University of Warsaw

Faculty
of Mathematics, Informatics
and Mechanics

The logical strength of Ramsey-theoretic principles over a weak base theory

Doctoral Dissertation in Mathematics

Katarzyna Wiesława Kowalik

Advisor
dr hab. Leszek Kołodziejczyk
Institute of Mathematics
University of Warsaw

May 2025

I hereby declare that the dissertation is my own work.

Katarzyna Wiesława Kowalik

Date

Signature

The dissertation is ready to be reviewed.

dr hab. Leszek Kołodziejczyk

Date

Signature

Abstract

The thesis contains results on the logical strength of some well-known combinatorial principles: Ramsey's theorem for n-tuples and k colours RT^n_k, the chain-antichain principle CAC, the ascending-descending sequence principle ADS, cohesive Ramsey's theorem CRT^2_2, and the cohesive set principle COH. We measure the strength of these statements in terms of reverse mathematics, that is, by formalizing them in second-order arithmetic and characterizing their consequences over a fixed base theory. We work over a relatively weak base theory RCA^*_0, which is obtained from the usual base theory for reverse mathematics, RCA_0, by restricting the scheme of mathematical induction from $\mathsf{I}\Sigma^0_1$ to $\mathsf{I}\Delta^0_0$. The weakening of induction allows for a finer analysis of the principles considered, but at the same time leads to some conceptual challenges related to the notion of infinity. Namely, it is consistent with RCA^*_0 that there exists an unbounded set $X \subseteq \mathbb{N}$ which is not in bijective correspondence with $\mathbb{N}$. This is of special importance for our analysis, as a typical Ramsey-theoretic principle asserts the existence of an infinite set $X \subseteq \mathbb{N}$ with some specific structure. Thus, one can consider at least two variants of such a principle over RCA^*_0: 'normal' and 'long', depending on whether the witnessing set X is required only to be unbounded or to be in bijective correspondence with the whole set $\mathbb{N}$.

In the first part of the thesis we study the strength of normal and long variants of RT^n_k for $n, k \geqslant 2$, CAC, ADS and CRT^2_2 in terms of their (first-order) arithmetical consequences. We prove that all the normal principles that we consider are $\forall\Pi^0_3$-conservative but not arithmetically conservative over RCA^*_0. This follows from two general conservativity criteria that we prove for a broad syntactic class of sentences. Moreover, for $n \geqslant 3$ we give a complete axiomatization of the arithmetical consequences of the normal version of RT^n_k over RCA^*_0 and compare them with the usual fragments of Peano Arithmetic. For the long principles, we show that they split into two very different groups: some of them turn out to be strong, because they imply $\mathsf{I}\Sigma^0_1$, whereas the others are $\forall\Pi^0_3$-conservative over RCA^*_0 and therefore are much closer to their normal counterparts.

The principle COH does not fit well into the classification into normal and long principles and thus requires a separate analysis. We show that its model-theoretic behaviour is very different from that of the normal principles. As a result, unlike over RCA_0, it is not the case that COH follows from RT^2_2 over RCA^*_0. We also show that COH is not arithmetically conservative over RCA^*_0, answering a question of Belanger.

The second part of the thesis is concerned with proof size. We refine our conservation result for CAC and show that $\mathsf{RCA}^*_0+\mathsf{CAC}$ is polynomially simulated by RCA^*_0 with respect to $\forall\Pi^0_3$ formulas. To this end we use the method of forcing interpretations and syntactically simulate a two-step model-theoretic argument, which involves construction of a computable ultrapower and a generic cut satisfying CAC. This polynomial simulation result sharply contrasts with the previously known fact that $\mathsf{RCA}^*_0+\mathsf{RT}^2_2$ has non-elementary speedup over RCA^*_0.

Streszczenie

Niniejsza rozprawa zawiera wyniki na temat siły logicznej pewnych dobrze znanych twierdzeń (zasad) kombinatorycznych typu ramseyowskiego: twierdzenia Ramseya dla n-tek i k kolorów RT^n_k, twierdzenia o łańcuchach i antyłańcuchach CAC, twierdzenia o ciągach wstępujących i zstępujących ADS, kohezywnego twierdzenia Ramseya CRT^2_2 oraz twierdzenia o zbiorze kohezywnym COH. Siłę tych twierdzeń mierzymy w sposób właściwy dla matematyki odwrotnej, tj. formalizujemy je w arytmetyce drugiego rzędu, a następnie charakteryzujemy ich konsekwencje logiczne nad ustaloną teorią bazową. Pracujemy w stosunkowo słabej teorii bazowej RCA^*_0, którą otrzymuje się z używanej zwykle w matematyce odwrotnej teorii bazowej RCA_0 poprzez ograniczenie schematu indukcji matematycznej z $\mathsf{I}\Sigma^0_1$ do $\mathsf{I}\Delta^0_0$. Osłabienie indukcji pozwala na dokładniejszą analizę badanych zasad, ale z drugiej strony prowadzi do pewnych trudności związanych z pojęciem nieskończoności. Mianowicie, jest niesprzeczne z RCA^*_0, że istnieje nieograniczony zbiór $X \subseteq \mathbb{N}$, który nie jest równoliczny z $\mathbb{N}$. Fakt ten ma kluczowe znaczenie dla naszej analizy, ponieważ typowa zasada ramseyowska stwierdza istnienie nieskończonego zbioru $X \subseteq \mathbb{N}$ o strukturze określonego typu. Pracując w RCA^*_0 można zatem rozważać co najmniej dwa warianty takiej zasady: „normalny” oraz „długi”, w zależności od tego, czy zbiór X ma być jedynie nieograniczony, czy też ma być równoliczny z całym $\mathbb{N}$.

W pierwszej części rozprawy badamy siłę normalnych i długich wariantów RT^n_k dla $n, k \geqslant 2$, CAC, ADS oraz CRT^2_2 charakteryzując ich konsekwencje arytmetyczne (tj. pierwszego rzędu). Dowodzimy, że wszystkie rozważane przez nas normalne zasady są $\forall\Pi^0_3$-konserwatywne, ale nie arytmetycznie konserwatywne nad RCA^*_0. Wynika to z dwóch ogólnych kryteriów na konserwatywność dotyczących pewnej szerokiej klasy syntaktycznej zdań. Dla $n \geqslant 3$ podajemy ponadto pełną aksjomatyzację arytmetycznych konsekwencji RT^n_k nad RCA^*_0 i porównujemy je ze zwykłymi fragmentami Arytmetyki Peana. Długie zasady dzielą się z kolei na dwie bardzo odmienne grupy: niektóre z nich są silne, jako że implikują $\mathsf{I}\Sigma^0_1$, podczas gdy pozostałe są $\forall\Pi^0_3$-konserwatywne nad RCA^*_0, a zatem są dużo bliższe swoim normalnym odpowiednikom.

Zasada COH słabo pasuje do podziału na zasady normalne oraz długie i wymaga odrębnej analizy. Dowodzimy, że jej zachowanie teoriomodelowe jest bardzo różne od zachowania zasad normalnych. W konsekwencji, inaczej niż nad RCA_0, nad RCA^*_0 zasada COH nie wynika z RT^2_2. Pokazujemy również, odpowiadając na pytanie postawione przez Belangera, że COH nie jest arytmetycznie konserwatywna nad RCA^*_0.

W drugiej części rozprawy zajmujemy się rozmiarami dowodów. Wzmacniamy nasz wynik o konserwatywności dla CAC pokazując, że RCA^*_0 wielomianowo symuluje $\mathsf{RCA}^*_0 + \mathsf{CAC}$ dla formuł złożoności $\forall\Pi^0_3$. W tym celu stosujemy metodę interpretacji forsingowych, dzięki której możemy syntaktycznie odtworzyć dwuetapowe rozumowanie teoriomodelowe, oparte na konstrukcji obliczalnej ultrapotęgi, a następnie przekroju generycznego spełniającego CAC. Nasz wynik o wielomianowej symulacji wyraźnie kontrastuje ze znanym wcześniej faktem, że $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2$ nieelementarnie skraca dowody w stosunku do RCA^*_0.

Dziękuję – Thank you

Like many other stories in Mathematics, this one also started in a pub. It was an autumn evening in 2018 when I met with Mateusz Łełyk and Bartosz Wcisło after the thesis defence of one of them. I asked whether they thought I could one day, perhaps in a few years, pursue my PhD in Logic at MIM UW. Without hesitation, they immediately replied: 'Of course! Go and talk to Leszek Kołodziejczyk'. So I did, and less than a year later I became his PhD student. In the meantime, Mateusz saved my life when he became my master's thesis advisor. Thus, I address the first 'thank you' to Mateusz and Bartek, without whom this PhD would not have started. Thank you for being my teachers, friends and older brothers in Logic.

There are no right words to express my gratitude to Leszek Kołodziejczyk. Thank you for everything you taught me, which is way more than Mathematics. Thank you for being a model of wisdom, kindness, rationality and responsibility. Thank you for the great amount of time you gave me and for always being there. Thank you for giving me hope in the human mind.

Many thanks to my coauthors, Marta Fiori Carones and Keita Yokoyama, as well as to Tin Lok Wong, whose ideas and expertise were invaluable for our papers. It was an honour and pleasure to work with and learn from each of you. I am also very grateful to Tin Lok Wong for his mathematical writings, especially for his great lecture notes, full of rigour and clarity. I have learned so much from him that I feel almost like one of his students.

Thank you to all my wonderful teachers of Mathematics, philosophy and languages at the University of Warsaw. Thank you for teaching me how to ask questions.

Thank you to my amazing friends in Mathematics: Agnieszka Widz, Damian Głodkowski, Jarek Swaczyna, Oriola Gjetaj and Ziemek Kostana. Thank you for all the mathematical discussions and for all the colours you brought into my life. It was a privilege and a great joy to walk part of this journey with you.

Thank you to all my friends who did not forget about me, even though I kept saying I did not have time because I was writing my thesis.

I am very grateful to Anton Freund for showing me lots of beautiful Mathematics and for making my stay in Würzburg so fruitful and enjoyable. Thank you to everyone else who helped me during my stay in Würzburg.

Thank you to Andreas Weiermann, Fedor Pakhomov and other members of the Logic group in Ghent for hosting me at their faculty and for introducing me to fascinating areas of proof theory.

Thank you to Paul Shafer for inviting me to Leeds many times and for the great mathematical exchanges.

Thank you to Ali Enayat for his time, support and the most beautiful way he presents Mathematics to others.

Finally, Thank you to my parents and grandparents. Their unconditional love and unwavering support are hard to comprehend. I think Mathematical Logic could be briefly described as a study of the language of Mathematics. So, I thank my Mom Elżbieta for being my first language teacher, and my Dad Krzysztof for being my first Mathematics teacher. Thank you for never asking what all this might be useful for.

Thank you to my grandma Grażyna and my grandpa Mieczysław for being my second home.

***

W bardzo dużym skrócie można by powiedzieć, że Logika Matematyczna zajmuje się badaniem języka Matematyki. Dziękuję zatem mojej Mamie Elżbiecie, która jest moją pierwszą nauczycielką języka, oraz mojemu Tacie Krzysztofowi, który jest moim pierwszym nauczycielem Matematyki. Dziękuję, że nigdy nie pytaliście, do czego mi się to wszystko przyda.

Dziękuję mojej Babci Grażynie i Dziadkowi Mieczysławowi - za to, że jesteście moim drugim domem.

Contents

Introduction **7**

1 Preliminaries **12**

1.1 First- and second-order arithmetic . . . 12
1.2 RCA_0^* as a base theory . . . 22
1.3 Combinatorial principles . . . 25

2 Normal versions **30**

2.1 Basic observations over RCA_0^* . . . 30
2.2 Between a model and its cuts . . . 33
2.3 Arithmetical consequences of Ramsey-like principles . . . 36
2.4 Ramsey for triples and beyond . . . 39
2.4.1 Provability of computational bounds on RT_2^n . . . 39
2.4.2 First-order consequences of RT_2^n for $n \geqslant 3$ . . . 42

3 Other principles **46**

3.1 Long versions . . . 46
3.2 The cohesive set principle . . . 53

4 A non-speedup result for CAC **57**

4.1 Basic definitions and earlier results . . . 57
4.2 Forcing interpretations . . . 61
4.2.1 Main definitions . . . 63
4.2.2 Polynomial forcing interpretations . . . 66
4.3 A two-step forcing construction . . . 71
4.3.1 Model-theoretic intuition . . . 71
4.3.2 Computable ultrapower . . . 75
4.3.3 Generic cut . . . 81
4.3.4 Completing the proof . . . 88

Bibliography **94**

Introduction

The problems that we study in this thesis have their origin in reverse mathematics, a programme in mathematical logic which aims to classify mathematical theorems according to the strength of axioms necessary to prove them. In order to do so, one translates a given mathematical theorem to a formal language of sufficiently large expressive power and, over some base theory, tries to prove an equivalence between the theorem and some well-understood axioms.

The usual logical setting for reverse mathematics is second-order arithmetic, which is formulated in a language rich enough to speak directly about natural numbers and sets of natural numbers and, by means of coding, about essentially any finite, countable and countably representable objects encountered in ordinary mathematics. There are two main groups of axioms in second-order arithmetic: induction axioms and set existence axioms. The logical strength of these axioms is closely related to their syntactic complexity.

The traditional base theory is RCA_0, whose axioms state a finite list of basic properties of natural numbers, the principle of mathematical induction for computably enumerable properties (henceforth: $\mathsf{I}\Sigma^0_1$) and the existence of computable sets of natural numbers. It is a remarkable phenomenon that a great number of theorems from many different areas of mathematics have turned out to be either provable in RCA_0 or equivalent over RCA_0 to one of four other theories: WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0. These theories are called the *Big Five* of reverse mathematics and form a strict linear order with respect to logical implication.

However (and fortunately), not all of mathematics can be arranged on just five shelves. Countable combinatorics is one branch of mathematics that provides a large number of theorems with logical strength that is particularly hard to determine. Arguably, the most famous example of such a theorem is Ramsey's theorem for pairs and two colours, RT^2_2, which says that for every function c mapping unordered pairs of natural numbers to $\{0, 1\}$ there exists an infinite set $H \subseteq \mathbb{N}$ such that the restriction of c to pairs from H is constant. For decades, the theorem has attracted the attention of many logicians, as it is a very natural principle with multiple applications in combinatorics itself as well as in other areas of mathematics.

A long line of research has shown that RT^2_2 is not equivalent over RCA_0 to any of the well-understood induction or set existence axioms. That is, $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ proved to be yet another theory of second-order arithmetic deserving independent study. In such a case, one of the first and most natural questions is what the theory can prove about finite objects. In particular, one would like to know whether $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is *conservative* over some well-studied theory of second-order arithmetic, that is, whether it proves exactly the same theorems in the language of first-order arithmetic as that other theory does. This is a long-standing open problem that has led to the discovery of intriguing connections between different topics and many new proof techniques. It turned out to be a particularly successful strategy to split RT^2_2 into a conjunction of some weaker combinatorial principles and then, by studying them separately, learn something new about RT^2_2 itself. The most well-known decompositions of RT^2_2 are $\mathsf{SRT}^2_2 + \mathsf{COH}$, introduced by Cholak, Jockusch and Slaman in [5], and $\mathsf{ADS} + \mathsf{EM}$, originating from work of Bovykin and Weiermann [4] and Hirschfeldt and Shore [24]. (All definitions relevant to our current work will be provided in Chapter 1.)

However, it is not only their role in research on RT^2_2 that makes other Ramsey-theoretic statements an interesting subject for logicians. They form a fascinating and complex world of pairwise inequivalent principles, full of challenging problems that require tools from various areas of logic, with computability theory having a prominent role. This complicated structure stands in stark contrast to the realm of the Big Five and provides formal evidence for the intuition that combinatorics is about solving diverse and nonequivalent problems, each time requiring finding an original solution rather than applying a single general pattern of reasoning.

This thesis is a part of a larger project, including [15, 16, 31, 32], to study the logical strength of some well-known Ramsey-theoretic principles but over a weaker base theory RCA^*_0, which differs from RCA_0 in that it assumes mathematical induction only for computable properties.

An important part of our initial motivation to work over RCA^*_0 was the hope that the behaviour of RT^2_2 and various related principles over RCA^*_0 would shed some light on the strength of RT^2_2 over RCA_0. This seemed to be a very promising direction of research, since the work of Belanger [3] demonstrated that in some contexts the situation over the weaker base theory might be relevant for the usual setting of RCA_0.

The system RCA^*_0 was introduced by Simpson and Smith in [48]. The weaker base theory makes it possible to calibrate the logical strength of mathematical theorems that are provable in RCA_0 and to track uses of Σ^0_1-induction in mathematical proofs. For example, over RCA^*_0 some theorems of countable algebra are equivalent to $\mathsf{I}\Sigma^0_1$ and thus to RCA_0, see e.g. [48, 20]. Another advantage of working in RCA^*_0 is that one can strengthen some reversals known to hold over RCA_0 by proving them using weaker assumptions. Also, over RCA^*_0 one can make some fine distinctions that disappear in the presence of $\mathsf{I}\Sigma^0_1$.

On the other hand, working over the weaker set of assumptions comes at the cost of having to face some conceptual challenges that do not occur over RCA_0. For us, of special importance is the fact that without assuming $\mathsf{I}\Sigma^0_1$ the notion of an infinite set becomes less robust. Namely, it is consistent with RCA^*_0 that there exists an unbounded subset of $\mathbb{N}$ which does not contain arbitrarily large finite subsets or, equivalently, that there exists an unbounded subset of $\mathbb{N}$ which is not in bijective correspondence with all of $\mathbb{N}$. This has a major impact on the study of Ramsey-like principles, because such a principle P typically takes the form 'for every relation X on $\mathbb{N}$ of a given type there exists an infinite set $Y \subseteq \mathbb{N}$ such that $\Phi(X, Y)$ holds', where Φ is expressible in first-order arithmetic. In this context X is usually called the *instance* and Y the *solution* of P. Thus, there are at least two natural formulations of a Ramsey-theoretic statement P in RCA_0^*: one that we will call *normal*, in which we allow the solution set Y to be merely unbounded, and the second one, called *long*, in which we require the solution set to have the size of the entire $\mathbb{N}$.

At first glance, it might seem that RCA_0^* is not the right setting for studying infinite combinatorics, as the lack of $\mathsf{I}\Sigma^0_1$ apparently leads to a proliferation of distinct versions of what is intuitively the same principle. Nevertheless, the picture turns out to be much tidier than might have been expected, since we have discovered just a few patterns which can be used to classify normal and long variants of Ramsey-like statements. Namely, in models of RCA_0^* that are not models of RCA_0, all the normal principles we consider exhibit a distinctive behaviour related to proper Σ^0_1-definable cuts – objects whose existence is equivalent to the failure of Σ^0_1-induction. Because of this, we can apply to them some general techniques from the theory of models of arithmetic and learn that these principles form a rather uniform group with respect to arithmetical consequences. In particular, they are all partially conservative over RCA_0^*. For the long principles, we show that each of them has just one of two contrasting properties: either it is strong in the sense that it implies $\mathsf{I}\Sigma^0_1$, or it has the same amount of conservativity over RCA_0^* as its normal counterpart.

We believe that the thesis provides, together with other papers of the project, strong evidence that reverse mathematics over the weaker base theory is an interesting and rewarding endeavour. As in the traditional framework of RCA_0, one needs to combine facts and techniques from different areas of logic, but in this case with a relatively strong emphasis on nonstandard models of arithmetic. Moreover, we feel that this new direction of research may bring fresh inspiration and new insights into more classical topics. For instance, one may encounter very interesting but previously unstudied fragments of first-order arithmetic, as we will see in Chapter 2, or consider some general properties of models of arithmetic that are also highly interesting in other contexts, e.g. [16, 31].

As for the strength of RT^2_2 over RCA_0, at the time of starting our project, the strongest known results on the first-order consequences of $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ were: $\forall\Pi^0_3$-conservativity over RCA_0, proved by Patey and Yokoyama [41], and an earlier theorem of Chong et al. [9] that could be seen as $\forall\Pi^0_4$-conservativity over an extension of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. Only in 2024 Le Houérou, Patey and Yokoyama managed to strengthen the above results by showing in [35] that RT^2_2 is $\forall\Pi^0_4$-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$, and in [36] that the result of Chong et al. can be improved to full Π^1_1-conservativity over the extension of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$, giving a new upper bound on the first-order consequences of $\mathsf{RCA}_0 + \mathsf{RT}^2_2$. However, the study of RT^2_2 over RCA_0^* has also brought new information: by the results of [16], RT^2_2 is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ if and only if it is $\forall\Pi^0_5$-conservative over

$\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. In general, [16] and our results from Chapter 2 indicate that it is at least reasonable to look for analogies in the behaviour of various combinatorial principles between the two base theories.

There is yet another important aspect to the change of the base theory. Namely, one can observe curious phenomena related to *proof size*, which can be used to compare axiomatic theories in a more quantitative way. Such a comparison is especially relevant when considering a conservation result, saying that a theory T_2 extending some theory T_1 proves the same sentences from a given syntactic class as T_1. In such a case one would like to know whether T_2 is able to provide essentially shorter proofs of those sentences than T_1. In our setting, a natural and interesting question is whether a given axiom with the same amount of conservativity over both base theories behaves in the same way over each of them also in terms of proof size.

Perhaps it is not a great surprise that, also in this context, RT^2_2 is a particularly interesting case and an inspiration for further research. By [41] and our results from Chapter 2, RT^2_2 is $\forall\Pi^0_3$-conservative over both RCA_0 and RCA^*_0. However, the quantitative analysis of these conservation results conducted by Kołodziejczyk, Wong and Yokoyama in [32] showed that the behaviour of RT^2_2 with respect to proof size changes dramatically depending on the base theory. Namely, for every proof of a $\forall\Pi^0_3$ sentence in $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ there exists an at most polynomially larger proof of this sentence in RCA_0. On the other hand, the proof size of Σ_1 theorems of $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2$ can grow superexponentially when they have to be derived in RCA^*_0. The arguments used to obtain these results rely heavily on bounds on Ramsey numbers for the finite version of RT^2_2. This makes one wonder whether the conservation results for other Ramsey-theoretic principles can also be refined by looking at their finite versions. Not all infinitary Ramsey-like statements have obvious finite counterparts (consider for instance principles that speak about stable colourings), but many do. Using those counterparts we will be able to compare in terms of proof size some Ramsey-theoretic principles that are partially conservative over RCA^*_0 and, as a result, we will see that their behaviour over the weaker base theory is more diverse than might initially seem. In particular, we will obtain an important distinction between RT^2_2 and CAC, the strongest principle below RT^2_2 that we will study.

Structure of the thesis. Chapter 1 has a preliminary character. It provides necessary background on first- and second-order arithmetic, explains the most important features of RCA^*_0 and introduces the Ramsey-theoretic principles that will be studied in the following chapters: RT^n_k for fixed $n, k \geqslant 2$, CAC, ADS, CRT^2_2 and COH.

In Chapter 2 we consider the normal versions of RT^n_k, for $n, k \geqslant 2$, CAC, ADS and CRT^2_2, with a focus mostly on their consequences in the first-order language. First, we show that the principles we study belong to a broad syntactic class of second-order sentences that are equivalent to their relativizations to proper Σ^0_1-definable cuts. Using this technical result we give two simple conservativity criteria for such sentences and apply them to the case of our principles, showing that they are $\forall\Pi^0_3$- but not arithmetically conservative over the weak base theory.

Then we focus more closely on RT^n_k for $n, k \geqslant 2$. We determine the amount of induction needed to prove the classical computability-theoretic lower bounds on complexity of solutions for RT^n_k. This allows us to give an axiomatization of the arithmetical consequences of $\mathsf{RCA}^*_0 + \mathsf{RT}^n_k$ for $n \geqslant 3$, which turn out to form an interesting, non-finitely axiomatizable fragment of PA.

Chapter 3 is devoted to principles that the methods from the previous chapter do not apply to. In the first part of the chapter we study the long variants of our Ramsey-like statements. We show that the long versions of RT^n_k for $n, k \geqslant 2$ and of CAC, and one formulation of long ADS, imply RCA_0. On the other hand, another possible formulation of long ADS and the long version of CRT^2_2 are $\forall\Pi^0_3$-conservative over RCA^*_0.

In the second part of Chapter 3 we study COH, which due to its syntactic form cannot be easily analyzed as either a normal or a long principle. We show that certain well-known computational properties of COH can be proved already in RCA^*_0, which implies that the model-theoretic behaviour of COH is quite different from that of the normal principles studied in Chapter 2. As a consequence, we learn that the implication $\mathsf{RT}^2_2 \Rightarrow \mathsf{COH}$, known to hold over RCA_0, is not provable in RCA^*_0. Additionally, we answer negatively the question whether COH is Π^1_1-conservative over RCA^*_0, asked by Belanger in [3].

The aim of Chapter 4 is to strengthen the conservation result for CAC from Chapter 2 by showing that proofs of $\forall\Pi^0_3$ sentences in RCA^*_0 grow at most polynomially relative to $\mathsf{RCA}^*_0 + \mathsf{CAC}$. We begin the chapter with some additional background needed to study proof size. In particular, to keep the thesis relatively self-contained, we take some time to give an exposition of the method of forcing interpretations. We obtain our result by transforming a two-step model-theoretic construction that can be used in an alternative proof of $\forall\Pi^0_3$-conservativity of $\mathsf{RCA}^*_0 + \mathsf{CAC}$ over RCA^*_0 into a syntactical forcing argument. In the first step we construct a restricted definable ultrapower of a model of RCA^*_0. Then, inside the ultrapower, we build a proper cut satisfying CAC, using bounds on the finite version of CAC.

Sources and contribution. Chapter 2 and Chapter 3 are based on [15] and [31, Sections 1-3], though the bounds in Lemma 2.19 are more general than those in [31, Lemma 3.3]. The results of those chapters should be viewed as the joint work of the coauthors of the papers, but the author of the thesis had a leading role in obtaining the general criterion in Theorem 2.9 based on examples for specific principles, deriving the generalized bounds in Lemma 2.19 and obtaining the results of Section 3.2. Chapter 4 contains so far unpublished results of the author [34].

Chapter 1

Preliminaries

1.1 First- and second-order arithmetic

This first section is a compromise between the desire to make the thesis comprehensible and self-contained and the need for the right proportion between background material and original results. Thus we explain notation that might not be completely standard in logic and recall some classical results about arithmetical theories. Nevertheless, we do assume some acquaintance with formal arithmetic. For a detailed introduction to first-order arithmetic we refer the reader to [18] and [29]. A standard monograph on second-order arithmetic is [47], see also [13]. We also assume familiarity with the most basic concepts of computability theory. The material presented in [49, Chapters 1-4] is more than enough.

Language. Our official proof system is a Hilbert-style calculus with equality whose only logical connectives are $\neg$ and $\Rightarrow$ and only quantifier is $\forall$. However, we will need to work with a fixed proof system only in the last chapter, and in the other parts of the thesis we will freely use the other connectives and the existential quantifier. The logical constants $\top$ and $\bot$ denote some fixed tautology and contradiction, e.g. $0 = 0$ and $\neg(0 = 0)$, respectively.

The language of first-order arithmetic $\mathcal{L}_{\mathrm{I}}$ has one sort of variables $x, y, z, \ldots$ and non-logical symbols $0, 1, +, \cdot, \leqslant$ with the obvious intended meaning. We use $2, 3, 4, \ldots$ to abbreviate numerals, that is terms of the form $1 + 1 + \cdots + 1$. We also use the symbols $<$, $>$, $\geqslant$ and $\neq$ as the usual abbreviations. The language of second-order arithmetic $\mathcal{L}_{\mathrm{II}}$ is a two-sorted language which is obtained from $\mathcal{L}_{\mathrm{I}}$ by adding variables of a second sort $X, Y, Z, \ldots$ and a relational symbol $\in$. The language $\mathcal{L}_{\mathrm{II}}$ does not contain an equality symbol for the second sort. Variables of the first and the second sort are called *first-* and *second-order variables*, respectively. The intended meaning of the formula $x \in X$ is 'the number x belongs to the set X'. Formulas without any second-order quantifiers are called *arithmetical*. The letters a, b and A, B are often used to denote first- and second-order parameters, respectively. We will write $\overline{X}$, $\overline{a}$ etc. for tuples of variables or parameters of some finite length appropriate in a given context. We abbreviate blocks of quantifiers of the same type by $\forall\overline{X}$, $\exists\overline{y}$ etc. For an $\mathcal{L}_{\mathrm{II}}$-theory T, the set of its consequences in the language $\mathcal{L}_{\mathrm{I}}$ is called *the first-order part* of T.

Arithmetical hierarchy. Bounded quantifiers $\forall x \leqslant t$, $\exists x \leqslant t$ are shorthand for $\forall x\,(x \leqslant t \Rightarrow ...)$ and $\exists x\,(x \leqslant t \wedge ...)$, respectively, where the number variable x does not occur in the term t. The class $\Delta^0_0 = \Sigma^0_0 = \Pi^0_0$ is the set of all $\mathcal{L}_{\mathrm{II}}$-formulas that contain only bounded quantifiers. For $n > 0$, the classes Σ^0_n and Π^0_n are defined dually: an $\mathcal{L}_{\mathrm{II}}$-formula φ is in the class Σ^0_n (Π^0_n) if it has the form $\exists\overline{x}\,\psi$ ($\forall\overline{x}\,\psi$), where ψ is Π^0_{n-1} (Σ^0_{n-1}). We will also consider the classes Δ^0_n, which, in contrast to Σ^0_n and Π^0_n, for $n \geqslant 1$ do not have syntactic definitions: the class Δ^0_n consists of those Σ^0_n formulas that are equivalent to a Π^0_n formula, where the equivalence depends on the theory or structure at hand. The classes Σ_n, Π_n, Δ_n are restrictions of Σ^0_n, Π^0_n and Δ^0_n, respectively, to the language $\mathcal{L}_{\mathrm{I}}$, that is, they consist of formulas without any second-order variables. The classes Σ^1_n and Π^1_n are defined analogously – here the subscript n counts the number of alternations of second-order quantifiers, followed by an arithmetical formula. Given a set parameter A, we use the notation $\Sigma_n(A)$ for the set of those Σ^0_n formulas that contain A as the only second-order parameter; similarly for $\Pi_n(A)$ and $\Delta_n(A)$. These are basically first-order formulas in the language $\mathcal{L}' := \mathcal{L}_{\mathrm{I}} \cup \{A\}$, where A is a fresh unary predicate symbol. If Γ is a class of formulas, then $\forall\Gamma$ denotes the class of universal closures of formulas from Γ. For example $\forall\Sigma^0_n$ and $\forall\Pi^0_{n+1}$ are the same class. The class $\exists\Gamma$ is defined dually.

Up to logical equivalence, negations of Σ^0_n formulas are Π^0_n formulas, and vice versa. Similarly, conjunctions and disjunctions of Σ^0_n (resp. Π^0_n) formulas can be regarded as Σ^0_n (resp. Π^0_n) formulas. In this sense one can speak of closure properties of the above formula classes: each class is closed under conjunction and disjunction, existential classes like Σ^0_n are closed under existential quantifiers of the appropriate sort, and universal classes like Π^0_n are closed under universal quantifiers of the appropriate sort. Another closure property related to bounded quantification requires assuming some arithmetical axioms and will be stated below.

Models. A model for the language $\mathcal{L}_{\mathrm{I}}$ is a tuple $(M, 0^M, 1^M, +^M, \cdot^M, \leqslant^M)$, where M is called the *universe* of the model. To simplify notation, we will follow common practice and denote the whole model also by M. Models for the language $\mathcal{L}_{\mathrm{II}}$ can be seen as pairs $(M, \mathcal{X})$, where $\mathcal{X} \subseteq \mathcal{P}(M)$. We call M (with the interpretations of all symbols of $\mathcal{L}_{\mathrm{I}}$) the *first-order part* of $(M, \mathcal{X})$, and $\mathcal{X}$ is called its *second-order universe*. We use ω to denote the standard natural numbers and $\mathbb{N}$ to denote the set of natural numbers as formalized within an arithmetic theory. Any structure having ω as its first-order part is called an *ω-model*.

Let us stress that we always work in (possibly two-sorted) first-order predicate logic. We use the so-called Henkin semantics for set quantifiers $\forall X$ and $\exists X$, that is, they always range over a given family $\mathcal{X}$ of subsets of some first-order structure M, not over the entire $\mathcal{P}(M)$ (unless $\mathcal{X} = \mathcal{P}(M)$). Throughout the thesis we will use the completeness, compactness and Löwenheim-Skolem theorems for first-order logic, usually without mention. In particular, we often need to work with countable models, that is structures $(M, \mathcal{X})$ such that both M and $\mathcal{X}$ are countable.

We will consider only models that satisfy PA^-, a finite list of axioms expressing that a structure M is a discretely ordered commutative semiring with 0 as the least element, see [29, Chapter 2].

Let M and K be $\mathcal{L}_{\mathrm{I}}$-structures such that M is a substructure of K. We say that K is an *end-extension* of M if $\forall x \in M\, \forall y \in K \setminus M\, (x < y)$. We call K a *cofinal extension* of M if $\forall y \in K\, \exists x \in M\, (y < x)$. An end- or a cofinal extension is *proper* if $M \neq K$. We denote end-extensions by $M \subseteq_{\mathrm{e}} K$ and cofinal extensions by $M \subseteq_{\mathrm{cf}} K$. We will use this notation also for other ordered structures, not necessarily closed under arithmetical operations.

For a class of formulas Γ we say that M is a Γ-*elementary substructure* of K if $M \subseteq K$ and for every Γ-formula φ and every tuple of elements $\overline{a} \in M$ it holds that $M \vDash \varphi(\overline{a})$ if and only if $K \vDash \varphi(\overline{a})$.

Given a first-order structure M we will often consider a second-order structure $(M, \mathcal{X})$, where $\mathcal{X}$ is some family of definable subsets of M. Often $\mathcal{X}$ will be Γ-Def(M), which is the family of all Γ-definable (with parameters) subsets of M for a formula class Γ. An element of Γ-Def(M) is often called a Γ-*set*.

A structure $(N, \mathcal{Y})$ is called an ω-*extension* of a structure $(M, \mathcal{X})$ if $M = N$ and $\mathcal{X} \subseteq \mathcal{Y}$.

Induction and collection. Mathematical induction is usually formalized by an axiom scheme. For a syntactically defined class of formulas Γ (e.g. Σ^0_n or Π^0_n), we define the theory $\mathsf{I}\Gamma$ to be PA^- together with universal closures of formulas from the set:

$$\big\{\big(\varphi(0) \wedge \forall x\big(\varphi(x) \Rightarrow \varphi(x+1)\big)\big) \Rightarrow \forall x\, \varphi(x) :\ \varphi \in \Gamma\big\}\,. \tag{1.1}$$

If the class Γ is not syntactically defined, so that it may depend on a given theory or model, then one needs to slightly modify the above axiom scheme. For example, $\mathsf{I}\Delta_n$ is defined to be the set of universal closures of formulas from the set:

$$\{\forall x\big(\varphi(x) \Leftrightarrow \psi(x)\big) \ \Rightarrow\ \big(\varphi(0) \wedge \forall x\big(\varphi(x) \Rightarrow \varphi(x+1)\big) \Rightarrow \forall x\, \varphi(x)\big) : \\ \varphi \in \Sigma_n,\, \psi \in \Pi_n\}. \tag{1.2}$$

Peano arithmetic, PA, is axiomatized by PA^- and the induction scheme for all formulas of $\mathcal{L}_{\mathrm{I}}$.

We will work with theories like $\mathsf{I}\Sigma_n$, $\mathsf{I}\Sigma^0_n$ and $\mathsf{I}\Sigma_n(A)$, which are not the same but in many contexts exhibit essentially the same behaviour. Therefore, in the rest of this section on arithmetical axioms, we will state some properties only for theories formulated in the language $\mathcal{L}_{\mathrm{I}}$, but it should be clear that the same holds also for the respective formulations using a set parameter or all set variables from the language $\mathcal{L}_{\mathrm{II}}$.

Note that $\mathsf{I}\Delta_0$ and $\mathsf{I}\Sigma_0$ are the same set of axioms. For all $n \in \omega$, $\mathsf{I}\Sigma_n$ is equivalent to $\mathsf{I}\Pi_n$. Also, $\mathsf{I}\Sigma_n$ is equivalent to the least number principle for Σ_n

formulas. The latter is axiomatized by universal closures of formulas from the set:

$$\{\exists x\, \varphi(x) \;\Rightarrow\; \exists x\big(\varphi(x) \,\wedge\, \forall y{<}x\, \neg\varphi(y)\big) :\; \varphi \in \Sigma_n\}\,. \tag{1.3}$$

Another important principle is *collection* (also called *bounding*). For a syntactically defined class of formulas Γ, the theory $\mathsf{B}\Gamma$ is axiomatized by universal closures of formulas from the set:

$$\{\forall x{\leqslant}u\, \exists y\, \varphi(x,y) \Rightarrow \exists v\, \forall x{\leqslant}u\, \exists y{\leqslant}v\, \varphi(x,y) :\; \varphi \in \Gamma\}, \tag{1.4}$$

together with $\mathsf{I}\Delta_0$. The theories $\mathsf{B}\Sigma_0$ and $\mathsf{B}\Sigma_1$ are equivalent, because with the usual Cantor pairing function $\langle x, y\rangle$, which is available in $\mathsf{I}\Delta_0$, one can contract a few unbounded quantifiers of the same kind into one, e.g. $\exists x\, \exists y\, \varphi$ is equivalent to some $\exists x\, \psi$, where φ, ψ are Δ_0. Collection guarantees that formula classes of the arithmetical hierarchy are closed under bounded quantification:

Proposition 1.1. *Let $n \geqslant 1$ and let φ be a Σ_n formula and ψ be a Π_n formula. Then, there exist a Σ_n formula φ' and a Π_n formula ψ' such that $\mathsf{B}\Sigma_n$ proves the equivalences $\forall x{\leqslant}t\, \varphi \Leftrightarrow \varphi'$ and $\exists x{\leqslant}t\, \psi \Leftrightarrow \psi'$.*

We have the following hierarchy of theories, originally studied by Paris and Kirby in [40], where each inclusion is proper:

$$\mathsf{I}\Delta_0 \subseteq \mathsf{B}\Sigma_1 \subseteq \mathsf{I}\Sigma_1 \subseteq \mathsf{B}\Sigma_2 \subseteq \mathsf{I}\Sigma_2 \subseteq \ldots \mathsf{PA}. \tag{1.5}$$

The theory $\mathsf{I}\Delta_0$ is Π_1-axiomatizable. For each $n \geqslant 1$, both $\mathsf{I}\Sigma_n$ and $\mathsf{B}\Sigma_n$ are Π_{n+2}-axiomatizable. For $n \geqslant 1$, $\mathsf{I}\Sigma_n$ is equivalent over $\mathsf{I}\Delta_0$ to the so-called *strong Σ_n-collection*:

$$\{\forall u\, \exists v\, \forall x{\leqslant}u\, \big(\exists y\, \varphi(x,y) \Rightarrow \exists y{\leqslant}v\;\, \varphi(x,y)\big) :\; \varphi \in \Sigma_n\}\,. \tag{1.6}$$

We will often use the following simple observation.

Proposition 1.2. *Suppose that $M \vDash \mathsf{I}\Delta_0$ and $n \geqslant 1$. Then the following holds.*

(a) $M \vDash \mathsf{B}\Sigma_n$ if and only if $(M, \Delta_n\text{-Def}(M)) \vDash \mathsf{B}\Sigma^0_1$.

(b) If $A \subseteq M$ is Σ_n-definable in M, then it is Σ^0_1-definable in $(M, \Delta_n\text{-Def}(M))$.

The exponential function. There exist Δ_0 formulas which define in ω the graphs of the exponential and superexponential functions, where the latter is defined recursively by $2_0(x) = x$ and $2_{y+1}(x) = 2^{2_y(x)}$. We denote these formulas simply by '$2^x = y$' and '$2_y(x) = z$', respectively. For both functions $\mathsf{I}\Delta_0$ proves the recursive equations implicitly defining them as well as the uniqueness of their values for all arguments. However, $\mathsf{I}\Sigma_1$ is the weakest theory in the hierarchy (1.5) that proves the Π_2 sentences $\mathsf{exp} := \forall x\, \exists y\, (x^y = z)$ and $\mathsf{supexp} := \forall x\, \forall y\, \exists z\, (2_y(x) = z)$, which express the totality of the exponential and superexponential functions, respectively.

With the exponential function one can easily interpret natural numbers as finite sets, and thus represent in arithmetic all finite objects as it would be done in set theory. We use the so-called Ackermann interpretation of finite set theory in arithmetic and write $x \in_{\mathrm{Ack}} c$ for the Δ_0 formula expressing that the x-th position in the binary expansion of c is 1. We say that a number c is a code for the set $(c)_{\mathrm{Ack}} := \{x \in \mathbb{N} \colon x \in_{\mathrm{Ack}} c\}$ but, following common practice, we will often identify sets with their codes. By $|c| = y$ we denote a Δ_0 formula saying that the cardinality of $(c)_{\mathrm{Ack}}$ (or simply c) is y. Within a formal theory all coded sets are, by definition, *finite*. However, note that if y is nonstandard, then the set $(c)_{\mathrm{Ack}}$ is an infinite subset of the universe of a given model M.

We will use the standard symbols for some well-known functions as abbreviations for formulas defining them (or their obvious variants for integers), for example log, or $\lfloor x \rfloor$ and $\lceil x \rceil$ for the floor and ceiling functions.

Turing machines and their computations, being finite objects, can also be formalized in first-order arithmetic in a natural way as soon as we have exp. We write $\varphi_e(x) = y$ if the e-th Turing machine halts on input x and outputs y. If this happens in fewer than s steps and also $x, y, e < s$, then we write $\varphi_{e,s}(x) = y$. If we just want to say that the e-th Turing machine halts (resp. in fewer than s steps) on input x, then we write $\varphi_e(x)\downarrow$ (resp. $\varphi_{e,s}(x)\downarrow$).

Since exp is necessary to represent finite objects in first-order arithmetic in a natural way, the weakest theories that we work with are $\mathsf{I}\Delta_0 + \mathsf{exp}$ and $\mathsf{B}\Sigma_1 + \mathsf{exp}$, which are different from each other and do not imply $\mathsf{I}\Sigma_1$.

Already $\mathsf{I}\Delta_0 + \mathsf{exp}$ supports partial satisfaction predicates. Namely, for every $n \in \omega$ there exist universal formulas $\mathrm{Sat}_{\Sigma_n}(e, x)$ and $\mathrm{Sat}_{\Pi_n}(e, x)$ such that $\mathsf{I}\Delta_0 + \mathsf{exp}$ proves a finite list of conditions expressing that $\mathrm{Sat}_{\Sigma_n}(e, x)$ and $\mathrm{Sat}_{\Pi_n}(e, x)$ satisfy the inductive (compositional) definition of satisfaction of a formula e by an assignment x. Crucially, for every $\Gamma \in \{\Sigma_n, \Pi_n \colon n \in \omega\}$ and every formula $\varphi(x) \in \Gamma$, the theory $\mathsf{I}\Delta_0 + \mathsf{exp}$ proves Tarski's biconditional:

$$\forall x \big(\mathrm{Sat}_\Gamma(\ulcorner \varphi(y) \urcorner, x) \Leftrightarrow \varphi(x)\big), \tag{1.7}$$

where $\ulcorner \varphi(y) \urcorner$ is (the numeral of) the Gödel number of $\varphi(y)$. If $n > 0$, then $\mathrm{Sat}_{\Sigma_n}(e, x)$ has complexity Σ_n, and $\mathrm{Sat}_{\Pi_n}(e, x)$ has complexity Π_n. For $n = 0$ it is Δ_1 provably in $\mathsf{I}\Delta_0 + \mathsf{exp}$. For every $n \in \omega$, one can use universal formulas to give a finite axiomatization of the theories $\mathsf{I}\Sigma_n + \mathsf{exp}$ and $\mathsf{B}\Sigma_n + \mathsf{exp}$ (cf. Section 4.1). We will also use satisfaction predicates with a free set variable, $\mathrm{Sat}_{\Sigma^0_n}(e, x, X)$ and $\mathrm{Sat}_{\Pi^0_n}(e, x, X)$, which have analogous properties. Depending on the context, we will substitute for X either a set parameter or a formula defining some set.

Sets and set existence axioms. Theories of second-order arithmetic are usually axiomatized by three groups of axioms: a finite list of basic properties of natural numbers like PA^-, induction axioms and set existence axioms. It is the last group that usually has the largest influence on the logical strength of $\mathcal{L}_{\mathrm{II}}$-theories studied in reverse mathematics. Here the basic axiom scheme is set comprehension, expressing that subsets of natural numbers defined by some distinguished formulas do exist. That is, for a class of formulas Γ whose definition is independent of any theory, we define Γ-*comprehension* to be the set of universal closures of formulas from the set:

$$\big\{\exists X \forall x\, \big(x \in X \Leftrightarrow \varphi(x)\big) :\ \varphi \in \Gamma\big\}\,, \tag{1.8}$$

where X does not appear freely in φ. If the formula class Γ is not syntactically defined, then the above axiom scheme needs to be modified (cf. the Δ_n-induction scheme (1.2)). For example, Δ^0_1-comprehension is axiomatized by universal closures of formulas from the set:

$$\big\{\forall x\big(\varphi(x) \Leftrightarrow \psi(x)\big) \Rightarrow \exists X \forall x\, \big(x \in X \Leftrightarrow \varphi(x)\big) :\ \varphi \in \Sigma^0_1, \psi \in \Pi^0_1\big\}\,. \tag{1.9}$$

Let us stress that when working in an $\mathcal{L}_{\mathrm{II}}$-theory T by a *set* (without any qualifiers) we mean an object of the second-order sort, thus in model-theoretic terms a set is an element of the second-order universe. Occasionally, we may use the word 'set' in the metatheory but it should always be clear what kind of object we refer to. On the other hand, we will formally reason about and quantify over Γ-sets, where Γ is a class from the arithmetical hierarchy, which will often not be sets in the above sense. Indeed, the assertion that every Γ-set exists as a set is equivalent to Γ-comprehension.

In the second-order context we define a set X to be *finite* if it is *bounded*, that is, if $\exists x\, \forall y\, (y \in X \Rightarrow y < x)$ holds. Note that we have already defined a finite set as a set coded by a first-order element. However, the two notions coincide over very weak axioms. Namely, by Δ_0-comprehension, for every number c it holds that $\exists X \forall x\, \big(x \in X \Leftrightarrow x \in_{\mathrm{Ack}} c\big)$. Conversely, $\mathsf{I}\Delta^0_0 + \mathsf{exp}$ guarantees that for every bounded set X there exists a number c that codes precisely X. Since all $\mathcal{L}_{\mathrm{II}}$-theories that we will consider contain Δ^0_0-comprehension and $\mathsf{I}\Delta^0_0 + \mathsf{exp}$, we will identify these two notions of a finite set, and also say that y is the cardinality of a bounded set X if $X = (c)_{\mathrm{Ack}}$ and $y = |c|$. Note that, by $\mathsf{I}\Delta^0_0$, every bounded set has a greatest element.

A set X is *infinite* if it *unbounded*, that is, $\forall x\, \exists y\, (x < y \land y \in X)$. We write $A \subseteq_{\mathrm{cf}} B$ if A is an infinite subset of B. Clearly, our definitions of finite and infinite sets guarantee that these notions are antonyms in a formal theory. However, we will work with bounded (and, in fact, definable) subsets of first-order universes that are not really finite, as seen in the metatheory.

In second-order arithmetic we can directly speak only about subsets of the set of natural numbers, but using appropriate coding we can represent essentially all objects of countable mathematics (including some uncountable but countably presentable structures like separable metric spaces). However, in what follows we will mostly work with objects that have rather simple definitions.

The complement of a set A is denoted $\overline{A}$ (it should always be clear from the context whether we mean a complement of a set or a tuple of sets). If A and B are sets such that $A \setminus B$ is finite, then we write $A \subseteq^* B$. We use the notation $A < B$ if every element of A is smaller than every element of B and, similarly, $A < a$ if a number a is greater than all elements of A.

A set F is a *function* if it satisfies $\forall x, x' \in F\, \big((x = \langle y, z\rangle \land x' = \langle y, z'\rangle) \Rightarrow z = z'\big)$. We will often denote functions by lower case letters such as f, g, h. We define the *domain* and *image* of a function in the obvious way. Note that for a function F its image is a Σ^0_1-set, so in general one cannot expect that it is a set. Monotone functions are an important exception, for their images are Δ^0_1-definable. For the restriction of a function F to a subset S of its domain we write $F \upharpoonright S$. For a set A, by $A \upharpoonright m$ we mean the intersection $A \cap [0, m)$. A *sequence* is a function whose domain is downward closed. For a sequence with the whole first-order universe as domain we write $(a_n)_{n \in \mathbb{N}}$, when we work in a formal theory, and $(a_m)_{m \in M}$, when we work with a model M (but see also Proposition 1.14 in the next section). We say that *a sequence is unbounded* if the collection of its terms is an unbounded subset of the first-order universe. By a *sequence of sets* $(A_n)_{n \in \mathbb{N}}$ we mean a set S such that $\langle n, x\rangle \in S$ iff $x \in A_n$.

For a set A and $k \in \omega$, the set $[A]^k$ consists of all (codes for) finite subsets of A of cardinality k. A set T is a *tree* if it consists of (codes for) finite sequences, which we call *nodes*, and is closed under taking initial segments of sequences. It is a *binary tree* if it consists only of sequences with terms from $\{0, 1\}$. A set P is an *infinite path* in a tree T if it is a sequence $(a_n)_{n \in \mathbb{N}}$ such that for each $n \in \mathbb{N}$ the finite sequence $\langle a_0, \ldots, a_n\rangle$ is a node of T.

For our applications it will be more convenient to define the Turing jump operator using satisfaction predicates rather than oracle machines. That is, for a set A and $n \geqslant 1$, we define the n-th Turing jump $A^{(n)}$ to be the $\Sigma_n(A)$-definable set $\{x \in \mathbb{N} \colon \mathrm{Sat}_{\Sigma^0_n}(x, x, A)\}$. For $n = 1$, our definition and the one using oracle machines are equivalent over RCA^*_0, that is, both jumps are mutually Δ^0_1-definable. To obtain equivalence for $n > 1$ one needs stronger collection axioms. We write $0^{(n)}$ for the n-th jump of the empty set. Also, A' stands for $A^{(1)}$.

Most of the notions concerning sets defined in the present section easily generalize to the case of definable sets. For example, in Section 2.4.1 we will reason in the first-order language about Turing jumps of definable sets.

Theories of second-order arithmetic. *Full second-order arithmetic* Z_2 consists of PA^- together with the induction and comprehension schemes for all formulas of $\mathcal{L}_{\mathrm{II}}$. This is a powerful system, so to calibrate the logical strength of different mathematical theorems one works only with fragments of Z_2, which are usually defined by weakening the induction and comprehension schemes.

Traditionally, the weakest considered fragment of Z_2, used as a base theory for reverse-mathematical studies, is the theory RCA_0, axiomatized by PA^-, $\mathsf{I}\Sigma^0_1$ and Δ^0_1-comprehension. In this thesis, for reasons indicated in the introduction and to be further discussed in the next section, we will work with a weaker base theory RCA^*_0, which is obtained from RCA_0 by replacing $\mathsf{I}\Sigma^0_1$ with $\mathsf{I}\Delta^0_0 + \mathsf{exp}$. By [48, Lemma 4.1], RCA^*_0 proves $\mathsf{B}\Sigma^0_1$.

Both RCA_0 and RCA^*_0 guarantee only the existence of computable sets. Namely, every structure of the form $(M, \Delta_1\text{-}\mathrm{Def}(M))$ satisfies RCA_0 (resp. RCA^*_0) if and only if its first-order part M satisfies $\mathsf{I}\Sigma_1$ (resp. $\mathsf{B}\Sigma_1 + \mathsf{exp}$). Thus the minimal ω-model for both the theories is (ω, REC), where REC is the family of computable subsets of natural numbers. The existence of more complex sets follows from the existence of appropriate iterations of Turing jump, as every Δ_{n+1}-definable set is $\Delta_1(0^{(n)})$-definable.

Proposition 1.3. *If* $(M, \mathcal{X}) \vDash \mathsf{RCA}^*_0$ *and* $0^{(n)} \in \mathcal{X}$, *then* $\Delta_{n+1}\text{-}\mathrm{Def}(M) \subseteq \mathcal{X}$.

A basic feature of RCA_0 is that it lets us define functions by primitive recursion. In fact, the provably total computable functions of RCA_0 are precisely the primitive recursive ones. In contrast, the provably total computable functions of RCA_0^* are the elementary computable functions.

The systems WKL_0 and WKL_0^* are axiomatized by RCA_0 and RCA_0^*, respectively, together with weak König's lemma WKL, which is a Π^1_2 sentence expressing that every infinite binary tree has an infinite path. Although WKL is not satisfied by the structure (ω, REC) (in fact, there is no minimal ω-model of WKL), there are ω-models of WKL with the second-order universe consisting entirely of low sets, where a set $X \subseteq \mathbb{N}$ is *low* if X' is computable in $0'$. This follows from the low basis theorem which says that every computable infinite binary tree has an infinite path that is low.

The strongest subsystem of Z_2 that will appear in this thesis is ACA_0, which extends RCA_0 by adding Σ^0_1-comprehension. This apparently small strengthening of the comprehension scheme has very strong consequences: ACA_0 proves induction and comprehension axioms for all arithmetical formulas. Moreover, for every $M \vDash \mathsf{PA}$ the structure $(M, \mathrm{Def}(M))$ satisfies ACA_0, where $\mathrm{Def}(M)$ is the family of all $\mathcal{L}_{\mathrm{I}}$-definable subsets of M.

Over RCA_0, the system ACA_0 is equivalent to the statement that for every function F the image of F exists (that is, is a set), and also to the assertion that for every set A, the Turing jump of A exists.

The theories RCA_0^*, RCA_0, WKL_0, WKL_0^*, ACA_0, as well as $\mathsf{I}\Sigma_n + \mathsf{exp}$ and $\mathsf{B}\Sigma_n + \mathsf{exp}$ for $n \geqslant 0$, are finitely axiomatized. This will play a role and will be further discussed in Chapter 4.

Conservativity. One of the very first questions one would like to ask about a given $\mathcal{L}_{\mathrm{II}}$-theory T is this: what consequences for finite objects does T have? Or more precisely: what $\mathcal{L}_{\mathrm{I}}$-sentences does T prove? The set of these sentences is called *the first-order part of* T. To answer this question one often tries to prove a conservation result. We say that a theory T_1 is Γ-*conservative* over a theory T_2, where Γ is a set of sentences in their common language $\mathcal{L}(T_1) \cap \mathcal{L}(T_2)$, if for every sentence $\gamma \in \Gamma$ such that $T_1 \vdash \gamma$, it also holds that $T_2 \vdash \gamma$.

It is often very interesting to find a pair of theories $T_2 \subsetneq T_1$ such that T_1 is arithmetically conservative over T_2, which means that they have the same first-order part or, at least, that T_1 is Π_2-conservative over T_2, which implies that they have the same provably total computable functions. To show that T_1 is arithmetically conservative over T_2, one frequently uses ω-extensions and actually proves something stronger, namely Π^1_1-conservativity. Typically, one takes a Σ^1_1 sentence φ consistent with T_2 and shows that every countable structure from a general enough class satisfying $T_2 + \varphi$ has an ω-extension satisfying T_1. Since ω-extensions preserve Σ^1_1 sentences, one obtains a model of $T_1 + \varphi$.

By what we have said above, it follows that the first-order parts of RCA_0^*, RCA_0 and ACA_0 are, respectively, $\mathsf{B}\Sigma_1 + \mathsf{exp}$, $\mathsf{I}\Sigma_1$ and PA. Also, RCA_0^* is Π_2-conservative over elementary function arithmetic EFA (which is a definitional extension of $\mathsf{I}\Delta_0 + \mathsf{exp}$), and RCA_0 is Π_2-conservative over primitive recursive arithmetic PRA.

The following is one of the most remarkable theorems of reverse mathematics, as WKL_0 proves many more theorems of ordinary mathematics than RCA_0 does. It was proved independently by Harrington, see [47, Corollary IX.2.6], and Ratajczyk [44].

Theorem 1.4 (Harrington, Ratajczyk)**.** *WKL_0 is Π^1_1-conservative over RCA_0. In fact, for every countable model of RCA_0 there exists an ω-extension satisfying WKL_0.*

The usual proof uses the method of forcing (cf. Section 4.2). Given a countable model $(M, \mathcal{X})$ of RCA_0, one constructs its ω-extension in countably many steps, where at each step an infinite path is added to $\mathcal{X}$ for an infinite binary tree that is already in $\mathcal{X}$. Such a path is a generic object that is forced to satisfy $\mathsf{I}\Sigma^0_1$.

The system WKL^*_0 was introduced by Simpson and Smith in [48] and more recently studied extensively in [14] and [16]. Its relation to RCA^*_0 is in an important way analogous to that of WKL_0 to RCA_0.

Theorem 1.5 (Simpson-Smith [48])**.** *For every model of RCA^*_0 there is an ω-extension satisfying WKL^*_0. In particular, WKL^*_0 is Π^1_1-conservative over RCA^*_0.*

The following result due to Yokoyama is a useful tool to combine conservation results.

Theorem 1.6 (Yokoyama [57])**.** *Let T_1 and T_2 be $\mathcal{L}_{\mathrm{II}}$-theories that are Π^1_2-axiomatizable. If both T_1 and T_2 are Π^1_1-conservative extensions of $T_0 \supseteq \mathsf{RCA}_0$, then $T_1 + T_2$ is also Π^1_1-conservative over T_0.*

Cuts. A nonempty subset I of a model of arithmetic M is a *cut* if it is an initial segment of M closed under the successor function. If $I \neq M$ then I is a *proper cut*. For a nonempty subset $A \subseteq M$ without a maximal element, we denote by $\sup_M A$ the cut $I = \{i \in M \colon \exists a \in A\, (i \leqslant a)\}$. If M is a model of PA^-, then $M \vDash \mathsf{I}\Sigma_n$ if and only if there is no Σ_n-definable proper cut.

If a cut $I \subseteq M$ is also closed under multiplication, then it a substructure of M, and thus it makes sense to ask about the amount of elementarity between I and M. As the following theorem shows, this can often say something about the amount of induction and collection satisfied by a given cut. Part (a) is folklore, part (b) is due to [40] and [11].

Theorem 1.7. *Let $M \vDash \mathsf{I}\Delta_0$ and let $I \subseteq M$ be a proper cut.*

(a) If I is closed under multiplication, then $I \vDash \mathsf{B}\Sigma_1$.

(b) For $n \geqslant 0$, if $M \vDash \mathsf{I}\Sigma_n$ and I is a Σ_{n+1}-elementary substructure of M, then $I \vDash \mathsf{B}\Sigma_{n+2}$.

Given a proper cut $I \subseteq M$ we can consider the second-order structure $(I, \mathrm{Cod}(M/I))$, where $\mathrm{Cod}(M/I)$ is the family of subsets of I that are *coded in M on I*:

$$\mathrm{Cod}(M/I) = \{X \in \mathcal{P}(I) \colon \exists c \in M\, \forall x\, (x \in X \Leftrightarrow (x \in_{\mathrm{Ack}} c \wedge x \in I))\}.$$

If $(M, \mathcal{X})$ is a model of a very weak second-order theory (RCA_0^* is more than enough), then $\mathrm{Cod}(M/I)$ coincides with the family of traces of elements of $\mathcal{X}$ on I, i.e. $X \in \mathrm{Cod}(M/I)$ if and only if there exists a set $X' \in \mathcal{X}$ satisfying $X = X' \cap I$.

Sometimes (for example, in order to state Theorem 2.9 in full generality), we abuse notation slightly and use $\mathrm{Cod}(M/I)$ to stand also for the collection of k-ary relations on I coded in M, that is for the family $\{\{\langle i_1, \ldots, i_k\rangle \in_{\mathrm{Ack}} s : i_1, \ldots, i_k \in I\} : s \in M\}$, where $1 < k \in \omega$. Here $\langle i_1, \ldots, i_k\rangle$ is defined in terms of the usual Cantor pairing function. If I is not closed under multiplication, then not all such coded k-ary relations will be elements of $\mathrm{Cod}(M/I)$ in the strict sense, but that should not lead to any confusion.

A cut I that is closed under exponentiation is called *exponential*. For every such cut I and every $X \in \mathrm{Cod}(M/I)$, codes for X can be found arbitrarily low above the cut I.

Proposition 1.8. *Let M be a nonstandard model of $\mathsf{I}\Delta_0 + \mathsf{exp}$ and I a proper exponential cut in M. For every number $a > I$ and every set $X \in \mathrm{Cod}(M/I)$ there exists $c < a$ such that $(c)_{\mathrm{Ack}} \cap I = X$.*

In every model M of arithmetic the smallest cut is ω, which may or may not be definable. The family of subsets of ω coded in M on ω is called the *standard system* of M and denoted $\mathrm{SSy}(M)$. Note that by Proposition 1.8, for every $M \subsetneq_{\mathrm{e}} K$ with $M \neq \omega$, we have $\mathrm{SSy}(M) = \mathrm{SSy}(K)$.

Exponential cuts will play a special role because of the following property.

Theorem 1.9 (Simpson-Smith [48])**.** *Let I be a proper exponential cut in a model $M \vDash \mathsf{I}\Delta_0$. Then the structure $(I, \mathrm{Cod}(M/I))$ satisfies WKL_0^*.*

The next three theorems can be seen as converses to Theorem 1.9. The first one is a classical result [45, 56], see also [29, Section 13.1]. The second one is an immediate consequence of Tanaka's self-embedding theorem [54]. The third one follows easily from Tanaka's theorem and some standard arguments.

Theorem 1.10 (Scott [45], Wilmers [56])**.** *Let T be a computably axiomatized theory containing $\mathsf{I}\Delta_0 + \mathsf{exp}$ and let $\mathcal{X}$ be a countable subset of $\mathcal{P}(\omega)$ such that $(\omega, \mathcal{X}) \vDash \mathsf{WKL}$. Then there exists a countable model $M \vDash T$ with $\mathrm{SSy}(M) = \mathcal{X}$.*

Theorem 1.11 (Tanaka [54])**.** *Every countable nonstandard model $(M, \mathcal{X})$ satisfying WKL_0 has an extension $(N, \mathcal{Y})$ also satisfying WKL_0 such that $M \subseteq_e N$ and $\mathcal{X} = \mathrm{Cod}(N/M)$.*

Theorem 1.12. *For every countable $(M, \mathcal{X}) \vDash \mathsf{WKL}_0$ there exists a proper end-extension $K \supseteq_e M$ such that $K \vDash \mathsf{B}\Sigma_1 + \mathsf{exp}$, M is a Σ_1-definable cut in K, and $\mathrm{Cod}(K/M) = \mathcal{X}$.*

Proof. Let $(M, \mathcal{X})$ be a countable model of WKL_0. By Theorem 1.11, there exists an end-extension $N \supsetneq_{\mathrm{e}} M$ with $(N, \Delta_1\text{-}\mathrm{Def}(N)) \vDash \mathsf{RCA}_0$ and $\mathrm{Cod}(N/M) = \mathcal{X}$. Fix some $a \in N \setminus M$. Note that since N satisfies $\mathsf{I}\Sigma_1$ and therefore supexp, the value $2_b(a)$ exists in N for each $b \in N$. Define a cut $K \subseteq_{\mathrm{e}} N$ by $K =$

$\sup\{2_m(a)\colon m \in M\}$. Then K is closed under exponentiation and, by Theorem 1.7 (a), it satisfies $\mathsf{B}\Sigma_1$. Also, M is a Σ_1-cut in K, because $m \in M$ if and only if $K \vDash \exists y\,(y = 2_m(a))$. Furthermore, by Proposition 1.8, $\mathrm{Cod}(K/M) = \mathrm{Cod}(N/M) = \mathcal{X}$. □

1.2 RCA_0^* as a base theory

The system RCA_0^* is strictly weaker than RCA_0, that is, it does not imply $\mathsf{I}\Sigma_1^0$. Thus, we will have to work with structures satisfying $\neg\mathsf{I}\Sigma_1^0$. In model-theoretic terms failure of Σ_1^0-induction means that there exists a Σ_1^0-definable proper cut. Namely, if the induction scheme fails for a Σ_1^0 formula $\exists y\,\varphi(x,y)$, then the Σ_1^0-definable set:

$$I = \{i \in \mathbb{N}\colon\ \exists w\, \forall x \leqslant i\ \exists y \leqslant w\ \varphi(x,y)\} \tag{1.10}$$

is a proper cut.

Speaking very intuitively, such a cut is a formal object corresponding to a process which takes an infinite amount of time but for some large number k never reaches the k-th stage. A simple situation of this kind is counting elements of some unbounded set S: having enumerated the first x elements of S, one can always enumerate its $(x+1)$-th element but may never get to enumerate its k-th element, for some very large number k. In fact, for every Σ_1^0-definable proper cut one can find such a 'short' infinite set.

Proposition 1.13. *Let $(M,\mathcal{X}) \vDash \mathsf{RCA}_0^*$. For every Σ_1^0-definable cut I there exists an unbounded set $A \in \mathcal{X}$ which can be enumerated in increasing order as $A = \{a_i\}_{i\in I}$.*

Proof. The statement of the proposition is obvious for $I = \mathbb{N}$, so assume that I is a proper Σ_1^0-cut defined in a model $(M,\mathcal{X}) \vDash \mathsf{RCA}_0^*$ by a Σ_1^0 formula $\exists y\,\varphi(x,y)$. Using Δ_1^0-comprehension one can define the set consisting of finite sequences of the smallest witnesses y to the formula $\exists y\,\varphi(x,y)$:

$$A = \big\{s = \langle s_0,\ldots,s_i\rangle\colon\ \forall j \leqslant i\ \big(\varphi(j,s_j)\ \wedge\ \forall y < s_j\ \neg\varphi(j,y)\big)\big\}\,. \tag{1.11}$$

The set A is unbounded because if not, then there would be some number b above the set A such that the cut I would be Δ_0^0-definable by the formula $\exists y \leqslant b\ \varphi(x,y)$, violating Δ_0^0-induction. It follows immediately from the definition of A that it can be enumerated in increasing order precisely by the cut I, and not the whole $\mathbb{N}$: $A = \{a_i\}_{i\in I}$, where $a_i = \langle s_0,\ldots,s_i\rangle$. □

Working in a model of $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0$ we will often use an unbounded set like (1.11) in Proposition 1.13 to split the first-order universe into I-many finite intervals $(a_{i-1},a_i] = \{x \in \mathbb{N}\colon a_{i-1} < x \leqslant a_i\}$. We will use the convention that $a_{-1} = -1$ to get $(a_{-1},a_0] = \{x \in \mathbb{N}\colon x \leqslant a_0\}$. We write $x \in (a_{i-1},a_i]$ for the formula expressing 'x belongs to the i-th interval' and $x = a_i$ for 'x is the i-th element of A'. Note that these are $\Delta_1(A)$-definable binary relations, and the shape of the formulas does not depend on A.

Thus, in a model of $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0$ there exist sequences with arbitrarily large terms whose domain is a proper cut. In fact, we can state the following simple observation, the first part of which can be seen as a converse to the previous proposition. Note that in Proposition 1.13 and in Proposition 1.14 (a) the cut I does not have to proper. On the other hand, properness is required for Proposition 1.14 (b), as witnessed by any constant sequence of length $\mathbb{N}$.

Proposition 1.14. *Let* $(M, \mathcal{X}) \vDash \mathsf{RCA}_0^*$.

(a) Every unbounded set $S \in \mathcal{X}$ *is enumerated in increasing order as* $\{s_i\}_{i \in I}$ *for some* Σ_1^0*-definable cut* I.

(b) Every sequence $(s_i)_{i \in I} \in \mathcal{X}$ *indexed by some proper* Σ_1^0*-definable cut* I *is unbounded.*

Proof.

(a) An unbounded set S is enumerated in increasing order by the $\Sigma_1(S)$-definable cut consisting of those numbers i such that S has a finite subset of cardinality i:

$$I = \{i \in \mathbb{N} \colon \exists c \left(\forall x \, (x \in_{\mathrm{Ack}} c \Rightarrow x \in S) \wedge |c| = i\right)\}. \tag{1.12}$$

For each $i \in I$, let s_i be the i-th smallest element of S. The map $i \mapsto s_i$ exists by Δ_0^0-comprehension.

(b) If the set of terms of the sequence $(s_i)_{i \in I}$ were bounded by some number b, then the cut I would be defined by the formula $\exists s \leqslant b \left(\langle i, s\rangle \in (s_i)_{i \in I}\right)$, violating Δ_0^0-induction. □

The above two propositions show that over RCA_0^* infinite subsets of natural numbers may have different sizes: one can say that there are as many cardinalities of infinite sets as there are Σ_1^0-definable cuts. From this perspective, over $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0$, the cardinality of $\mathbb{N}$ behaves like a singular cardinal number. The following proposition states that $\mathsf{I}\Sigma_1^0$ is necessary for a robust notion of an infinite set.

Proposition 1.15. *Over* RCA_0^**, the following are equivalent:*

(i) $\mathsf{I}\Sigma_1^0$,

(ii) 'For every unbounded set X *there is an increasing bijection* $f \colon \mathbb{N} \to X$*',*

(iii) 'For every unbounded set X *there is a bijection* $f \colon \mathbb{N} \to X$*',*

(iv) 'Every unbounded set has arbitrarily large finite subsets'.

Proof. The implication (i) ⇒ (ii) was proved in [48, Lemma 2.5]. The implication (ii) ⇒ (iii) is trivial. Given a set X, a bijection $f \colon \mathbb{N} \to X$ and arbitrary number $x \in \mathbb{N}$, the set $\{f(y) \colon y < x\}$ is a subset of X by Δ_0^0-comprehension and has cardinality x, thus (iii) ⇒ (iv). Finally, if $\mathsf{I}\Sigma_1^0$ fails, then there exists a Σ_1^0-definable proper cut I, and the set defined as in (1.11) does not have a finite subset of cardinality b for any number $b > I$, so (iv) ⇒ (i). □

In fact, this is a more general phenomenon: by external induction on n one can show that, over $\mathsf{I}\Delta_0 + \mathsf{exp}$, the theory $\mathsf{I}\Sigma_n$ is equivalent to the statement that every unbounded Σ_n-set has arbitrarily large finite subsets, for $n \geqslant 1$.

As we already declared in the previous section, any unbounded set is called infinite. It will be convenient to define the *cardinality of an infinite set X* to be the cut I which enumerates X in increasing order. In particular, we say that a set X *has cardinality* $\mathbb{N}$ if there exists a bijection $f\colon \mathbb{N} \to X$.

Working in a formal theory we will use the words 'infinite', 'unbounded' and 'cofinal' interchangeably, depending on which one seems to be most natural in a given context.

In a model of RCA_0^* it is often useful to consider the following $\forall\Pi^0_3$-definable set:

$$\mathrm{I}^0_1 := \{x \in \mathbb{N}\colon \text{ every unbounded set } X \text{ has a finite subset of cardinality } x\}. \quad (1.13)$$

Note that by the correspondence between infinite subsets of $\mathbb{N}$ and Σ^0_1-definable cuts, I^0_1 can equivalently be defined as the intersection of all Σ^0_1-definable cuts. Obviously, intersections preserve the property of being an initial segment closed under successor, so I^0_1 is itself a cut. We observe that it is also closed under multiplication, which is not obvious at first glance, as in all models of $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma^0_1$ there exist Σ^0_1-definable cuts that are not closed even under addition.

Proposition 1.16. RCA_0^* *proves that the cut* I^0_1 *is closed under multiplication.*

Proof. It is enough to check that if every unbounded set has a finite subset of cardinality x, then every unbounded set has a finite subset of cardinality x^2. So suppose $a^2 \notin \mathrm{I}^0_1$, and let S be an unbounded set enumerated by a Σ^0_1-cut I that does not have a finite subset of cardinality a^2. Let R be the $\Delta_1(S)$-definable set consisting of every a-th element of S, that is

$$R = \{x \in S\colon \exists i \in I\,(\text{`}x \text{ is the } i\text{-th element of } S\text{'} \wedge \text{`}a \text{ divides } i\text{'})\}.$$

If R is finite, then the set $S \setminus [0, \max(R)]$ is unbounded and does not contain a finite subset of cardinality a. If R is unbounded, then it itself does not contain a finite subset of cardinality a, because otherwise S would have a finite subset of cardinality a^2, contrary to our assumption. In any case, $a \notin \mathrm{I}^0_1$. □

By Proposition 1.15, the cut I^0_1 is proper if and only if $\mathsf{I}\Sigma^0_1$ fails. In such a case one can ask whether I^0_1 is itself Σ^0_1-definable. Both possibilities are consistent with $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma^0_1$. For instance, if ω is Σ^0_1-definable, then clearly $\mathrm{I}^0_1 = \omega$ because ω is the smallest cut. For the other possibility, note that the smallest Σ^0_1-definable cut, if it exists, must necessarily satisfy $\mathsf{I}\Sigma_1$, so the theory $\mathsf{RCA}_0^* +$ 'I^0_1 is Σ^0_1-definable' interprets $\mathsf{I}\Sigma_1$. On the other hand, it is known that $\mathsf{I}\Sigma_1$ proves $\mathrm{Con}(\mathsf{RCA}_0^*)$, so for Gödel-style reasons we must have $\mathsf{RCA}_0^* \nvdash$ 'I^0_1 is Σ^0_1-definable'.

The next lemma is a special case of a more general result about coding in models of $\mathsf{B}\Sigma^0_n + \mathsf{exp}$ [7, Proposition 4]. We can view it as a generalization of the fact that in a model of $\mathsf{B}\Sigma_1 + \mathsf{exp}$ every bounded Δ_1-definable set is coded.

Lemma 1.17 (Chong-Mourad). *Let* $(M, \mathcal{X}) \models \mathsf{RCA}_0^*$ *and let* I *be a proper* Σ_1^0*-cut in* $(M, \mathcal{X})$*. If* $X \subseteq I$ *is such that both* X *and* $I \setminus X$ *are* Σ_1^0*-definable in* $(M, \mathcal{X})$*, then* $X \in \mathrm{Cod}(M/I)$*.*

1.3 Combinatorial principles

A typical Ramsey-theoretic statement P has the form

$$\forall X \exists Y \big(\text{`}Y \text{ is infinite'} \wedge \Psi(X, Y)\big), \tag{1.14}$$

where the initial quantifiers range over subsets of natural numbers and Ψ is a property which does not depend on any infinite sets other than X and Y. Here X and Y are seen as codes of relations of a certain type on $\mathbb{N}$, such as orderings or partitions. Such a statement P is usually called a (combinatorial or Ramsey-like) *principle*. The set X is an *instance* of P and Y is a *solution* of P for X.

A statement of the form (1.14) can be formalized in second-order arithmetic as a Π_2^1 sentence in which the property Ψ is expressed by an arithmetical formula. However, we have to face one conceptual challenge when working over the weak base theory RCA_0^*. As we have seen in the previous section, in the absence of $\mathsf{I}\Sigma_1^0$ infinite sets can be 'short', i.e. they can have cardinality strictly smaller than $\mathbb{N}$. This may seem unsatisfactory from the point of view of infinite Ramsey theory, where one usually seeks to find in a given mathematical structure a possibly large substructure with some desired property. In the case of infinite Ramsey theory on $\mathbb{N}$, one may prefer solution sets of cardinality equal to that of the whole set of natural numbers rather than that of some proper cut. Therefore, we will study at least two versions of each of the principles that we will consider in this thesis. We refer to a formalization of a principle P like in (1.14) in which 'infinite' = 'unbounded' as the *normal version* of P. The other formalization, with 'infinite' = 'of cardinality $\mathbb{N}$', is called a *long version* of P. 'Normal' suggests that this formulation is closer to the common way of defining infinity for subsets of $\mathbb{N}$, whereas 'long' refers to the fact that solution sets for P are required to be enumerated by the whole set of natural numbers, rather than just by some cut 'shorter' than $\mathbb{N}$. We denote the normal versions of various principles by their standard abbreviations and use the prefix ℓ- for the long versions.

In the rest of this subsection we introduce normal versions of all the Ramsey-like principles that we will study in the next chapters. We also recall some relevant results from the RCA_0 framework. To avoid confusion, the long versions will be introduced only in Chapter 3, in which they are studied.

The usual formulation of Ramsey's theorem for n-tuples and k colours in second-order arithmetic is as follows.

RT_k^n *For every function* $c\colon [\mathbb{N}]^n \to k$ *there exists an unbounded set* $H \subseteq \mathbb{N}$ *such that* c *is constant on* $[H]^n$*.*

The function c is called a *colouring* of $[\mathbb{N}]^n$ and the set H is said to be *homogeneous* for c. A colouring c is called a k-colouring if its range is contained in $k = \{0, 1, \ldots, k-1\}$. We will study RT^n_k only for fixed $n, k \geqslant 2$, since for every fixed $n, k \in \omega$ the sentence RT^n_1 is trivial and RT^1_k is provable without any induction axioms.

Clearly, for $n' \geqslant n$ and $k' \geqslant k$, RCA_0 proves that $\mathsf{RT}^{n'}_{k'} \Rightarrow \mathsf{RT}^n_k$. One can also easily verify in RCA_0 that for a fixed $n \geqslant 2$ the strength of RT^n_k does not increase if we consider a larger but fixed number of colours. We recall the standard proof of this fact as we will refer to it in the next chapter.

Proposition 1.18. *For $n, k \geqslant 2$, RCA_0 proves that $\mathsf{RT}^n_k \Rightarrow \mathsf{RT}^n_{k+1}$.*

Proof. Let $c\colon [\mathbb{N}]^n \to k+1$. Define a colouring $d\colon [\mathbb{N}]^n \to k$ by letting $d(\overline{x}) = \min\{c(\overline{x}), k-1\}$. Use RT^n_k to get a homogeneous set H for d. If H is homogeneous for d with colour $i < k-1$ then it is also homogeneous for c. Otherwise consider the restriction of c to $[H]^n$ which is now a colouring of an infinite set with colours $\{k-1, k\}$. Let $\{h_i\colon i \in \mathbb{N}\}$ be an enumeration of H in increasing order. Apply RT^2_2 to the 2-colouring of $\mathbb{N}$ given by $\widetilde{c}(i,j) = c(h_i, h_j) - k + 1$ to obtain an infinite homogeneous set $\widetilde{H}$. Transfer it back onto H by letting $H^* = \{h_i \in H\colon i \in \widetilde{H}\}$. If $\widetilde{H}$ is homogeneous for $\widetilde{c}$ with colour $i \in \{0, 1\}$, then H^* is homogeneous for c with colour $i + k - 1$. □

It was shown by Simpson that each RT^n_k is provable in ACA_0, for a proof see [47, Lemma III.7.4]. On the other hand, Jockusch proved in [28] that there is a computable instance of RT^3_2 for which every homogeneous set computes $0'$. By formalizing his construction in RCA_0 one proves that RT^3_2 implies ACA_0. However, one cannot 'code the jump' in the above sense using a 2-colouring of pairs of natural numbers: by Seetapun and Slaman [46] RT^2_2 does not imply ACA_0 over RCA_0. Thus, there are only two nontrivial RT^n_k principles up to equivalence over RCA_0: RT^3_2 and RT^2_2.

In addition to Ramsey's theorem, we will study some of its weakenings: the chain-antichain principle CAC, the ascending-descending sequence principle ADS, cohesive Ramsey's theorem for pairs and two colours CRT^2_2, and the cohesive set principle COH. Our choice of the principles is motivated by their prominent role in reverse mathematics, as well as the fact that they form a linear order with respect to logical strength over RCA_0, see Theorem 1.20 below. To define these principles we need a few more notions.

Let $(\mathbb{N}, \preccurlyeq)$ be a partial order. A set $S \subseteq \mathbb{N}$ is a *chain* in $\preccurlyeq$ if all its elements are $\preccurlyeq$-comparable, i.e. for all $x, y \in S$ either $x \preccurlyeq y$ or $x \succcurlyeq y$. A set $S \subseteq \mathbb{N}$ is an *antichain* in $\preccurlyeq$ if all its elements are $\preccurlyeq$-incomparable, i.e. for all distinct $x, y \in S$ neither $x \preccurlyeq y$ nor $x \succcurlyeq y$.

Let $(\mathbb{N}, \preccurlyeq)$ be a linear order. A set $S \subseteq \mathbb{N}$ is an *ascending sequence* in $\preccurlyeq$ if for all $x, y \in S$ it holds that $x \leqslant y$ iff $x \preccurlyeq y$. A set $S \subseteq \mathbb{N}$ is a *descending sequence* in $\preccurlyeq$ if for all $x, y \in S$ it holds that $x \leqslant y$ iff $x \succcurlyeq y$.

A colouring $c\colon [\mathbb{N}]^2 \to 2$ is *stable* if for every $x \in \mathbb{N}$ there exists $y \in \mathbb{N}$ such that for all $z \geqslant y$ it holds that $c(x,y) = c(x,z)$. In other words, for each number $x \in \mathbb{N}$ the colour of a pair (x, y) is the same for all but finitely many y.

A colouring $c\colon [\mathbb{N}]^2 \to 2$ is called *transitive* if for every $x, y, z \in \mathbb{N}$ such that $x < y < z$, from $c(x, y) = c(y, z) = i$ it follows that $c(x, z) = i$, for $i < 2$.

Let $(R_n)_{n\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{N}$. A set $C \subseteq \mathbb{N}$ is called *cohesive* for $(R_n)_{n\in\mathbb{N}}$ if for every $n \in \mathbb{N}$ either it holds that $C \subseteq^* R_n$ or it holds that $C \subseteq^* \overline{R_n}$.

CAC *For every partial order $(\mathbb{N}, \preccurlyeq)$ there exists an unbounded set $S \subseteq \mathbb{N}$ which is either a chain or an antichain in $\preccurlyeq$.*

ADS *For every linear order $(\mathbb{N}, \preccurlyeq)$ there exists an unbounded set $S \subseteq \mathbb{N}$ which is either an ascending or a descending sequence in $\preccurlyeq$.*

CRT^2_2 *For every $c\colon [\mathbb{N}]^2 \to 2$ there exists an unbounded set $S \subseteq \mathbb{N}$ such that c is stable when restricted to $[S]^2$.*

COH *For every sequence $(R_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{N}$ there exists an unbounded set $C \subseteq \mathbb{N}$ which is cohesive for $(R_n)_{n\in\mathbb{N}}$.*

CAC is proved with one simple application of RT^2_2: suppose that $(\mathbb{N}, \preccurlyeq)$ is a partial order. Define a colouring c on $[\mathbb{N}]^2$ by letting $c(x, y) = 0$ if x and y are comparable according to $\preccurlyeq$ and $c(x, y) = 1$ otherwise. Then every set H homogeneous for $i = 0$ is a chain in $\preccurlyeq$ and every set H homogeneous for $i = 1$ is an antichain in $\preccurlyeq$.

ADS can be seen as a weakening of RT^2_2 to transitive colourings [24], since every linear ordering $(\mathbb{N}, \preccurlyeq)$ induces a transitive 2-colouring c on $[\mathbb{N}]^2$ as follows: for $x < y$ let $c(x, y) = 0$ if $x \preccurlyeq y$ and let $c(x, y) = 1$ otherwise. ADS also immediately follows from CAC: for a linear order $(\mathbb{N}, \preccurlyeq)$ define a partial order $\preccurlyeq'$ by letting $x \preccurlyeq' y$ iff $x \leqslant y$ and $x \preccurlyeq y$. Then every chain for $\preccurlyeq'$ is an ascending sequence for $\preccurlyeq$ and every antichain for $\preccurlyeq'$ is a descending sequence for $\preccurlyeq$.

We note that ADS could alternatively be formulated in terms of sequences in the strict sense (i.e. functions with domains downward closed) rather than sets:

For every linear order $(\mathbb{N}, \preccurlyeq)$ there exists an unbounded strictly increasing sequence $s_0 \prec s_1 \prec s_2 \prec \ldots$ or an unbounded strictly decreasing sequence $s_0 \succ s_1 \succ s_2 \succ \ldots$. (*)

Here there is no requirement on how the elements s_i are ordered by the natural order $\leqslant$. To emphasize the difference between our official formulation of ADS and the one in (*), we will refer to solutions of the first one as *set solutions*, and to those of the second one as *sequence solutions*.

Both formulations of ADS are equivalent over RCA_0, where the domains of unbounded sequences must be the whole $\mathbb{N}$. That is, from every set solution S one obtains a sequence solution just by taking the increasing enumeration of S. On the other hand, given a sequence solution $(s_n)_{n\in\mathbb{N}}$, by Δ^0_1-comprehension we can also obtain a set solution by taking the set of those numbers s_n that are $\leqslant$-greater than all s_m for $m < n$. We will see in Proposition 2.4 that this argument still works over RCA_0^*, where the domains of unbounded sequences may be proper cuts. Therefore, we do not give any official name to $(*)$ and stay with only one normal formulation of ADS. However, in Chapter 3 the distinction will matter for the long counterparts.

COH is implied by RT^2_2 over RCA_0, as was proved by Mileti [39] (the earlier proof from [5] requires $\mathsf{I}\Sigma^0_2$, cf. [6]). Hirschfeldt and Shore [24] improved this result and showed that RCA_0 proves $\mathsf{ADS} \Rightarrow \mathsf{COH}$. The proof makes essential use of Σ^0_1-induction in the base theory.

CRT^2_2 is clearly just a weakening of RT^2_2: a colouring c only has to be stable on a solution set rather than constant. To see that COH implies CRT^2_2 note that every colouring $c\colon [\mathbb{N}]^2 \to 2$ is stable on any set C that is cohesive for the sequence $(\{y\colon\ c(n,y)=1\})_{n\in\mathbb{N}}$.

For completeness, let us recall two other principles that appear in the well-known decompositions of RT^2_2. Stable Ramsey's theorem for pairs SRT^2_2 is the restriction of RT^2_2 to colourings that are stable on $\mathbb{N}$. The Erdös-Moser principle EM is a weakening of RT^2_2 which requires a colouring c only to be transitive on a solution set, rather than constant. Both SRT^2_2 and EM can be formulated as normal and long principles in a natural way. We skip the formal definitions as these principles will play no role in the rest of the thesis. The equivalences $\mathsf{RT}^2_2 \Leftrightarrow \mathsf{SRT}^2_2{+}\mathsf{CRT}^2_2$ and $\mathsf{RT}^2_2 \Leftrightarrow \mathsf{ADS}{+}\mathsf{EM}$ are both provable in RCA_0^*. The $(\Rightarrow)$ implications follow immediately from the definitions, while the reverse implications can be shown using a version of Lemma 2.2 below. However, we will see in Section 3.2 that RCA_0^* does not prove the equivalence $\mathsf{RT}^2_2 \Leftrightarrow \mathsf{SRT}^2_2 + \mathsf{COH}$, the first splitting of RT^2_2 that appeared in the reverse-mathematical literature in [5].

Remark 1.19. Our formulations of the combinatorial principles require solutions to colourings, orders etc. defined only on the whole $\mathbb{N}$ rather than on any unbounded subset of $\mathbb{N}$. However, over RCA_0 no generality is lost, since for every unbounded subset A there is a bijection $f\colon A \to \mathbb{N}$. Thus, as in the proof of Proposition 1.18, one can 'transfer' an instance X of a principle P from any unbounded subset A to $\mathbb{N}$, apply P to $f[X]$ to get a solution Y and then transfer it back onto A to obtain a solution $f^{-1}[Y]$ for the original X. We will see in Lemma 2.2 that the same is true over RCA_0^*, though the argument is a little bit less obvious.

In the last three theorems of this section we summarize the most important characteristics of the principles we will study in the following chapters. First we note that they form a linearly ordered hierarchy with respect to implication over RCA_0.

Theorem 1.20. *Over RCA_0, the following sequence of implications holds:*

$$\mathsf{RT}^3_2 \Rightarrow \mathsf{RT}^2_2 \Rightarrow \mathsf{CAC} \Rightarrow \mathsf{ADS} \Rightarrow \mathsf{COH} \Rightarrow \mathsf{CRT}^2_2. \tag{1.15}$$

The first four implications on the left cannot be provably reversed in RCA_0.

Proof. We have already discussed all the implications and the nonimplication from RT^2_2 to RT^3_2. Hirschfeldt and Shore [24] proved strictness of $\mathsf{RT}^2_2 \Rightarrow \mathsf{CAC}$

and $\mathsf{ADS} \Rightarrow \mathsf{COH}$. Lerman, Solomon and Towsner [37] separated CAC from ADS. □

For the last implication in (1.15), it is known from [24] that CRT^2_2 implies COH over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$, but it is still an open question whether one can prove the implication without assuming $\mathsf{B}\Sigma^0_2$.

In the next chapter, however, we will need even stronger separation properties of the principles considered.

Theorem 1.21. *All the principles* RT^3_2*,* RT^2_2*,* CAC*,* ADS*,* CRT^2_2 *do not follow from and are pairwise distinct over* WKL_0*.*

Proof. By Patey and Yokoyama [41], $\mathsf{WKL}_0 + \mathsf{RT}^2_2$ is Π^0_3-conservative over RCA_0 and thus does not imply RT^3_2, which is equivalent to ACA_0 over RCA_0.

It was proved by Towsner in [55] that $\mathsf{WKL}_0 + \mathsf{CAC}$ does not prove RT^2_2 and that $\mathsf{WKL}_0 + \mathsf{ADS}$ does not prove CAC.

The nonimplication from $\mathsf{WKL}_0 + \mathsf{CRT}^2_2$ to ADS follows from conservation results. Namely, by Theorem 1.4 and Theorem 1.22 (d) below, WKL_0 and CRT^2_2 are Π^1_1-conservative over RCA_0. From Theorem 1.6 we get that $\mathsf{WKL}_0 + \mathsf{CRT}^2_2$ is also Π^1_1-conservative over RCA_0. On the other hand, by Theorem 1.22 (a), $\mathsf{RCA}_0 + \mathsf{ADS}$ implies $\mathsf{B}\Sigma_2$.

Finally, to see that WKL_0 does not imply any of the other principles from the statement of the theorem it is enough to check that $\mathsf{WKL}_0 \nvdash \mathsf{CRT}^2_2$. Actually, one can separate these principles already on ω-models, using the fact proved by Hirschfeldt and Shore in [24] that in such models CRT^2_2 is equivalent to COH. Namely, as we mentioned in Section 1.1, there exists an ω-model of WKL_0 consisting entirely of low sets. However, no such model can satisfy COH, because there exists a computable instance of COH without a low solution, as shown by Jockusch and Stephan [27] (see also [5]). □

Let us also note that none of the principles listed in the previous theorem, with the obvious exception of RT^3_2, implies WKL_0 over RCA_0. This was shown by Liu in [38].

The last theorem relates the logical strength of our principles to induction and collection axioms.

Theorem 1.22.

(a) RT^2_2*,* CAC *and* ADS *imply* $\mathsf{B}\Sigma^0_2$ *over* RCA_0*.*

(b) CAC *and* ADS *are* Π^1_1*-conservative over* $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$*.*

(c) RT^2_2 *is* $\forall\Pi^0_4$*-conservative over* $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$*.*

(d) COH *and* CRT^2_2 *are* Π^1_1*-conservative over* RCA_0*.*

Proof. Part (a) for RT^2_2 was proved by Hirst [25], and for CAC and ADS by Hirschfeldt and Shore [24]. Part (b) was shown by Chong, Slaman and Yang [8]. Part (c) is the recent result by Le Houérou, Patey and Yokoyama [35]. Part (d) is due to Cholak, Jockusch and Slaman [5]. □

Chapter 2

Normal versions

In the present chapter we study RT^n_k for $n, k \geqslant 2$, CAC, ADS and CRT^2_2 in their normal formulations, which require solution sets to be merely unbounded. For illustrative purposes we find it useful to focus on just a few well-known principles but, as we will see in Theorems 2.12 and 2.16, our results easily generalize to a broader syntactic class of sentences.

2.1 Basic observations over RCA^*_0

In this section we verify that some well-known and useful properties of the normal versions of Ramsey-theoretic principles we consider still hold over RCA^*_0.

Firstly we note that some of the implications between our principles known to hold over RCA_0 transfer immediately to RCA^*_0. Indeed, it is very easy to check that the arguments given in Section 1.3 for the implications from RT^2_2 to CAC and CRT^2_2, and for the one from CAC to ADS, do not require any induction axioms. Also, those implications that are known to be strict over RCA_0 remain strict over the weaker base theory, cf. Theorem 1.20. We can thus state the following proposition (see Proposition 3.1 for its 'long' counterpart).

Proposition 2.1. *Over RCA^*_0, the following sequences of implications hold:*

(a) $\mathsf{RT}^{n'}_{k'} \Rightarrow \mathsf{RT}^n_k$, *for* $n' \geqslant n \geqslant 2$ *and* $k' \geqslant k \geqslant 2$,

(b) $\mathsf{RT}^2_2 \Rightarrow \mathsf{CAC} \Rightarrow \mathsf{ADS}$,

(c) $\mathsf{RT}^2_2 \Rightarrow \mathsf{CRT}^2_2$.

None of the implications in (b) and (c) can be provably reversed in RCA^*_0.

Next we show that even though over RCA^*_0 it is not always the case that there exists a bijection between an unbounded set and $\mathbb{N}$, no generality is lost by restricting our principles to instances defined only on all of $\mathbb{N}$ rather than on an arbitrary unbounded set (cf. Remark 1.19).

Lemma 2.2. *Over* RCA_0^**, each of* RT^n_k*, for* $n, k \geqslant 2$*,* CAC*,* ADS *and* CRT^2_2 *is equivalent to its generalization to orderings/colourings defined on an arbitrary unbounded subset of* $\mathbb{N}$*.*

Proof. The proofs are similar for all principles; we sketch them for ADS and CRT^2_2. Working in RCA_0^*, assume ADS and let $(A, \preccurlyeq)$ be a linear order, where A is an unbounded subset of $\mathbb{N}$. By Proposition 1.13, the set A can be enumerated in increasing order by some Σ^0_1-cut I as $A = \{a_i\}_{i \in I}$.

Define a linear order $\widetilde{\preccurlyeq}$ on $\mathbb{N}$ by

$$x \mathrel{\widetilde{\preccurlyeq}} y \Leftrightarrow \exists i, j \in I \big(x \in (a_{i-1}, a_i] \wedge y \in (a_{j-1}, a_j] \wedge \\ \big((i \neq j \wedge a_i \preccurlyeq a_j) \vee (i = j \wedge x \leqslant y)\big)\big).$$

That is, elements are $\widetilde{\preccurlyeq}$-ordered according to the $\preccurlyeq$-ordering between the nearest elements of A $\preccurlyeq$-above them, if that makes sense, and according to the usual natural number ordering otherwise. Since $\widetilde{\preccurlyeq}$ is $\Delta_1(A, \preccurlyeq)$-definable, it exists as a set. By ADS, there is a set $\widetilde{S} \subseteq_{\mathrm{cf}} \mathbb{N}$ which is either an ascending or a descending sequence in $\widetilde{\preccurlyeq}$.

Now we can use $\Delta_1(\widetilde{S}, A)$-comprehension to define a solution set $S \subseteq A$ for $\preccurlyeq$ by taking those elements a_i of A for which the set $(a_{i-1}, a_i] \cap \widetilde{S}$ is nonempty:

$$S = \{a \in A \colon \exists x \leqslant a \ (x \in \widetilde{S} \wedge [x, a) \cap A = \emptyset)\}.$$

It is easy to check that S is unbounded and it is either an ascending or a descending sequence in $\preccurlyeq$.

The argument for CRT^2_2 is very similar. Given a colouring $c \colon [A]^2 \to 2$ we use Δ^0_1-comprehension to define $\widetilde{c} \colon [\mathbb{N}]^2 \to 2$ by:

$$\widetilde{c}(x, y) = \begin{cases} c(a_i, a_j) & \text{if } \exists i, j \in I \ (i \neq j \wedge x \in (a_{i-1}, a_i] \wedge y \in (a_{j-1}, a_j])\,, \\ 0 & \text{otherwise.} \end{cases}$$

If $\widetilde{S} \subseteq_{\mathrm{cf}} \mathbb{N}$ is a set on which $\widetilde{c}$ is stable, then by $\Delta_1(\widetilde{S}, A)$-comprehension we define in the same way as above a set $S \subseteq_{\mathrm{cf}} A$ on which c is stable. □

Now we can verify that also over RCA_0^* the strength of Ramsey's Theorem for n-tuples does not depend on the number of colours as long as it is fixed (recall that in Section 1.3 to prove this fact over RCA_0 we assumed that every unbounded set can be enumerated by $\mathbb{N}$).

Lemma 2.3. *For each* $n, k \geqslant 2$*,* $\mathsf{RCA}_0^* \vdash (\mathsf{RT}^n_k \Leftrightarrow \mathsf{RT}^n_{k+1})$*.*

Proof. Given a colouring $c \colon [\mathbb{N}]^n \to k + 1$ we repeat the argument for Proposition 1.18, making the following change. The homogeneous set H might now be of smaller cardinality than that of $\mathbb{N}$ but by the previous proposition we may freely apply RT^n_2 to the 2-colouring $c \restriction [H]^n$. □

Thus from now on we will speak only about RT^n_2, for $n \geqslant 2$.

Recall our discussion in Section 1.3 about two different possible formulations of ADS. We check that for normal versions they are equivalent also over RCA_0^*.

Proposition 2.4. *Let $(s_i)_{i\in I}$ be a sequence, indexed by some cut I, that is strictly increasing (or strictly decreasing) with respect to a linear order $(\mathbb{N}, \preccurlyeq)$. Then, provably in RCA_0^*, there is an unbounded set S such that for all $x, y \in S$ it holds that $x \leqslant y$ iff $x \preccurlyeq y$ (or for all $x, y \in S$ it holds that $x \leqslant y$ iff $x \succcurlyeq y$).*

Proof. Let $(s_i)_{i\in I}$ be a strictly increasing sequence in $(\mathbb{N}, \preccurlyeq)$ indexed by some cut I (the case of a decreasing sequence is treated the same way). Note that the collection of terms of the sequence $(s_i)_{i\in I}$ is unbounded in the sense of $\leqslant$ – this follows from Proposition 1.14 (b) in the case $I \neq \mathbb{N}$, and from the finite pigeonhole principle (provable in $\mathsf{I}\Delta_0 + \mathsf{exp}$) in the case $I = \mathbb{N}$. So, for any $i \in I$, we can find a term s_j satisfying $s_k < s_j$ for every $k \leqslant i$ (otherwise the set $\{s_k \colon k \leqslant i\}$ would be unbounded, violating Σ_1^0-collection). Thus, the following set is unbounded:

$$S = \left\{x \in \mathbb{N} \colon \exists i \in I\big(x = s_i \wedge \forall j{<}i\,(s_j < s_i)\big)\right\}.$$

Clearly, this is a Σ_1^0 definition. Intuitively, one goes through the enumeration of the sequence $(s_i)_{i\in I}$ and puts a term s_i into S if it is $\leqslant$-greater than all the terms that were listed into the sequence so far.

Note that the complement of S is also Σ_1^0-definable:

$$\overline{S} = \left\{x \in \mathbb{N} \colon \exists i \in I\big(x < s_i \wedge \forall j{<}i\,(x \neq s_j)\big)\right\}.$$

Thus, by Δ_1^0-comprehension, S exists as a set and for all $x, y \in S$ it holds that $x \leqslant y$ iff $x \preccurlyeq y$. □

It will later be useful to view ADS as a restriction of RT_2^2 to transitive colourings. We verify that the argument of [24] goes through in RCA_0^*.

Lemma 2.5 (Hirschfeldt-Shore [24])**.** *Over RCA_0^*, ADS is equivalent to RT_2^2 restricted to transitive 2-colourings.*

Proof. The implication from RT_2^2 for transitive 2-colourings to ADS is immediate and does not require any induction axioms (cf. Section 1.3). The other direction is [24, Theorem 5.3], which requires a comment. Given a transitive colouring $c \colon [\mathbb{N}]^2 \to 2$, we build a linear order $\preccurlyeq$ by inserting numbers $0, 1, \ldots$ into it one-by-one. When $\preccurlyeq$ is already defined on $\{0, \ldots, n-1\}$, we insert n into the order directly above the $\preccurlyeq$-largest $k < n$ such that $c(k, n) = 0$; if there is no such k, we place n at the bottom of $\preccurlyeq$. Then we claim that any ascending or descending sequence (given as a set) in $\preccurlyeq$ is a homogeneous set for c. The only thing to check is that the ordering $\preccurlyeq$ agrees with c in the sense that for any $i < j$, we have $i \prec j$ iff $c(i, j) = 0$. One proceeds by induction on n for the formula $\forall i < j \leqslant n\,\big(i \prec j \Leftrightarrow c(i, j) = 0\big)$. In [24], $\mathsf{I}\Sigma_1^0$ is invoked for this purpose, but it will be clear from the above description that the induction formula is actually bounded. The induction step uses the transitivity of c. □

Remark 2.6. The principle CAC can also be seen as a restriction of RT_2^2. A colouring $c \colon [\mathbb{N}]^2 \to k$ is *semitransitive* if for all colours $i < k$ except at most one, for every $x, y, z \in \mathbb{N}$ such that $x < y < z$, from $c(x, y) = c(y, z) = i$ it follows that $c(x, z) = i$. Hirschfeldt and Shore [24, Theorem 5.2] showed that over RCA_0, CAC is equivalent to RT^2_2 restricted to semitransitive 2-colourings. In [15] we verify that this equivalence still holds over RCA^*_0.

2.2 Between a model and its cuts

The main result of this section is Theorem 2.9, which says that in a model of $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$ each of the principles RT^n_k, CAC, ADS and CRT^2_2 is equivalent to its own relativization to any Σ^0_1-definable proper cut of the model. This rather technical result will be used repeatedly in the following sections to characterize first-order consequences of our principles. The gist of the proof of Theorem 2.9 is a certain syntactical property of many Ramsey-like statements that we now identify.

Definition 2.7. The $\mathcal{L}_{\mathrm{II}}$-sentence χ belongs to the class of sentences pSO if there exists a sentence γ of second-order logic in a language $(\leqslant, R_1, \ldots, R_k)$, where $k \in \omega$ and each R_i is a relation symbol of arity $m_i \in \omega$, such that χ expresses:

> For any relations $R_1, \ldots, R_k$ on $\mathbb{N}$ and for each $D \subseteq_{\mathrm{cf}} \mathbb{N}$,
> there exists $H \subseteq_{\mathrm{cf}} D$ such that $(H, \leqslant, R_1, \ldots, R_k) \vDash \gamma$.

We slightly abuse notation in this definition by writing $(H, \leqslant, R_1, \ldots, R_k)$ instead of the more cumbersome $(H, \leqslant \cap H^2, R_1 \cap H^{m_1}, \ldots, R_k \cap H^{m_k})$. Let us stress that pSO sentences are genuine sentences in the language of second-order arithmetic, thus both the compactness theorem and the Löwenheim-Skolem theorem apply to them. The fact that $(H, \leqslant, R_1, \ldots, R_k)$ satisfies a sentence of second-order logic γ is expressed by relativizing each first-order quantifier in γ to H and restricting each m-ary second-order quantifier to m-ary relations on H. The latter, when interpreted in a model of arithmetic $(M, \mathcal{X})$, are understood as elements of $\mathcal{X} \cap \mathcal{P}(H^m)$.

The abbreviation pSO stands for 'pseudo–second-order': pSO sentences appear to use both first- and second-order quantification of $\mathcal{L}_{\mathrm{II}}$, but they are relativized to arbitrarily small unbounded subsets of $\mathbb{N}$ in such a way that in cases where $\mathsf{I}\Sigma^0_1$ fails their behaviour is closer to that of arithmetical statements; cf. Corollary 2.10 below.

Lemma 2.8. *Let* P *be one of the principles* RT^n_2*, for* $n \geqslant 2$, CAC*,* ADS *or* CRT^2_2*. Then there exists a* pSO *sentence* χ *which is provably in* RCA^*_0 *equivalent to* P*, both in the entire universe and on any proper* Σ^0_1*-cut.*

Proof. The proofs are similar for all the above principles P and rely on Lemma 2.2. We give a somewhat detailed argument for ADS and restrict ourselves to stating the appropriate χ for the other principles.

Let γ be the sentence

either R is not a linear order
or for every x, y it holds that $R(x, y)$ iff $x \leqslant y$
or for every x, y it holds that $R(x, y)$ iff $x \geqslant y$,

and let χ say that for every binary relation R and every unbounded set D, there is $H \subseteq_{\mathrm{cf}} D$ such that $(H, \leqslant, R)$ satisfies γ. We claim that ADS is equivalent to χ provably in RCA_0^*. Clearly, if $\preccurlyeq$ is a linear order on $\mathbb{N}$, then χ applied with $D = \mathbb{N}$ and $R = \preccurlyeq$ implies the existence of a set H witnessing ADS for $\preccurlyeq$. Thus, χ implies ADS. In the other direction, given a relation R and an unbounded set D, either R is a linear order on D or not. In the latter case, $H = D$ witnesses χ. In the former, Lemma 2.2 lets us apply ADS to obtain either an ascending or a descending sequence in $R \cap D^2$, which witnesses χ. Thus, ADS implies χ.

The above argument also works in a structure of the form $(I, \mathrm{Cod}(M/I))$ for I a proper Σ_1^0-cut I in a model of RCA_0^*. To verify this, one has to check that an analogue of Lemma 2.2 holds in $(I, \mathrm{Cod}(M/I))$, which is unproblematic.

For CAC, the corresponding pSO sentence χ says that for every binary relation R and every unbounded set D there exists an unbounded $H \subseteq_{\mathrm{cf}} D$ such that $(H, \leqslant, R) \vDash \gamma$, where γ states that if R is a partial order, then it is a chain or antichain. For RT_k^n, the sentence γ states that if $R_1, \ldots, R_k$ form a colouring of unordered n-tuples, i.e. they are disjoint n-ary relations whose union is the set of all n-tuples that are strictly increasing with respect to $\leqslant$, then all but one of the relations R_j are in fact empty. For CRT_2^2, the appropriate γ says that the binary relation R is a stable colouring when restricted to the set of unordered pairs. □

Theorem 2.9. *If χ is a* pSO *sentence, then for every $(M, \mathcal{X}) \vDash \mathsf{RCA}_0^*$ and every proper Σ_1^0-cut I in $(M, \mathcal{X})$, it holds that $(M, \mathcal{X}) \vDash \chi$ if and only if $(I, \mathrm{Cod}(M/I)) \vDash \chi$.*

Proof. Let γ be a second-order sentence and assume for notational simplicity that it contains only one unary relation symbol R in addition to $\leqslant$, and that all second-order quantifiers are unary. Let χ be a pSO sentence stating that for every set R and every unbounded set D there exists an unbounded subset $H \subseteq_{\mathrm{cf}} D$ such that $(H, \leqslant, R) \vDash \gamma$. Let $(M, \mathcal{X}) \vDash \mathsf{RCA}_0^* + \neg \mathsf{I}\Sigma_1^0$, and let $A \in \mathcal{X}$ be a unbounded subset of M enumerated as $A = \{a_i\}_{i \in I}$, as in Proposition 1.13.

Suppose first that $(M, \mathcal{X}) \vDash \chi$. Let $R, D \in \mathrm{Cod}(M/I)$ be such that $D \subseteq_{\mathrm{cf}} I$. Define $\widetilde{R}, \widetilde{D} \subseteq M$ by:

$$x \in \widetilde{R} \Leftrightarrow \exists i \in I\, (x = a_i \wedge i \in R),$$
$$x \in \widetilde{D} \Leftrightarrow \exists i \in I\, (x = a_i \wedge i \in D).$$

Since both $\widetilde{R}$ and $M \setminus \widetilde{R}$ are Σ_1-definable in A and (the code for) R, we know that $\widetilde{R} \in \mathcal{X}$. Similarly, $\widetilde{D} \in \mathcal{X}$. Notice that $\widetilde{D} \subseteq_{\mathrm{cf}} M$, because we have $D \subseteq_{\mathrm{cf}} I$ and $A \subseteq_{\mathrm{cf}} M$.

By our assumption that $(M, \mathcal{X}) \vDash \chi$, there exists $\widetilde{H} \in \mathcal{X}$ such that $\widetilde{H} \subseteq_{\mathrm{cf}} \widetilde{D}$ and $(\widetilde{H}, \leqslant, \widetilde{R}) \vDash \gamma$. Let $H = \{i \in I \colon\ a_i \in \widetilde{H}\}$. Notice that both H and $I \setminus H$ are Σ_1-definable in $\widetilde{H}$ and A, so $H \in \mathrm{Cod}(M/I)$ by Lemma 1.17. Moreover, $H \subseteq_{\mathrm{cf}} D$.

To prove that the structure $(H, \leqslant, R)$ satisfies γ, we show that the assignment $\widetilde{H} \ni a_i \mapsto i \in H$ induces an isomorphism between the structures $(\widetilde{H}, \leqslant, \widetilde{R}; \mathcal{X} \cap \mathcal{P}(\widetilde{H}))$ and $(H, \leqslant, R; \mathrm{Cod}(M/I) \cap \mathcal{P}(H))$. The fact that this map is an isomorphism between $(\widetilde{H}, \leqslant, \widetilde{R})$ and $(H, \leqslant, R)$ follows directly from the definitions. Thus, we only need to argue that this map also induces an isomorphism of the second-order structures $\mathcal{X} \cap \mathcal{P}(\widetilde{H})$ and $\mathrm{Cod}(M/I) \cap \mathcal{P}(H)$. If $\widetilde{X} \in \mathcal{X}$ is a subset of $\widetilde{H}$, then $\{i \in I \colon\ a_i \in \widetilde{X}\}$ is in $\mathrm{Cod}(M/I)$ by Lemma 1.17. If $\widetilde{X}, \widetilde{Y} \in \mathcal{X}$ are distinct subsets of $\widetilde{H}$, then $\{i \in I \colon\ a_i \in \widetilde{X}\}$ and $\{i \in I \colon\ a_i \in \widetilde{Y}\}$ are clearly distinct. Finally, if $X \in \mathrm{Cod}(M/I)$ is a subset of H, then $\widetilde{X} = \{a_i \colon\ i \in X\}$ is in $\mathcal{X}$ by Δ^0_1-comprehension, and it is a subset of $\widetilde{H}$.

Now suppose that $(I, \mathrm{Cod}(M/I)) \vDash \chi$. Let $R, D \in \mathcal{X}$ be such that $D \subseteq_{\mathrm{cf}} M$. By replacing D with an appropriate unbounded subset if necessary, we may assume w.l.o.g. that $D \cap (a_{i-1}, a_i]$ has at most one element for each $i \in I$. We now transfer R, D to $\widetilde{R}, \widetilde{D} \subseteq I$ defined as follows:

$$i \in \widetilde{R} \Leftrightarrow i \in I \ \wedge\ \exists x \in (a_{i-1}, a_i] \cap R,$$
$$i \in \widetilde{D} \Leftrightarrow i \in I \ \wedge\ \exists x \in (a_{i-1}, a_i] \cap D.$$

By Lemma 1.17, $\widetilde{R}, \widetilde{D} \in \mathrm{Cod}(M/I)$. Notice that $\widetilde{D} \subseteq_{\mathrm{cf}} I$, given that $D \subseteq_{\mathrm{cf}} M$.

Since $(I, \mathrm{Cod}(M/I)) \vDash \chi$, there exists $\widetilde{H} \subseteq_{\mathrm{cf}} \widetilde{D}$ such that $(\widetilde{H}, \leqslant, \widetilde{R}) \vDash \gamma$. Define

$$H = \{x \in D \colon\ \exists i \in \widetilde{H}\, (x \in (a_{i-1}, a_i])\}.$$

Clearly $H \in \mathcal{X}$ and $H \subseteq_{\mathrm{cf}} D$. To show that $(H, \leqslant, R) \vDash \gamma$, it remains to prove that the structures $(\widetilde{H}, \leqslant, \widetilde{R}; \mathrm{Cod}(M/I) \cap \mathcal{P}(\widetilde{H}))$ and $(H, \leqslant, R; \mathcal{X} \cap \mathcal{P}(H))$ are isomorphic. The isomorphism is induced by the map that takes $i \in \widetilde{H}$ to the unique element of $H \cap (a_{i-1}, a_i]$. The verification that this is indeed an isomorphism is similar to the one in the proof of the other direction. □

Note that the equivalence in Theorem 2.9 does not depend on the choice of the cut I. Moreover, once I is fixed, the equivalence does not depend on the second-order universe $\mathcal{X}$, as long as I is Σ^0_1-definable in $\mathcal{X}$. If I is $\Sigma_1(A)$-definable for $A \in \mathcal{X}$, then I has the same Σ^0_1 definition in $(M, \Delta_1\text{-}\mathrm{Def}(M, A))$. Thus we can state the following:

Corollary 2.10. *Let χ be a* pSO *sentence, $(M, \mathcal{X}) \vDash \mathsf{RCA}^*_0$ and $A \in \mathcal{X}$ such that $(M, \mathcal{X}) \vDash \neg\mathsf{I}\Sigma_1(A)$. Then $(M, \mathcal{X}) \vDash \chi$ if and only if $(M, \Delta_1\text{-}\mathrm{Def}(M, A)) \vDash \chi$. In particular, if $(M, \mathcal{X}) \vDash \neg\mathsf{I}\Sigma_1$ then $(M, \mathcal{X}) \vDash \chi$ if and only if $(M, \Delta_1\text{-}\mathrm{Def}(M)) \vDash \chi$.*

One can informally say that in the absence of $\mathsf{I}\Sigma^0_1$ Ramsey-like statements become first-order properties (cf. Definition 2.11 and equivalence (2.1) below).

The special case of the above corollary suggests that it can even happen that they are satisfied by models of the form $(M, \Delta_1\text{-Def}(M))$. Indeed, we will see in the next section that such structures exist and thus make our Ramsey-theoretic statements 'computably true'.

2.3 Arithmetical consequences of Ramsey-like principles

Theorem 2.9 and Corollary 2.10 make it possible to prove a very simple criterion for Π^1_1-conservativity over RCA^*_0 for pSO sentences. Before we state the result, let us introduce the following definition of what we will call Δ_ℓ *relativizations* to simplify notation in this and the next section.

Definition 2.11. Given an $\mathcal{L}_{\mathrm{II}}$-sentence σ, let Δ_ℓ-σ be the $\mathcal{L}_{\mathrm{I}}$-sentence obtained from σ by replacing all the second-order quantifiers with first-order quantifiers ranging over Δ_ℓ-definable sets. We write $\Delta_\ell(A)$-σ if the introduced first-order quantifiers range over sets that are Δ_ℓ-definable with a set parameter A.

By Proposition 1.2 (a) and a straightforward induction on the complexity of σ one can show that for a model $M \vDash \mathsf{B}\Sigma_\ell + \mathsf{exp}$ it holds that

$$M \vDash \Delta_\ell\text{-}\sigma \quad \textit{if and only if} \quad (M, \Delta_\ell\text{-Def}(M)) \vDash \mathsf{RCA}^*_0 + \sigma. \tag{2.1}$$

We are especially interested in Δ_ℓ relativizations of Π^1_2 combinatorial principles. If P is such a principle, then Δ_ℓ-P says that for every Δ_ℓ-definable instance of P there exists a Δ_ℓ-definable solution. It was first observed by Specker [50] that Δ_1-RT^2_2 is false in the standard model, that is, there exists a computable instance of RT^2_2 without a computable solution. The same holds for all other Ramsey-like Π^1_2 statements that we study here. For the weakest of them, that is CRT^2_2, recall that in ω-models it is equivalent to COH. By [27] there exists a computable instance of COH without a low solution, and a fortiori without a computable one.

The following theorem reduces Π^1_1-conservativity of pSO sentences over RCA^*_0, which at first glance looks like a Π_2 property ('for every proof from $\mathsf{RCA}^*_0 + \psi$ there exists a proof from RCA^*_0'), to provability from WKL^*_0 or $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$, which is a Σ_1 property.

Theorem 2.12. *Let ψ be a* pSO *sentence. Then the following are equivalent:*

(i) $\mathsf{RCA}^*_0 + \psi$ *is Π^1_1-conservative over* RCA^*_0*,*

(ii) $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1 \vdash \psi$*,*

(iii) $\mathsf{WKL}^*_0 \vdash \psi$*.*

Moreover, if $\mathsf{WKL}_0 \nvdash \psi$*, then* $\mathsf{RCA}^*_0 + \psi$ *is not arithmetically conservative over* RCA^*_0*.*

Proof. The implication (iii) $\Rightarrow$ (i) is immediate from Theorem 1.5.

Assume that (i) holds. Note that by Corollary 2.10, $\mathsf{RCA}_0^* + \psi$ proves the Π^1_1 statement $\forall X\,(\neg\mathsf{I}\Sigma_1(X) \Rightarrow \Delta_1(X)\text{-}\psi)$. Thus, by (i), this statement is provable in RCA_0^*. However, again by Corollary 2.10, in each model of $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma^0_1$ this Π^1_1 statement is equivalent to ψ. This proves that (i) implies (ii).

Now assume that (iii) fails, and let $(M, \mathcal{X})$ be a countable model of $\mathsf{WKL}_0^* + \neg\psi$. If $(M, \mathcal{X}) \models \neg\mathsf{I}\Sigma^0_1$, then clearly $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma^0_1 \nvdash \psi$. Otherwise, $(M, \mathcal{X})$ is a model of WKL_0, so by Theorem 1.12 there exists a structure $(K, \Delta_1\text{-Def}(K)) \models \mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma^0_1$ in which M is a proper Σ_1-cut and $\mathrm{Cod}(K/M) = \mathcal{X}$. By Theorem 2.9, we get $(K, \Delta_1\text{-Def}(K)) \models \neg\psi$. This proves that (ii) implies (iii).

For the 'moreover' part, note that if we do have a countable model $(M, \mathcal{X})$ of $\mathsf{WKL}_0 + \neg\psi$, then the structure $(K, \Delta_1\text{-Def}(K))$ constructed as in the previous paragraph satisfies RCA_0^* but does not satisfy the first-order statement $\neg\mathsf{I}\Sigma_1 \Rightarrow \Delta_1\text{-}\psi$, which is provable from $\mathsf{RCA}_0^* + \psi$ by Corollary 2.10. This proves that if $\mathsf{WKL}_0 \nvdash \psi$, then $\mathsf{RCA}_0^* + \psi$ is not arithmetically conservative over RCA_0^*. $\square$

It follows that all the principles $\mathsf{RT}^3_2, \mathsf{RT}^2_2, \mathsf{CAC}, \mathsf{ADS}, \mathsf{CRT}^2_2$ are arithmetically nonconservative over RCA_0^*, because none of them is implied by WKL_0. Moreover, since they are known to be pairwise distinct over WKL_0 (Theorem 1.21), they differ in strength over $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1$ and can be distinguished by their first-order consequences over RCA_0^*.

Corollary 2.13. *Let* P *be one of the principles* RT^3_2, RT^2_2, CAC, ADS, CRT^2_2, *and let* Q *be a principle to the right of* P *in this sequence or the constant* $\top$. *Then:*

(a) Q *does not imply* P *over* $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma^0_1$,

(b) $\neg\mathsf{I}\Sigma_1 \Rightarrow \Delta_1\text{-}\mathsf{P}$ *is an arithmetical sentence provable in* $\mathsf{RCA}_0^* + \mathsf{P}$ *but not in* $\mathsf{RCA}_0^* + \mathsf{Q}$.

In particular, $\mathsf{RCA}_0^* + \mathsf{P}$ *is not arithmetically conservative over* RCA_0^*.

Proof. Let P and Q be as above. By Lemma 2.8 we can assume that both P and Q are pSO sentences. By Theorem 1.21 there exists a countable model $(M, \mathcal{X}) \models \mathsf{WKL}_0 + (\mathsf{Q} \wedge \neg\mathsf{P})$. Take an end-extension $K \supseteq_e M$ as in the proof of Theorem 2.12. Then by Corollary 2.10 the structure $(K, \Delta_1\text{-Def}(K))$ is a model of $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1$ satisfying $\mathsf{Q} \wedge \neg\mathsf{P}$. Thus we have (a).

For (b) note that by Corollary 2.10, in any model of $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1$ with the second-order universe consisting exactly of the Δ_1-definable sets, P is equivalent to Δ_1-P, so the structure $(K, \Delta_1\text{-Def}(K))$ from the previous paragraph does not satisfy Δ_1-P and hence $\mathsf{RCA}_0^* + \mathsf{Q} \nvdash \neg\mathsf{I}\Sigma_1 \Rightarrow \Delta_1\text{-}\mathsf{P}$. On the other hand, it is immediate from Corollary 2.10 that $\mathsf{RCA}_0^* + \mathsf{P} \vdash \neg\mathsf{I}\Sigma_1 \Rightarrow \Delta_1\text{-}\mathsf{P}$. $\square$

Clearly, for all $n > 3$, $\mathsf{RCA}_0^* + \mathsf{RT}^n_2$ proves the sentence $\neg\mathsf{I}\Sigma_1 \Rightarrow \Delta_1\text{-}\mathsf{RT}^n_2$, so RT^n_2 is also arithmetically nonconservative over RCA_0^*. However, Theorem 2.12 does not allow us to separate RT^{n+1}_2 from RT^n_2 for $n \geqslant 3$, because they are equivalent over RCA_0. We do not know whether this equivalence still holds over RCA_0^*.

Question 2.14. Does RT^3_2 imply RT^4_2 over RCA^*_0? More generally, does RT^n_2 imply RT^{n+1}_2 over RCA^*_0 for $n \geqslant 3$?

The second question about provability of implications between our principles in RCA^*_0 and $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$ that is not answered by Proposition 2.1 and Corollary 2.13 is this:

Question 2.15. Does $\mathsf{RCA}^*_0 + \mathsf{ADS}$ or $\mathsf{RCA}^*_0 + \mathsf{CAC}$ prove CRT^2_2?

We note that over RCA_0, the principles CAC and ADS not only imply CRT^2_2 but are also arithmetically nonconservative over the base theory (they imply $\mathsf{B}\Sigma_2$), whereas $\mathsf{RCA}_0 + \mathsf{CRT}^2_2$ is Π^1_1-conservative over RCA_0.

Although each of our principles proves more arithmetical sentences than the base theory, the next theorem shows that all of them are logically weak. In particular, they do not prove totality of any computable function that is not already total in RCA^*_0.

Theorem 2.16. *Let χ be a* pSO *sentence such that there exists an ω-model of the theory $\mathsf{WKL}_0 + \chi$. Then $\mathsf{WKL}^*_0 + \chi$ is $\forall\Pi^0_3$-conservative over RCA^*_0.*

Proof. Let χ be a pSO sentence and $(\omega, \mathcal{X}) \vDash \mathsf{WKL}_0 + \chi$. By the downward Löwenheim-Skolem theorem we can assume that $\mathcal{X}$ is countable. Let $\varphi :=$ $\exists X\, \exists x\, \forall y\, \exists z\, \theta(X, x, y, z)$ be a $\exists\Sigma^0_3$ sentence consistent with RCA^*_0. We construct a model satisfying the theory $\mathsf{WKL}^*_0 + \chi + \varphi$. By Theorem 1.10 there exists a nonstandard model $(K, \mathcal{Y}) \vDash \mathsf{RCA}^*_0 + \varphi$ with $\mathrm{SSy}(K) = \mathcal{X}$. Take $A \in \mathcal{Y}$ and $a \in K$ such that $(K, \mathcal{Y}) \vDash \forall y\, \exists z\, \theta(A, a, y, z)$. Consider the following function:

$$f_\theta(y) = \min\{z > 2^y :\ \forall y' \leqslant y\ \exists z' \leqslant z\ \theta(A, a, y', z')\},$$

which is $\Delta_0(A, a)$-definable and by $\mathsf{B}\Sigma^0_1+\mathsf{exp}$ total in K. Pick some nonstandard $c \geqslant a$ and let M be the cut given by the first ω iterations of the function f on c, that is, $M := \sup_K(\{f^{(n)}(c) : n \in \omega\})$. The construction of M guarantees that ω is a $\Sigma_1(M \cap A)$-definable proper cut in M: a number $x \in M$ belongs to ω if and only if there exist the first x iterations of the function f on c, where f is $\Delta_0(M \cap A, a)$-definable.

Now we consider two cases. If $M = K$, then by Theorem 1.5 we can find an ω-extension $(K, \mathcal{Y}')$ satisfying WKL^*_0. Clearly, $(K, \mathcal{Y}')$ satisfies φ, because Σ^1_1 sentences are preserved in ω-extensions. Now we can apply Theorem 2.9 to learn that $(K, \mathcal{Y}') \vDash \chi$, because ω is Σ^0_1-definable and $\mathrm{Cod}(K/\omega) = \mathrm{SSy}(K) \vDash \chi$.

Otherwise, M is a proper exponential cut in K with $M \cap A \in \mathrm{Cod}(K/M)$ so, by Theorem 1.9, the structure $(M, \mathrm{Cod}(K/M))$ satisfies WKL^*_0. By the definition of f we have that $M \vDash \forall y\, \exists z\, \theta(M \cap A, a, y, z)$, and so $(M, \mathrm{Cod}(K/M)) \vDash \varphi$. Finally, by Proposition 1.8, $\mathrm{SSy}(K) = \mathrm{SSy}(M) = \mathrm{Cod}(M/\omega) = \mathcal{X}$, so by Theorem 2.9, $(M, \mathrm{Cod}(K/M)) \vDash \chi$. □

The last theorem of this section summarizes what we have learnt about arithmetical consequences of our principles.

Theorem 2.17. *The principles* RT^n_2 *for* $n \geqslant 2$, CAC, ADS, *and* CRT^2_2 *are all* $\forall\Pi^0_3$*-conservative over* RCA^*_0. *Each of* RT^n_2 *for* $n \geqslant 2$, CAC *and* ADS *is not* Π_4*-conservative and* CRT^2_2 *is not* Π_5*-conservative over* RCA^*_0.

Proof. The conservation result is immediate from Lemma 2.8 and Theorem 2.16, and the fact that all the above principles are true in $(\omega, \mathcal{P}(\omega))$.

For the nonconservativity, note that a special case of Corollary 2.13 is the fact that RCA^*_0 does not prove $\neg\mathsf{I}\Sigma_1 \Rightarrow \Delta_1\text{-}\mathsf{P}$, for any P from among our principles. If P is RT^n_2 for $n \geqslant 2$, CAC or ADS, then the sentence $\neg\mathsf{I}\Sigma_1 \Rightarrow \Delta_1\text{-}\mathsf{P}$ has complexity Π_4, and for CRT^2_2 it is Π_5. □

Thus, we have tight bounds on the amount of conservativity of RT^2_2, CAC, and ADS over RCA^*_0. The following question remains open:

Question 2.18. Is $\mathsf{RCA}^*_0 + \mathsf{CRT}^2_2$ $\forall\Pi^0_4$-conservative over RCA^*_0?

2.4 Ramsey for triples and beyond

In this section we extend the results of the previous one in the case of RT^n_2, where $n \geqslant 3$. Namely, we give a full axiomatization of its first-order consequences over RCA^*_0. It turns out that these have a very close connection to the amount of induction needed to prove classical computational bounds on solutions to RT^n_2.

2.4.1 Provability of computational bounds on RT^n_2

We prove two technical lemmas that will be crucial in characterizing the first-order consequences of RT^n_2. They establish the amount of mathematical induction needed to prove classical computability-theoretic results about complexity of solution sets for RT^n_2. Those results were obtained by Jockusch [28], who gives two kinds of lower bounds on solutions to RT^n_2. On the one hand, some computable instances of RT^n_2 do not have simple solutions in the sense of the arithmetical hierarchy. On the other hand, for $n \geqslant 3$ there are computable instances of RT^n_2 all of whose solutions are hard in that they compute $0'$.

Lemma 2.19. *For every* $n \geqslant 2$ *and* $\ell \geqslant 1$, *it is provable in* $\mathsf{I}\Sigma_\ell$ *that there exists a* Δ_p*-definable* 2*-colouring of* $[\mathbb{N}]^n$ *with no* Σ_ℓ*-definable infinite homogeneous set, where* $p = \max\{\ell - n + 1, 1\}$. *In particular,* $\mathsf{I}\Sigma_\ell \vdash \neg\Delta_\ell\text{-}\mathsf{RT}^n_2$.

Proof. Firstly we give a proof for $\ell \geqslant 2$, by formalizing in $\mathsf{I}\Sigma_\ell$ the usual argument due to Jockusch [28]. The case of $\ell = 1$ is proved similarly, but we need to weaken the statement of Jockusch's original theorem so that the construction can be carried out in $\mathsf{I}\Sigma_1$ only. Let us note that in each case it is enough to show that there is no Δ_ℓ-definable infinite homogenous set. This is because for $\ell \geqslant 1$, $\mathsf{I}\Sigma_\ell$ proves that every unbounded Σ_ℓ-set has an unbounded Δ_ℓ-subset [18, Theorem I.3.22]. We will also use a result by Švejdar [53], who showed that for each $\ell \geqslant 1$, $\mathsf{B}\Sigma_\ell$ proves the limit lemma relativized to $0^{(\ell-1)}$. As observed in [18, Theorem I.3.2], Švejdar's proof works in $\mathsf{I}\Sigma_1$ also for the uniform limit lemma: there is a total Δ_1-function f of three arguments such that for each total Δ_2-function h of one argument we have $\forall x\big(h(x) = \lim_s f(e, x, s)\big)$, where e is an index of h. We will think about f as a uniform sequence of functions $(h_e(x, s))_{e\in\mathbb{N}}$. By relativization to $0^{(\ell-1)}$, one can prove the uniform limit lemma for $\Delta_{\ell+1}$-functions in $\mathsf{I}\Sigma_\ell$, where $\ell \geqslant 2$.

So, let $\ell \geqslant 2$. We firstly show that $\mathsf{I}\Sigma_\ell$ proves that there is a $\Delta_{\ell-1}$-definable colouring $c\colon [\mathbb{N}]^2 \to 2$ with no Δ_ℓ-definable infinite homogeneous set. We follow the exposition of Jockusch's proof by Hirschfeldt [23, Theorem 6.11]. By the uniform limit lemma relativized to $0^{(\ell-2)}$, there is a $\Delta_{\ell-1}$-sequence $h_1, h_2, \ldots$ of functions from $\mathbb{N}\times\mathbb{N}$ to 2 such that for every Δ_ℓ-definable set A there exists an index e such that for each $x \in \mathbb{N}$ the limit $\lim_s h_e(x, s)$ exists and $x \in A$ if and only if $\lim_s h_e(x, s) = 1$.

We define the colouring c in stages. At stage s we define $c(x, s)$ for all $x < s$. We proceed in substages from $e = 0$ to $\lfloor (s-2)/2\rfloor$. At substage e define the set $B_{e,s} = \{x < s\colon h_e(x, s) = 1\}$. If $|B_{e,s}| \geqslant 2e + 2$ then there exist at least two numbers $x_1, x_2 \in B_{e,s}$ such that $c(x_1, s)$ and $c(x_2, s)$ have not yet been defined, since we define at most two values of c per substage, so at earlier substages we have defined $c(x, s)$ for at most $2e$ numbers $x < s$. Pick the least such x_1, x_2 and put $c(x_1, s) = 0$ and $c(x_2, s) = 1$. If $|B_{e,s}| < 2e + 2$, do nothing. After the final substage, put $c(x, s) = 0$ for all $x < s$ for which $c(x, s)$ has not yet been defined.

It follows from the construction that the colouring c is computable in $0^{(\ell-2)}$, and therefore it is $\Delta_{\ell-1}$-definable. Now suppose that A is an infinite Δ_ℓ-definable homogeneous set. Then A is given as $\lim_s h_e(x, s)$ for some index e. We can use $\mathsf{I}\Sigma_\ell$ to claim that for each w, A has a finite subset with w elements; in particular, A has a finite subset with $2e + 2$ elements. Let z be the smallest number such that $|A \restriction z| = 2e + 2$. Now by $\mathsf{B}\Sigma_\ell$ there exists $s > z$ such that for all $s' \geqslant s$ we have $B_{e,s'} \restriction z = B_{e,s} \restriction z$, so in fact $B_{e,s} \restriction z = A \restriction z$. Since A is infinite, we can assume that $s \in A$. But then at substage e of stage s we have defined $c(x_1, s) \neq c(x_2, s)$ for some $x_1, x_2 \in A\restriction z$, contradicting the assumption that the set A is homogeneous for c.

To finish the proof of the case $\ell \geqslant 2$ we assume first that $n \leqslant \ell$. We verify in $\mathsf{I}\Sigma_\ell$ that for each k, each r with $\ell \geqslant r \geqslant 2$ and each Δ_r-colouring $c\colon [\mathbb{N}]^k \to 2$, there exists a Δ_{r-1}-colouring $d\colon [\mathbb{N}]^{k+1} \to 2$ such that any infinite set (definable or not) that is homogeneous for d is also homogeneous for c. Then starting with the above constructed $\Delta_{\ell-1}$-definable 2-colouring of $[\mathbb{N}]^2$ without a Δ_ℓ-definable infinite homogeneous subset, one proceeds by external induction from $k = 2$ to $n - 1$ to obtain the desired $\Delta_{\ell-n+1}$-definable 2-colouring of $[\mathbb{N}]^n$ with no Δ_ℓ-definable infinite homogeneous subset.

So let c be a Δ_r-definable 2-colouring of $[\mathbb{N}]^k$. By the limit lemma relativized to $0^{(r-2)}$ we can see the colouring c as the limit of some Δ_{r-1}-definable function $h\colon [\mathbb{N}]^k \times \mathbb{N} \to 2$. From h we obtain a colouring $d\colon [\mathbb{N}]^{k+1} \to 2$ by restricting the domain of h to those $\langle x_1, \ldots, x_k, x_{k+1}\rangle \in [\mathbb{N}]^k \times \mathbb{N}$ such that $x_{k+1} > \max\{x_1, \ldots, x_k\}$. Now let H be an infinite set homogeneous for d with colour $i \in \{0, 1\}$ and let $\overline{x} \in [H]^k$. By the property of the function h, there is some $s_0 \in \mathbb{N}$ such that for all $s \geqslant s_0$, $c(\overline{x}) = d(\overline{x}, s)$. Since H is infinite we can assume that s_0 is in H. Hence $c(\overline{x}) = d(\overline{x}, s_0) = i$, so H is also homogeneous for c with colour i.

For fixed $\ell \geqslant 2$, the remaining case of $n > \ell$ is straightforward. Namely, we already have a Δ_1-definable 2-colouring c of $[\mathbb{N}]^\ell$ with no Δ_ℓ-definable infinite homogeneous set, so we can define a Δ_1-definable 2-colouring d of $[\mathbb{N}]^n$ by $d(x_1, \ldots, x_\ell, \ldots, x_n) = c(x_1, \ldots, x_\ell)$. Clearly, every infinite set homogeneous for d is also homogeneous for c.

We now prove the lemma for $\ell = 1$. Working only in $\mathsf{I}\Sigma_1$, we cannot repeat the argument proving that there exists a Δ_1-definable 2-colouring of $[\mathbb{N}]^2$ with no Δ_2-definable infinite homogeneous set. Firstly, we can no longer claim that every infinite Δ_2-definable set has arbitrarily large finite subsets. Secondly, we cannot use $\mathsf{B}\Sigma_2$ to argue that for every z there is a step s from which onwards some computable approximation of a given Δ_2-set is correct about elements of $[0, z]$.

We define a 2-colouring c of $[\mathbb{N}]^2$ as in the case of $\ell \geqslant 2$ with the only difference that now we set $B_{e,s} = \{x < s \colon \varphi_{e,s}(x)\downarrow\}$, where $\varphi_{e,s}(x)\downarrow$ is as in Section 1.1. Suppose that A is an infinite Σ_1-set which is homogeneous for c. There exists a number e such that A is the domain of the e-th partial computable function φ_e. By $\mathsf{I}\Sigma_1$, A has at least $2e + 2$ elements, so let z be its $(2e + 2)$-th smallest element. By another application of $\mathsf{I}\Sigma_1$ in the form of strong Σ_1-collection, there exists $s > z$ such that for all $x \leqslant z$, if $\varphi_e(x)$ halts, then it halts in fewer than s steps, i.e. $B_{e,s} \restriction z = A \restriction z$. Since there are arbitrarily large elements in A, we can assume that $s > z$ and s belongs to A. As previously, we conclude that in stage s we defined $c(x_1, s) \neq c(x_2, s)$ for some $x_1, x_2 \in A \restriction z$, so A cannot be homogeneous for c.

The statement of the lemma for $\ell = 1$ and $n > 2$ follows immediately from the previous paragraph: just as in the case of $n > \ell \geqslant 2$, we define a 2-colouring d of $[\mathbb{N}]^n$ by letting $d(x_1, x_2, \ldots, x_n) = c(x_1, x_2)$, where c is the Δ_1-definable 2-colouring of $[\mathbb{N}]^2$ defined above. □

Lemma 2.20. *Let $n \geqslant 3$ and $\ell \geqslant 1$. Suppose that $(M, \mathcal{X}) \models \mathsf{RCA}_0^* + \mathsf{RT}_2^n$ and $M \models \mathsf{I}\Sigma_\ell$. Then $0^{(\ell)} \in \mathcal{X}$. As a consequence, $\Delta_{\ell+1}$-Def$(M) \subseteq \mathcal{X}$ and $M \models \mathsf{B}\Sigma_{\ell+1}$.*

Proof. By Proposition 2.1 (a) it is enough to consider $n = 3$. Let $M \models \mathsf{RCA}_0^* + \mathsf{RT}_2^3 + \mathsf{I}\Sigma_\ell$, where $\ell \geqslant 1$. We will prove by induction on $j \leqslant \ell$ that $0^{(j)} \in \mathcal{X}$. For $j = \ell$, this, together with Proposition 1.3 and Proposition 1.2 (a), will immediately imply $\Delta_{\ell+1}$-Def$(M) \subseteq \mathcal{X}$ and $M \models \mathsf{B}\Sigma_{\ell+1}$, because $(M, \mathcal{X})$ satisfies Δ_1^0-comprehension and $\mathsf{B}\Sigma_1^0$.

The base step of the induction holds by Δ_1^0-comprehension in $(M, \mathcal{X})$. So let $j < \ell$ and assume that $0^{(j)} \in \mathcal{X}$. We have to prove that $0^{(j+1)} \in \mathcal{X}$.

Consider the usual computable instance of RT_2^3 from [28] whose all solutions compute $0'$ and relativize it to $0^{(j)}$:

$$c(x,y,z) = \begin{cases} 0 & \text{if there is a } \Sigma_{j+1} \text{ sentence } \exists v\, \varphi(v) \text{ with code at most } x \\ & \text{such that } \forall v \leqslant y\, \neg\, \mathrm{Sat}_{\Pi_j}(\ulcorner\varphi\urcorner, v) \wedge \exists v \leqslant z\, \mathrm{Sat}_{\Pi_j}(\ulcorner\varphi\urcorner, v), \\ 1 & \text{otherwise.} \end{cases}$$

The colouring c is $\Delta_1(0^{(j)})$-definable, so $c \in \mathcal{X}$. By RT^3_2, there exists an infinite $H \in \mathcal{X}$ homogeneous for c. We claim that H cannot be 0-homogeneous for c. To see this, note that by $\mathsf{I}\Sigma_\ell$ we have strong Σ_{j+1}-collection, so for any given x there is a bound w such that for any Σ_{j+1} sentence with code below x, if this sentence is true, then there is a witness for it below w. Thus, for any $z > y \geqslant w$, we must have $c(x,y,z) = 1$, which implies that no infinite set can be 0-homogeneous for c.

So, H is 1-homogeneous for c. We can now compute $0^{(j+1)}$ with oracle access to $0^{(j)} \oplus H$ as follows: given a Σ_{j+1} sentence $\exists v\, \varphi(v)$, find some $x \in H$ above the code for the sentence, find $y \in H$ above x, and use $0^{(j)}$ to determine whether $\exists v \leqslant y\, \varphi(v)$ holds; if it does not, then neither does $\exists v\, \varphi(v)$. Both $0^{(j)}$ and H are in $\mathcal{X}$, so by Δ^0_1-comprehension $0^{(j+1)}$ is in $\mathcal{X}$ as well. □

2.4.2 First-order consequences of RT^n_2 for $n \geqslant 3$

The two lemmas from the previous section together with the analysis from Section 2.2 allow us to give a full axiomatization of the first-order part of $\mathsf{RCA}^*_0 + \mathsf{RT}^n_2$ for $n \geqslant 3$. In the rest of this chapter, $\mathsf{I}\Sigma_\ell$ and $\mathsf{B}\Sigma_\ell$ stand for the usual finite axiomatizations (in the presence of exp) of the theories defined by induction and collection schemes in Section 1.1; for more details on those axiomatizations, see the discussion in Section 4.1.

Theorem 2.21. *Let $n \geqslant 3$ and let R^n be the theory axiomatized by $\mathsf{B}\Sigma_1 + \mathsf{exp}$ and the set of sentences:*

$$\{\mathsf{B}\Sigma_\ell \Rightarrow (\mathsf{B}\Sigma_{\ell+1} \vee \Delta_\ell\text{-}\mathsf{RT}^n_2) :\ \ell \geqslant 1\}. \tag{2.2}$$

Then R^n axiomatizes the first-order consequences of $\mathsf{RCA}^*_0 + \mathsf{RT}^n_2$.

Proof. Fix $n \geqslant 3$ and let R^n be as above. Note that R^n follows from PA. We first argue that for every $M \vDash \mathsf{R}^n$ there is a family of sets $\mathcal{X} \subseteq \mathcal{P}(M)$ such that $(M, \mathcal{X}) \vDash \mathsf{RCA}^*_0 + \mathsf{RT}^n_2$, which will mean that R^n proves all arithmetical consequences of $\mathsf{RCA}^*_0 + \mathsf{RT}^n_2$.

So, let $M \models \mathsf{R}^n$. If $M \models \mathsf{PA}$, then $(M, \mathrm{Def}(M))$ is a model of ACA_0 and, a fortiori, of $\mathsf{RCA}^*_0 + \mathsf{RT}^n_2$. Otherwise, let $\ell \in \omega$ be such that $M \vDash \mathsf{I}\Sigma_\ell \wedge \neg \mathsf{I}\Sigma_{\ell+1}$. Then, clearly, $M \vDash \mathsf{B}\Sigma_\ell$ and, by Lemma 2.19, $M \nvDash \Delta_\ell\text{-}\mathsf{RT}^n_2$ if $\ell \geqslant 1$. By the assumption that $M \vDash \mathsf{R}^n$, we have $M \vDash \mathsf{B}\Sigma_{\ell+1}$. Now, it also follows from R^n that $M \vDash \mathsf{B}\Sigma_{\ell+2} \vee \Delta_{\ell+1}\text{-}\mathsf{RT}^n_2$. We assumed that $M \nvDash \mathsf{I}\Sigma_{\ell+1}$, so $M \nvDash \mathsf{B}\Sigma_{\ell+2}$ as well. Thus we have $M \vDash \Delta_{\ell+1}\text{-}\mathsf{RT}^n_2$, and therefore, by the equivalence (2.1), $(M, \Delta_{\ell+1}\text{-}\mathrm{Def}(M)) \models \mathsf{RCA}^*_0 + \mathsf{RT}^n_2$.

In the other direction, we assume that $(M, \mathcal{X}) \models \mathsf{RCA}_0^* + \mathsf{RT}_2^n$ and prove that $M \models \mathsf{R}^n$. Of course, $M \vDash \mathsf{B}\Sigma_1 + \mathsf{exp}$. If $M \models \mathsf{PA}$, then we are done. Otherwise, let $\ell \geqslant 1$ be such that $M \models \mathsf{B}\Sigma_\ell \wedge \neg \mathsf{B}\Sigma_{\ell+1}$. We need to show that $M \vDash \Delta_\ell\text{-}\mathsf{RT}_2^n$. Clearly, $M \vDash \mathsf{I}\Sigma_{\ell-1}$ so, by Lemma 2.20, $\Delta_\ell\text{-Def}(M) \subseteq \mathcal{X}$. By another application of Lemma 2.20, M cannot satisfy $\mathsf{I}\Sigma_\ell$, since otherwise it would also satisfy $\mathsf{B}\Sigma_{\ell+1}$. Hence, there exists a Σ_ℓ-definable proper cut I in M. By Proposition 1.2 (b), the cut I is Σ_1^0-definable in $(M, \Delta_\ell\text{-Def}(M))$, and thus also in $(M, \mathcal{X})$. Moreover, both of these structures satisfy RCA_0^* so, by our assumption that $(M, \mathcal{X}) \models \mathsf{RCA}_0^* + \mathsf{RT}_2^n$ and a twofold application of Theorem 2.9, we infer that $(M, \Delta_\ell\text{-Def}(M)) \models \mathsf{RT}_2^n$. Therefore, by the equivalence (2.1), we conclude that $M \vDash \Delta_\ell\text{-}\mathsf{RT}_2^n$. □

The last theorem of this chapter describes the relationship of the theories R^n to the usual fragments of first-order arithmetic. We also show a link to the theory IB studied by Kaye in [30].

Definition 2.22. The theory IB is axiomatized by $\mathsf{B}\Sigma_1$ and the set of sentences

$$\{\mathsf{I}\Sigma_\ell \Rightarrow \mathsf{B}\Sigma_{\ell+1} : \ell \geqslant 1\}. \tag{2.3}$$

Note that IB is not contained in any $\mathsf{I}\Sigma_\ell$ because the hierarchy (1.5) is strict. Kaye showed that $\mathsf{IB} + \mathsf{exp}$ implies the theory of all κ-like models of arithmetic (for κ possibly singular). It is now known (see [19, Section 3.3] and [2, Section 6]) that $\mathsf{IB} + \mathsf{exp}$ is actually strictly stronger than the theory of all κ-like models.

Theorem 2.23. *Let $n \geqslant 3$. Then:*

(a) the first-order consequences of $\mathsf{RCA}_0^* + \mathsf{RT}_2^n$ are strictly in between $\mathsf{IB} + \mathsf{exp}$ and PA; as a result, they are not finitely axiomatizable.

(b) the Π_3 consequences of $\mathsf{RCA}_0^* + \mathsf{RT}_2^n$ coincide with $\mathsf{B}\Sigma_1 + \mathsf{exp}$; for $\ell \geqslant 1$, the $\Pi_{\ell+3}$ consequences are strictly in between

$$\mathsf{B}\Sigma_1 + \exp + \bigwedge_{1 \leqslant j \leqslant \ell} (\mathsf{I}\Sigma_j \Rightarrow \mathsf{B}\Sigma_{j+1})$$

and $\mathsf{B}\Sigma_{\ell+1}$.

Proof. We first prove (b). As in the previous theorem, we let R^n stand for the first-order consequences of $\mathsf{RCA}_0^* + \mathsf{RT}_2^n$.

Recall from Section 1.1 that $\mathsf{B}\Sigma_\ell + \mathsf{exp}$ is $\Pi_{\ell+2}$-axiomatizable for $\ell \geqslant 1$, and that the first-order consequences of RCA_0^* are axiomatized by $\mathsf{B}\Sigma_1 + \mathsf{exp}$. Thus, by Theorem 2.16, it is clear that the Π_3 consequences of R^n coincide with $\mathsf{B}\Sigma_1 + \mathsf{exp}$.

To see that all $\Pi_{\ell+3}$ consequences of R^n follow from $\mathsf{B}\Sigma_{\ell+1}$ for $\ell \geqslant 1$, we modify the proof of Π_3-conservativity over RCA_0^* (Theorem 2.16) as follows. Given a $\Sigma_{\ell+3}$ formula $\varphi := \exists x \, \forall y \, \exists z \, \theta(x, y, z)$ consistent with $\mathsf{B}\Sigma_{\ell+1}$, by Theorem 1.10 we can take a model K satisfying $\mathsf{B}\Sigma_{\ell+1} + \varphi$ with $(\omega, \mathrm{SSy}(K)) \models \mathsf{RT}_2^n$.

Let $a \in K$ be a witness for the outer existential quantifier in φ. We define the function

$$f(y) = \min\{z > y\colon\ \forall y' \leqslant y\ \exists z' \leqslant z\ \theta(a, y', z') \\ \wedge\ \text{`true } \Sigma_\ell \text{ sentences with codes} \leqslant y \text{ are witnessed} \leqslant z\text{'}\}, \quad (2.4)$$

which by $\mathsf{B}\Sigma_{\ell+1}$ is total, and it is $\Delta_{\ell+1}$-definable in K using $\mathrm{Sat}_{\Sigma_\ell}(e, x)$. Note that the existence of z satisfying the second conjunct in the above definition is guaranteed by strong Σ_ℓ-collection. The change in the definition of f compared to the proof of Theorem 2.16 is due to the fact that for $\ell \geqslant 1$ we have to care about Σ_ℓ-elementarity between K and the constructed model M so that we can apply Theorem 1.7 (b) to get $M \vDash \mathsf{B}\Sigma_{\ell+1}$. On the other hand, the totality of exponentiation follows immediately since $\mathsf{B}\Sigma_{\ell+1} \supseteq \mathsf{I}\Sigma_1 \vdash \mathsf{exp}$.

We define M to be the cut $\sup_K\{f^m(c) : m \in \omega\}$, where c is some nonstandard number not less than a. By the construction, M satisfies $\mathsf{B}\Sigma_{\ell+1} + \varphi$, and ω is a $\Sigma_{\ell+1}$-definable proper cut in M. By Proposition 1.2, ω is Σ^0_1-definable in the structure $(M, \Delta_{\ell+1}\text{-Def}(M))$ which satisfies RCA^*_0. Since $\mathrm{SSy}(M) = \mathrm{SSy}(K)$, we can apply Theorem 2.9 to learn that $(M, \Delta_{\ell+1}\text{-Def}(M)) \vDash \mathsf{RT}^n_2$, and thus $M \vDash \mathsf{R}^n + \varphi$.

A similar construction shows that R^n does not prove $\mathsf{B}\Sigma_{\ell+1}$ for $\ell \geqslant 1$. Namely, we consider a model $M \models \mathsf{B}\Sigma_\ell + \mathsf{exp} + \neg\mathsf{I}\Sigma_\ell$ such that ω is Σ_ℓ-definable in M and $(\omega, \mathrm{SSy}(M)) \vDash \mathsf{RT}^n_2$. We obtain such an M using Theorem 1.10 as above but with a modified definition (2.4) of the function f to guarantee only $\Sigma_{\ell-1}$-elementarity. Again, by the choice of M, the structure $(M, \Delta_\ell\text{-Def}(M))$ satisfies $\mathsf{RCA}^*_0 + \mathsf{RT}^n_2$, and hence $M \vDash \mathsf{R}^n + \neg\mathsf{B}\Sigma_{\ell+1}$.

It follows immediately from the definition of RCA^*_0 and Lemma 2.20 that the $\Pi_{\ell+3}$ consequences of R^n include $\mathsf{B}\Sigma_1 + \mathsf{exp}$ and $\mathsf{I}\Sigma_j \Rightarrow \mathsf{B}\Sigma_{j+1}$ for each $j \leqslant \ell$. For $\ell \geqslant 1$, the inclusion is strict, because the sentence

$$\mathsf{B}\Sigma_\ell \Rightarrow (\mathsf{B}\Sigma_{\ell+1} \vee \Delta_\ell\text{-}\mathsf{RT}^n_2) \quad (2.5)$$

is $\Pi_{\ell+3}$ but it is not provable in $\mathsf{B}\Sigma_1 + \mathsf{exp} + \bigwedge_{1\leqslant j\leqslant \ell}(\mathsf{I}\Sigma_j \Rightarrow \mathsf{B}\Sigma_{j+1})$. For the unprovability, we consider a model $M \vDash \mathsf{B}\Sigma_\ell + \mathsf{exp} + \neg\mathsf{I}\Sigma_\ell$ chosen as in the previous paragraph but this time with $(\omega, \mathrm{SSy}(M)) \nvDash \mathsf{RT}^n_2$. Here we additionally use Theorem 1.21. Clearly, $M \models \mathsf{I}\Sigma_j \Rightarrow \mathsf{B}\Sigma_{j+1}$ for each $j \leqslant \ell$ and, in fact, M is a model of IB. Reasoning as before we learn that $(M, \Delta_\ell\text{-Def}(M)) \vDash \mathsf{RCA}^*_0 + \neg\mathsf{RT}^n_2$, and so $M \vDash \neg\Delta_\ell\text{-}\mathsf{RT}^n_2$. Finally, M was chosen not to satisfy $\mathsf{I}\Sigma_\ell$ so it does not satisfy $\mathsf{B}\Sigma_{\ell+1}$ either. Therefore, (2.5) fails in M, as required.

We have thus proved (b). Regarding (a), note that the containments

$$\mathsf{IB} + \mathsf{exp} \subseteq \mathsf{R}^n \subsetneq \mathsf{PA}$$

follow directly from the statement of (b), and in the proof of (b) we constructed a model of $\mathsf{IB} + \mathsf{exp}$ not satisfying R^n. Finally, since IB is not contained in any $\mathsf{I}\Sigma_\ell$, no subtheory of PA extending IB can be finitely axiomatizable. □

Note that the proof of Theorem 2.23 immediately gives the following statement, which says essentially that Lemma 2.19 is optimal with respect to the amount of induction used to prove the existence of colourings without homogeneous sets of a given arithmetical complexity.

Corollary 2.24. *For each* $\ell \geqslant 1, n \geqslant 2$, *the theory* $\mathsf{B}\Sigma_\ell + \mathsf{exp} + \Delta_\ell\text{-}\mathsf{RT}^n_2$ *is consistent.*

Remark 2.25. The above methods do not allow for a complete axiomatization of the first-order consequences of $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2$, because we do not have Lemma 2.20 for $n = 2$. However, one can observe the following. By [5], the first-order consequences of $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2$ follow from $\mathsf{I}\Sigma_2$. The same argument as in the proof of Theorem 2.23 shows that their Π_3 part coincides with $\mathsf{B}\Sigma_1 + \mathsf{exp}$ and their Π_4 part is strictly weaker than $\mathsf{B}\Sigma_2$. In [31] we provide an example of a Π_4 sentence, namely the cardinality scheme $\mathsf{C}\Sigma_2$, which is provable from $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2$ but does not follow from $\mathsf{I}\Sigma_1$. The proof relies on the fact that models of $\mathsf{B}\Sigma_n + \neg\mathsf{I}\Sigma_n$ have many automorphisms.

Remark 2.26. One of our initial motivations to study RT^2_2 over RCA^*_0 was to get some insight into the problem of characterizing the first-order consequences of $\mathsf{RCA}_0 + \mathsf{RT}^2_2$. The methods of the present chapter indeed suggested a possible way of attack on this challenge. In [31] we study a Π^1_2 statement $\Delta^0_2\text{-}\mathsf{RT}^2_2$, which says that for every set X and every $\Delta^0_2(X)$-definable colouring $c\colon [\mathbb{N}]^2 \to 2$ there exists a set Y and a $\Delta^0_2(Y)$-definable homogeneous set for c. Using the techniques from the present chapter one readily shows that $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \Delta^0_2\text{-}\mathsf{RT}^2_2$ is Π_4- but not Π_5-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. In fact, this theory proves the Π_5 sentence $\neg\mathsf{I}\Sigma_2 \Rightarrow \Delta_2\text{-}\mathsf{RT}^2_2$, which does not follow from $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. Thus, it is very natural to ask whether this sentence follows from $\mathsf{RCA}_0 + \mathsf{RT}^2_2$. A positive answer would immediately give arithmetical nonconservativity of $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. However, in [31] it is shown that this is not the case: $\mathsf{RCA}_0 + \mathsf{B}\Sigma_2 + (\neg\mathsf{I}\Sigma_2 \Rightarrow \Delta_2\text{-}\mathsf{RT}^2_2)$ has non-elementary proof speedup over RCA_0 with respect to Σ_1 sentences, whereas by [32] $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is polynomially simulated by RCA_0 with respect to $\forall\Pi^0_3$ sentences.

The speedup result depends on the exponential lower bound on Ramsey numbers for the finite version of RT^2_2. Therefore, one could still try to prove nonconservativity of RT^2_2 over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ by weakening the sentence $\neg\mathsf{I}\Sigma_2 \Rightarrow \Delta_2\text{-}\mathsf{RT}^2_2$ to $\neg\mathsf{I}\Sigma_2 \Rightarrow \Delta_2\text{-}\mathsf{P}$, where P is a combinatorial principle whose finite version has a polynomial upper bound. One possibility would be $\neg\mathsf{I}\Sigma_2 \Rightarrow \Delta_2\text{-}\mathsf{CAC}$, which is also easily seen to be unprovable in $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$. Unfortunately, this approach too was recently ruled out by the work of Le Houérou et al. [36], who showed that $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathrm{WO}(\epsilon_0)$. Namely, it is enough to construct a model $(M, \mathcal{X})$ satisfying $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathrm{WO}(\epsilon_0)$ with Σ_2-definable ω and $\mathrm{SSy}(M) \vDash \neg\mathsf{P}$. This can be readily done for any principle P which does not follow from WKL_0 on ω-models, using standard techniques from the theory of models of arithmetic. Thus, for any such P the sentence $\neg\mathsf{I}\Sigma_2 \Rightarrow \Delta_2\text{-}\mathsf{P}$ does not follow from $\mathsf{RCA}_0 + \mathsf{RT}^2_2$.

Nonetheless, the considerations about $\Delta_2\text{-}\mathsf{CAC}$ turned our attention to the problem of proof size for CAC and were part of the inspiration for Chapter 4.

Chapter 3

Other principles

The present chapter studies principles which elude the methods developed for the normal versions. Firstly we discuss long versions of the principles studied in the previous chapter. Then we will focus on COH, which requires yet another set of techniques.

3.1 Long versions

Let us officially introduce the long versions of our combinatorial principles. 'Long' stands for the fact that now we require solution sets to be of cardinality $\mathbb{N}$ rather than just unbounded.

ℓ-RT^n_k *For every function $c\colon [\mathbb{N}]^n \to k$ there exists a set $H \subseteq \mathbb{N}$ of cardinality $\mathbb{N}$ such that c is constant on $[H]^n$.*

ℓ-CAC *For every partial order $(\mathbb{N}, \preccurlyeq)$ there exists a set $S \subseteq \mathbb{N}$ of cardinality $\mathbb{N}$ which is either a chain or an antichain in $\preccurlyeq$.*

ℓ-$\mathsf{ADS}^{\mathrm{set}}$ *For every linear order $(\mathbb{N}, \preccurlyeq)$ there exists a set $S \subseteq \mathbb{N}$ of cardinality $\mathbb{N}$ such that either for all $x, y \in S$ it holds that $x \leqslant y$ iff $x \preccurlyeq y$ or for all $x, y \in S$ it holds that $x \leqslant y$ iff $x \succcurlyeq y$.*

ℓ-$\mathsf{ADS}^{\mathrm{seq}}$ *For every linear order $(\mathbb{N}, \preccurlyeq)$ there exists a sequence $(s_i)_{i\in\mathbb{N}}$ which is either strictly $\preccurlyeq$-increasing or strictly $\preccurlyeq$-decreasing.*

ℓ-CRT^2_2 *For every $c\colon [\mathbb{N}]^2 \to 2$ there exists a set $S \subseteq \mathbb{N}$ of cardinality $\mathbb{N}$ such that for every $x \in S$ there exists $y \in S$ such that $c(x, y) = c(x, z)$ holds for all $z \in S$ with $z \geqslant y$.*

Recall that the set and sequence formulations of ADS are equivalent over RCA_0, and by Proposition 2.4, also over RCA^*_0 if we consider their normal versions. However, the proof of Proposition 2.4 cannot be repeated here: the way we obtained a set solution S from a sequence solution $(s_i)_{i\in I}$ does not exclude the possibility that S is enumerated in increasing order by a cut J strictly shorter than I. Thus we have to start our analysis with two long variants of ADS: ℓ-$\mathsf{ADS}^{\mathrm{set}}$ and ℓ-$\mathsf{ADS}^{\mathrm{seq}}$. It is straightforward to check that ℓ-$\mathsf{ADS}^{\mathrm{set}}$ implies ℓ-$\mathsf{ADS}^{\mathrm{seq}}$ but, as we will see from Theorems 3.2 and 3.7, this implication cannot be reversed.

As for Ramsey's theorem, for a fixed $n \geqslant 2$ the strength of its long version does not depend on the number of colours – the argument is just the same as for Proposition 1.18. Theorem 3.2 will show that also the length of tuples is unimportant for $n \geqslant 3$. Thus, there are in fact just two long variants of Ramsey's theorem: ℓ-RT^2_2 and ℓ-RT^3_2.

One could expect that the requirement on solution sets to be of cardinality $\mathbb{N}$ would make the long versions logically stronger than the normal versions. However, this will prove correct only for some of our principles. In fact, we will see that the long versions behave in one of two contrasting ways. Some of them are strong and imply $\mathsf{I}\Sigma^0_1$, whereas the other ones are partially conservative over RCA^*_0.

We start with easy implications which are known to hold over RCA_0 (cf. Proposition 2.1). Their proofs do not make use of $\mathsf{I}\Sigma^0_1$ (cf. Section 1.3), so they transfer immediately to RCA^*_0. Also, they are strict over RCA_0 and thus they remain strict over weaker sets of axioms.

Proposition 3.1. *Over* RCA^*_0*, the following sequences of implications hold:*

$$\ell\text{-}\mathsf{RT}^3_2 \Rightarrow \ell\text{-}\mathsf{RT}^2_2 \Rightarrow \ell\text{-}\mathsf{CAC} \Rightarrow \ell\text{-}\mathsf{ADS}^{\mathrm{set}} \Rightarrow \ell\text{-}\mathsf{ADS}^{\mathrm{seq}},$$
$$\ell\text{-}\mathsf{RT}^2_2 \Rightarrow \ell\text{-}\mathsf{CRT}^2_2.$$

None of the implications ℓ-$\mathsf{RT}^3_2 \Rightarrow \ell$-$\mathsf{RT}^2_2 \Rightarrow \ell$-$\mathsf{CAC} \Rightarrow \ell$-$\mathsf{ADS}^{\mathrm{set}}$ *and* ℓ-$\mathsf{RT}^2_2 \Rightarrow \ell$-$\mathsf{CRT}^2_2$ *can be provably reversed in* RCA^*_0*.*

In the rest of this section, we describe some results obtained in an attempt to answer questions left open by Proposition 3.1. It will follow from these results (specifically from Theorem 3.2 and Theorem 3.7) that also the implication ℓ-$\mathsf{ADS}^{\mathrm{set}} \Rightarrow \ell$-$\mathsf{ADS}^{\mathrm{seq}}$ cannot be provably reversed in RCA^*_0, and that ℓ-$\mathsf{ADS}^{\mathrm{set}}$ implies ℓ-CRT^2_2.

Observe that Theorem 2.9, which was our main technical tool in Chapter 2, cannot be applied to long versions. The reason is that we cannot transfer solution sets from a Σ^0_1-cut back to the whole universe because these solutions would be too 'short' to witness the long version of a principle P. However, the next theorem, which is a minor generalization of [58, Proposition 2.3], shows that the strength of some of the long principles can be determined quite easily.

Theorem 3.2. *Over* RCA^*_0*, each of the principles* ℓ-RT^3_2*,* ℓ-RT^2_2*,* ℓ-CAC*,* ℓ-$\mathsf{ADS}^{\mathrm{set}}$ *implies* $\mathsf{I}\Sigma^0_1$*.*

Proof. The proof for ℓ-RT^2_2 was given by Yokoyama in [58], and it uses a transitive colouring, so essentially the same argument works for each of the principles listed above. By the previous proposition it is enough to give a proof for ℓ-$\mathsf{ADS}^{\mathrm{set}}$.

Working in RCA_0^*, suppose that $\mathsf{I}\Sigma_1^0$ fails, and let $A = \{a_i\}_{i \in I}$ be an unbounded set enumerated in increasing order by a proper Σ_1^0-cut I, as in Proposition 1.13. We define a linear order $\preccurlyeq$ on $\mathbb{N}$ in the following way:

$$x \preccurlyeq y \Leftrightarrow \begin{array}{l} \exists i \in I\,(x \in (a_{i-1}, a_i] \wedge y \in (a_{i-1}, a_i] \wedge x \geqslant y) \\ \vee\ \exists i, j \in I\,(i < j \wedge x \in (a_{i-1}, a_i] \wedge y \in (a_{j-1}, a_j]). \end{array}$$

That is, we invert the usual ordering $\leqslant$ on each interval $(a_{i-1}, a_i]$, but we compare elements from different intervals in the usual way. The order $\preccurlyeq$ is a set by $\Delta_1(A)$-comprehension, because for each x the interval $(a_{i-1}, a_i]$ it belongs to is uniquely determined.

If $S \subseteq \mathbb{N}$ is such that any two elements $x, y \in S$ satisfy $x \preccurlyeq y \leftrightarrow y \leqslant x$, then S has to be contained in an interval of the form $(a_{i-1}, a_i]$, so it is finite. On the other hand, if all $x, y \in S$ satisfy $x \preccurlyeq y \leftrightarrow x \leqslant y$, then S can contain at most one element from each $(a_{i-1}, a_i]$, so the cardinality of S is some proper cut I and not the whole $\mathbb{N}$. □

Now it follows that over RCA_0^*, $\ell\text{-}\mathsf{RT}_2^3$ implies $\ell\text{-}\mathsf{RT}_2^n$ for each $n \geqslant 3$. This is because for $n \geqslant 3$, $\mathsf{RCA}_0^* + \ell\text{-}\mathsf{RT}_2^n$ is the same theory as $\mathsf{RCA}_0 + \mathsf{RT}_2^n$, which is in turn equivalent to ACA_0.

We now aim to show that the long versions of other principles are logically weak. For this purpose, we introduce an auxiliary statement, a version of the *grouping principle* GP_2^2 considered in [41]. The original grouping principle is a weakening of RT_2^2 stating that, for any 2-colouring of pairs and any notion of largeness of finite sets (suitably defined), there is an infinite sequence of large finite sets $G_0, G_1, \ldots$ (the *groups*) such that for each $i < j$ the colouring is constant on $G_i \times G_j$. We consider a weaker version tailored to RCA_0^*, in which the number of groups can be a proper cut, but the cardinality of individual groups should eventually exceed any finite number.

Definition 3.3. The *growing grouping principle* GGP_2^2 states that for every colouring $c\colon [\mathbb{N}]^2 \to 2$ there exists a sequence of finite sets $(G_i)_{i \in I}$, which is called a *grouping*, such that

(i) for every $i < j \in I$ and every $x \in G_i$, $y \in G_j$ it holds that $x < y$,

(ii) for every $i < j \in I$, the colouring $c \restriction (G_i \times G_j)$ is constant,

(iii) for every $i \in I$, $|G_i| \leqslant |G_{i+1}|$, and $\sup_{i \in I} |G_i| = \mathbb{N}$.

Note that over RCA_0, GGP_2^2 follows immediately from RT_2^2, because every homogeneous set can be split into a grouping of length $\mathbb{N}$. The next lemma is a possibly surprising result on the behaviour of GGP_2^2 under $\neg\mathsf{I}\Sigma_1^0$. It says that no typical Ramsey-like principle is needed to prove GGP_2^2 over $\neg\mathsf{I}\Sigma_1^0$.

Lemma 3.4. $\mathsf{WKL}_0^* + \neg\mathsf{I}\Sigma_1^0$ *implies* GGP_2^2. *Moreover,* GGP_2^2 *restricted to transitive colourings is provable in* $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0$.

Remark 3.5. The principle GGP^2_2 exhibits quite unusual behaviour in terms of conservativity. Namely, by Lemma 3.4, GGP^2_2 is Π^1_1-conservative over $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$. On the other hand, Yokoyama [private communication] has pointed out that GGP^2_2 is not arithmetically conservative over RCA_0. This can be seen as follows. It is shown in [41, Theorem 5.7 and Corollary 5.9] that RCA_0 extended by a statement $\mathsf{GP}(\mathsf{L}_\omega)$ intermediate between GGP^2_2 and GP^2_2 proves the principle known as 2-DNC and, as a consequence, an arithmetical statement $\mathsf{C}\Sigma_2$ unprovable in RCA_0. However, it is clear from the proof of [41, Theorem 5.7] that $\mathsf{RCA}_0 + \mathsf{GGP}^2_2$ is enough for the argument to go through.

This contrasts with the behaviour of e.g. CRT^2_2 and COH (see the next section), which are both Π^1_1-conservative over RCA_0 but not over $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$.

Proof of Lemma 3.4. The proof uses the technique of building a grouping by thinning out a family of finite sets first 'from below' and then 'from above'. This method was applied to construct large finite groupings in [33].

Assume $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1$ and let $A = \{a_i\}_{i \in I}$ be an unbounded set enumerated in increasing order by a Σ^0_1-definable proper cut I as in Proposition 1.13. By possibly thinning out A (which can only shorten I), we may also assume that for each $i \in I$, $a_0 > 2^i$ and $|(a_i, a_{i+1}]| \geqslant a_0 a_i 2^{a_i+1}$.

Let $c\colon [\mathbb{N}]^2 \to 2$. We want to obtain a sequence of sets $(G_i)_{i\in I}$ witnessing GGP^2_2 such that $G_i \subseteq (a_{i-1}, a_i]$ for each i. We proceed in two main stages.

(1) We stabilize the colour 'from below' by thinning out each interval $(a_{i-1}, a_i]$ to a set G'_i with the property that for every $x \leqslant a_{i-1} < \min(G'_i)$ the colouring c is constant on $\{x\} \times G'_i$.

For each $i \in I$, build a finite sequence of finite sets $B^i_{-1} \supseteq B^i_0 \supseteq \ldots \supseteq B^i_{a_{i-1}}$ in the following way. Let $B^i_{-1} = (a_{i-1}, a_i]$, and for each $0 \leqslant x \leqslant a_{i-1}$ let $B^i_x = \{y \in B^i_{x-1} \colon c(x,y) = k\}$, where $k \in \{0,1\}$ is such that $|\{y \in B^i_{x-1} \colon c(x,y) = k\}| \geqslant |\{y \in B^i_{x-1} \colon c(x,y) = 1-k\}|$. We choose $k = 0$ if the two values are equal. In other words, we throw out from B^i_{x-1} those elements which have the 'less popular' colour paired with x. Let $G'_i = B^i_{a_{i-1}}$.

The sequence $(G'_i)_{i\in I}$ is $\Delta_1(A)$-definable and clearly has the desired colouring property. We also have $G'_0 = [0, a_0]$ and $|G'_i| \geqslant a_0 a_{i-1} 2^{a_{i-1}+1-(a_{i-1}+1)} = a_0 a_{i-1}$ for each $0 < i \in I$.

(2) We stabilize the colour 'from above'. For each $i \in I$, we can construct an infinite sequence of finite sets $G'_i = D^i_i \supseteq D^i_{i+1} \supseteq D^i_{i+2} \supseteq \ldots$ indexed by $i \leqslant j \in I$, with a single step of the construction essentially like in stage (1). That is, given $j > i$, we let D^i_j be $\{x \in D^i_{j-1} \colon c(x, \min(G'_j)) = k\}$ for that k for which this set is larger. We only need to compare each $x \in D^i_{j-1}$ with one element of G'_j, because we have already arranged for c to be constant on $\{x\} \times G'_j$. Note that for each $i < j \in I$ the colouring c is constant on $D^i_j \times G'_j$. Also, each D^0_j is nonempty, while for $0 < i \leqslant j \in I$ we have $|D^i_j| \geqslant a_0 a_{i-1} 2^{-j} \geqslant a_{i-1}$, because we chose $A = \{a_i\}_{i\in I}$ to satisfy $a_0 > 2^i$, for each $i \in I$.

Intuitively, we would want to define $G_i = \bigcap_{i \leqslant j \in I} D^i_j$, but then being a member of G_i might not be Δ^0_1-definable. However, if we fix $m \in I$ and consider only the sets $\bigcap_{j=i}^m D^i_j$ for $i \leqslant m$, we obtain a node of length $a_m + 1$ in the computable binary tree T defined as follows. A finite 0-1 sequence τ belongs to T if the largest m such that $\mathrm{lh}(\tau) > a_m$ satisfies (if we identify τ with the finite set it codes):

(i) $\tau \cap [0, a_m] \subseteq \bigcup_{i=0}^{m} G'_i$,

(ii) for every $i < j \leqslant m$, the colouring c is constant on $(G'_i \cap \tau) \times (G'_j \cap \tau)$,

(iii) $|\tau \cap G'_i| \geqslant a_{i-1}$ for every $i \leqslant m$.

Thus each node $\tau \in T$ is a finite approximation to an unbounded grouping $(G_i)_{i \in I}$ that we are looking for. The tree T is infinite because for arbitrary $m \in I$ there exists a node in T of length a_m (for instance $\bigcup_{i \leqslant m} \bigcap_{j=i}^{m} D^i_j$) and the set $A = \{a_m \colon m \in I\}$ is unbounded. By WKL, T has an infinite path G, and we get the desired grouping $(G_i)_{i \in I}$ by taking $G_i = G \cap (a_{i-1}, a_i]$.

Now assume only $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma^0_1$ and let $c\colon [\mathbb{N}]^2 \to 2$ be a transitive colouring. By the argument from the proof of Lemma 2.5, we can think of c as given by a linear ordering $(\mathbb{N}, \preccurlyeq)$. The first stage of the construction, 'from below', is exactly as before. In the 'from above' stage, we will make a small change. If we built the sets D^i_j for c as in the previous construction, then, in terms of $\preccurlyeq$, we would look at the position of $\min(G'_j)$ in the $\preccurlyeq$-ordering relative to the elements of D^i_{j-1}. This would split D^i_{j-1} into a 'top part' and a 'bottom part' with respect to $\preccurlyeq$, and we would take whichever of these two parts were larger. Now, we will take the $\preccurlyeq$-bottom *half* of D^i_{j-1} if $\min(G'_j)$ lies above it, and the $\preccurlyeq$-top *half* if it does not. (Do this in a way that includes the $\preccurlyeq$-midpoint in case $|D^i_{j-1}|$ is odd, so that $|D^i_j|$ is exactly $\lceil |D^i_{j-1}|/2 \rceil$.)

Now consider the set

$$S = \{\langle i, j\rangle \in [I]^2 \colon i < j \wedge D^i_j \text{ is the } \preccurlyeq\text{-top half of } D^i_{j-1}\}.$$

It is easy to see that both are S and $[I]^2 \setminus S$ are Σ^0_1-definable, so by Lemma 1.17, there is an element $s > I$ coding S on I. We can think of s as a subset of $[0, b] \times [0, b]$ for some $I < b < \log a_0$. Thus we can use s to generalize the new definition of D^i_j to $i \in I$ and $j \in [i, b]$: D^i_j is the $\preccurlyeq$-top half of D^i_{j-1} if $\langle i, j\rangle \in s$, and the $\preccurlyeq$-bottom half otherwise. Let $G_i = \bigcap_{j=i}^{b} D^i_j$. It is easy to check that $(G_i)_{i \in I}$ is Δ^0_1-definable and that it witnesses GGP^2_2 for $\preccurlyeq$. □

Remark 3.6. Clearly, for $I < j \leqslant b$ the set D^i_j does not really exist, and the number s contains only an 'instruction' how to thin out G'_i in $(b-i)$-many steps as if there were more than I-many groups G'_j. The reason why the proof of GGP^2_2 for transitive colourings does not obviously generalize to arbitrary ones is that in general, for $i \in I < j$, it is not clear how to split a subset of $(a_{i-1}, a_i]$ into a 'more red' and a 'more blue' half with respect to a (nonexistent) element of G'_j. However, if the colouring is transitive and given by an ordering $\preccurlyeq$, then even though we cannot actually compare the elements of $(a_{i-1}, a_i]$ to a nonexistent element, we can say which ones form the $\preccurlyeq$-top and $\preccurlyeq$-bottom half and so we can follow the 'instruction' given by a code s.

Theorem 3.7. RCA_0^* *proves* ℓ-$\mathsf{ADS}^{\mathrm{seq}} \Leftrightarrow \mathsf{ADS}$*, and* WKL_0^* *proves* ℓ-$\mathsf{CRT}^2_2 \Leftrightarrow \mathsf{CRT}^2_2$*.*

Proof. Let us first consider the case of ADS. Clearly, ℓ-$\mathsf{ADS}^{\mathrm{seq}}$ implies ADS, and the two principles are equivalent over RCA_0. So, we only need to prove ℓ-$\mathsf{ADS}^{\mathrm{seq}}$ from ADS working in $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma^0_1$.

Let $(\mathbb{N}, \preccurlyeq)$ be an instance of ℓ-$\mathsf{ADS}^{\mathrm{seq}}$. By Lemma 3.4, we can apply GGP^2_2 to the transitive colouring given by $\preccurlyeq$, obtaining a grouping $(G_i)_{i\in I}$ indexed by some Σ^0_1-definable cut I. By Lemma 2.2, we can apply ADS to the order $\preccurlyeq$ restricted to the set $A = \{\min(G_i)\colon\ i \in I\}$. Without loss of generality, assume that this gives us an unbounded set $S \subseteq_{\mathrm{cf}} A$ such that for any $x, y \in S$, $x \preccurlyeq y$ iff $x \geqslant y$. Assume $S = \{\min(G_{i_j})\colon\ j \in J\}$ for some cut $J \subseteq I$. Now consider the strictly decreasing sequence in $(\mathbb{N}, \preccurlyeq)$ defined as follows: first list the elements of G_{i_0} in $\preccurlyeq$-descending order, then the elements of G_{i_1} in $\preccurlyeq$-descending order, and so on. This sequence can be obtained using $\Delta_1(S, \preccurlyeq)$-comprehension, and it has length $\mathbb{N}$, because $S \subseteq_{\mathrm{cf}} A \subseteq_{\mathrm{cf}} \mathbb{N}$, so $\sup_{j\in J} |G_{i_j}| = \sup_{i\in I} |G_i| = \mathbb{N}$.

Basically the same argument shows that $\mathsf{RCA}_0^* + \mathsf{GGP}^2_2$ proves $\mathsf{CRT}^2_2 \Rightarrow \ell$-$\mathsf{CRT}^2_2$: one firstly applies GGP^2_2 to some $c\colon [\mathbb{N}]^2 \to 2$ to get a grouping $(G_i)_{i\in I}$, and then uses CRT^2_2 for c restricted to the set $A = \{\min(G_i)\colon\ i \in I\}$ to obtain a set $S = \{\min(G_{i_j})\colon\ j \in J\}$, on which c is stable. It is immediate to check that c is also stable on the set $\bigcup_{j\in J} G_{i_j}$, which has cardinality $\mathbb{N}$.

However, the instance to which we apply GGP^2_2 in the second argument is not necessarily transitive, so Lemma 3.4 only implies $\mathsf{WKL}_0^* + \neg\mathsf{I}\Sigma^0_1 \vdash \mathsf{CRT}^2_2 \Rightarrow \ell$-$\mathsf{CRT}^2_2$ and thus $\mathsf{WKL}_0^* \vdash \mathsf{CRT}^2_2 \Leftrightarrow \ell$-$\mathsf{CRT}^2_2$. □

We have already seen that the other long principles we are considering imply $\mathsf{I}\Sigma^0_1$, but let us note which step precisely in the above proof would fail for them. Consider e.g. ℓ-RT^2_2. Given the set $S = \{\min(G_{i_j})\colon\ j \in J\}$ as above we have to construct a long solution set for some $c\colon [\mathbb{N}]^2 \to 2$ using all elements of each group G_{i_j}, $j \in J$. However, we have no control over c within individual groups, so it may well happen e.g. that $c \restriction [G_{i_j}]^2 = 0$ but $c \restriction [S]^2 = 1$.

Theorem 3.7 allows us to show that ℓ-$\mathsf{ADS}^{\mathrm{seq}}$ and ℓ-CRT^2_2 are weak principles in the sense that they are partially conservative over RCA_0^*.

Corollary 3.8. $\mathsf{RCA}_0^* + \ell$-$\mathsf{ADS}^{\mathrm{seq}}$ *and* $\mathsf{RCA}_0^* + \ell$-CRT^2_2 *are* $\forall\Pi^0_3$*-conservative over* RCA_0^**. In particular, they do not imply* $\mathsf{I}\Sigma^0_1$*.*

Proof. The $\forall\Pi^0_3$-conservativity is immediate from Theorems 2.8, 2.16 and 3.7.

Every theory that is at least Π_1-conservative over RCA_0^* is consistent with $\neg\mathrm{Con}(\mathsf{I}\Delta_0)$ and thus cannot imply even $\mathsf{I}\Delta_0 + \mathsf{supexp} \subsetneq \mathsf{I}\Sigma_1$. □

Our results from Chapter 2 and the present section on the relationships between the normal and long versions of $\mathsf{RT}^2_2, \mathsf{CAC}, \mathsf{ADS}$, and CRT^2_2 are summarized in Figure 1. One phenomenon apparent from the figure is that all of the principles considered up to this point either imply $\mathsf{I}\Sigma^0_1$ or are $\forall\Pi^0_3$-conservative over RCA_0^*.

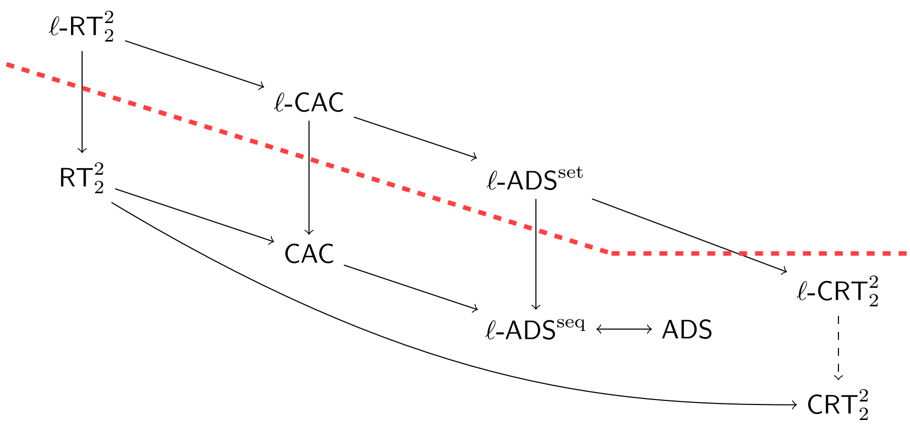

Figure 3.1: Summary of relations between the various versions of RT^2_2, CAC, ADS and CRT^2_2 over RCA^*_0. Solid arrows represent implications provable in RCA^*_0 that do not provably reverse in RCA^*_0. The dashed arrow represents an implication for which the reversal is open over RCA^*_0 but known over WKL^*_0. Also the implications from CAC and ADS to CRT^2_2 and from any of $\mathsf{RT}^2_2, \mathsf{CAC}, \mathsf{ADS}$ to ℓ-CRT^2_2 are open. All theories above the thick dashed line imply $\mathsf{I}\Sigma^0_1$, and all theories below the line are $\forall\Pi^0_3$-conservative over RCA^*_0.

The main open problems related to normal versions of our principles concern RT^n_k and CRT^2_2 and have already been stated in Chapter 2. Among the long principles, questions about those that imply $\mathsf{I}\Sigma^0_1$ move us back to the traditional realm of reverse mathematics over RCA_0. As for the weaker long principles, an important matter is to settle the status of GGP^2_2.

Question 3.9. Does $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$ imply GGP^2_2? Is GGP^2_2 equivalent to WKL^*_0 over $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$?

A more specialized but related group of problems concerns ℓ-CRT^2_2.

Question 3.10. Is ℓ-CRT^2_2 equivalent to CRT^2_2 over RCA^*_0? Does it follow from $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2$?

By the argument used to prove Theorem 3.7, if GGP^2_2 is provable in $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$, then both parts of Question 3.10 have a positive answer.

3.2 The cohesive set principle

The cohesive set principle COH has already been defined in Section 1.3. We start with conservativity issues. In [3], Belanger extended results from [5] and [8] in the following theorem.

Theorem 3.11 (Belanger, [3])**.** *For any $n \geqslant 2$, COH is Π^1_1-conservative over both $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_n$ and $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_n$.*

Together with Theorem 1.22 (d) proved in [5], this determines the first-order part of COH over each level of induction and collection axioms, except for the lowest one, that is RCA^*_0. Thus, Belanger asked whether COH is also Π^1_1-conservative over RCA^*_0. A negative answer to this question follows immediately from our Theorem 2.17 and the fact that COH implies CRT^2_2 over RCA^*_0: it is easy to see that the standard proof in RCA_0 of the implication $\mathsf{COH} \Rightarrow \mathsf{CRT}^2_2$ mentioned in Section 1.3 does not use any induction axioms.

Corollary 3.12. $\mathsf{RCA}^*_0 + \mathsf{COH}$ *is not Π_5-conservative over* RCA^*_0*.*

In the rest of this section we demonstrate that the behaviour of COH over RCA^*_0 is much different from that of the normal principles studied in Chapter 2. In fact, in terms of our classification of Ramsey-theoretic statements into normal and long principles, COH has some aspects of both. On the one hand, the solution set C is only required to be unbounded but not to have cardinality $\mathbb{N}$. On the other hand, C is required to behave in a certain way with respect to each element of the sequence $(R_n)_{n\in\mathbb{N}}$, which obviously has length $\mathbb{N}$. We will show that the latter feature of COH has an interesting consequence for models of $\mathsf{RCA}^*_0 + \mathsf{COH}$. Namely, in contrast to all the normal principles considered in Section 2.3, COH is never 'computably true', i.e. it never holds in a model of the form $(M, \Delta_1\text{-Def}(M))$. Thus, we will obtain a possibly surprising result that, over RCA^*_0, COH does not follow from RT^n_k for any $n, k \in \omega$. This stands in stark contrast to the situation over RCA_0 where COH is considered a rather weak principle: it is Π^1_1-conservative over the base theory and follows even from ADS.

We prove the result by means of a detour through what is called the Σ^0_2-separation principle in [3].

Σ^0_2-separation *For every two disjoint Σ^0_2-sets A_0, A_1 there exists a Δ^0_2-set B such that $A_0 \subseteq B$ and $A_1 \subseteq \overline{B}$.*

Note that this is a Π^1_2 sentence: we quantify over $\mathcal{L}_{\mathrm{II}}$-definitions which may contain set parameters, so, in fact, we say something like 'for every set A and every Σ^0_2 definition using A as a parameter...'. Let us stress that the first-order statement which we could call Σ_2-separation is false: there exist two disjoint Σ_2-sets that cannot be separated by any Δ_2-set.

It was shown in [3] that COH is equivalent to Σ^0_2-separation over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ and that the implication from COH to Σ^0_2-separation works over RCA_0. Below, we verify that this implication remains valid over RCA^*_0. On the other hand, we show that $\mathrm{B}\Sigma_1+\mathsf{exp}$ is enough to prove the existence of two disjoint Σ_2-sets that are Δ_2-inseparable. Then, we will conclude that for any structure M satisfying $\mathsf{B}\Sigma_1+\mathsf{exp}$, the second-order structure $(M, \Delta_1\text{-Def}(M))$ satisfies the negation of the Σ^0_2-separation principle and hence also satisfies $\neg\mathsf{COH}$.

Lemma 3.13. RCA^*_0 *proves that* COH *implies* Σ^0_2*-separation.*

Proof. We will follow the structure of the proof in RCA_0 described in [3] (which is based on [27]), pointing out where we have to depart from it. We work in $\mathsf{RCA}^*_0+\mathsf{COH}$ and prove the dual formulation of Σ^0_2-separation: if A_0 and A_1 are Π^0_2-sets such that $A_0\cup A_1=\mathbb{N}$, then there exists a Δ^0_2-set B such that $B\subseteq A_0$ and $\overline{B}\subseteq A_1$.

Assume that:

$$A_0 = \{n\in\mathbb{N}\colon \forall y\,\exists z\,\theta_0(n,y,z)\},$$
$$A_1 = \{n\in\mathbb{N}\colon \forall y\,\exists z\,\theta_1(n,y,z)\},$$

where θ_0, θ_1 are Δ^0_0, and for each $n\in\mathbb{N}$ it holds that $n\in A_0$ or $n\in A_1$.

The argument in RCA_0 would now make use of a Δ^0_1-definable function $f\colon\mathbb{N}\times\mathbb{N}\to 2$ such that for every n,

$$\{s\in\mathbb{N}\colon f(n,s)=i\}\text{ is unbounded iff } n\in A_i.$$

It seems unclear whether we can have access to such a function in RCA^*_0. However, we can use a witness comparison argument to find a Δ^0_1-definable $f\colon\mathbb{N}\times\mathbb{N}\to 2$ such that for every n,

$$\text{if } \{s\in\mathbb{N}\colon f(n,s)=i\}\text{ is unbounded, then } n\in A_i.$$

Namely, for every n at least one of $\forall y\,\exists z\,\theta_0(n,y,z)$ and $\forall y\,\exists z\,\theta_1(n,y,z)$ holds. So, by $\mathsf{B}\Sigma_1$, for every n and s there must exist some number w_0 such that $\forall y\leqslant s\,\exists z\leqslant w_0\,\theta_0(n,y,z)$ or some w_1 such that $\forall y\leqslant s\,\exists z\leqslant w_1\,\theta_1(n,y,z)$. Hence we can define a total function f as follows. Let $f(n,s)=0$ if there is such w_0 and there is no such w_1 with $w_1<w_0$. Otherwise put $f(n,s)=1$.

Now consider the Δ^0_1-definable sequence of sets $(R_n)_{n\in\mathbb{N}}$, where for every index $n\in\mathbb{N}$ we define $R_n=\{s\in\mathbb{N}\colon f(n,s)=0\}$. Let C be a cohesive set for this sequence. Notice that if $C\subseteq^* R_n$, then R_n is unbounded and hence $n\in A_0$, and analogously if $C\subseteq^*\overline{R}_n$ then $n\in A_1$. Let B be the set of indices of those sets R_n for which $C\subseteq^* R_n$:

$$B=\{n\in\mathbb{N}\colon \exists k\,\forall\ell\geqslant k\,(\ell\in C\Rightarrow \ell\in R_n)\}.$$

Clearly, B is Σ_2-definable in C, but since C is cohesive for $(R_n)_{n\in\mathbb{N}}$, $\overline{B}$ is also Σ_2-definable in C in an analogous way. Thus, B is Δ^0_2. Moreover, it follows from the construction that if $n\in B$ then $n\in A_0$ and if $n\in\overline{B}$ then $n\in A_1$. ☐

Lemma 3.14. $\mathsf{B}\Sigma_1+\mathsf{exp}$ *proves that there exist two disjoint* Σ_2*-sets that cannot be separated by a* Δ_2*-set.*

Proof. We verify that an essentially standard proof of the existence of two Δ_2-inseparable disjoint Σ_2-sets (which is just a relativization to $0'$ of the usual proof of an analogous theorem for Σ_1-sets) goes through in $\mathsf{B}\Sigma_1 + \mathsf{exp}$. The computability-theoretic facts and notions needed for the proof to work were formalized within $\mathsf{B}\Sigma_1 + \mathsf{exp}$ in [10].

A *Turing functional* Φ is a Σ_1-set of tuples $\langle x, y, P, N\rangle$, where $x, y \in \mathbb{N}$ and P, N are disjoint finite sets. Turing functionals are constrained to be well-defined in the sense that for fixed x, P, N there is at most one y such that $\langle x, y, P, N\rangle \in \Phi$, and to be monotone in the sense that increasing P or N preserves membership in Φ. Intuitively, the sets P and N contain those numbers whose membership in an oracle was queried during a computation. Namely, given a Turing functional Φ and an oracle A, we say that $\Phi^A(x) = y$ if there exist $P \subseteq A$ and $N \subseteq \overline{A}$ such that $\langle x, y, P, N\rangle \in \Phi$.

We work in $\mathsf{B}\Sigma_1 + \mathsf{exp}$. Let $(\Phi_e)_{e\in\mathbb{N}}$ be an effective listing of all Turing functionals. Let A_0 be the Σ_2-set $\{e \in \mathbb{N} : \Phi_e^{0'}(e) = 0\}$, and let A_1 be the Σ_2-set $\{e \in \mathbb{N} : \Phi_e^{0'}(e) = 1\}$. Clearly, A_0 and A_1 are disjoint. We claim that they cannot be separated by a Δ_2-set.

Suppose that B is a Δ_2-set such that $A_0 \subseteq B$ and $A_1 \subseteq \overline{B}$. By [10, Corollary 3.1], provably in $\mathsf{B}\Sigma_1 + \mathsf{exp}$ the Δ_2-set B is *weakly recursive* in $0'$ in the following sense: there is some Turing functional Φ_{e_0} such that for every x, if $x \in B$ then $\Phi_{e_0}^{0'}(x) = 1$, and if $x \notin B$ then $\Phi_{e_0}^{0'}(x) = 0$. By the definition of A_0 and A_1, this implies that $\Phi_{e_0}^{0'}(e_0) = 0$ iff $\Phi_{e_0}^{0'}(e_0) = 1$, which is a contradiction, because $\Phi_{e_0}^{0'}$ is defined on every input and takes 0/1 values. ☐

Theorem 3.15. *Any model of* RCA_0^* *of the form* $(M, \Delta_1\text{-}\mathrm{Def}(M))$ *satisfies* $\neg\mathsf{COH}$.

Proof. This is an immediate consequence of Lemma 3.13 and Lemma 3.14. Lemma 3.13 says that if the structure $(M, \Delta_1\text{-}\mathrm{Def}(M))$ satisfied COH, then it would also satisfy the Σ_2^0-separation principle. The latter would contradict Lemma 3.14, because in $(M, \Delta_1\text{-}\mathrm{Def}(M))$ the Σ_2^0-sets are exactly the Σ_2-definable sets and the Δ_2^0-sets are exactly the Δ_2-definable sets. ☐

We note that the above theorem relativizes straightforwardly to arbitrary $A \subseteq M$, that is, COH fails in any model of RCA_0^* of the form $(M, \Delta_1\text{-}\mathrm{Def}(M, A))$.

Corollary 3.16. *For any* $n, k \geqslant 2$, $\mathsf{RCA}_0^* + \mathsf{RT}_k^n$ *does not imply* COH.

Proof. By Theorem 3.15, it is enough to note that there exists a structure satisfying $\mathsf{RCA}_0^* + \mathsf{RT}_k^n$ of the form $(M, \Delta_1\text{-}\mathrm{Def}(M))$. The existence of such a model follows from Theorems 1.12 and 2.9 and Lemma 2.8. In fact, we constructed such a model in the proof of Theorem 2.23. ☐

Corollary 3.17. RT_k^n, *for* $n, k \geqslant 2$, CAC, *and* ADS *are incomparable with* COH *with respect to implications over* RCA_0^*.

Another consequence of Theorem 3.15 is that an analogue of Theorem 2.9 does not hold for COH. In particular, it is not true that if $(M, \mathcal{X}) \vDash \mathsf{RCA}_0^*$ and

$(I, \mathrm{Cod}(M/I)) \vDash \mathsf{COH}$ for some Σ^0_1-cut I of M, then $(M, \mathcal{X}) \vDash \mathsf{COH}$, since in a model of $\neg\mathsf{I}\Sigma_1$ this would work in particular for $\mathcal{X} = \Delta_1\text{-Def}(M)$. On the other hand, using methods in the style of Section 2.2 it is easy to show that the converse implication still holds.

Proposition 3.18. *For every $(M, \mathcal{X}) \vDash \mathsf{RCA}^*_0$ and every proper Σ^0_1-cut I in $(M, \mathcal{X})$, if $(M, \mathcal{X}) \vDash \mathsf{COH}$, then $(I, \mathrm{Cod}(M/I)) \vDash \mathsf{COH}$.*

Proof. Suppose $(M, \mathcal{X}) \vDash \mathsf{RCA}^*_0 + \mathsf{COH}$ and I is a proper Σ^0_1-cut in $(M, \mathcal{X})$. Let $A \in \mathcal{X}$ be a cofinal subset of M enumerated in increasing order $A = \{a_i\}_{i \in I}$, as in Proposition 1.13.

Let $(R_i)_{i \in I}$ be a sequence of subsets of I that belongs to $\mathrm{Cod}(M/I)$. We define a sequence $(\widetilde{R}_n)_{n \in M}$ in the following way. If $n \in (a_{i-1}, a_i]$ for some $i \in I$, let

$$\widetilde{R}_n = \{x \in M \colon \exists j \in I \, (x \in (a_{j-1}, a_j] \wedge j \in R_i)\}.$$

The sequence $(\widetilde{R}_n)_{n \in M}$ is Δ_1-definable in A and the code for $(R_i)_{i \in I}$, so it belongs to $\mathcal{X}$. By COH in $(M, \mathcal{X})$, there exists an unbounded set $\widetilde{C} \in \mathcal{X}$ which is cohesive for $(\widetilde{R}_n)_{n \in M}$. Define $C = \{i \in I \colon \widetilde{C} \cap (a_{i-1}, a_i] \neq \emptyset\}$. Both C and $I \setminus C$ are Σ_1-definable in $\widetilde{C}$ and A, so $C \in \mathrm{Cod}(M/I)$ by Lemma 1.17. Moreover, $C \subseteq_{\mathrm{cf}} I$ and it is easy to check that C is cohesive for $(R_i)_{i \in I}$. □

Remark 3.19. The fact that COH is not implied over RCA^*_0 by any of the $\forall\Pi^0_3$-conservative principles, even RT^n_k for $n \geqslant 3$, suggests that it may be a rather strong principle over the weaker base theory. This conjecture was recently confirmed by Mengzhou Sun [52], who showed, building on our methods from Section 2.2, that $\mathsf{RCA}^*_0 + \mathsf{COH}$ implies $\mathsf{I}\Sigma^0_1$.

Remark 3.20. Our formulation of COH could be called a normal version, as the solution set is only required to be unbounded. However, by the result of Sun mentioned above, there is no point in considering a 'long' variant of COH separately, because in the presence of $\mathsf{I}\Sigma^0_1$ every cohesive set that is unbounded in fact has cardinality $\mathbb{N}$.

Chapter 4

A non-speedup result for CAC

In Chapter 2 we saw that RT^n_k for $n, k \geqslant 2$, CAC, ADS and CRT^2_2 have very similar logical strength when considered over the weak base theory RCA^*_0: they are all $\forall\Pi^0_3$- but not arithmetically conservative over RCA^*_0. Then, in Chapter 3, we saw that these principles can be divided into two groups based on the logical strength of their long versions.

In the present chapter we will see that there is yet another method of comparing our principles, namely proof size. Our main result is that $\mathsf{RCA}^*_0 + \mathsf{CAC}$, and hence $\mathsf{RCA}^*_0 + \mathsf{ADS}$, does not allow for essentially shorter proofs of $\forall\Pi^0_3$ sentences than are already available in RCA^*_0. This strongly contrasts with an earlier result on RT^2_2 in [32].

Section 4.1 provides basic background on proof size and relevant results about RT^2_2. Section 4.2 introduces the technique of forcing interpretations, an important tool in the study of proof size. In Section 4.3 we prove the main result of this chapter: $\mathsf{RCA}^*_0 + \mathsf{CAC}$ is polynomially simulated by RCA^*_0 with respect to $\forall\Pi^0_3$ sentences.

4.1 Basic definitions and earlier results

A conservation result saying that T_1 *is* Γ*-conservative over* T_2, where Γ is a class of sentences in the common language of T_1 and T_2, naturally leads to a question about proof size: does T_1 have smaller proofs of formulas from Γ than T_2? To make this question precise one has to fix a formal definition of a proof and its size. We will provide only the most basic information, and for a comprehensive introduction to the topic of proof size, we refer the reader to the survey by Pudlák [42].

We declare that our proof system is a Hilbert-style calculus with $\neg$ and $\Rightarrow$ as the only logical connectives and $\forall$ as the only quantifier. We will take advantage of this convention when proving theorems by induction on formula complexity.

However, to enhance readability, we will often use the other connectives and existential quantifier as abbreviations for their usual translations, e.g. $\varphi \wedge \psi$ is shorthand for $\neg(\varphi \Rightarrow \neg\psi)$. When we say '$\theta$ contains φ as a conjunct' we mean that θ is of the above form. Also, we adopt in the obvious way the definition of formula classes from Section 1.1 to our proof system with only one quantifier.

For a given language $\mathcal{L}$, our logical axioms are generated by finitely many schemes as in [18, Definition 0.10] with the difference that our only inference rule is modus ponens, and the rule for quantifiers is replaced with the additional axiom schemes $\forall x(\varphi \Rightarrow \psi) \Rightarrow (\forall x\varphi \Rightarrow \forall x\psi)$ and $\theta \Rightarrow \forall x\theta$, where x is not free in θ. All the languages that we work with have equality, and thus we also have among our logical axioms the usual equality axioms expressing that equality is symmetric, reflexive, transitive and that it is a congruence with respect to all function and relation symbols of a given language.

We say that $\delta = \langle \varphi_0, \ldots, \varphi_{n-1} \rangle$ is a proof of a formula φ in a theory T, if $\varphi_{n-1} = \varphi$ and for every $i < n$, the formula φ_i is either a logical axiom, an axiom of T or is obtained by modus ponens from formulas with smaller indices in δ. Here and in the rest of this chapter, by a theory we mean just a set of sentences. In particular, we do not require a theory to be deductively closed, since otherwise comparisons of proof size would become trivial.

We adopt the usual measure of the size of a formal proof, namely the number of symbols occurring in it (under some fixed natural representation of syntactic objects as words over a finite alphabet). We use vertical lines $|\cdot|$ to denote the size of proofs and other syntactic objects.

Throughout this chapter, we will use the word 'proof' both for syntactic objects of the formal languages that we study and for our arguments in the metalanguage to justify our lemmas and theorems. It should always be clear which meaning of 'proof' we have in mind.

It is an intriguing fact that many important results of the form 'T_1 is Γ-conservative over T_2' that were studied quantitatively fit one of just two scenarios described by the following definitions.

Definition 4.1. Let T_1 and T_2 be theories and let Γ be a set of sentences in their common language $\mathcal{L}(T_1) \cap \mathcal{L}(T_2)$. We say that T_1 *is polynomially simulated by* T_2 *with respect to* Γ if there exists a polynomial-time algorithm which, given as input a proof in T_1 of a sentence $\gamma \in \Gamma$, outputs a proof of γ in T_2.

Note that if T_1 is polynomially simulated by T_2 with respect to a set of sentences Γ, then in particular T_1 is Γ-conservative over T_2, and for every proof δ in T_1 of a sentence $\gamma \in \Gamma$ there exists a proof of γ in T_2 of size at most polynomially larger than $|\delta|$. Well-known conservation results that were strengthened to polynomial simulation are, for instance, arithmetical conservativity of RCA_0 over $\mathsf{I}\Sigma_1$ [26] and Π^1_1-conservativity of WKL_0 over RCA_0 [1, 17].

A contrasting situation is described by the next definition.

Definition 4.2. Let T_1 and T_2 be theories and let Γ be a set of sentences in their common language $\mathcal{L}(T_1) \cap \mathcal{L}(T_2)$. We say that T_1 *has non-elementary* *speedup over T_2 with respect to* Γ if for every elementary computable function f there exists a sentence $\gamma \in \Gamma$ and a proof δ of γ in T_1 such that every proof of γ in T_2 has size greater than $f(|\delta|)$.

Classical examples for non-elementary speedup are the following two conservative extensions: ACA_0 over PA and GB over ZF (the latter result is [43, Theorem 4.2], and the former is proved in the same way, cf. [26, Theorem 2.2]), as well as $\mathsf{I}\Sigma_1$ being Π_2-conservative over PRA [26, Theorem 2.9]. We will often omit the adjective 'non-elementary' and speak just about speedup.

We say that two theories T_1 and T_2 of the same language $\mathcal{L}$ are *polynomially equivalent* if they polynomially simulate each other with respect to all sentences of $\mathcal{L}$. Note that to prove this property for such theories it is enough to construct a polynomial-time algorithm which, given as input an axiom of T_1, outputs its proof in T_2, and vice versa. Clearly then, any two finite theories with the same deductive closure are polynomially equivalent.

In the rest of this chapter it will be convenient for us to work with finite theories. Since we will consider $\mathsf{I}\Delta_0 + \mathsf{exp}$, RCA_0^*, RCA_0 and similar theories that are naturally axiomatized by infinite schemes, let us briefly explain that all these theories (as defined in Section 1.1) are polynomially equivalent to finite sets of sentences.

For RCA_0, one uses the standard satisfaction predicates $\mathrm{Sat}_{\Sigma^0_1}(e, x, X)$ and $\mathrm{Sat}_{\Pi^0_1}(e, x, X)$, as described in Section 1.1. Let Θ be a finite fragment of RCA_0, containing PA^-, exp and finitely many instances of the Σ^0_1-induction scheme, such that it proves the compositional conditions for the predicates $\mathrm{Sat}_{\Sigma^0_1}(e, x, X)$ and $\mathrm{Sat}_{\Pi^0_1}(e, x, X)$ as in [18, Definitions I.1.71 and I.1.74], as well as basic properties of the exponential function. It follows that for every Γ-formula φ, where Γ is Σ^0_1 or Π^0_1, the finite theory Θ proves Tarski's biconditional:

$$\forall X\, \forall x \big(\mathrm{Sat}_\Gamma(\ulcorner \varphi \urcorner, x, X) \Leftrightarrow \varphi(x, X)\big). \tag{4.1}$$

The usual proof of (4.1) goes by induction on formula complexity, and each step of induction relies on the corresponding clause of the compositional definition of satisfaction. One can easily verify that such a proof in Θ can be constructed in time polynomial in $|\varphi|$. We skip the details of the algorithm, as we will see very similar arguments later on, e.g. the one for Lemma 4.26.

Let us write Θ' for Θ extended by the following instance of the Σ^0_1-induction scheme:

$$\forall Y\, \forall e \big(\mathrm{Sat}_{\Sigma^0_1}(e, 0, Y) \wedge \forall x \big(\mathrm{Sat}_{\Sigma^0_1}(e, x, Y) \Rightarrow \mathrm{Sat}_{\Sigma^0_1}(e, x+1, Y)\big) \Rightarrow \forall x\, \mathrm{Sat}_{\Sigma^0_1}(e, x, Y)\big), \tag{4.2}$$

and of the instance of Δ^0_1-comprehension scheme:

$$\forall Y\, \forall e, e' \big(\forall x \big(\mathrm{Sat}_{\Sigma^0_1}(e, x, Y) \Leftrightarrow \mathrm{Sat}_{\Pi^0_1}(e', x, Y)\big) \Rightarrow \exists X\, \forall x \big(x \in X \Leftrightarrow \mathrm{Sat}_{\Sigma^0_1}(e, x, Y)\big)\big), \tag{4.3}$$

where e and e' are codes for Σ^0_1 and Π^0_1 formulas, respectively. Now it is straightforward to describe a polynomial-time algorithm using (4.1) that constructs proofs in Θ' of every instance of $\mathsf{I}\Sigma^0_1$ or Δ^0_1-comprehension.

The cases of $\mathsf{I}\Delta_0 + \mathsf{exp}$ and RCA^*_0 are similar, though one needs a little bit more care without assuming Σ_1-induction. Namely, the usual satisfaction predicate $\mathrm{Sat}_{\Delta_0}(e, x)$ is not Δ_0 as a formula of e and x, so the replacement of the Δ_0-induction scheme by its single instance is not as straightforward as for $\mathsf{I}\Sigma_1$. However, by [18, Theorem V.5.4], there is a Δ_0 formula $\eta(e, x, u)$ that defines satisfaction for a Δ_0 formula e evaluated on x, provided that the extra argument u is sufficiently large with respect to e and x, as given by an exponential term. There is a fixed fragment Ξ of $\mathsf{I}\Delta_0 + \mathsf{exp}$ such that one can find a proof in Ξ of (a version of) Tarski's biconditional (4.1) using η in time polynomial in the size of a given Δ_0 formula φ. Let us agree that Ξ contains, in addition to $\mathsf{PA}^- + \mathsf{exp}$, finitely many instances of the following Π_1 formulation of the Δ_0-induction scheme (rather then the usual scheme (1.1) from Section 1.1):

$$\Big\{\forall x\, \forall \overline{z}\, \Big(\big(\varphi(0, \overline{z}) \wedge \forall y\!<\!x\, \big(\varphi(y, \overline{z}) \Rightarrow \varphi(y+1, \overline{z})\big)\big) \Rightarrow \varphi(x, \overline{z})\Big) : \ \varphi \in \Delta_0\Big\}. \tag{4.4}$$

Then Ξ together with induction for η proves each axiom of $\mathsf{I}\Delta_0$, and the proof can be constructed in polynomial time. A similar process can be applied to RCA^*_0, this time allowing a second-order variable in the definition of satisfaction and in the analogous Π^0_1 formulation of $\mathsf{I}\Delta^0_0$. This assumption on the form of bounded induction axioms will be useful later in order to simplify proofs of Lemmas 4.27 and 4.32.

From now on, we will use the names $\mathsf{I}\Delta_0 + \mathsf{exp}$, RCA^*_0, RCA_0, WKL^*_0, WKL_0 for the finite theories described above that are polynomially equivalent to the usual axiomatizations introduced in Section 1.1.

The chief motivation for this chapter is earlier work of Kołodziejczyk, Wong and Yokoyama on RT^2_2. In [32] they showed that the behaviour of RT^2_2 with respect to proof size depends on the base theory.

Theorem 4.3 ([32, Theorem 3.1])**.** $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2$ *has non-elementary speedup over* RCA^*_0 *with respect to* Σ_1 *sentences.*

Theorem 4.4 ([32, Theorem 2.1])**.** $\mathsf{WKL}_0 + \mathsf{RT}^2_2$ *is polynomially simulated by* RCA_0 *with respect to* $\forall\Pi^0_3$ *sentences.*

The strengthening of the simulated theory from $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ to $\mathsf{WKL}_0 + \mathsf{RT}^2_2$ in Theorem 4.4 comes essentially for free, and we will comment on that in Section 4.3.1.

Now, it is very natural to ask about proof size for the other combinatorial principles that we have studied. Clearly, over RCA_0 there is nothing left to prove because for every consequence P of $\mathsf{WKL}_0 + \mathsf{RT}^2_2$, the theory $\mathsf{WKL}_0 + \mathsf{P}$ is also polynomially simulated by RCA_0 with respect to $\forall\Pi^0_3$ sentences. Except for RT^2_2, the situation over RCA^*_0 has not been studied until now. We will focus on CAC since it is stronger than ADS and, as opposed to CRT^2_2, it has a well-defined finite version that will play a key role in our analysis.

There is a clue that the behaviour of CAC with respect to proof size might be quite different than that of RT^2_2. Namely, the proof of Theorem 4.3 relies heavily on the classical exponential lower bound on Ramsey numbers for the finite version of RT^2_2: there exists a 2-colouring of $[2^{\frac{k}{2}}]^2$ without a homogeneous subset containing k elements. This fact is used in [32] to show that over RCA^*_0, RT^2_2 implies that the cut I^0_1 is closed under exponentiation. Then, by some classical results including shortening of cuts and the so-called finitistic Gödel theorem, one can witness the speedup of $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2$ over RCA^*_0 by either sentences stating the existence of the value of a fast-growing function on large numbers or by finite consistency statements. On the contrary, the upper bound on the finite version of CAC is only polynomial due to the following theorem by Dilworth [12].

Theorem 4.5 (Dilworth)**.** *In every finite partial order $(P, \leqslant)$ the size of the largest antichain is equal to the smallest number n such that $(P, \leqslant)$ is a union of n chains.*

As an easy consequence of Dilworth's theorem we get the following upper bound on the finite version of CAC: in every partial order on a set of size $k(k-1)$ there exists a chain or an antichain of size k. This has the effect that CAC does not imply the closure of the cut I^0_1 under any super-polynomially growing function [15, Theorem 3.16].

Thus, the argument from [32] used to prove Theorem 4.3 will not go through for CAC. On the other hand, one may try to use the polynomial upper bound for the finite version of CAC analogously to how an upper bound on finite Ramsey's theorem expressed in terms of so-called α-large sets was used in the proof of Theorem 4.4 in [32]. That proof involves constructing a forcing interpretation that simulates the model-theoretic argument from [33]. As we will see, one can take advantage of this strategy also in the case of CAC over the weaker base theory RCA^*_0. Our goal for the rest of this chapter is to prove the following result.

Theorem 4.6. $\mathsf{WKL}^*_0 + \mathsf{CAC}$ *is polynomially simulated by* RCA^*_0 *with respect to* $\forall\Pi^0_3$ *sentences.*

In the next section we will introduce the technique of forcing interpretation and then, in Section 4.3, we will continue our discussion of CAC.

4.2 Forcing interpretations

The general idea of forcing is to prove the existence of a mathematical object X with some desired properties not by a direct construction but by arguing that every typical object of a given type satisfies these properties. A common scenario is as follows. One defines an infinite partial order $(P, \leqslant)$, whose elements are called *conditions* and seen as approximations to X. One also specifies a family $\{D_k\}_{k\in K}$ of dense subsets of P, where a set $D \subseteq P$ is called *dense* if for every $p \in P$ there exists $q \in D$ with $q \leqslant p$. Each set from such a family corresponds to some property that the desired object X should satisfy. Eventually, X is obtained from a so-called *generic filter* $G \subseteq P$, where 'generic' refers to the fact that G has a nonempty intersection with all members of the family $\{D_k\}_{k\in K}$. The final object X obtained from such a generic filter is also called generic. When a structure N is obtained by adding a generic object to another structure M (and possibly closing under appropriate operations), then the former is called a *generic model* and the latter a *ground model*.

Forcing arguments are especially well-known for establishing independence results in set theory, but they are also very fruitful in many other areas of mathematical logic. In the context of second-order arithmetic forcing is often used for proving conservation results by means of ω-extensions. An early example of this technique is Harrington's proof of Theorem 1.4. In its main step one defines a partial order of infinite subtrees of a given infinite binary tree T, with the property that every generic filter is an infinite branch in T. In [1] Avigad introduced the method of *forcing interpretation*, by which he transformed the above model-theoretic forcing construction into a syntactical one, and thus obtained a quantitative strengthening of Harrington's theorem (an alternative argument was given by Hájek in [17]):

Theorem 4.7 (Avigad, Hájek)**.** WKL_0 *is polynomially simulated by* RCA_0 *with respect to* Π^1_1 *sentences.*

The general structure of a forcing interpretation is similar to that of a usual interpretation of a theory T_1 in a theory T_2. Recall that an interpretation in the usual sense consists of an $\mathcal{L}(T_2)$ definition of the domain of the interpretation and a function that translates each $\mathcal{L}(T_1)$ formula φ to an $\mathcal{L}(T_2)$ formula φ^*, in such a way that T_2 proves the translations of all axioms of T_1. The function $\varphi \mapsto \varphi^*$ is defined by recursion on formula complexity so that it commutes with logical connectives, and quantifiers are translated to their relativisations to the domain of the interpretation (cf. [18, Definition III.1.2]).

For a forcing interpretation, one firstly defines a partially ordered set Cond of *conditions*, a set Name of *names*, and a relation $s \Vdash v\!\downarrow$ on Cond×Name, pronounced 's forces v to be a valid name'. Then, for every formula φ of $\mathcal{L}(T_1)$, one recursively defines the relation $s \Vdash \varphi$, read as 's forces that φ holds'. Intuitively, $s \Vdash \varphi$ means that any model provided by a generic filter containing s satisfies φ. Finally, to obtain a forcing interpretation of T_1 in T_2, one has to show that T_2 proves that every condition forces all axioms of T_1.

Just like in the case of an ordinary interpretation, a forcing interpretation can be used to prove a conservation result. To this aim, one has to prove in T_2 a reflection scheme expressing 'if φ is forced by every condition then φ holds', for φ from a certain class of formulas. As we will see in Section 4.2.2, if one also shows that there is a polynomial-time algorithm which constructs such a forcing interpretation as well as proofs in T_2 of the reflection scheme, then one obtains a strengthening of the conservation result to one of polynomial simulation.

4.2.1 Main definitions

Our presentation of forcing interpretations follows closely the one from [32]. Definition 4.8 below is basically [32, Definitions 1.5 and 1.6], which in turn is based on [1, Definition 4.2]. The most important difference compared to [32] is that we define the forcing translation for all formulas of a given language straightforwardly without a detour through so-called simple formulas, which are roughly translations into a relational language. The detour was made in order to avoid potential ambiguities related to translating formulas with complex terms. Here, without simple formulas, we will not be able to introduce the method of forcing interpretation in full generality, but we feel that this way of presentation is more intuitive. Moreover, it will be clear from our constructions that one does not need simple formulas in many natural cases, especially in a situation like ours when terms of the interpreted language are in direct correspondence with terms of the interpreting one. In the following, we will comment on those fragments of our presentation whose counterparts in [32] make substantial use of simple formulas.

We follow standard conventions regarding forcing notation and use lower-case letters s, s', s'' etc. as variables for forcing conditions. Our default variables for names are both lower- and (when the interpreted theory is in the two-sorted language) upper-case letters such as v, w, V, W. We write $\overline{v}$, $\overline{V}$ for finite tuples of names.

Definition 4.8 ([32, Definitions 1.5 and 1.6])**.** A *forcing translation* τ *of a language* $\mathcal{L}_1$ *to a language* $\mathcal{L}_2$ consists of $\mathcal{L}_2$-formulas:

$$s \in \mathrm{Cond}_\tau, \quad s' \trianglelefteq_\tau s, \quad v \in \mathrm{Name}_\tau, \quad s \Vdash_\tau v\!\downarrow, \quad s \Vdash_\tau \alpha(\overline{v}), \tag{4.5}$$

for every atomic $\mathcal{L}_1$ formula $\alpha(\overline{x})$, such that the following conditions are satisfied.

(FT1) For every formula in (4.5), the only free variables are exactly those shown.

(FT2) $s' \trianglelefteq_\tau s$ contains $s' \in \mathrm{Cond}_\tau \wedge s \in \mathrm{Cond}_\tau$ as a conjunct.

(FT3) $s \Vdash_\tau v\!\downarrow$ contains $s \in \mathrm{Cond}_\tau \wedge v \in \mathrm{Name}_\tau$ as a conjunct.

(FT4) $s \Vdash_\tau \alpha(\overline{v})$ contains $s \Vdash_\tau \overline{v}\!\downarrow$ as a conjunct, for every atomic formula $\alpha(\overline{v})$.

(FT5) If $\alpha(\overline{u}, v)$ is an atomic $\mathcal{L}_1$ formula and w is a variable, then the formulas

$$\big(s \Vdash_\tau \alpha(\overline{u}, v)\big)[w/v] \quad \text{and} \quad s \Vdash_\tau \big(\alpha(\overline{u}, v)[w/v]\big)$$

are the same.

For complex formulas the forcing relation $\Vdash_\tau$ is defined inductively by the following clauses, where all formulas have exactly the free variables shown.

(FT6) $s \Vdash_\tau \neg\varphi(\overline{v})$ is

$$s \Vdash_\tau \overline{v}\!\downarrow \ \wedge \ \forall s' \trianglelefteq_\tau s \ \big(s' \nVdash_\tau \varphi(\overline{v})\big).$$

(FT7) $s \Vdash_\tau \varphi(\overline{v}, \overline{u}) \Rightarrow \psi(\overline{v}, \overline{w})$ is

$$s \Vdash_\tau \overline{v}\!\downarrow \;\wedge\; s \Vdash_\tau \overline{u}\!\downarrow \;\wedge\; s \Vdash_\tau \overline{w}\!\downarrow \;\wedge\; \forall s' \trianglelefteq_\tau s\; \exists s'' \trianglelefteq_\tau s' \big(s' \Vdash_\tau \varphi(\overline{v}, \overline{u}) \Rightarrow s'' \Vdash_\tau \psi(\overline{v}, \overline{w})\big).$$

(FT8) $s \Vdash_\tau \forall w\, \varphi(w, \overline{v})$ is

$$s \Vdash_\tau \overline{v}\!\downarrow \;\wedge\; \forall w\, \forall s' \trianglelefteq_\tau s\, \exists s'' \trianglelefteq_\tau s' \big(s' \Vdash_\tau w\!\downarrow \Rightarrow\; s'' \Vdash_\tau \varphi(w, \overline{v})\big).$$

In the above definition $s \Vdash_\tau \overline{v}\!\downarrow$ is shorthand for the formula $\bigwedge_{i=1}^{n} s \Vdash_\tau v_i\!\downarrow$, where n is the length of the tuple $\overline{v}$. We write $s \nVdash_\tau \varphi$ for $\neg(s \Vdash_\tau \varphi)$. Expressions like $\forall s \in \mathrm{Cond}_\tau$ and $\forall v \in \mathrm{Name}_\tau$ are abbreviations for $\forall s\,(s \in \mathrm{Cond}_\tau \Rightarrow \dots)$ and $\forall v\,(v \in \mathrm{Name}_\tau \Rightarrow \dots)$. Later, to simplify notation, we will often relax the requirement on displayed free variables and will not distinguish between the free variables occurring in the antecedent and the consequent of an implication.

By straightforward induction on formula complexity one verifies that the technical properties (FT4) and (FT5) essentially hold for all formulas of a given language $\mathcal{L}_1$.

Lemma 4.9 ([32, Lemma 1.7])**.** *Let τ be a forcing translation of a language $\mathcal{L}_1$ to a language $\mathcal{L}_2$. Then:*

(a) $s \Vdash_\tau \varphi(\overline{v})$ contains $s \Vdash_\tau \overline{v}\!\downarrow$ as a conjunct, for every $\mathcal{L}_1$ formula φ.

(b) If $\varphi(\overline{u}, v)$ is an $\mathcal{L}_1$ formula and w is a variable, then

$$\big(s \Vdash_\tau \varphi(\overline{u}, v)\big)[w/v] \quad \textit{and} \quad s \Vdash_\tau \big(\varphi(\overline{u}, v)[w/v]\big)$$

differ only up to renaming bound variables, and the variable w is substitutable for v in $\varphi(\overline{u}, v)$ if and only if it is so in $s \Vdash_\tau \varphi(\overline{u}, v)$.

The next definition states what is needed for a forcing translation to be a forcing interpretation. To put it simply, the interpreting theory has to prove that the set of forcing conditions is a preorder, that the rules of first-order logic are preserved and that the axioms of an interpreted theory are always forced.

Definition 4.10 ([32, Definition 1.8])**.** Let T_1 and T_2 be theories and let τ be a forcing translation of $\mathcal{L}(T_1)$ to $\mathcal{L}(T_2)$. Then τ is a *forcing interpretation of T_1 in T_2* if the following properties are provable in T_2.

The relation $\trianglelefteq_\tau$ is a nonempty preorder:

(FI1) $\exists s\big(s \in \mathrm{Cond}_\tau\big)$,

(FI2) $\forall s \in \mathrm{Cond}_\tau \big(s \trianglelefteq_\tau s\big)$,

(FI3) $\forall s, s', s'' \in \mathrm{Cond}_\tau \big(s'' \trianglelefteq_\tau s' \wedge s' \trianglelefteq_\tau s \Rightarrow s'' \trianglelefteq_\tau s\big)$.

Any generic model is nonempty:

(FI4) $\forall s\in\mathrm{Cond}_\tau\ \exists s'\trianglelefteq_\tau s\,\exists v\in\mathrm{Name}_\tau\ s'\Vdash_\tau v\downarrow$.

The forcing relation $\Vdash_\tau$ is monotone:

(FI5) $\forall s,s'\in\mathrm{Cond}_\tau\ \forall v\in\mathrm{Name}_\tau\big(s\Vdash_\tau v\downarrow\wedge s'\trianglelefteq_\tau s\ \Rightarrow\ s'\Vdash_\tau v\downarrow\big)$,

(FI6) $\forall s,s'\in\mathrm{Cond}_\tau\ \forall\overline{v}\in\mathrm{Name}_\tau\big(s\Vdash_\tau\alpha(\overline{v})\ \wedge\ s'\trianglelefteq_\tau s\ \Rightarrow\ s'\Vdash_\tau\alpha(\overline{v})\big)$,
for each atomic formula $\alpha(\overline{x})$ of $\mathcal{L}(T_1)$.

The axioms of equality are forced, and the values of functions are defined:

(FI7) $\forall s\in\mathrm{Cond}_\tau\ \forall v\in\mathrm{Name}_\tau\ \big(s\Vdash_\tau v\downarrow\Rightarrow\ s\Vdash_\tau v=v\big)$,

(FI8) $\forall s\in\mathrm{Cond}_\tau\ \forall v,v'\in\mathrm{Name}_\tau\ \big(s\Vdash_\tau v=v'\ \Rightarrow\ s\Vdash_\tau v'=v\big)$,

(FI9) $\forall s\in\mathrm{Cond}_\tau\ \forall v,v',v''\in\mathrm{Name}_\tau\ \big(s\Vdash_\tau v=v'\ \wedge\ v'=v''\ \Rightarrow\ s\Vdash_\tau v=v''\big)$,

(FI10) $\forall s\in\mathrm{Cond}_\tau\ \forall\overline{v}\in\mathrm{Name}_\tau\ \Big(s\Vdash_\tau\overline{v}\downarrow\Rightarrow$
$\big(\forall s'\trianglelefteq_\tau s\,\exists s''\trianglelefteq_\tau s'\,\exists w\in\mathrm{Name}_\tau\ (s''\Vdash_\tau f(\overline{v})=w)\ \wedge$
$s\Vdash_\tau\forall w,w'\,(w=f(\overline{v})\ \wedge\ w'=f(\overline{v})\ \Rightarrow\ w=w')\big)\Big)$,
for each function symbol f of $\mathcal{L}(T_1)$,

(FI11) $\forall s\in\mathrm{Cond}_\tau\ \forall\overline{u},\overline{v},w\in\mathrm{Name}_\tau\ \big(s\Vdash_\tau w=t(\overline{v})\ \Rightarrow$
$\big(s\Vdash_\tau\alpha(\overline{u},w)\ \Leftrightarrow\ s\Vdash_\tau\alpha(\overline{u},t(\overline{v}))\big)\big)$,
for each term $t(\overline{x})$ and each atomic formula $\alpha(\overline{y},z)$ of $\mathcal{L}(T_1)$.

Density conditions:

(FI12) $\forall s\in\mathrm{Cond}_\tau\,\forall v\in\mathrm{Name}_\tau\ \big(\forall s'\trianglelefteq_\tau s\,\exists s''\trianglelefteq_\tau s'\ (s''\Vdash_\tau v\downarrow)\ \Rightarrow\ s\Vdash_\tau v\downarrow\big)$,

(FI13) $\forall s\in\mathrm{Cond}_\tau\,\forall\overline{v}\in\mathrm{Name}_\tau\ \Big(\forall s'\trianglelefteq_\tau s\,\exists s''\trianglelefteq_\tau s'\ (s''\Vdash_\tau\alpha(\overline{v}))\ \Rightarrow\ s\Vdash_\tau\alpha(\overline{v})\downarrow\Big)$,
for each atomic formula $\alpha(\overline{x})$ of $\mathcal{L}(T_1)$.

The axioms of T_1 are forced:

(FI14) $\forall s\in\mathrm{Cond}_\tau\ s\Vdash_\tau\sigma$,
for each axiom σ of T_1.

A forcing translation τ that satisfies conditions (FI1)-(FI13) is called a *forcing interpretation of* $\mathcal{L}(T_1)$ *in* T_2. It will follow from Lemma 4.17 that such a τ is a forcing interpretation of pure logic formulated in the language $\mathcal{L}(T_1)$.

Let us comment on the density conditions. If $\forall s'\trianglelefteq_\tau s\,\exists s''\trianglelefteq_\tau s'\ (s''\Vdash_\tau\varphi(\overline{v}))$ holds, then the set $\{s^*\in\mathrm{Cond}_\tau\colon s^*\trianglelefteq_\tau s\ \wedge\ s^*\Vdash_\tau\varphi\}$ is dense in the preorder $\trianglelefteq_\tau$ restricted to the set $\{s^*\in\mathrm{Cond}_\tau\colon s^*\trianglelefteq_\tau s\}$. In such a case we say that $\varphi(\overline{v})$ *is densely forced below* s. Observe that this is equivalent to $s\Vdash_\tau\neg\neg\varphi$ (just unfold the definition of forcing negation (FT6) twice). Therefore, the density conditions (FI12) and (FI13), together with Lemma 4.14 (b) from the next section, guarantee that forcing the double negation of a formula implies forcing the formula.

If the set of forcing conditions has a greatest element, we denote it by $\mathbf{1}$, skipping the implicit subscript τ, as it should always be clear from the context. By Lemma 4.14 below, it follows that $\mathbf{1} \Vdash_\tau \varphi$ is equivalent to $\forall s \in \mathrm{Cond}_\tau (s \Vdash_\tau \varphi)$. Note that if φ is a universal statement $\forall x\, \psi(x)$, then $\forall s \in \mathrm{Cond}_\tau\, (s \Vdash_\tau \varphi)$ is implied by $\forall s \in \mathrm{Cond}_\tau\, \forall v \in \mathrm{Name}_\tau\, \big(s \Vdash_\tau v\!\downarrow \Rightarrow\ s \Vdash_\tau \psi(v)\big)$. In fact, the opposite implication holds as well, as can be shown using Lemma 4.14.

The next lemma states some general properties of forcing interpretations.

Lemma 4.11 ([32, Example 1.9 and the paragraph below it])**.** *Let T_1 and T_2 be theories and let σ be a sentence in $\mathcal{L}(T_2)$.*

(a) If there exists an interpretation of T_1 in T_2 in the usual sense, then there exists a forcing interpretation of T_1 in T_2.

(b) If there exist forcing interpretations of T_1 in $T_2 + \sigma$ and $T_2 + \neg\sigma$, then there exists a forcing interpretation of T_1 in T_2.

The first item says that interpretations between theories in the usual sense can be seen as a special case of forcing interpretations. The second claim says that forcing interpretations are closed under definition by cases.

4.2.2 Polynomial forcing interpretations

In the present section we explain how a forcing interpretation can be used to obtain a non-speedup result. There are two things that one needs to care about. Firstly, one has to devise a polynomial-time algorithm constructing a forcing interpretation. Then, one needs a kind of a reflection theorem, i.e. one has to show that if some formula is forced, then it is actually provable in the interpreting theory. Moreover, proofs of the instances of such a reflection scheme should also be constructed by a polynomial-time algorithm.

Let us firstly make precise what it means that a forcing interpretation is polynomial. One could simply modify Definition 4.10 by demanding that there exists a polynomial-time algorithm which constructs proofs in the interpreting theory of conditions (FI1)-(FI14) for the appropriate inputs. Note, however, that we need to describe such an algorithm only for conditions (FI6), (FI10), (FI11), (FI13), (FI14), which are given by schemes. The other conditions are expressed by single sentences, so the construction of their proofs takes a fixed amount of time. Thus, we will work with the following definition.

Definition 4.12. A forcing interpretation τ of a theory T_1 in a theory T_2 is called *polynomial* if there exists a polynomial-time algorithm which:

(i) given as input an atomic formula α of $\mathcal{L}(T_1)$, outputs proofs in T_2 of the instances of (FI6) and (FI13) for α;

(ii) given as input a function symbol f of $\mathcal{L}(T_1)$, outputs a proof in T_2 of the instance of (FI10) for f;

(iii) given as input a term t and an atomic formula α of $\mathcal{L}(T_1)$, outputs a proof in T_2 of the instance of (FI11) for t and α;

(iv) given as input an axiom σ of T_1 outputs a proof in T_2 of the instance of (FI14) for σ.

A forcing interpretation of $\mathcal{L}(T_1)$ in T_2 is *polynomial* if it satisfies conditions (i)-(iii) from the above definition.

Let us note that Definition 1.8 from [32] of a polynomial forcing interpretation consists only of condition (iv) of our definition. In that exposition this is enough: because of the restriction to simple formulas and the assumption that all languages have finitely many non-logical symbols, there are de facto only finitely many terms and atomic formulas to consider, so all the clauses of the definition of a forcing interpretation, except for the last one about forcing non-logical axioms, are given by single sentences.

Also, note that in Definition 4.12 we do not need to assume that τ is a 'polynomial forcing translation', which would mean that the translation $\varphi \mapsto (s \Vdash_\tau \varphi)$ is constructed by a polynomial-time algorithm. This property holds for any forcing translation satisfying clause (i) from Definition 4.12. Indeed, for an atomic formula α, the translation $s \Vdash_\tau \alpha$ can be read off from a proof of (FI6) for α, and for a complex formula φ one proceeds by induction, according to clauses (FT6)-(FT8) from Definition 4.8, and this clearly takes time polynomial in $|\varphi|$. Thus we will always assume tacitly that each algorithm that works with a polynomial forcing interpretation τ has a built-in polynomial-time procedure which, given as input a formula φ, outputs a forcing translation $s \Vdash_\tau \varphi$.

The following lemmas show that for any polynomial forcing interpretation, some basic properties can be verified uniformly in polynomial time. We provide only sketches of proofs, mainly to discuss the complexity of the verification procedures. For more details we refer the reader to [32, Section 1.3]. We will often use these lemmas without explicit reference.

Let us firstly observe that for any polynomial forcing interpretation we can feasibly construct a proof that a contradiction is never forced.

Lemma 4.13 ([32, Lemma 1.10])**.** *Let τ be a polynomial forcing interpretation of $\mathcal{L}(T_1)$ in T_2. Then there exists a polynomial-time algorithm which, given as input a formula φ of $\mathcal{L}(T_1)$, outputs a proof in T_2 of the sentence:*

$$\forall s \in \mathrm{Cond}_\tau\, \forall \overline{v} \in \mathrm{Name}_\tau\, \neg\big(s \Vdash \varphi(\overline{v}) \,\wedge\, s \Vdash \neg\varphi(\overline{v})\big). \tag{4.6}$$

Proof sketch. Given an $\mathcal{L}(T_1)$ formula φ, the algorithm constructs a proof of (4.6) using the definition of forcing negation (FT6). The proof does not depend on the shape of the formula φ in a substantial way: it uses just one template for all formulas of $\mathcal{L}(T_1)$, so it has size linear in $|s \Vdash_\tau \varphi|$. Thus, the whole procedure takes time polynomial in $|\varphi|$. □

The next two lemmas generalize the monotonicity (FI6) and density (FI13) conditions as well as the substitution property (FI11) to all formulas of the language of the interpreted theory.

Lemma 4.14 ([32, Lemma 1.11])**.** *Let τ be a polynomial forcing interpretation of $\mathcal{L}(T_1)$ in T_2. Then there exists a polynomial-time algorithm, which given as input a formula φ of $\mathcal{L}(T_1)$, outputs proofs in T_2 of the sentences:*

(a) $\forall s, s' \in \mathrm{Cond}_\tau \, \forall \overline{v} \, \big((s' \trianglelefteq_\tau s \, \wedge \, s \Vdash_\tau \varphi(\overline{v})) \, \Rightarrow \, s' \Vdash_\tau \varphi(\overline{v}) \big)$,

(b) $\forall s \in \mathrm{Cond}_\tau \, \forall \overline{v} \, \big(\forall s' \trianglelefteq_\tau s \, \exists s'' \trianglelefteq_\tau s' (s'' \Vdash \varphi(\overline{v})) \, \Rightarrow \, s \Vdash \varphi(\overline{v}) \big)$.

Proof sketch. Given a formula φ of $\mathcal{L}(T_1)$, the algorithm constructs proofs of (a) and (b) simultaneously by induction on formula complexity. If φ is atomic, then the algorithm simply applies conditions (FI6) and (FI13). If φ is a complex formula, then the algorithm uses the previously constructed proofs of (a) and (b) for the immediate subformulas of φ to run one of three fixed procedures, depending on whether φ is a negation, an implication or a universal formula. Each of these procedures relies on (FT6), (FT7) or (FT8), and depends only on the outermost connective or quantifier of φ, and thus takes time linear in $|s \Vdash_\tau \varphi|$. Since for every subformula of φ the proofs of (a) and (b) are constructed precisely once, the whole procedure takes time polynomial in $|\varphi|$. □

Lemma 4.15 ([32, Lemma 1.12])**.** *Let τ be a polynomial forcing interpretation of $\mathcal{L}(T_1)$ in T_2. Then there exists a polynomial-time algorithm which, given as input a term $t(\overline{x})$ of $\mathcal{L}(T_1)$ with exactly the free variables shown, and a formula $\varphi(\overline{y}, z)$ of $\mathcal{L}(T_1)$ such that $t(\overline{x})$ is substitutable for z in $\varphi(\overline{y}, z)$ - outputs a proof in T_2 of the sentence:*

$$\forall s \in \mathrm{Cond}_\tau \, \forall \overline{u}, \overline{v}, w \, \big(s \Vdash_\tau w = t(\overline{v}) \, \Rightarrow \, \big(s \Vdash_\tau \varphi(\overline{u}, w) \Leftrightarrow s \Vdash_\tau \varphi(\overline{u}, t(\overline{v})) \big) \big). \quad (4.7)$$

Proof sketch. The scheme of this proof is as for the previous lemma. Given an $\mathcal{L}(T_1)$ term $t(\overline{x})$ and an $\mathcal{L}(T_1)$ formula $\varphi(\overline{y}, z)$, the algorithm constructs a proof of (4.7) by induction on formula complexity. The base step simply uses (FI11). For the induction step, the algorithm runs one of three fixed procedures corresponding to the outermost connective or quantifier of φ and, depending on the case, possibly also applies Lemma 4.14 (a) for the immediate subformulas of φ. As in the previous proof, one easily observes that the running time of the algorithm is polynomial in $|\varphi|$. □

As mentioned in the previous section, our official proof system has only two logical connectives $\neg$ and $\Rightarrow$ and the universal quantifier, so $\varphi \wedge \psi$ and $\exists x \, \varphi(x)$ are shorthand for $\neg(\varphi \Rightarrow \neg\psi)$ and $\neg\forall x \neg\varphi(x)$, respectively. The following simple lemma will be useful later when we force axioms that are expressed most naturally with logical symbols other than our official ones.

Lemma 4.16. *Let τ be a polynomial forcing interpretation of $\mathcal{L}(T_1)$ in T_2. Then there exists a polynomial-time algorithm which, given as input formulas φ and ψ of $\mathcal{L}(T_1)$, outputs a proof in T_2 of the sentences:*

(a) $\forall s \in \mathrm{Cond}_\tau \big(s \Vdash_\tau \neg(\varphi \Rightarrow \neg\psi) \, \Leftrightarrow \, (s \Vdash_\tau \varphi \, \wedge \, s \Vdash_\tau \psi) \big)$;

(b) $\forall s \in \mathrm{Cond}_\tau \big(s \Vdash_\tau \neg\forall x \neg\varphi(x) \;\Leftrightarrow$
$$\forall s' \trianglelefteq_\tau s \,\exists s'' \trianglelefteq_\tau s' \,\exists v\, (s'' \Vdash_\tau v\!\downarrow\, \wedge s'' \Vdash_\tau \varphi(v))\big).$$

Proof sketch. The arguments for both items are very similar, so we discuss only the first one. Given formulas φ and ψ of $\mathcal{L}(T_1)$, the algorithm constructs the proof of (a) as follows. Firstly, it uses the definition of forcing negation (FT6) and then the definition of forcing implication (FT7) to rewrite the left-hand side of the equivalence in (a) as

$$\forall s' \trianglelefteq_\tau s \,\exists s'' \trianglelefteq_\tau s' \,\forall s''' \trianglelefteq_\tau s'' \,\exists s'''' \trianglelefteq_\tau s''' \,(s'' \Vdash_\tau \varphi \,\wedge\, s'''' \Vdash_\tau \psi).$$

Then, it uses polynomial-time procedures given by Lemma 4.14 to construct proofs of monotonicity and density for φ and ψ. For the less obvious direction ($\Rightarrow$), it uses the density property twice to derive $s \Vdash_\tau \varphi$ and $s \Vdash_\tau \psi$.

The construction goes according to one and the same template for all formulas of $\mathcal{L}(T_1)$. It only requires substituting, into a fixed number of blanks, formulas or other proofs that can be found in time polynomial in $|\varphi|$ and $|\psi|$. Therefore, the running time of our algorithm is polynomial in $|\varphi|$ and $|\psi|$. □

The next lemma states that forcing interpretations preserve axioms and rules of first-order logic.

Lemma 4.17 ([32, Proposition 1.14])**.** *Let τ be a polynomial forcing interpretation of $\mathcal{L}(T_1)$ in T_2. Then:*

(a) there exists a polynomial-time algorithm which, given as input an instance $\varphi(\overline{x})$ of a logical axiom in $\mathcal{L}(T_1)$, outputs a proof in T_2 of the sentence:

$$\forall s \in \mathrm{Cond}_\tau \,\forall \overline{v}\, \big(s \Vdash_\tau \overline{v}\!\downarrow \Rightarrow\; \big(s \Vdash_\tau \varphi(\overline{v})\big)\big); \tag{4.8}$$

(b) there exists a polynomial-time algorithm which, given as input formulas φ and ψ of $\mathcal{L}(T_1)$, outputs a proof in T_2 of the sentence:

$$\forall s \in \mathrm{Cond}_\tau \,\forall \overline{v}\, \big(\big(s \Vdash_\tau \varphi(\overline{v}) \Rightarrow \psi(\overline{v}) \,\wedge\, s \Vdash_\tau \varphi(\overline{v})\big) \;\Rightarrow\; s \Vdash_\tau \psi(\overline{v})\big). \tag{4.9}$$

Proof sketch. Part (a). Given a logical axiom φ, the algorithm first checks which of the finitely many axiom schemas it instantiates, and then it runs one of finitely many fixed procedures, each corresponding to one logical axiom scheme. The procedures use only definitions of forcing negation (FT6), implication (FT7) and a universal formula (FT8), properties (FI7)-(FI11) of forcing equality, and proofs constructed by polynomial-time algorithms given by Lemma 4.14 and, in case of the scheme '$\forall x\varphi \Rightarrow \varphi(t)$', also by Lemma 4.15. Each procedure constructs a proof of (4.8) according to a fixed template, in which one has to substitute a fixed number of times formulas like '$s \Vdash_\tau \varphi(\overline{v})$' or some auxiliary proofs that can be found in time polynomial in $|\varphi|$. Putting all these pieces together, we see that the running time of our algorithm is polynomial in $|\varphi|$.

The argument for part (b) is similar: for all formulas φ and ψ, the algorithm again uses a single fixed template of a proof of (4.9) which formalizes the following reasoning in T_2.

Suppose that $s \Vdash_\tau \varphi(\overline{v}) \Rightarrow \psi(\overline{v})$ and $s \Vdash_\tau \varphi(\overline{v})$. By the definition of forcing implication (FT7), we have $\forall s' \trianglelefteq_\tau s\ \exists s'' \trianglelefteq_\tau s' \big(s' \Vdash_\tau \varphi(\overline{v}) \Rightarrow s'' \Vdash_\tau \psi(\overline{v})\big)$. On the other hand, by Lemma 4.14 (a) applied to $\varphi(\overline{v})$, we get $\forall s' \trianglelefteq_\tau s \big(s' \Vdash_\tau \varphi(\overline{v})\big)$, so we conclude that $\forall s' \trianglelefteq_\tau s\ \exists s'' \trianglelefteq_\tau s' \big(s'' \Vdash_\tau \psi(\overline{v})\big)$. Now, by Lemma 4.14 (b) applied to $\psi(\overline{v})$, we obtain $s \Vdash_\tau \psi(\overline{v})$, as required.

To construct a formal proof of (4.9) for φ and ψ based on the previous paragraph, the algorithm needs only to fill the fixed template with some formulas and auxiliary proofs (of instances of Lemma 4.14) that can be constructed in time polynomial in $|\varphi|$ and $|\psi|$. This all takes time polynomial in $|\varphi|$ and $|\psi|$. □

With the above lemma we are ready to prove the main property of polynomial forcing interpretations.

Corollary 4.18 ([32, Corollary 1.15]). *Let τ be a polynomial forcing interpretation of T_1 in T_2. Then there exists a polynomial-time algorithm which, given as input a proof δ in T_1 of a formula $\varphi(\overline{x})$, outputs a proof in T_2 of the sentence:*

$$\forall s \in \mathrm{Cond}_\tau\, \forall \overline{v} \in \mathrm{Name}_\tau \big(s \Vdash_\tau \overline{v}\downarrow \Rightarrow\ s \Vdash_\tau \varphi(\overline{v})\big). \tag{4.10}$$

Proof sketch. Given a proof $\delta = \langle \varphi_0, \varphi_1, \ldots, \varphi_{n-1} \rangle$ in T_1 of the formula $\varphi = \varphi_{n-1}$, the algorithm goes through n stages. In stage i, it constructs a proof of (an analogue of) (4.10) for φ_i, depending on whether φ_i is an axiom of T_1 or a logical axiom, or whether it was derived by modus ponens from some formulas appearing earlier in δ. In each case the algorithm uses one of three fixed polynomial-time procedures that are guaranteed to exist by Definition 4.12 (iv), Lemma 4.17 (a) and Lemma 4.17 (b), respectively. For every $0 \leqslant i < n$, the proof of (4.10) for φ_i is constructed exactly once, so the running time of the algorithm is polynomial in $|\delta|$. □

Finally, we define the reflection property, which expresses the connection between interpreted and interpreting theories.

Definition 4.19 ([32, Definition 1.16]). Let τ be a polynomial forcing interpretation of T_1 in T_2 and let Γ be a set of sentences in their common language $\mathcal{L}(T_1) \cap \mathcal{L}(T_2)$. Then τ is *polynomially Γ-reflecting* if there exists a polynomial-time algorithm which, given as input a sentence $\gamma \in \Gamma$, outputs a proof in T_2 of the sentence:

$$\forall s \in \mathrm{Cond}_\tau (s \Vdash_\tau \gamma)\ \Rightarrow\ \gamma. \tag{4.11}$$

Theorem 4.20. *[Essentially [1, Section 10]] Let T_1 and T_2 be theories and let Γ be a set of sentences in their common language $\mathcal{L}(T_1) \cap \mathcal{L}(T_2)$. If τ is a polynomial forcing interpretation of T_1 in T_2 that is polynomially Γ-reflecting, then T_1 is polynomially simulated by T_2 with respect to Γ.*

Proof. This follows easily from Definition 4.19 and Corollary 4.18. □

4.3 A two-step forcing construction

4.3.1 Model-theoretic intuition

Our general strategy for strengthening the $\forall\Pi^0_3$-conservativity of $\mathsf{RCA}^*_0 + \mathsf{CAC}$ over RCA^*_0 to polynomial simulation is to define a forcing interpretation that follows a model-theoretic construction that can be used to prove the conservation theorem, as in the case of non-speedup results for WKL_0 [1] and RT^2_2 [32]. To improve the readability of the technical description of the formalized forcing in the following sections, we now discuss the model-theoretic perspective in some detail.

Recall that we have proved $\forall\Pi^0_3$-conservativity of CAC over RCA^*_0 as an immediate consequence of Theorem 2.16. The proof of the theorem relies on a proper cut construction, a widely applicable method for proving $\forall\Pi^0_3$-conservativity of a theory T_1 over a theory T_2, where the latter is usually assumed to contain $\mathsf{B}\Sigma^0_1 + \mathsf{exp}$. Typically, one proceeds as follows. For an arbitrary $\exists\Sigma^0_3$ sentence $\varphi := \exists X\, \exists x\, \forall y\, \exists z\, \theta(X, x, y, z)$ consistent with T_2, one takes a countable model $(M, \mathcal{X})$ satisfying $T_2 + \varphi$ together with parameters $B \in \mathcal{X}$ and $b \in M$ such that $\forall y\, \exists z\, \theta(B, b, y, z)$ holds in $(M, \mathcal{X})$. Then, one constructs a proper cut $I \subseteq M$ such that the structure $(I, \mathrm{Cod}(M/I))$ satisfies $T_1 + \forall y\, \exists z\, \theta(B \cap I, b, y, z)$. To this end, one may consider the following computable function:

$$f_\theta(y) = \min\{z > 2^y \colon\ \forall y' \leqslant y\ \exists z' \leqslant z\ \theta(B, b, y', z')\}, \tag{4.12}$$

which is total because φ, exp and $\mathsf{B}\Sigma^0_1$ hold in $(M, \mathcal{X})$. If the cut I is chosen in such a way that it satisfies T_1 and the set $\{x \in I \colon \exists y\, (x = f_\theta^{(y)}(b))\}$ is cofinal in I, then $(I, \mathrm{Cod}(M/I)) \vDash T_1 + \varphi$. The additional requirement that $z > 2^y$ guarantees that the cut is also closed under exponentiation. Thus, by Theorem 1.9, we learn that $(I, \mathrm{Cod}(M/I)) \vDash \mathsf{WKL}^*_0$. If the construction of the cut I can be performed for any $\exists\Sigma^0_3$ sentence φ, we obtain a conservation result for T_1, and in fact for $T_1 + \mathsf{WKL}^*_0$, over T_2.

However, forcing constructions often formalize the process of approximation of some generic object, and the construction from Section 2.3 has a different character: the structure obtained in the proof of Theorem 2.16 satisfies a given pSO sentence χ for a general reason expressed by Theorem 2.9, rather than as a result of an approximation process. Moreover, Theorem 2.16 is quite general and covers also the case of RT^2_2, which has non-elementary speedup over RCA_0, so there is little hope that the proper cut construction from the proof of that theorem can be useful for our present purposes.

Luckily, one can prove the $\forall\Pi^0_3$-conservativity of $\mathsf{RCA}^*_0 + \mathsf{CAC}$ over RCA^*_0 using an alternative proper cut construction, like the one used to show the $\forall\Pi^0_3$-conservativity of $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ over RCA_0 in [33] (this would also work for the $\forall\Pi^0_3$-conservativity of $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2$ over RCA^*_0, but see Remark 4.22 below). The argument would go as follows. As above, for an arbitrary $\exists\Sigma^0_3$ sentence $\varphi := \exists X\, \exists x\, \forall y\, \exists z\, \theta(X, x, y, z)$ consistent with RCA^*_0, we would take a countable nonstandard model $(M, \mathcal{X}) \vDash \mathsf{RCA}^*_0 + \varphi$, together with witnesses $B \in \mathcal{X}$ and

$b \in M$ to the initial quantifiers '$\exists X\, \exists x$'. Then, using the function f_θ defined as in (4.12), we define the following set:

$$Y := \{x \in M : \exists y\, (x = f_\theta^{(y)}(b))\}, \tag{4.13}$$

which is unbounded because f_θ is a total function. To obtain a cut I such that $(I, \mathrm{Cod}(M/I)) \models \mathsf{WKL}_0^* + \mathsf{CAC} + \varphi$, we would construct (in the metatheory) a sequence $X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$ of length ω of subsets of Y that are finite in the sense of M and satisfy the following condition: the set X_0 is an arbitrary finite subset of Y with nonstandard cardinality; and for some enumeration $\{P_i\}_{i\in\omega}$ of all partial orders defined on X_0 and coded in M (which is fixed in the metatheory and uses countability of M), for every $i \in \omega$ the set $X_{i+1} \neq \emptyset$ is a chain or an antichain in $P_i \restriction X_i$. At each step of the construction the condition can be met by applying Dilworth's theorem (Theorem 4.5) and maintaining the invariant that the sets X_i are 'sufficiently large', which in this case simply means that they have a nonstandard number of elements. By the polynomial bounds implicit in Theorem 4.5, in the i-th step we can take X_{i+1} of size roughly $\sqrt{|X_i|}$, so if $|X_i| > \omega$, then also $|X_{i+1}| > \omega$, because ω is closed under multiplication. Now, we define the following proper cut:

$$I := \sup\{\min(X_i) :\ i \in \omega\}, \tag{4.14}$$

and it can be verified that the structure $(I, \mathrm{Cod}(M/I))$ satisfies $\mathsf{WKL}_0^* + \mathsf{CAC} + \varphi$.

There is, however, a major problem which does not appear in [33], where the base theory is RCA_0. Namely, even though the set Y is unbounded it may not contain finite subsets of nonstandard cardinalities that are needed for the construction described above. This is possible since we do not assume $\mathsf{I}\Sigma_1^0$ in our base theory, and this may have the effect that some total computable functions, such as f_θ or even exp, can be iterated on some numbers only standardly many times. Thus, for instance, in some models of $\mathsf{RCA}_0^* + \varphi$ the set $\{2_n(b) : n \in \omega\}$ might be cofinal in M, where b is a witness to '$\exists x$' in φ. In such a case no proper cut in M containing b satisfies exp, and hence neither can it satisfy RCA_0^*.

In a model-theoretic argument one can easily get around this obstacle and use compactness to get a countable model satisfying $\mathsf{RCA}_0^* + \forall y\, \exists x\, \theta(B, b, y, z)$ together with the set of sentences $\{\exists y\, (y = f_\theta^{(c)}(b) \wedge c > n) :\ n \in \omega\}$. Unfortunately, from a syntactical point of view, this kind of solution looks rather like an ad hoc trick: one works only with those models of RCA_0^* that are convenient to work with. In contrast, a syntactical construction must be uniform, i.e., it has to provide a uniform description in RCA_0^* of a generic proper cut satisfying $\mathsf{CAC} + \varphi$. This description has to make sense in any model of RCA_0^*, including the standard model or a model violating $\mathsf{I}\Sigma_1^0$.

One way to circumvent this issue is to work with an auxiliary theory T^*, satisfiable only in 'convenient models', and to define two forcing interpretations: τ_1 of T^* in RCA_0^* (more precisely, in a slight extension of RCA_0^* – see below) and τ_2 of $\mathsf{WKL}_0^* + \mathsf{CAC}$ in T^*. We choose our auxiliary theory T^* to be $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$, with the axiom SC defined as follows:

SC $\mathbb{I}$ is a proper cut closed under multiplication such that for every number x, the value $2_c(x)$ exists for some $c > \mathbb{I}$,

where $\mathbb{I}$ is a fresh unary predicate symbol (not to be confused with the italic 'I'). The abbreviation SC stands for 'short cut' because, intuitively, the cut $\mathbb{I}$ is defined to play the role of ω: it is required to have those properties of ω that are crucial for the model-theoretic construction described above.

Let us first see how the axiom SC is used in the second forcing interpretation τ_2 and what the connection is with the model-theoretic argument. The forcing conditions of τ_2 will be finite sets of cardinality greater than the cut $\mathbb{I}$ that are exponentially sparse, where a set $s = \{x_0, \ldots, x_n\}$ is called *exponentially sparse* if for each $i < n$ it holds that $2^{x_i} < x_{i+1}$. These forcing conditions correspond to the finite sets X_i which approximated the cut I discussed above. The names for potential elements of the generic cut will be simply all natural numbers, and a condition s will force a name v to be valid if the intersection $s \cap [0, v]$ is not a condition. This intuitively means that in the generic filter there will appear a condition $s' \trianglelefteq_{\tau_2} s$ such that $v < \min(s')$, and thus v will indeed belong to the cut defined as in (4.14). The exponential sparsity of the forcing conditions guarantees that the generic cut will satisfy exp. Finally, the closure of $\mathbb{I}$ under multiplication makes it possible to keep adding to the generic filter conditions of cardinalities decreasing roughly by taking square root, so that by Dilworth's theorem CAC will eventually be forced.

Remark 4.21. To prove their results on the strength of RT^2_2 over RCA_0, the authors of [33, 32] used an upper bound for the finite version of RT^2_2 expressed in terms of so-called α-largeness (the bound was also obtained in [33]): if a finite set X is ω^{300x}-large, then every colouring $f\colon [X]^2 \to 2$ has an ω^x-large homogeneous set. Thus, the sets X_i used to construct a cut satisfying RT^2_2 in [33] had to be ω^c-large for some nonstandard number c and then, in [32], the forcing conditions were ω^c-large finite sets for some c above the cut $\mathbb{I}$. The auxiliary theory T^* only required $\mathbb{I}$ to be closed under addition, as it was enough to decrease exponents determining the size of the forcing conditions only by the constant factor of 300. On the other hand, to have enough forcing conditions, it was necessary for T^* to assume that every unbounded set has an ω^c-large finite subset for some c greater than $\mathbb{I}$. Note that the latter condition immediately implies that the exponential function can be iterated on every number more than $\mathbb{I}$-many times.

For the first forcing interpretation τ_1 of $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$, it is enough to define it in $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$ rather then in RCA^*_0. This is because we already know, by Theorem 4.4, that $\mathsf{RCA}^*_0 + \mathsf{I}\Sigma^0_1$ polynomially simulates $\mathsf{WKL}^*_0 + \mathsf{CAC}$, and polynomial simulations are easily seen to be closed under case distinction.

Working under the assumption that $\mathsf{I}\Sigma^0_1$ fails, one has natural candidates that can serve as an interpretation of $\mathbb{I}$: Σ^0_1-definable proper cuts. However, the axiom SC requires the cut $\mathbb{I}$ to be 'short': for every number x, the cut $\mathbb{I}$ must be properly contained in the Σ^0_1-cut $J_x = \{i \in \mathbb{N}\colon \exists y\, (y = 2_i(x))\}$. Thus, we will use the cut I^0_1, which is the intersection of all Σ^0_1-definable cuts (see Section 1.2 for its definition and basic properties).

However, the cut I^0_1 may or may not be Σ^0_1-definable itself. If it is not, then we have a straightforward (with no forcing) interpretation of $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$ in $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$: all atomic formulas of $\mathcal{L}_{\mathrm{I}}$ translate to themselves and '$t(\overline{x}) \in \mathbb{I}$' is translated to '$t(\overline{x}) \in \mathrm{I}^0_1$'. Indeed, this interpretation satisfies the axiom SC because, by Proposition 1.16, the cut I^0_1 is closed under multiplication and, for all numbers x, we have $\mathrm{I}^0_1 \neq J_x$.

In the other case the situation is not that simple: if the cut I^0_1 is itself Σ^0_1-definable, then it might happen that it is J_x for some number x. Then the axiom SC would obviously fail under the above interpretation. On the model-theoretic level, our solution is to non-cofinally extend the ground model M to some M' so that for each number $x \in M'$ there exists a new element c above the old cut I^0_1 such that the value $2_c(x)$ exists. We can achieve this by the Δ^0_1 ('computable') ultrapower construction, and then interpret $\mathbb{I}$ as the following cut:

$$\sup_{M'}\big((\mathrm{I}^0_1)^M\big) = \{x \in M' \colon \exists y \in M\,(x < y \wedge M \vDash y \in \mathrm{I}^0_1)\}. \tag{4.15}$$

Then, one can actually find a single element d in the ultrapower that can be thought of as a diagonal of $(\mathrm{I}^0_1)^M$, and show that for all numbers $x \in M'$, the value $2_d(x)$ exists. As we will see in the next section, the construction of such an ultrapower can naturally be simulated by syntactical forcing.

Remark 4.22. As we already mentioned, the $\forall\Pi^0_3$-conservativity of $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2$ over RCA^*_0 can also be proved using the alternative proper cut construction described above. The only difference with the case of CAC is that in the i-th step the finite set $|X_{i+1}|$ has size roughly $\log|X_i|$, due to the classical exponential upper bound on Ramsey numbers for the finite version of RT^2_2 (note that without $\mathsf{I}\Sigma^0_1$ one cannot work effectively with the upper bound using α-largeness). However, by Theorem 4.3 we a priori know that this construction cannot be presented as a formalized forcing argument. Now we can precisely point to the place where the strategy to prove non-speedup for CAC would fail for RT^2_2. Namely, one would have to modify the axiom SC so that the cut $\mathbb{I}$ is closed under exponentiation. But then, one could not use the cut I^0_1 as an interpretation of $\mathbb{I}$ because RCA^*_0 does not prove I^0_1 to be closed under exponentiation (on the other hand, it is shown in [32] that I^0_1 is closed under 2^x provably in $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2$).

Let us summarize our strategy for proving Theorem 4.6. We make two case distinctions. First, we consider $\mathsf{RCA}^*_0 + \mathsf{I}\Sigma^0_1$ and $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1$ separately. In the first case we use Theorem 4.4 from [32]. In the second case we make another case distinction and work with the theories $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1 + \Sigma^0_1\text{-}\mathsf{LPC}$ and $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1 + \neg\Sigma^0_1\text{-}\mathsf{LPC}$, where $\Sigma^0_1\text{-}\mathsf{LPC}$ is the following sentence ('LPC' stands for 'least proper cut'):

$\Sigma^0_1\text{-}\mathsf{LPC}$ The cut I^0_1 is Σ^0_1-definable.

The case of $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1 + \neg\Sigma^0_1\text{-}\mathsf{LPC}$ is simpler, as this theory admits an almost trivial (non-forcing) interpretation of the auxiliary theory $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$, which is the identity on $\mathcal{L}_{\mathrm{I}}$ and interprets $\mathbb{I}$ as I^0_1. Thus we can define a polynomial forcing interpretation of $\mathsf{WKL}_0^* + \mathsf{CAC}$ directly in $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \neg\Sigma_1^0$-$\mathsf{LPC}$.

In the case of $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \Sigma_1^0$-$\mathsf{LPC}$ we prove polynomial simulation of $\mathsf{WKL}_0^* + \mathsf{CAC}$ by composing two forcing interpretations. Firstly, we construct a forcing interpretation τ_1 of $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$ in $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \Sigma_1^0$-$\mathsf{LPC}$, which simulates the model-theoretic construction of a computable ultrapower. Then, we define a forcing interpretation τ_2 of $\mathsf{WKL}_0^* + \mathsf{CAC}$ in $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$, which follows the model-theoretic proof of $\forall\Pi_3^0$-conservativity of $\mathsf{RCA}_0^* + \mathsf{CAC}$ over RCA_0^*.

Let us note that we cannot, at least if we work with the specific interpretations τ_1 and τ_2, avoid the second case distinction – it will be clear from our construction that the assumption that I_1^0 is Σ_1^0-definable is crucial for τ_1 to work.

4.3.2 Computable ultrapower

Ultrapowers are one of the basic tools in logic for constructing new models of a given theory T from an already constructed model of T. This is possible due to Łoś's theorem, which implies that an ultrapower of a given model M is elementarily equivalent to M. The usual ultrapower construction often produces a model of cardinality greater than that of the initial model M. To control the cardinality, one often constructs a so-called *definable ultrapower*. Here, the set of indices is the universe of M itself, and instead of all total functions $f\colon M \to M$ one considers only definable ones. This is a classical way of obtaining a countable nonstandard model of true arithmetic. One can modify this construction further and consider only those total functions that are definable by formulas of a restricted complexity. In such a situation only a restricted version of the Łoś theorem will be available, but this is often enough for some interesting applications, cf. e.g. [18, Theorem IV.1.53(1)].

Our first forcing interpretation is based on an arithmetical ultrapower construction restricted to Δ_1-definable total functions. Such ultrapowers were first systematically studied by Hirschfeld in [21, 22]. He focused only on the standard model ω and considered ultrapowers of the form R/U, where R is the semiring of all computable functions and U is a nonprincipal ultrafilter in the Boolean algebra of all computable sets. In [22] Hirschfeld proved some important properties of such ultrapowers, most of which easily generalize to nonstandard models. For us, the following result, which can be seen as a restricted version of the Łoś theorem, is of special importance.

Theorem 4.23 (essentially [22, Theorem 2.3])**.** *Let M be a model of $\mathsf{B}\Sigma_1 + \mathsf{exp}$ and let M' be of the form $R(M)/U$, where $R(M)$ is the set of Δ_1-definable functions of M and U is an ultrafilter in the Boolean algebra Δ_1-$\mathrm{Def}(M)$. Then, for every Δ_0 formula $\varphi(x_1, \ldots, x_n)$ and every tuple of Δ_1-definable functions $f_1, \ldots, f_n$, the following holds:*

$$M' \vDash \varphi([f_1], \ldots, [f_n]) \quad \textit{iff} \quad \{x \in M\colon\ M \vDash \varphi(f_1(x), \ldots, f_n(x))\} \in U, \tag{4.16}$$

where $[f_i]$ are elements of the quotient $R(M)/U$.

In the following we will consider a 'computable ultrapower' of a second-order structure rather than that of a first-order one. We keep the intuitive name 'computable ultrapower' but let us stress that 'computable' will actually mean 'computable in a given model', i.e. with the use of an arbitrary set from the second-order universe of a given model as an oracle. However, for our purposes it would essentially be fine to construct the ultrapower from functions computable relative to a fixed set oracle. For a situation where noncomputable elements of the ultrapower are genuinely needed, see [51].

The model-theoretic construction of a computable ultrapower easily translates to a syntactical forcing argument. Our forcing conditions are unbounded sets that are computable according to a given model $(M, \mathcal{X})$, which are simply unbounded elements of $\mathcal{X}$, and they are ordered by inclusion. The unboundedness requirement guarantees that the generic ultrapower is not a cofinal extension of M. The set of names consists of those elements of $\mathcal{X}$ which are (graphs of) total functions. Every name is forced to be valid by any condition. We keep the usual forcing notation and use lower-case letters to denote metavariables for conditions (s, s', ...) and names (v, w, ...), but let us stress that they are always second-order objects, as described above. For a tuple of names $\overline{v}$, we abbreviate the tuple of their values on some number x by $\overline{v}(x)$. An atomic formula $\alpha(\overline{v})$ of $\mathcal{L}_{\mathrm{I}}$ is forced by a condition s if $\alpha(\overline{v}(x))$ holds for all but finitely many $x \in s$. A condition s forces the value of a term $t(\overline{v})$ to be in the cut $\mathbb{I}$ if there is an element $i \in \mathrm{I}^0_1$ such that the value $t(\overline{v}(x))$ is smaller than i for all but finitely many $x \in s$.

The following definition formalizes what we have just said. To simplify notation, we use abbreviations $\forall^\infty x\, \varphi$ and $\exists^\infty x\, \varphi$ for $\exists y\, \forall x\, (y < x \Rightarrow \varphi)$ and $\forall y\, \exists x\, (y < x \wedge \varphi)$, respectively.

Definition 4.24. The following list of clauses defines a forcing translation τ_1 from the language $\mathcal{L}_{\mathrm{I}} \cup \{\mathbb{I}\}$ to the language $\mathcal{L}_{\mathrm{II}}$.

(i) $s \in \mathrm{Cond}_{\tau_1}$ is
$$\exists^\infty x\, (x \in s);$$

(ii) $s' \trianglelefteq_{\tau_1} s$ is
$$s \in \mathrm{Cond}_{\tau_1} \wedge s' \in \mathrm{Cond}_{\tau_1} \wedge \forall x\, (x \in s' \Rightarrow x \in s);$$

(iii) $v \in \mathrm{Name}_{\tau_1}$ is
$$\text{`}v\text{ is a total function'};$$

(iv) $s \Vdash_{\tau_1} v\!\downarrow$ is
$$s \in \mathrm{Cond}_{\tau_1} \wedge v \in \mathrm{Name}_{\tau_1};$$

(v) if $\alpha(\overline{x})$ is an atomic formula of the form $t_1(\overline{x}) = t_2(\overline{x})$ or $t_1(\overline{x}) \leqslant t_2(\overline{x})$, then $s \Vdash_{\tau_1} \alpha(\overline{v})$ is
$$s \Vdash_{\tau_1} \overline{v}\!\downarrow \wedge \forall^\infty x\, \big(x \in s \Rightarrow \alpha(\overline{v}(x))\big);$$

(vi) if $\alpha(\overline{x})$ is an atomic formula of the form $t(\overline{x}) \in \mathbb{I}$, then $s \Vdash_{\tau_1} \alpha(\overline{v})$ is

$$s \Vdash_{\tau_1} \overline{v}\!\downarrow \;\wedge\; \exists i \in \mathrm{I}_1^0 \; \forall^\infty x \left(x \in s \Rightarrow t(\overline{v}(x)) \leqslant i\right).$$

Let us make a few simple observations about τ_1 that will be used later often without mention. Firstly, there is a largest forcing condition, namely $\mathbf{1} = \{x \in \mathbb{N} \colon x = x\}$. Secondly, if $s' \trianglelefteq_{\tau_1} s$, then s' is an unbounded subset of s. Thirdly, it follows immediately from the definition of τ_1 that for every condition s, every term $t(\overline{x})$ and every tuple of names $\overline{v}$ there exists a name w such that $s \Vdash_{\tau_1} w = t(\overline{v})$, namely $w(x) = t(\overline{v}(x))$. Lastly, the definition (FT8) of forcing a universal formula simplifies because every condition forces every name to be valid.

Lemma 4.25. *The forcing translation τ_1 is a polynomial forcing interpretation of the language $\mathcal{L}_{\mathrm{I}} \cup \{\mathbb{I}\}$ in $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \Sigma_1^0\text{-}\mathsf{LPC}$.*

Proof. Let us first note that the conditions from Definition 4.10 given by single sentences, i.e. (FI1)-(FI5), (FI7)-(FI9) and (FI12), follow immediately from the definition of τ_1. Since there are only three function symbols in the interpreted language, the condition (FI10) is also given by a single sentence and follows easily from the definition of τ_1 as well.

The schematic conditions (FI6) and (FI11) are also unproblematic and their proofs can be constructed by a polynomial-time algorithm. Namely, given an atomic formula α, the algorithm first finds its forcing translation according to clauses (v) and (vi) of Definition 4.24. The condition (FI6) follows immediately from the definition of τ_1, so the algorithm constructs its proof using a single fixed template in which it substitutes expressions like $s \Vdash_\tau \alpha$ into finitely many blanks. This clearly takes time polynomial in $|\alpha|$.

For (FI11), the proof goes by induction on the complexity of terms occurring in α and uses the equality axioms in the interpreting theory. For each subterm r of a term occurring in α, the proof of the formula $\forall^\infty x \in s \left(w(x) = t(\overline{v}(x))\right) \Rightarrow \forall^\infty x \in s \left(r(\overline{u}(x), w(x)) = r(\overline{u}(x), t(\overline{v}(x)))\right)$ is constructed just once, so the whole construction of the proof of (FI11) is polynomial in $|\alpha|$.

The only nonobvious condition is the density property for atomic formulas (FI13). We have two cases and we show how to prove both of them by contraposition. It should be clear that the proofs do not depend substantially on the terms occurring in an atomic formula α and can be constructed in time polynomial in $|\alpha|$.

So, let s be a condition and $\overline{v}$ be some names. Firstly, let $\alpha(\overline{x})$ be an atomic formula of the form $t_1(\overline{x}) = t_2(\overline{x})$ (the case when $\alpha(\overline{x})$ is $t_1(\overline{x}) \leqslant t_2(\overline{x})$ is treated in the same way). Suppose that $s \nVdash_{\tau_1} t_1(\overline{v}) = t_2(\overline{v})$. This means that there are unboundedly many $x \in s$ such that $t_1(\overline{v}(x)) \neq t_2(\overline{v}(x))$. Clearly, the set $s' := \{x \in s \colon t_1(\overline{v}(x)) \neq t_2(\overline{v}(x))\}$ is a condition below s. Obviously, s' does not force $t_1(\overline{v}) = t_2(\overline{v})$ and neither does any condition below it. Thus, we have $\exists s' \trianglelefteq_{\tau_1} s \, \forall s'' \trianglelefteq_{\tau_1} s' \, (s'' \nVdash_{\tau_1} \alpha(\overline{v}))$.

In the second case, let $\alpha(\overline{x})$ be an atomic formula of the form $t(\overline{x}) \in \mathbb{I}$. Suppose that $s \nVdash_{\tau_1} t(\overline{v}) \in \mathbb{I}$. This means that for every $i \in \mathrm{I}_1^0$ there are unboundedly many $x \in s$ such that $t(\overline{v}(x)) > i$. Recall that we work under the assumption that the cut I^0_1 is Σ^0_1-definable, so by Proposition 1.13 there exists an unbounded set $A = \{a_i\}_{i \in \mathrm{I}^0_1}$ indexed by I^0_1. We define recursively a set $s' = \{x_0, x_1, \dots\}$ as follows:

$$\begin{aligned} x_0 &= \min(s), \\ x_{i+1} &= \min\{y \in s\colon\ y > x_i \wedge y > a_i \wedge t(\overline{v}(y)) > i\}. \end{aligned}$$

Note that the set s' is unbounded and has I^0_1-many elements. Indeed, by the assumption that $s \nVdash_{\tau_1} t(\overline{v}) \in \mathbb{I}$, for every $i \in \mathrm{I}^0_1$ there exist arbitrarily large numbers $x \in s$ with $t(\overline{v}(x)) > i$. Also, for every $i \in \mathrm{I}^0_1$, the i-th step of the recursive construction of s' can be performed, since otherwise the set $\{i \in \mathbb{N}\colon \exists x\,(x = x_i)\}$ would be a Σ^0_1-cut properly contained in I^0_1, contradicting Σ^0_1-LPC.

Therefore, the set s' is a condition below s and, by its definition, for every $i \in \mathrm{I}^0_1$ it holds that $\forall x \in s'(x > x_i \Rightarrow t(\overline{v}(x)) > i)$. Thus s' does not force $t(\overline{v}) \in \mathbb{I}$ and neither does any condition below it. Hence, we obtain $\exists s' \trianglelefteq_{\tau_1} s\, \forall s'' \trianglelefteq_{\tau_1} s'$ $(s'' \nVdash_{\tau_1} \alpha(\overline{v}))$, as required. □

The following lemma is a forcing analogue of Theorem 4.23. Intuitively, it says that Δ_0 formulas are absolute between the ground model and the generic ultrapower.

Lemma 4.26. *There exists a polynomial-time algorithm which, given as input a Δ_0 formula $\varphi(\overline{z})$ of $\mathcal{L}_{\mathrm{I}}$, outputs a proof in $\mathsf{RCA}^*_0 + \neg\mathsf{I}\Sigma^0_1 + \Sigma^0_1$-$\mathsf{LPC}$ of the sentence:*

$$\forall s \in \mathrm{Cond}_{\tau_1} \forall \overline{v} \in \mathrm{Name}_{\tau_1} \big(s \Vdash_{\tau_1} \varphi(\overline{v}) \Leftrightarrow \forall^\infty x \in s\ \varphi(\overline{v}(x))\big). \tag{4.17}$$

Proof. For a given Δ_0 formula φ, we show how to construct a proof of (4.17) by recursion on its subformulas. The construction we describe can be performed in time polynomial in $|\varphi|$, which can be seen as follows. The algorithm starts with atomic subformulas of φ and uses just one proof template for all atoms. To obtain a proof of (4.17) for an atomic formula α, the algorithm simply inserts α into a fixed number of blanks in this template.

Then, given a complex subformula ψ, the algorithm takes already produced proofs of (4.17) for the immediate subformulas of ψ, and merges these proofs into a proof of (4.17) for ψ. This is done by executing one of three fixed subroutines corresponding to the syntactic form of ψ, i.e. whether ψ is a negation, an implication or a universally quantified formula. For example, if ψ is $\theta \Rightarrow \zeta$, then the algorithm constructs a proof of (4.17) for ψ by inserting θ, ζ and ψ into a constant number of blanks in the fixed proof template for implication, and then adjoins the template filled in this way to the previously constructed proofs of (4.17) for θ and ζ.

To summarize, for a given Δ_0 formula φ with k subformulas, our algorithm goes through k stages, where each stage has one of four types, and the time needed to perform a stage of a given type is polynomial in the size of a given subformula of φ. Therefore, the proof of (4.17) for φ is constructed in time polynomial in $|\varphi|$.

Having discussed the time complexity of our algorithm, let us describe its subroutines informally but in some detail. The base case of atomic subformulas follows immediately from Definition 4.24 (v). For each type of complex subformula, we build the required proof by contraposition. When we consider a formula with free variables $\overline{z}$, we assume that the tuple of names $\overline{v}$ has the same length as $\overline{z}$.

Suppose that $\psi(\overline{z})$ has the form $\neg\theta(\overline{z})$ and we already have a proof of (4.17) for $\theta(\overline{z})$. For the ($\Rightarrow$) direction assume that $\exists^\infty x \in s\ \theta(\overline{v}(x))$. By Δ^0_1-comprehension we can define the unbounded set $s' = \{x \in s\colon \theta(\overline{v}(x))\}$, which is a condition below s. By the proof constructed in a previous step of the recursion, we have that $s' \Vdash_{\tau_1} \theta(\overline{v})$. From the definition of forcing negation (FT6) we get $s \nVdash_{\tau_1} \neg\theta(\overline{v})$.

For the other direction, assume that $s \nVdash_{\tau_1} \neg\theta(\overline{v})$. Again, by the definition of forcing negation (FT6), there is a condition $s' \trianglelefteqslant_{\tau_1} s$ such that $s' \Vdash_{\tau_1} \theta(\overline{v})$. By the proof constructed in a previous step of the recursion, we have $\forall^\infty x \in s'\, \theta(\overline{v}(x))$. Since s' is a forcing condition, s' is unbounded in s. Thus $\neg\forall^\infty x \in s\, \neg\theta(\overline{v}(x))$.

Suppose that $\psi(\overline{z})$ has the form $\theta(\overline{z}) \Rightarrow \zeta(\overline{z})$ and we already have proofs of (4.17) for $\theta(\overline{z})$ and $\zeta(\overline{z})$. Firstly, let us assume that $\exists^\infty x \in s\, \big(\theta(\overline{v}(x)) \wedge \neg\zeta(\overline{v}(x))\big)$. Define a condition below s by $s' := \{x \in s\colon\ \theta(\overline{v}(x)) \wedge \neg\zeta(\overline{v}(x))\}$. By the proofs constructed in previous steps of the recursion and the definition of s' we have that for all conditions $s'' \trianglelefteqslant_{\tau_1} s'$ it holds that $s'' \Vdash_{\tau_1} \theta(\overline{v})$ and $s'' \nVdash_{\tau_1} \zeta(\overline{v})$. In particular, we get $\exists s' \trianglelefteqslant_{\tau_1} s\ \forall s'' \trianglelefteqslant_{\tau_1} s' \big(s' \Vdash_{\tau_1} \theta(\overline{v}) \wedge s'' \nVdash_{\tau_1} \zeta(\overline{v})\big)$. By the definition of forcing implication (FT7) we obtain $s \nVdash_{\tau_1} \theta(\overline{v}) \Rightarrow \zeta(\overline{v})$.

Conversely, suppose that $s \nVdash_{\tau_1} \theta(\overline{v}) \Rightarrow \zeta(\overline{v})$. Then, there exists a condition $s' \trianglelefteqslant_{\tau_1} s$ such that $s' \Vdash_{\tau_1} \theta(\overline{v})$ but no s'' below s' forces $\zeta(\overline{v})$. In particular, $s' \nVdash_{\tau_1} \zeta(\overline{v})$. By the proofs of (4.17) for $\theta(\overline{z})$ and $\zeta(\overline{z})$ constructed in previous steps of the recursion, we learn that $\exists^\infty x \in s'\ \big(\theta(\overline{v}(x)) \wedge \neg\zeta(\overline{v}(x))\big)$. Since s' is a condition below s, it is unbounded in s. Thus, we obtain $\exists^\infty x \in s\ \neg\big(\theta(\overline{v}(x)) \Rightarrow \zeta(\overline{v}(x))\big)$.

Suppose that $\psi(\overline{z})$ has the form $\forall y \leqslant t(\overline{z})\ \theta(y, \overline{z})$, which is shorthand for $\forall y\big(y \leqslant t(\overline{z}) \Rightarrow \theta(y, \overline{z})\big)$, where y is not among $\overline{z}$, and that we have already constructed a proof of (4.17) for $\theta(y, \overline{z})$. For the ($\Rightarrow$) direction, let us assume that $\exists^\infty x \in s\ \exists y\, \big(y \leqslant t(\overline{v}(x)) \wedge \neg\theta(y, \overline{v}(x))\big)$. We define the unbounded set $s' := \{x \in s\colon \exists y\, \big(y \leqslant t(\overline{v}(x)) \wedge \neg\theta(y, \overline{v}(x))\big)\}$ and a function w as follows:

$$w(x) = \begin{cases} \text{least } y \leqslant t(\overline{v}(x)) \text{ such that } \neg\theta(y, \overline{v}(x)) & \text{if } x \in s', \\ 0 & \text{otherwise.} \end{cases}$$

Both s' and w are Δ^0_1-definable, so s' is a condition below s and w is a valid name. From the definitions of s' and the function w we obtain that $\forall^\infty x \in s' \big(w(x) \leqslant t(\overline{v}(x)) \wedge \neg\theta(w(x), \overline{v}(x))\big)$. By the proof of (4.17) for the atomic subformula $y \leqslant t(\overline{z})$ constructed in the base step of the recursion, and the proof of (4.17) for $\theta(y, \overline{z})$ constructed in a previous step, we get proofs of $s' \Vdash_{\tau_1} w \leqslant t(\overline{v})$ and $s' \nVdash_{\tau_1} \theta(w, \overline{v})$. But clearly no condition below s' forces

$\theta(w,\overline{v})$, so $s' \Vdash_{\tau_1} \neg\theta(w,\overline{v})$. Thus, by the definition of forcing implication (FT7) and a universal formula (FT8), we learn that $s \nVdash_{\tau_1} \forall y\,\big(y \leqslant t(\overline{v}) \Rightarrow \theta(y,\overline{v})\big)$.

For the other direction, assume that $s \nVdash_{\tau_1} \forall y\,\big(y \leqslant t(\overline{v}) \Rightarrow \theta(y,\overline{v})\big)$. Then, by (FT8), there exist a name w and a condition $s' \trianglelefteq_{\tau_1} s$ such that no condition s'' below s' satisfies $s'' \Vdash_{\tau_1} \big(w \leqslant t(\overline{v}) \Rightarrow \theta(w,\overline{v})\big)$. In particular, $s' \nVdash_{\tau_1} \big(w \leqslant t(\overline{v}) \Rightarrow \theta(w,\overline{v})\big)$. By the definition of forcing implication (FT7), we learn that there exists a condition $s'' \trianglelefteq_{\tau_1} s'$ such that $s'' \Vdash_{\tau_1} w \leqslant t(\overline{v})$ and $s'' \nVdash_{\tau_1} \theta(w,\overline{v})$. By the proof of (4.17) for the atomic subformula $y \leqslant t(\overline{z})$ constructed in the base step of the recursion, and the proof of (4.17) for $\theta(y,\overline{z})$ constructed in a previous step, we obtain that $\exists^\infty x \in s''\,\big(w(x) \leqslant t(\overline{v}(x)) \wedge \neg\theta(w(x),\overline{v}(x))\big)$. Since s'' is unbounded in s, we can conclude that $\exists^\infty x \in s\,\big(w(x) \leqslant t(\overline{v}(x)) \wedge \neg\theta(w(x),\overline{v}(x))\big)$, and therefore $\exists^\infty x \in s\; \neg\forall y\,\big(y \leqslant t(\overline{v}(x)) \Rightarrow \theta(y,\overline{v}(x))\big)$. □

We finish this section by showing that τ_1 determines a polynomial forcing interpretation of $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$.

Lemma 4.27. *The forcing translation τ_1 is a polynomial forcing interpretation of $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$ in $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \Sigma_1^0\text{-}\mathsf{LPC}$.*

Proof. It is enough to show that $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \Sigma_1^0\text{-}\mathsf{LPC}$ proves that $\mathbf{1}$ forces each axiom of $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$. Then, the polynomiality of τ_1 will follow by Lemma 4.25 and the fact that $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$ is finitely axiomatized.

So, let us work in $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \Sigma_1^0\text{-}\mathsf{LPC}$. Using Lemma 4.26, it is straightforward to show that $\mathbf{1}$ forces $\mathsf{PA}^- + \mathsf{exp}$. We argue only for exp. Take any name v. Clearly, the function w defined by $w(x) = 2^{v(x)}$ is total, so $\mathbf{1} \Vdash_{\tau_1} w\!\downarrow$. Since '$z = 2^y$' is a Δ_0 formula, we can use Lemma 4.26 to get $\mathbf{1} \Vdash_{\tau_1} w = 2^v$.

By our discussion of finite axiomatization of $\mathsf{I}\Delta_0 + \mathsf{exp}$ in Section 4.1, we can assume that Δ_0-induction is given just by one sentence $\forall \overline{x}\, \varphi(\overline{x})$, where φ is Δ_0. We show that $\mathbf{1} \Vdash_{\tau_1} \forall \overline{x}\, \varphi(\overline{x})$. So, let $\overline{v}$ be any names. By Definition 4.24 (iv), $\mathbf{1} \Vdash_{\tau_1} \overline{v}\!\downarrow$. Since $\mathsf{I}\Delta_0$ holds in the interpreting theory we know that for every number x the formula $\varphi(\overline{v}(x))$ also holds. Thus, by Lemma 4.26, we obtain $\mathbf{1} \Vdash_{\tau_1} \forall \overline{x}\, \varphi(\overline{x})$.

Now we show that the axiom SC is forced. We check first that the set $\mathbb{I}$ is forced to be an initial segment, that is, $\mathbf{1}$ forces the sentence $\forall x, y\,\big((x \in \mathbb{I} \wedge y < x) \Rightarrow y \in \mathbb{I}\big)$. So, take some names v and w and let s be a condition such that $s \Vdash_{\tau_1} (v \in \mathbb{I} \wedge w < v)$. This means that $\forall^\infty x \in s\,(w(x) < v(x))$ and that there is $i \in \mathrm{I}_1^0$ such that $\forall^\infty x \in s\,(v(x) < i)$. Then, clearly, it holds that $\forall^\infty x \in s\,(w(x) < i)$, so by Definition 4.24 (vi) we obtain $s \Vdash_{\tau_1} w \in \mathbb{I}$.

Next we check that $\mathbb{I}$ is forced to be closed under multiplication, i.e. $\mathbf{1}$ forces the sentence $\forall y, z\big((y \in \mathbb{I} \wedge z \in \mathbb{I}) \Rightarrow yz \in \mathbb{I}\big)$. So, let v, w be any names and suppose that s is a condition such that $s \Vdash_{\tau_1} (v \in \mathbb{I} \wedge w \in \mathbb{I})$. Then, there are $i, j \in \mathrm{I}_1^0$ such that $\forall^\infty x \in s\,\big(v(x) \leqslant i \wedge w(x) \leqslant j\big)$. By Proposition 1.16, we know that the cut I_1^0 is closed under multiplication, so we have $ij \in \mathrm{I}_1^0$. Then, $\forall^\infty x \in s\,\big(v(x)w(x) \leqslant ij\big)$ so, by Definition 4.24 (vi), we obtain $s \Vdash_{\tau_1} vw \in \mathbb{I}$.

To show that $\mathbb{I}$ is forced to be a proper cut we find a name d such that $\mathbf{1}$ forces that d is strictly greater than any element of $\mathbb{I}$, i.e. $\mathbf{1} \Vdash_{\tau_1} \forall y\big(y \in \mathbb{I} \Rightarrow y < d\big)$.

We use the unbounded set $A = \{a_i\}_{i \in \mathrm{I}_1^0}$ as in Proposition 1.13 to define the following total function d:

$$d(x) = i, \tag{4.18}$$

where i is the unique element of I_1^0 such that $x \in (a_{i-1}, a_i]$. Clearly, the function d is $\Delta_1(A)$-definable so it is a valid name. Now, let v be a name and let s be any condition such that $s \Vdash_{\tau_1} v \in \mathbb{I}$. Then there exists a number $i \in \mathrm{I}_1^0$ such that $\forall^\infty x \in s\,(v(x) \leqslant i)$. The definition (4.18) of d guarantees that for all $x > a_i$ it holds that $d(x) > i$, so we get $\forall^\infty x \in s\,\big(v(x) < d(x)\big)$ and then, by Definition 4.24 (v), we have $s \Vdash_{\tau_1} v < d$. By the definition (FT8) of forcing a universal formula, we obtain $\mathbf{1} \Vdash_{\tau_1} \forall y\big(y \in \mathbb{I} \Rightarrow y < d\big)$.

Finally, we show that $\mathbf{1}$ forces a strengthening of the last property of $\mathbb{I}$ mentioned by the axiom SC: there exists $z > \mathbb{I}$ such that for all x the value $2_z(x)$ exists. Using the function d defined above, for every name v we can define a function w by $\Delta_1(A)$-comprehension as follows:

$$w(x) = 2_{d(x)}(v(x)).$$

The function w is total since each value of d is in I_1^0, and for each number x and each $i \in \mathrm{I}_1^0$ the value of $2_i(x)$ exists. Otherwise, for some number x the set $J = \{j \in \mathbb{N} \colon \exists y\,(y = 2_j(x))\}$ would be a Σ_1^0-definable cut strictly contained in I_1^0 (recall from Section 1.1 that $y = 2_j(x)$ is a Δ_0 formula), which is impossible since I_1^0 is the intersection of all Σ_1^0-definable cuts.

Thus, w is a valid name, and given its definition we can apply Lemma 4.26 to learn that $\mathbf{1} \Vdash_{\tau_1} w = 2_d(v)$. Since v is an arbitrary name, we obtain that $\mathbf{1} \Vdash_{\tau_1} \forall x\,\exists y\,(y = 2_d(x))$. Finally, we can conclude that $\mathbf{1} \Vdash_{\tau_1} \exists z > \mathbb{I}\ \forall x\,\exists y\,(y = 2_z(x))$, because we have already shown that $\mathbf{1} \Vdash_{\tau_1} d > \mathbb{I}$. ☐

4.3.3 Generic cut

We start directly with definitions as we have already discussed the motivation and intuitions for the second forcing interpretation in Section 4.3.1.

Since the language $\mathcal{L}_{\mathrm{II}}$ has two sorts of variables, the following definition specifies two sets of names and distinguishes two cases for the relation of validity. Both first- and second-order names are just arbitrary numbers, but in the latter case we think about names as codes for finite sets and denote them by capital letters.

Definition 4.28. The forcing translation τ_2 from the language $\mathcal{L}_{\mathrm{II}}$ to the language $\mathcal{L}_{\mathrm{I}} \cup \{\mathbb{I}\}$ is defined as follows:

(i) $s \in \mathrm{Cond}_{\tau_2}$ is

$$\forall x, y \in_{\mathrm{Ack}} s\,(x < y \Rightarrow 2^x < y)\ \wedge\ |s| > \mathbb{I};$$

(ii) $s' \trianglelefteq_{\tau_2} s$ is

$$\forall x\,(x \in_{\mathrm{Ack}} s' \Rightarrow x \in_{\mathrm{Ack}} s);$$

(iii) $v \in \mathrm{Name}_{\tau_2}$ is
$$v = v;$$

(iv) $V \in \mathrm{Name}_{\tau_2}$ is
$$V = V;$$

(v) $s \Vdash_{\tau_2} v\!\downarrow$ is
$$s \in \mathrm{Cond}_{\tau_2} \ \wedge\ v \in \mathrm{Name}_{\tau_2} \ \wedge\ s \cap [0, v] \notin \mathrm{Cond}_{\tau_2};$$

(vi) $s \Vdash_{\tau_2} V\!\downarrow$ is
$$s \in \mathrm{Cond}_{\tau_2} \ \wedge\ V \in \mathrm{Name}_{\tau_2};$$

(vii) if $\alpha(\overline{v})$ is an atomic formula of the form $t_1(\overline{v}) = t_2(\overline{v})$ or $t_1(\overline{v}) \leqslant t_2(\overline{v})$, then $s \Vdash_{\tau_2} \alpha(\overline{v})$ is
$$s \Vdash_{\tau_2} \overline{v}\!\downarrow \ \wedge\ t_1(\overline{v}) = t_2(\overline{v})$$
or
$$s \Vdash_{\tau_2} \overline{v}\!\downarrow \ \wedge\ t_1(\overline{v}) \leqslant t_2(\overline{v}),$$
respectively;

(viii) if $\alpha(\overline{v}, V)$ is an atomic formula of the form $t(\overline{v}) \in V$, then $s \Vdash_{\tau_2} \alpha(\overline{v}, V)$ is
$$s \Vdash_{\tau_2} \overline{v}\!\downarrow \ \wedge\ s \Vdash_{\tau_2} V\!\downarrow \ \wedge\ t(\overline{v}) \in_{\mathrm{Ack}} V.$$

We will often omit the subscript '$_{\mathrm{Ack}}$' when we work in $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$. This should not lead to any confusion, by our convention to use capital letters to denote names for sets.

Later it will be convenient to have the following list of simple properties of the set of forcing conditions of τ_2.

Lemma 4.29. *Let $s = \{s_1 < \cdots < s_c\}$ be a forcing condition of τ_2. Then $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$ proves the following.*

(a) s can be split into a disjoint union of two conditions $s = s_1 \sqcup s_2$ such that $\max(s_1) < \min(s_2)$.

(b) Any subset of s with at least $\lceil \sqrt{c}\,\rceil$-many elements is also a condition.

(c) For all names v and w, if $s \Vdash_{\tau_2} v\!\downarrow$ and $w < v$, then $s \Vdash_{\tau_2} w\!\downarrow$.

(d) For every name v, if $s \Vdash_{\tau_2} v\!\downarrow$, then $v < \max(s)$.

Proof. We reason in $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$.

(a) Let $s_1 := \{s_1 < \ldots s_{\lfloor \frac{c}{2} \rfloor}\}$ and $s_2 := s \setminus s_1$. Clearly, both s_1 and s_2 are exponentially sparse. Since $|s| = c \leqslant \lfloor \frac{c}{2} \rfloor + \lfloor \frac{c}{2} \rfloor + 1$ and $\mathbb{I}$ is an initial segment closed under addition, we must have $|s_1| = \lfloor \frac{c}{2} \rfloor \notin \mathbb{I}$, so both s_1 and s_2 are conditions.

(b) Let s' be a subset of s with $\lceil\sqrt{c}\,\rceil$-many elements. Since $|s| < (\lceil\sqrt{c}\,\rceil)^2$ and $\mathbb{I}$ is an initial segment closed under multiplication, we have $\lceil\sqrt{c}\,\rceil \notin \mathbb{I}$. Clearly, s' is also exponentially sparse, so it is a condition.

(c) If $w < v$, then $s \cap [0, w] \subseteq s \cap [0, v]$, and thus $|s \cap [0, w]| \leqslant |s \cap [0, v]| \in \mathbb{I}$, so w is not a condition.

(d) If $v \geqslant \max(s)$, then $s \cap [0, v] = s \in \mathrm{Cond}_{\tau_2}$. □

Note that all the clauses of the above lemma are formulated as single sentences and not as schemes, so including (the formal version of) their proofs in any other proof we construct will only increase the time complexity of the relevant algorithm by a fixed additive constant.

Lemma 4.30. *τ_2 is a polynomial forcing interpretation of the language $\mathcal{L}_{\mathrm{II}}$ in the theory $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$.*

Proof. We first check that $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$ proves that τ_2 satisfies the conditions from Definition 4.10 given by single sentences, that is, (FI1)-(FI5), (FI7)-(FI9) and (FI12).

The relation $\trianglelefteq_{\tau_2}$ is clearly a preorder because it coincides with set inclusion. To see that its field is nonempty, note that the axiom SC guarantees that there are in fact unboundedly many forcing conditions: for any number x, if y is a number above the cut $\mathbb{I}$ such that the value $2_y(x)$ exists, then for $y' := 2 \cdot \lfloor \frac{y}{2} \rfloor$ the set $s = \{2_2(x), 2_4(x), \ldots, 2_{y'}(x)\}$ is exponentially sparse and has y'-many elements, where $y' > \mathbb{I}$ because $\mathbb{I}$ is closed under multiplication. The code for s exists because it is bounded by the value $2_{y'+1}(x)$. Thus, the conditions (FI1)-(FI3) are satisfied.

For (FI4), which says that any generic model is nonempty, note that every condition forces each standard natural number to be a valid name. To see that the monotonicity condition for names (FI5) holds, assume that $s' \trianglelefteq_{\tau_2} s$ and $s \Vdash_{\tau_2} v\downarrow$, which means that $|s \cap [0, \ldots, v]| \in \mathbb{I}$. Then, by the definition of $\trianglelefteq_{\tau_2}$, we get $s' \cap [0, \ldots, v] \subseteq s \cap [0, \ldots, v]$, so $|s' \cap [0, \ldots, v]| \in \mathbb{I}$ and thus $s' \Vdash_{\tau_2} v\downarrow$, as required.

The conditions (FI7)-(FI9) expressing that equality is an equivalence relation follow immediately from the definition of forcing atomic formulas. Concerning the density condition (FI12), we only need some care in the case of first-order names, because the definition of $s \Vdash_{\tau_2} v\downarrow$ is nontrivial. We reason by contraposition. Let s and v be such that $s \nVdash_{\tau_2} v$. This means that the set $s \cap [0, v]$ is a condition, i.e. it has more than $\mathbb{I}$ many elements. But then for any condition $s' \trianglelefteq_{\tau_2} s \cap [0, v] \trianglelefteq s$ it holds that $s' \cap [0, v] = s'$, so it cannot force v to be a valid name. Hence, it is not densely forced below s that v is a valid name.

The condition (FI10), saying that the values of functions are well-defined, is also given by a single sentence because there are only three function symbols in the interpreted language. In fact, we can show something stronger than (FI10): there exists a polynomial-time algorithm which, given as input a term $t(\overline{x})$ of

$\mathcal{L}_{\mathrm{II}}$, outputs a proof of the sentence:

$$\forall s \in \mathrm{Cond}_\tau \ \forall \overline{v} \in \mathrm{Name}_\tau \Big(s \Vdash_\tau \overline{v}\!\downarrow \,\Rightarrow \big(\exists w \in \mathrm{Name}_\tau \left(s \Vdash_{\tau_2} w\!\downarrow \,\wedge\, s \Vdash_{\tau_2} t(\overline{v}) = w\right) \\ \wedge\ s \Vdash_\tau \forall w, w' \left(w = t(\overline{v}) \,\wedge\, w' = t(\overline{v}) \ \Rightarrow\ w = w'\right)\big)\Big) \tag{4.19}$$

We describe how to construct a proof of the above sentence for any term $t(\overline{x})$ of $\mathcal{L}_{\mathrm{II}}$. One picks arbitrary s and $\overline{v}$ and assumes that $s \Vdash_{\tau_2} \overline{v}\!\downarrow$. The proof of the uniqueness of the value $t(\overline{v})$ requires only invoking the definition of forcing for atomic formulas as well as clauses (FT7) and (FT8). This is a single template in which one has to substitute the term t a fixed number of times.

The proof of the existence of the value $t(\overline{v})$ is constructed by recursion on subterms of t. The base step is trivial as we have to consider subterms which are either variables or numerals. For the recursive step, assume that the algorithm has already constructed proofs of the existence of the values v_1, v_2 for subterms $r_1(\overline{v})$, $r_2(\overline{v})$. By Lemma 4.29 (c), it is enough to check that there is a valid name w for the complex term $v' \cdot v' + 1$, where $v' = \max\{v_1, v_2\}$. So, from the assumption $s \Vdash_{\tau_2} v'\!\downarrow$ we know that $|s \cap [0, v']|$ is in $\mathbb{I}$. Since s is exponentially sparse, it holds that $|s \cap [0, v'^2 + 1]| \leqslant |s \cap [0, v']| + 1 \in \mathbb{I}$, so $s \cap [0, v'^2 + 1]$ is not a condition either and thus $s \Vdash_{\tau_2} v'^2 + 1\!\downarrow$. The proof of (4.19) for t is finished by recalling Lemma 4.16 (a) and noting that s and $\overline{v}$ were arbitrary.

The schematic conditions (FI6), (FI11) and (FI13), which concern forcing atomic formulas, follow immediately from the definition of τ_2. The algorithm constructs their proofs using three fixed templates, in which it substitutes expressions like $s \Vdash_{\tau_2} \alpha(\overline{v})$ and, in case of (FI6) and (FI13), also previously constructed proofs for (FI5) and (FI12), respectively. This clearly takes time polynomial in the size of a given atomic formula α. □

The next lemma extends clause (viii) of Definition 4.28 and says that a Δ^0_0 formula is forced if and only if its Ackermann translation holds. Here, by *the Ackermann translation* of a Δ^0_0 formula φ we mean an $\mathcal{L}_{\mathrm{I}}$-formula φ_{Ack} which is obtained from φ by replacing all atomic subformulas of the form '$t \in X$' with '$t \in_{\mathrm{Ack}} X$', where the number variable X does not occur in φ – recall that here we use capital letters for numbers that occur as names for sets. To keep the notation simple, we do not distinguish between an $\mathcal{L}_{\mathrm{II}}$-formula and its Ackermann translation.

Lemma 4.31. *There exists a polynomial-time algorithm which, given as input a Δ^0_0 formula $\varphi(\overline{x}, \overline{X})$ of $\mathcal{L}_{\mathrm{II}}$, outputs a proof in $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$ of the sentence:*

$$\forall s \in \mathrm{Cond}_{\tau_2} \ \forall \overline{v}, \overline{V} \in \mathrm{Name}_{\tau_2} \big(s \Vdash_{\tau_2} \overline{v}\!\downarrow \Rightarrow \big(s \Vdash_{\tau_2} \varphi(\overline{v}, \overline{V}) \Leftrightarrow \varphi(\overline{v}, \overline{V})\big)\big). \tag{4.20}$$

Proof. We show how to construct a proof of (4.20) for a given Δ^0_0 formula by recursion on its subformulas. The argument that the construction can be performed by a polynomial-time algorithm is similar to the one in the proof of Lemma 4.26.

So, let $\varphi(\overline{x}, \overline{X})$ be a Δ^0_0 formula. The base step for its atomic subformulas is guaranteed by clauses (vii) and (viii) of Definition 4.28. We consider three cases of a complex subformula $\psi(\overline{x}, \overline{X})$. We always assume that tuples of names $\overline{v}$ and $\overline{V}$ have the same length as $\overline{x}$ and $\overline{X}$, respectively.

Suppose that $\psi(\overline{x}, \overline{X})$ has the form $\neg\theta(\overline{x}, \overline{X})$ and we have already constructed a proof of (4.20) for $\theta(\overline{x}, \overline{X})$. Take any s, $\overline{v}$ and $\overline{V}$ such that $s \Vdash_{\tau_2} \overline{v}\downarrow$. Assume first that $s \Vdash_{\tau_2} \neg\psi(\overline{v}, \overline{V})$. By Lemma 4.13, we have $s \nVdash_{\tau_2} \psi(\overline{v}, \overline{V})$, so by the proof constructed in a previous step of the recursion we get $\neg\psi(\overline{v}, \overline{V})$.

Conversely, assume that $\neg\psi(\overline{v}, \overline{V})$ holds. By the previously constructed proof for $\psi(\overline{x}, \overline{X})$, no condition forces $\psi(\overline{v}, \overline{V})$. In particular, no condition below s forces this sentence so, by the definition of forcing negation (FT6), we obtain $s \Vdash_{\tau_2} \neg\psi(\overline{v}, \overline{V})$.

Suppose that $\psi(\overline{x}, \overline{X})$ has the form $\theta(\overline{x}, \overline{X}) \Rightarrow \zeta(\overline{x}, \overline{X})$ and we have already constructed proofs of (4.20) for $\theta(\overline{x}, \overline{X})$ and $\zeta(\overline{x}, \overline{X})$. Let s, $\overline{v}$ and $\overline{V}$ be such that $s \Vdash_{\tau_2} \overline{v}\downarrow$. Assume that $s \Vdash_{\tau_2} \theta(\overline{v}, \overline{V}) \Rightarrow \zeta(\overline{v}, \overline{V})$ and that $\theta(\overline{v}, \overline{V})$ holds. By the proof constructed in a previous step, we have $s \Vdash_{\tau_2} \theta(\overline{v}, \overline{V})$. Thus, by the definition of forcing implication (FT7), we obtain a condition $s' \trianglelefteq_{\tau_2} s$ such that $s' \Vdash_{\tau_2} \zeta(\overline{v}, \overline{V})$. Hence, by the previously constructed proof of (4.20) for $\zeta(\overline{x}, \overline{X})$, we get $\zeta(\overline{v}, \overline{V})$.

For the other direction, assume that $\theta(\overline{v}, \overline{V}) \Rightarrow \zeta(\overline{v}, \overline{V})$ holds and let $s' \trianglelefteq_{\tau_2} s$ be such that $s' \Vdash_{\tau_2} \theta(\overline{v}, \overline{V})$. By the proof constructed in a previous step, we have $\theta(\overline{v}, \overline{V})$ so, by our assumption, $\zeta(\overline{v}, \overline{V})$ holds as well. Thus, by the proof of (4.20) for $\zeta(\overline{x}, \overline{X})$, we get $s' \Vdash_{\tau_2} \zeta(\overline{v}, \overline{V})$. Therefore, by the definition of forcing implication (FT7), we conclude that $s \Vdash_{\tau_2} \theta(\overline{v}, \overline{V}) \Rightarrow \zeta(\overline{v}, \overline{V})$.

Suppose that $\psi(\overline{x}, \overline{X})$ has the form $\forall y \leqslant t(\overline{x})\, \theta(y, \overline{x}, \overline{X})$, which is shorthand for $\forall y\big(y \leqslant t(\overline{x}) \Rightarrow \theta(y, \overline{x}, \overline{X})\big)$, where y is not among $\overline{x}$, and that we have already constructed a proof of (4.20) for $\theta(y, \overline{x}, \overline{X})$. Let s and $\overline{v}$ be such that $s \Vdash_{\tau_2} \overline{v}\downarrow$ and assume that $s \Vdash_{\tau_2} \forall y\big(y \leqslant t(\overline{v}) \Rightarrow \theta(\overline{v}, \overline{V})\big)$. Take some $u \leqslant t(\overline{v})$. By the proof of Lemma 4.30, specifically by the proof of (4.19), there exists a name w such that $s \Vdash_{\tau_2} w\downarrow$ and $s \Vdash_{\tau_2} w = t(\overline{v})$. By the definition of forcing for atomic formulas, we know that $w = t(\overline{v})$ so, by Lemma 4.29 (c), we get $s \Vdash_{\tau_2} u\downarrow$. Again, by the definition of forcing for atomic formulas we get $s \Vdash_{\tau_2} u \leqslant t(\overline{v})$. By (FT7) and (FT8), there exists $s' \trianglelefteq_{\tau_2} s$ such that $s' \Vdash_{\tau_2} \theta(u, \overline{v}, \overline{V})$ so, by the previously constructed proof of (4.20) for $\theta(y, \overline{x}, \overline{X})$, we obtain $\theta(u, \overline{v}, \overline{V})$. Since u was arbitrary, we conclude that $\forall y \leqslant t(\overline{x})\, \theta(y, \overline{x}, \overline{X})$ holds.

Conversely, assume that $\forall y\big(y \leqslant t(\overline{v}) \Rightarrow \theta(\overline{v}, \overline{V})\big)$ holds. Take a name u and a condition $s' \trianglelefteq_{\tau_2} s$ such that $s' \Vdash_{\tau_2} u\downarrow$ and $s' \Vdash_{\tau_2} u \leqslant t(\overline{v})$. By the definition of forcing for atomic formulas we have $u \leqslant t(\overline{v})$ so, by our assumption, $\theta(u, \overline{v}, \overline{V})$ holds. By the previously constructed proof of (4.20) for $\theta(y, \overline{x}, \overline{X})$, we get $s' \Vdash_{\tau_2} \theta(u, \overline{v}, \overline{V})$ so, by the definitions of forcing implication (FT7) and a universal sentence (FT8), we obtain that $s \Vdash_{\tau_2} \forall y\big(y \leqslant t(\overline{v}) \Rightarrow \theta(\overline{v}, \overline{V})\big)$. $\square$

It remains to prove that all the axioms of $\mathsf{RCA}^*_0 + \mathsf{CAC}$ are forced. In fact, we can show without any additional effort that τ_2 is a polynomial forcing interpretation of $\mathsf{WKL}^*_0 + \mathsf{CAC}$ (cf. Theorem 1.9).

Lemma 4.32. *The forcing translation τ_2 is a polynomial forcing interpretation of* $\mathsf{WKL}_0^* + \mathsf{CAC}$ *in the theory* $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$.

Proof. It is enough to show that $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$ proves that every condition of τ_2 forces every axiom of $\mathsf{WKL}_0^* + \mathsf{CAC}$. Then the polynomiality of τ_2 will follow by Lemma 4.30 and the fact that $\mathsf{WKL}_0^* + \mathsf{CAC}$ is finitely axiomatized, as explained in Section 4.1.

So, we reason in $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$. It follows immediately from Lemma 4.31 and the definition of forcing a universal formula (FT8) that the axioms of PA^- are forced by every condition. Let us check that exp is also forced. Suppose that $s \Vdash_{\tau_2} v\downarrow$, where $s = \{s_1 < \cdots < s_c\}$. Then $s \cap [0, v] = \{s_1, \ldots, s_i\}$ is not a condition, so $i \in \mathbb{I}$. Since $v < s_{i+1}$ and s is exponentially sparse, we have $2^v < s_{i+2}$, and hence $s \cap [0, 2^v] \subseteq \{s_1, \ldots, s_{i+1}\}$. Thus, we get $|s \cap [0, 2^v]| \leqslant i + 1 \in \mathbb{I}$, and so $s \Vdash_{\tau_2} 2^v\downarrow$. By Lemma 4.31, this implies $s \Vdash_{\tau_2} \mathsf{exp}$ because '$y = 2^x$' is a Δ_0 formula.

For Δ_0^0-induction, by the discussion of finite axiomatization of RCA_0^* in Section 4.1, we can assume that it is given by a single sentence $\forall \overline{X}\, \forall \overline{x}\, \varphi(\overline{X}, \overline{x})$, where φ is Δ_0^0. Let $\overline{V}$ and $\overline{v}$ be tuples of names of the same length as $\overline{X}$ and $\overline{x}$, and let s be a condition such that $s \Vdash_{\tau_2} \overline{v}\downarrow$ holds. Since we work under $\mathsf{I}\Delta_0$, we know that (the Ackermann translation of) $\varphi(\overline{V}, \overline{v})$ holds. Therefore, by Lemma 4.31, we obtain $s \Vdash_{\tau_2} \varphi(\overline{V}, \overline{v})$, as required.

To prove that WKL and Δ_1^0-comprehension are forced we follow a fragment of the proof of Lemma 2.15 in [32]. Recall that the Σ_1^0-separation principle is the following scheme:

$$\forall \overline{z}\, \forall \overline{Z} \big(\forall x \big(\exists y\, \varphi_1(x, y, \overline{z}, \overline{Z}) \Rightarrow \neg \exists y\, \varphi_2(x, y, \overline{z}, \overline{Z})\big) \Rightarrow \\ \exists X \forall x \big((\exists y\, \varphi_1(x, y, \overline{z}, \overline{Z}) \Rightarrow x \in X) \wedge (\exists y\, \varphi_2(x, y, \overline{z}, \overline{Z}) \Rightarrow x \notin X)\big)\big),$$

where φ_1, φ_2 are Δ_0^0. By [47, Lemma IV.4.4]), this principle implies over RCA_0 both WKL and Δ_1^0-comprehension, and it is easy to check that the implication remains valid over RCA_0^* (cf. the proof of Lemma 3.2 in [16]). Note that we only need to show that every condition forces a fixed finite number of instances of Σ_1^0-separation which is needed to prove WKL_0 and the finitely many instances of Δ_1^0-comprehension occurring in our axiomatization of RCA_0^*. Then, by Corollary 4.18, we can conclude that every condition forces WKL and Δ_1^0-comprehension as well.

So, let s, $\overline{v}$ and $\overline{V}$ be such that $s \Vdash_{\tau_2} \overline{v}\downarrow$ and $s \Vdash_{\tau_2} \forall x \big(\exists y\, \varphi_1(x, y, \overline{v}, \overline{V}) \Rightarrow \neg \exists y\, \varphi_2(x, y, \overline{v}, \overline{V})\big)$. Define the following finite set:

$$W := \{x < \max(s) \colon \exists y < \max(s)\, \big(\varphi_1(x, y, \overline{v}, \overline{V}) \,\wedge\, \forall z < y\, \neg \varphi_2(x, z, \overline{v}, \overline{V})\big)\}.$$

We check that already s forces the following universal sentence:

$$\forall x \big((\exists y\, \varphi_1(x, y, \overline{v}, \overline{V}) \Rightarrow x \in W) \,\wedge\, (\exists y\, \varphi_2(x, y, \overline{v}, \overline{V}) \Rightarrow x \notin W)\big). \tag{4.21}$$

Let s' and u be such that $s' \trianglelefteq_{\tau_2} s$ and $s' \Vdash_{\tau_2} u\downarrow$. By Lemma 4.16 (a), it is enough to show that s' forces each of the conjuncts of the instance of (4.21) for

u. For the first one, we assume, purely for simplicity of notation, that already $s' \Vdash_{\tau_2} \exists y\, \varphi_1(u, y, \overline{v}, \overline{V})$. We show that $u \in W$ which, by Lemma 4.31, implies that also $s' \Vdash_{\tau_2} u \in W$.

By Lemma 4.16 (b), there exist a condition $s'' \trianglelefteq_{\tau_2} s'$ and a name w such that $s'' \Vdash_{\tau_2} w\downarrow$ and $s'' \Vdash_{\tau_2} \varphi_1(u, w, \overline{v}, \overline{V})$. Then, by Lemma 4.31, $\varphi_1(u, w, \overline{v}, \overline{V})$ holds. By Lemma 4.29 (d), it holds that $w < \max(s'')$. On the other hand, by Lemma 4.14 (a), $s'' \Vdash_{\tau_2} \exists y\, \varphi_1(u, y, \overline{v}, \overline{V}) \Rightarrow \neg\exists y\, \varphi_2(u, y, \overline{v}, \overline{V})$. Thus, any z such that $\varphi_2(u, z, \overline{v}, \overline{V})$ holds cannot be forced by s'' to be a valid name, because otherwise s'' would also force $\varphi_2(u, z, \overline{v}, \overline{V})$. In particular, any such z has to be greater than w, so u satisfies the condition defining W, as required.

For the second conjunct of (4.21), we assume as previously that $s' \Vdash_{\tau_2} \exists y\, \varphi_2(u, y, \overline{v}, \overline{V})$, and we show that $u \notin W$, which implies $s' \Vdash_{\tau_2} u \notin W$ by Lemma 4.31. As above, by Lemma 4.16 (b) and Lemma 4.29 (d), we get a condition $s'' \trianglelefteq_{\tau_2} s'$ and a name $z < \max(s'')$ such that $s'' \Vdash_{\tau} z\downarrow$ and $s'' \Vdash_{\tau_2} \varphi_2(u, z, \overline{v}, \overline{V})$. Thus, by Lemma 4.31, we have that $\varphi_2(u, z, \overline{v}, \overline{V})$ holds.

Towards a contradiction, suppose that $u \in W$. Then, there must be some number $w < \max(s)$ such that $\varphi_1(u, w, \overline{v}, \overline{V})$ holds but for every $k < w$ we have $\neg\varphi_2(u, k, \overline{v}, \overline{V})$. In particular, we have $w \leqslant z$ so, by Lemma 4.29 (c), we get $s'' \Vdash_{\tau_2} w\downarrow$. But now, by Lemma 4.31, we learn that $s'' \Vdash_{\tau_2} \varphi_1(u, w, \overline{v}, \overline{V})$ so, obviously, $s'' \Vdash_{\tau_2} \exists y\, \varphi_1(u, y, \overline{v}, \overline{V})$. This is a contradiction as, by Lemma 4.14 (a), $s'' \Vdash_{\tau_2} \exists y\, \varphi_1(u, y, \overline{v}, \overline{V}) \Rightarrow \neg\exists y\, \varphi_2(u, y, \overline{v}, \overline{V})$. Therefore, $u \notin W$. This completes the proof that WKL and Δ^0_1-comprehension are forced.

Finally, we check that CAC is forced. We will use the more intuitive symbol $\preccurlyeq$ for a finite set that codes a partial order. Let s be such that $s \Vdash_{\tau_2}$ '$\preccurlyeq$ is a partial order on $\mathbb{N}$'. We will find a condition $s^* \trianglelefteq_{\tau_2} s$ which forces the sentence 'There exists an unbounded chain or antichain in $\preccurlyeq$'. By Lemma 4.29 (a), we can split s into a disjoint union $s = s_1 \sqcup s_2$ such that $\max(s_1) < \min(s_2)$ and both s_1, s_2 are conditions. Now, for every $v \in s_1$ we have $s_2 \cap [0, v] = \emptyset$, so $s_2 \Vdash_{\tau_2} v\downarrow$. By Lemma 4.14 (a), s_2 also forces '$\preccurlyeq$ is a partial order on $\mathbb{N}$' so, by Lemma 4.31, we learn that $\preccurlyeq$ is a partial order on $s_1 \times s_1$. Let c be the largest number such that $|s_1| \geqslant c(c-1)$. Then, from Theorem 4.5 (Dilworth's theorem), which is easily provable in $\mathsf{I}\Delta_0 + \mathsf{exp}$ by elementary finite combinatorics, it follows that there exists $s^* \subseteq s_1$ such that $|s^*| = c$ and s^* is a chain or an antichain in $\preccurlyeq\restriction s_1 \times s_1$. Note that, by Lemma 4.29 (b), s^* is also a condition. By Lemma 4.31, we obtain $s^* \Vdash_{\tau_2}$'s^* is a chain or antichain in $\preccurlyeq$'. Let us stress that 's^*' on the right of the previous formula occurs as a second-order name, so clearly $s^* \Vdash_{\tau_2} s^*\downarrow$.

The last thing to show is that s^* forces itself to be an unbounded set, that is, $s^* \Vdash_{\tau_2} \forall x\, \exists y \in s^*(x < y)$. So, pick some condition $s' \trianglelefteq_{\tau_2} s^*$ and a name v such that $s' \Vdash_{\tau_2} v\downarrow$. Then $s' \cap [0, v]$ is not a condition. By Lemma 4.29 (d), there are some $s_i, s_{i+1} \in s' = \{s_1, \ldots, s_c\}$ such that $s_i \leqslant v < s_{i+1}$. Note that $|s' \cap [0, v]| = i \in \mathbb{I}$, so clearly $|s' \cap [0, s_{i+1}]| = i + 1 \in \mathbb{I}$ and thus $s' \Vdash_{\tau_2} s_{i+1}\downarrow$. Since $s' \subseteq s^*$, we have $s_{i+1} \in s^*$, and so, by Lemma 4.31, we get $s' \Vdash_{\tau_2} (s_{i+1} \in s^* \wedge v < s_{i+1})$. This finishes the proof that CAC is forced by every condition. $\square$

4.3.4 Completing the proof

In this last section we finally prove Theorem 4.6. Our main task is to combine the forcing interpretations τ_1 and τ_2 to obtain a polynomial simulation of $\mathsf{WKL}_0^* + \mathsf{CAC}$ in $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \Sigma_1^0\text{-}\mathsf{LPC}$. Then we will complete the proof by adding the remaining cases of $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \neg\Sigma_1^0\text{-}\mathsf{LPC}$ and $\mathsf{RCA}_0^* + \mathsf{I}\Sigma_1^0$.

We need two technical lemmas. The first one says that our two forcing interpretations τ_1 and τ_2 can be composed. The second one states that the composition is $\forall\Pi_3^0$-reflecting.

Lemma 4.33. *There exists a polynomial-time algorithm which, given as input a proof δ of a sentence σ in $\mathsf{WKL}_0^*{+}\mathsf{CAC}$, outputs a proof in $\mathsf{RCA}_0^*{+}\neg\mathsf{I}\Sigma_1^0{+}\Sigma_1^0\text{-}\mathsf{LPC}$ of the sentence:*

$$\mathbf{1} \Vdash_{\tau_1} (\forall s \in \mathrm{Cond}_{\tau_2}\, (s \Vdash_{\tau_2} \sigma)).$$

Proof. Let δ be a proof of a sentence σ in $\mathsf{WKL}_0^* + \mathsf{CAC}$. By Lemma 4.32 and Corollary 4.18, there exists a polynomial-time algorithm which, given as input δ, returns a proof δ' in $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$ of the sentence $\forall s \in \mathrm{Cond}_{\tau_2}(s \Vdash_{\tau_2} \sigma)$. Now, by Lemma 4.27 and again Corollary 4.18, one can apply another polynomial-time algorithm which on input δ' outputs a proof in $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \Sigma_1^0\text{-}\mathsf{LPC}$ of the sentence $\mathbf{1} \Vdash_{\tau_1} (\forall s \in \mathrm{Cond}_{\tau_2}\, (s \Vdash_{\tau_2} \sigma))$, as required. □

Lemma 4.34. *There exists a polynomial-time algorithm which, given as input an $\mathcal{L}_{\mathrm{II}}$-sentence $\exists X \exists x\, \varphi(x)$, where $\varphi(X,x)$ is Π_2^0, outputs a proof in $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \Sigma_1^0\text{-}\mathsf{LPC}$ of the sentence:*

$$\exists X \exists x\, \varphi(X,x) \Rightarrow \mathbf{1} \Vdash_{\tau_1} \big(\exists s \in \mathrm{Cond}_{\tau_2}\, \exists V, v \in \mathrm{Name}_{\tau_2}\, (s \Vdash_{\tau_2} \varphi(V,v))\big). \tag{4.22}$$

Proof. We describe informally how to construct a proof of (4.22) for an arbitrary sentence $\exists X \exists x\, \varphi(X,x)$ with $\varphi(X,x)$ being Π_2^0. It should be easily seen that the main part of the proof is constructed from a single template for all such sentences, into which one substitutes a fixed number of times the sentence '$\exists X \exists x\, \varphi(X,x)$' or other expressions obtained from it in polynomial-time. This is complemented by a few auxiliary procedures that can be seen to be polynomial-time, which we will comment on in appropriate places below.

So, let $\exists X\, \exists x\, \varphi(X,x)$ be an $\mathcal{L}_{\mathrm{II}}$-sentence, where $\varphi(X,x)$ is Π_2^0. For technical reasons that will become clear below, we replace the subformula $\varphi(X,x) := \forall y\, \exists z\, \theta(X,x,y,z)$ with a Π_2^0 formula $\varphi^*(X,x) := \forall y\, \exists z\, \exists z' \leqslant z\ \theta^*(X,x,y,z,z')$, where all quantifiers in θ^* are bounded by z. The algorithm building the proof of (4.22) starts its work with the construction of a proof in RCA_0^* of the equivalence:

$$\forall X\, \forall x\, \big(\varphi(X,x) \Leftrightarrow \varphi^*(X,x)\big) \tag{4.23}$$

by recursion on subformulas of $\theta(X,x,y,z)$. We skip the details of this procedure, but it should be easily seen that it can be carried out in time polynomial in $|\theta(X,x,y,z)|$. To simplify notation, from now on we will assume that already $\varphi(X,x)$ is in the above form where each quantifier in the Δ_0^0 matrix is bounded by a variable.

Next, the algorithm writes the following definition of a (possibly partial) function of the variable y with parameters X and x:

$$f_\theta(X,x,y) = \min\{z > 2^y\colon\ \forall y' \leqslant y\ \exists z' \leqslant z\ \theta(X,x,y',z')\}. \tag{4.24}$$

Note that (4.24), as a definition of the graph of f_θ, is a $\Delta_0(X)$ definition. We emphasize that the first two arguments of the function f_θ are always some fixed parameters, so by iterations of f_θ on some number y we mean the values $f_\theta(X,x,y)$, $f_\theta(X,x,f_\theta(X,x,y))$, and so on.

Then, the algorithm finds the Ackermann translations (as described before Lemma 4.31) of the formula θ and of the definition of f_θ. This will be needed for the construction of proofs in the first-order theory $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$. The translations are readily constructed in polynomial time. Let us note that the first-order version of f_θ takes as its first parameter, instead of a set X, some number u seen as a code for a (finite) set. To enhance readability, we will slightly abuse notation and denote these translations also by θ and f_θ, respectively, which should not lead to any confusion.

The main part of the proof of (4.22) consists of proving two claims. The first one guarantees that, provably in $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma^0_1 + \Sigma^0_1\text{-}\mathsf{LPC}$, if the sentence $\exists X\,\exists x\,\forall y\,\exists z\,\theta(X,x,y,z)$ holds, then $\mathbf{1}$ forces that there exists a number y such that for some number u, the function $f_\theta(u,y,\cdot)$ can be iterated on y more than $\mathbb{I}$-many times (note that y occurs also as a parameter). Then, by the second claim, it will follow that the existence of these iterations allows to find a condition s and names V, v of τ_2 such that s forces $\forall y\,\exists z\,\theta(V,v,y,z)$, provably in $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$.

Claim 1. $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma^0_1 + \Sigma^0_1\text{-}\mathsf{LPC}$ proves the following sentence:

$$\exists X\exists x\,\forall y\,\exists z\,\theta(X,x,y,z) \Rightarrow \mathbf{1} \Vdash_{\tau_1} \exists y\,\exists u\,\exists z\,\exists c\,\big(z = f_\theta^{(c)}(u,y,y)\ \wedge\ c > \mathbb{I}\big), \tag{4.25}$$

and the proof can be constructed in time polynomial in $|\theta|$.

Proof of Claim 1. We will describe informally a proof in $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma^0_1 + \Sigma^0_1\text{-}\mathsf{LPC}$ of (4.25). Its main part can be obtained from a single template into which one has to substitute θ or other expressions or proofs obtained from θ in polynomial time. The only part of the proof which is not immediately seen to be constructible in time polynomial in $|\theta|$ is the proof of statement (4.27) which will be discussed below.

Assume the antecedent of the implication (4.25) and pick B and b such that $\forall y\,\exists z\,\theta(B,b,y,z)$ holds. Fix an unbounded set $A = \{a_i\}_{i\in \mathrm{I}^0_1}$ whose existence is guaranteed by the axiom Σ^0_1-LPC and Proposition 1.13. Recall from the remarks after Proposition 1.13 that for each number x there is a unique $i \in \mathrm{I}^0_1$ such that $x \in (a_{i-1}, a_i]$, and that the statement '$x \in (a_{i-1}, a_i]$' is expressed by a $\Delta_1(A)$ formula of x and i, the shape of which does not depend on the specific set A.

To witness the existential quantifier '$\exists u$' in the consequent of the implication (4.25) we will define a total function $\hat{B}$ that maps each number x to the initial segment of the set B needed to compute the first i iterations of the function f_θ on b, where i is such that $x \in (a_{i-1}, a_i]$. Before we give a definition of $\hat{B}$ let us make two observations.

Firstly, for every $i \in \mathrm{I}^0_1$, the value $f^{(i)}_\theta(B,b,b)$ exists. Otherwise, the set $\{x \in \mathbb{N}\colon \exists z\bigl(z = f^{(x)}_\theta(B,b,b)\bigr)\}$ would be a Σ^0_1-definable cut properly contained in I^0_1, contradicting the assumption that Σ^0_1-LPC holds.

Secondly, let $\psi(B,A,r,b,x,i)$ be a Δ^0_0 formula expressing that $x \in (a_{i-1}, a_i]$ and r is the sequence of the first i iterations of $f_\theta(B,b,\cdot)$ starting at b:

$$\begin{aligned}\psi(B,A,r,b,x,i) :=\; & x \in (a_{i-1}, a_i] \,\wedge\, |r| = i+1 \,\wedge\, (r)_0 = b \\ & \wedge\, \forall j < i\, ((r)_{j+1} = f_\theta(B,b,(r)_j)).\end{aligned} \tag{4.26}$$

By Kleene's normal form theorem for Σ^0_1 formulas (cf. [23, Lemma 7.13]), there is a Δ^0_0 formula ψ', a term $t(r,x_1,x_2,x_3)$ and a proof in RCA^*_0 of the following sentence:

$$\begin{aligned}\forall X,Y\,\forall r\,\forall x_1,x_2,x_3\bigl(\,&\psi(X,Y,r,x_1,x_2,x_3) \\ &\Leftrightarrow \forall u \geqslant t(r,x_1,x_2,x_3)\, \psi'(X{\restriction}u, Y, r, x_1,x_2,x_3)\bigr).\end{aligned} \tag{4.27}$$

In our case, because of the assumption that all quantifiers in θ are bounded by a variable, we can make the same assumption about ψ, and thus t can be found in polynomial time, say as $(r + x_1 + x_2 + x_3)^{|\psi|}$ (note that such a term has size polynomial in $|\psi|$). Then one can construct in polynomial time a routine proof that t majorizes any value needed to evaluate ψ. Let us note that for a general Δ_0 formula ψ it would not be possible to build t in polynomial time due to the possibility of simulating exponentiation by repeated squaring in quantifier bounds.

Now we can define the following $\Delta_1(A,B)$-functions, the first of which is constant:

$$\hat{b}(x) = b;$$

$$\hat{B}(x) = \text{the code for } B{\restriction}u,$$
$$\text{where } u = t(r,b,x,i), \text{ with } r \text{ unique satisfying (4.26) and } t \text{ as in (4.27)}.$$

We show that $\hat{B}$ and $\hat{b}$ witness the outer existential quantifiers '$\exists y\,\exists u$' in (4.25). As in the proof of Lemma 4.27, let the function d be defined by $d(x) = i$, where $x \in (a_{i-1}, a_i]$. We define a total $\Delta_1(A,B)$-function w as follows:

$$w(x) = f^{d(x)}_\theta(\hat{B}(x), \hat{b}(x), \hat{b}(x)). \tag{4.28}$$

The formula $z = f^{(v)}_\theta(u,y,y)$ is Δ^0_0 and (4.28) holds for every number x. Thus, we can apply Lemma 4.26 to get a proof that $\mathbf{1}$ forces $w = f^{(d)}_\theta(\hat{B},\hat{b},\hat{b})$. Finally, we construct a proof of the fact that $\mathbf{1} \Vdash_{\tau_1} d > \mathbb{I}$ in the same way as we did when proving Lemma 4.27, and thus we obtain a proof of (4.25). ⊣

Claim 2. $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$ proves the following sentence:

$$\begin{aligned}\exists y\,\exists u\,\exists z\,\exists c\,&\bigl(z = f^{(c)}_\theta(u,y,y) \,\wedge\, c > \mathbb{I}\bigr) \Rightarrow \\ &\exists s \in \mathrm{Cond}_{\tau_2}\,\exists v, V \in \mathrm{Name}_{\tau_2}\,\bigl(s \Vdash_{\tau_2} \forall y\,\exists z\,\theta(V,v,y,z)\bigr),\end{aligned} \tag{4.29}$$

and the proof can be constructed in time polynomial in $|\theta|$.

Proof of Claim 2. We describe informally a proof of (4.29) in $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$. As in the case of the first claim, it should be easily seen that the proof follows one and the same template for all θ, into which one has to substitute expressions and auxiliary proofs that can be obtained from θ in polynomial time.

Assume that the antecedent of the implication holds, and let b, B and $c > \mathbb{I}$ be such that the value $z = f_\theta^{(c)}(B, b, b)$ exists (note that here B is a number seen as a code for a finite set). Define the finite set:

$$s = \{x \in \mathbb{N} \colon \exists k < c \left(x = f_\theta^{(k)}(B, b, b)\right)\}. \tag{4.30}$$

By (the first-order translation of) the definition of the function f_θ (4.24), it follows that s is exponentially sparse. By our assumption, s has c elements and $c > \mathbb{I}$, so s is a condition of τ_2. Since b is the smallest element of s (for $k = 0$), we have that $|s \cap [0, b]| = 1 \in \mathbb{I}$, so $s \Vdash_{\tau_2} b\downarrow$. Clearly, B is a valid name for a second-order object.

We show that $s \Vdash_{\tau_2} \forall y\, \exists z\, \theta(B, b, y, z)$. Take any condition $\{s_1 < \cdots < s_k\} = s' \trianglelefteq_{\tau_2} s$ and a name v such that $s' \Vdash_{\tau_2} v\downarrow$. It follows that the set $s' \cap [0, v]$ is not a condition, so it has j elements $s_1 < \cdots < s_j$ for some $j \in \mathbb{I}$. Since $\mathbb{I}$ is a cut, it also holds that $j + 2 \in \mathbb{I} < k$. Thus, we can take the next two elements $s_{j+1}, s_{j+2} \in s'$ and, clearly, the set $s' \cap [0, s_{j+2}]$ is not a condition. Hence, s' forces both s_{j+1} and s_{j+2} to be valid names. By the definitions of s and the function f_θ, we learn that $\forall y \leqslant s_{j+1}\, \exists z \leqslant s_{j+2}\; \theta(B, b, y, z)$. Since $v < s_{j+1}$, we obtain that $\exists z \leqslant s_{j+2}\; \theta(B, b, v, z)$. The last formula is Δ_0, so by Lemma 4.31 we get $s' \Vdash_{\tau_2} \exists z\, \theta(B, b, v, z)$. By the definition of forcing a universal formula (FT8), this shows that $s \Vdash_{\tau_2} \forall y\, \exists z\, \theta(B, b, y, z)$. ⊣

The algorithm finishes constructing the proof of (4.22) as follows. It simulates the algorithm provided by Corollary 4.18 to transform the proof of (4.29) guaranteed by Claim 2 into a proof in $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \Sigma_1^0\text{-}\mathsf{LPC}$ of the sentence:

$$\begin{aligned} \mathbf{1} \Vdash_{\tau_1} \Big(\exists u\, \exists y\, \exists z\, \exists c \left(z = f_\theta^{(c)}(u, y, y) \wedge c > \mathbb{I}\right) \Rightarrow \\ \exists s \in \mathrm{Cond}_{\tau_2}\, \exists V, v \in \mathrm{Name}_{\tau_2} \left(s \Vdash_{\tau_2} \forall y\, \exists z\, \theta(V, v, y, z)\right)\Big). \end{aligned} \tag{4.31}$$

Then it applies the procedure given by Lemma 4.17 (b) to the antecedent and the consequent of the implication in (4.31) and, combining it with (4.31), obtains a proof of the sentence:

$$\begin{aligned} \mathbf{1} \Vdash_{\tau_1} \exists u\, \exists y\, \exists z\, \exists c \left(z = f_\theta^{(c)}(u, y, y) \wedge c > \mathbb{I}\right) \Rightarrow \\ \mathbf{1} \Vdash_{\tau_1} \exists s \in \mathrm{Cond}_{\tau_2}\, \exists V, v \in \mathrm{Name}_{\tau_2} \left(s \Vdash_{\tau_2} \forall y\, \exists z\, \theta(V, v, y, z)\right). \end{aligned} \tag{4.32}$$

Finally, the algorithm derives (4.22) in a fixed number of steps from (4.25) and (4.32), as required. □

Lemma 4.35. $\mathsf{WKL}_0^* + \mathsf{CAC}$ *is polynomially simulated by* $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \Sigma_1^0\text{-}\mathsf{LPC}$ *with respect to* $\forall\Pi_3^0$ *sentences.*

Proof. Let δ be a proof in $\mathsf{WKL}_0^* + \mathsf{CAC}$ of a sentence $\forall X \forall x\, \varphi(X,x)$, where $\varphi(X,x)$ is Σ_2^0. To avoid complicating the proof, we will ignore the distinction between $\neg\varphi$ and the Π_2^0 formula equivalent to it (by a polynomial-time constructible proof).

By Lemma 4.33 and Lemma 4.34, there exist algorithms that output proofs in $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \Sigma_1^0\text{-}\mathsf{LPC}$ of the sentences:

$$\mathbf{1} \Vdash_{\tau_1} \big(\forall s \in \mathrm{Cond}_{\tau_2}\, (s \Vdash_{\tau_2} \forall X \forall x\, \varphi(X,x))\big) \tag{4.33}$$

and

$$\exists X \exists x\, \neg\varphi(X,x) \Rightarrow \\ \mathbf{1} \Vdash_{\tau_1} (\exists s \in \mathrm{Cond}_{\tau_2}\, \exists V, v \in \mathrm{Name}_{\tau_2}\, (s \Vdash_{\tau_2} v\!\downarrow \wedge\, s \Vdash_{\tau_2} \neg\varphi(V,v))) \tag{4.34}$$

in time polynomial in $|\delta|$ and $|\varphi(X,x)|$, respectively. Note that one can construct in polynomial time (cf. a remark in the paragraph before Lemma 4.11) a proof that infers from (4.33) the sentence:

$$\mathbf{1} \Vdash_{\tau_1} \big(\forall s \in \mathrm{Cond}_{\tau_2}\, \forall V, v \in \mathrm{Name}_{\tau_2}\, \big(s \Vdash_{\tau_2} v\!\downarrow \Rightarrow\ s \Vdash_{\tau_2} \varphi(V,v)\big)\big). \tag{4.35}$$

On the other hand, by combining Lemma 4.13 and Corollary 4.18, there is a polynomial-time algorithm which, given as input the formula $\varphi(X,x)$, outputs a proof in $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \Sigma_1^0\text{-}\mathsf{LPC}$ of the sentence:

$$\mathbf{1} \Vdash_{\tau_1} \forall s \in \mathrm{Cond}_{\tau_2}\, \forall V, v \in \mathrm{Name}_{\tau_2} \neg\big(s \Vdash_{\tau_2} \varphi(V,v)\ \wedge\ s \Vdash_{\tau_2} \neg\varphi(V,v)\big). \tag{4.36}$$

Now, in polynomial time one can obtain a proof in $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \Sigma_1^0\text{-}\mathsf{LPC}$ of the sentence $\forall X \forall x\, \varphi(X,x)$, by combining (4.34), (4.35) and (4.36) in a fixed number of inferences using the algorithms from Section 4.2.2. □

The case when $\Sigma_1^0\text{-}\mathsf{LPC}$ fails is much simpler: as mentioned in Section 4.3.1, we have an almost trivial interpretation (in the usual non-forcing sense) of $\mathsf{I}\Delta_0 + \mathsf{exp} + \mathsf{SC}$ in $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \neg\Sigma_1^0\text{-}\mathsf{LPC}$. Therefore, we can define a forcing interpretation of $\mathsf{WKL}_0^* + \mathsf{CAC}$ directly in $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \neg\Sigma_1^0\text{-}\mathsf{LPC}$ and prove the reflection property as stated in Definition 4.19.

Lemma 4.36. $\mathsf{WKL}_0^* + \mathsf{CAC}$ *is polynomially simulated by* $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \neg\Sigma_1^0\text{-}\mathsf{LPC}$ *with respect to* $\forall\Pi_3^0$ *sentences.*

Proof. We appeal to Theorem 4.20. To define a polynomial forcing interpretation τ_3 of $\mathsf{WKL}_0^* + \mathsf{CAC}$ in $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \neg\Sigma_1^0\text{-}\mathsf{LPC}$ one repeats the definition of τ_2 in Section 4.3.3 replacing each occurrence of the predicate $\mathbb{I}$ with the definition of the cut I_1^0. The analogues of Lemmas 4.29-4.32 hold and are proved in the same way because I_1^0 satisfies the properties expressed by the axiom SC.

To see that τ_3 is polynomially $\forall\Pi_3^0$-reflecting we only need to adapt (in fact, simplify) the proof of Lemma 4.34. So, let us work in $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \neg\Sigma_1^0\text{-}\mathsf{LPC}$ and assume that a $\exists\Sigma_3^0$ sentence $\exists X\, \exists x\, \forall y\, \exists z\, \theta(X,x,y,z)$ holds, where θ is Δ_0^0. Take a set B and a number b that witness the existential quantifiers

'$\exists X\, \exists x$'. Define a $\Delta_0(B)$-function f_θ and a finite set s just as in (4.24) and (4.30), respectively. Note that with the current assumptions the definition of s makes sense as well: the value $f_\theta^{(c)}(B, b, b)$ exists for some number $c > \mathrm{I}_1^0$ because the set $\{x \in \mathbb{N} \colon \exists z \big(z = f_\theta^{(x)}(B, b, b)\big)\}$ is a Σ_1^0-definable cut which, by $\neg\Sigma_1^0$-LPC, is a proper superset of I_1^0. Now one can check like in the proof of Lemma 4.34 that the condition s forces $\forall y\, \exists z\, \theta(V, v, y, z)$, for the names $V :=$ *the initial segment of B needed to compute* $f_\theta^{(c)}(B, b, b)$ and $v := b$. This clearly implies (the contraposition of) the instance of the reflection property (4.11) for the $\forall\Pi_3^0$ sentence $\forall X\, \forall x\, \exists y\, \forall z\, \neg\theta(X, x, y, z)$. □

Finally, we are ready to prove the main theorem of this chapter.

Proof of Theorem 4.6. Let δ be a proof in $\mathsf{WKL}_0^* + \mathsf{CAC}$ of a $\forall\Pi_3^0$ sentence σ. By Lemma 4.35, Lemma 4.36 and Theorem 4.4, in time polynomial in $|\delta|$ one can find proofs of σ in, respectively, $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \Sigma_1^0$-$\mathsf{LPC}$, $\mathsf{RCA}_0^* + \neg\mathsf{I}\Sigma_1^0 + \neg\Sigma_1^0$-$\mathsf{LPC}$ and RCA_0. Combining these three proofs by means of a simple case distinction, one obtains a proof of σ in RCA_0^* in time polynomial in $|\delta|$. □

We conclude by some comments on the other combinatorial principles that we have studied in Chapter 2. By Proposition 2.1 (b), CAC implies ADS over RCA_0^*, and ADS is a single axiom, so we immediately obtain the following.

Corollary 4.37. $\mathsf{RCA}_0^* + \mathsf{ADS}$ *is polynomially simulated by* RCA_0^* *with respect to* $\forall\Pi_3^0$ *sentences.*

For CRT_2^2, we only know that it is implied by RT_2^2 over RCA_0^*. As opposed to RT_2^2 and CAC, there is no natural finite version of CRT_2^2 which could be used to construct a forcing interpretation analogous to the one from Section 4.3.3. As of now, we have no strong evidence concerning the question whether $\mathsf{RCA}_0^* + \mathsf{CRT}_2^2$ has speedup over RCA_0^*. For instance, [15] left open the question about CRT_2^2 implying any nontrivial closure properties of the cut I_1^0. Thus, we leave the following question.

Question 4.38. Is $\mathsf{RCA}_0^* + \mathsf{CRT}_2^2$ polynomially simulated by RCA_0^* with respect to $\forall\Pi_3^0$ sentences?

Bibliography

[1] Jeremy Avigad. Formalizing forcing arguments in subsystems of second-order arithmetic. *Annals of Pure and Applied Logic*, 82(2):165–191, 1996.

[2] David Belanger, C.T. Chong, Wei Wang, Tin Lok Wong, and Yue Yang. Where pigeonhole principles meet König lemmas. *Transactions of the American Mathematical Society*, 374(11):8275–8303, 2021.

[3] David R. Belanger. Conservation theorems for the cohesiveness principle, 2022. Preprint, available at arXiv:2212.13011.

[4] Andrey Bovykin and Andreas Weiermann. The strength of infinitary Ramseyan principles can be accessed by their densities. *Annals of Pure and Applied Logic*, 168(9):1700–1709, 2017.

[5] Peter A. Cholak, Carl G. Jockusch, and Theodore A. Slaman. On the strength of Ramsey's theorem for pairs. *Journal of Symbolic Logic*, 66(1):1–55, 2001.

[6] Peter A. Cholak, Carl G. Jockusch, Jr., and Theodore A. Slaman. Corrigendum to: "On the strength of Ramsey's theorem for pairs". *Journal of Symbolic Logic*, 74(4):1438–1439, 2009.

[7] C. T. Chong and K. J. Mourad. The degree of a Σ_n cut. *Annals of Pure and Applied Logic*, 48(3):227–235, 1990.

[8] C. T. Chong, Theodore A. Slaman, and Yue Yang. Π^1_1-conservation of combinatorial principles weaker than Ramsey's theorem for pairs. *Advances in Mathematics*, 230:1060–1077, 2012.

[9] C. T. Chong, Theodore A. Slaman, and Yue Yang. The inductive strength of Ramsey's theorem for pairs. *Advances in Mathematics*, 308:121–141, 2017.

[10] C. T. Chong and Yue Yang. The jump of a Σ_n-cut. *Journal of the London Mathematical Society (2)*, 75(3):690–704, 2007.

[11] Peter G. Clote. Partition relations in arithmetic. In: Carlos Augusto Di Prisco, editor. *Methods in Mathematical Logic*, pages 32–68. Springer, 1985.

[12] Robert P. Dilworth. A decomposition theorem for partially ordered sets. *Annals of Mathematics*, 51(1):161–166, 1950.

[13] Damir D. Dzhafarov and Carl Mummert. *Reverse Mathematics*. Springer, 2022.

[14] Ali Enayat and Tin Lok Wong. Unifying the model theory of first-order and second-order arithmetic via WKL_0^*. *Annals of Pure and Applied Logic*, 168(6):1247–1283, 2017.

[15] Marta Fiori-Carones, Leszek A. Kołodziejczyk, and Katarzyna W. Kowalik. Weak cousins of Ramsey's theorem over a weak base theory. *Annals of Pure and Applied Logic*, 172(10):article no. 103028, 22 pages, 2021.

[16] Marta Fiori-Carones, Leszek A. Kołodziejczyk, Tin Lok Wong, and Keita Yokoyama. An isomorphism theorem for models of Weak König's Lemma without primitive recursion. *Journal of the European Mathematical Society*, 2024. Online first, DOI:10.4171/JEMS/1522.

[17] Petr Hájek. Interpretability and fragments of arithmetic. In: Peter Clote and Jan Krajíček, editors. *Arithmetic, Proof Theory, and Computational Complexity*, pages 185–196. Clarendon Press, 1993.

[18] Petr Hájek and Pavel Pudlák. *Metamathematics of First-Order Arithmetic*. Association for Symbolic Logic, 2016.

[19] Ian Robert Haken. *Randomizing Reals and the First-Order Consequences of Randoms*. PhD thesis, University of California, Berkeley, 2014.

[20] Kostas Hatzikiriakou. Algebraic disguises of Σ_n^0-induction. *Archive for Mathematical Logic*, 29(1):47–51, 1989.

[21] Joram Hirschfeld. Models of arithmetic and the semi-ring of recursive functions. In: A. Hurd and P. Loeb, editors. *Victoria Symposium on Nonstandard Analysis*, pages 99–105. Springer, 1974.

[22] Joram Hirschfeld. Models of arithmetic and recursive functions. *Israel Journal of Mathematics*, 20(2):111–126, 1975.

[23] Denis R. Hirschfeldt. *Slicing the Truth*. World Scientific, 2015.

[24] Denis R. Hirschfeldt and Richard A. Shore. Combinatorial principles weaker than Ramsey's theorem for pairs. *Journal of Symbolic Logic*, 72:171–206, 2007.

[25] Jeffry L. Hirst. *Combinatorics in Subsystems of Second Order Arithmetic*. PhD thesis, The Pennsylvania State University, 1987.

[26] Aleksandar D. Ignjatović. *Fragments of First and Second Order Arithmetic and Length of Proofs*. PhD thesis, University of California, Berkeley, 1990.

[27] Carl Jockusch and Frank Stephan. A cohesive set which is not high. *Mathematical Logic Quarterly*, 39(4):515–530, 1993.

[28] Carl G. Jockusch. Ramsey's theorem and recursion theory. *Journal of Symbolic Logic*, 37(2):268–280, 1972.

[29] Richard Kaye. *Models of Peano Arithmetic*. Oxford University Press, 1991.

[30] Richard Kaye. Constructing κ-like models of arithmetic. *Journal of the London Mathematical Society*, 55(1):1–10, 1997.

[31] Leszek A. Kołodziejczyk, Katarzyna W. Kowalik, and Keita Yokoyama. How strong is Ramsey's theorem if infinity can be weak? *Journal of Symbolic Logic*, 88(2):620–639, 2023.

[32] Leszek A. Kołodziejczyk, Tin Lok Wong, and Keita Yokoyama. Ramsey's theorem for pairs, collection, and proof size. *Journal of Mathematical Logic*, 24(2):article no. 2350007, 40 pages, 2024.

[33] Leszek A. Kołodziejczyk and Keita Yokoyama. Some upper bounds on ordinal-valued Ramsey numbers for colourings of pairs. *Selecta Mathematica (N.S.)*, 26(4):article no. 56, 18 pages, 2020.

[34] Katarzyna W. Kowalik. A non speed-up result for the chain-antichain principle over a weak base theory. In preparation.

[35] Quentin Le Houérou, Ludovic Levy Patey, and Keita Yokoyama. Π^0_4 conservation of Ramsey's theorem for pairs, 2024. Preprint, available at arXiv:2404.18974.

[36] Quentin Le Houérou, Ludovic Levy Patey, and Keita Yokoyama. Conservation of Ramsey's theorem for pairs and well-foundedness. *Transactions of the American Mathematical Society*, 378:2157–2186, 2025.

[37] Manuel Lerman, Reed Solomon, and Henry Towsner. Separating principles below Ramsey's theorem for pairs. *Journal of Mathematical Logic*, 13(2):article no. 1350007, 44 pages, 2013.

[38] Jiayi Liu. RT^2_2 does not imply WKL_0. *Journal of Symbolic Logic*, 77(2):609–620, 2012.

[39] Joseph R. Mileti. *Partition theorems and computability theory*. PhD thesis, University of Illinois at Urbana-Champaign, 2004.

[40] Jeff B. Paris and Laurence A. B. Kirby. Σ_n-collection schemas in arithmetic. In: Angus Macintyre, Leszek Pacholski, Jeff B. Paris, editors. *Logic Colloquium '77*, pages 199–209. North-Holland, 1978.

[41] Ludovic Patey and Keita Yokoyama. The proof-theoretic strength of Ramsey's theorem for pairs and two colors. *Advances in Mathematics*, 330:1034–1070, 2018.

[42] Pavel Pudlák. The Lengths of Proofs. In: Samuel R. Buss, editor. *Handbook of Proof Theory*, pages 547–637. Elsevier, 1998.

[43] Pavel Pudlák. On the length of proofs of finitistic consistency statements in first order theories. In: J. B. Paris, A. J. Wilkie, G. M. Wilmers, editors. *Logic Colloquium '84*, pages 165–196. North-Holland, 1986.

[44] Zygmunt Ratajczyk. Traces of models on intial segments. *The Journal of Symbolic Logic*, 50(1):273–274, 1985. Abstract of a Logic Colloquium presentation.

[45] Dana Scott. Algebras of Sets Binumerable in Complete Extensions of Arithmetic. In: Jacob C. E. Dekke, editor. *Recursive Function Theory*, pages 117–121. American Mathematical Society, 1962.

[46] David Seetapun and Theodore A. Slaman. On the strength of Ramsey's theorem. *Notre Dame Journal of Formal Logic*, 36(4):570–582, 1995.

[47] Stephen G. Simpson. *Subsystems of Second Order Arithmetic*. Association for Symbolic Logic, 2009.

[48] Stephen G. Simpson and Rick L. Smith. Factorization of polynomials and Σ^0_1 induction. *Annals of Pure and Applied Logic*, 31(2–3):289–306, 1986.

[49] Robert I. Soare. *Turing Computability*. Springer, 2016.

[50] Ernst Specker. Ramsey's theorem does not hold in recursive set theory. In: R. O. Gandy and C. E. M. Yates, editors. *Logic Colloquium '69*, pages 439–442. North-Holland, 1971.

[51] Mengzhou Sun. The Kaufmann-Clote question on end extensions of models of arithmetic and the weak regularity principle. *Journal of Symbolic Logic*, 2025. Online first, DOI:10.1017/jsl.2025.15.

[52] Mengzhou Sun. On finite cohesiveness principle, 2025. Preprint, available at arXiv:2503.18383.

[53] Vítězslav Švejdar. The limit lemma in fragments of arithmetic. *Commentationes Mathematicae Universitatis Carolinae*, 44(3):565–568, 2003.

[54] Kazuyuki Tanaka. The self-embedding theorem of WKL_0 and a non-standard method. *Annals of Pure and Applied Logic*, 84(1):41–49, 1997.

[55] Henry Towsner. Constructing sequences one step at a time. *Journal of Mathematical Logic*, 20(3):article no. 2050017, 43 pages, 2020.

[56] George M. Wilmers. *Some Problems in Set Theory: Non-standard models and their Application to Model Theory*. PhD thesis, Oxford University, 1975.

[57] Keita Yokoyama. On Π^1_1 conservativity of Π^1_2 theories in second order arithmetic. In: C. T. Chong et al., editors. *Proceedings of the 10th Asian Logic Conference*, pages 375–386. World Scientific, 2010.

[58] Keita Yokoyama. On the strength of Ramsey's theorem without Σ_1-induction. *Mathematical Logic Quarterly*, 59(1-2):108–111, 2013.