{e}$ such that the Julia set is the whole plane. The sequence can be chosen to satisfy $\limsup_{n\to\infty}(\lambda_n - e^{-1})n^{\frac{1}{2}} < \infty$.*

Proof. Let us denote

$$P = \{z \in \mathbb{C} : \frac{1}{2} < \mathrm{Re}(z) < \frac{3}{2}, 0 < \mathrm{Im}(z) < \frac{1}{2}\} \tag{3.7}$$

$$S_n = \{z :\in \mathbb{C} : \frac{1}{2} < \mathrm{Re}(z) < \frac{3}{2}, 0 < \mathrm{Im}(z) < \epsilon_n\} \subset P \tag{3.8}$$

Consider a sequence $(m_i)_{i=1}^\infty$ of natural numbers and put $M_n = \sum_{i=1}^n m_i$. We will set $\lambda_k = \frac{1}{e} + \frac{1}{n}$ for $k \in [M_{n-1}, M_n)$, and the construction will boil down to choosing the right sequence $(m_i)_{i=1}^\infty$.

Fix n and let $\lambda = \frac{1}{e} + \frac{1}{n}$. We will use the notation $f_\lambda(z) = \lambda e^z$. Let us choose ϵ_n such that $f_\lambda(z)$ increases the real part by a fixed amount β_n on S_n, that is

$$\exists_{\beta_n > 0} \forall_{z \in S_n} \mathrm{Re}(f_\lambda(z)) > \mathrm{Re}(z) + \beta_n. \tag{3.9}$$

Since the strip S_n has finite size, this means the iterates of any $z \in P$ will leave the set S_n after some number of iterations. More precisely, we have:

$$\exists_{K_n \in \mathbb{N}} \forall_{z \in P} \exists_{k \in \mathbb{N}, 0 < k \leqslant K_n} F_\omega^k(z) \in \mathbb{C} \setminus S_n.$$

In particular, if we assume that all the iterates of f_λ lie in P, then this means that at least once every K_n iterates a point from P lands in $P \setminus S_n$.

Now let $\varrho_S|dz|$ be the hyperbolic metric on $S = \{0 < \mathrm{Im}(z) < \pi\}$ and let the hyperbolic derivative of f_λ be denoted by $|f'_\lambda(z)|_{\varrho_S}$. Then there exists α_n such that for all $z \in P$ for which $f_\lambda(z) \in P \setminus S_n$ we have

$$|f'_\lambda(z)|_{\varrho_S} > \alpha_n > 1, \tag{3.10}$$

and $|f'_\lambda(z)|_{\varrho_S} \geqslant 1$ for z such that $f_\lambda(z) \in P$. This is an immediate consequence of the Schwarz-Pick Lemma applied to $f_\lambda^{-1} : S \to S$. Combined with the previous observation, if we assume that all iterates lie in P, then this implies

$$\forall_{M>K_n} |f_\lambda^M(z)'|_{\varrho_S} > \alpha_n^{\lfloor \frac{M}{K_n} \rfloor}. \tag{3.11}$$

Recall that we have the following formula

$$|f'(z)| = |f'(z)|_{\varrho_S} \frac{\varrho_S(z)}{\varrho_S(f(z))}$$

which yields the following bound:

$$\exists_{C_n>0} \forall_{z: f(z) \in P \setminus S_n} |f'(z)| > |f'(z)|_{\varrho_S} C_n \tag{3.12}$$

since the metric ϱ_S is bounded from above on $P \setminus S_n$ and bounded from below everywhere in P (so we just take $C_n = \inf\limits_{z: z\in P, f(z) \in P\setminus S_n} \frac{\varrho_S(z)}{\varrho_S(f(z))}$). The estimate (3.12) holds for all holomorphic functions between these respective domains, in particular any compositions of the exponential functions that are of interest to us.

Consider a sequence $(\delta_n)_{n=1}^\infty$ with $\lim\limits_{n\to\infty} \delta_n = 0$. For each δ_n let us pick m_n large enough so that

$$\delta_n > 4 \frac{\mathrm{diam}(P)}{C_n \prod\limits_{k=1}^{n} \alpha_k^{\frac{m_k}{K_k}}}, \tag{3.13}$$

and further enlarge m_n by an additional K_n (in particular this means now that that $m_n > K_n$ holds for all n). We claim that the sequence ω in which each $\frac{1}{e} + \frac{1}{n}$ is repeated m_n times is our desired sequence for which the Julia set is the whole plane.

Indeed, assume that there exists an open set $V \subset \mathcal{F}_\omega$. Then without loss of generality we have

$$\forall_{n \geqslant 0} F_\omega^n(V) \subset P$$

since we may assume that F_ω^n converge on V to 1 by Proposition 3.7. The assumptions of Proposition 3.7 are satisfied, since iterates of any point on the real line at some point are larger than 1 (by our choice of the numbers β_n, m_n and by the fact that (3.9) holds on the real line as well) which is enough to conclude that the iterates converge to infinity, and this reasoning works when we begin the iteration from an arbitrary index k. Now let $z \in V$ and let $D(z, r) \subset V$ be an open disk around z with radius r. We denote

$$V_n = F_\omega^{M_n}(V).$$

Now let us note that the Koebe one quarter theorem combined with

$$\forall_{n\in\mathbb{N}} V_n \subset P$$

and (3.13) implies that

$$\forall_{n\in\mathbb{N}}\, r < \delta_n.$$

Indeed, let us assume that $r \geqslant \delta_n$. By Koebe one quarter theorem we have (since in P all the exponential functions we consider are univalent)

$$F_\omega^{M_n}(D(z,r)) \supset D\left(F_\omega^{M_n}(z), \frac{1}{4}r|(F_\omega^{M_n})'(z)|\right)$$

and now (3.11), (3.12) and (3.13) yield

$$\frac{1}{4}r|(F_\omega^{M_n})'(z)| \geqslant \frac{1}{4}\left(4\frac{\operatorname{diam}(P)}{C_n\prod_{k=1}^n \alpha_k^{\frac{m_k}{K_k}}}\right)|(F_\omega^{M_n})'(z)|_{\varrho_S} C_n > \operatorname{diam}(P).$$

The second inequality is just (3.11). The first inequality is the comparison $r \geqslant \delta_n$ combined with (3.12) and (3.13). Applying (3.12) is valid here, since without loss of generality we may assume that the iterates $F_\omega^{M_n}(z)$ leave S_n, otherwise we can consider a number $N_n(z) \in [m_n - K_n, m_n]$ such that $F_\omega^{N_n(z)+M_{n-1}}(z)$ leaves S_n. Indeed, by construction such a choice of $N_n(z)$ is always possible (for all z), and then we have the following inequality

$$\begin{aligned}\frac{1}{4}r|(F_\omega^{N_n(z)+M_{n-1}})'(z)| &\geqslant \frac{1}{4}\left(4\frac{\operatorname{diam}(P)}{C_n\alpha_n^{\frac{N_n}{K_k}}\prod_{k=1}^{n-1} \alpha_k^{\frac{m_k}{K_k}}}\right)|(F_\omega^{N_n(z)+M_{n-1}})'(z)|_{\varrho_S} C_n \\ &> \operatorname{diam}(P).\end{aligned}$$

This yields the desired contradiction, since we have shown that if $r \geqslant \delta_n$, then V_n contains a ball too large to fit in P.

Since n was arbitrary, and we have picked the numbers δ_n to converge to zero, there can be no open disk around z in V, and thus V is not an open set which yields the contradiction.

Now let us calculate exactly how quickly does the example sequence ω converges to 0. To make any estimate we first need to set appropriate $\epsilon_n, K_n, C_n, \alpha_n, \delta_n$ and choose the sequence $(m_i)_{i=1}^\infty$. Note that the hyperbolic metric ϱ_S is given by the density $\varrho_S(z) = \frac{1}{\sin(\operatorname{Im}(z))}$. Indeed, since the exponential function sends the strip $\{0 < \operatorname{Im}(z) < \pi\}$ onto the upper halfplane, which has the hyperbolic metric with density $\varrho_H(z) = \frac{1}{\operatorname{Im}(z)}$, this yields:

$$\varrho_S(z) = \varrho_H(\exp(z))|\exp'(z)| = \frac{|\exp'(z)|}{\operatorname{Im}(\exp(z))} = \frac{1}{\sin(\operatorname{Im}(z))}$$

.

Now let us take $\epsilon_n = \frac{1}{\sqrt{n}}$. For $z \in S_n$ we have the following inequality for the function $e^{z-1}(1+\frac{1}{n})$ for sufficiently large n:

$$\begin{aligned}\operatorname{Re}(e^{z-1}(1+\frac{1}{n})) &= e^{\operatorname{Re}(z)-1}\cos(\operatorname{Im}(z))(1+\frac{1}{n}) \\ &> \operatorname{Re}(z)(1-\frac{\operatorname{Im}(z)^2}{2})(1+\frac{1}{n}) \\ &> \operatorname{Re}(z)(1-\frac{1}{2n})(1+\frac{1}{n}) \\ &> \operatorname{Re}(z)(1+\frac{1}{3n}) \\ &> \operatorname{Re}(z)+\frac{1}{6n}.\end{aligned}$$

The above inequality along with the fact that the rectangle S_n has length 1, means we can pick $K_n = 6n$.

We move on to the choice of α_n. Since the exponential function is an isometry between the two metrics ϱ_S and ϱ_H, we have

$$|f'_\lambda(z)|\frac{\varrho_H(f_\lambda(z))}{\varrho_S(z)} = 1.$$

This yields:

$$|f'_\lambda(z)|_{\varrho_S} = |f'_\lambda(z)|\frac{\varrho_S(f_\lambda(z))}{\varrho_S(z)} = |f'_\lambda(z)|\frac{\varrho_H(f_\lambda(z))}{\varrho_S(z)}\frac{\varrho_S(f_\lambda(z))}{\varrho_H(f_\lambda(z))} = \frac{\varrho_S(f_\lambda(z))}{\varrho_H(f_\lambda(z))} = \frac{\mathrm{Im}(f_\lambda(z))}{\sin(\mathrm{Im}(f_\lambda(z)))}.$$

Now, since $f'_\lambda = f_\lambda$ we can write (for sufficiently large n)

$$\inf_{f_\lambda(z)\in P\setminus S_n}\left|e^{z-1}(1+\frac{1}{n})\right|\left|\frac{\varrho_S(f_\lambda(z))}{\varrho_S(z)}\right| = \inf_{f_\lambda(z)\in P\setminus S_n}\left|\frac{\mathrm{Im}(f_\lambda(z))}{\sin(\mathrm{Im}(f_\lambda(z)))}\right|$$
$$> \frac{\epsilon_n}{\epsilon_n - \frac{\epsilon_n^3}{7}} > \frac{1}{1-\frac{1}{7n}} > 1 + \frac{1}{7n}.$$

The first inequality is simply estimating the numerator and denominator by the end value of the interval. This means $\alpha_n = 1 + \frac{1}{7n}$ is a valid choice for the estimate in (3.10).

Lastly let us note that since the hyperbolic metric is bounded by a constant times distance from the boundary of the set, we can write $C_n > \frac{C'}{\sqrt{n}}$ for some constant C' independent of n.

If we now set $m_n = 7K_n + K_n = 48n$, we get

$$C''\frac{\sqrt{n}}{\prod_{k=1}^n(1+\frac{1}{7k})^7} > 4\frac{\mathrm{diam}P}{C_n\prod_{k=1}^n \alpha_k^{\frac{m_k}{K_k}}}$$

for some constant C'' independent of n. Note that the left hand side converges to 0. Indeed, we have

$$\prod_{k=1}^n(1+\frac{1}{7k})^7 > \prod_{k=1}^n(1+\frac{1}{k}) = n+1.$$

This means we can choose δ_n such that $\delta_n > C''\frac{\sqrt{n}}{\prod_{k=1}^n(1+\frac{1}{7k})^7}$ while keeping $\lim_{n\to\infty}\delta_n = 0$, and thus our choice of all the numbers $\epsilon_n, K_n, C_n, \alpha_n, \delta_n$ and of the sequence $(m_i)_{i=1}^\infty$ is valid. To summarize, the sequence ω in which each $\lambda_k = \frac{1}{e} + \frac{1}{n}$ is repeated $48n$ times was shown to satisfy $\mathcal{J}_\omega = \mathbb{C}$. Since $48\sum_{k=1}^n k \approx n^2$, the rate of convergence of ω to $\frac{1}{e}$ is $\frac{1}{n^{\frac{1}{2}}}$. □

Actually, the above construction works without any modification if we take any $\lambda_k \geqslant \frac{1}{n}$ repeated $48n$ times, equality is not necessary. Indeed, all that is needed is some bound for the expansion of the hyperbolic metric. Thus the proof yields the following corollary:

Corollary 3.16. *There exists a constant $C > 0$ such that if $\omega = (\lambda_n)_{n=1}^\infty$ is a sequence satisfying $\lambda_n = \frac{1}{e} + \frac{C}{n^p}$ for $p < \frac{1}{2}$, then $\mathcal{J}_\omega = \mathbb{C}$.*

We do not conclude whether the Julia set of $\omega = (\lambda_n)_{n=1}^\infty$ where $\lambda_n = \frac{1}{e} + \frac{1}{n^p}$ for $\frac{1}{2} < p < 2$ is the whole plane, but if not, then the behaviour on the Fatou set is vastly different from the autonomous case for e^{z-1}, or the case for $p > 2$. For instance, we know that the real line along with its preimages lies in the Julia set. In the autonomous system the iterations on the left half-plane keep getting closer to the real line, never leaving a certain cone. This is not true for the system we are considering, as indicated by the following theorem (which is from the author's master thesis). For the proof we refer to [20].

Theorem 3.17. *Let $S_\theta = \{z \in \mathbb{C} : \arg(z) \in (\frac{\pi}{2} + \theta, \frac{3\pi}{2} - \theta)\}$ where $\theta \in (0, \frac{\pi}{2})$ and let $f_n = e^z - 1 + \frac{1}{n^p}$ where $p < 2$. Denote $F_n(z) = f_n \circ f_{n-1} \circ ... \circ f_1(z)$. Then for every $z \in S_\theta$ there exists n such that $F_n(z) \in \mathbb{C} \setminus S_\theta$.*

Appendix A

Appendix: Proof of Propositions 3.6 and 3.7

For sake of completeness we provide the following proof of Proposition 3.6, which is just a minor modification of the proof of Theorem 7 given in [30]. The proof of Proposition 3.7 is analogous.

Proof of Proposition 3.6. Let $M, \bar{\lambda}$ be positive constants and let ω be a sequence satisfying $M > \lambda_n > \bar{\lambda} > 0$. Assume that $\forall_{x\in\mathbb{R}}\forall_{k\in\mathbb{N}} \lim_{n\to\infty} F_\omega^{k,n}(x) = \infty$. Finally assume there is no open set U, such that there exists a subsequence of $(F_\omega^n)_{n=1}^\infty$ that converges on U to a constant $a \in \mathbb{R}$ uniformly on compact sets. We make all these assumptions in all the subsequent Lemmas required for the proof of Proposition 3.6, but for brevity we don't state them each time.
We begin by noting that the real line lies in the Julia set.

Lemma A.1. $\mathbb{R} \subset \mathcal{J}_\omega$.

Proof. Let us assume that a point $w \in \mathbb{R}$ is in the Fatou set. Then there exists an open set V such that $w \in V$ and the family $(F_\omega^n|_V)$ is normal. Since $F_\omega^n|_{\mathbb{R}\cap V} \to \infty$ as $n \to \infty$, we conclude F_ω^n converges to infinity uniformly on compact subsets of V. Now consider a ball $\overline{B(w,r)} \subset V$. We have

$$(F_\omega^n)'|_{B(w,r)} \to \infty$$

uniformly as $n \to \infty$. Thus by Bloch's Theorem, the image $F_\omega^n(B(w,r))$ for sufficiently large n contains a ball of radius 2π. This implies there exists a sequence of points $z_n \in B(w,r)$ such that

$$\lim_{n\to\infty} |\mathrm{Re}(F_\omega^n(z_n))| = \infty$$

and

$$\mathrm{Im}(F_\omega^n(z_n)) \in \pi + 2\pi\mathbb{Z}.$$

Thus $F_\omega^{n+1}(z_n) \in (-\infty, 0)$, which contradicts the convergence $F_\omega^n|_{B(w,r)} \to \infty$, and concludes the proof. □

A straightforward consequence of this Lemma is the following corollary:

Corollary A.2. *If $V \subset \mathbb{C}$ is an open set and $V \cap J(\{\lambda_n\}) = \emptyset$, then $V \cap \mathbb{R} = \emptyset$. Furthermore,*

$$\left(\bigcup_{k\in\mathbb{Z}} \mathbb{R} + k\pi i\right) \cap \bigcup_{n=0}^{\infty} F_\omega^n(V) = \emptyset$$

As the original authors note, the following Lemma is actually due to Misiurewicz, and can be found in [25].

Lemma A.3. *For every $z \in \mathbb{C}$ and every $n \geqslant 1$ we have*

$$|(F_\omega^n)'(z)| > |\mathrm{Im} F_\omega^n(z)|$$

Proof. For any n we have $f_{\lambda_n}(z) = \lambda_n e^x(\cos(y) + i\sin(y))$. Since $|\sin(y)| \leqslant |y|$ we have $|\mathrm{Im}(f_{\lambda_n}(z))| \leqslant \lambda_n e^x |y| = |f_{\lambda_n}(z)||\mathrm{Im}(z)|$. So,

$$\frac{|\mathrm{Im}(f_{\lambda_n}(z))|}{|\mathrm{Im}(z)|} \leqslant |f_{\lambda_n}(z)|. \tag{A.1}$$

Therefore we have

$$\begin{aligned} \mathrm{Im}(F_\omega^n(z)) &= \Big(\prod_{k=2}^{n} \frac{|\mathrm{Im}(F_\omega^k(z))|}{|\mathrm{Im}(F_\omega^k(z))|}\Big) \cdot \mathrm{Im}(f_{\lambda_1}(z)) \\ &\leqslant \Big(\prod_{k=2}^{n} |F_\omega^k(z)|\Big)|\mathrm{Im}(f_{\lambda_1}(z))| \\ &\leqslant \prod_{k=1}^{n} |F_\omega^k(z)| \\ &= |(F_\omega^n)'(z)| \end{aligned}$$

□

The authors of [30] also make the following observation, with regards to the above computation: if we denote by $\mathbb{H}^+$ and $\mathbb{H}^-$, respectively, the upper and lower half-plane, then the branches of inverse maps $f_{\lambda_n}^{-1}$ are well defined on $\mathbb{H}^+$ and $\mathbb{H}^-$. Since the hyperbolic metric on $\mathbb{H}^+, \mathbb{H}^-$ is given by $\frac{|dz|}{|\mathrm{Im}(z)|}$, the inequality (A.1) is an expression of the fact that $f_{\lambda_n}^{-1}$ is a contraction in the hyperbolic metric.

Lemma A.4. *If $V \subset \mathbb{C}$ is an open connected set and $V \subset \overline{V} \subset \mathbb{C} \setminus \mathcal{J}_\omega$, then there exists an integer $N \geqslant 0$ such that for all $n \geqslant N$,*

$$F_\omega^n(V) \subset S := \{z \in \mathbb{C} : |\mathrm{Im}(z)| < \pi\}.$$

Proof. By corollary A.2, for every $n \in \mathbb{N}$, either the set $F_\omega^n(V)$ is contained in S, or it is disjoint from S. If $F_\omega^n(V) \cap S = \emptyset$ for infinitely many integers n, then using Lemma A.3 and the Chain Rule we obtain

$$\limsup_{n\to\infty} |(F_\omega^n)'|_V = \infty.$$

This by Bloch's Theorem implies $F_\omega^n(V)$ contains a ball of radius 2π for infinitely many n, which is a contradiction with corollary A.2. The contradiction concludes the proof.

□

Write S as

$$S = S^+ \cup S^- \cup \mathbb{R},$$

where

$$S^+ := \{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < \pi\}$$

and

$$S^- := \{z \in \mathbb{C} : -\pi < \mathrm{Im}(z) < 0\}.$$

For a given $f_\lambda = \lambda e^z$ denote by g_λ the branch of the holomorphic inverse of f_λ which maps S^+ to S^+. Let ρ denote the hyperbolic metric on S^+.

Lemma A.5. *For every $\lambda \in [\bar{\lambda}, M]$ and for all $z, w \in S^+$, we have that*

$$\rho(g_\lambda(z), g_\lambda(w)) \leqslant \rho(z, w). \tag{A.2}$$

Also, for every compact subset $K \subset S^+$ there exists $\kappa \in (0,1)$ such that for any $\lambda \in [\bar{\lambda}, \infty)$ and for all $z, w \in K$, we have

$$\rho(g_\lambda(z), g_\lambda(w)) \leqslant \kappa \rho(z, w). \tag{A.3}$$

Proof. The inequality (A.2) is an immediate consequence of Schwarz Lemma. Since the map $g_\lambda : S^+ \to S^+$ is not bi-holomorphic, it also follows from Schwarz lemma that

$$\rho(g_\lambda(z), g_\lambda(w)) < \rho(z, w), \tag{A.4}$$

whenever $z, w \in S^+$ and $z \neq w$, and in addition,

$$\limsup_{\substack{z,w \to \xi \\ z \neq w}} \frac{\rho(g_\lambda(z), g_\lambda(w))}{\rho(z, w)} < 1 \tag{A.5}$$

for every $\xi \in S^+$. In order to prove (A.3), fix $\lambda_2 \geqslant \lambda_1 \geqslant \bar{\lambda}$. Since $g_{\lambda_2}(z) = g_{\lambda_1}(z) - \log(\frac{\lambda_2}{\lambda_1})$ and $g_{\lambda_2}(w) = g_{\lambda_1}(w) - \log(\frac{\lambda_2}{\lambda_1})$, and since the metric ρ is invariant under horizontal translation, we have

$$\rho(g_{\lambda_2}(z), g_{\lambda_2}(w)) = \rho(g_{\lambda_1}(z), g_{\lambda_1}(w)).$$

Thus it is enough to check (A.3) for $f_{\bar{\lambda}}$. But this follows immediately from (A.4), (A.5) and the compactness of K. Indeed, denote by $|f'|_\rho$ the derivative with respect to the metric ρ, and consider the function $G : K \times K \to \mathbb{R}$ defined by:

$$G(z, w) = \begin{cases} \frac{\rho(g_{\bar{\lambda}}(z), g_{\bar{\lambda}}(w))}{\rho(z,w)} & \text{for } z \neq w \\ |f'|_\rho & \text{for } z = w \end{cases}$$

Then G is continuous in $K \times K$ and $G(z, w) < 1$ for all $(z, w) \in K \times K$, and (A.3) follows. □

The following Lemma will complete the proof of the theorem.

Lemma A.6. *The interior of the set*

$$\Lambda := \bigcap_{n=0}^{\infty} (F_\omega^n)^{-1}(S)$$

is empty.

Proof. Since for any n we have

$$f_{\lambda_n}(S^+) = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\},$$

$$f_{\lambda_n}(S^-) = \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$$

and

$$f_{\lambda_n}(\mathbb{R}) = (0,\infty)$$

it follows that

$$\bigcap_{n=0}^{\infty}(F_\omega^n)^{-1}(S) = \bigcap_{n=0}^{\infty}(F_\omega^n)^{-1}(S^+) \cup \bigcap_{n=0}^{\infty}(F_\omega^n)^{-1}(S^-) \cup \mathbb{R}.$$

We shall prove that $\bigcap_{n=0}^{\infty}(F_\omega^n)^{-1}(S^+)$ has empty interior. The case of S^- can be done in an analogous way.

Let us assume the opposite, that is suppose there exists $V \subset \mathbb{C}$, a nonempty, open, connected and bounded set with

$$V \subset \overline{V} \subset \bigcap_{n=0}^{\infty}(F_\omega^n)^{-1}(S^+).$$

Then of course the family $(F_\omega^n|_V)_{n=0}^{\infty}$ is normal. Now let us fix a disk W contained with its closure in V. Put $\delta := \operatorname{dist}(W, \partial V) > 0$. Let N be an integer large enough so that

$$\left(\frac{\pi}{2}\right)^N \cdot \frac{\delta}{72} > 2\pi.$$

Now, seeking a contradiction, assume that there exists $\xi \in W$ such that for at least N integers $n_1, n_2, ..., n_N \geqslant 0$ we have

$$F_\omega^{n_i}(\xi) \in \{z \in \mathbb{C} : \operatorname{Im}(z) > \frac{\pi}{2}\}.$$

Then $|(F_\omega^{n_N})'(\xi)| > \left(\frac{\pi}{2}\right)^N$, and again Bloch's Theorem implies that $F_\omega^{n_N}(W)$ contains some ball of radius 2π. Since $F_\omega^{n_N}(W)$ does not intersect the Julia set for the sequence $\{\lambda_i\}_{i=n_N}^{\infty}$, but the copies of the real line $\mathbb{R} + 2\pi i\mathbb{Z}$ lie in this Julia set, so we arrive at a contradiction. Thus we conclude that for any $z \in W$ the trajectory $F_\omega^n(z)$ visits the set

$$\{z \in \mathbb{C} : \operatorname{Im}(z) > \frac{\pi}{2}\}$$

at most N times. For every integer $k \geqslant 0$ let

$$W_k := W \cap \bigcap_{n=k}^{\infty}(F_\omega^n)^{-1}\left(\{z \in \mathbb{C} : \operatorname{Im}(z) \leqslant \frac{\pi}{2}\}\right).$$

Each set W_k is closed in W, and as we have just proved

$$W = \bigcup_{k=0}^{\infty} W_k.$$

Since W is an open subset of $\mathbb{C}$ it is completely metrizable, and the Baire Category Theorem holds for it. Thus there exists $q_1 \geqslant 0$ such that

$$W^* := \operatorname{Int}_{\mathbb{C}}(W_{q_1}) \neq \emptyset.$$

This means that for all integers $n \geqslant q_1 \geqslant 0$, we have

$$F_\omega^n(W^*) \subset \{z \in \mathbb{C} : 0 < \operatorname{Im}(z) < \frac{\pi}{2}\}. \tag{A.6}$$

Consequently,

$$F_\omega^n(W^*) \subset \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$$

for all $n > q_1$. Finally note that there exists a constant M (dependent on $\bar{\lambda}$) such that, if $\operatorname{Re}(z) > M$, $\operatorname{Im}(z) \in (0, \frac{\pi}{2})$, and $f_{\lambda_n}(z) \in S$ (for all $\lambda_n > \bar{\lambda} > 0$) then

$$\mathrm{Re}(f_{\lambda_n}) > \mathrm{Re}(z) + 1.$$

We shall now finish the proof by excluding all possible limits of subsequences of F^n_ω. Firstly, assume that there is a subsequence n_k such that $((F^{n_k}_\omega)|_{W^*})_{k=1}^\infty$ converge to infinity. This implies that $((F^{n_k}_\omega)'|_{W^*})_{k=1}^\infty$ converge to infinity, which once again can be excluded by a combination of Bloch's Theorem and (A.6).
There can also be no subseqence converging to a point in S^+, as all the maps $f_{\lambda_n}|_{W^*}$, $n > q_1$ expand the hyperbolic metric ρ.
Thus let g be a non-constant limit of some subsequence $(F^{n_k}_\omega)_{k=1}^\infty$ converging uniformly. Shrinking W^* if necessary, one can assume that $g(W^*)$ is contained in some compact subset $K \subset S^+$. Putting

$$\widetilde{K} := \{z \in S^+ : \rho(z, K) \leqslant 1\},$$

we see that there is $q_2 > q_1$ such that for every $k \geqslant q_2$

$$F^{n_k}_\omega(W^*) \subset \widetilde{K}.$$

Note that $\widetilde{K}$ has finite hyperbolic diameter, let us denote $D := \mathrm{diam}_\rho(\widetilde{K}) < \infty$. Let $z, w \in W^*$ with $z \neq w$. Then, using (A.2) and (A.3), we see that $\rho(z, w) \leqslant \kappa^{k-q_2} D$ for every $k \geqslant q_2$, which is a contradiction. Thus there can also be no subsequences with non-constant limits in S^+.
Since all limits of subsequences of $(F^n_\omega)_{n=0}^\infty$ with values in S^+ have been excluded, the only possibility left is the convergence to a constant on the real line, but this contradicts the assumptions of Proposition 3.6, which concludes the proof.

□

Proposition 3.7 can be proved analogously except we also have to prove that no subsequence can converge to a real constant (by assumption it cannot converge to 1, we have to also exclude all other real numbers). We do this with the help of the following Lemma.

Lemma A.7. *For every sufficiently small $\varepsilon > 0$ there exists $\beta > 0$ such that for every*

$$z \in \{z \in \mathbb{C} : 0 < \mathrm{Im}(z) < \epsilon\} \setminus D(1, \epsilon)$$

and every $\lambda_n \geqslant \frac{1}{e}$ we have

$$\mathrm{Re}(f_{\lambda_n}(z)) > \mathrm{Re}(z) + \beta.$$

The above is based on Lemma 14 from [30], and as the authors of [30] point out, the proof is just a simple calculation (in fact it is even simpler for the version we require here). This way we can exclude the possibility of any real constant limits other than 1. Indeed, assume $\{F^n_\omega\}$ is normal on an open set W. Since the only possible limit of a subsequence of iterates is a real number, then for any ε there exists N such that for all $n > N$ we have

$$\mathrm{Im}(F^n_\omega(W)) < \varepsilon.$$

But now any limit of a subsequence other than 1 is impossible by Lemma A.7. The possibility of convergence to 1 is explicitly forbidden in the assumptions of Proposition 3.7.

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