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Singular $\mathbb{Q}$-homology planes

PhD dissertation

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Author's declaration:
aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Abstract

The thesis is devoted to studying normal complex $\mathbb{Q}$-acyclic algebraic surfaces S'. Let S_0 be the smooth locus of such a surface. The following results have been obtained. If S' has non-negative Kodaira dimension then it is logarithmic, i.e. its singularities are of quotient type. We classify possible S' with non-quotient singularities. S' can be nonrational. The completion of the resolution of S' is birationally ruled, i.e. it is a a blowup of a $\mathbb{P}^1$-bundle over some smooth complete curve. We classify possible S' for which $\overline{\kappa}(S_0) = 0$ and S_0 does not admit a $\mathbb{C}^*$-fibration. The main result is the theorem saying that if $\overline{\kappa}(S') = -\infty$ then $\overline{\kappa}(S_0) \neq 2$. The full description of possible singular $\mathbb{Q}$-homology planes S' of negative Kodaira dimension is given.

Keywords: Q-homology plane, acyclic surface, Kodaira dimension,
quotient singularities;

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Streszczenie

Rozprawa poświęcona jest badaniu normalnych, zespolonych, $\mathbb{Q}$-acyklicznych powierzchni algebraicznych S'. Niech S_0 będzie częścią gładką takiej powierzchni. Uzyskano następujące wyniki. Jeśli S' ma nieujemny wymiar Kodairy, to S' jest logarytmiczna, tzn. jej osobliwości są ilorazowe. Sklasyfikowano możliwe S' z osobliwościami nieilorazowymi. S' może być niewymierne. Uzupełnienie rezolwenty S' jest jest rozdmuchaniem pewnej $\mathbb{P}^1$-wiązki nad gładką krzywą zupełną. Sklasyfikowano możliwe S', dla których $\overline{\kappa}(S_0) = 0$ i S_0 nie posiada $\mathbb{C}^*$-rozwłóknienia. Głównym wynikiem jest twierdzenie mówiące, że jeśli $\overline{\kappa}(S') = -\infty$, to $\overline{\kappa}(S_0) \neq 2$. Podano pełny opis możliwych osobliwych płaszczyzn $\mathbb{Q}$-homologicznych S' o ujemnym wymiarze Kodairy.

Słowa kluczowe: płaszczyzna Q-homologiczna, powierzchnia acykliczna, wymiar Kodairy, osobliwości ilorazowe;

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Introduction

We consider the problem of classifying normal $\mathbb{Q}$-acyclic singular surfaces defined over $\mathbb{C}$, we call them *singular $\mathbb{Q}$-homology planes*. (For convenience we exclude the smooth case by definition). This generalizes the notion of a *logarithmic* $\mathbb{Q}$-homology plane by relaxing the assumption on the type of the singular locus, i.e. we do not assume that it is of quotient type. Let S' be such a surface and let S_0 be its smooth locus. Denote the desingularization of S' by S. By definition we have $\overline{\kappa}(S') = \overline{\kappa}(S)$, we have also $\overline{\kappa}(S) \leq \overline{\kappa}(S_0)$. The main goal of this paper is to give the classification of singular $\mathbb{Q}$-homology planes S' satisfying $\overline{\kappa}(S') = -\infty$.

In the logarithmic case, under the assumption that S' (and hence S_0) is $\mathbb{C}^1$- or $\mathbb{C}^*$-ruled, a structure theorem was obtained in [MS91]. The assumption that the singular locus of S' is of quotient type simplifies calculations and excludes some exotic situations one has to deal with in general. Some results known for logarithmic $\mathbb{Q}$-homology planes do not hold in general. In particular, the theorem [PS97, Theorem 1.1] saying that logarithmic singular $\mathbb{Q}$ - homology planes are rational does not hold for general singular $\mathbb{Q}$-homology planes (cf. 5.4).

In chapter 1 we give basic definitions and recall well-known facts from the theory of open surfaces, we state a lemma 1.7.1 which helps us later to obtain some singular $\mathbb{Q}$-homology planes. Let $\widehat{E} \subset S$ be the exceptional divisor of the resolution. Let $\overline{S}$ be the completion of S, denote the boundary divisor of $S \subset \overline{S}$ by D.

In chapter 2 we study the topology of the pair $(\overline{S}, D + \widehat{E})$. We show that S' is affine and $\overline{S}$ is $\mathbb{P}^1$-ruled (2.2.3), this generalizes [PS97, Theorem 1.1]. We prove also that if $\overline{\kappa}(S') \geq 0$ then the singularities of S' are of quotient type (2.2.4).

In chapter 3 by studying various $\mathbb{P}^1$-rulings of $\overline{S}$ induced by some 0-curves contained in D we prove that if $\overline{\kappa}(S_0) = 0$ then with two exceptions (cf. 3.2.7) S_0 is $\mathbb{C}^*$-ruled. By general structure theorems it is known that if $\overline{\kappa}(S_0) = -\infty$ or 1 then S_0 has a $\mathbb{C}^1$- or a $\mathbb{C}^*$-ruling. Therefore our result complements these theorems allowing to study S''s with smooth locus of non-general type in a unified manner.

In the simplest case, when $\overline{\kappa}(S_0) = -\infty$ (chapter 4) S' has to be logarithmic, hence only well known examples appear (cf. [MS91]).

If $\overline{\kappa}(S_0) = 0, 1$ and $\overline{\kappa}(S') \geq 0$ then by the results of chapter 2 and 3 S' is logarithmic and (again with two exceptions) S_0 is $\mathbb{C}^*$-ruled, hence we reduce the analysis to the one done in [MS91]. Therefore the analysis of the following cases is needed:

(A) $\overline{\kappa}(S_0) = 0, 1$, $\overline{\kappa}(S') = -\infty$,

(B) $\overline{\kappa}(S_0) = 2$, any $\overline{\kappa}(S')$.

In chapter 5 we study the case (A) by analyzing various $\mathbb{C}^*$-rulings of S_0. There are three possible types: (1) gyoza - with one 2-section, which is contained in D, (2) sandwich of type II - with two 1-sections contained in D and (3) sandwich of type I - with one 1-section in D and one 1-section in $\widehat{E}$. The type (3) (for which almost by definition $\overline{\kappa}(S') = -\infty$) was not studied before, up to now only $\mathbb{C}^*$-rulings of S_0 induced by a $\mathbb{C}^*$-ruling of S' were considered (cf. [MS91]). We reduce the case (1) to the case (2) by finding another $\mathbb{C}^*$-ruling. In cases (2) and (3) we obtain a full description of possible S''s. We show how to construct them starting from a $\mathbb{P}^1$-ruled surfaces by blowing up, contracting some divisors and throwing out others. In case (3) we obtain new examples of singular $\mathbb{Q}$-homology planes with non-quotient or non-rational singularities (cf. 5.4.6).

The case (B) is the most difficult, since there are no structure theorems for open surfaces of general type. It is easy to show that then S' has exactly one singular point and it is of quotient type (cf. 2.2.1). For $\overline{\kappa}(S') = -\infty$ (still case (B)) it is the main result of [KR07] that S' cannot be topologically contractible. Modifying the methods developed in [KR99] and [KR07] we deal with case (B) for $\overline{\kappa}(S') = -\infty$ for general

$\mathbb{Q}$-homology planes in chapter 6 reproving the theorem of Koras and Russell as a special case (cf. 6.6.5). The analysis of the case (B) for $\overline{\kappa}(S')=0$ is possible. For $\overline{\kappa}(S')=1$ this will be more difficult, and for $\overline{\kappa}(S')=2$ the problem of classification is rather hopeless.

For clarity we state the main result:

Theorem: Let S be a desingularization of a singular $\mathbb{Q}$-homology plane S'. Let $S_0=S'-Sing\ S'$.

(1) The completion of S is $\mathbb{P}^1$-ruled (cf. 2.2.3).

(2) If $\overline{\kappa}(S')\geq 0$ then S' is logarithmic (cf. 2.2.4).

(3) If $\overline{\kappa}(S_0)=0$ then with two exceptions S_0 is $\mathbb{C}^*$-ruled (cf. 3.2.7 and 3.2.2).

(4) If $\overline{\kappa}(S')=-\infty$ then $\overline{\kappa}(S_0)\neq 2$ (cf. 6.6.5).

(5) Assume $\overline{\kappa}(S')=-\infty$ and $\overline{\kappa}(S_0)<2$. All such surfaces S' are classified (see 4.1.3, 4.2.1, 3.2.2, 5.2.1, 5.3.3 and 5.4.5). They can be obtained in a precisely described way by blowing up generalized Hirzebruch surfaces and contracting some exceptional divisors to singular points. The boundary divisors and the exceptional divisors are described. If $\overline{\kappa}(S_0)\geq 0$ then $\operatorname{Sing} S'$ consist precisely of one point, which does not have to be a rational singularity.

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Contents

Chapter 1

Definitions and general results

We consider algebraic varieties defined over $\mathbb{C}$. In this chapter we set up the notation and collect basic facts from the theory of open surfaces we will use.

1.1 Generalities on divisors

Let $T = \sum_{i=1}^{n} m_i T_i$ with T_i irreducible and $m_i \in \mathbb{Z} \setminus \{0\}$ (or more generally $m_i \in \mathbb{Q} \setminus \{0\}$) be a simple normal crossing divisor (snc-divisor) on a smooth complete surface, i.e. all its components are smooth, intersect transversally, at most two in one point (nc-divisor). Notice that by a result of Zariski a smooth complete surface is projective. By a component we always mean an irreducible component. Let

$$Q(T) = (m_i m_j T_i T_j)_{1 \le i,j \le n}$$

and let

$$d(T) = det(-Q(T)).$$

We put $d(\emptyset) = 1$. We define the reduction of T as $\underline{T} = \sum T_i$ and denote the number of components of T by $\#T$. We say that T is *rational* if all its components are rational. If we refer to a divisor as a subset of a surface, we refer really to its support. For example, writing $T \subseteq T'$ we mean that T and T' satisfy $\operatorname{Supp} T \subseteq \operatorname{Supp} T'$. We will denote the free abelian group generated by irreducible components of T by $\mathcal{L}(T)$. The numerical equivalence of divisors will be denoted by $\equiv$. We write $T \geq 0$ for effective ($\mathbb{Z}$- and $\mathbb{Q}$-) divisors and for $\mathbb{Z}$-divisors linearly equivalent to effective $\mathbb{Z}$-divisors. Two $\mathbb{Q}$-divisors T, U are linearly equivalent if rT and rU are linearly equivalent $\mathbb{Z}$-divisors for some nonzero integer r. If T is a $\mathbb{Q}$-divisor linearly equivalent to some effective $\mathbb{Q}$-divisor then we write $T \geq_{\mathbb{Q}} 0$.

Let $DGraph(T)$ be a dual graph of T, i.e. a weighted one-dimensional simplicial complex with one vertex v_i for each irreducible component T_i of T and one edge between v_i and v_j for each point of intersection of T_i with T_j. The weight assigned to v_i is $-T_i^2$. Let $|DGraph(T)|$ be the geometric realization of $DGraph(T)$. Consider T as a topological subspace of a surface with its analytical topology. The natural map $\coprod_{i=1}^{n} T_i \to T$ identifies some pairs of points, which homotopically is the same as adding a cone over them. It is an exercise in homotopy theory to see that for a connected T this gives

$$T \underset{htp}{\approx} \bigvee_{i=1}^{n} T_i \vee \ |DGraph(T)|.$$

In particular,

$$\widetilde{H}_j(T, \mathbb{Z}) = \oplus_{i=1}^{n} \widetilde{H}_j(T_i, \mathbb{Z}) \oplus \widetilde{H}_j(|DGraph(T)|, \mathbb{Z}),$$

where $\widetilde{H}_j(|DGraph(T)|, \mathbb{Z}) = 0$ for $j \neq 1$ ($\widetilde{H}_j$'s are the reduced homology groups). We say that T is a *tree* if each connected component of $DGraph(T)$ contains no loops, i.e $\pi_1(|DGraph(T)|) = 0$.

We define the *branching number* of T_i as $\beta_T(T_i) = T_i\,(\underline{T} - T_i)$. We say that T_i is a *tip* of T if $\beta_T(T_i) = 1$. It is a *branching component* if $\beta_T(T_i) \geq 3$. If T is connected and does not have any branching components then it is a *chain*. An snc-chain T is *admissible* if it is rational and $T_i^2 \leq -2$ for every i. A curve L is a (b)*-curve* if and only if $L \cong \mathbb{P}^1$ and $L^2 = b$.

Lemma 1.1.1. *([KR07, 2.1.1]). Let $T = \underline{T}$ be a connected snc-tree. The following formulas hold:*

(i) Let C be a component of T and let $T_1 \ldots, T_\beta$ be the connected components of $T - C$. If C_i is the component of T_i meeting C then

$$d(T) = -C^2 \prod_i d(T_i) - \sum_i d(T_i - C_i) \prod_{i \neq j} d(T_j).$$

(ii) Let $T = T_1 + T_2$, where T_1, T_2 are connected and intersect in one point. Let C_1, C_2 be the intersecting components, then

$$d(T) = d(T_1)d(T_2) - d(T_1 - C_1)d(T_2 - C_2).$$

Suppose T is a chain and a tip T_1 of T is fixed. This choice induces a unique linear order on the set of irreducible components of T with T_1 as a first component. We write $T = T_1 + T_2 + \ldots + T_n$, where T_i's are irreducible components of T. We write also $T = [-T_1^2, -T_2^2, \ldots, -T_n^2]$. We denote a chain of (-2)-curves of length k by $[(k)]$. For example, $[3, (4)]$ is just $[3, 2, 2, 2, 2]$.

Lemma 1.1.2. *Let T be an admissible chain. For every $d > 2$ there exist at least two T's with $d(T) = d$: $[d]$ and $[(d-1)]$. This is a full list of all other T's for $d \leq 11$:*

$d = 5$ *:* $[3, 2]$,

$d = 7$ *:* $[4, 2]$, $[3, (2)]$,

$d = 8$ *:* $[3, 3]$, $[2, 3, 2]$,

$d = 9$ *:* $[5, 2]$, $[3, (3)]$,

$d = 10$ *:* $[4, (2)]$,

$d = 11$ *:* $[6, 2]$, $[4, 3]$, $[3, (4)]$, $[2, 3, (2)]$.

1.2 Pairs

An snc-pair (W, D) is a pair consisting of a smooth complete surface W and a reduced snc-divisor D on W. D is *snc-minimal* if for every (-1)-curve in D the direct image of D after its contraction is not an snc-divisor. The pair (W, D) is snc-minimal if D is. If D is a tree then this is equivalent to the property that each (-1)-curve in D has $\beta_D > 2$. The blowup and blowdown of an snc-pair (W, D) are defined as appropriate transformation of W with the divisorial part defined as a full preimage and proper image of D with reduced structure. The divisorial part is assumed to remain snc. We identify isomorphic pairs. We have a natural partial order: $(W', D') \prec (W, D)$ if and only if there exists a birational regular morphism $\eta : W' \to W$, such that $\eta_* D' = D$. We say that the snc-pair (W, D) is minimal with respect to some property if it is a minimal element of the set of snc-pairs satisfying this property. The modification of an snc-pair is just a birational transformation of snc-pairs, i.e. a sequence of blowdowns and blowups. If (W, D) is an snc-pair then we will write $W - D$ for $W \setminus \operatorname{Supp} D$.

Let X be a smooth surface. If X is not complete then by Nagata's embedding theorem and Hironaka's theorem on resolution of singularities we can embed X into a smooth complete surface $\overline{X}$, such that $D = \overline{X} \setminus X$ is an snc-divisor. Moreover, $\overline{X}$ is projective by Zariski's theorem. We call the pair $(\overline{X}, D)$ an *snc-completion* of X.

If $p : Y \to X$ is a dominating morphism of surfaces and D a divisor on X we write $p^{-1}(D)$ for the reduced full preimage of D.

Example 1.2.1. $\mathbb{C}^2 \setminus \{0\}$ does not have an snc-minimal completion.

Definition 1.2.2. The sequence of blowups is *connected* if for every $i > 0$ the center of the $(i+1)$-th blowup belongs to the exceptional locus of the i-th blowup. The sequence of blowdowns is *connected* if the sequence of blowups reversing it is connected.

Let D be an snc-divisor. The blowup of D is *sprouting* if its center belongs to exactly one component of D. In other case it is *subdivisional*.

Corollary 1.2.3. *Assume that X' is a normal affine surface. Let $X_0 = X' \setminus \operatorname{Sing} X' - D'$ for some divisor D' on X'. Then the snc-minimal completion of X_0 exists.*

Proof. Let $p : X \to X'$ be a desingularization of X', such that $\widehat{E} = p^{-1}(Sing\ X')$ is an snc-divisor. Let $\overline{X}$ be a smooth completion of X. Let D be a reduced divisor with support $p^{-1}(D') \cup \overline{X} \setminus X$. We can assume that D is an snc-divisor. It is connected, which follows from X' being affine. Let E be the sum of components of $\widehat{E}$ not contained in D. We see that $(\overline{X}, D + E)$ is an snc-completion of X_0. Clearly, we can assume that E is snc-minimal. Moreover, since there exists an ample divisor with support contained in D, $Q(D)$ is not negative definite, so in the process of snc-minimalization the divisor D cannot be contracted to a point (cf. [Goo69, Gra62]).

□

1.3 Barks

We recall basic definitions from the theory of peeling (see [Miy01, §2.3] for a complete discussion). In this paragraph we consider only reduced snc-divisors.

Assume that T is a chain with $Q(T)$ negative definite (this holds for example for admissible T). Fix an ordering of components of T induced by choosing some tip T_1. We define some numbers describing $T = T_1 + T_2 + \ldots + T_n$ as follows:

$$d'(T) = d(T - T_1), d'(\emptyset) = 0,\ e(T) = \frac{d'(T)}{d(T)},\ \widetilde{e}(T) = e(T^t),$$

where $T^t = T_n + \ldots + T_1$. *Bark* of T is a $\mathbb{Q}$-divisor $\operatorname{Bk} T = \sum \alpha_i T_i$ satisfying

$$T_1 \operatorname{Bk} T = -1,\ T_i \operatorname{Bk} T = 0 \text{ for } i > 0.$$

It is well defined, since $d(T) \neq 0$. We put

$$\operatorname{Bk}^* T = \operatorname{Bk} T + \operatorname{Bk} T^t.$$

Let's fix a divisor T'. Suppose that T is a rational chain contained in T', such that T does not contain any branching component of T'. If T is a connected component of T' and $Q(T)$ is negative definite then we call T a *rod* of T'. In this case let $T_1 + T_2 \ldots + T_n$ be any linear ordering of T, we put $\operatorname{Bk} T = \operatorname{Bk}(T_1 + \ldots + T_n) + \operatorname{Bk}(T_n + \ldots + T_1)$. Clearly, this does not depend on the ordering chosen. If T contains exactly one tip U of T' then T is called a *twig* of T'. If additionally $Q(T)$ is negative definite then we write $\operatorname{Bk} T$ for a bark of T considered with a linear ordering induced by U. A *maximal twig* of T' is a twig which is maximal with respect to $T \subseteq T'$. Similarly, a *maximal admissible twig* is an admissible twig, which is maximal (among admissible twigs of T) with respect to $T \subseteq T'$.

Assume that the divisor V is not a chain and let $V_1, \ldots, V_k$ be all its maximal admissible twigs. We define

$$\delta(V) = \sum_{i=1}^{k} \frac{1}{d(V_i)},\ e(V) = \sum_{i=1}^{k} e(V_i) \text{ and } \widetilde{e}(V) = \sum_{i=1}^{k} \widetilde{e}(V_i).$$

We say that V is a *fork* (*wide fork*) if V is connected, has a unique branching component and three (three or more) maximal twigs. The fork V is *admissible* if it is rational, with maximal twigs being admissible, $\delta(V) > 1$ and the branching component B satisfies $B^2 \leq -2$. It is easy to check that, assuming the remaining conditions, the condition $B^2 \leq -2$ is equivalent to negative definiteness of $Q(V)$. Let $F = B + T(1) + T(2) + T(3)$ be an admissible fork with maximal twigs $T(i)$. We define

$$\operatorname{Bk} F = \frac{\delta(F) - 1}{-B^2 - \widetilde{e}(F)}(B + \sum_{i=1}^{3} \operatorname{Bk} T(i)^t) + \sum_{i=1}^{3} \operatorname{Bk} T(i).$$

For a general reduced snc-divisor D let $\{T_\alpha\}, \{R_\beta\}$ and $\{F_\gamma\}$ be the sets of its maximal admissible twigs that are not contained in some admissible forks of D, a set of admissible rods of D and a set of admissible forks of D. Define

$$\operatorname{Bk} D = \sum_{\alpha} \operatorname{Bk} T_\alpha + \sum_{\beta} \operatorname{Bk} R_\beta + \sum_{\gamma} \operatorname{Bk} F_\gamma,$$

and set $D^\# = D - BkD$. The following propositions describe important properties of $\operatorname{Bk} D$ ([Miy01, §2.3]).

Proposition 1.3.1. *Let D be a reduced snc-divisor, then:*

(i) $\operatorname{Bk} D$ *is effective and either* $Q(\operatorname{Bk} D)$ *is negative definite or* $\operatorname{Bk} D = 0$,

(ii) $(K_X + D^{\#})Z = 0$ *for every* $Z \subseteq \operatorname{Bk} D$,

(iii) $\operatorname{Supp} D \setminus \operatorname{Supp} D^{\#}$ *consists of (-2)-rods and (-2)-forks, i.e. rods and forks consisting of components with self-intersection -2.*

Proposition 1.3.2. *Let $T = T_1 + \ldots + T_n$ be an admissible ordered snc-chain, let* $\operatorname{Bk} T = \sum_{i=1}^n m_i T_i$ *and* $\operatorname{Bk}^* T = \sum_{i=1}^n m_i^* T_i$, *then:*

(i) $d'(T) \leq d(T) - 1$, $e(T) = \frac{1}{-T_1^2 - e(T - T_1)}$, $\frac{1}{d(T)} \leq e(T) \leq 1 - \frac{1}{d(T)}$,

(ii) $m_i = \frac{d(T_{i+1} + \ldots + T_n)}{d(T)}$,

(iii) $0 < m_i < 1$ *and* $0 < m_i^* \leq 1$ *(in particular* $\operatorname{Supp} \operatorname{Bk} T = \operatorname{Supp} \operatorname{Bk}^* T = \operatorname{Supp} T$*). Moreover, if $m_i^* = 1$ for some i then $T = [2, 2, \ldots, 2]$ and $m_i^* = 1$ for each i,*

(iv) $\operatorname{Bk}^2 T = -e(T)$, $(\operatorname{Bk}^* T)^2 = -e(T) - \widetilde{e}(T) - \frac{2}{d(T)} = -\frac{d'(T) + d'(T^t) + 2}{d(T)} \geq -2$.

Remark 1.3.3. The function $e(\ )$, called *inductance* or *capacity*, gives in terms of Hirzebruch-Jung continued fractions (see 1.3.4) a one-to-one correspondence between weighted ordered dual graphs of ordered admissible chains and points in $\mathbb{Q} \cap (0,1)$ ([Miy01, 2.3.3(3)]). However, although $e(T)$ determines the chain T, hence also $d(T)$ and $\widetilde{e}(T)$, there is no simple formula for $d(T)$ or $\widetilde{e}(T)$ as a function of $e(T)$. In fact, the graph of $\widetilde{e}(T)$ as a function of $e(T)$ is dense in $[0,1]^2$.

Example 1.3.4. Let $e = \frac{11}{19}$. Then we can write e as $\frac{11}{19} = \cfrac{1}{2 - \cfrac{1}{4 - \frac{1}{3}}}$, and for $T = [2, 4, 3]$ we have $e(T) = e$. We have also $\widetilde{e}(T) = \frac{7}{19}$ and $d(T) = 19$.

Proposition 1.3.5. *Let $F = B + T(1) + T(2) + T(3)$ be a reduced, admissible fork with maximal twigs $T(i)$. Let* $\operatorname{Bk} F = \sum_{i=1}^n m_i F_i$ *and $d_i = d(T_i)$, then:*

(i) $0 < m_i \leq 1$ *(in particular* $\operatorname{Supp} \operatorname{Bk} F = \operatorname{Supp} F$*). Moreover, if $m_i = 1$ for some i then F is a (-2)-fork and $m_i = 1$ for each i,*

(ii) (d_1, d_2, d_3) *is one of the* Platonic triples*:* $(2,3,3)$, $(2,3,4)$, $(2,3,5)$ *or* $(2,2,k)$ *for some* $k \geq 2$,

(iii) $1 < \widetilde{e}(F) < 2 \leq -B^2$,

(iv) $d(F) = d_1 d_2 d_3 (-B^2 - \widetilde{e}(F))$,

(v) $\operatorname{Bk}^2 F = -\frac{(\delta(F) - 1)^2}{-B^2 - \widetilde{e}(F)} - e(F) < -e(F) < -1$.

Remark 1.3.6. Notice that since $e(T) + \delta(T) \leq 1$ for an admissible chain T, we have $(\operatorname{Bk}^* T)^2 = -2$ if and only if T consists of (-2)-curves. For an admissible fork F we get also by 1.3.5(iii) that $\frac{\delta - 1}{-B^2 - \widetilde{e}} \leq 1$, so $-\operatorname{Bk}^2 F \leq \delta - 1 + e \leq 2$ and again the equality occurs if and only if F consists of (-2)-curves.

1.4 Singularities

Let q be a singular point on a normal surface X. Let $p : \widetilde{X} \to X$ be a desingularization of $q \in X$. Put $\widehat{E} = p^{-1}(q)$. The matrix $Q(\widehat{E})$ is negative definite (i.e $\widehat{E}$ is *algebraically contractible* or *contractible* for short). We will always assume that resolutions are *good*, i.e. $\widehat{E}$ is an snc-divisor. We say that p is a *minimal good resolution* if it is good and $\widehat{E}$ is snc-minimal.

We say that $q \in X$ is *topologically rational* if $\widehat{E}$ is a rational tree. It is *rational* if $R^1 p_* \mathcal{O}_{\widetilde{X}} = 0$. It is of *quotient type* if there exists an analytical neighborhood N of q and a small (i.e. not containing any pseudo-reflections) finite subgroup G of $GL(2, \mathbb{C})$, such that (N, q) is analytically isomorphic to $(\widetilde{N}/G, 0)$ for some ball $\widetilde{N}$ around 0 in $\mathbb{C}^2$. It follows that $G = \pi_1(N \setminus \{q\})$. All these notions are well-known to be independent of the choice of a resolution.

Proposition 1.4.1. *Assume $q \in X$ is of quotient type. Let G be as above. Assume that $\widehat{E}$ is the exceptional divisor of a minimal good resolution. Then ([Art66, Bri68]):*

(i) G is cyclic if and only if $\widehat{E}$ is an admissible chain, moreover then $d(\widehat{E}) = |G|$,

(ii) G is non-cyclic if and only if it is non-abelian if and only if $\widehat{E}$ is an admissible fork, moreover then $d(\widehat{E}) = |G/[G,G]|$,

(iii) if $q \in X$ is of quotient type then it is rational,

(iv) if $q \in X$ is rational then it is topologically rational.

Example 1.4.2. ([Abh79]). Let $V \subseteq \mathbb{C}^3$ be given by $x^2 + y^3 + z^7 = 0$. Then the blowup of V in 0 has an exceptional line contained in the singular locus, hence is not normal. Since the blowup of a normal surface with rational singularity remains normal ([Lip69, 8.1]), $0 \in V$ is not a rational singularity. On the other hand, it is topologically rational. More generally, let $V(p_1, p_2, p_3) \subseteq \mathbb{C}^3$ be a Pham-Brieskorn surface given by the equation $x_1^{p_1} + x_2^{p_2} + x_3^{p_3} = 0$, where $p_1, p_2, p_3 \geq 2$. If one of p_1, p_2, p_3 is relatively prime with two others then $0 \in V(p_1, p_2, p_3)$ is topologically rational (see [Ore95] for an easy proof). The rationality of $0 \in V(p_1, p_2, p_3)$ is equivalent to each of the following conditions ([FZ03, 2.21]): (i) $0 \in V(p_1, p_2, p_3)$ is of quotient type, (ii) $\sum_{i=1}^{3} \frac{1}{p_i} > 1$, (iii) $\overline{\kappa}(V \setminus \{0\}) = -\infty$.

1.5 Rulings

We say that the surface X is $\mathbb{P}^1$-*ruled* (respectively $\mathbb{C}^1$-*ruled*, $\mathbb{C}^*$-*ruled*, $\mathbb{C}^{**}$-*ruled*) if there exists a curve B and a regular dominating map $p : X \to B$, such that the generic fiber F of p is isomorphic to $\mathbb{P}^1$ (respectively to $\mathbb{C}^1$, $\mathbb{C}^*$ and $\mathbb{C}^{**}$). We call also the $\mathbb{C}^1$-ruling an *affine* ruling. We say that X is $\mathbb{C}^{***}$-ruled if for generic fiber there exists an isomorphism with $\mathbb{C} \setminus \{p_1, p_2, p_3\}$, where p_1, p_2, p_3 are different points of $\mathbb{C}$ (they can be different for different fibers). If X is normal then B has to be smooth.

Suppose that X is smooth and has a ruling as above. Then for some snc-completion $(\overline{X}, D)$ this ruling can be extended to a $\mathbb{P}^1$-ruling $\overline{p} : \overline{X} \to \overline{B}$, where $\overline{B}$ is a smooth completion of B. Depending on the type of the ruling of X we will say that $(\overline{X}, D)$ is affine- (respectively $\mathbb{C}^*$-, $\mathbb{C}^{**}$-, etc.) ruled. Let F denote a generic fiber of $\overline{p}$. An irreducible curve C on $\overline{X}$ is called a D-*component* if $C \subseteq D$. It is called an X-*component* if it is not a D-component. C is an n-*section* if $FC = n$. We will say just *section* for a 1-section. C is *horizontal* if $n > 0$, otherwise it is *vertical*. The divisor is horizontal (vertical) if all its components are such. The snc-completion $(\overline{X}, D)$ is p-*minimal* if it is minimal with respect to the property that the extension of p from X to $\overline{X}$ exists. This is equivalent to the property that every (-1)-curve in D with $\beta_D \leq 2$ is horizontal. If X has an snc-minimal completion then it has also a p-minimal completion.

Lemma 1.5.1. *Let F be a singular fiber of a $\mathbb{P}^1$-ruling of a smooth complete surface. We denote by $\mu(C)$ the multiplicity of an irreducible curve C in the fiber containing it. One has (cf. [Fuj82, §4]):*

(i) F is a connected rational snc-tree containing a (-1)-curve,

(ii) each (-1)-curve of F intersects at most two other components of F,

(iii) if a contraction of some (-1)-curve of F increases the number of (-1)-curves in the induced fiber then $F = [2, 1, 2]$,

(iv) F is produced from a smooth 0-curve by a sequence of blowups. If the (-1)-curve of F is unique then the sequence is connected (cf. 1.2),

Suppose further that F as above has a unique (-1)-curve C. Let $B_1, \ldots, B_n$ be the branching components of F written in order in which they are produced in the sequence of blowups as in (iv) and let $B_{n+1} = C$. We can write $\underline{F}$ as $\underline{F} = T_1 + T_2 + \ldots + T_{n+1}$, where the divisors T_i are connected chains consisting of all components of $\underline{F} - T_1 - \ldots - T_{i-1}$ created not later than B_i. We call T_i the i-th branch *of F. We say that F is* branched *if $n \neq 0$.*

(v) $\mu(C) > 1$ and there are exactly two components of F with multiplicity one. They are tips of the fiber and lie on the first branch,

(vi) if $\mu(C)=2$ then either $F=[2,1,2]$ or C is a tip of F and $\underline{F}-C$ is a (-2)-chain or a (-2)-fork of type $(2,2,n)$

(vii) if F is branched then the connected component of $\underline{F}-C$ not containing curves of multiplicity one is a chain (possibly empty).

Definition 1.5.2. For an snc-pair $(\overline{X},D)$ put $X=\overline{X}-D$. Let π be a $\mathbb{P}^1$-ruling of $\overline{X}$. Following [Fuj82] we introduce some characteristic numbers of the triple $\tau=(\overline{X},D,\pi)$:

(i) h_τ is the number of horizontal D-components,

(ii) $\sigma_\tau(F)$ is the number of X-components contained in F,

(iii) $\Sigma_\tau=\sum\limits_{F\not\subseteq D}(\sigma_\tau(F)-1)$,

(iv) ν_τ is the number of fibers contained in D,

(v) we also define a *rivet* as an intersection point of at least two different horizontal components of D or a connected component of $F\cap D$ which meets horizontal component(s) of D at more than one point.

If there is no danger of confusion we omit indices writing h for h_τ, $\sigma(F)$ for $\sigma_\tau(F)$, etc. If $\overline{X}$ and π are fixed but more than one choice of D is possible we write $\Sigma_{\overline{X}-D}$ instead of $\Sigma_{(\overline{X},D,\pi)}$.

Lemma 1.5.3. *(cf. [Fuj82, 4.16]) If $\pi:\overline{X}\to C$ is a $\mathbb{P}^1$-ruling as above then*

$$\Sigma=h+\nu+b_2(\overline{X})-b_2(D)-2.$$

Proof. If we contract a vertical (-1)-curve and change $\overline{X}$ and D for their images then one checks easily that the numbers $b_2(\overline{X})-b_2(D)-\Sigma+\nu$ and h do not change, so we can assume that all fibers of π are smooth. Then $b_2(D)=h+\nu$, $\Sigma=0$ and $b_2(\overline{X})=2$. □

Definition 1.5.4. Let φ be a $\mathbb{C}^*$-ruling from a smooth open surface X onto $\mathbb{P}^1$. It is a *Platonic fibration* if and only if two conditions are satisfied:

(i) φ has precisely three singular fibers which are equal to $\mu_i F_i$, where $F_i\cong\mathbb{C}^*$ and (μ_1,μ_2,μ_3) is a Platonic triple (cf. 1.3.5(ii)),

(ii) there exists an snc-completion $(\overline{X},D_1+D_2)$ of X with $D_1\cap D_2=\emptyset$ and an extension $\overline{\varphi}:\overline{X}\to\mathbb{P}^1$, such that every fiber of $\overline{\varphi}$ is a chain and each D_i contains a section of $\overline{\varphi}$.

1.6 Minimal models

In this section X is a smooth open surface and $(\overline{X},D)$ is its snc-completion.

Definition 1.6.1. A smooth open surface X is *almost minimal* if it has an snc-completion $(\overline{X},D)$ for which there does not exist a *log-exceptional curve of the first kind* on $\overline{X}$, i.e. an irreducible curve C, such that

$$(K_{\overline{X}}+D^{\#})C<0 \quad\text{and}\quad Q(\operatorname{Bk}D+C) \quad\text{is negative definite.}$$

The pair $(\overline{X},D)$ is then called *almost minimal.*

Remark. If $\overline{\kappa}(X)\geq 0$ then from the Zariski decomposition it follows that the condition for C can be changed for $(D^{\#}+K_{\overline{X}})C<0$ and $C^2<0$.

Lemma 1.6.2. *([Miy01, 2.3.8, 2.3.4]). Let C be a log-exceptional curve of the first kind, then:*

(i) if $C\subseteq D$, then C is a (-1)-curve and $\beta_D(C)\leq 2$,

(ii) if $C\not\subseteq D$, then C is a (-1)-curve, intersects D transversally and the points of intersection belong to different components of $\operatorname{Bk}D$. Moreover, either $CD=1$ or $CD=2$ and one of the connected components of $\operatorname{Bk}D$ intersecting C is a rod of D. In particular, C intersects each connected component of D at most once.

Proposition 1.6.3. *([Miy01, 2.3.11]). The construction of an* almost minimal model *(which does not have to be unique) for a given snc-pair $(\overline{X}, D)$ goes by repeating operations (1) and (2) alternately:*

(1) snc-minimalize D, i.e. contract subsequently all non-branching (-1)-curves in D,

(2) find and contract a log-exceptional curve of the first kind C, such that $C \nsubseteq D$.

After each step change D for its proper image. After finite number of steps the resulting pair is an snc-pair and is almost minimal.

Definition 1.6.4. For a smooth open surface X let $(\overline{X}, D)$ be its snc-completion and let $(\overline{X}_m, D_m)$ be the almost minimal model. The connected components of $\operatorname{Bk} D_m$ can be contracted to quotient singularities. The resulting pair $(\overline{X}_r, D_r)$ is called a *relatively minimal model* of $(\overline{X}, D)$. We define the *almost minimal* and the *relatively minimal* model of X to be respectively $\overline{X}_m - D_m$ and $\overline{X}_r - D_r$. If $\overline{\kappa}(X) \geq 0$ then these models are unique. Clearly, the almost minimal model of X is a smooth locus of the relatively minimal model of X.

Example 1.6.5. An almost minimal model of $\mathbb{C}^2 \setminus \{0\}$ is $\mathbb{C}^2$.

Let T be a $\mathbb{Q}$-divisor on a smooth projective surface. T is *nef* if $TC \geq 0$ for every irreducible curve C. T is *pseudoeffective* if $TH \geq 0$ for every nef divisor H. Effective and nef divisors are pseudoeffective.

Proposition 1.6.6. *(Zariski-Fujita decomposition; cf. [Miy01, 2.1.19]). Let T be a pseudo-effective $\mathbb{Q}$-divisor on a smooth projective surface V. There exists a unique effective divisor $T^- = \sum_{i=1}^r a_i N_i$ with N_i irreducible, such that:*

(i) either $Q(T^-)$ is negative definite or $T^- = 0$,

(ii) $T^+ := T - T^-$ is nef (hence pseudoeffective),

(iii) $T^+ N_i = 0$ for every $1 \leq i \leq r$.

Remark. It follows from the lemma stated below that if T is effective then T^+ is effective.

Lemma 1.6.7.

(i) Let A and B be some ($\mathbb{Z}$- or $\mathbb{Q}$-) divisors, such that $A + B$ is effective and $Q(B)$ is negative definite. If $AB_i = 0$ for each irreducible component B_i of B then A is effective.

(ii) For every natural n one has $h^0(n(K_{\overline{X}} + D)) = h^0([n(K_{\overline{X}} + D^\#)])$, where $[\ ]$ denotes the integer part of a $\mathbb{Q}$-divisor.

Proof. (i) We can assume that A and B are $\mathbb{Z}$-divisors and B is effective and nonzero. Write $B = \sum b_i B_i$ for some positive integers b_i and irreducible components B_i of B. Choose $b'_i \in \mathbb{N}$, such that the sum $\sum b'_i$ is the smallest possible among divisors $\sum b'_i B_i$, such that $A + \sum b'_i B_i$ is effective. If $b'_i > 0$ for some i then $(A + \sum b'_i B_i)(\sum b'_i B_i) = (\sum b'_i B_i)^2 < 0$ by the assumptions. Hence $\operatorname{Supp}(A + \sum b'_i B_i)$ contains some B_i, a contradiction with the definition of b'_i. Thus A is effective.

(ii) Let $\{T\}$ denote the fractional part of a $\mathbb{Q}$-divisor T, i.e. $T = [T] + \{T\}$. Let T be some effective divisor, such that $n(K_{\overline{X}} + D) \sim T$. Then $T - n \operatorname{Bk} D$ is effective by (i) and $n(K_{\overline{X}} + D^\#) \sim T - n \operatorname{Bk} D$. This gives $[T - n \operatorname{Bk} D] \geq -\{T - n \operatorname{Bk} D\}$, and since $[T - n \operatorname{Bk} D]$ is a $\mathbb{Z}$-divisor and components of $\{T - n \operatorname{Bk} D\}$ appear in $\{T - n \operatorname{Bk} D\}$ with proper fractional coefficients, we get that $[T - n \operatorname{Bk} D]$ is effective. □

Proposition 1.6.8. *(Kawamata, cf. [Fuj82, 6.11]). Let $(\overline{X}, D)$ be an snc-completion of X, such that $\overline{\kappa}(X) \geq 0$. For $\mathcal{P} = (K_{\overline{X}} + D)^+$ one has:*

(i) $\mathcal{P} \equiv 0$ if and only if $\overline{\kappa}(X) = 0$,

(ii) $\mathcal{P} \not\equiv 0$ and $\mathcal{P}^2 = 0$ if and only if $\overline{\kappa}(X) = 1$,

(iii) $\mathcal{P}^2 > 0$ if and only if $\overline{\kappa}(X) = 2$.

Notice that by the remark after 1.6.6, in case (i) $\mathcal{P} \sim 0$ as a $\mathbb{Q}$-divisor.

Lemma 1.6.9. *Assume $\kappa(K_{\overline{X}} + D) \geq 0$. One has:*

(i) The maximal twigs of D are contained in Supp $(K_{\overline{X}}+D)^-$. If D is snc-minimal then the maximal twigs of D are admissible ([Fuj82, 6.13]).

(ii) If $(\overline{X},D)$ is almost minimal then $(K_{\overline{X}}+D)^+ = K_{\overline{X}}+D^{\#}$ and $(K_{\overline{X}}+D)^- = \operatorname{Bk} D$ (cf. [Miy01, §2.3]).

Proof. (i) Let $T = C_1 + \ldots + C_n$ be a maximal twig of D. We have $C_i(K_{\overline{X}}+D) = \beta_D(C_i) - 2 \leq 0$. Clearly, $C_1 \subseteq D^-$ and since $C_iC_{i+1} = 1$, we get $C_i \subseteq (K_{\overline{X}}+D)^-$ by induction. □

Proposition 1.6.10. *(Iitaka, Kawamata). Let $\varphi : X \to Y$ be a fibration, i.e. a dominating morphism with irreducible and reduced generic fiber. Assume that X is smooth. Then for a general $y \in Y$:*

(i) $\overline{\kappa}(X) \leq \overline{\kappa}(\varphi^{-1}(y)) + \dim Y$ ([Iit82, Theorem 10.4]),

(ii) if Y is smooth and $\dim X - \dim Y \leq 1$ then $\overline{\kappa}(\varphi^{-1}(y)) + \overline{\kappa}(Y) \leq \overline{\kappa}(X)$ ([Kaw78]).

Remark. For *dim Y = dim X* (ii) implies $\overline{\kappa}(Y) \leq \overline{\kappa}(X)$. For a proof of (ii) in case X is a surface see [Miy01, 2.1.14].

Theorem 1.6.11. *(Structure theorem).*

(i) If $\overline{\kappa}(X) = -\infty$ and D is connected or X is not rational then X is affine-ruled ([Rus81], [Miy01, 2.2.1]).

(ii) Assume that $\overline{\kappa}(X) = -\infty$ and X is not affine-ruled. There exists a smooth surface $\widetilde{X}$ dominating X, which is affine-ruled ([KM99, Theorem 1.1]). If $Q(D)$ is not negative definite then the almost minimal model of X has a Platonic fibration, hence is isomorphic with $(\mathbb{C}^2 - \{0\})//G$ for some small finite non-abelian subgroup of $GL(2,\mathbb{C})$ (cf. [Miy01, 2.5.1] and [MT84a]).

(iii) If $\overline{\kappa}(X) = 0$ and $(\overline{X},D)$ is almost minimal then for every connected component J of D either J is a smooth elliptic curve or it is rational and is one of the following ([Fuj82, 8.8]):

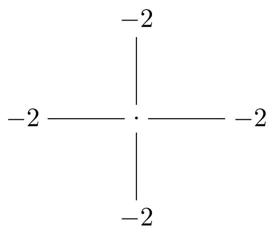
(I) an admissible chain or an admissible fork,

(O) a cycle, i.e. every component C of J satisfies $\beta_J(C) = 2$,

(Y) a fork F satisfying $\delta(F) = 1$,

(H) has dual graph

(X) has dual graph



(iv) If $\overline{\kappa}(X) = 1$ then X is either $\mathbb{C}^*$-ruled or elliptically ruled, i.e. it has a fibration with a generic fiber isomorphic to an elliptic curve (cf. [Fuj82, 6.11]).

Remark. A surface, which is affine-ruled or is dominated by an affine-ruled surface has $\overline{\kappa} = -\infty$. Elliptically- and $\mathbb{C}^*$-ruled surfaces have $\overline{\kappa} < 2$ by 1.6.10, but do not have to satisfy $\overline{\kappa} = 1$. For a more detailed structure theorems in cases (iii) and (iv) see [Miy01, §2.6].

We state a version of Bogomolov-Miyaoka-Yau inequality proved by Langer ([Lan03, Corollary 5.2]), which generalizes the inequalities of Miyaoka [Miy84, Theorem 1.1] and Kobayashi [Kob90, Theorem 2]:

Theorem 1.6.12. *Let (X, D) be a normal projective surface together with a $\mathbb{Q}$-divisor $D = \sum a_i D_i$ with $0 \leq a_i \leq 1$. Assume that the pair is log-canonical and a multiple of $K_X + D$ is effective. Then*

$$3\chi_{orb}(X, D) + \frac{1}{4}((K_X + D)^-)^2 \geq (K_X + D)^2,$$

where $\chi_{orb}(X, D)$ is the orbifold Euler number *(see [Lan03, 3.4] for a general definition and [Lan03, §9] for computations in special cases).*

Corollary 1.6.13. *Let (X, D) be an snc-pair with $\kappa(K_X + D) \geq 0$. Then:*

(1)

$$3\chi(X - D) + \frac{1}{4}((K_X + D)^-)^2 \geq (K_X + D)^2.$$

(2) Let $D_1, D_2, \ldots, D_n$ be all the connected components of D which are also connected components of $\operatorname{Bk} D$. In particular, D_i's are contractible to quotient singularities (cf. [Miy01, 2.3.14]). Denote the respective local fundamental groups by $G_1, \ldots, G_n$. Then

$$\chi(X - D) + \sum_{i=1}^{n} \frac{1}{|G_i|} \geq \frac{1}{3}(K_X + D^{\#})^2.$$

Proof. According to [Lan03, 3.4, 7.6] if (X, D) is a pair as in 1.6.12 and D is reduced then for $x \in D$ the local orbifold numbers $\chi_{orb}(x; X, D)$ vanish, hence

$$\chi_{orb}(X, D) = \chi(X - \operatorname{Sing} X - D) + \sum_{x \in \operatorname{Sing} X} \chi_{orb}(x; X, D).$$

This already proves (1), where X is smooth. Let $\pi : (X, D) \to (X', D')$ be a morphism contracting the connected components of $\operatorname{Bk} D$ to quotient points. Then $K_X + D^{\#} \equiv \pi^*(K_{X'} + D')$ by [Miy01, 2.3.14.1]. We need to know $\chi_{orb}(x; X', D')$. If $x \notin D'$ then the preimage of x is a connected component of D (and of $\operatorname{Bk} D$), so by [Lan03, 3.7] we have $\chi_{orb}(x; X', D') = \frac{1}{|G|}$, where G is the local fundamental group of x. We have $\chi(X' - \operatorname{Sing} X' - D') = \chi(X - D)$. Since $((K_X' + D')^-)^2 \leq 0$, (2) follows from 1.6.12 applied to (X', D').

□

Remark. Part (2) generalizes the Kobayashi inequality for the case $\overline{\kappa}(X - D) = 0, 1$, it is stronger than the original Miyaoka inequality (there is no $\frac{1}{4}N^2$ term, using the notation of [Miy84, Theorem 1.1]). If $\overline{\kappa}(X - D) = 2$ then to get the original Kobayashi inequality one has to apply 1.6.12 to the *strongly minimal model* of (X, D) (cf. [Miy01, 2.4.12, 2.6.6]).

Lemma 1.6.14. *Let X_0 be as in 1.2.3. Then there exists an open subset $X_m \subseteq X_0$, such that $\chi(X_m) \leq \chi(X_0)$ and X_m is isomorphic to an almost minimal model of X.*

Proof. Let $(\overline{X}, D + E)$ be an snc-minimal completion of X_0 as in the proof of 1.2.3. Consider the process of producing an almost minimal model of $(\overline{X}, D + E)$. If we contract a curve as in 1.6.3(2), then the lemma 1.6.2 implies that it causes a subtraction of a curve with $\chi = 1$ or $\chi = 0$ from X_0. Contractions of (-1)-curves contained in the boundary divisor do not affect X_0, unless some connected component of the boundary is eventually contracted to a smooth point which does not belong to the proper image of the boundary divisor. Then this point adds to an almost minimal model of X_0. This cannot happen for D. Indeed, since X' is affine, there exists an ample divisor with support contained in D, so $Q(D)$ is not negative definite. Affiness of X' implies that each curve intersects D or its image. Thus the snc-minimality of E implies that the above contraction to a smooth point cannot happen for E also. □

Remark. If $\overline{\kappa}(X_0) = 2$ then analogously the smooth part of the strongly minimal model X_{sm} of X_0 is an open subset of X_m with $\chi(X_{sm}) \leq \chi(X_m)$.

1.7 Quotients

The following lemma will be used to construct some (singular) $\mathbb{Q}$-homology planes. It is also useful in considering the question of affiness of singular $\mathbb{Q}$-homology planes.

Lemma 1.7.1. *(Contraction lemma). Let A and B be effective snc-divisors on a smooth complete surface X. Assume that $A \cap B = \emptyset$ and that for every irreducible curve $C \nsubseteq B$ on X one has $AC > 0$. Then for sufficiently large and sufficiently divisible n one has:*

(i) $|nA|$ has no base points,

(ii) $\varphi_{|nA|}$ is birational and contracts exactly the curves in B,

(iii) $Im\ \varphi_{|nA|}$ is normal, projective and is isomorphic to $\operatorname{Proj} \bigoplus_{n\geq 0} H^0(\mathcal{O}_X(nA))$.

The proof of (i) is a part of the proof of Nakai's criterion in [Har77, V.1.10]. One shows that $\mathcal{O}_X(A)\otimes\mathcal{O}_A$ is ample on A and then using the exact sequence $0 \to \mathcal{O}(-A) \to \mathcal{O}_X \to \mathcal{O}_A \to 0$ that $\mathcal{O}(nA)$ is generated by global sections for $n \gg 0$.

Statements (ii) and (iii) are proved for example in [Rei87, 2.3, 2.4].

Definition 1.7.2. Let $(\overline{X}, D)$ be an snc-completion of a smooth surface X and let $NS(\overline{X})$ be the Neron-Severi group $\overline{X}$. Define $NS(X)$ as a cokernel of the natural map $\mathcal{L}(D) \to NS(\overline{X})$ (cf. 1.1). This does not depend on an snc-completion of X (cf. [Fuj82, 1.19]). We denote $NS(X) \otimes \mathbb{Q}$ by $NS_{\mathbb{Q}}(X)$.

Remark. Assume X is complete. Since a homology class of a numerically trivial divisor on X is torsion (cf. [Laz04, 1.1.21]), there is a natural map $j : NS(X) \to H_2(X, \mathbb{Q})$. Since $NS(X)$ is torsionless, j is a monomorphism. On the other hand, if X is not complete then $NS(X)$ can have torsion.

Corollary 1.7.3. *Let A and B be effective snc-divisors on a smooth complete surface X. Assume that $A \cap B = \emptyset$, A is connected, $Q(B)$ negative definite and $NS_{\mathbb{Q}}(X - A - B) = 0$. Then there exists a normal affine surface Y and a morphism $\zeta : X - A \to Y$ contracting connected components of B, such that $\zeta : X - A - B \to Y - \zeta(B)$ is an isomorphism.*

Proof. Since $NS_{\mathbb{Q}}(X - A - B) = 0$, there exists a divisor $H = H_A + H_B$ with $H_A \subseteq A$ and $H_B \subseteq B$, which is numerically equivalent to an ample divisor on X. Then H is ample, because ampleness is a numerical property by Nakai's criterion. To use 1.7.1 we need to show that there exists a divisor F, such that $\operatorname{Supp} F = \operatorname{Supp} A$ and $FC > 0$ for all irreducible curves $C \nsubseteq B$. To deal with curves $C \subseteq A$ we use Fujita's argument ([Fuj82, 2.4]). Let $\mathcal{U}$ consist of all effective divisors T, such that $T \subseteq A$ and $TT_i > 0$ for any prime component T_i of T. Writing $H_A = H_+ - H_-$, where H_+, H_- are effective and have no common component, we see that $\mathcal{U}$ is nonempty because $H_+ \in \mathcal{U}$. Suppose F is an element of $\mathcal{U}$ with maximal number of components. For an irreducible curve $C \nsubseteq F$ satisfying $CF > 0$ one would get $tF + C \in \mathcal{U}$ for $t > max(0, -C^2)$, hence $\operatorname{Supp} F = \operatorname{Supp} A$ by connectedness of A.

Suppose an irreducible curve $C \nsubseteq B$ satisfies $CF = 0$. Since $F \in U$, we have $C \nsubseteq F$. We can choose some reduced divisor $F' \subseteq F$, such that the irreducible components of $F' + B$ give a basis of $NS_{\mathbb{Q}}(X)$. Let's write $C \equiv \sum_i \alpha_i F_i + B^+ - B^-$, where $F_i \subseteq F'$, the divisors $B^+, B^- \subseteq B$ are effective and have no common component. For each j we have $CF_j = 0$, so $(\sum_i \alpha_i F_i)F_j = CF_j = 0$, hence $\sum_i \alpha_i F_i = 0$ because $d(F') \neq 0$. We have $(B^+)^2 = B^+C + B^+B^- \geq 0$, so $B^+ = 0$. Thus the divisor $C + B^-$ is nonzero, effective and numerically trivial, a contradiction. Let $\zeta = \varphi_{|nF|}$ for n as in lemma 1.7.1. Then $\zeta : X - A \to Im\ \zeta$ contracts connected components of B. We have also $nF = \zeta^* H$, where H is a very ample divisor on $Im\ \zeta$, which implies that $Im\ \zeta$ is affine. □

Chapter 2

Topology of $\mathbb{Q}$-homology planes

2.1 Homology groups

2.1.1. Notation. Let S' be a *singular* $\mathbb{Q}$*-homology plane*, i.e. an irreducible normal surface, which is $\mathbb{Q}$-acyclic and not smooth. We assume nothing more about the type of singularities. In particular, S' does not have to be a *logarithmic* $\mathbb{Q}$*-homology plane*, i.e. its singularities do not have to be of quotient type. If $\epsilon\colon S \to S'$ is a good resolution and $(\overline{S}, D)$ is an *snc*-completion of S then by definition $\overline{\kappa}(S') = \overline{\kappa}(S) = \kappa(K_{\overline{S}} + D)$, where $K_{\overline{S}}$ stands for a canonical divisor on $\overline{S}$ (see [Iit82] for the definition and properties of Kodaira dimension of a divisor). Let $\{p_1, \ldots, p_q\}$ be the singular locus of S' and let $\widehat{E}_i = \epsilon^{-1}(p_i)$. We assume that $\widehat{E} = \widehat{E}_1 + \widehat{E}_2 + \ldots + \widehat{E}_q$ is *snc*-minimal. The intersection matrix $Q(\widehat{E})$ is negative definite. We put $S_0 = S \setminus \widehat{E} \cong S' \setminus Sing\ S'$. We define $M_i = \partial Tub(\widehat{E}_i)$, where $Tub(\widehat{E}_i)$ is a tubular neighborhood of $\widehat{E}_i$ in S. There exists a deformation retraction $Tub(\widehat{E}_i) \to \widehat{E}_i$. We can assume that $Tub(\widehat{E}_i) \cap Tub(\widehat{E}_j) = \emptyset$ for $i \neq j$ and that every M_i is a closed oriented 3-manifold. Put $M = \bigcup_{i=1}^{q} M_i$. The construction of $Tub(\widehat{E}_i)$ can be found in [Mum61].

Convention. We write $H_i(X, A)$ for $H_i(X, A; \mathbb{Q})$ and define $b_i(X, A) = \dim H_i(X, A)$.

Lemma 2.1.2. *([Mum61]). There exist exact sequences*

$$0 \longrightarrow K_i \longrightarrow H_1(M_i, \mathbb{Z}) \xrightarrow{j} H_1(\widehat{E}_i, \mathbb{Z}) \longrightarrow 0,$$

where K_i are finite groups, $|K_i| = |d(\widehat{E}_i)|$ and j is induced by a composition of inclusion $M_i \to cl(Tub(\widehat{E}_i))$ with retraction onto $\widehat{E}_i$.

Remark. Since $H_1(\widehat{E}, \mathbb{Z})$ is free abelian, it follows that $H_1(M_i, \mathbb{Z}) = H_1(\widehat{E}_i, \mathbb{Z}) \oplus K_i$. Clearly, Betti numbers of M are: $b_0(M) = b_3(M) = q$ and $b_2(M) = b_1(M) = b_1(\widehat{E})$.

Proposition 2.1.3. *Let $j_{\widehat{E}} : \widehat{E} \to S$, $j_M : M \to S_0$, $i_D : D \to \overline{S}$ and $i_{D\cup\widehat{E}} : D \cup \widehat{E} \to \overline{S}$ be the inclusion maps. One has:*

(i) $H_k(j_{\widehat{E}})$ *is an isomorphism for positive* k,

(ii) $H_k(j_M)$ *is an isomorphism for positive* k,

(iii) D *is connected,*

(iv) $H_1(i_D)$ *is an isomorphism,*

(v) $H_2(i_{D\cup\widehat{E}})$ *is an isomorphism,*

(vi) $b_1(\widehat{E}) = b_1(D) = b_1(\overline{S})$,

(vii) $H_k(S', \mathbb{Z}) = 0$ *for* $k \geq 2$,

(viii) $\pi_1(\epsilon) : \pi_1(S) \to \pi_1(S')$ *is an epimorphism, it is an isomorphism if* $b_1(\widehat{E}) = 0$.

(ix) if $b_1(\widehat{E})=0$ *then* $|d(D)|=|d(\widehat{E})|\cdot|H_1(S',\mathbb{Z})|^2$.

Proof. (i) We look at the homology exact sequence of a pair $(S,\widehat{E})$. The pairs $(S,\widehat{E})$ and $(S',\operatorname{Sing} S')$ are 'good CW-pairs' (see [Hat02, Thm 2.13]), so for $k>1$ we have $H_k(S,\widehat{E})=H_k(S',Sing\ S')=0$ and then $H_k(j_{\widehat{E}})\colon H_k(\widehat{E})\to H_k(S)$ induced by inclusion $j_{\widehat{E}}$ is an isomorphism for $k>1$. Now $b_1(S,\widehat{E})=b_1(S',\operatorname{Sing} S')=q-1=b_0(\widehat{E})-1$, so $H_1(j_{\widehat{E}})$ is also an isomorphism.

(ii) Let $k>0$. We know that $H_k(j_{\widehat{E}})$ is an epimorphism, so the Mayer-Vietories sequence for $S=S_0\cup\bigcup_{i=1}^q Tub(\widehat{E}_i)$ splits into exact sequences:

$$0\longrightarrow H_k(M)\longrightarrow H_k(S_0)\oplus H_k(\widehat{E})\longrightarrow H_k(S)\longrightarrow 0.$$

Now by (i) $H_k(j_M)$ is a homomorphism between spaces of the same dimension and it is injective, because $H_k(j_{\widehat{E}})$ is.

(iii) $\overline{S}$ is connected, so by (ii) the Lefschetz duality (see [Dol80]) $H^0(D)=H_4(\overline{S},S)$ gives a connectedness of D:

$$0=H_4(S)\longrightarrow H_4(\overline{S})\longrightarrow H_4(\overline{S},S)\longrightarrow H_3(S)=0.$$

(iv) The Neron-Severi group of a smooth complete surface X embeds into $H^2(X)\cong H_2(X)$. Since $d(\widehat{E})\neq 0$, we see that the inclusion $j:\widehat{E}\hookrightarrow\overline{S}$ induces a monomorphism on H_2. Using (i) we can write the exact sequence of a pair $(\overline{S},S)$ as:

$$\ldots\longrightarrow H_3(\widehat{E})\longrightarrow H_3(\overline{S})\longrightarrow H_3(\overline{S},S)\longrightarrow H_2(\widehat{E})\longrightarrow H_2(\overline{S})\to\ldots.$$

Now $H_2(j)$ is a monomorphism, so $H_3(\overline{S})\to H_3(\overline{S},S)$ is an epimorphism. Hence it is an isomorphism, because $H_3(\widehat{E})=0$. By Poincare and Lefschetz duality we get $b_1(\overline{S})=b_1(D)$. Now looking at the exact sequence of the pair $(\overline{S},D)$:

$$\ldots\longrightarrow H_1(D)\longrightarrow H_1(\overline{S})\longrightarrow H_1(\overline{S},D)\longrightarrow\ldots$$

we see that $H_1(i_D)$ is an epimorphism, because by (ii) and Lefschetz duality $H_1(\overline{S},D)=H^3(S)=H^3(\widehat{E})=0$. It is therefore an isomorphism.

(v) Let $\gamma=H_2(i_{D\cup\widehat{E}})$. Consider the exact sequence of a pair $(\overline{S},D\cup\widehat{E})$:

$$0\longrightarrow H_3(\overline{S})\xrightarrow{\alpha}H_3(\overline{S},D\cup\widehat{E})\xrightarrow{\beta}H_2(D\cup\widehat{E})\xrightarrow{\gamma}H_2(\overline{S})\longrightarrow H_2(\overline{S},D\cup\widehat{E})$$

$$\xrightarrow{\delta}H_1(D\cup\widehat{E})\xrightarrow{\epsilon}H_1(\overline{S})\longrightarrow H_1(\overline{S},D\cup\widehat{E})\xrightarrow{\zeta}\widetilde{H}_0(D\cup\widehat{E})\longrightarrow 0.$$

Since $b_1(\overline{S},D\cup\widehat{E})=b_3(S_0)=q$ by (ii) and $b_0(D\cup\widehat{E})=q+1$ by (iii), we get that ζ is a monomorphism, hence ϵ is an epimorphism. Therefore by (v) $\dim\operatorname{Im}\delta=\dim\operatorname{Ker}\epsilon=b_1(\widehat{E})+b_1(D)-b_1(\overline{S})=b_1(\widehat{E})$. However, $b_2(\overline{S},D\cup\widehat{E})=b_2(S_0)=b_1(\widehat{E})$ by (ii), so δ is a monomorphism. We infer that γ is an epimorphism. We compute $b_2(\overline{S})=b_2(D\cup\widehat{E})-\dim\operatorname{Im}\beta$ and $\dim\operatorname{Im}\beta=b_3(\overline{S},D\cup\widehat{E})-b_3(\overline{S})=b_1(S_0)-b_1(\overline{S})=b_1(\widehat{E})-b_1(D)$ by (ii) and (iv). Hence $b_2(\overline{S})=b_2(D\cup\widehat{E})+b_1(D)-b_1(\widehat{E})$.

We will now prove that γ is a monomorphism. By the above computation of $b_2(\overline{S})$ this is equivalent to the equality $b_1(D)=b_1(\widehat{E})$. Consider the case when $\widehat{E}$ is a rational tree. Then $H_3(\overline{S},D)=H^1(S)=0$ by (i). The homology exact sequence of a pair $(\overline{S},D)$ then gives $H_3(\overline{S})=0$. Since $H_2(S_0)=0$ by (ii) and 2.1.2, the homology exact sequence of a pair $(\overline{S},S_0)$ gives $H_3(\overline{S},S_0)=0$. By Lefschetz duality the last group is isomorphic to $H^1(D)$, so the statement is proved. Now assume that $\widehat{E}$ is not a rational tree. This implies that $\overline{\kappa}(S_0)\leq 1$ by 2.2.1. If $\overline{\kappa}(S_0)=1$ then S_0 is either elliptically ruled or $\mathbb{C}^*$-ruled (cf. 1.6.11(iv)). Since modifications of $D+\widehat{E}$ do not change $b_1(D)$ and $b_1(\widehat{E})$, we can assume that this ruling extends to $\overline{S}$. In the case of elliptically ruled S_0 the divisor $D+\widehat{E}$ is vertical, hence $Q(D+\widehat{E})$ is semi-negative definite, but since γ is an epimorphism, we know that $NS(\overline{S})$ is generated by classes of irreducible components of $D+\widehat{E}$, so this contradicts the Hodge index theorem. Thus S_0 is $\mathbb{C}^*$-ruled and there are unique sections contained in $\widehat{E}$ and in D, because $\widehat{E}$ cannot be vertical, otherwise would be a rational tree. It follows that $b_1(\widehat{E})=b_1(D)=b_1(B)$, where B is the base curve of the ruling. Hence we can assume $\overline{\kappa}(S_0)\leq 0$. First we will obtain a contradiction in the case $\overline{\kappa}(S)=\overline{\kappa}(S_0)=0$ by showing that $\widehat{E}$ is a rational tree. Indeed, in the above case we can assume that S_0 is almost minimal, because the minimalization does not effect the rationality of $\widehat{E}$. We have $K + D^{\#} + \widehat{E}^{\#} \equiv 0$ by 1.6.8 and $K + D^{\#} \geq_{\mathbb{Q}} 0$ by 1.6.7. Therefore in this case $\widehat{E}^{\#} = 0$, so $\widehat{E}$ is a rational tree by 1.3.1(iii), a contradiction. Thus we get $\overline{\kappa}(S) < 0$, so S is affine-ruled (cf. 1.6.11(i)). Let π be the extension of this ruling to $\overline{S}$. Consider a divisor $T = \sum_i d_i D_i + \sum_j e_j E_j \equiv 0$ with distinct irreducible components $D_i \subseteq D$ and $E_j \subseteq \widehat{E}$. To finish the proof that γ has no kernel it is enough to show that $T = 0$. Using negative definiteness of $Q(\widehat{E})$ we see that each e_j vanishes, otherwise $0 > (\sum_j e_j E_j)^2 = T(\sum_j e_j E_j)$. Intersecting T with a fiber we see that the horizontal component of D does not occur in the sum $T = \sum_j d_j D_j$ with nonzero coefficient, therefore $\operatorname{Supp} T$ is contained in fibers of the $\mathbb{P}^1$-ruling of $\overline{S}$. If $T \neq 0$ then T, and hence D, has to contain at least one fiber, otherwise $T^2 < 0$. However, this implies that $\widehat{E}$ is vertical, hence is a rational tree, a contradiction.

(vi) It was shown in the proof of (v) that $b_2(\overline{S}) = b_2(D \cup \widehat{E}) + b_1(D) - b_1(\widehat{E})$, hence by (v) $b_1(\widehat{E}) = b_1(D)$.

(vii) Let $3 \leq k \leq 4$. Since $H_k(j_{\widehat{E}})$ is an isomorphism by (i), the groups $H_k(S, \widehat{E}, \mathbb{Z})$ are torsion. We have $H_k(S, \widehat{E}, \mathbb{Z}) \cong H_k(S', \mathbb{Z})$, so the exact sequence of a pair $(S, \widehat{E})$ with coefficients in $\mathbb{Z}$ gives $H_k(S', \mathbb{Z}) \cong H_k(S, \widehat{E}, \mathbb{Z}) \cong H_k(S, \mathbb{Z})$. However, since $H_k(S, \mathbb{Z})$ are torsion, by the universal coefficient formula and Lefschetz duality we get $H_k(S, \mathbb{Z}) \cong H^{k+1}(S, \mathbb{Z}) \cong H_{3-k}(\overline{S}, D, \mathbb{Z}) = 0$. The vanishing of $H_2(S', \mathbb{Z})$ is more subtle (it will not be used until chapter 6). The generalization of Andreotti-Frankel theorem proved by Karchyauskas says that an affine variety X of complex dimension n has the homotopy type of a CW-complex of real dimension not greater than n (see [GM88] for proofs and generalizations). In particular, $H_n(X, \mathbb{Z})$ is torsionless. Knowing that S' is affine (cf. 2.2.3(iii)) we get that $H_2(S', \mathbb{Z})$ is torsionless, hence vanishes.

(viii) For simplicity we assume that $\widehat{E}$ is connected. In general the proof is by induction on the number of connected components of $\widehat{E}$. Let $B \subseteq S'$ be a contractible neighborhood of p_1. We can assume that the preimage of B under $\epsilon : S \to S'$ is $Tub(\widehat{E})$ and that the boundaries $\partial Tub(\widehat{E})$ and ∂B are homeomorphic. Put $G = \pi_1(S' \setminus B) \cong \pi_1(S \setminus Tub(\widehat{E}))$ and $H = \pi_1(\partial B) \cong \pi_1(\partial Tub(\widehat{E}))$. Then by van Kampen's theorem $\pi_1(S) \cong G \underset{H}{*} \pi_1(\widehat{E})$ and $\pi_1(S') \cong G \underset{H}{*} \{1\}$. Clearly, $\pi_1(\widehat{E})$ is in the kernel of $\pi_1(\epsilon)$.

(ix) Let $M_D = \partial Tub(D)$ be the boundary of the tubular neighborhood of D. We can assume that M_D is a 3-manifold disjoint from $M = \partial Tub(\widehat{E})$. D is a rational tree and $d(D) \neq 0$, because by (v) the components of D are independent in $H_2(\overline{S})$. Thus we can use Mumford's result 2.1.2. Notice that $H_2(M_D, \mathbb{Z})$ and $H_2(M, \mathbb{Z})$ are free abelian groups by Poincare duality. We know also that $H_2(S_0)$ is finite in this case. Consider an exact sequence of a pair (K, M_D), where $K = \overline{S} \setminus (Tub(D) \cup Tub(\widehat{E}))$:

$$0 \longrightarrow H_2(K, \mathbb{Z}) \longrightarrow H_2(K, M_D, \mathbb{Z}) \longrightarrow H_1(M_D, \mathbb{Z}) \longrightarrow H_1(K, \mathbb{Z}) \longrightarrow H_1(K, M_D, \mathbb{Z}) \longrightarrow 0.$$

By Lefschetz duality (cf. [Hat02, 3.43]) $H_i(K, M_D, \mathbb{Z}) \cong H^{4-i}(K, M, \mathbb{Z}) = H^{4-i}(S', Sing\ S', \mathbb{Z})$, and for $i > 1$ we get $H_i(K, M_D, \mathbb{Z}) \cong H^{4-i}(S', \mathbb{Z}) \cong H_{3-i}(S', \mathbb{Z})$ by universal coefficient formula. This gives an exact sequence:

$$0 \longrightarrow H_2(K, \mathbb{Z}) \longrightarrow H_1(S', \mathbb{Z}) \longrightarrow H_1(M_D, \mathbb{Z}) \longrightarrow H_1(K, \mathbb{Z}) \longrightarrow H_2(S', \mathbb{Z}) \longrightarrow 0.$$

Consider the reduced exact sequence of a pair (K, M):

$$0 \longrightarrow H_2(K, \mathbb{Z}) \longrightarrow H_2(K, M, \mathbb{Z}) \longrightarrow H_1(M, \mathbb{Z}) \longrightarrow H_1(K, \mathbb{Z}) \longrightarrow H_1(K, M, \mathbb{Z}) \longrightarrow \widetilde{H}_0(M, \mathbb{Z}) \longrightarrow 0.$$

Since $H_i(K, M, \mathbb{Z}) \cong H_i(S', Sing\ S', \mathbb{Z})$ and $H_1(S', Sing\ S', \mathbb{Z}) = H_1(S', \mathbb{Z}) \oplus \widetilde{H}_0(\operatorname{Sing} S', \mathbb{Z})$ we get:

$$0 \longrightarrow H_2(K, \mathbb{Z}) \longrightarrow H_2(S', \mathbb{Z}) \longrightarrow H_1(M, \mathbb{Z}) \longrightarrow H_1(K, \mathbb{Z}) \longrightarrow H_1(S', \mathbb{Z}) \longrightarrow 0.$$

Since $H_2(S', \mathbb{Z}) = 0$ by (vii), we get $H_2(K, \mathbb{Z}) = 0$. Now $|H_1(M, \mathbb{Z})| = |d(\widehat{E})|$ and $|H_1(M_D, \mathbb{Z})| = |d(D)|$ by 2.1.2, so we get the thesis easily.

□

Corollary 2.1.4.

(i) $b_1(S_0) = b_2(S_0) = b_1(\widehat{E}),\ b_3(S_0) = q,\ b_4(S_0) = 0,$

(ii) $\chi(S_0) = 1 - q,\ \chi(S) = \#\widehat{E} + 1 - b_1(\widehat{E}),\ \chi(\overline{S}) = \#D + \#\widehat{E} + 2 - 2b_1(\widehat{E}),$

(iii) $\Sigma_{S_0} = h + \nu - 2$ *and* $\nu \leq 1$.

Proof. (i) follows from 2.1.3(ii) and 2.1.2.

(ii) $\chi(S_0) = \chi(S') - q = 1 - q$ and other equalities are follow from (i) and 2.1.3(vi).

(iii) By 1.5.3 and 2.1.3(v) $\Sigma_{S_0} = h + \nu - 2$. Suppose $\nu > 1$. Then the numerical equivalence of fibers of a $\mathbb{P}^1$-ruling gives a numerical dependence of components of $D + \widehat{E}$, hence $Q(D + \widehat{E})$ is not of full rank and we get $d(D + \widehat{E}) = 0$. This contradicts 2.1.3(v). □

2.2 Algebraic properties

It is known ([PS97, Theorem 1.1]) that logarithmic $\mathbb{Q}$-homology planes are rational. We will see that this is not true for a general $\mathbb{Q}$-homology plane S', so the description of birational type of S' is of interest. We describe also general properties of the singularities of S'.

Lemma 2.2.1.

(i) if $\overline{\kappa}(S_0) = 2$ then S' is logarithmic and $\# \operatorname{Sing} S' = 1$,

(ii) if $\overline{\kappa}(S_0) = 0$ or 1 then either $\# \operatorname{Sing} S' = 1$ or $\# \operatorname{Sing} S' = 2$ and $\widehat{E}_1 = \widehat{E}_2 = [2]$.

Proof. We assume additionally that S' is affine, this will be proved in 2.2.3(iii). Let (S_m, D_m) be the almost minimal model of $(\overline{S}, D + \widehat{E})$. By 1.6.14 the almost minimal model $S_m - D_m$ of S_0 is isomorphic to an open subset of S_0 satisfying $\chi(S_m - D_m) \leq \chi(S_0) = 1 - q$. By 1.6.13(2) $\frac{1}{3}((K_{S_m} + D_m)^+)^2 \leq \chi(S_m - D_m) + \sum_{P \in Q} \frac{1}{|G_P|} \leq 1 - q + \frac{\#Q}{2} \leq 1 - \frac{q}{2}$, where Q is the set of singular points of $S_m - D_m$. If $\overline{\kappa}(S_0) = 2$ then we get $q = 1$ and $0 < \sum_{P \in Q} \frac{1}{|G_P|}$, so there is a unique singular point on S' and it is of quotient type. If $\# \operatorname{Sing} S' > 1$ then we get $q = 2$ and $1 \leq 1/|G_{P_1}| + 1/|G_{P_2}|$, so $|G_{P_1}| = |G_{P_2}| = 2$. □

Remark 2.2.2. If $\kappa(\overline{S}) = -\infty$ then by modifying the pair $(\overline{S}, D)$ we can assume that there exists a $\mathbb{P}^1$-ruling $\overline{p} : \overline{S} \to B$, such that B is a smooth complete curve. It is easy to see that topologically B is determined uniquely. Indeed, since blowup does not change the fundamental group of a surface, we can assume that all fibers are smooth. Applying the exact sequence of a fibration we get $\pi_1(\overline{S}) = \pi_1(B)$. This determines B. If S or S_0 are $\mathbb{C}^1$- or $\mathbb{C}^*$-ruled we can always assume that $\overline{p}$ extends the given ruling.

Proposition 2.2.3.

(i) $NS_{\mathbb{Q}}(S_0) = 0$,

(ii) $d(D) < 0$, and $Q(D)$ has signature $(1^+, (\#D - 1)^-)$,

(iii) S' is affine,

(iv) $\overline{S}$ is $\mathbb{P}^1$-ruled over a curve of genus $\frac{1}{2} b_1(D) = \frac{1}{2} b_1(\widehat{E})$ (hence $\kappa(\overline{S}) = -\infty$),

(v) if $\overline{\kappa}(S') \geq 0$ then $\overline{S}$ is rational and S' has topologically rational singularities,

(vi) $\widehat{E}$ and D are trees with at most one nonrational component,

(vii) $\pi_1(i_D) : \pi_1(D) \to \pi_1(\overline{S})$ is an isomorphism,

(viii) if $\widehat{E}$ consist only of (-2)-curves then $\overline{\kappa}(S') = \overline{\kappa}(S_0)$.

Proof. (i) follows from 2.1.3(v) and the inclusion $NS(\overline{S}) \hookrightarrow H^2(\overline{S}) \cong H_2(\overline{S})$.

(ii) Since by 2.1.3(v) the components of $D + \widehat{E}$ form a basis of $H_2(\overline{S})$ we get $d(D) \neq 0$. By Hodge's index theorem we get that the signature of $Q(D)$ is $(1^+, (\#D - 1)^-)$, because $Q(\widehat{E})$ is negative definite. It follows that $d(D) = det(Q(-D)) < 0$.

(iii) 2.1.3(iii) and (i) imply that $A = D$ and $B = \widehat{E}$ satisfy the assumptions of 1.7.3, so S' is affine. Notice that by 1.7.1(ii) the boundary divisor D of S can be identified with the boundary divisor of S' in the image of appropriate $\varphi_{|nD|}$.

(iv) Assume on the contrary that $\overline{S}$ is not $\mathbb{P}^1$-ruled, or equivalently that $\kappa(\overline{S}) \geq 0$. $\overline{S}$ cannot be rational, so S' is not logarithmic by [PS97, Theorem 1.1], hence $0 \leq \kappa(\overline{S}) \leq \overline{\kappa}(S) \leq \overline{\kappa}(S_0) < 2$ by 2.2.1. Affiness of S' implies that S cannot contain complete curves not contained in $\widehat{E}$, hence S cannot be elliptically ruled. It follows from 1.6.11(iv) that if $\overline{\kappa}(S)=1$ then S is $\mathbb{C}^*$-ruled, so $\overline{S}$ is $\mathbb{P}^1$-ruled, a contradiction with $\kappa(\overline{S})\geq 0$. Therefore we have $\overline{\kappa}(S)=0$. We will prove that D is algebraically contractible. We can assume that $(\overline{S},D)$ is almost minimal, so $(K_{\overline{S}}+D)^+=K_{\overline{S}}+D^{\#}\equiv 0$ by 1.6.9(ii). Now $K_{\overline{S}}\geq_{\mathbb{Q}}0$ implies $D^{\#}=0$, so $D=\operatorname{Bk}D$ and $Q(D)$ is negative definite, which contradicts (ii).

(v) By (iv) we can assume that there exists a $\mathbb{P}^1$-ruling $\overline{p}:\overline{S}\to B$ as in 2.2.2. We have $b_1(B)=b_1(\overline{S})=b_1(\widehat{E})$ by 2.1.3(vi), so $\overline{S}$ is rational if and only if $b_1(\widehat{E})=0$. By 2.2.3 we can assume that $\overline{\kappa}(S_0)\leq 1$. If S_0 is $\mathbb{C}^*$-ruled then from $\overline{\kappa}(S)\geq 0$ we get that $\widehat{E}$ has to be contained in some fibers of $\overline{p}$, so it is a rational tree. We have left with the case $\overline{\kappa}(S)=\overline{\kappa}(S_0)=0$. We can assume that S_0 is relatively minimal, because the minimalization does not effect the rationality of $\widehat{E}$. We have $K+D^{\#}+\widehat{E}^{\#}\equiv 0$ by 1.6.8 and $K+D^{\#}\geq_{\mathbb{Q}}0$ by 1.6.7. Therefore $\widehat{E}^{\#}=0$, so $\widehat{E}$ is a rational tree by 1.3.1(iii).

(vi) This is clear if $\overline{S}$ is rational (cf. 2.1.3(vi)), so by (v) we can assume that $\overline{\kappa}(S)=-\infty$, so S is affine-ruled. Let $\overline{p}:\overline{S}\to B$ be the extension to a $\mathbb{P}^1$-ruling of $\overline{S}$. Then D is a tree and has exactly one irreducible component - the horizontal section. Now $\widehat{E}$ is not a rational tree, so it has a horizontal component E_0. Then $g(E_0)\geq g(B)$, so $b_1(E_0)\geq b_1(B)$. However, $b_1(B)=b_1(D)=b_1(\widehat{E})$, so $b_1(E_0)=b_1(\widehat{E})$, hence $\widehat{E}$ is a tree by 1.1.

(vii) If $\overline{\kappa}(S)\geq 0$ then the statement follows from (v) and 2.1.3(iv). If $\overline{\kappa}(S)=-\infty$ then S is affine-ruled by 2.1.3(iv). Let D_h be the horizontal component of D, then $\pi_1(D)=\pi_1(D_h)$, so the composition $\overline{p}\circ i_D:D\to\overline{S}\to B$ induces an isomorphism on π_1. The exact sequence of fibration gives that $\pi_1(\overline{p})$ is an isomorphism.

(viii) We have to prove $\overline{\kappa}(S_0)\leq\overline{\kappa}(S)$. If $\widehat{E}$ consists of (-2)-curves then $(K+D)E_i=0$ for each irreducible component E_i of $\widehat{E}$. If T is an effective divisor linearly equivalent to $n(K+D+\widehat{E})$ then, since $Q(\widehat{E})$ is negative definite, $T-n\widehat{E}$ is effective by 1.6.7 and we are done. □

We now state a theorem strengthening the proposition 2.2.3(v). As we will see later it does not generalize to the case $\overline{\kappa}(S')=-\infty$.

Theorem 2.2.4. *Singular $\mathbb{Q}$-homology planes of non-negative Kodaira dimension are rational and logarithmic, i.e. the singularities are of quotient type. If the singular locus is disconnected then it consists of two points of type A_1.*

Proof. We only need to prove the logarithmicity of S'. By 2.2.1 we can assume that $\overline{\kappa}(S_0)=0$ or 1 and that $\widehat{E}$ is connected. If $\overline{\kappa}(S_0)=1$ then S_0 is $\mathbb{C}^*$-ruled by 1.6.11(iv). It will be proved in chapter 3 (cf. 3.2.2) that with two exceptions (for which $\widehat{E}$ is a (-2)-chain), if $\overline{\kappa}(S_0)=0$ then S_0 is $\mathbb{C}^*$-ruled as well. Therefore, we can assume that S_0 is $\mathbb{C}^*$-ruled. Consider an extension $\pi:\overline{S}\to B$ of this ruling to an snc-completion $(\overline{S},D+\widehat{E})$ with $D+\widehat{E}$ being π-minimal. Denote the set of horizontal components of $D+\widehat{E}$ by D_h. Since $\overline{\kappa}(S')\geq 0$, $D_h\subseteq D$ and D_h consists of at most two components. It consists either of two 1-sections or of one 2-section, hence it can intersect only these fiber components which have multiplicity not greater than two. Let F be a singular fiber containing $\widehat{E}$ and let D_v be the divisor of D-components of F. We use 1.5.1 without comments. By 2.1.4(iii) we get $\nu\leq 1$ and $\Sigma_{S_0}=\#D_h+\nu-2\leq 1$, so $\sigma\leq 2$ for every fiber of π. Suppose $\widehat{E}$ is not a resolution of a quotient singularity, in particular it is not an admissible chain (cf. 1.4.1). We obtain a successive restrictions on F eventually leading to a contradiction.

(1) (-1)-curves of F are S_0-components.

Proof. Suppose F contains a (-1)-curve $D_0\subseteq D$. We have $\Sigma_{S_0}=0$. Indeed, if $\Sigma_{S_0}>0$ then $\#D_h=2$ and $\nu=1$, so by simply connectedness of D at most one horizontal component of D intersects D_0. However, in this case $\mu(D_0)=1$, so D_0 is a tip of F, which contradicts the π-minimality of D. First we prove that D_v contains components of multiplicity one. If $\#D_h=2$ then π-minimality of D implies that D_0 intersects both horizontal components of D, hence $\mu(D_0)=1$. If $\#D_h=1$ then simply connectedness of D implies that $D_h\cap F$ is a branching point of $\pi_{|D_h}$ and by π-minimality D_0 intersects two other D-components of F, which have multiplicity one, because $\mu(D_0)=2$. Thus we are done. Let C be the unique S_0-component of F. We see easily that D_0 cannot be the unique (-1)-curve of F, hence $C^2=-1$ and there are no more (-1)-curves in F. Let's make a connected sequence of blowdowns starting from D_0 until the number of (-1)-curves decreases. Clearly, since $\widehat{E}\cap D=\emptyset$, in this process we do not touch $C+\widehat{E}$ (first we would touch C, and then C becomes a 0-curve). Let F' be the image of F, we can write $\underline{F'}-C=D'+\widehat{E}$, where

D' is the image of D_v. Since $C + \widehat{E}$ is not touched, $D' \neq 0$. Notice that D_v contains a component of multiplicity one, hence the same is true for D'. Since C is the unique (-1)-curve of F', it follows that $\widehat{E}$ is a chain, a contradiction. □

(2) F contains two (-1)-curves.

Proof. Suppose F has a unique (-1)-curve C. Write $\underline{F} - C = A + B$, where A and B are disjoint, connected, and B is a chain (possibly empty). By our assumption on $\widehat{E}$ we have $\widehat{E} \subseteq A$, hence B can contain only S_0- and D-components. Notice that by 2.2.3(iii) each S_0-component intersects D. Since D is connected, this implies that either $BD_h > 0$ or $B = 0$. If $B \neq 0$ we get that B contains a curve with $\mu \leq 2$, so then F consist of two branches with the first being equal to $[2, k, 2]$ for some $k > 1$, hence $\widehat{E}$ is an admissible fork of type $(2, 2, n)$, a contradiction. Thus $B = 0$. If $\mu(C) \leq 2$ then again $\widehat{E}$ would be an admissible fork, so we get $\mu(C) > 2$. If follows that $D_hC = 0$, so there is a unique D-component D_1 intersecting C. Since D is connected, there is a chain $T \subseteq F$ of D-components containing D_1 and some D-component D_2 with $\mu(D_2) \leq 2$. If D_2 lies on the first branch of F then T contains all branching components of F, so $\widehat{E}$ is a chain, a contradiction. If D_2 lies on the second branch then $\widehat{E}$ is an admissible fork of type $(2, 2, n)$, a contradiction. □

(3) Both (-1)-curves of F intersect $\widehat{E}$.

Proof. Let C_1 and C_2 be the (-1)-curves of F. They are S_0 components by (1). We get $\Sigma_{S_0} > 0$, so D_h consists of two 1-sections, which can intersect F only in components of multiplicity one. Suppose one of C_i's, say C_2, does not intersect $\widehat{E}$. Then $D_v \neq 0$, because C_2 has to intersect some component of F. We make a connected sequence of blowdowns starting from C_2 until there is only one (-1)-curve left, we denote the image of F by F'. In this process we do not touch $C_1 + \widehat{E}$, so we can write $\underline{F}' - C_1 = D' + \widehat{E}$, where D' is the image of $D_v + C_2$. Since D' intersects the image of D_h, it contains a component of multiplicity one. It follows that $\widehat{E}$ is a chain, a contradiction. □

(4) There are no D-components in F.

Proof. We can write $\underline{F} - C_1 - C_2 = \widehat{E} + D' + D''$, where $D_v = D' + D''$, D' and D'' are connected and $D' \cap D'' = \emptyset$. Suppose $D_v \neq 0$, say $D' \neq 0$. One of C_i's, say C_1, intersects D'. Contract C_2 and subsequent (-1)-curves until the number of (-1)-curves decreases. Clearly, $C_1 + D'$ is not touched in this process. Denote the image of F by F' and let U be the image of $D'' + C_2 + \widehat{E}$. Now F' is a fiber with a unique (-1)-curve and since both $C_2 + D''$ and $C_1 + D'$ intersect D_h, we infer that both U and $D' + C_1$ contain components of multiplicity one. Thus F' is a chain. Consider the reverse sequence of blowups recovering F from F'. The fiber F is not a chain, so a branching curve is produced. It follows that $D'' \neq 0$, otherwise C_2 is a tip of F with multiplicity greater than one, hence $DC = D_hC = 0$, which is impossible. Now it is easy to see that one of the connected components of $\underline{F} - C_2$ is a chain not containing curves of multiplicity one, a contradiction. □

$D_v = 0$ implies that D_h intersects both C_i's, so they have multiplicity one, hence are tips of F. It follows that F is a chain, a contradiction.

□

Chapter 3

S_0 not $\mathbb{C}^*$-ruled, $\overline{\kappa}(S_0) = 0$

In this chapter we assume that $\overline{\kappa}(S_0) = 0$, hence $\overline{\kappa}(S') \leq 0$. We assume that $\widehat{E}$ is snc-minimal and that S_0 does not admit any $\mathbb{C}^*$-ruling. We prove that there are exactly two surfaces S' satisfying these conditions, for this surfaces $\overline{\kappa}(S') = 0$ (cf. 3.2.7).

3.1 Description of the boundary

Lemma 3.1.1. *The divisor D is rational.*

Proof. Suppose D is not rational. Then $\widehat{E}$ is not rational by 2.1.3(vi). Let $(\overline{S}, D + \widehat{E}) \to (\widetilde{S}, \widetilde{D} + \widetilde{E})$ be a modification of $(\overline{S}, D + \widehat{E})$, such that $(\widetilde{S}, \widetilde{D} + \widetilde{E})$ is almost minimal. By 1.6.11(iii) $\widetilde{D}$ and $\widetilde{E}$ are disjoint smooth elliptic curves. By 2.2.3(iv) we can assume that $\overline{S}$ is $\mathbb{P}^1$-ruled over a smooth elliptic curve, so Lüroth theorem implies that every rational curve in $\overline{S}$ is vertical. In particular, (-1)-curves contracted in the process of minimalization are vertical, hence the number of horizontal components of $D+\widehat{E}$ and $\widetilde{D}+\widetilde{E}$ is the same. Thus by 1.6.8(i) and 1.6.9(ii) for a generic fiber F we get $-2+F(D+\widehat{E}) = FK_{\widetilde{S}}+F\widetilde{D}+F\widetilde{E} = F\,\mathrm{Bk}(\widetilde{D}+\widetilde{E}) = 0$, because all components contained in $\mathrm{Supp}\,\mathrm{Bk}(\widetilde{D}+\widetilde{E})$ are rational, hence vertical. We get $F(D + \widehat{E}) = 2$, so S_0 is $\mathbb{C}^*$-ruled, a contradiction. □

From now on we assume that D is rational. In particular, $\overline{S}$ and $\widehat{E}$ are rational by 2.2.3(iv).

Lemma 3.1.2. *Every irreducible curve L, such that $L \not\subseteq D \cup \widehat{E}$ satisfies $\overline{\kappa}(S_0 - L) = 2$.*

Proof. Suppose $\overline{\kappa}(S_0 - L) = 1$. Since S_0 does not contain complete curves, 1.6.11(iv) implies that $S_0 - L$ is $\mathbb{C}^*$-ruled. S_0 is not $\mathbb{C}^*$-ruled, so it is affine-ruled, a contradiction with $\overline{\kappa}(S_0) = 0$. Suppose $\overline{\kappa}(S_0 - L) = 0$. Since $\overline{S}$ is rational, we have $Pic(S_0) \otimes \mathbb{Q} \cong NS_{\mathbb{Q}}(S_0) = 0$ by 2.2.3(i), so there exists a rational function f such that $(f) = kL$ for some $k > 0$. We get a morphism $f : S_0 - L \to \mathbb{C}^*$. If $S_0 - L \to B \to \mathbb{C}^*$ is its Stein factorization then $\overline{\kappa}(B) \geq \overline{\kappa}(\mathbb{C}^*) = 0$ and $0 \geq \overline{\kappa}(f^{-1}(b)) + \overline{\kappa}(B)$ for a generic $b \in B$ by 1.6.10. Since $S_0 - L$ is not affine ruled, we get $\overline{\kappa}(f^{-1}(b)) = 0$, i.e. f is a $\mathbb{C}^*$-ruling, a contradiction. □

Definition 3.1.3. Let (X, B) be an snc-pair. A smooth curve C on X is a *simple curve on* (X, B) if it is rational and for any J, a connected component of B, satisfies $|C \cap J| \leq 1$. If $C^2 = -1$ then we say that it is *exceptional*.

Corollary 3.1.4. *There is no simple curve on $(\overline{S}, D + \widehat{E})$. If D is snc-minimal then the pair $(\overline{S}, D + \widehat{E})$ is almost minimal.*

Proof. Let L be a simple curve on $(\overline{S}, D + \widehat{E})$. By 2.2.3(iii) S' is affine, so $L \cap D \neq \emptyset$. By 1.6.14 the almost minimal model X_m of $S_0 - L$ is an open subset of $S_0 - L$ satisfying $\chi(X_m) \leq \chi(S_0 - L)$. By 1.6.13(2) and 3.1.2 it satisfies $0 < \chi(X_m) + \sum_{P\in Q} \frac{1}{|G_P|}$, where Q is the set of singular points of the relatively minimal model X_r of $S_0 - L$. Put $s = |L \cap \widehat{E}|$. Observe that $Q(D)$ is not negative definite, so by the construction of X_r we have $|Q| \leq q - s$. This gives $\sum_{P\in Q} \frac{1}{|G_P|} \leq \frac{q-s}{2}$, so $\chi(S_0 - L) \geq \chi(X_m) > -\sum_{P\in Q} \frac{1}{|G_P|} \geq \frac{s-q}{2}$. We compute $\chi(S_0 - L) = \chi(S_0) - \chi(L) + |L \cap D| + s = 1 - q + s - 2 + |L \cap D|$, hence $|L \cap D| = \chi(S_0 - L) + 1 + q - s > \frac{q-s}{2} + 1$, i.e. $|L \cap D| > 1$, a contradiction.

If the pair $(\overline{S}, D+\widehat{E})$ is not almost minimal then by 1.6.2 there exists an exceptional simple curve on $(\overline{S}, D+\widehat{E})$, a contradiction. □

Let $T_1,\dots,T_n$ be the maximal twigs of D. The following technical lemma, which is a small generalization of [Kor93, 6.2] allows to bound from below the self-intersection of one of the branching components of D having four maximal twigs.

Lemma 3.1.5. *Let T be an snc-divisor with two branching components B_1, B_s with branching numbers $\beta_T(B_1)=\beta_T(B_s)=3$. Let T_1, T_2 and T_3, T_4 be maximal twigs of T intersecting B_1 and B_s respectively. Write $T-T_1-T_2-T_3-T_4=B_1+B_2+\ldots+B_s$. Assume that $T-B_1-B_2$ is contractible, T is not negative definite and $d(T)\neq 0$. Then $\widetilde{e}(T_1)+\widetilde{e}(T_2)>-B_1^2-1$ or $\widetilde{e}(T_3)+\widetilde{e}(T_4)>-B_s^2-1$.*

Proof. Put $b_i=-B_i^2$. Assume that $\widetilde{e}(T_3)+\widetilde{e}(T_4)\leq b_s-1$. Define $D^{(i)}=T_3+T_4+B_s+B_{s-1}+\ldots+B_i$. Put $d_i=d(D^{(i)})$ and $\Delta_i=d_{i+1}-d_i$. By 1.1.1(ii) applied to $D^{(i)}$ for $i=2,\dots,s-2$ we can write $d_i=d_{i+1}b_i-d_{i+2}$ (we put $d_{s+2}=0$), so $\Delta_{i+1}=d_{i+1}(b_i-2)+\Delta_i$. Put $T_0=B_{s-1}+\ldots+B_2$. Since $\widetilde{e}(T_0)<1$ we have $d_2=d(T_3)d(T_4)d(T_0)(b_s-\widetilde{e}(T_3)-\widetilde{e}(T_4)-\widetilde{e}(T_0))>0$, so $D^{(2)}$ is negative definite. In particular $d_i>0$ for $i=2,\dots,s$. Hence $\Delta_2\leq\Delta_3\leq\ldots\leq\Delta_{s-1}$. Since T is not negative definite, we have $d(T)<0$ by Sylvester's criterion. Applying 1.1.1(ii) for D we get $0>d(D)=d_2d(T_1+B_1+T_2)-d_3d(T_1+T_2)$, so $\Delta_2d(T_1+T_2)>d_2(d(T_1+B_1+T_2)-d(T_1+T_2))=d_2d(T_1)d(T_2)(b_1-1-\widetilde{e}(T_1)-\widetilde{e}(T_3))$. Suppose $b_1-1\geq\widetilde{e}(T_1)-\widetilde{e}(T_2)$. Then $\Delta_{s-1}\geq\Delta_2>0$. By 1.1.1(i) applied to $T_3+B_s+T_4$ we get $(b_{s-1}-1)(b_s-e(T_3)-e(T_4))<1$, hence $b_s-1<\widetilde{e}(T_3)+\widetilde{e}(T_4)$, a contradiction. □

From now on up to the end of this chapter we will assume that D is snc-minimal. Recall that the connected components of $D+\widehat{E}$ are described in 1.6.11(iii). We use the notation of 1.6.11(iii) below.

Lemma 3.1.6. *D can be only of type (X) or (Y). If it is of type (X) then its branching component B is either a 0-curve or a (-1)-curve. In case (Y) it is a (-1)-curve and the triple $(d(T_1),d(T_2),d(T_3))$ is up to permutation one of the following: $(3,3,3)$, $(2,3,6)$, $(2,4,4)$.*

Proof. By 2.2.3(ii) D is not negative definite, so the case (I) is impossible. Case (O) is excluded by 2.2.3(vi). In case (H) write $D-T_1-T_2-T_3-T_4=B_1+\ldots+B_s$. The chain $B_2+\ldots+B_{s-1}$ is admissible, otherwise after some modifications gives a $\mathbb{C}^*$-ruling of S_0 (cf. 5.1.2(4)). By 3.1.5 we can assume that $B_1^2>-2$. Assume T_1 and T_2 meet B_1. Blow up on the intersection of B_1 with $D-T_1-T_2-B_1$ until $B_1^2=-1$. Then $T_1+2B_1+T_2$ gives a $\mathbb{C}^*$-ruling of S_0, a contradiction. Thus only (X) and (Y) remain.

We have $d(D)<0$ by 2.2.3(ii), so by 1.1.1(i) $B^2\geq-1$. In case (Y) we have $\delta(D)=1$ by definition, so we need only to prove that $B^2=-1$. Suppose $B^2>0$ in case (X) or $B^2\geq 0$ in case (Y). Let $\pi:(\widetilde{S},\widetilde{D})\to(\overline{S},D)$ be the modification obtained by blowing up the point of intersection of T_1 with B until $B^2=0$. Consider the $\mathbb{P}^1$-ruling of $\widetilde{S}$ given by B. We see that $\widetilde{D}$ contains no vertical (-1)-curves. Let D_h be the divisor of horizontal components of $\widetilde{D}$ (these are disjoint sections of the ruling). Put $D_v=\widetilde{D}-D_h-B$. Notice that if some component H of D_h intersects a vertical curve T then $\mu(T)=1$ and H does not intersect any other component lying in the fiber containing T.

We prove that S_0-components of singular fibers are (-1)-curves. Let C be an S_0-component of some fiber. We have $K_{\overline{S}}+D+\widehat{E}\equiv\operatorname{Bk}D+\operatorname{Bk}\widehat{E}$, so $K_{\widetilde{S}}+\widetilde{D}+\widehat{E}\equiv\pi^*\operatorname{Bk}D+\operatorname{Bk}\widehat{E}$. We have $L^2=-2-LK_{\widetilde{S}}=-2+L(\widetilde{D}-\pi^*\operatorname{Bk}D)+L(\widehat{E}-\operatorname{Bk}\widehat{E})\geq-2+L(\widetilde{D}-\pi^*\operatorname{Bk}D)$. Since $B\not\subseteq\operatorname{Bk}D$ and since π is obtained by blowing up in the tips of the subsequent reduced full preimages of $D-T_1-T_2-B$, the components in $\pi^*\operatorname{Bk}D\subseteq\widetilde{D}$ have multiplicities smaller than one ($\operatorname{Bk}D\neq D$ because $Q(D)$ is not negative definite). Thus $L^2>-2$ and we are done.

Let F be a fiber containing some connected component of $\widehat{E}$. If F contains some $\widetilde{D}$-components, then there exists a chain of S_0-components in F connecting $\widehat{E}\cap F$ with some $\widetilde{D}$-component of F. In fact this chain consists of a unique (-1)-curve L, since all S_0-components are (-1)-curves and two of them cannot meet. By 3.1.4 $D_hL>0$, so $\mu(L)=1$, a contradiction. Therefore there are no $\widetilde{D}$-components in F, hence each S_0-component intersects D_h, so it has $\mu=1$. We have $\#D_h\leq 4$, so from 3.1.4 it follows that there are exactly two S_0-components, each intersecting two components of D_h. This eliminates the case (Y). Notice that it follows also that these two (-1)-curves are tips of F and $\widehat{E}\cap F$ is a (-2)-chain.

Consider the case (X). We have $D_v\neq 0$, because $B^2>0$. We can write $D_v=D_0+D_1+\ldots+D_n$, where $D_0^2=-3$, $n\geq 0$ and $D_i^2=-2$ for every $1\leq i\leq n$. Let F' be a fiber containing D_v. Connectedness of D_v implies that each (-1)-curve of F' intersects D_h. In particular, the (-1)-curves, and hence all components of F' have $\mu=1$. We have $KD_v=1$ and $KF=-2$, so there are exactly three (-1)-curves in F', call them L_2, L_3 and L_4. We have $\sigma(F') = 3$, $\sigma(F) = 2$ and $\Sigma_{S_0} = 3$ by 1.5.3, so any other singular fiber has $\sigma = 1$. However, the unique (-1)-curve of such a fiber has $\mu > 1$, so cannot intersect D_h, hence cannot intersect $\widetilde{D}$, a contradiction. Thus F and F' are the only singular fibers, which implies that $\widehat{E}$ is connected. Since $\mu(L_i) = 1$ and F' cannot contain a 0-curve as a proper subdivisor, we get that one of L_i's, say L_4, intersects D_n and two others intersect D_0 (it is possible that $n = 0$). Each L_i intersects exactly one T_i, so by renaming we can assume that for $i = 2, 3, 4$ we have $T_i^2 = -2$ and $L_iT_i = 1$. The remaining section contained in D_h, call it T_1', is a (-1)-curve and intersects D_n. Let M_2 be the (-1)-curve of F intersected by T_4. Denote the second (-1)-curve of F by M_1. If $T_1'M_2 > 0$ then the contraction of $F - M_2 + F' - L_4$ does not touch T_4 and touches T_1' once. Therefore the images of T_4 and T_1' are disjoint sections of a $\mathbb{P}^1$-ruling of a Hirzebruch surface and have self-intersections -2 and 0. This is impossible. We infer that $T_1'M_2 = 0$ and $T_1'M_1 = 1$. Now by symmetry we can assume that T_2 intersects M_2 and T_3 intersects M_1. The contraction of $F - M_1 + F' - L_3$ does not touch T_3 and touches T_1' exactly $n+1$ times. Thus as above we get a $\mathbb{P}^1$-ruling of a Hirzebruch surface with two disjoint sections having self-intersections -2 and n. It follows from the properties of a Hirzebruch surface that $n = 2$. Now observe that $T_4 + 2L_4 + D_2$ and $T_3 + 2L_3 + D_0 + L_2$ are disjoint 0-divisors, so they are fibers of the same $\mathbb{P}^1$-ruling of $\widetilde{S}$. This contradicts the fact that T_2 intersects the second one and not the first one.

□

3.2 Rulings of S_0 with $\nu > 0$

We need couple of remarks about rulings of special type on $\overline{S}$. All the characteristic numbers used below refer to the pair $(\overline{S}, D + \widehat{E})$.

Lemma 3.2.1. *With the assumptions as above, let $p : \overline{S} \to \mathbb{P}^1$ be a $\mathbb{P}^1$-ruling, such that $\nu > 0$ (cf. 1.5). Let F_∞ be a fiber contained in D and D_h be the divisor of horizontal components of D. One has:*

(i) $\widehat{E}$ is vertical, $\nu = 1$ and $\Sigma = \#D_h - 1$,

(ii) components of D_h are disjoint and each ot them intersects F_∞ in a point,

(iii) a component of a singular fiber is an S_0-component if and only if it is a (-1)-curve.

Proof. (i) $D \cap \widehat{E} = \emptyset$, so $\widehat{E}$ is vertical. By 2.1.4(iii) $\nu = 1$ and $\Sigma = h - 1$.

(ii) Since D does not contain any loops, this is obvious.

(iii) Since $(\overline{S}, D + \widehat{E})$ is almost minimal by 3.1.4, we have $K_{\overline{S}} + D + \widehat{E} \equiv \operatorname{Bk} D + \operatorname{Bk} \widehat{E}$, i.e. $K_{\overline{S}} + D^\# + \widehat{E}^\# \equiv 0$. D is connected and not negative definite, so $\operatorname{Supp} D = \operatorname{Supp} D^\#$. Hence for any S_0-component L we have $L^2 = -2 - K_{\overline{S}}L = -2 + LD^\# + L\widehat{E}^\# \geq -2 + LD^\# > -2$, so $L^2 = -1$, because L is contained in some singular fiber. On the other hand, a vertical (-1)-curve is an S_0-component, because $D + \widehat{E}$ is snc-minimal. □

Theorem 3.2.2. *Let S_0 be the smooth locus of a singular $\mathbb{Q}$-homology plane S', such that $\overline{\kappa}(S_0) = 0$ and S_0 is not $\mathbb{C}^*$-ruled. Then $\overline{\kappa}(S') = 0$ and S' has a unique singular point. Moreover, either (i) S' (hence S_0) is $\mathbb{C}^{**}$-ruled, its singularity is of type A_1 and its snc-minimal boundary D is a fork with branching (-1)-curve and three maximal twigs: $[2]$, $[2,2,2]$ and $[2,2,2]$ (cf. 3.2.4) or (ii) S' (hence S_0) is $\mathbb{C}^{***}$-ruled, its singularity is of type A_2 and its snc-minimal boundary D is a fork with branching (-1)-curve and three maximal twigs: $[2,2]$, $[2,2]$ and $[2,2]$. (cf. 3.2.6).*

Proof. Suppose S_0 is not $\mathbb{C}^*$-ruled. By 3.1.6 we have only 13 cases to consider:

(X0) $T_1 = T_2 = T_3 = T_4 = [2]$ and $B^2 = 0$,

(X1) $T_1 = T_2 = T_3 = T_4 = [2]$ and $B^2 = -1$,

D is of type (Y) with $B^2 = -1$ and:

(Y1a) $T_1 = [3], T_2 = [3], T_3 = [3]$,

(Y1b) $T_1 = [3], T_2 = [3], T_3 = [2,2]$,

(Y1c) $T_1 = [3], T_2 = [2,2], T_3 = [2,2]$, (Y1d) $T_1 = [2,2], T_2 = [2,2], T_3 = [2,2]$,

(Y2a) $T_1 = [2], T_2 = [4], T_3 = [4]$,

(Y2b) $T_1 = [2], T_2 = [4], T_3 = [2,2,2]$,

(Y2c) $T_1 = [2], T_2 = [2,2,2], T_3 = [2,2,2]$,

(Y3a) $T_1 = [2], T_2 = [3], T_3 = [6]$,

(Y3b) $T_1 = [2], T_2 = [3], T_3 = [2,2,2,2,2]$,

(Y3c) $T_1 = [2], T_2 = [2,2], T_3 = [6]$,

(Y3d) $T_1 = [2], T_2 = [2,2], T_3 = [2,2,2,2,2]$.

Write each T_i as $T_i = T_{i,1} + T_{i,2} + \ldots + T_{i,k_i}$, where $T_{i,1}$ is a tip of D. In cases (Y1a), (Y2a) and (Y3a) we compute $d(D) = 0$, so these are excluded by 2.2.3(ii). In each other case we specify a $\mathbb{P}^1$-ruling $\pi : \overline{S} \to \mathbb{P}^1$ with $\nu > 0$ defined by some 0-divisor (F_∞) in D. Below we list appropriate quadruples $(F_\infty, FD, \Sigma_{S_0}, D_v)$, where F is the generic fiber and $D_v = D - \underline{F}_\infty - D_h$. In fact in case (Y2c) we consider two rulings simultaneously.

(X0) $(B, 4, 3, 0)$,

(X1) $(T_1 + 2B + T_2, 4, 1, 0)$,

(Y1b) $(T_1 + 3B + 2T_{3,2} + T_{3,1}, 3, 0, 0)$,

(Y1c) $(T_1 + 3B + 2T_{3,2} + T_{3,1}, 3, 0, T_{2,1})$,

(Y1d) $(T_{1,2} + 2B + T_{3,2}, 4, 2, T_{2,1})$,

(Y2b) $(T_1 + 2B + T_{3,3}, 3, 1, T_{3,1})$,

(Y2c) $(T_1 + 2B + T_{3,3}, 3, 1, T_{3,1} + T_{2,1} + T_{2,2})$,

(Y2c)' $(T_{2,3} + 2B + T_{3,3}, 4, 2, T_{3,1} + T_{2,1})$,

(Y3b) $(T_1 + 2B + T_{3,5}, 3, 1, T_{3,1} + T_{3,2} + T_{3,3})$,

(Y3c) $(T_1 + 2B + T_{2,2}, 3, 1, 0)$,

(Y3d) $(T_1 + 2B + T_{3,5}, 3, 1, T_{2,1} + T_{3,1} + T_{3,2} + T_{3,3})$.

Notice that D_v has at most two connected components and each of them is a chain of (-2)-curves. Let F be some singular fiber of π. The S_0-components of F are (-1)-curves by 3.2.1(iii), denote them by L_i, $i = 1, \ldots, \sigma(F)$. We prove successive statements. We use 3.1.4 repeatedly.

(1) Every S_0-component intersects D_h.

Proof. If L is an S_0-component then $L^2 = -1$ by 3.2.1(iii). Suppose $LD_h = 0$. Then L intersects two D-components by 3.1.4, which are (-2)-curves, so $F = [2,1,2]$. Both these D-components must be tips of D. Since $LD_h = 0$ and $\nu > 0$, we obtain $FD = 2$, otherwise D would contain a loop. This is a contradiction. □

(2) If $\mu(L) > 1$ for some S_0-component L of F then $\sigma(F) = 1$ and $\mu(L) = 2$.

Proof. Suppose $\sigma(F) \geq 2$. The curve $L = L_1$ intersects some D-component of F, otherwise $D_h L_1 \geq 2$ and $D_h F \geq D_h(\mu(L_1)L_1 + L_2) > 4$, which is impossible. Thus $D_v \cap F \neq \emptyset$ and we get that $4 \geq D_h F \geq D_h(\mu(L_1)L_1 + D_v \cap F + \mu(L_2)L_2) \geq 2 + D_h(D_v \cap F) + \mu(L_2) D_h L_2$, so by (1) $\mu(L_2) = D_h L_2 = 1$ and $D_v \cap F$ is connected. We get $L_2 D_v > 0$, because L_2 cannot be simple. It follows that $F = [1,(k),1]$ for some $k > 0$, a contradiction.

Suppose $\sigma(F) = 1$ and $\mu(L_1) > 2$. Since $D_h L_1 > 0$, this is possible only for (Y1b) or (Y1c). Moreover, then $|D_h \cap L_1| = 1$ and the point of intersection does not belong to any other component of F. However, $FD = 3$ for (Y1b) and (Y1c), so there are no D-components in F. Thus L_1 is simple, a contradiction. □

(3) If $\sigma(F) > 1$ then $F = [1,(k),1]$ for some $k \geq 0$. If $\sigma(F) = 1$ then in cases other than (X1) $F = [2,1,2]$ and F contains a D-component.

Proof. If $\sigma(F) > 1$ then all L_i's are tips of F by (2). Suppose $\sigma(F) > 2$. Then there are some D-components in F, otherwise $FD \geq 6$ by 3.1.4. The divisor $F - \sum_i L_i$ is connected and contains a D-component, so there are no $\widehat{E}$-components in F. Since D_v consists of (-2)-curves, we get $-2 = K_{\overline{S}}F = \sum_i K_{\overline{S}}L_i = -\sigma(F)$, a contradiction. Thus $\sigma(F) = 2$ and both (-1)-curves have multiplicities one by (2), so $F = [1,(k),1]$ for some $k \geq 0$.

Assume now $\sigma(F) = 1$ and consider cases different from (X1). We have $\mu(L_1) = 2$ by (2). There are some D-components in F, otherwise by 3.1.4 L would meet two 2-sections contained in D_h, which is possible in case (X1) only. Suppose F is branched. Then L_1 is a tip of F and $\underline{F} - L_1$ is one of the connected components of D_v, hence it must be $[2,2,2]$, which is possible for (Y3b) only. In this case D_v is connected, $FD = 3$ and $\Sigma_{S_0} = 1$. In particular, there exists a fiber F' with $\sigma(F') = 2$ and it does not have any D-components, so both S_0-components of F' meet D_h at least twice, which contradicts $FD = 3$. Thus F is a chain, so $F = [2,1,2]$. □

(4) $\overline{\kappa}(S) = 0$ and $K_{\overline{S}} + D^{\#} \equiv 0$.

Proof. By (2), (3) and 1.5.1(vi) every singular fiber consists of (-1)- and (-2)-curves. $\widehat{E}$ is vertical, so 2.2.3(viii) implies $\overline{\kappa}(S) = \overline{\kappa}(S_0) = 0$. The pair $(\overline{S}, D + \widehat{E})$ is almost minimal, so by 1.6.8(i) and 1.6.9(ii) we get $K_{\overline{S}} + D^{\#} + \widehat{E}^{\#} \equiv 0$. Since by (2) and (3) $\widehat{E}$ consists of (-2)-chains and admissible (-2)-forks, $\widehat{E} = \operatorname{Bk}\widehat{E}$, so $\widehat{E}^{\#} = 0$. □

(5) Cases other than (X0), (X1), (Y1d) and (Y2c) are impossible. $\#\widehat{E} = 8 - B^2 - \#D$.

Proof. By (4) we have $K_{\overline{S}}\operatorname{Bk}D = K_{\overline{S}}^2 + K_{\overline{S}}D$, so $K_{\overline{S}}\operatorname{Bk}D \in \mathbb{Z}$. This excludes (Y1b), (Y1c), (Y2b), (Y3b) and (Y3c). In the remaining cases (X0), (X1), (Y1d), (Y2c) and (Y3d) the maximal twigs of D are (-2)-chains, so by (4) $K_{\overline{S}}(K_{\overline{S}} + B) = 0$. Noether's formula and 2.1.4(ii) give $12 = K_{\overline{S}}^2 + 2 + \#D + \#\widehat{E}$, so $\#\widehat{E} = 8 - B^2 - \#D$. For (Y3d) we get $\#\widehat{E} = 0$, a contradiction. □

(6) Case (X0) is impossible. $\widehat{E}$ is connected.

Proof. By (5) we have $\#\widehat{E} = 3 - B^2 \geq 3$ for (X1) and (X0), so by 2.2.1(ii) $\widehat{E}$ is connected. Consider the case (Y1d). Suppose there exists a singular fiber F with $\sigma(F) = 1$, let L be its (-1)-curve. By (3) $F = [2,1,2]$ and there is a D-component in F, so $D_v = T_{2,1} \subseteq F$ and F contains an $\widehat{E}$-component. It follows that the sections $T_{1,1}$ and $T_{3,1}$ intersect L, a contradiction. By (3) there are only two singular fibers and they are of type $[1,(k),1]$, so $\widehat{E}$ is connected, since $D_v \neq 0$.

Suppose that the case (X0) occurs. Since $\Sigma_{S_0} = 3$, there is a singular fiber F with $\sigma(F) > 1$, hence by (3) $F = [1,(k),1]$ for some $k \geq 0$. It is easy to see that for every such fiber $k > 0$. In fact, if $k = 0$ then take any (-1)-curve $L \subseteq F$ and two components $H_1, H_2 \subseteq D_h$ intersecting L. Since $D_v = 0$, $H_1 + 2L + H_2$ gives a $\mathbb{C}^*$-ruling of S_0, a contradiction. Since $\widehat{E}$ is connected, we see that there is only one fiber with $\sigma > 1$. This contradicts $\Sigma_{S_0} = 3$. □

(7) Case (X1) is impossible.

Proof. Suppose the case (X1) occurs. We have $\Sigma_{S_0} = 1$, so there is a fiber $F_1 = [1,(k),1]$, where $k \geq 0$. Suppose $k > 0$. We have $D_v = 0$, so $\widehat{E} \subseteq F_1$ by (6) and F_∞ and F_1 are the only singular fibers. By (5) we can write $F_1 = L_1 + E_1 + E_2 + E_3 + E_4 + L_2$. Notice that D_h consists of two 2-sections, T_3 and T_4, and by 3.1.4 D_h intersects $F_1 - \widehat{E}$ in four points. If L_1 intersects both 2-sections then the contraction of $\underline{F}_\infty - T_2 + F_1 - L_1$ touches T_3 seven times, so the image of T_3 is a smooth 2-section on a Hirzebruch surface with self-intersection 5, a contradiction. Thus L_1 intersects only one component of D_h, say T_3, hence L_2 intersects T_4. After the contraction of $\underline{F}_\infty - T_1 + F_1 - L_1$ the surface becomes a Hirzebruch surface and the images of the 2-sections, T_3' and T_4', satisfy $T_3'T_4' = 2$, $T_3'^2 = 0$ and $T_4'^2 = 20$. However, $T_3' - T_4' \equiv \alpha F$ for some $\alpha \in \mathbb{Z}$ and a generic fiber F, because T_3' and T_4' are 2-sections. Thus $(T_3' - T_4')^2 = 0$, which is a contradiction. Thus $k = 0$ and $\widehat{E} \subseteq F_0$, where F_0 is a singular fiber with $\sigma(F_0) = 1$. By (5) and 1.5.1(vi) $\widehat{E}$ is a (-2)-fork with four components. Let M be the (-1)-curve of F_0. Denote the $\widehat{E}$-component intersecting M by E_0 and the branching component of $\widehat{E}$ by E_1. Consider a new $\mathbb{P}^1$-ruling of $\overline{S}$ given by the 0-divisor $T_3 + 2M + T_4$. For this ruling we have $\Sigma_{S_0} = 0$. Let F' be a fiber containing $\widehat{E} - E_0$. There is exactly one (-1)-curve $U \subseteq F'$, which is the unique S_0-component of F'. Notice that T_1 and T_2 are now the only possible vertical D-components and they are (-2)-curves. It follows that U cannot intersect them, hence $\underline{F}'$ contains no D-components. Hence U intersects E_1 and $\mu(E_1) = \mu(U) = 2$. It follows that

E_0 intersects F' only in E_1 and B intersects U in one point. Thus U is a simple curve on $(\overline{S}, D + \widehat{E})$, a contradiction.

□

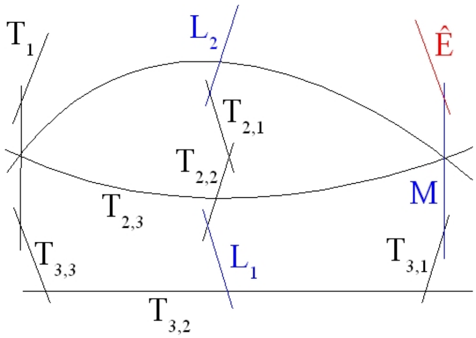


Figure 3.1: (Y2c), ruling

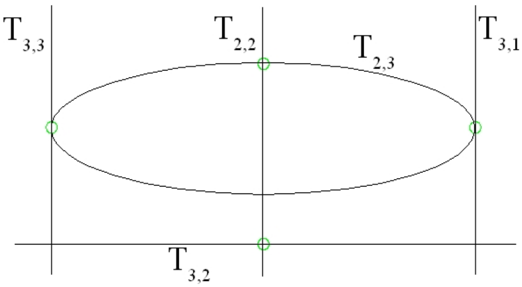


Figure 3.2: (Y2c), contraction

Lemma 3.2.3. *In the case (Y2c) there are three singular fibers (see Fig. 3.1): $F_\infty = T_1 + 2B + T_{3,3}$, $F_1 = L_1 + T_{2,2} + T_{2,1} + L_2$ and $F_0 = T_{3,1} + M + \widehat{E}$, where $\widehat{E} = [2]$ and L_1, L_2, M are (-1)-curves. $L_1 T_{3,2} = 1$ and the 2-section $T_{2,3}$ meets L_2. The divisor $D + L_1 + L_2 + M + \widehat{E}$ can be contracted to a sum of three lines and a smooth conic in $\mathbb{P}^2$, where the lines intersect in one point and exactly two of them are tangent to the conic (see Fig. 3.3).*

Proof. We use the facts showed in the proof of 3.2.2. We have $\Sigma_{S_0} = 1$, so by (3) there exists a fiber $F_1 = [1, (k), 1]$ for some $k \geq 0$ and this is the unique fiber with $\sigma > 1$. There exists also a singular fiber F_0 with $\sigma(F_0) = 1$. Indeed, otherwise F_1 would contain $\widehat{E}$, hence would not contain any D-component and this contradicts $D_v \neq 0$. We have $F_0 = [2, 1, 2]$ by (3). Since $\#D_v = 3$ and $\#\widehat{E} = 1$ by (5), F_1 contains two components of D_v and F_0 contains $\widehat{E}$ and one D-component. Besides F_∞, F_0 and F_1 there are no singular fibers. Notice that $T_{2,3}$ is a 2-section intersecting the unique (-1)-curve of F_0, call it M, in a branching point of $\pi_{|T_{2,3}}$. Let $L_1 \subset F_1$ be the (-1)-curve meeting $T_{2,2}$. Suppose L_1 meets the 2-section $T_{2,3}$ also. Then L_2, the second (-1)-curve of F_1, meets $T_{2,1}$ and $T_{3,2}$. Contraction of $F_\infty - T_{3,3} + F_1 - T_{2,2} + F_0 - T_{3,1}$ touches $T_{3,2}$ twice and $T_{2,3}$ five times. Therefore we get a ruling of a Hirzebruch surface having a section with self-intersection 0 and a disjoint 2-section with self-intersection 3. This is a contradiction, hence L_1 meets the section $T_{3,2}$. Contraction of $F_\infty - T_{3,3} + F_1 - T_{2,2} + F_0 - T_{3,1}$ touches $T_{3,2}$ once, so its image is a section of a $\mathbb{P}^1$-ruling of a Hirzebruch surface having self-intersection -1. Moreover, the image of $T_{2,3}$ is a smooth 2-section tangent to the images of $T_{3,3}$ and $T_{3,1}$ (see Fig. 3.2). Contracting the 1-section we get a divisor as in the thesis. □

We recover the situation of case (Y2c).

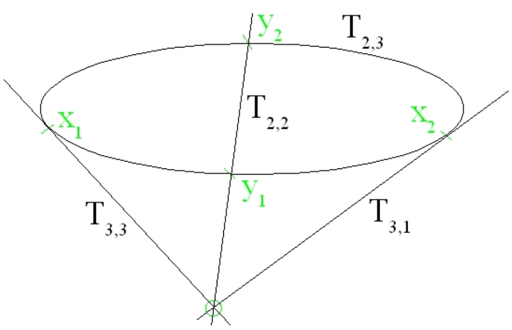


Figure 3.3: (Y2c), final configuration

Example 3.2.4. Let $x_1, x_2, y_1 \in T_{2,3}$ be three points lying on a smooth conic in $\mathbb{P}^2$. This choice is unique up to an automorphism of $\mathbb{P}^2$. (This can be seen as follows. Using an automorphism of $\mathbb{P}^2$ we can assume that these points are $([1:0:0],[0:1:0],[0:0:1])$, hence the conic is $\{[x:y:z] \in \mathbb{P}^2 : axy+byz+czx = 0\}$ for some $a, b, c \in \mathbb{C}$, such that $abc \neq 0$. Automorphisms of $\mathbb{P}^2$ which are diagonal in chosen coordinates fix the chosen points and act transitively on the set of described conics.) Let $T_{3,3}, T_{3,1}$ be two lines tangent to $T_{2,3}$ at x_1 and x_2 respectively. Let $T_{2,2}$ be a line through $T_{3,3} \cap T_{3,1}$ intersecting $T_{2,3}$ in y_1, denote the second point of intersection by y_2 ($y_2 \neq y_1$, because $T_{2,3}$ is non-degenerate). We use the same names for divisors and their birational transforms. Blow once in each of $T_{3,3} \cap T_{3,1}$, x_1, x_2, y_1 and denote the respective exceptional curves by $T_{3,2}$, T_1, $\widehat{E}$ and $T_{2,1}$. Now blow once in each of $T_{3,3} \cap T_1$, $T_{2,2} \cap T_{3,2}$, $T_{2,1} \cap T_{2,3}$ and $T_{3,1} \cap \widehat{E}$ and denote the respective exceptional curves by B, L_1, L_2 and M. Denote the resulting complete surface by $\overline{S}$. Define $D = T_{3,1} + T_{3,2} + T_{3,3} + T_{2,1} + T_{2,2} + T_{2,3} + T_1 + B$, $S = \overline{S} - D$ and $S' = S/\widehat{E}$. Clearly, D is a fork with $\delta(D) = 1$, $B^2 = -1$ and other components of D are (-2)-curves.

Lemma 3.2.5. *In the case (Y1d) there are three singular fibers (see Fig. 3.4): $F_\infty = T_{1,2}+2B+T_{3,2}$, $F_1 = L_1 + E_1 + E_2 + L_2$ and $F_2 = M + T_{2,1} + L_3$, where $\widehat{E} = E_1 + E_2 = [2,2]$ and L_1, L_2, L_3, M are (-1)-curves. $T_{3,1}M = T_{3,1}L_1 = 1$, $T_{1,1}L_2 = T_{1,1}M = 1$, $T_{2,2}L_1 = T_{2,2}L_2 = T_{2,2}L_3 = 1$ and $T_{2,2} \cap T_{2,1} \neq T_{2,2} \cap L_3$. There exists regular morphism $\theta : \overline{S} \to \mathbb{P}^2$ contracting the divisor $B + M + L_1 + L_2 + L_1' + L_2' + L_1'' + L_2''$ consisting of disjoint (-1)-curves, such that the image of $T_{1,2}+T_{2,2}+T_{3,2}$ is a triple of lines intersecting in $\theta(B)$ and the image of $T_{1,1} + T_{2,1} + T_{3,1}$ is a triple of lines intersecting in $\theta(M)$ (see Fig. 3.6). Moreover, $\theta(T_{1,2})\cap\theta(T_{2,1})$, $\theta(T_{2,2})\cap\theta(T_{3,1})$, $\theta(T_{3,2})\cap\theta(T_{1,1})$ lie on a line $\theta(E_1)$ and $\theta(T_{1,2})\cap\theta(T_{3,1})$, $\theta(T_{2,2})\cap\theta(T_{1,1})$, $\theta(T_{3,2}) \cap \theta(T_{2,1})$ lie on a line $\theta(E_2)$.*

Proof. We have $\Sigma_{S_0} = 2$, so by (3) there exist fibers $F_1 = [1,(k_1),1]$ and $F_2 = [1,(k_2),1]$ and since by (6) $\widehat{E}$ is connected, (3) implies that F_∞, F_1 and F_2 are the only singular fibers of π. We can assume that $T_{2,1}$ is contained in F_2, so $k_2 = 1$, $\widehat{E} \subseteq F_1$ and $k_1 = 2$ by (5). There is a (-1)-curve in F_2, call it M, such that $T_{2,2}M = 0$. By 3.1.4 $T_{1,1} + T_{3,1}$ intersects M, so by symmetry we can assume that $T_{3,1}$ does. Let L_1 be the (-1)-curve of F_1 intersecting $T_{3,1}$. The contraction of $F_\infty - T_{3,2} + F_1 - L_1 + F_2 - M$ does not touch $T_{3,1}$ and the images of $T_{3,1}$ and $T_{1,1}$ are two disjoint sections on a Hirzebruch surface, hence the image of $T_{1,1}$ must have self-intersection 2 and we infer that the contraction touches $T_{1,1}$ exactly four times. Since $k_2 = 2$, it follows that $T_{1,1}$ does not intersect L_1 and intersects M (see Fig. 3.4). Clearly, the analogous rulings of $\overline{S}$ induced by $F_\infty' = T_{1,2} + 2B + T_{2,2}$ or $F_\infty'' = T_{2,2} + 2B + T_{3,2}$ have the same structure of singular fibers. Denote the (-1)-curves of the fibers of these rulings containing $\widehat{E}$ as L_1', L_2' and L_1'', L_2'' respectively. It is easy to see that $L_1, L_1', L_1'', L_2, L_2', L_2''$ are disjoint. For example, for $i = 1, 2$ we have $L_i F_\infty' = 1$, so $L_i(L_1' + L_2') = 0$. Let $\omega : \overline{S} \to \widetilde{S}$ be the contraction of all these exceptional curves. For any $i, j, k \in \{1,2\}$ we have $\omega(T_{i,2})\omega(T_{j,1}) = 1$, $\omega(T_{i,j})^2 = 0$ and $\omega(E_k)^2 = 1$. We see also that $\omega(E_k)$ meets each $T_{i,j}$ once and only in points being images of curves contracted by ω (see Fig. 3.5). Now since $b_2(\widetilde{S}) = b_2(\overline{S}) - 6 = 3$, the $\mathbb{P}^1$-ruling $\widetilde{p} : \widetilde{S} \to \mathbb{P}^1$ induced by $\omega(T_{1,2})$ has only one singular fiber $\widetilde{F}$. Furthermore, $FM = F'M = F''M = 0$ implies that $\widetilde{F} = M + N$, where N is the birational transform of some S_0-component (see Fig. 3.5). We have $\omega(T_{i,j})N = 0$ and $BN = 1$. If we define θ as the composition of ω with the contraction of $B + M$, then the properties of θ stated in the thesis follow (see Fig. 3.6). □

We recover the situation of case (Y1d).

Example 3.2.6. Let $P_1 = [0,1,1], P_2 = [1,1,0], Q_1 = [1,0,0], Q_2 = [0,0,1]$ be points in $\mathbb{P}^2_{(x,y,z)}$. The lines $\overline{Q_1P_1}$, $\overline{Q_1P_2}$, $\overline{Q_2P_1}$ and $\overline{Q_2P_2}$ have equations $y = z$, $z = 0$, $x = 0$ and $x = y$. Put $P_3 = [1, \epsilon, \epsilon - 1]$, where $\epsilon = -\zeta_3$ for some primitive third root of unity ζ_3. Then following condition is satisfied: the points $\overline{Q_1P_1} \cap \overline{Q_2P_2} = \{[1,1,1]\}$, $\overline{Q_1P_2} \cap \overline{Q_2P_3} = \{[\epsilon, \epsilon - 1, 0]\}$, $\overline{Q_1P_3} \cap \overline{Q_2P_1} = \{[0,1,\epsilon]\}$ lie on a common line E_2 (having equation $(1-\epsilon)x + \epsilon y = z$) and the points $\overline{Q_1P_1} \cap \overline{Q_2P_3} = \{[1,\epsilon,\epsilon]\}$, $\overline{Q_1P_2} \cap \overline{Q_2P_1} = \{[0,1,0]\}$, $\overline{Q_1P_3} \cap \overline{Q_2P_2} = \{[\epsilon,\epsilon,\epsilon-1]\}$ lie on a common line E_1 (having equation $z = \epsilon x$). Blow once in Q_1 and Q_2 and denote the exceptional curve of the first blowup by B. Blow once in each of the six points of intersection of lines $\overline{Q_i, P_j}$ with $E_1 + E_2$. Let D be the divisor consisting of the proper transforms of B and of lines $\overline{Q_iP_j}$. Denote the resulting surface by $\overline{S}$ and put $S = \overline{S} \setminus D$, $S' = S/\widehat{E}$, where $\widehat{E} = E_1 + E_2$. Clearly, D is a fork with $\delta(D) = 1$, $B^2 = -1$ and $D - B + \widehat{E}$ consists of (-2)-curves.

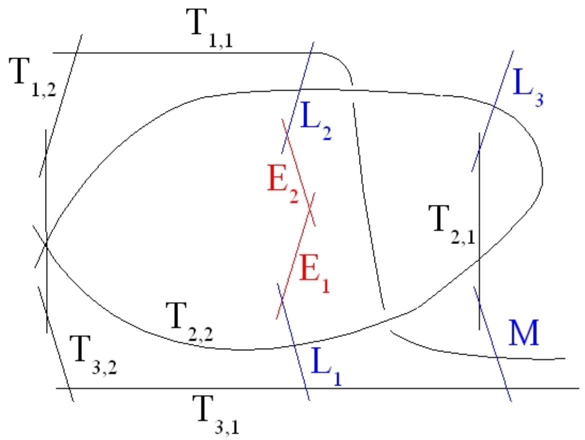


Figure 3.4: (Y1d), ruling

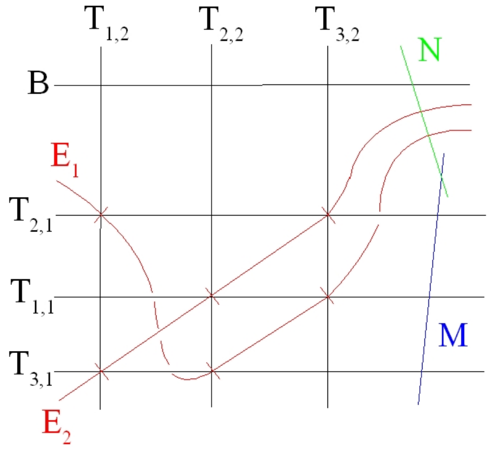


Figure 3.5: (Y1d), contraction

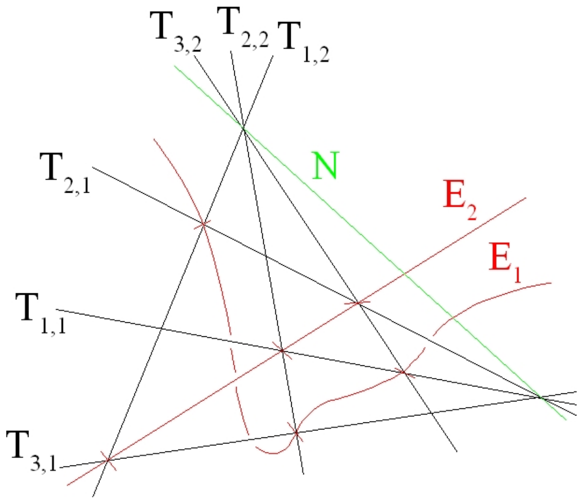


Figure 3.6: (Y1d), final configuration

Corollary 3.2.7. *There are exactly two non-isomorphic singular $\mathbb{Q}$-homology planes S', such that their smooth parts have Kodaira dimension zero and do not admit $\mathbb{C}^*$-rulings. These surfaces have Kodaira dimension zero. Their construction is given in 3.2.4 and 3.2.6.*

Proof. It follows from 3.2.2 that S' as above can be only of type (Y2c) or (Y1d). If it is of type (Y2c) then 3.2.3 implies that it can be constructed as in 3.2.4. The construction was determined uniquely by a choice of a smooth conic in $\mathbb{P}^2$ and a triple of different points on it, hence S' with S_0 of type (Y2c) is unique up to isomorphism. Clearly, the surfaces S' with S_0 of type (Y2c) and of type (Y1d) are non-isomorphic, because their singularities are of different type. We now prove that if S' is of type (Y1d) then it can be constructed as in 3.2.6. Let $\theta : \overline{S} \to \mathbb{P}^2$ be as in 3.2.5, put $Q_1 = \theta(B)$, $Q_2 = \theta(M)$, $P_1 = \theta(T_{1,2} \cap T_{1,1})$ and $P_2 = \theta(T_{3,2} \cap T_{3,1})$, we can assume that their coordinates are as in 3.2.5. Since $P_3 = \theta(T_{2,2} \cap T_{2,1}) \notin \overline{P_1 Q_2}$, we can write $P_3 = [1, \epsilon, u]$ for some $\epsilon, u \in \mathbb{C}$. The condition of collinearity of $\theta(T_{1,2}) \cap \theta(T_{2,1}) = [1, \epsilon, \epsilon]$, $\theta(T_{2,2}) \cap \theta(T_{3,1}) = [\epsilon, \epsilon, u]$, $\theta(T_{3,2}) \cap \theta(T_{1,1}) = [0, 1, 0]$ implies $u = \epsilon^2$ and the condition of collinearity of $\theta(T_{1,2}) \cap \theta(T_{3,1}) = [1, 1, 1]$, $\theta(T_{2,2}) \cap \theta(T_{1,1}) = [0, \epsilon, u]$, $\theta(T_{3,2}) \cap \theta(T_{2,1}) = [1, \epsilon, 0]$ implies $\epsilon^2 - \epsilon + 1 = 0$, hence $-\epsilon$ is a primitive third root of unity. Therefore for fixed choice of points P_1, P_2, Q_1, Q_2 there are two choices for P_3, we call them P_3 and P_3'. This implies that up to isomorphism there are at most two surfaces S' of type (Y1d) as above. Notice that the change of P_1 with P_2 gives rise to the same set of points $\{P_3, P_3'\}$. Now the automorphism $\sigma \in \operatorname{Aut}\mathbb{P}^2$ given by

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

fixes Q_1, Q_2 and changes P_1 with P_2. Since σ changes P_3 with P_3' we conclude that the choices of P_3 and P_3' are equivalent.

We now check that constructions 3.2.6 and 3.2.4 result with singular $\mathbb{Q}$-homology planes with prescribed properties. In each case we have $b_1(\overline{S}) = 0$, $b_2(\overline{S}) = 9$ and it is easy to check that the components of $D + \widehat{E}$ are independent in $NS(\overline{S})$, hence $H_2(D + \widehat{E}) \to H_2(\overline{S})$ is a monomorphism. The homology exact sequence of a pair $(\overline{S}, D)$ and the Lefschetz duality give $b_1(S) = b_3(S) = b_4(S) = 0$ and $b_2(S) = \#\widehat{E}$. Then the homology exact sequence of a pair $(S, \widehat{E})$ gives that S' is $\mathbb{Q}$-acyclic. Since $\widehat{E}$'s are resolutions of singular points of type A_1 and A_2 respectively, the constructed S''s are normal. We get $\overline{\kappa}(S) = \overline{\kappa}(S_0)$ by 2.2.3(viii). We check easily that $K_{\overline{S}} + D^{\#}$ intersects trivially with all components of $D + \widehat{E}$, hence $K_{\overline{S}} + D^{\#} \equiv 0$. Thus $\overline{\kappa}(S) = 0$.

Suppose that for one of S' as above the smooth locus S_0 admits a $\mathbb{C}^*$-ruling. There exists a modification $(\widetilde{S}, \widetilde{D} + \widetilde{E}) \to (\overline{S}, D + \widehat{E})$, such that this ruling extends to a $\mathbb{P}^1$-ruling $\pi : \widetilde{S} \to \mathbb{P}^1$. We can assume that $\widetilde{D} + \widetilde{E}$ is π-minimal. Since $\overline{\kappa}(S') \neq -\infty$, there are no sections contained in $\widetilde{E}$, hence $\widetilde{E} = \widehat{E}$. Since D does not contain components with non-negative self-intersection, the same holds for $\widetilde{D}$. Suppose $h = 1$, let D_h be the horizontal section of $\widetilde{D}$. We have $\nu = 1$ by 2.1.4(iii), so there exists a fiber $F_\infty \subseteq \widetilde{D}$. Since $\widetilde{D}$ is simply connected, F_∞ can intersect D_h only in a branching point of $\pi_{|D_h}$, hence by π-minimality $F_\infty = [2, 1, 2]$. The contractions minimalizing $\widetilde{D}$ cannot touch F_∞, hence D contains two (-2)-tips as maximal twigs, a contradiction. Therefore $h = 2$ and we get $\Sigma_{S_0} = \nu \leq 1$ by 2.1.4(iii). Denote the horizontal components of $\widetilde{D}$ by D_0 and D_∞. If $\nu > 0$ then $D_0 + D_\infty$ intersects the fiber contained in $\widetilde{D}$ in two different points, hence the fiber is smooth by the π-minimality of $\widetilde{D}$, so $\widetilde{D}$ contains a 0-curve, a contradiction. Thus $\Sigma_{S_0} = \nu = 0$. Now $\overline{\kappa}(S_0) = 0$ implies that $F(K_{\overline{S}} + \widetilde{D} + \widehat{E})^- = F(K_{\widetilde{S}} + \widetilde{D} + \widehat{E}) = 0$, so D_0 and D_∞ cannot be contained in maximal twigs of $\widetilde{D}$ by 1.6.9)(i). There is a unique singular fiber F_0 containing a rivet, other fibers are chains intersected by D_0 and D_∞ in tips (i.e. they are *column fibers*, cf. 5.1.7 and 5.1.8(ii)). It follows that there are at least two such fibers, otherwise D_0 and D_∞ would be contained in maximal twigs of $\widetilde{D}$. Thus D_0 and D_∞ are branching in $\widetilde{D}$ and since (-1)-curves contained in $\widetilde{D}$ can appear only in F_0, after minimalization of $\widetilde{D}$ they images are branching in D, a contradiction. □

Remark 3.2.8. Using the description 3.2.5 it is easy to compute $\operatorname{Aut} S'$. Let η be an automorphism of a surface S' of type (Y1d). Since $D + \widehat{E}$ does not contain curves with non-negative self-intersection, $\eta_{|S_0}$ extends to $\overline{\eta} \in \operatorname{Aut}(\overline{S}, D + \widehat{E})$. We proved in 3.2.7 that one can assume that θ maps B, M to fixed points $Q_1, Q_2 \in \mathbb{P}^2$ and maps the set of nodes of maximal twigs of D to the fixed set of three points $\{P_1, P_2, P_3\} \subseteq \mathbb{P}^2$ (P_1, P_2 can be fixed arbitrarily and then up to automorphism of $\mathbb{P}^2$ fixing Q_1, Q_2 and $\{P_1, P_2\}$ there is only one choice for P_3). Notice that $\overline{\eta}$ fixes B and M and acts on $\{L_1, L_1', L_2, L_2', L_3, L_3'\}$, hence descends to $\widetilde{\eta} \in \operatorname{Aut}\mathbb{P}^2 = \theta(\overline{S})$ fixing Q_1, Q_2 and $\{P_1, P_2, P_3\}$. The automorphism of $\mathbb{P}^2$ is defined uniquely by specifying the images of four points in general position, so $\operatorname{Aut} S'$ is isomorphic with a subgroup of the group of permutations of three elements. However, σ defined in 3.2.7, which fixes Q_1, Q_2 and changes P_1 with P_2, does not fix P_3, hence $\operatorname{Aut} S' < \mathbb{Z}_3$. We conclude that $\operatorname{Aut} S' \cong \mathbb{Z}_3$, where the generator in the coordinates as before is given by

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & -\epsilon & 0 \\ 0 & -\epsilon & 1 \end{pmatrix},$$

where $\epsilon = -\zeta_3$ for some primitive third root of unity ζ_3.

Remark. Let S' be of type (Y2c). Notice that using the ruling given by $F_\infty = T_1 + 2B + T_{3,3}$ we found an exceptional S_0-component M intersecting $T_{3,1}$ and $T_{2,3}$. Similarly using the ruling given by $F_\infty' = T_1 + 2B + T_{2,3}$ we find an exceptional S_0-component M' intersecting $T_{2,1}$ and $T_{3,3}$. Now one can check that the ruling of type (Y2c)' given by $T_{2,3} + 2B + T_{3,3}$ has precisely five exceptional S_0-components and precisely three of them, say L_1, L_2 and L_3, are disjoint from M, M', $T_{2,1}$, $T_{3,1}$ and from each other. After contracting the divisor $B + M + M' + L_1 + L_2 + L_3$ the image of $\overline{S}$ has $b_2 = 3$ and $T_{2,1}$ and $T_{3,1}$ became exceptional. Contracting them we get a morphism $\theta : \overline{S} \to \mathbb{P}^2$, such that $\theta(\widehat{E})$ is a conic and $\theta(D)$ is a sum of five lines. Moreover, θ is $\operatorname{Aut} S'$-equivariant. Then one shows as above that $\operatorname{Aut} S' \cong \mathbb{Z}_2$.

Chapter 4

$\overline{\kappa}(S_0) = -\infty$

In this chapter we assume that $\overline{\kappa}(S_0) = -\infty$, which implies that $\overline{\kappa}(S') = -\infty$. This is the simplest case and it was analyzed before (assuming affiness and logarithmicity) by Miyanishi and Sugie ([MS91, 2.5-2.8]). For completeness we recover their results.

Remark. We warn that in [MS91] an unusual definition of the Kodaira dimension of a singular surface is used, i.e. it is identified with the Kodaira dimension of the smooth locus, not with the Kodaira dimension of the resolution.

By 1.6.11(i) there is an snc-completion $(\overline{S}, D)$ with a $\mathbb{P}^1$-ruling $p : \overline{S} \to B$ onto some smooth complete curve B with D being p-minimal. If S_0 is $\mathbb{C}^1$- or $\mathbb{C}^*$-ruled we assume that p extends this ruling and $\widehat{E}$ is p-minimal.

4.1 Affine-ruled S_0

Lemma 4.1.1. *If S_0 is affine-ruled then S' is rational and there exists exactly one fiber of p contained in D. Each other singular fiber has a unique (-1)-curve, which is an S_0-component. S' has only cyclic singularities.*

Proof. Clearly, the section D_h of p contained in $D + \widehat{E}$ is in fact contained in D, otherwise D is contained in some fiber and $Q(D)$ is negative definite. Hence $\widehat{E}$ is vertical, so it is a rational tree (not necessarily connected). Then $\overline{S}$, D and B are rational by 2.1.3(vi). We have $\Sigma_{S_0} = \nu - 1$ and $\nu \leq 1$ by 2.1.4(iii), hence $\Sigma_{S_0} = 0$ and there exists exactly one fiber F_∞ contained in D, which is smooth by p-minimality of D. Each singular fiber of p contains exactly one (-1)-curve. Indeed, if $D_0 \subseteq D$ is a (-1)-curve contained in some fiber then by p-minimality of D it intersects D_h and two D-components contained in a fiber. But then $\mu(D_0) > 1$, so for any fiber F we get $FD_h \geq \mu(D_0)D_0D_h > 1$, a contradiction. Thus a singular fiber F has exactly one (-1)-curve, say C, which is the unique S_0-component, hence $\mu(C) > 1$. There are exactly two components of multiplicity one in F and they are tips of F. The section D_h intersects one of them. If $\underline{F} - C$ is connected, then C is a tip of F and F is branched. If $\underline{F} - C$ is not connected, then its connected component not contained in D is just some connected component of $\widehat{E}$. Hence $\widehat{E}$ is a sum of admissible chains, so S' has only cyclic singularities. □

Remark. In fact, singularities of any normal surface containing a cylinder (a product of a curve with $\mathbb{C}^1$) are cyclic by [MS80].

To not to introduce additional symbols, for the needs of the construction and lemma below we cancel the assumptions made about S, S_0, etc.

Construction 4.1.2. Take $\widetilde{S} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ for some $n \in \mathbb{Z}$ and denote the section of the projection $\widetilde{p} : \widetilde{S} \to B$, where $B \cong \mathbb{P}^1$, corresponding to the inclusion of the second summand by D_h. Then $D_h^2 = -n$. Let F_∞ be a smooth fiber and D_0 some section disjoint from D_∞. Choose k distinct points $x_1, \ldots, x_k \in D_0 \setminus F_\infty$ and blow each of them once. For each i make a connected sequence of blowups subdivisional for the respective fiber. This produces fibers $F_1, \ldots, F_k$ with unique (-1)-curves $C_i \subseteq F_i$. Let D_i be the connected component of $\underline{F}_i - C_i$ intersecting D_h. By renumbering we can assume there is $m \leq k$, such that C_i is a tip of F_i if and only if $i > m$. Assume also that $m \geq 1$ (for $m = 0$ we would get a smooth surface). For $i \leq m$ put $\widehat{E}_i = \underline{F}_i - D_i - C_i$. Clearly, each $\widehat{E}_i$ is a chain. Let $\overline{S}$ be the preimage of $\widetilde{S}$ under all these blowups and $p : \overline{S} \to B$ be the induced $\mathbb{P}^1$-ruling. Put $D = \underline{F}_\infty + D_h + \sum_{i=1}^k D_i$, $S = \overline{S} - D$ and $\widehat{E} = \sum_{i=1}^m \widehat{E}_i$. Let $S \to S'$ be the morphism contracting $\widehat{E}_i$'s.

Lemma 4.1.3. *The surface S' constructed in 4.1.2 is a singular, normal $\mathbb{Q}$-homology plane of negative Kodaira dimension. Its smooth locus is affine-ruled. Conversely, each singular, normal $\mathbb{Q}$-homology plane with affine-ruled smooth locus (hence of negative Kodaira dimension) can be obtained by construction 4.1.2.*

Proof. The last part of the statement is a consequence of lemma 4.1.1. By definition $\widehat{E}_i$'s are admissible chains, so S' is normal and has only cyclic singularities. We have $\overline{\kappa}(S') = -\infty$, because S is affine ruled (cf. 1.6.10). We have $d(D) = -\prod_i d(D_i)$ by 1.1.1(i), so $d(D) \neq 0$. This shows that the natural map $\mathcal{L}(D + \widehat{E}) \otimes \mathbb{Q} \to NS(\overline{S})$ (cf. 1.1) is injective. Hence the homomorphism $H_2(D \cup \widehat{E}) \to H_2(\overline{S})$ induced by inclusion is injective. Since $b_2(\widetilde{S}) = 2$, we have $b_2(\overline{S}) = \#D + \#\widehat{E}$, so it is an isomorphism. It follows from 1.7.3 that S' is affine. Since $H_2(\widehat{E}) \to H_2(\overline{S})$ and $H_2(D) \to H_2(\overline{S})$ induced by inclusions are injective, from the exact sequence of the pair $(\overline{S}, D)$ we get $b_1(S) = b_3(S) = b_4(S) = 0$ and $b_2(S) = \#\widehat{E}$. Using the exact sequence of a pair $(S, \widehat{E})$ we conclude that $b_i(S') = 0$ for $i > 0$. □

4.2 Non affine-ruled S_0

Lemma 4.2.1. *If $\overline{\kappa}(S_0) = -\infty$ and S_0 is not affine-ruled then S_0 has a structure of a Platonic fibration. Moreover, $S' \cong \mathbb{C}^2//G$ for some small, noncyclic group $G < GL(2, \mathbb{C})$.*

Proof. We follow the arguments of [KR07, §3]. Assume that S_0 is not affine-ruled. The boundary divisor $D + \widehat{E}$ is not connected and by 2.2.3(ii) not negative definite, so 1.6.11(ii) implies that it contains a Platonically fibred open subset U, which is its almost minimal model. By 1.6.14 we have $\chi(U) \leq \chi(S_0)$. Furthermore, $S_0 - U$ is a disjoint sum of s curves isomorphic to $\mathbb{C}$ and s' curves isomorphic to $\mathbb{C}^*$ for some $s, s' \in \mathbb{N}$ (cf. 1.6.2). It follows that $0 = \chi(U) = \chi(S_0) - s = \chi(S') - q - s = 1 - q - s$, so $s = 0$ and $q = 1$. Then $s' \leq 1$, so we get $s' = 0$, otherwise S_0 is affine-ruled, a contradiction. Thus $S_0 = U$ and by 1.6.11(ii) $S' \cong \mathbb{C}^2/G$, where G is a small noncyclic subgroup of $GL(2, \mathbb{C})$. □

Remark. In the next chapter we give a general construction of S''s with $\mathbb{C}^*$-ruled smooth locus. In particular, we will reconstruct all S''s with Platonic fibration on S_0 (see 5.4.5).

Chapter 5

$\mathbb{C}^*$-rulings on S_0, $\overline{\kappa}(S') = -\infty$

5.1 Generalities on $\mathbb{C}^*$-rulings on S_0

In this chapter we assume that S_0 is $\mathbb{C}^*$-ruled, $\overline{\kappa}(S_0) \geq 0$ and $\overline{\kappa}(S') = -\infty$. By Iitaka's easy addition theorem 1.6.10(i) we have $\overline{\kappa}(S_0) \neq 2$. We describe such S''s and show how to construct them. To give a construction we describe singular fibers of the extension of the ruling to some completion $\overline{S}$ of S_0 (cf. 5.2.1, 5.3.3, 5.4.5).

Remark 5.1.1. We comment on the assumption $\overline{\kappa}(S') = -\infty$. Consider the problem of classification of singular $\mathbb{Q}$-homology planes S' with smooth locus S_0 of non-generic Kodaira dimension. By the results of chapter 4 we can assume that $\overline{\kappa}(S_0) \geq 0$. If $\overline{\kappa}(S_0) = 1$ then by the structure theorem 1.6.11(iv) S_0 is $\mathbb{C}^*$-ruled. By the results of chapter 3, excluding two well described exceptions (cf. 3.2.7), this is also the case if $\overline{\kappa}(S_0) = 0$ (cf. 3.2.2). Therefore without loss of generality we can assume that S_0 is $\mathbb{C}^*$-ruled. Let $p : \overline{S} \to B$ be an extension of this ruling. Since $\overline{\kappa}(S') \leq \overline{\kappa}(S_0)$, $\overline{\kappa}(S') \neq 2$. If $\overline{\kappa}(S') \geq 0$ then $\widehat{E}$ cannot be a section of p, hence $p_{|S}$ gives a $\mathbb{C}^*$-ruling of S'. Moreover, by 2.2.4 S' is logarithmic in this case. These are exactly the assumptions made in [MS91, 2.9 - 2.17], where the possible fibers of $\mathbb{C}^*$-rulings are described and a structure theorem (without construction) is given. Notice that in [MS91] $\overline{\kappa}(S')$ is defined as $\overline{\kappa}(S_0)$, and not as $\overline{\kappa}(S)$.

First we collect some well-known results about linear systems of divisors.

Proposition 5.1.2. *Let D be an effective divisor on a complete smooth surface X.*

(1) Assume $\kappa(K_X + D) = -\infty$. If D is snc and reduced or X is rational then for every divisor F and $n \gg 0$ one has $\kappa(F + n(K_X + D)) = -\infty$ ([Fuj79, 2.5], cf. [Miy01, 2.2.7]).

(2) If $h^0(D) \geq 2$ then the generic member of $|D|$ can be written as $R + A_1 + \ldots + A_n$, where R is the fixed part of $|D|$, A_i's are irreducible and $A_i^2 \geq 0$.

(3) If D is snc and C is a (-1)-curve, such that $CD \leq 1$, then $\kappa(K_X + D + C) = \kappa(K_X + D)$.

(4) If X is rational and D is a smooth 0-curve then there exists a $\mathbb{P}^1$-ruling of X, such that D is a fiber.

(5) If X is rational and $|K_X + D| = \emptyset$ then $p_a(D) := \frac{1}{2}D(K_X + D) + 1 \leq 0$. In particular, D is a rational snc-tree and if $D = D_1 + D_2$ for some reduced connected divisors D_1, D_2 then $D_1 D_2 \leq 1$ ([Rus80]).

(6) If X rational, not isomorphic to a Hirzebruch surface or $\mathbb{P}^2$ and D is a rational snc-divisor then there exist smooth rational curves A_i, such that $D \sim A_1 + \ldots + A_n$ and $A_i^2 < 0$ for every i ([KR99, 4.1]).

Proof. (2) We can assume that $|D|$ has no fixed components. Blow up in base points of $|D|$ until $|D'|$, where D' is the proper inverse image of D, is base-point free. It is enough to prove the statement for D', because blowdown can only increase the self-intersection numbers of curves. Bertini's theorem (cf. [Har77, III.10.9]) implies that the generic member of $|D'|$ is smooth. We can write $D' \sim A_1 + \ldots + A_n$ for disjoint A_i's, so $A_i^2 = 0$.

(3) Since $C(K_X + D + C) \leq -1$, nC is contained in the fixed part of $n(K_X + D + C)$, so $\dim|n(K_X + D + C)| \leq \dim|n(K_X + D)|$. The opposite is obvious.

(4) Since $D^2 = 0$ and X is rational, the Riemann-Roch theorem gives $h^0(D) = h^0(D)+h^0(K_X-D) \geq 2$. Thus $|D|$ contains a pencil $\{F_t : t \in \mathbb{P}^1\}$ containing D and for this reason it does not have fixed components. It follows from the equality $F_{t_1}F_{t_2} = 0$ that it does not have fixed points too. Thus F_t's are disjoint for different $t \in \mathbb{P}^1$. Let $F_0 = D$. If $D - F_\infty = (\varphi)$ then φ defines a morphism $\varphi : X \to \mathbb{P}^1$. Its generic fiber F is smooth by Bertini's theorem, hence it is isomorphic to $\mathbb{P}^1$.

(6) We can assume that D is a smooth rational curve. Using induction the proof reduces easily to the case $A^2 = 0$. Then by (4) A gives a $\mathbb{P}^1$-ruling and there exists a singular fiber of this ruling linearly equivalent to A.

□

Definition 5.1.3. Let V, W and $W + V$ be connected snc-trees on a smooth complete surface X, such that V and W have no common components. We say that W *contracts to* W' *using* V if and only if there exists a birational morphism $\alpha : X \to X'$, such that $\alpha_* W = W'$, W' is snc and all contractions in α take place inside $V + W$. If W contracts to W' using $V = 0$ then we simply say that W contracts to W'. If W contracts to 0 we will say also that it contracts to a point.

Example 5.1.4. Let F be a fiber of a $\mathbb{P}^1$-ruling of a smooth complete surface. Then F contracts to a smooth 0-curve (using 0). Assume that F has a unique (-1)-curve C. Then $\underline{F} - C = F_0 + F_1$ where F_0 and F_1 are disjoint connected rational snc-trees. If F is branched let's assume also that F_0 is the part containing curves with multiplicity $\mu = 1$. Then F_1 contracts to a point using $F_0 + C$.

Remark. It is clear, that 5.1.2(4) works also for D which contracts to a smooth 0-curve.

Lemma 5.1.5. *(Pac-man lemma). Let V, W be as in 5.1.3. Assume that W contracts to a point using V. Then $\kappa(K_X + V + W) = \kappa(K_X + V)$.*

Proof. Contraction of W to a point is obtained by a sequence of contractions of (-1)-curves that are non-branching in successive images of $V + W$, so $\kappa(K_X + V + W)$ does not change in this process. By 5.1.2(3) $\kappa(K_X + V)$ does not change too. □

Let $p : \overline{S} \to B$ be an extension of a $\mathbb{C}^*$-ruling of S_0. We assume that the boundary divisor $D + \widehat{E}$ is p-minimal (cf. 1.5). Let D_h and E_h be the divisors of horizontal components of D and $\widehat{E}$ respectively. We have $D_h \neq 0$, otherwise D would be contained in a fiber, which contradicts 2.2.3(ii).

Definition 5.1.6. After Fujita, we say that p is a *gyoza* if D_h is a 2-section, $E_h = 0$ in this case. If D_h+E_h consists of two 1-sections we say that p is a *sandwich.* (Gyoza and sandwich are called respectively *twisted* and *untwisted* $\mathbb{C}^*$-fibrations in [MS91].) This second kind of a $\mathbb{C}^*$-ruling can be of two types: type (I) when E_h and D_h are 1-sections and type (II) with two sections in D_h. In case of a gyoza and sandwich (II) $\widehat{E}$ is vertical, so it is snc-minimal.

Definition 5.1.7. A singular fiber F of p will be called a *column fiber* if and only if it is a chain $\underline{F} = A_n + \ldots + A_1 + C + B_1 + \ldots + B_m$ with a unique (-1)-curve C, such that $D_h + E_h$ intersects F exactly in A_n and B_m, in each once and transversally. A and B are called *adjoint chains.* Now $A = A_1 + \ldots + A_n$ and $B = B_1 + \ldots + B_m$ are admissible chains, so from 1.1.1(i) and the fact that $d(A)$ and $d'(A)$ are coprime we get easily that $e(A) + e(B) = 1$ and $d(A) = d(B) = \mu_F(C)$. In fact, we have also $\widetilde{e}(B) + \widetilde{e}(A) = 1$ (see [Fuj82, 3.7]). We will say that F has *weight* $w = \widetilde{e}(A)$ with respect to the the component of $D_h + E_h$ intersecting A. (It is therefore of weight $\widetilde{e}(B) = 1 - w$ with respect to the second component of $D_h + E_h$).

Remark. In our considerations the middle (-1)-curve will be always an S_0-component.

We state an easy lemma describing singular fibers with $\sigma \leq 1$:

Lemma 5.1.8. *([Fuj82, 7.5, 7.6]). Let F be a singular fiber of p. One has:*

(i) if $\sigma(F) = 0$ then $F = [2, 1, 2]$, p is a gyoza and $p(F)$ is a branching point of $p_{|D_h}$,

(ii) if $\sigma(F) = 1$ and F does not contain a rivet (cf. 1.5.2) then either F is a column fiber or p is a gyoza and $p(F)$ is a branching point of $p_{|D_h}$.

(iii) if $\sigma(F) = 1$ and F contains a rivet then D_h meets F in two different points.

5.2 Gyoza

Theorem 5.2.1. *If S' is a singular, normal $\mathbb{Q}$-homology plane of negative Kodaira dimension with $\mathbb{C}^*$-ruled smooth locus of non-negative Kodaira dimension then this ruling can be assumed to be of type sandwich (I) or (II).*

Proof. Assume that $p : \overline{S} \to B$ is a gyoza. Then $\widehat{E}$ is vertical, hence rational, so by 2.2.3 D is a rational tree and $\overline{S}$ and B are rational. By 2.1.3(v) and 2.1.4(iii) $\Sigma_{S_0} = \nu - 1$ and $\nu \leq 1$, hence $\Sigma_{S_0} = 0$ and $\nu = 1$. Let F_∞ be the unique fiber contained in D. Let $F_1, \ldots, F_n$ be all column fibers of p. Since no F_i can contain components of $\widehat{E}$, there is another singular fiber, call it F_0. We state and prove successive statements.

(1) $F_\infty = [2,1,2]$, F_0 is unique and contains $\widehat{E}$. D_h is not contained in any maximal twig of D.

Proof. By 5.1.8(i) $F_\infty = [2,1,2]$ and F_∞ contains a branching point of $p_{|D_h}$. By simply connectedness of D, F_0 does not contain a rivet, so by 5.1.8(ii) it contains a branching point of $p_{|D_h}$ too. Since $p_{|D_h}$ is a 2-covering, it has exactly two branching points by Hurwitz formula, so F_0 is unique, hence contains $\widehat{E}$.

Since $\mathcal{N} = (K_{\overline{S}} + D + \widehat{E})^-$ is effective and any fiber F of p satisfies $F\mathcal{N} = F(K_{\overline{S}} + D + \widehat{E}) - F(K_{\overline{S}} + D + \widehat{E})^+ = -F(K_{\overline{S}} + D + \widehat{E})^+ \leq 0$, we get $F\mathcal{N} = 0$, so $\mathcal{N}$ is vertical. By 1.6.9(i) it follows that D_h is not contained in any maximal twig of D. □

(2) $n = 0$.

Proof. Let's contract successively all (-1)-curves C in F_0 satisfying $CD \leq 1$ (if there are any). This includes non-branching (-1)-curves in D. At each step of the contraction process the image of D remains snc. Denote the images of D, F_0 and $\overline{S}$ by $\widetilde{D}$, $\widetilde{F}_0$ and $\widetilde{S}$ respectively. Let $\widetilde{p}$ be the ruling induced from p. We have $\kappa(K_{\widetilde{S}} + \widetilde{D}) = \kappa(K_{\overline{S}} + D)$ by 5.1.5. Let F be a smooth fiber of $\widetilde{p}$. By 5.1.2(1) $|F + k(K_{\widetilde{S}} + \widetilde{D})| = \emptyset$ for $k \gg 0$ and simultaneously $|F + K_{\widetilde{S}} + \widetilde{D}| \neq \emptyset$, because $F + \widetilde{D}$ contains a loop (cf. 5.1.2(5)). Let m be a maximal natural number, such that $|F + m(K_{\widetilde{S}} + \widetilde{D})| \neq \emptyset$. There exist curves A_l, $l = 1, \ldots, t$, such that $F + m(K_{\widetilde{S}} + \widetilde{D}) \sim A_1 + \ldots + A_t$. Since $|\sum_{i=1}^t A_i + K_{\widetilde{S}} + \widetilde{D}| = \emptyset$ by maximality of m, for each i we get $|A_i + K_{\widetilde{S}}| = \emptyset$ and $|A_i + K_{\widetilde{S}} + \widetilde{D}| = \emptyset$, so A_i are smooth rational curves and $A_i\widetilde{D} \leq 1$ by 5.1.2(5). Now by 5.1.2(6) we can assume $A_i^2 < 0$. We have $F(\sum_{i=1}^t A_i) = F(F + m(K_{\widetilde{S}} + \widetilde{D})) = 0$, which implies that every A_i is vertical. If $K_{\widetilde{S}}(K_{\widetilde{S}} + \widetilde{D}) \leq 0$ then $-2 = K_{\widetilde{S}}F \geq K_{\widetilde{S}}(F + m(K_{\widetilde{S}} + \widetilde{D})) = \sum_{i=1}^t KA_i$, so for some A_i, say A_1, $K_{\widetilde{S}}A_1 < 0$. Then A_1 would be a vertical (-1)-curve and $A_1\widetilde{D} \leq 1$, which is not possible for column fibers and F_∞. By the definition of $\widetilde{F}_0$ this is in fact a contradiction, so we get $K_{\widetilde{S}}(K_{\widetilde{S}} + \widetilde{D}) > 0$. Since $\widetilde{D}$ is a rational tree, Riemann-Roch's theorem for a divisor $-(K_{\widetilde{S}} + \widetilde{D})$ implies that $h^0(-(K_{\widetilde{S}} + \widetilde{D})) + h^0(2K_{\widetilde{S}} + \widetilde{D}) \geq K_{\widetilde{S}}(K_{\widetilde{S}} + \widetilde{D}) > 0$. This gives $-(K_{\widetilde{S}} + \widetilde{D}) \geq 0$. Suppose $n \neq 0$. Let C be a (-1)-curve of some column fiber. Then from $C(-(K_{\widetilde{S}} + \widetilde{D})) = -1$ we get $-(K_{\widetilde{S}} + \widetilde{D} + C) \geq 0$. Now by 5.1.2(5) $C\widetilde{D} = 2$ implies $|K_{\widetilde{S}} + \widetilde{D} + C| \neq \emptyset$, hence $K_{\widetilde{S}} + \widetilde{D} + C = 0$. From $D_h(K_{\widetilde{S}} + \widetilde{D} + C) = 0$ we obtain $\beta_{\widetilde{D}}(D_h) = 2$. However, a contribution of a column fiber and of F_∞ to $\beta_{\widetilde{D}}(D_h)$ is equal to two and one appropriately, a contradiction. □

(3) F_0 is a chain.

Proof. Let $T \subseteq F_0$ be a component intersecting D_h. Since F_0 contains a branching point of $p_{|D_h}$, T is unique and has $\mu(T) = 2$. Moreover, $T \subseteq D$, otherwise D_h would be a tip of D, contradicting (1). Let L be the unique S_0-component of F_0. We have $L^2 = -1$, otherwise by p-minimality T would be the unique (-1)-curve of F_0, hence by 1.5.1(vi) $\widehat{E}$ would consists of (-2)-curves in contradiction to 2.2.3(viii). Let ϕ be the composition of contractions of (-1)-curves in F_0 starting from L until T becomes the unique (-1)-curve. Denote the images of F_0 and T by F_0' and T'. If in the backward process ϕ^{-1} recovering F_0 from F_0' some sprouting blowup is made then $L + \widehat{E}$ contracts to a point using D, a contradiction with 5.1.5. Suppose F_0' is branched. Then by 1.5.1(vi) T' is a tip of F_0'. Since ϕ consists only of sprouting blowups, T is a tip of F_0 and $D \cap F_0$ is a chain, a contradiction with (1). Thus F_0 is a chain and $F_0' = [2,1,2]$. □

(4) $D_h^2 = 0$.

Proof. Let D_h' be the image of D_h after contraction of F_0 and F_∞ to smooth fibers. Since ϕ does not touch D_h, $D_h'^2=D_h^2+4$. Now the image of $\overline{S}$, call it $\widetilde{S}$, is a Hirzebruch surface, thus $K_{\widetilde{S}}^2=8$. Let F be a fiber of the induced $\mathbb{P}^1$-ruling of $\widetilde{S}$. Since $D_h'F=2$ and D_h' is smooth and rational, we check easily that $K_{\widetilde{S}}+D_h'\equiv -F$, so $K_{\widetilde{S}}D_h'=-6$ and we get $D_h^2=D_h'^2-4=0$. □

Now 5.1.2(4) gives another $\mathbb{P}^1$-ruling of $\overline{S}$ with D_h as a fiber. This is a $\mathbb{C}^*$-ruling (sandwich of type (II)) of S_0 with T and the (-1)-curve of F_∞ as sections.

□

5.3 Sandwich II

Assume $p\colon \overline{S}\to B$ is a sandwich of type (II) (cf. 5.1.6). $\widehat{E}$ is vertical, so by 2.2.3(iv) D is a rational tree and $\overline{S}$ and B are rational. Write $D_h=D_0+D_\infty$. By 2.1.3(v) and 2.1.4(iii) $\Sigma_{S_0}=\nu\leq 1$.

Lemma 5.3.1. *There is a unique smooth fiber F_∞ contained in D and there are two singular fibers (see Fig. 5.1), F_0 and F_1. F_1 is a column fiber, F_0 is a chain with two (-1) curves, none of which is a tip of F_0, and $\widehat{E}$ is a chain between them. The sections D_0 and D_∞ are disjoint and intersect F_0 in tips. At least one of them has negative self-intersection.*

Proof. Let $F_1,F_2,\dots,F_n$ be all column fibers of p. They do not contain components of $\widehat{E}$, so there exists another singular fiber F_0. We state and prove successive statements.

(1) F_0 is unique and contains $\widehat{E}$. D_0 and D_∞ are not contained in maximal twigs of D.

Proof. If $\Sigma_{S_0}=\nu=1$ then F_0 does not contain a rivet by simply connectedness of D, so $\sigma(F_0)=2$ by 5.1.8, hence F_0 is unique. If $\Sigma_{S_0}=\nu=0$ then F_0 contains a rivet by 5.1.8, hence is unique by simply connectedness of D.

Since $F\mathcal{N}=-F(K+D+\widehat{E})^+\leq 0$ and $\mathcal{N}$ is effective, we get $F\mathcal{N}=0$, so $\mathcal{N}$ is vertical. By 1.6.9(i) we get that D_0 and D_∞ cannot be contained in maximal twigs of D. □

(2) $\Sigma_{S_0}=\nu=1$. If $n>0$ then $n=1$, $\widetilde{F}_0$ (defined below) is smooth and does not contain a rivet.

Proof. Clearly, $\Sigma_{S_0}=\nu=n=0$ is impossible by (1), so we can assume $n>0$. Let ϕ be the composition of subsequent contractions of (-1)-curves C in F_0 satisfying $CD\leq 1$ (if there are any). Let $\widetilde{F}_0$, $\widetilde{D}$, $\widetilde{S}$ be the images of F_0, D and $\overline{S}$ and $\widetilde{p}$ the induced ruling of $\widetilde{S}$. Then $\widetilde{D}$ is $\widetilde{p}$-minimal and $\widetilde{F}_0$ contains a rivet if and only of F_0 does. By the definition of $\widetilde{F}_0$ we have $\kappa(K_{\widetilde{S}}+\widetilde{D})=\kappa(K_{\overline{S}}+D)$. Since $\overline{S}$ is rational, repeating word by word the arguments from 5.2.1(2) we get $-K_{\widetilde{S}}-\widetilde{D}>0$. Let C be a (-1)-curve of some column fiber. From $C(-(K_{\widetilde{S}}+\widetilde{D}))=-1$ we get $-(K_{\widetilde{S}}+\widetilde{D}+C)\geq 0$. Now $C\widetilde{D}=2$ implies $|K_{\widetilde{S}}+\widetilde{D}+C|\neq\emptyset$, hence $K_{\widetilde{S}}+\widetilde{D}+C=0$. We obtain $0=D_0(K_{\widetilde{S}}+\widetilde{D}+C)=D_0(\widetilde{D}-D_0)-2$, so $\beta_{\widetilde{D}}(D_0)=2$ (similarly $\beta_{\widetilde{D}}(D_\infty)=2$). We argue that $\widetilde{F}_0$ is smooth. If $\nu=0$ then $n\geq 2$ by (1), so $\widetilde{F}_0$ cannot contain any $\widetilde{D}$-components, hence $\widetilde{F}_0$ is smooth by 5.1.8. On the other hand, if $\nu=1$ then the assumption $n>0$ implies that again $\widetilde{F}_0$ contains no $\widetilde{D}$-components, hence by the definition of $\widetilde{F}_0$ every (-1)-curve contained in $\widetilde{F}_0$ intersects both sections contained in $\widetilde{D}$. Therefore, if F_0 is singular then it contains exactly one (-1)-curve L, so $\mu(L)>1$ and L intersects 1-sections of p, which is impossible. Now we need only to prove $\Sigma_{S_0}=\nu=1$.

Suppose $\Sigma_{S_0}=\nu=0$. Consider the image F_0' of F_0 before the last contraction of ϕ. Write $F_0'=U_1+U_1$, where U_1 is a birational transform of $\widetilde{F}_0$. Since F_0' contains a rivet, U_2 is an image of some D-component. Now instead of contracting U_2 we can contract U_1, which shows that $L+\widehat{E}$, where L is the unique S_0-component of F_0, contracts to a point using D, a contradiction. □

(3) $n=1$. F_0 is a chain with two (-1)-curves, they are S_0-components and are not tips of F_0.

Proof. Let T_0 and T_∞ be the components of F_0 meeting D_0 and D_∞ respectively. They are both D-components, otherwise D_0 or D_∞ would be contained in some maximal twig of D by (2). They have $\mu(T_0)=\mu(T_\infty)=1$ and since F_0 does not contain a rivet, they are contained in different connected components of $D\cap F_0$, W_0 and W_∞ respectively. By p-minimality of D it follows that (-1)-curves in F_0 are S_0-components. Now if there is only one (-1)-curve in F_0 then T_0 and T_∞ are tips of F_0 and at least one of W_0 or W_∞ is a chain, which contradicts (1). Denote the (-1)-curves of F_0 by C_0 and C_∞, they are not tips of F_0, because $\underline{F}_0-C_0-C_\infty$ has exactly three connected components. F_0 is simply connected, so one of the S_0-components, say C_0, satisfies $C_0 D = 1$, say $C_0 W_0 = 1$. Let's make a connected sequence of contractions of (-1)-curves in F_0 starting from C_0 until the number of (-1)-curves in the fiber decreases. This process is connected, so it cannot touch C_∞, because $W_\infty \neq 0$. Hence W_∞ is not touched and we get that all contracted curves have intersection with the proper image of D smaller than two, so the image of F_0, denote it by F_0', does not contain a rivet. It follows that F_0' is a column fiber, so T_∞ is a tip of W_∞, which implies that $n \neq 0$ by (1).

Suppose F_0 is not a chain. Consider the backward connected sequence of blowups recovering F_0 from F_0'. Let R be the last curve produced in this sequence, such that the respective preimage of F_0', call it F_0'', is not branched. Then $\mu(R) > 1$, so the points of intersection of birational transforms of D_0 and D_∞ with F_0'' do not belong to R. It follows that $\widehat{E}$ and R are contained in different connected components of $\underline{F}_0 - C_0$, so $R \subseteq D$ and $\widehat{E} + C_0$ contracts to a point using D, a contradiction. □

(4) T_0 and T_∞ are tips of F_0.

Proof. Since F_0 is a chain, the backward sequence of blowups as above begins on a tip of F_0'. Moreover, since C_0 and C_∞ are not tips of F_0, the set Λ of components of multiplicity one contained in F_0 has three connected components, two of them are tips of F_0. These are exactly T_0 and T_∞, otherwise $\widehat{E} + C_0$ would contract to a point using D. □

(5) At least one of D_0 or D_∞ has negative self-intersection.

Proof. Assume $D_0^2 \geq 0$. We can contract F_1 and F_0 to smooth fibers without touching D_0. Let D_∞' be the proper image of D_∞ after contractions. From (4) it follows that D_0 and D_∞' are disjoint. We get on the Hirzebruch surface $D_0 - D_\infty' \equiv F$, so $D_\infty^2 < D_\infty'^2 = -D_0^2 \leq 0$.

□

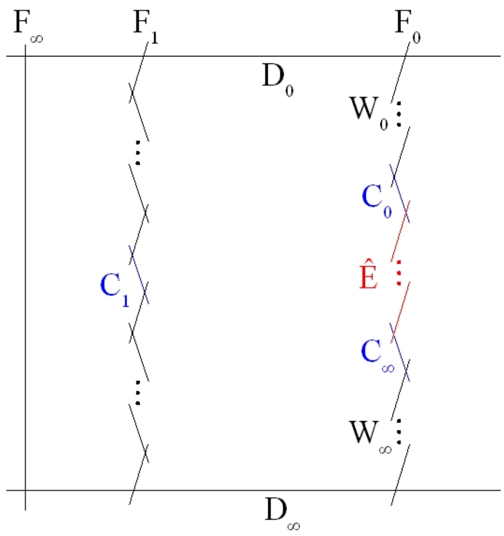


Figure 5.1: sandwich II

We will show now that there are no more restrictions. As we did in chapter 4, for the needs of the construction and lemma below we cancel the assumptions made about S, S_0, etc.

Construction 5.3.2. Pick $n \in \mathbb{N}$, $s \in \mathbb{N}_+$ and $w_1, w_0, w_\infty \in \mathbb{Q} \cap (0,1)$. Let $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$ be the Hirzebruch surface and $\widetilde{p} : \mathbb{F}_n \to \mathbb{P}^1$ the $\mathbb{P}^1$-ruling. Let D_0 and D_∞ be two sections corresponding to the first and the second summand of the bundle. We have $D_0^2 = n$ and $D_\infty^2 = -n$. Choose three different points on D_0 and denote the fibers of $\widetilde{p}$ containing them by F_∞, $\widetilde{F}_1$ and $\widetilde{F}_0$. By a connected sequence of blowups starting from $D \cap \widetilde{F}_1$ produce a column fiber F_1 which has weight w_1 with respect to the birational transform of D_0. Let C_1 be its (-1)-curve. Proceed analogously with $\widetilde{F}_0$ and produce a column fiber with weight w_∞ with respect to D_∞, denote its (-1)-curve by C_∞. Make a sequence of s sprouting blowups in the same fiber, each time on the point of intersection with the birational transform of D_0. Denote the produced curves subsequently by $U_1,\dots,U_s$. Finally, make a connected sequence of subdivisional blowups in F_0' starting from $U_{s-1}\cap U_s$, denote the resulting singular fiber by F_0 and its second (-1)-curve (i.e. different form C_∞) by C_0. By construction $\underline{F}_0-C_0-C_\infty$ has three connected components. Denote the one intersecting C_0 and C_∞ by $\widehat{E}$, and the one intersecting D_i by W_i ($i\in\{0,\infty\}$). We give a natural order to W_0 and W_∞, such that their components intersecting D_0 and D_∞ are the last ones. (Hence $\widetilde{e}(W_\infty)=w_\infty$). Since we can obtain any positive non-integral proper fraction as $\widetilde{e}(W_0)$, we can assume that $\widetilde{e}(W_0)=w_0$. We have $D_0^2=n-2-s$.

Let $\overline{S}$ be the obtained surface and $p:\overline{S}\to\mathbb{P}^1$ the induced ruling. Let $D=D_0+D_\infty+\underline{F}_\infty+W_0+W_\infty+(\underline{F}_1-C_1)$. Define $S=\overline{S}-D$, $S_0=S-\widehat{E}$ and $S'=S/\widehat{E}$ (as a topological space). We will show below that $NS_{\mathbb{Q}}(S_0)=0$, hence by 1.7.3 S' and the quotient morphism can be realized in the algebraic category.

The order given to W_0 and W_∞ agrees with their order as twigs of D (cf. 1.3). Let's fix a natural order on $\widehat{E}$, such that C_0 intersects the first curve of $\widehat{E}$. Define $\alpha=1-\frac{1}{\mu_1}-\frac{1}{min(\mu_0,\mu_\infty)}$, where $\mu_i=\mu(C_i)$ is the multiplicity of a respective curve in a fiber. Of course, α is determined by w_1,w_0,e_∞ and s and it can be computed easily in each particular case.

Theorem 5.3.3. *The surface S' constructed in 5.3.2 is a singular, normal $\mathbb{Q}$-homology plane of negative Kodaira dimension. $\overline{\kappa}(S_0)$ is 0 or 1 and it is determined by the sign of the number $\alpha=1-\frac{1}{\mu_1}-\frac{1}{min(\mu_0,\mu_\infty)}$ (i.e. $\overline{\kappa}(S_0)=\operatorname{sgn}\alpha$). Moreover, each singular, normal $\mathbb{Q}$-homology plane S' with $\overline{\kappa}(S')=-\infty$ and $\overline{\kappa}(S_0)\geq 0$, which has a $\mathbb{C}^*$-ruling of type sandwich (II) can be obtained by construction 5.3.2.*

Proof. As for the last part of the statement notice that we can contract both F_0 and F_1 to smooth fibers without touching the negative section. Then it is clear, that the construction 5.3.2 is forced by lemma 5.3.1.

By definition $\widehat{E}$ is contained in a fiber, so $Q(\widehat{E})$ is negative definite. To apply 1.7.3 and infer that S' is normal and affine one needs to prove that $NS_{\mathbb{Q}}(S_0)=0$. Since $b_2(\mathbb{F}_n)=2$, it follows from the construction that $b_2(\overline{S})=\#D+\#\widehat{E}$, so it is enough to show that the classes of irreducible components of $D+\widehat{E}$ are independent in $NS(\overline{S})$. If there exists a divisor $T=\sum_i d_iD_i+\sum_j e_jE_j$ for $D_i\subseteq D$ and $E_j\subseteq\widehat{E}$ which is numerically trivial, then $T=\sum_i d_iD_i$, otherwise $0=T\sum_j e_jE_j=(\sum_j e_jE_j)^2<0$ by negative definiteness of $Q(\widehat{E})$. Assume $d_i\neq 0$. None of the components of W_0+W_∞ can be a component of T. Indeed, for example let $U\subseteq W_0$ be the smallest (with the respect to the natural ordering of a maximal twig) component contained in T. If U is the first component of W_0 we get a contradiction by multiplying by C_0, otherwise by a component of W_0 smaller than U and intersecting U. Now multiplying T by the last components of W_0 and W_∞ we see that T is vertical. From the properties of the intersection matrix of a fiber it follows that $T^2=0$ implies $T=fF_\infty$ for some $f\in\mathbb{Q}$, because F_∞ is the only fiber contained in D. Intersecting with D_0 we get $T=0$. We infer that $H_2(D\cup\widehat{E})\to H_2(\overline{S})$ is a monomorphism, hence an isomorphism.

We check that S' is $\mathbb{Q}$-acyclic. We know that $H_2(D)\to H_2(\overline{S})$ and $H_2(\widehat{E})\to H_2(\overline{S})$ induced by inclusions are monomorphisms. We see that $b_1(D)=b_1(\widehat{E})=0$ and $b_3(\overline{S})=b_1(\overline{S})=0$, because $\overline{S}$ is rational. The exact sequence of a pair $(\overline{S},D)$ together with Lefschetz duality give $b_1(S)=b_3(S)=b_4(S)=0$ and $b_2(S)=\#\widehat{E}$. Then the exact sequence of a pair $(\widehat{E},S)$ gives $b_1(S')=b_2(S')=b_3(S')=b_4(S')=0$.

We analyze the Kodaira dimension. For $\xi\in\mathbb{Q}$ define a divisor $X_\xi=(\xi+1)\operatorname{Bk}W_0+\xi C_0+(\xi+1)\operatorname{Bk}\widehat{E}+(\rho+1)\operatorname{Bk}\widehat{E}^t+\rho C_\infty+(\rho+1)\operatorname{Bk}W_\infty$ (notice the ordering of $\widehat{E}$ defined above), where $\rho=(\xi+1)\frac{\mu_\infty}{\mu_0}-1$. For all irreducible components T of F_0 we have $T(K+D+\widehat{E})=TX_\xi=0$. This is clear for F_0 and follows from the definition of a bark for all non-exceptional curves in F_0, so we only check it for C_0: by 1.3.2(ii) we get $C_0X_\xi=(\xi+1)e(W_0)-\xi+(\xi+1)e(\widehat{E})+(\rho+1)\frac{1}{d(\widehat{E})}$. By 5.1.7 and 1.1.1(i) $\mu_\infty=d(W_\infty)=d(W_0+C_0+\widehat{E})=\mu_0d(\widehat{E})(1-e(W_0)-e(\widehat{E}))$, so $C_0X_\xi=1=L(K+D+\widehat{E})$. We now define the effective $\mathbb{Q}$-divisor X as $X=X_{\frac{\mu_0}{\mu_\infty}-1}$ if $\mu_0\geq\mu_\infty$ and as $X=X_0$ if not. In this way $\operatorname{Supp}X\subsetneq\operatorname{Supp}F_0$.

Let Y be the sum of barks of maximal twigs of D contained in F_1. We define $\mathcal{P}=K+D+\widehat{E}-Y-X$, then $T\mathcal{P}=0$ for all vertical curves. Since $NS(\overline{S})$ is generated by fiber components together with D_0 then it follows that $\mathcal{P}-(\mathcal{P}D_0)F\equiv 0$. We compute $\mathcal{P}D_0=1-D_0(Y+X)=1-\frac{1}{\mu_1}-\frac{\xi+1}{\mu_0}=\alpha$. Clearly, $\alpha\geq 0$, so $K+D+\widehat{E}\equiv\alpha F+Y+X$ and in fact $K+D+\widehat{E}\sim\alpha F+Y+X$, because $\overline{S}$ is rational.

Thus $\overline{\kappa}(S_0) \geq 0$. Moreover, $Y + X$ is effective and negative definite, so $(K + D + \widehat{E})^+ = \mathcal{P}$ from the uniqueness of Zariski decomposition. We see that $\widetilde{F}_0$ defined in construction 5.3.2 is the same as defined in 5.3.1(3) if we put $C = C_0$. Let $\widetilde{S}$ and $\widetilde{D}$ be as in 5.3.1(3). If $K_{\widetilde{S}} + \widetilde{D}$ has a Zariski decomposition then $F_\infty \subseteq \mathrm{Supp}(K_{\widetilde{S}} + \widetilde{D})^-$ by 1.6.9(i), because D_0 is non-branching in $\widetilde{D}$. This gives a contradiction, because $F_\infty^2 = 0$. Thus $\overline{\kappa}(S) = \kappa(K_{\overline{S}} + D) = \kappa(K_{\widetilde{S}} + \widetilde{D}) = -\infty$. □

Remark. Since $\widehat{E}$ is a chain, the unique singular point of S' is a cyclic singularity, i.e. it is of type $\mathbb{C}^2//G$ for a small cyclic group $G < GL(2,\mathbb{C})$. We have $|G| = d(\widehat{E})$ by 1.4.1(i).

Remark. Our result expressing the $\overline{\kappa}(S_0)$ in terms of α contradicts the result of Miyanishi and Sugie [MS91, Lemma 2.15(2)]. This is because their formula computing the Kodaira dimension ($\overline{\kappa}(X)$ is our $\overline{\kappa}(S_0)$) is wrong, it does not take into account the fiber F_1 (their F_0).

5.4 Sandwich I

5.4.1. Notice that if $p : \overline{S} \to B$ is a sandwich of type (I), then $p_{|S}$ is a $\mathbb{C}^1$-ruling, so the assumption $\overline{\kappa}(S') = -\infty$ is satisfied automatically. Let $E_h^2 = -N < 0$. Let $F_1, F_2, \ldots, F_n$ be all the column fibers of p. Denote their weights with respect to E_h (cf. 5.1.7) by $w_1, w_2, \ldots, w_n$. Let C_i be the unique (-1)-curve of F_i, put $\mu_i = \mu(C_i)$.

Notice that by 2.2.3(iv) the rationality of one of $\overline{S}$, $\widehat{E}$, D or B implies the rationality of all others. We remind that the rationality of $\widehat{E}$ as a divisor does not imply that the singularities of S' are rational (cf. 1.4.2).

Lemma 5.4.2. *If the morphism p is a sandwich of type (I), then its singular fibers are column fibers with weights satisfying* $\sum_{i=1}^n w_i < N$ *(see Fig. 5.2). There exists a linear bundle* $\mathcal{L}$ *over* B *with* $\deg \mathcal{L} = -N < 0$, *such that* $\overline{S}$ *is a blowup of* $\mathbb{P}(\mathcal{O}_{\mathcal{B}} \oplus \mathcal{L})$ *and p is the morphism induced by the projection onto* B.

Proof. Since $\widehat{E} \cap D = \emptyset$, $\nu = 0$ and there are no rivets in $D + \widehat{E}$. By 2.1.4(iii) $\Sigma_{S_0} = 0$, hence every fiber has exactly one S_0-component. By 5.1.8(ii) every singular fiber is a column fiber. We contract all singular fibers to smooth fibers (i.e. contract subsequently their (-1)-curves) without touching E_h. Denote the image of $\overline{S}$ by $\widetilde{S}$ and the image of D_h by $\widetilde{D}_h$. Then E_h is disjoint from $\widetilde{D}_h$. Since $E_h^2 = -N < 0$, we can write $\widetilde{S} = \mathbb{P}(\mathcal{O}_B \oplus \mathcal{L})$ for a line bundle $\mathcal{L}$ with *deg* $\mathcal{L} = -N < 0$ (see [Har77, V.2]). Now $\widetilde{D}_h$ and E_h are sections coming from the linear summands of the bundle. Let $E_i \subseteq F_i$ be the maximal twig of $\widehat{E} - E_h$ with its natural ordering as a twig of $\widehat{E}$. The matrix $Q(\widehat{E})$ is negative definite, so $0 < \det Q(-\widehat{E}) = d(E_1)d(E_2)\ldots d(E_n)(-E_h^2 - \sum_{i=1}^n \widetilde{e}(E_i))$ (cf. 1.1.1(i)), hence $\sum_{i=1}^n w_i < N$, because $w_i = \widetilde{e}(E_i)$. □

Corollary 5.4.3. *If p is a sandwich of type (I) then* S' *is contractible.*

Proof. By 5.4.2 singular fibers are column fibers, so in each fiber there is a component of $\widehat{E}$ of multiplicity one, hence by [Fuj82, 4.19] $\pi_1(S) = \pi_1(B)$. We can assume that the generators are contained in E_h, hence they are contracted when creating S', so $\pi_1(S') = 0$. Thus by 2.1.3(vii)-(viii) and Whitehead's theorem S' is contractible. □

Again, for the needs of the construction and lemma below we cancel the assumptions made about S, S_0, etc.

Construction 5.4.4. Pick $n \in \mathbb{N}$ and for each $i = 1, \ldots, n$ choose a number $w_i \in \mathbb{Q} \cap (0,1)$. Choose a positive integer N, such that $\sum_{i=1}^n w_i < N$. Let B be a complete curve of genus $g(B)$, such that $g(B) > 0$ if n was chosen smaller than 3. Define $\widetilde{S} = \mathbb{P}(\mathcal{L} \oplus \mathcal{O}_B)$, where $\mathcal{L}$ is a line bundle over B of degree $\deg \mathcal{L} = -N$. Let $\widetilde{p} : \widetilde{S} \to B$ be the induced $\mathbb{P}^1$-fibration. Denote the sections induced by inclusions of the direct summands $\mathcal{O}_B$ and $\mathcal{L}$ by $\widetilde{D}_h$ and E_h. Then $\widetilde{D}_h^2 = N$ and $E_h^2 = -N$. Choose n distinct points $x_1, \ldots, x_n \in \widetilde{D}_h$ and blow up each point once. For each i make a connected sequence of subdivisional blowups creating over x_i a column fiber F_i, such that its weight with respect to E_h is w_i. Denote the birational transform of $\widetilde{D}_h$ by D_h. Write $\underline{F}_i = E_i + C_i + D_i$ where $C_i^2 = -1$, E_i and D_i are connected chains and $D_i \cap E_h = \emptyset$. Let μ_i be the multiplicity of C_i in F_i. Fix a natural order on each E_i and D_i

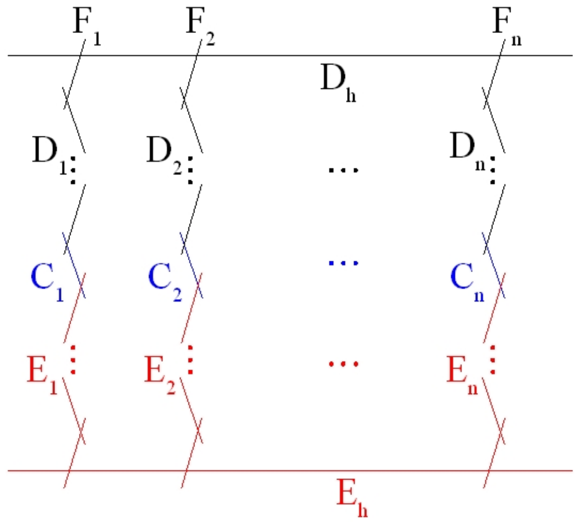


Figure 5.2: sandwich I

treated as a twigs of $\widehat{E} = E_1 + \ldots + E_n + E_h$ and $D = D_1 + \ldots + D_n + D_h$ respectively. Denote the obtained surface by $\overline{S}$ and the induced $\mathbb{P}^1$-ruling by p. Define $S = \overline{S} \setminus D$, $S_0 = S - \widehat{E}$ and $S' = S/\widehat{E}$ (as a topological space). We will show below that $NS_{\mathbb{Q}}(S_0) = 0$, hence by 1.7.3 S' and the quotient morphism can be realized in the algebraic category.

Remark. The additional assumption that $g(B) > 0$ if $n < 3$ is justified as follows. If $g(B) = 0$ and $n < 3$, then $\widehat{E}$ is a chain, so it contracts either to a smooth point or to a cyclic singularity. Moreover, $\overline{\kappa}(S_0) = -\infty$ in this case (see the proof of 5.4.5). But then S_0 is affine ruled (see 4.2.1), and appropriate S''s were described in 4.1.1.

Theorem 5.4.5. *The surface S' constructed in 5.4.4 is a singular, normal, contractible surface of negative Kodaira dimension. $\overline{\kappa}(S_0)$ is determined by the sign of the number $\alpha = n - 2 + 2g(B) - \sum_{i=1}^{n} \frac{1}{\mu_i}$ (i.e. it is $-\infty$ for negative α, zero for $\alpha = 0$ and one for $\alpha > 0$). Moreover, each singular, normal $\mathbb{Q}$-homology plane of negative Kodaira dimension with smooth locus having a $\mathbb{C}^*$-ruling of type sandwich (I) can be obtained by construction 5.4.4.*

Proof. The last part of the statement is a consequence of the lemma 5.4.2. (Notice that the assumption $\overline{\kappa}(S_0) \geq 0$ was not used there.)

The matrix $Q(\widehat{E} - E_h)$ is negative definite and $d(\widehat{E}) = d(E_1)d(E_2)\cdots d(E_n)(N - \sum_{i=1}^{n} w_i) > 0$, so by Sylvester's theorem $Q(\widehat{E})$ is negative definite. We have $d(D) = d(D_1)d(D_2)\cdots d(D_n)(-N + n - \sum_{i=1}^{n}(1 - w_i))$, so $d(D) \neq 0$. It follows that the classes of irreducible components of $D + \widehat{E}$ are independent in $NS(\overline{S})$, hence are a basis, because $b_2(\overline{S}) = \#D + \#E$. We apply 1.7.3 and infer that S' is normal and affine. By Iitaka's easy addition theorem 1.6.10 we have $\overline{\kappa}(S_0) \leq 1$. Define an effective divisor $X = \sum_{i=1}^{n}(\operatorname{Bk} D_i + \operatorname{Bk} E_i)$ and put $\mathcal{P} = K_{\overline{S}} + D + \widehat{E} - X$. For every irreducible curve T contained in the fibers of p the divisor X satisfies $T(K_{\overline{S}} + D + \widehat{E}) = TX$. For a general fiber F the divisor $\mathcal{P} - (\mathcal{P}D_h)F$ intersects trivially with D_h and all fiber components, hence $\mathcal{P} \equiv \alpha F$, because $\mathcal{P}D_h = n - 2 + 2g(D_h) - XD_h = \alpha$ by 1.3.2(ii) and 5.1.7. We get $K_{\overline{S}} + D + \widehat{E} \equiv \alpha F + X$. Now if $\alpha \geq 0$ then $K_{\overline{S}} + D + \widehat{E}$ is pseudo-effective, so $\overline{\kappa}(S_0) \geq 0$ by [Miy01, 2.2.6]. Conversely, if $\overline{\kappa}(S_0) \geq 0$ then from $F(K_{\overline{S}} + D + \widehat{E}) = 0$ it follows that the supports of $(K_{\overline{S}} + D + \widehat{E})^-$ and $(K_{\overline{S}} + D + \widehat{E})^+$ are contained in the fibers of p. Since $\mathcal{P}^2 \geq 0$, we get $(K_{\overline{S}} + D + \widehat{E})^+ \equiv \beta F$ for some $\beta \geq 0$ and $\operatorname{Supp}(K_{\overline{S}} + D + \widehat{E})^- = \operatorname{Supp} X$ by 1.6.9(i). Using the equivalence $\mathcal{P} + X \equiv (K_{\overline{S}} + D + \widehat{E}) \equiv (K_{\overline{S}} + D + \widehat{E})^+ + (K_{\overline{S}} + D + \widehat{E})^-$ we get $(\alpha - \beta)F \equiv (K_{\overline{S}} + D + \widehat{E})^- - X$. The divisor on the right hand side is supported on $\operatorname{Supp} X$ and since $F^2 = 0$, its intersection matrix is not negative definite. Thus $(K_{\overline{S}} + D + \widehat{E})^- = X$ and $\alpha = \beta \geq 0$. By 1.6.8 this proves that $\overline{\kappa}(S_0)$ is determined by the sign of α as stated.

Now we check that S' is a $\mathbb{Q}$-acyclic (then it is contractible by 5.4.3). We know from the above that that the map $H_2(D + \widehat{E}) \to H_2(\overline{S})$ induced by inclusion is an isomorphism. Clearly, $H_1(D) \to H_1(\overline{S})$ and

$H_1(\widehat{E}) \to H_1(\overline{S})$ are monomorphism, because they are monomorphisms after composing with $H_1(p)$. The exact sequence of a pair $(D, \overline{S})$ gives $b_4(S) = b_3(S) = 0$, $b_2(S) = \#\widehat{E}$ and $b_1(S) = b_1(\overline{S}) = b_1(B)$. Then the exact sequence of a pair $(\widehat{E}, S)$ gives $b_1(S') = b_2(S') = b_3(S') = b_4(S') = 0$. Since we assumed that $g(B) > 0$ if $n < 3$, S' is singular. □

Corollary 5.4.6. *Let P be the image of $\widehat{E}$ under the contraction morphism $S \to S'$ as above. It is a topologically rational singularity if and only if $B \cong \mathbb{P}^1$. Furthermore:*

(i) $\overline{\kappa}(S_0) = -\infty$ if and only if $\alpha < 0$, $g(B) = 0$, $n = 3$ and (μ_1, μ_2, μ_3) is up to order one of the Platonic triples (cf. 1.3.5). Moreover, the smooth locus of S' has a structure of a Platonic fibration and P is a noncyclic singularity of quotient type. Conversely, each such S' can be obtained by the construction above. (This complements the description given in 4.2.1).

(ii) Assume $\overline{\kappa}(S_0) \geq 0$. Then P is not of quotient type.

(iii) $\overline{\kappa}(S_0) = 0$ if and only if either

(a) $g(B) = 1$ and $n = 0$ or

(b) $g(B) = 0$, $n = 4$ and $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 2$ or

(c) $g(B) = 0$, $n = 3$ and (μ_1, μ_2, μ_3) is up to order one of $(2,3,6), (2,4,4), (3,3,3)$.

Proof. (i) If $\alpha < 0$ then $\frac{n}{2} \leq \sum_{i=1}^{n}(1 - \frac{1}{\mu_i}) < 2(1 - g(B))$, so $g(B) = 0$ and $n \leq 3$, hence $n = 3$ by the assumptions of the construction. Then $\sum_{i=1}^{3} \frac{1}{\mu_i} > 1$, so (μ_1, μ_2, μ_3) is up to order one of the Platonic triples. Thus S_0 has a structure of a Platonic fibration and $\widehat{E}$ is an admissible rational fork, because $N > \sum_{i=1}^{3} w_i \geq \sum_{i=1}^{3} \frac{1}{\mu_i} > 1$. Conversely, a Platonic $\mathbb{C}^*$-fibration of S_0 can be extended to a $\mathbb{P}^1$-fibration of some snc-completion of S_0. The sections contained in the boundary have to be contained in different connected components of the boundary, so the extension is of type sandwich (I). Moreover, $g(B) = 0$ by the definition of a Platonic $\mathbb{C}^*$-fibration, so $\alpha < 0$.

(ii) If P is of quotient type then by 1.3.5(ii) $\alpha = 1 - \sum_{i=1}^{3} \frac{1}{\mu_i} < 0$, so $\overline{\kappa}(S_0) = -\infty$.

(iii) Assume $\alpha = 0$. For $n = 0$ we get $g(B) = 1$. Assume $n > 0$. We have $\frac{n}{2} \leq \sum_{i=1}^{n}(1 - \frac{1}{\mu_i}) = 2(1 - g(B))$, so we get $g(B) = 0$ and $n \leq 4$. We have $n \geq 3$ by the assumptions of the construction. For $n = 3$ and $n = 4$ we get $\sum_{i=1}^{3} \frac{1}{\mu_i} = 1$ and $\sum_{i=1}^{4} \frac{1}{\mu_i} = 2$, so we get the case (b) or (c) respectively. Conversely, in each case $\alpha = 0$. □

From theorems 2.2.4, 5.4.5 and 5.3.3 we have the following

Corollary 5.4.7. *If the singular $\mathbb{Q}$-homology plane is nonrational or has singularities which are not of quotient type, then it is contractible and has negative Kodaira dimension. Moreover, the singularity is unique and the smooth locus is $\mathbb{C}^*$-ruled.*

For $g(B) = 0$ the singularity $P \in S'$ is topologically rational. It does not have to be rational, as follows from the following example.

Example 5.4.8. Assume $g(B) = 0$. Then:

(i) if $N > n$ then P is a rational singularity,

(ii) if $N = 1$, $n = 3$ and $E_1 = [a]$, $E_2 = [b]$ and $E_3 = [c]$ with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$ then $P \in S'$ is a topologically rational but not a rational singularity.

Proof. A criterion of Artin [Art66, Theorem 3] says that $P \in S'$ is a rational singularity if and only if $p_a(Z) = 0$ for a fundamental cycle Z of $\widehat{E}$. A fundamental cycle is the smallest nonzero effective divisor $Z \subseteq \widehat{E}$, such that $ZE_i \leq$ for each irreducible $E_i \subseteq \widehat{E}$. Now if $N > n$ then the fundamental cycle is reduced, and since $\widehat{E}$ is a rational snc-tree, its arithmetic genus vanishes. In case (ii) the fundamental cycle is $Z = 3E_h + E_1 + E_2 + E_3$, hence $p_a(Z) = 1$. □

Remark. In more complicated cases the fundamental cycle can be computed easily using [Lau72, Proposition 4.1].

Remark 5.4.9. It is well known that a $\mathbb{P}^1$-bundle over a complete curve is a projectivization of a $\mathbb{C}^2$-bundle. If s_1, s_2 are disjoint sections of this $\mathbb{P}^1$-bundle then the action of $\mathbb{C}^*$ given by $t * [\alpha s_1 + \beta s_2] = [\alpha s_1 + t\beta s_2]$ fixes precisely s_1 and s_2. Notice that if we have a smooth surface with a $\mathbb{C}^*$-action and we blow up in such a way that the center is contained in the fixed point locus then the action extends to the blowup. It follows that all surfaces S' appearing in 5.4.4 admit a $\mathbb{C}^*$-action with $\operatorname{Sing} S'$ as the fixed point locus. The weighted graphs of exceptional loci for the resolutions of normal surfaces with good $\mathbb{C}^*$-action (i.e. having positive weights for some equivariant embedding in some affine space) were described in [OW71]. It follows from 5.4.5 that all of them can be realized as graphs of exceptional loci for the resolutions of singular contractible $\mathbb{Q}$-homology planes.

Chapter 6

$\overline{\kappa}(S_0) = 2$ and $\overline{\kappa}(S') = -\infty$

In this chapter we assume that $\overline{\kappa}(S_0) = 2$, $\overline{\kappa}(S') = -\infty$ and that $D + \widehat{E}$ is snc-minimal. Although we do not assume that S' is contractible, but only that it is $\mathbb{Q}$-acyclic, still part of the methods from [KR07] work. We adapt them to our situation. We do not hesitate to use computer programs if necessary. Finally, by careful analysis of numerical and geometrical properties of S' we show that $\overline{\kappa}(S_0) = 2$ is impossible.

6.1 Preliminary results

Definition 6.1.1. Decompose $\widehat{E}$ as $\widehat{E} = E + \Delta$, where Δ is the divisor of external (-2)-curves in $\widehat{E}$, i.e. Δ is a reduced divisor with the smallest support, such that E does not contain a (-2)-tip. By 2.2.3(viii) $\Delta \neq \widehat{E}$, so $K\widehat{E} = KE > 0$. Define ϵ by the equality $(K + D + \widehat{E})^2 = -1 - \epsilon$. Since $\overline{\kappa}(S_0) = 2$ and S_0 is rational, the snc-minimal completion $(\overline{S}, D + \widehat{E})$ is unique, hence ϵ is an invariant of S.

We begin with a lemma which mainly collects some results obtained in the previous chapters.

Lemma 6.1.2.

(i) *S' has exactly one singular point and it is of quotient type,*

(ii) *there is no simple curve on $(\overline{S}, D + \widehat{E})$,*

(iii) *S_0 and the pair $(\overline{S}, D + \widehat{E})$ are almost minimal, in particular*
$K + D + \widehat{E} \equiv (K + D + \widehat{E})^+ + \operatorname{Bk} D + \operatorname{Bk} \widehat{E}$,

(iv) $\epsilon \geq 0$,

(v) *$\widehat{E}$ and D are rational trees and $\overline{S}$ is rational,*

(vi) $\#\widehat{E} + \#D = 7 + \epsilon + KD + KE$,

(vii) *if $\epsilon < 2$ then* $|2K + D + E| \neq \emptyset$,

(viii) $KE + \epsilon \geq 3$,

(ix) *if D has a component with nonnegative self-intersection then this component is branching in D and D is not a fork.*

Proof. (i) is just 2.2.1(i). The proof of (ii) is the same as in 3.1.4. (iii) follows from (ii) and 1.6.9(ii). (iv) is a consequence of 1.6.13(1). (v) Since the unique singular point of S' is of quotient type, $\widehat{E}$ is a rational tree, so we are done by 2.2.3(iv).

(vi) Since D and $\widehat{E}$ are connected rational trees, we have $K(K+D+\widehat{E}) = 3-\epsilon$, so $K^2 = 3-\epsilon-KD-KE$ and the formula follows from the Noether formula and 2.1.4(ii).

(vii) Riemann-Roch's theorem for a divisor $-K-D-E$ gives $h^0(-K-D-E)+h^0(2K+D+E) \geq 2-\epsilon$. If $-K - D - E \geq 0$ then $-K - D - \widehat{E} \geq 0$, which contradicts $\kappa(K + D + \widehat{E}) = 2$. Thus $2K + D + E \geq 0$.

(viii) Suppose $KE + \epsilon \leq 2$. By Riemann-Roch's theorem $h^0(-K - D) + h^0(2K + D) \geq K(K + D) = 3 - \epsilon - EK > 0$, so $-K - D \geq 0$, otherwise we would have $\kappa(K + D) \geq 0$. By (vii) this gives $K + E \geq 0$. Maximal twigs of E are in the fixed part of $K+E$, so E cannot be a chain. Let's write $\widehat{E} = B+E_1+E_2+E_3$, where B is the branching component of $\widehat{E}$, and E_i's are its maximal twigs. Since E is not a chain, $d(E_i) \geq 3$ for all i, so $\delta(\widehat{E}) \leq 1$. This is a contradiction, because $\widehat{E}$ is a resolution of a quotient singularity (cf. 1.4.1(ii)).

(ix) Let D_0 be a component of D with $D_0^2 = -b \geq 0$. After some connected modification $(\widetilde{S}, \widetilde{D}) \to (\overline{S}, D)$ we can assume that $b = 0$. In fact we can assume that this modification is subdivisional for D, unless $D = D_0$. In any case, if $\beta_D(D_0) < 3$ we get a $\mathbb{C}^1$- or $\mathbb{C}^*$- ruling of S_0. Then $\overline{\kappa}(S_0) \leq 1$ by Iitaka's addition theorem (cf. 1.6.10(i)), a contradiction. Suppose now D_0 is the branching component of a fork. Now D_0 gives a $\mathbb{C}^{**}$- ruling of $\widetilde{S}$. We have $\Sigma_{S_0} = 2$, because $\widehat{E} \subseteq F_0$ for some fiber F_0. Notice that since there are no vertical (-1)-curves in $\widetilde{D}$, every vertical (-1)-curve is an S_0-component. Let D_h be the divisor of horizontal sections of $\widetilde{D}$, it consists of three sections. Denote the divisor of $\widetilde{D}$-components contained in F_0 by D_v. Suppose F is a singular fiber with the unique (-1)-curve L. D_h can intersect F only in components of multiplicity one, which in this case are two tips of the first branch of F (cf. 1.5.1(v)). $\widetilde{D}$ does not contain loops, so at most one of these tips is a $\widetilde{D}$-component, hence L is simple, a contradiction with (ii). Thus every singular fiber has at least two (-1)-curves. Since $D_hF < 4$, by (ii) this implies that $D_v \neq 0$. Notice that any exceptional S_0-component intersecting $\widehat{E}$ is a tip of F_0, otherwise it would have $\mu > 1$ and it could not intersect D_h, which contradicts (ii). Hence some S_0-component $M \subseteq F_0$ intersecting $\widehat{E}$ is not exceptional and intersects D_v. We conclude that F_0 contains precisely two exceptional components, L_1 and L_2, and $\sigma(F_0) = 3$, hence F_0 is the only singular fiber.

Suppose F_0 is branched. Then at least for one of L_1 or L_2, say for L_1, after making successive contractions of L_1 (i.e. after making a connected sequence of contractions starting from L_1) the number of branching components in the fiber decreases. It follows that $\mu(L_1) > 1$. Indeed, if T is the maximal twig of $\widetilde{D}$ containing L_1 and T intersects $\widetilde{D} - T$ in T_0 then we see that the equality $\mu(L_1) = 1$ would imply that after contraction of T the component T_0 becomes a non-exceptional component of a fiber with unique (-1)-curve and satisfies $\beta \geq 2$ and $\mu = 1$, a contradiction with 1.5.1(v). We infer that $D_hL_1 = 0$, so by (ii) L_1 is not a tip of F_0. Moreover, one of the connected components of $\underline{F}_0 - L_1$ does not contain curves with multiplicity one, so it is not intersected by D_h, which implies that it does not contain any $\widetilde{D}$-component. Hence L_1 is simple, a contradiction with (ii).

Since F_0 is a chain, M is not branching, so (ii) implies that it intersects D_h, hence $\mu(M) = 1$. Now $D_v \neq 0$ implies $D_h(L_1 + L_2) \leq 1$, so by (ii) $L_1\widehat{E} = L_2\widehat{E} = 0$. Therefore we can successively contract L_1 and L_2 without touching $\widehat{E}$ until M becomes the unique exceptional component of the fiber. This contradicts $\mu(M) = 1$.

□

Remark. In fact, S_0 is not only almost minimal, but also strongly minimal (cf. [Miy01, 2.4.12]). From (ix) we see that the maximal twigs of D are admissible, in particular D is not a chain by 2.2.3(ii).

Definition 6.1.3. We denote the local fundamental group of the unique singular point of S' by G and write K for $K_{\overline{S}}$. Let T_i for $i = 1, \ldots, s$ be all maximal twigs of D and let $T = T_1 + \ldots + T_s$. We put $d_i = d(T_i)$, $e_i = e(T_i)$ (recall that by our convention from 1.3 tip of a maximal twig is its first component), $\widetilde{e}_i = e(T_i^t)$, $\delta = \delta(D)$, $e = e(D)$ and $\widetilde{e} = \widetilde{e}(D)$. We write $\mathcal{P}$ for $(K_{\overline{S}} + D + \widehat{E})^+$ and $\mathcal{N}$ for $(K_{\overline{S}} + D + \widehat{E})^-$.

Lemma 6.1.4. *(Koras-Russell, [KR07, 5.3, 5.15])*

(i) $\delta \leq e = -\operatorname{Bk}^2 D \leq 1 + \epsilon + \operatorname{Bk}^2 \widehat{E} + \frac{3}{|G|}$,

(ii) If $\epsilon < 2$ *then* $s - 2 - \frac{6}{|G|} \leq \delta$,

(iii) If $\epsilon < 2$ *then* $s - 3 \leq \epsilon + \operatorname{Bk}^2 \widehat{E} + \frac{9}{|G|}$,

(iv) If $\epsilon < 2$ *and* $\Delta = \emptyset$ *then* $e + \delta \geq s + \epsilon - \frac{5}{2} + \frac{KE}{4}$.

Proof. (i) We have $e = -\operatorname{Bk}^2 D$ by 1.3.2(iv). Computing a square of 6.1.2(iii) gives $-1-\epsilon = \mathcal{P}^2 + \operatorname{Bk}^2 D + \operatorname{Bk}^2 \widehat{E}$, so (i) follows from the Kobayashi inequality.

(ii) By 6.1.2(vii) we have $0 \leq \mathcal{P}(2K+D+E) = 2\mathcal{P}(K+D+E) - \mathcal{P}(D+E) = 2\mathcal{P}^2 - \mathcal{P}R \leq \frac{6}{|G|} - \mathcal{P}R$, where $R = D - T$. Now R is a rational connected tree, so $\mathcal{P}R = (K+D-\operatorname{Bk} D)R = -2 + (T - \operatorname{Bk} D)R = -2 + s - \delta$ by 1.3.2(ii).

(iii) is a consequence of (i) and (ii).

(iv) From the proof of [KR07, 5.15] it follows that if $\Delta = \emptyset$ then $e + \delta \geq s - \frac{v}{4}$, where $v = (E+2K)(E+2(K+D)) = -2 + KE + 2(E+2K)(K+D) = 10 - 4\epsilon - EK$. This gives (iv).

□

6.2 Bounding the shape of $\widehat{E}$

The following theorem is the key result in case $\overline{\kappa}(S_0) = 2$. It is a modification of [KR07, 5.10].

Proposition 6.2.1. *Either $KE + 2\epsilon \leq 5$ or $\epsilon = 2$, $\widehat{E} = [4]$ and D consists of (-2)-curves.*

Proof. The idea is to use (1) and (4) of 5.1.2 to find and contract an exceptional simple curve on $(\overline{S}, D + \widehat{E})$. Notice that $(2K+E)(K+D) = 6 - 2\epsilon - EK$. Suppose there exists a (-1)-curve $A \subseteq \overline{S}$, such that $A\widehat{E} \leq 1$. Under this assumption it is proved in [KR07, 5.10, 5.11] that if S' is contractible then the inequality $KE + 2\epsilon > 5$ would imply that the process of finding and contracting exceptional simple curves could be iterated to infinity, which is impossible. The proof of [KR07, 5.10] does not require the contractibility, but only the $\mathbb{Q}$-acyclicity of S', so it can be applied in our situation. However, the existence of the curve A, which is assured by lemma [KR07, 5.7] in case S' is topologically contractible, has to be reconsidered in our situation.

Suppose $KE + 2\epsilon > 5$. From the above remarks it follows that we can assume that there is no (-1)-curve $A \subseteq \overline{S}$, such that $A\widehat{E} \leq 1$. It appears that the only point where the proof of existence of the curve A given in [KR07, 5.7] does not work in our more general situation is the case [KR07, 5.7.4(ii)], where $KD = 0$, $K + \widehat{E}^{\#} \equiv 0$ and $\operatorname{Bk}^2 \widehat{E}$ is an integer. Then by 6.1.4(i) we get $\operatorname{Bk} \widehat{E}^2 = -1$, so $\widehat{E}$ is a chain by 1.3.5(iii). We have $D^2 = -2 - DK = -2$, so $-1 - \epsilon = (K + D + \widehat{E})^2 = (D + \operatorname{Bk} \widehat{E})^2 = D^2 - 1$, hence $\epsilon = 2$ and $K(K + \widehat{E}) = 3 - \epsilon - DK = 1$. Moreover, any (-1)-curve contained in D could serve as the curve A, so we get that $D_i^2 \leq -2$ for every $D_i \subseteq D$, hence D consists of (-2)-curves. Further arguments have to be modified as follows. By Riemann-Roch's theorem $h^0(\widehat{E} + 2K) + h^0(-K - \widehat{E}) \geq K(K + \widehat{E}) = 1$. If $-K - \widehat{E} \sim U$ for an effective divisor U then $K + \widehat{E}^{\#} \equiv 0$ implies $U + \operatorname{Bk} \widehat{E} \equiv 0$, hence $\operatorname{Bk} \widehat{E} = 0$, which is impossible. We get $2(K + \widehat{E}) \geq 0$, which by 1.6.7(ii) implies that $[2(K + \widehat{E}^{\#})] \sim U$ for some effective divisor U. Now $K + \widehat{E}^{\#} \equiv 0$ implies that $U + \{2 \operatorname{Bk} \widehat{E}\} = 0$, hence $2 \operatorname{Bk} \widehat{E}$ is a $\mathbb{Z}$-divisor. Since $\widehat{E}$ is not a (-2)-chain we obtain $2 \operatorname{Bk} \widehat{E} = \widehat{E}$ and $2K + \widehat{E} = 0$. It follows that $\Delta = 0$ and $KE = 2$. Moreover, since $E_i(2K + E) = 0$ for each component E_i of E, we get that either $\widehat{E} = [4]$ or $\widehat{E} = [3, (k), 3]$ for some $k \geq 0$. To finish the proof we need to exclude cases other than $\widehat{E} = [4]$.

Suppose $\widehat{E} = [3, (k), 3]$ for some $k \geq 0$. We have $\#D = 9 - k$ by 6.1.2(vi), so there are only finitely many possibilities for the weighted dual graph of D. Notice that the inequality 6.1.4(i) gives $e(D) \leq 1 + \epsilon + \operatorname{Bk}^2 \widehat{E} + \frac{3}{|G|} = 2 + \frac{3}{d(E)}$. We have $d(E) = 4(k+2)$ and D consists of (-2)-curves, so $e(D) = s - \delta$. Computing the square of 6.1.2(iii) we get $-1 - \epsilon = \mathcal{P}^2 - e(D) - 1$, so $\mathcal{P}^2 = s - 2 - \delta$. Since $\mathcal{P}^2 > 0$, we obtain:

$$0 < s - 2 - \delta \leq \frac{3}{4(11 - \#D)} = \frac{3}{4(k+2)}.$$

In particular, $s - 2 \leq \delta + \frac{3}{8} \leq \frac{s}{2} + \frac{3}{8}$, so $s \leq 4$. Another condition is given by 2.1.3(ix):

$$\sqrt{-\frac{d(D)}{d(E)}} = |H_1(S', \mathbb{Z})| \in \mathbb{N}.$$

We check by straight computations that up to permutation of maximal twigs there are only two pairs of weighted dual graphs of $(D,\widehat{E})$ satisfying both conditions (taking into account that D consists of (-2)-curves one checks first that the first condition implies that $k \leq 1$ for $s = 3$ and $k \leq 2$ for $s = 4$):

(1) $s = 3$, $T_1 = [2, 2]$, $T_2 = [2, 2, 2]$, $T_3 = [2, 2, 2]$, $\widehat{E} = [3, 3]$,

(2) $s = 4$, $T_1 = [2]$, $T_2 = [2]$, $T_3 = [2]$, $T_4 = [2, 2, 2]$, $\widehat{E} = [3, 3]$.

Notice that in case (2) $D - T_1 - T_2 - T_3 - T_4$ has three components. In both cases $-d(D) = d(\widehat{E}) = 8$, so $H_1(S', \mathbb{Z}) = 0$ by 2.1.3(ix).

To deal with these cases consider an affine ruling of S. Let $\pi : (\widetilde{S}, \widetilde{D}) \to \mathbb{P}^1$ be an extension of this ruling to some snc-completion of S. W can assume that $\widetilde{D}$ is π-minimal. Let $F_1, F_2, \ldots, F_r$ be all the singular fibers of π. Each F_i has $\sigma(F_i) > 0$, otherwise S_0 would be affine ruled, which is impossible. Furthermore, each F_i contains come $\widetilde{D}$-component. Indeed, if some F_i has no $\widetilde{D}$-components, then $\sigma(F_i) = 1$, because each S_0-component intersects $\widetilde{D}$ by affiness of S'. Then the S_0-component of F_i is the unique (-1)-curve of F_i, hence cannot intersect $\widetilde{D}$, because has multiplicity greater than one, a contradiction. Let μ_i be the greatest common divisor of multiplicities of S-components contained in F_i. Using van Kampen's theorem one shows that $\pi_1(S) = \langle \sigma_1, \sigma_2, \ldots, \sigma_r : \sigma_1^{\mu_1} = \ldots = \sigma_1^{\mu_r} = \sigma_1\sigma_2\ldots\sigma_r = 1\rangle$ (cf. [Fuj82, 4.19]). We have also $\pi(S') = \pi_1(S)$ by 2.1.3(viii).

Suppose $r > 2$. Then $(\widetilde{S}, \widetilde{D}) = (\overline{S}, D)$ and since in both cases the branching curves of D have $\beta_D \leq 3$, we get $r = 3$. If $\widehat{E}$ is horizontal then $\Sigma_{S_0} = 1$ and if not then $\Sigma_{S_0} = 0$. In any case there are at least two of F_i's, say F_1 and F_2, without $\widehat{E}$-components and satisfying $\sigma(F_i) = 1$ (here σ is the number of S_0-components). Since D is connected and each component of D is a (-2)-curve, each such a fiber F_i has two branches, the first equal to $[2, 2, 2]$, and the unique (-1)-curve of F_i is its tip. This implies that at least two maximal twigs of D are not its tips, which excludes (2), hence $s = 3$ and F_3 contains two D-components. If both components of $\widehat{E}$ are horizontal then $\Sigma_{S_0} = 1$, so $\sigma(F_3) = 2$ and we see that at least one S_0-component has multiplicity one, so $\mu_3 = 1$. We compute $\pi_1(S) = \langle \sigma_1, \sigma_2 : \sigma_1^2 = \sigma_2^2 = \sigma_1\sigma_2 = 1\rangle = \mathbb{Z}_2$, a contradiction with $H_1(S', \mathbb{Z}) = 0$. Thus exactly one component of E is vertical. Now $\sigma(F_3) = 1$, so $F_3 = [3, 1, 2, 2]$ and $\mu_3 = 3$. We compute $\pi_1(S) = \langle \sigma_1, \sigma_2, \sigma_3 : \sigma_1^2 = \sigma_2^2 = \sigma_3^3 = \sigma_1\sigma_2\sigma_3 = 1\rangle = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, a contradiction with $H_1(S', \mathbb{Z}) = 0$. Therefore $r \leq 2$. It follows that $\pi_1(S)$ is abelian, hence vanishes, because $H_1(S', \mathbb{Z}) = 0$. By 2.1.3(vii) and Whitehead's theorem it implies that S' is contractible. In this case the proof of [KR07, 5.7] works.

□

Corollary 6.2.2. *If $\epsilon = 0$ then $KE \in \{3, 4, 5\}$. If $\epsilon = 1$ then $KE \in \{2, 3\}$. If $\epsilon = 2$ then either $KE = 1$ or $\widehat{E} = [4]$.*

Proposition 6.2.3.

(i) If $\epsilon = 0$ then $\#\widehat{E} = 1$ and D is a fork,

(ii) If $\widehat{E}$ is a fork then $\epsilon = 2$,

(iii) Δ does not contain a fork.

Proof. (i) For $\epsilon = 0$ lemma 6.1.4(iii) gives $0 \leq s - 3 \leq \mathrm{Bk}^2\,\widehat{E} + \frac{9}{|G|}$. If $\widehat{E}$ is a fork then $\mathrm{Bk}^2\,\widehat{E} < -1$ by 1.3.5(v), so $|G| \leq 8$. Since G is small, G is the quaternion group, for which the resolution consist of (-2)-curves, a contradiction with 2.2.3(viii). Thus $\widehat{E}$ is a chain, so $d(\widehat{E}) = |G|$ and we get $d'(\widehat{E}) + d'(\widehat{E}^t) \leq 7$ (cf. 1.3.2(iv)). Suppose $\#\widehat{E} > 1$. Taking into account 6.2.2 there are two possibilities for $\widehat{E}$: $[3, 4]$ and $[2, 5]$. In both cases we obtain $\mathrm{Bk}^2\,\widehat{E} + \frac{9}{|G|} = 0$, so $s = 3$ and inequalities (i)-(iii) from 6.1.4 are replaced by equalities. We get $\widetilde{e} = \delta < 1$, so $d(D) = d_1d_2d_3(b - \widetilde{e}) < 0$ gives $b \leq 0$, a contradiction with 6.1.2(ix). Therefore $\#\widehat{E} = 1$. If $s \neq 3$ then 6.1.4(iii) and 6.2.2 give subsequently $(s - 3)|G| \leq 5$, $s = 4$ and $\widehat{E} = [5]$. In this case $e = \delta = \frac{4}{5}$, so inequality 6.1.4(iv) fails, a contradiction.

(ii) Let $\widehat{E}$ be a fork. By (i) $\epsilon \neq 0$. Suppose $\epsilon = 1$. A not so long numerical analysis of possible forks and its properties described in [Bri68, Satz 2.9] implies that in order to satisfy 6.1.4(iii) $\widehat{E}$ has to satisfy $\#E = 1$ and E has to be the branching curve of a fork, such that the determinants of its maximal twigs are $2, 2, n$. (see [KR07, 6.17] for a detailed proof). Since $KE \geq 2$ for $\epsilon = 1$ by 6.2.2, we have $E^2 \leq -4$, so by 1.3.5(iv) $|G/[G, G]| = 4n(-E^2 - 2) + 4 \geq 20$. Simultaneously 6.1.4(iii) gives $|G| < \frac{9}{e(\widehat{E}) - \epsilon} = \frac{9}{1 - \frac{1}{n}} \leq 18$, a contradiction.

(iii) Suppose Δ contains a fork. Then $\epsilon = 2$ by (ii), so $\#E = 1$ by 6.2.2. If $S \setminus \Delta$ is affine ruled then $\Sigma_{S_0} = 0$ implies that each fiber has only one (-1)-curve, hence each connected component of Δ is a chain, which contradicts our assumption. Since $\overline{\kappa}(S \setminus \Delta) = -\infty$ by 1.6.7, $S \setminus \Delta$ contains a Platonic fibration U as an open subset (cf. 1.6.14). An snc-minimal boundary of a Platonic fibration is a disjoint union of two forks. The description of $S \setminus (\Delta \cup U)$ given in [MT84b] implies that $U = S \setminus (\Delta \cup L)$ for a (-1)-curve L, such that $LD = 1$. It can be shown that $L\widehat{E} = 1$, i.e. L is simple on $(\overline{S}, D)$, which contradicts 6.1.2 (see [KR07, 6.1] for a detailed proof).

□

Corollary 6.2.4. *$S \setminus \Delta$ is affine ruled.*

Proof. Since $\overline{\kappa}(S \setminus \Delta) = -\infty$ then $S \setminus \Delta$ is affine ruled or it contains a Platonic fibration as an open subset. The last case is impossible by 6.2.3(iii). □

Corollary 6.2.5. *$\widehat{E}$ is of one of the following types:*

(a) $[5]$, $[6]$, $[7]$

(b1) fork:

$$\begin{array}{ccccc} A & \text{——} & -2 & \text{——} & B \\ & & | & & \\ & & -2 & & \end{array}$$

with (A, B) equal to one of: $([3], [2, 2])$, $([3], [2, 2, 2])$, $([3], [2, 2, 2, 2])$, $([2, 3], [2, 2])$ *or* $([(n), 3], [2])$, *where $n \geq 0$, (recall that a tip of the maximal twig is its first component),*

(b2) fork:

$$\begin{array}{ccccc} A & \text{——} & -3 & \text{——} & B \\ & & | & & \\ & & -2 & & \end{array}$$

with (A, B) equal to one of: $([2, 2], [2, 2])$, $([2, 2], [2, 2, 2])$, $([2, 2], [2, 2, 2, 2])$ *or* $([2], [(n)])$, *where $n \geq 0$,*

(b3) $[(r), 3, (x)]$ *for $r, x \geq 0$,*

(c1) $[(r), 4]$ *or* $[(r), 5]$ *for $r \geq 0$,*

(c2) $[(x), 3, (y), 3]$ *or* $[(x), 3, (y), 4]$ *or* $[(x), 4, (y), 3]$ *for $x, y \geq 0$,*

(c3) $[(r), 3, (x), 3, (y), 3]$ *for $r, x, y \geq 0$,*

(c4) $[2, 4, 2]$, $[2, 5, 2]$, $[2, 3, 3, 2]$, $[2, 3, 4, 2]$, $[2, 4, 2, 2]$, $[2, 5, 2, 2]$.

Proof. If $\widehat{E}$ is a fork then $\epsilon = 2$ by 6.2.3(ii), so $E = [3]$ by 6.2.2. We know that Δ does not contain a fork, so all possible $\widehat{E}$'s satisfying 1.3.5(ii)-(iii) are listed in (b1) and (b2). Chains for $\epsilon = 2$ other than $[4]$ are in (b3) and $\widehat{E}$'s for $\epsilon = 0$ are in (a) (cf. 6.2.2 and 6.2.3(i)). Now we can assume that $\widehat{E}$ is a chain and $\epsilon = 1$, so $KE \in \{2, 3\}$ by 6.2.2. For $E\Delta \leq 1$ all possible $\widehat{E}$'s are listed in (c1), (c2) and (c3), so we can assume $E\Delta = 2$. Using 6.1.4(iii) we get $d'(\widehat{E}) + d'(\widehat{E}^t) \leq d(\widehat{E}) + 7$ and since $d(\widehat{E}) = 2d'(\widehat{E}) - d''(\widehat{E}) = 2d'(\widehat{E}^t) - d''(\widehat{E}^t)$, we have $\frac{1}{2}(d(\widehat{E}) + d''(\widehat{E})) + \frac{1}{2}(d(\widehat{E}) + d''(\widehat{E}^t)) \leq d(\widehat{E}) + 7$, so $d''(\widehat{E}) + d''(\widehat{E}^t) \leq 14$. This gives six possibilities for $\widehat{E}$: $[2, 4, 2]$, $[2, 5, 2]$, $[2, 3, 3, 2]$, $[2, 3, 4, 2]$, $[2, 4, 2, 2]$ and $[2, 5, 2, 2]$, which are listed in (c4). □

6.3 Pre-minimal rulings

We recall the notion of Hamburger-Noether pairs. For details see [Rus80] and [KR99, Appendix].

Definition 6.3.1. Suppose we are given an irreducible germ of a singular analytic curve (χ_1, q_1) on a smooth algebraic surface and a curve C_1 passing through q_1, smooth at q_1. Put $c_1 = (C_1 \cdot \chi_1)_{q_1}$ and choose a coordinate y_1 in such a way that $\{y_1 = 0\}$ is transversal to C_1 at q_1 and for Y_1, defined as $Y_1 = \{y_1 = 0\}$, c_1 is not smaller than $p_1 = (Y_1 \cdot \chi_1)_{q_1}$. Blow up over q_1 until the proper transform χ_2 of χ_1 meets the reduced inverse image F_1 of C_1 in a point q_2, which does not belong to components of F_1 other than the exceptional component C_2 of F_1. We then say that C_2 *(and F_1) is produced from C_1 by the pair* $\binom{c_1}{p_1}$. This does not depend on the choice of y_1. Put $c_2 = (C_2 \cdot \chi_2)_{q_2}$. Then $c_2 = gcd(c_1, p_1)$. Notice that the pairs $\binom{c_1}{p_1}$ and $\binom{c_1/c_2}{p_1/c_2}$ give the same sequence of blowups. We repeat this procedure and define successively (χ_i, q_i) and C_i until χ_{h+1} is smooth for some $h \geq 1$. Then we refer to the sequence $\binom{c_1}{p_1}, \binom{c_2}{p_2}, \ldots, \binom{c_h}{p_h}$ as the sequence of *Hamburger-Noether pairs* (or *characteristic pairs* for short) *of the resolution of* (χ_1, q_1) or the sequence of *characteristic pairs of* F, where F is the reduced total transform of C_1.

Remark. We remind that since (χ_1, q_1) is singular and irreducible, there is a unique distinguished tangent direction at q_1, i.e. if z is a germ of a line in the distinguished direction then for any other germ of a line u one has $(\{u = 0\} \cdot \chi_1)_{q_1} < (\{z = 0\} \cdot \chi_1)_{q_1}$. Therefore, if there is no need to start with some given C_1 then it is natural to choose C_1 having distinguished tangent direction for (χ_1, q_1). However, making this choice one should remember that (assuming χ_2 is singular) (C_2, q_2) does not have to have distinguished tangent direction for (χ_2, q_2).

Definition 6.3.2. Let F be a singular fiber of a $\mathbb{P}^1$-ruling of some surface, such that L is the unique exceptional curve of F. Suppose some component U of F with $\mu_F(U) = 1$ is distinguished. Then there is precisely one way of contracting F to a smooth fiber without contracting U. For some $q \in L$ let (χ, q) be an irreducible germ of some analytical curve intersecting L transversally at q. Let (χ', q') be the image of (χ, q) after the above contractions. We take the image of U as C_1 (cf. 6.3.1). We then say that F is produced by the sequence of characteristic pairs of the resolution of (χ', q') and we refer to this sequence as *the sequence of the characteristic pairs of* F.

Example 6.3.3. Consider a $\mathbb{P}^1$-ruling of some complete surface. Let $F = A_n + \ldots + A_1 + L + B_1 + \ldots + B_m$ be some column fiber and let A_n be the distinguished component. Then F is produced by one characteristic pair $\binom{c}{p}$. Here are some examples. If $F = [k, 1, (k-1)]$ then $\binom{c}{p} = \binom{k}{1}$. If $F = [(k-1), 1, k)]$ then $\binom{c}{p} = \binom{k}{k-1}$. If $F = [5, 3, 1, (3), 3, 2]$ then $\binom{c}{p} = \binom{14}{3}$.

Notation 6.3.4. Assume that $\#E = 1$. Let f be an affine ruling of $S \setminus \Delta$. Let F be some singular fiber of f and let H be the section contained in the boundary. Put $\gamma = -E^2$, $n = -H^2$ and $d = E \cdot F$. Let h be the number of characteristic pairs of F. If $\Delta \cap F = \Delta_1 + \ldots + \Delta_k$ with Δ_k as a tip of F is the decomposition into irreducible components then the last pair of F is $\binom{c_h}{p_h} = \binom{k+1}{1}$. If $\Delta \neq \emptyset$ then $E\Delta_{i_0} = 1$ for a unique $i_0 \leq k$. Assume that F', defined as the fiber F with $\binom{c_h}{p_h}$ contracted, is produced by the pairs $(\underline{c}_i, \underline{p}_i)$ with $i = 1, \ldots, h-1$ (hence $gcd(\underline{c}_i, \underline{p}_i) = \underline{c}_{i+1}$ for $i = 1, \ldots, h-1$ and $gcd(\underline{c}_{h-1}, \underline{p}_{h-1}) = 1$). Define $c'_h = c_h - i_0$ and $\tau = c_h CE + c'_h$. Then $d = \underline{c}_1 \tau$. Notice that $c'_h = 0$ if and only if $c_h = 1$.

If f has precisely two singular fibers, we write the analogous quantities with $\widetilde{(\ )}$: $\widetilde{\tau}$, $\widetilde{C}$, $\widetilde{\underline{p}}_i$, $\widetilde{c}'_h$ etc. If f has more singular fibers then instead of C, $\underline{c}_i$, τ, etc. we write C_F, $\underline{c}_i(F)$, $\tau(F)$, etc.

Lemma 6.3.5. *With the assumptions as in 6.3.4 the following equations hold:*

$$d(n+2) + \gamma - 2 = \sum_F \tau(F)(\underline{c}_1(F) + \sum_{i=1}^{h(F)-1} \underline{p}_i(F)), \tag{6.1}$$

$$nd^2 + \gamma = \sum_F (\tau^2(F) \sum_{i=1}^{h(F)-1} \underline{c}_i(F)\underline{p}_i(F) + \tau(F)C_F E + c'_{h(F)}(F) C_F E + c'_{h(F)}(F)), \tag{6.2}$$

where the sum is taken over all singular fibers of f.

Proof. It is enough to consider one singular fiber. We first give a proof in the case $\Delta = 0$. We have $\Sigma_{S_0} = 0$. We distinguish the component of F intersecting H and contract F to a smooth 0-curve without touching H. We write this sequence of contractions as $\overline{S} = S^{(m)} \xrightarrow{\sigma_m} S^{(m-1)} \xrightarrow{\sigma_{m-1}} \ldots \xrightarrow{\sigma_1} S^{(0)}$, where $S^{(0)}$ is a Hirzebruch surface. Denote by $K^{(i)}$ and $E^{(i)}$ the canonical divisor and respectively the birational transform of E on $S^{(i)}$. For $i = 0, \ldots, m-1$ we have $K^{(i+1)}E^{(i+1)} - K^{(i)}E^{(i)} = \mu_i$ and $(E^{(i)})^2 - (E^{(i+1)})^2 = \mu_i^2$, where μ_i is the multiplicity of the center of σ_{i+1} on $E^{(i)}$. We have $E^{(0)} \equiv d(nF^{(0)} + H)$, where $F^{(0)}$ is some fiber of the induced $\mathbb{P}^1$-ruling of $S^{(0)}$ and $d = E^{(0)}F^{(0)} = EF$. We compute $K^{(m)}E^{(m)} - K^{(0)}E^{(0)} = KE + d(n+2) = \gamma - 2 + d(n+2)$ and $(E^{(0)})^2 - (E^{(m)})^2 = nd^2 + \gamma$, which gives left sides of the above equations. We need to compute $\sum \mu_i$ and $\sum \mu_i^2$. Let $F', \underline{c}_i, \underline{p}_i, \tau$ be as defined above. Since $\Delta \cap F = 0$, we have $\tau = CE$ and the sequence of characteristic pairs for F is $\binom{\underline{c}_1}{\underline{p}_1}, \ldots, \binom{\underline{c}_{h-1}}{\underline{p}_{h-1}}, \binom{1}{1}$. Let $\binom{c}{p}$ be one of these characteristic pairs and let $I(c, p)$ consist of these indices, for which the blowup σ_i is the part of the sequence of contractions determined by the characteristic pair $\binom{c}{p}$. If E intersects C transversally in one point (i.e. if $\tau = 1$) then it is easy to prove by induction on c that

$$\sum_{I(c,p)} \mu_i = c + p - gcd(c,p) \ \text{ and } \sum_{I(c,p)} \mu_i^2 = cp.$$

Now for $\tau > 1$ the multiplicity of each center is τ times bigger, hence for $CE = \tau$ we get

$$\sum_{I(c,p)} \mu_i = \tau(c + p - gcd(c,p)) \ \text{ and } \sum_{I(c,p)} \mu_i^2 = \tau^2 cp.$$

We have $c'_h = 0$ and $c_h = 1$, so this gives $\sum \mu_i = \tau \sum_{i=1}^{h}(\underline{c}_i + \underline{p}_i - gcd(\underline{c}_i, \underline{p}_i)) = \tau(\underline{c}_1 + \sum_{i=1}^{h} \underline{p}_i - 1) = \tau(\underline{c}_1 + \sum_{i=1}^{h-1} \underline{p}_i)$ and $\sum \mu_i^2 = \tau^2 \sum_{i=1}^{h} \underline{c}_i \underline{p}_i = \tau^2(\sum_{i=1}^{h-1} \underline{c}_i \underline{p}_i + 1)$, as required.

We now consider the case $\Delta \neq 0$. Let E' be the image of E after contracting F to F'. It follows from the arguments given above that

$$K^{(m)}E^{(m)} - K'E' = \tau(\underline{c}_1 + \sum_{i=1}^{h-1} \underline{p}_i - 1)$$

and

$$E'^2 - (E^{(m)})^2 = \tau^2 \sum_{i=1}^{h-1} \underline{c}_i \underline{p}_i.$$

We only need to compute $K'E' - KE$ and $E^2 - E'^2$. We are now left with the last pair $\binom{c_h}{p_h}$. The proper transform of E' after making first c'_h blowups (there is one center at each step) is $E^{(i_0)}$, where i_0 was defined by $E\Delta_{i_0} \neq 0$. The multiplicity of each of these centers is $CE + 1$, so $K'E' - K^{(i_0)}E^{(i_0)} = c'_h(CE + 1)$ and $(E^{(i_0)})^2 - E'^2 = c'_h(CE + 1)^2$. Now one has to be more careful, because $E^{(i_0)}$ can intersect the fiber in more than one point (in fact it intersects it in one point only if $i_0 = 1$ and $\Delta_1 \cap E \cap C \neq \emptyset$). One checks easily that $K^{(i_0)}E^{(i_0)} - KE = (c_h - c'_h)CE$ and $E^2 - (E^{(i_0)})^2 = (c_h - c'_h)CE^2$. This gives (6.1) and (6.2). □

Lemma 6.3.6. *If the sequence of pairs of positive integers $(c_1, p_1), (c_2, p_2), \ldots (c_h, p_h)$, such that $c_i \geq p_i$ and $gcd(c_i, p_i) = c_{i+1}$ for $i = 1, \ldots, h-1$ satisfies the equations*

$$c_1(n+1) + 1 = \sum_{i=1}^{h} p_i, \tag{6.3}$$

$$nc_1^2 = \sum_{i=1}^{h} c_i p_i. \tag{6.4}$$

then either

(i) $n = 1, h = 8, (c_1, p_1) = (4, 2), (c_2, p_2) = (2, 1)$ *or*

(ii) $n = 1, h = 7, (c_1, p_1) = (3, 1)$ *or*

(iii) $n = 2, h = 7, (c_1, p_1) = (2, 1)$.

Proof. If the sequence $(c_i, p_i)_{i=1}^{h}$ satisfies (6.3) and (6.4) together with the divisibility conditions as above then we will say that it is of type $*_n$. Multiplying the first equation by c_1 and subtracting the second one we obtain

$$c_1^2 + c_1 = \sum_{i=2}^{h} p_i(c_1 - c_i). \tag{6.5}$$

In particular $h \neq 1$. Put $c_1 = kc_2$ and $p_1 = k'c_2$. First we prove that the sequence $(c_i, p_i)_{i=1}^{h}$ of type $*_n$ satisfies one of the following:

(a) $n = 1$, $(c_1, p_1) = (kc_2, (k-1)c_2)$ for some $k, c_2 > 1$ and $(c_i, p_i)_{i=2}^{h}$ is of type $*_k$,

(b) $n=2$, $(c_1,p_1)=(2,2)$ and $(c_i,p_i)_{i=2}^h$ is of type $*_1$,

(c) $n=2$, $h=7$, $(c_1,p_1)=(2,1)$,

(d) $n=3$, $h=7$, $(c_1,p_1)=(3,1)$.

Suppose $c_2=1$. Equation (6.5) gives $k(k+1)=(k-1)(h-1)$, so $k\neq 1$ and $(k-1)|k(k+1)=(k-1)(k+2)+2$, hence $k\in\{2,3\}$ and $h=7$. It follows from (6.3) that we obtain case (c) or (d).

Suppose $c_2>1$. For $i\geq 2$ we have $c_1-c_i\geq (k-1)c_2$ and by (6.3) $\sum_{i=2}^h p_i=c_1(n+1)+1-p_1$, so equation (6.5) gives $1\geq c_2(k^2n-kn-k'k+k'-k)$ and then $k^2n-kn-k'k+k'-k\leq 0$, because $c_2>1$. If $k=k'$ then $k=c_2>1$ and since $h>1$, (6.4) implies $n>1$. In this case the inequality gives $(k-1)(n-1)\leq 1$, so $n=k=2$ and we get the case (b). We can therefore assume $k>k'\geq 1$. Writing the above inequality as $n\leq\frac{k'(k-1)+k}{k(k-1)}<1+\frac{1}{k-1}$ we see that $n=1$ and then $(k-1)(k-k'-1)\leq 1$, hence $k'=k-1$. One checks easily that this gives case (a).

Now it is easy to see that in fact case (b) cannot occur. Indeed, since in this case $(c_i,p_i)_{i=2}^h$ is of type $*_1$, then (c_2,p_2) can be only as in (a) (with respective renumbering), i.e. $(c_2,p_2)=(kc_3,(k-1)c_3)$ for some $k,c_3>1$, in particular $c_2=kc_3\geq 4$, a contradiction. Notice also that if (c_1,p_1) is as in (a) then $k>1$, so after renumbering (c_2,p_2) is as in (b) or (c). □

Lemma 6.3.7. *If $\#E=1$ then any affine ruling of $S\setminus\Delta$ has more than one singular fiber.*

Proof. Let $f:(\overline{S}^\dagger,D^\dagger)\to\mathbb{P}^1$ be some affine ruling of $S\setminus\Delta$ with one singular fiber F. Notice that $\tau>1$, otherwise the (-1)-curve of F, which is not touched when minimalizing $D^\dagger$ to D, would be simple on $(\overline{S},D)$. Using 6.3.5 we get

$$d(n+1)+\gamma-2=\tau\sum_{i=1}^{h-1}\underline{p}_i, \tag{6.6}$$

$$nd^2+\gamma=\tau^2\sum_{i=1}^{h-1}\underline{p}_i\underline{c}_i+\tau CE+c_h'CE+c_h'. \tag{6.7}$$

Computing the difference of the above equations modulo τ we see that $\tau|c_h'CE+c_h'-2$. Notice that if $c_h'\neq 0$ then $c_h'CE+c_h'=2$. Indeed, if $c_h'\neq 0$ then $c_h'CE+c_h'-2\geq 0$ and $c_h'CE+c_h'-2$ cannot be positive, otherwise $c_h'CE+c_h'-2\geq\tau=c_hCE+c_h'\geq c_h'CE+c_h'$, a contradiction. Therefore there are two cases to consider: (i) $c_h'CE+c_h'=2$ and (ii) $c_h'=0$. We show that both lead to equations

$$\underline{c}_1(n+1)+1=\sum_{i=1}^{h-1}\underline{p}_i,$$

$$n\underline{c}_1^2=\sum_{i=1}^{h-1}\underline{p}_i\underline{c}_i.$$

Suppose $c_h'CE+c_h'=2$. Then $CE=c_h'=1$, so $\tau=c_h+1$. Taking (6.7) modulo τ^2 we have $\tau^2|\gamma-2-\tau$, hence $\tau|\gamma-2$. If $\tau\neq\gamma-2$ then $\tau^2\leq\gamma-2-\tau\leq 5-\tau$ by 6.2.2, which contradicts $\tau>1$. Thus $\tau=\gamma-2$ and we are done. Now suppose $\Delta=\emptyset$. We have $c_h=1$ and taking (6.6) modulo τ and (6.7) modulo τ^2 we have $\tau|\gamma-2$ and $\tau^2|\gamma$, hence $\tau=2$ and $\gamma=4$ by 6.2.2. Thus again we get the above equations.

Using 6.3.6 we check that all three sequences of characteristic pairs satisfying these equations give rise to the same boundary D, which is a fork with branching (-2)-curve and maximal twigs $T_1=[2]$, $T_2=[2,2]$ and $T_3=[c_h+1,(5)]$. We compute $d(D)=-1$, a contradiction with 2.1.3(ix). □

Remark. If f has only one singular fiber F then $S\setminus F\cong\mathbb{C}^1\times\mathbb{C}^1$, so $\pi_1(S')=\pi_1(S)=0$ and by 2.1.3(vii) and Whitehead's theorem S' is contractible. Now the final result of [KR07] excludes contractible S' satisfying $\overline{\kappa}(S')=-\infty$ and $\overline{\kappa}(S_0)=2$, so by referring to it we could omit the proof of 6.3.7. However, the above independent arguments will allow us to obtain [KR07, Theorem 1.1(i)] as a special case (cf. 6.6.5).

Definition 6.3.8. Let $\pi:X\to C$ be a dominating morphism of a smooth surface to a smooth complete curve C. We say that π is *pre-minimal* if for some snc-completion $(\overline{X},\overline{X}\setminus X)$ it has an extension $\overline{\pi}:\overline{X}\to C$, such that the boundary divisor $\overline{X}\setminus X$ can be made snc-minimal using only subdivisional blowdowns. Then we will say also that $\overline{\pi}:(\overline{X},\overline{X}\setminus X)\to C$ is pre-minimal.

We now proceed to show that in some situations the affine ruling of $S \setminus \Delta$ can be chosen pre-minimal. We adapt a lemma [KR99, 5.3] to our situation. We follow the original notation.

Notation 6.3.9. Assume $\#E = 1$. Let $f : (\overline{S}^\dagger, D^\dagger + \Delta) \to \mathbb{P}^1$ be some affine ruling of $S \setminus \Delta$ with $D^\dagger$ being f-minimal (*good* affine ruling of S, using the terminology of [KR99]). We have $\Sigma_{S_0} = 0$ because $\#E = 1$. Let $H^2 = -n$, where H is the horizontal component of $D^\dagger$. If $\beta_{D^\dagger}(H) > 2$ then $(\overline{S}^\dagger, D^\dagger) = (\overline{S}, D)$ and the ruling is pre-minimal. Assume $\beta_{D^\dagger}(H) \leq 2$. If $n = 1$ then $D^\dagger$ is not snc-minimal. In any case by successive contractions of exceptional curves in $D^\dagger$ we obtain a morphism $\varphi_f : \overline{S}^\dagger \to \overline{S}$. Let F be a singular fiber of f, such that $F \cap D^\dagger$ is branched. Denote the component of F meeting H by G. Let $G + Z$ be the first branch of F and let Z_1 be the unique curve of highest multiplicity in Z. Let Z_u and Z_l (upper, lower) be the connected components of $Z - Z_1$ with Z_u meeting G (see Fig. 6.1). Let Z_{lu} be the component of Z_l meeting Z_1 and C the unique (-1)-curve of F. Let h be the number of sprouting blowups needed to produce F from a smooth 0-curve (number of characteristic pairs of F) and μ the multiplicity of C. If there is another singular fiber denote it by $\widetilde{F}$. Analogously for $\widetilde{F}$ define $\widetilde{G}, \widetilde{Z}_1, \widetilde{h}$, etc. Put $H^\dagger = Z_u + G + H + \widetilde{G} + \widetilde{Z}_u$. Define $\Delta' = \Delta \cap F$ and $\widetilde{\Delta} = \Delta \cap \widetilde{F}$. We introduce the following modification of definition [KR99, 5.1]:

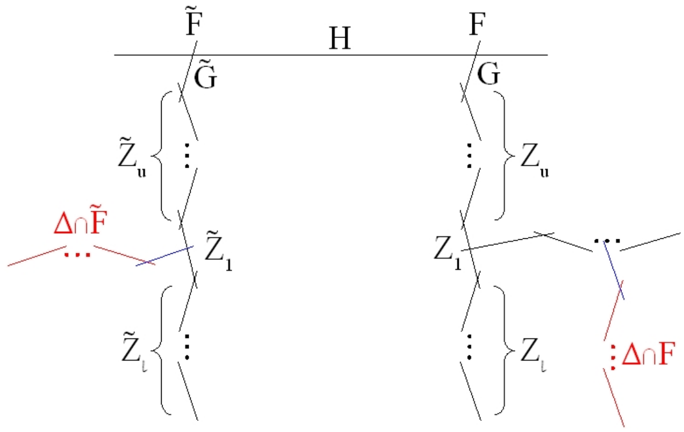


Figure 6.1: pre-minimal ruling

Definition 6.3.10. In the situation as above f is *almost minimal* if $D^\dagger$ is snc-minimal (i.e. $\varphi_f = id$) or there are exactly two singular fibers and contractions in φ_f do not touch their (-1)-curves.

Remark. If f has more than two singular fibers then $\beta_{D^\dagger}(H) > 2$ because each singular fiber contains some $D^\dagger$-components, hence $D^\dagger = D$ is snc-minimal and f is almost minimal. If f has only one singular fiber then it is almost minimal if and only if $n \neq 1$. Assume that f is almost minimal with two singular fibers. Then it follows from the definition that the contractions in φ_f take place within $H^\dagger$. Moreover, if $\widetilde{Z}_1 = C_1$ (this could not happen in [KR99]) then they are subdivisional with respect to $D^\dagger$. It follows that an almost minimal ruling is pre-minimal.

Lemma 6.3.11. *(Koras-Russell) Let C be a (-1)-curve in $\overline{S}$, such that $\kappa(K_{\overline{S}}+D+\Delta+C) = -\infty$. Then there exists a pre-minimal affine ruling of $S \setminus \Delta$ with C in a fiber, such that either*

(i) f is almost minimal or

(ii) f has exactly two singular fibers, $\widetilde{\Delta} = 0$ and φ_f contracts precisely $H^\dagger + \widetilde{Z}_1$. If Z_1 is touched x times in this process then $x \geq 4$ and $V^2 = 2 - x$, where $V \subseteq D$ is the birational transform of $\widetilde{Z}_{lu}$.

Remark. The lemma implies that we have a good control over the curves that are contracted when minimalizing the boundary. Notice that in case (ii) both fibers are branched and the second branch of $\widetilde{F}$ contains a (-1)-curve only.

The above lemma is essentially the lemma [KR99, 5.3]. We sketch the way the original arguments have to be modified if necessary. We write the references to numbering of [KR99] in square brackets.

Proof. The starting point is an affine ruling f of $S \setminus (\Delta \cup C)$. Notice that $CD > 0$, hence non-existence of such a ruling would imply that Δ contains a fork, which contradicts 6.2.3(iii). We can assume that f is not almost minimal, in particular $D^\dagger \neq D$. Since every singular fiber contains some $D^\dagger$-component, f has at most two singular fibers, by 6.3.7 it has precisely two. The idea is to improve f. As for the preliminary results used, the proofs of [4.2] and [5.2.2] go without modifications. The calculations in terms of characteristic pairs as [3.7] or [5.3.3](i) do not hold in our situation, but they can be ignored. If the improvement of f is found using [5.3.4] then it is almost minimal in the sense of 6.3.10. Therefore in [Case I] only the subcase (α), where the improvement is produced in other way, needs some care. Fortunately, the proof goes without modifications, giving part (ii) of the thesis. In cases [II(a),(b),(c)] the produced improvement has $D^\dagger = D$, so is almost minimal. Thus we are left with [Case II(d)]. If $\widetilde{F}$ is branched then the original proof works. Suppose $\widetilde{F}$ is a chain. Then $\widetilde{F} = D_0 + C + \widetilde{\Delta}$ with $C^2 = -1$ and $D_0 \subset D^\dagger$. Since G is not contracted by φ_f, D_0 cannot be contracted because T_0 is not a tip of T by the assumptions [Case II(d)]. □

Corollary 6.3.12. *If $\#E = 1$ then the affine ruling of $S \setminus \Delta$ can be chosen pre-minimal, exactly as in 6.3.11.*

Proof. Take an f-minimal completion of some affine ruling f of $S \setminus \Delta$. Since at least one of the branching components of $D^\dagger$ remains branching in D, there exists a vertical (-1)-curve, it is an S_0-component. Take it as C and apply 6.3.11. □

Corollary 6.3.13. *Let $\#E = 1$ and let f be a pre-minimal affine ruling of $S \setminus \Delta$ which has two singular fibers. One has:*

(i) $h + \widetilde{h} = n + 1 + \epsilon + EK$,

(ii) $d(D) = -d(\widehat{E}) \cdot gcd(\widetilde{\mu}, \mu)^2$.

Proof. (i) Since f is pre-minimal, contractions in φ_f are subdivisional with respect to $D^\dagger$, hence $K_{\overline{S}^\dagger}(K_{\overline{S}^\dagger} + D^\dagger) = K(K + D) = 3 - \epsilon - EK$. Contract singular fibers to smooth fibers without touching H, denote the image of D by $\widetilde{D}$ and the resulting surface by $\overline{S}$. Each sprouting blowdown in $D^\dagger$ increases $K(K + D)$ by one. At the end we have $K_{\widetilde{S}}(K_{\widetilde{S}} + \widetilde{D}) = 8 - 4 + n - 2 = n + 2$, so we get $K(K + D) + h - 1 + \widetilde{h} - 1 = n + 2$, hence $h + \widetilde{h} = n + 1 + \epsilon + EK$.

(ii) We have $\pi_1(S') = \langle \sigma_1, \sigma_2 : \sigma_1^\mu = \sigma_2^{\widetilde{\mu}} = \sigma_1\sigma_2 = 1 \rangle = \mathbb{Z}_{gcd(\widetilde{\mu},\mu)}$, so (ii) follows from 2.1.3(ix). □

6.4 D is a fork

Lemma 6.4.1. *If $\epsilon = 2$ then $KE = 1$.*

Proof. Suppose $\epsilon = 2$ and $KE \neq 1$, then $\widehat{E} = [4]$ by 6.2.2. Let $f : (\overline{S}^\dagger, D^\dagger) \to \mathbb{P}^1$ be a pre-minimal affine ruling (we use the notation of 6.3.9). Let $F_1, \ldots F_N$ be the singular fibers and let $U = D_h + \underline{F}_1 + \ldots + \underline{F}_N$, where D_h is the horizontal component of $D^\dagger$. We have $\Sigma_S = 0$ and by 6.3.7 $N \geq 2$. Suppose $N > 2$. Then $D^\dagger = D$. Let h_i be the numbers of sprouting blowups needed to produce F_i from a smooth 0-curve. If we contract all F_i's to smooth fibers without touching D_h we make $h_1 + h_2 + \ldots + h_N$ sprouting blowdowns inside U. We have $K(K + U) = K(K + D) - N$, so we get that $-1 - N + h_1 + \ldots + h_N = 8 - 2N$, because $K^2 = 8$ for a Hirzebruch surface and $KD_h = 0$ by 6.2.1. Notice that $h_i \neq 1$ because $\Delta = \emptyset$. We get $N = 3$ and $h_1 = h_2 = h_3 = 2$, hence $s = 3$ and since D consists of (-2)-curves by 6.2.1, maximal twigs of D are equal to $[2, 2, 2]$. We compute $\pi_1(S') = \langle \sigma_1, \sigma_2, \sigma_3 : \sigma_1\sigma_2\sigma_3 = 1, \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1 \rangle = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. However, $d(D) = -16$ and $d(\widehat{E}) = 4$, so $H_1(S', \mathbb{Z}) = \mathbb{Z}_2$ by 2.1.3(ix), a contradiction. Thus $N = 2$. Put $F = F_1$, $\widetilde{F} = F_2$ and $h = h_1$, $\widetilde{h} = h_2$. We have $h + \widetilde{h} = 5 + n$ and $h, \widetilde{h} \neq 1$.

Suppose f is not almost minimal. Then $\widetilde{h} = 2$, so $h = 4$. By 6.3.11 $\varphi_f : \overline{S}^\dagger \to \overline{S}$ contracts precisely $H^\dagger + \widetilde{Z}_1$ and Z_1 is touched exactly four times, hence $Z_1^2 = -6$. D consists of (-2)-curves, so it follows that the second branch of F is $[(5)]$ and the third is $[2, 1]$. We have also $Z_l = [(k)]$ and $\widetilde{Z}_l = [(m), -2 - p]$ for some non-negative integers k, m and p, hence $G = [k + 1]$ and $\widetilde{G} = [m + 2]$. If $k \neq 1$ then the chain $\widetilde{G}$ is contracted before G, so $m = 0$ and we see that Z_1 is touched at most once, a contradiction. Therefore

$k=1$ and we get $m=1$. We see that Z_{lu} is touched once by φ_f, so $p=1$. Therefore D has two branching components, B_1 and B_2, and $D-B_1-B_2=T_1+T_2+T_3+T_4$, where $T_1B_1=T_2B_1=1$, $T_1=[2,2]$, $T_2=[2]$, $T_3=[2]$ and $T_4=[2,2,2,2]$. We compute $d(D)=-25$, which is a contradiction by 2.1.3(ix). Thus f is almost minimal.

We have now $Z_l=[(k)]$ and $\widetilde{Z}_l=[(p)]$ for some positive integers k,p, so $Z_u=\widetilde{Z}_u=\emptyset$, $\widetilde{G}=[p+1]$ and $G=[k+1]$. We can assume that $h\geq\widetilde{h}$. Suppose $n=1$. Then $(\widetilde{h},h)=(2,4)$ or $(\widetilde{h},h)=(3,3)$. Consider the case $(\widetilde{h},h)=(2,4)$. Notice that $\widetilde{Z}_1^2=-2$, so $\widetilde{G}$ is not contracted by φ_f, hence $p>1$. If $k\neq 1$ then φ_f contracts only H, so $p=k=2$ and the second branch of F is $[2,2,1]$. In this case $d(D)=-9$, a contradiction with 2.1.3(ix). Therefore $k=1$. We get $p=3$ and $Z_1^2=-3$ and we infer that the second branch of F is $[2,2]$ and the third is $[1,2]$. Thus D has two branching components, B_1 and B_2, and $D-B_1-B_2=T_1+T_2+T_3+T_4$ with $T_1=[(5)]$, $T_2=[2]$, $T_3=[2]$ and $T_4=[2]$. We get $d(D)=-16$ and $gcd(\widetilde{\mu},\mu)=4$, a contradiction with 6.3.13(ii). Consider the case $(\widetilde{h},h)=(3,3)$. We can assume $k\geq p$. If $p=1$ and $k=2$ then the second branch of $\widetilde{F}$ is $[2,2,2]$ and the second branch of F is $[2,2]$, $gcd(\widetilde{\mu},\mu)=6$ and $d(D)=-36$, a contradiction with 6.3.13(ii). If $p=1$ and $k=3$ then the second branch of $\widetilde{F}$ is $[2,2]$ and the second branch of F is $[1,2]$, $gcd(\widetilde{\mu},\mu)=4$ and $d(D)=-16$, a contradiction with 6.3.13(ii). It follows that $p=k=2$. Then the second branches of $\widetilde{F}$ and F are equal to $[1,2]$, so $d(D)=-9$, again a contradiction with 6.3.13(ii).

We have now $n=2$, so $(\widetilde{h},h)=(2,5)$ or $(\widetilde{h},h)=(3,4)$. Now Z_l, $\widetilde{Z}_l$, G and $\widetilde{G}$ are irreducible (-2)-curves. If $(\widetilde{h},h)=(2,5)$ then $gcd(\widetilde{\mu},\mu)=2$ and the second branch of F is $[1,2,2,2]$, hence $d(D)=-4$. If $(\widetilde{h},h)=(3,4)$ then $gcd(\widetilde{\mu},\mu)=2$, the second branch of $\widetilde{F}$ is $[2,1]$ and the second branch of F is $[1,2,2]$, so $d(D)=-4$. In both cases we get a contradiction with 6.3.13(ii).

□

Lemma 6.4.2. *If $\#E=1$, $\#\Delta\leq 1$ and no maximal twig of D containing more than one component contains a (-2)-tip then $(\overline{S},D+\Delta)$ is affine ruled. If additionally $s=4$ then not all maximal twigs of D are tips.*

Proof. Let $f:(\overline{S}^\dagger,D^\dagger+\Delta)\to\mathbb{P}^1$ be a pre-minimal affine ruling. Suppose $D^\dagger\neq D$. Then f has two singular fibers, F and $\widetilde{F}$, and $n=1$ (cf. 6.3.9). Clearly, Z_l and Z_u are adjoint admissible chains. The components of Z_l are not contracted by φ_f by 6.3.11(ii). If $Z_l\subseteq\Delta$ then Z_l is irreducible, because $\#\Delta\leq 1$. By our assumption about maximal twigs of D if $Z_l\subseteq D^\dagger$ and Z_l is not irreducible then it has a $\leq(-3)$-curve as a tip. In any case it implies that the component of F intersecting H is a (-2)-curve. Analogous argument holds for $\widetilde{F}$, hence H meets two (-2)-curves in $D^\dagger$. Therefore D contains a non-branching component with non-negative self-intersection, a contradiction with 6.1.2(ix).

Suppose that $s=4$ and all maximal twigs of D are tips. Then $D^\dagger=D$ by the first part of the above lemma. If $\beta_D(H)\leq 2$ then there are two branching components in D, otherwise the maximal twig containing H would not be a tip. Then by 3.1.5 one of them is a (-1)-curve. However, branched (-1)-curve cannot be a component of a fiber, a contradiction. Thus H is a branching component of D and there are more than two singular fibers. At least two of them do not contain a branching component of D, hence contain unique D-components by our assumption. This implies that each of these two fibers contains a component of Δ, a contradiction with $\#\Delta\leq 1$.

□

Proposition 6.4.3. *D is a fork.*

Proof. Suppose D is not a fork. We will prove that $\widehat{E}=[5]$, $\epsilon=1$ and $s=4$ and then we will eliminate this case in several steps. We prove successive statements.

(1) $\#E=1$ and $\epsilon\neq 0$.

Proof. We have $\epsilon\neq 0$ by 6.2.3(i). To prove $\#E=1$ we can assume $\epsilon\neq 2$ by 6.4.1. Thus $\epsilon=1$, $\widehat{E}$ is a chain by 6.2.3(ii) and it satisfies $(s-4)|G|\leq 7-d'(\widehat{E})-d'(\widehat{E}^t)$ by 6.1.4(iii). Using $2\leq KE\leq 3$ this gives only two cases for which $\#E\neq 1$: $s=4$ and $\widehat{E}=[3,3]$ or $s=4$ and $\widehat{E}=[3,4]$. By 6.1.4(i) in both cases $e+\delta<3$, which contradicts 6.1.4(iv). □

(2) If $K(K+D)\neq 0$ then $\widehat{E}=[5]$, $\epsilon=1$ and $s=4$.

Proof. Assume $K(K+D)\neq 0$. For $\epsilon=2$ we have $K(K+D)=3-\epsilon-EK=0$ by 6.4.1, so $\epsilon=1$ by (1). Then $KE=3$, so by 6.1.4(iii) $s=4$ and $\widehat{E}=[2,5]$ or $s\leq 5$ and $\widehat{E}=[5]$. In the first case we have $e=\delta=\frac{4}{3}$ by 6.1.4, so maximal twigs of D are tips, a contradiction with 6.4.2. Suppose $s=5$ in the second case. Then $e+\delta=\frac{18}{5}<\frac{17}{4}$, which is impossible by 6.1.4(iv). □

We choose a pre-minimal affine ruling $\pi:(\overline{S}^{\dagger},D^{\dagger})\to C$. Subdivisional modifications of D do not change $K(K+D)$, so $K^{\dagger}(K^{\dagger}+D^{\dagger})=K(K+D)$, where $K^{\dagger}=K_{\overline{S}^{\dagger}}$. According to 6.3.7 π has at least two singular fibers. For some computations below it is useful to recall that if σ is a blowup of a smooth complete surface and σ', σ^* denote respectively the proper and the full preimages then for any two divisors A,B one has $A\cdot B=\sigma'A\cdot\sigma^*B$.

(3) If $D^{\dagger}\cap F$ is not a chain for some fiber F of π then $K(K+D)\neq 0$.

Proof. Suppose $F\cap D^{\dagger}$ is branched and $K(K+D)=0$. Write F as $F=F\cap D^{\dagger}+C+\Delta_1$, where C is a (-1)-curve, and $\Delta_1\subset\Delta$. We contract the chain $C+\Delta_1$ and successive (-1)-curves in F as long as they are subdivisional for $D^{\dagger}$. Denote the images of $D^{\dagger}$, E and F by $D^{(1)}$, $E^{(1)}$ and $F^{(1)}$. Let $K^{(1)}$ be the canonical divisor of the image of $\overline{S}$. In general, if after some sequence of contractions we define $D^{(i)}$ then we denote the appropriate images of E, F, etc. by $E^{(i)}$, $F^{(i)}$ etc. The contraction of $C+\Delta_1$ and contractions subdivisional with respect to the image of $D^{\dagger}$ do not change $K^{\dagger}(K^{\dagger}+D^{\dagger})$ and $E(K^{\dagger}+D^{\dagger})$, i.e. $K^{(1)}(K^{(1)}+D^{(1)})=K(K+D)=0$ and $E^{(1)}(K^{(1)}+D^{(1)})=E(K+D)=EK$. Moreover, $D^{(1)}$ has the same number of branching components as D, so $D^{(1)}$ is branched.

Let $D^{(1)}_{\alpha}$ be the (-1)-tip of $D^{(1)}$, and let $D^{(2)}$ be the image of $D^{(1)}$ after the contraction of $D^{(1)}_{\alpha}$. Let $D^{(1)}_{\beta}$ be the unique $D^{(1)}$-component intersecting $D^{(1)}_{\alpha}$. We have $h^0(-K^{(2)}-D^{(2)})+h^0(2K^{(2)}+D^{(2)})\geq K^{(2)}(K^{(2)}+D^{(2)})=1$, so $-K^{(2)}-D^{(2)}\geq 0$, otherwise $2(K^{(2)}+D^{(2)})\geq 0$, which is impossible, since $\kappa(K^{(2)}+D^{(2)})=-\infty$. For every component V of $D^{(2)}$ we have $V(-K^{(2)}-D^{(2)})=2-\beta_{D^{(2)}}(V)$. Since $s\geq 4$, $D^{(2)}$ is branched and every branching curve of $D^{(2)}$, and hence every component of $D^{(2)}$ which is not a tip, is in the fixed part of $-K^{(2)}-D^{(2)}$. Suppose $D^{(2)}_{\beta}$ is not a tip of $D^{(2)}$, then $-K^{(2)}-D^{(2)}-D^{(2)}_{\beta}\geq 0$, so $-K^{(1)}-D^{(1)}-D^{(1)}_{\beta}\geq 0$. Clearly, $E^{(1)}$ is in the fixed part of $-K^{(1)}-D^{(1)}-D^{(1)}_{\beta}$, so $-K^{(1)}-D^{(1)}-E^{(1)}\geq 0$. It follows that $-(K^{\dagger}+D^{\dagger}+E)\geq 0$, a contradiction with $\kappa(K^{\dagger}+D^{\dagger}+E)=2$. Thus $D^{(2)}_{\beta}$ is a tip of of $D^{(2)}$.

Let $D^{(3)}$ be the image of $D^{(2)}$ after the contraction of $D^{(2)}_{\beta}$. Since $D^{(2)}_{\beta}$ is a tip, $D^{(2)}$ has the same number of branching components as $D^{(1)}$ (greater than one by our assumptions about D), hence $D^{(3)}$ is not a chain. Moreover, $F^{(3)}$ is not a 0-curve, as the branching components of $D^{\dagger}\cap F$ have not been contracted. We made two sprouting blowdowns, so $K^{(3)}(K^{(3)}+D^{(3)})=K^{(1)}(K^{(1)}+D^{(1)})+2=K(K+D)+2=2$. Riemann-Roch's theorem gives $h^0(-K^{(3)}-D^{(3)})\geq 2$. Since π has at least two singular fibers, we have $\beta_D(H)>1$. Since $D^{(3)}$ is connected and is not a chain, H is in a fixed part of $-K^{(3)}-D^{(3)}$. Let's write $-K^{(3)}-D^{(3)}=H+R+\sum_{i=1}^{f}A_i$, where $H+R$ is a fixed part, $f>0$ and $A_i^2\geq 0$ (cf. 5.1.2(2)). Intersecting with a generic fiber F' we have $1=1+F'R+F'\sum_{i=1}^{f}A_i$, hence $F'A_i=0$ and $F'R=0$, so R is vertical and $A_i\sim F'$ for each i. We get that $K^{(3)}+D^{(3)}+H+fF'+R\sim 0$. Intersecting with $E^{(3)}$ we get $0\geq E^{(3)}(K^{(3)}+D^{(3)}+F')=E^{(2)}(K^{(2)}+D^{(2)}-D^{(2)}_{\alpha}+F')=E^{(1)}(K^{(1)}+D^{(1)})+E^{(1)}(F'-2D^{(1)}_{\alpha}-D^{(1)}_{\beta})=EK+E^{(1)}(F^{(1)}_0-2D^{(1)}_{\alpha}-D^{(1)}_{\beta})$, which implies $E^{(1)}(F^{(1)}-2D^{(1)}_{\alpha}-D^{(1)}_{\beta})<0$. This is a contradiction, because $F^{(1)}$ is branched, so the multiplicities of $D^{(1)}_{\alpha}$ and $D^{(1)}_{\beta}$ in it are greater than one. □

(4) $\widehat{E}=[5]$, $\epsilon=1$ and $s=4$.

Proof. Suppose (4) does not hold. Then by (2) and (3) H is the only branching curve in $D^{\dagger}$, so $D^{\dagger}=D$, every singular fiber F of π has at most one branching component and $F\cap D$ is a chain. In particular, there are exactly s singular fibers. Let c be the number of singular fibers which are chains. If F is such a fiber then $F\cap\Delta\neq\emptyset$ and $F\cap D$ is a tip, so $\widetilde{e}(F\cap D)\leq\frac{1}{2}$. Since $s\geq 4$ and Δ has at most three connected components, we see that $c<s$, so we have an inequality $\widetilde{e}(D)<(s-c)+\frac{c}{2}=s-\frac{c}{2}$. Let's contract all singular fibers to smooth 0-curves without touching H. The contraction of chain fibers does not affect $K(K+D)$ and the contraction of any other singular fiber increases $K(K+D)$ by one, so if $\widetilde{D}$ and $\widetilde{S}$ are the images of $D^{\dagger}$ and $\overline{S}^{\dagger}$ after contraction then $\widetilde{D}\equiv H+sF'$ for a generic fiber F' and $K_{\widetilde{S}}(K_{\widetilde{S}}+\widetilde{D})=s-c$. Putting $n=-H^2$ we get $s-c=K_{\widetilde{S}}(K_{\widetilde{S}}+\widetilde{D})=8+n-2-2s$, so $n=3s-c-6$.

Since $0 > d(D) = d_1 \dots d_s(n - \widetilde{e}(D))$ we get $s - \frac{c}{2} > \widetilde{e}(D) > 3s - c - 6$, so $12 \geq 6s - c > 3s$. Hence $s \leq 3$, a contradiction. □

Denote the set of irreducible components of a divisor W by $\mathcal{C}(W)$. We notice the following fact (recall that T is the sum of maximal twigs of D, cf. 6.1.3):

(5) If $R \subseteq D$ is a $\leq (-4)$-tip of D then $\sum_{V \in \mathcal{C}(T)} V(2K + R) \leq 1$ and each $V \in \mathcal{C}(T)$ satisfies $V(2K + R) \geq 0$.

Proof. Let m be a maximal natural number, such that $E + m(K + D) \geq 0$. It is greater than one by (4) and 6.1.2(vii). By (5) and (6) of 5.1.2 we can write $E + m(K + D) = \sum C_i$, where $C_i^2 < 0$. Multiplying both sides by $E + 2K + R$ we have $EK - 2 + m(4 - 2\epsilon - EK + R(D - R)) = \sum_i C_i(E + 2K + R)$, so $\sum_i C_i(E + 2K + R) = 1$ by (4). Suppose that $C_j(E + 2K + R) < 0$ for some j. Then $C_jK \geq 0$. Indeed, if $C_jK < 0$, then $C_j^2 = -1$ and $C_j(E + R) \leq 1$. Simultaneously $|K + D + C_j| = \emptyset$ by the definition of m, so either C_j is simple or it is a non-branching component of D, a contradiction. We get that $C_j = R$ and $KR - 2 = R(2K + R) < 0$, which is impossible by our assumption on R. Therefore $C_i(E + 2K + R) \geq 0$ for each i. If V is a component of T then $V(E + n(K + D)) = n(\beta_D(V) - 2)$, so tips of D, and hence all components of T, appear among C_i's and we are done. □

(6) There are no $\leq (-4)$-tips in D.

Proof. Suppose T_1 contains a ≤ -4-tip of D, denote it by R. By (5) we have $1 \geq \sum_{V \in \mathcal{C}(T)} V(2K + R)$. We have $0 \leq V(2K + R) \leq 1$ for every $V \in \mathcal{C}(T)$, so $T - R$ consists of (-2)-curves and $-5 \leq R^2 \leq -4$. Maximal twigs of D other than T_1 are tips, otherwise $e \geq \frac{1}{5} + \frac{1}{2} + \frac{1}{2} + \frac{2}{3} > \frac{9}{5}$, a contradiction with 6.1.4(i). If $R^2 = -5$ then $V(2K + R) = 0$ for every $V \in \mathcal{C}(T - R)$, so R is a maximal twig, a contradiction with 6.4.2. Thus $T_1 = [4, (k - 1)]$ for some positive integer k, hence by 6.1.4(i) $\frac{9}{5} \geq e = \frac{3}{2} + \frac{1}{3 + 1/k}$, so $k \leq 3$. By 6.4.2 there is an affine ruling f of $(\overline{S}, D)$. For every singular fiber F the divisor $F \cap D$ is branched, otherwise the maximal twig containing $D \cap F$ has more than three components, a contradiction. Thus by 6.3.7 f has two singular fibers and we have $h + \widetilde{h} = n + 5$ by 6.3.13(i). This implies that one of h or $\widetilde{h}$, say h, is at least 4, so the second branch of respective singular fiber F contains at least two D-components, hence includes T_1. Let L be the unique S_0-component of F. Now $T_1 + L$ should contract to a point. This is possible only for $k > 3$, a contradiction. □

(7) Maximal twigs of D are $[2]$, $[2]$, $[3]$ and $[3, 2]$.

Proof. We assume that $d_1 \leq d_2 \leq d_3 \leq d_4$. By 6.1.4(i) and (iv) we have $e \geq \frac{9}{5}$ and $\delta \geq \frac{13}{4} - e \geq \frac{13}{4} - \frac{9}{5} = \frac{29}{20}$, so $d_1 = 2$ and $2 \leq d_2 \leq 3$. If $d_2 = 3$ then the lower bound on δ gives $d_3 = d_4 = 3$, and since by 6.4.2 not all maximal twigs are tips, $e \geq \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{2}{3} > \frac{9}{5}$, a contradiction. Thus $d_2 = 2$ and we have $\frac{1}{d_3} + \frac{1}{d_4} \geq \frac{9}{20}$, so $d_3 \leq 4$. Since there are no (-4)-tips in D by (6), for $d_3 = 4$ we have $e \geq 1 + \frac{3}{4} + \frac{1}{4} > \frac{9}{5}$, which is impossible, hence $d_3 \leq 3$. T_3 is a (-3)-tip, otherwise $e \geq \frac{3}{2} + \frac{1}{3} > \frac{9}{5}$. We get $d_4 \leq 8$ and $e_4 \leq \frac{9}{5} - \frac{4}{3} < \frac{1}{2}$, so T_4 contains a (-3)-tip, hence $T_4 = [3, 3]$ or $T_4 = [3, (k)]$ for some $k \in \{0, 1, 2\}$. Only $T_4 = [3]$ and $T_4 = [3, 2]$ satisfy 6.1.4(iv), so other cases are excluded. The case $T_4 = [3]$ is excluded by 6.4.2. □

Now we see by 6.4.2 that there is an affine ruling f of $(\overline{S}, D)$. Exactly as in (6) we obtain that f has two singular fibers and the second branch of one of them consists of an S_0-component L and all components of T_4. Now again $T_4 + L$ should contract to a point, and we obtain a contradiction by checking that for $T_4 = [3, 2]$ this is impossible. □

Lemma 6.4.4. *Let $\mathcal{P} \equiv (K + D + \widehat{E})^+$ and let B be the branching component of D. Put $b = -B^2$. Then:*

(i) $b \in \{1, 2\}$ *and* $b < \widetilde{e}$,

(ii) $\delta < 1$,

(iii) $\mathcal{P} \equiv \frac{1 - \delta}{\widetilde{e} - b}(B + \sum_{i=1}^{3} \operatorname{Bk} T_i^t)$,

(iv) $\operatorname{Bk}^2 \widehat{E} = -\frac{(1 - \delta)^2}{\widetilde{e} - b} + e - 1 - \epsilon$.

Proof. (i) $0 > d(D) = d_1 d_2 d_3 (b - \widetilde{e}) \geq b - \widetilde{e}$ by 1.1.1(i) and 2.2.3(ii). Now $\widetilde{e}_i < 1$, so $b < \widetilde{e} < 3$ and we get $b \in \{1, 2\}$ by 6.1.2(ix).

(ii) $\mathcal{P}V = 0$ for every component V of $T + \widehat{E}$, because $T + \widehat{E} \subset (B + D + \widehat{E})^-$. Components of $D + \widehat{E}$ generate $NS(\overline{S}) \otimes \mathbb{Q}$, so $\mathcal{P}B \neq 0$, otherwise $\mathcal{P} \equiv 0$ and hence $\overline{\kappa}(S_0)$ would be smaller than two. We infer that $0 < B\mathcal{P} = B(K + D - \operatorname{Bk} D) = 1 - \delta$.

(iii) Both $\mathcal{P}$ and $B + \sum_{i=1}^{3} \operatorname{Bk} T_i^t$ intersect trivially with all components of $T + \widehat{E}$, so they are linearly dependent in $NS(\overline{S}) \otimes \mathbb{Q}$, moreover $\mathcal{P}B = 1 - \delta$ and $(B + \sum_{i=1}^{3} \operatorname{Bk} T_i^t)B = \widetilde{e} - b$.

(iv) We compute $\mathcal{P}^2 = (1 - \delta)^2/(\widetilde{e} - b)^2 (B^2 + \sum_{i=1}^{3} \widetilde{e}_i) = (1 - \delta)^2/(\widetilde{e} - b)$, so now (iv) follows from 6.1.2(iii).

□

Remark 6.4.5. If KT is bounded (for example this is the case when we can bound the determinants d_1, d_2, d_3) then there is only finitely many possibilities for the dual graphs of D and $\widehat{E}$. Indeed, by 6.2.1 $KE + \epsilon \leq 5$ and by 6.4.4(i) $b \in \{1, 2\}$. Now it is enough to bound $\#\widehat{E} + \#D$, and this is done using 6.1.2(vi).

Lemma 6.4.6. *If $b = \#E = 1$ then any affine ruling of $S \setminus \Delta$ has two singular fibers.*

Proof. Let $f : (\overline{S}^{\dagger}, D^{\dagger} + \Delta) \to \mathbb{P}^1$ be an affine ruling of $S \setminus \Delta$. We have $\Sigma_{S_0} = 0$, because $\#E = 1$. By 6.3.7 f has more than one singular fiber. Suppose it has more than two singular fibers. Clearly, each fiber contains some D-components, so we infer that $D^{\dagger} = D$, B is horizontal and f has three singular fibers F_1, F_2, F_3. Let L_i and Δ_i for $i = 1, 2, 3$ be respectively the S_0-component and the connected component of Δ contained in F_i (it is possible that $\Delta_i = 0$). Let m be the greatest integer, such that $B + m(K + D) \geq 0$. By 5.1.2(5) $m > 0$, because $BD = 3 - b > 1$. Write $B + m(K + D) = \sum_j c_j C_j$ for $c_j > 0$ and $C_j^2 < 0$. Multiplying by the generic fiber F' we get $1 - m = \sum_j c_j F' C_j$, so $m = 1$ and $F' C_j = 0$, hence all C_j's are vertical. Let D' be the divisor consisting of vertical components of D not intersecting B. For any component $D_0 \subseteq D'$ we have $D_0(K + D + B) = \beta_D(D_0) - 2$. Since for each F_i the divisor $F_i \cap D$ is a chain, the components of D' are in the fixed part of $K + D + B$. Each L_i intersects D', so it follows that L_i's, and hence all components of Δ are in the fixed part of $K + D + B$. Now for each i we have $E(L_i + \Delta_i) \geq 2$, otherwise L_i would be simple. Thus we get $EK = E(K + D + B) = E(\sum_j c_j C_j) \geq \sum_i E(L_i + \Delta_i) \geq 6$, a contradiction with 6.2.2.

□

Corollary 6.4.7. *If Δ has three connected components then $b = \epsilon = 2$.*

Proof. If Δ has three connected components then $\widehat{E}$ is a fork, so $\epsilon = 2$ by 6.2.3(ii) and we get $\#E = 1$. Since Δ does not contain a fork, $S \setminus \Delta$ is affine ruled. We have $\Sigma_{S_0} = 0$, so singular fibers have unique (-1)-curves. It follows that each connected component of Δ is contained in a different fiber, hence $b = 2$ by 6.4.6. □

6.5 Surface W.

We define $W = \overline{S} - (T + \widehat{E})$ ($T = D - B$, where B is the branching component of D). Clearly, $S_0 \subset W \subset S \subset \overline{S}$ and $\chi(W) = \chi(S_0) + \chi(\mathbb{C}^{**}) = -1$. Our goal is to prove that $\overline{\kappa}(W) = 2$. To achieve this we give couple of technical lemmas (combining arguments which are often subtle with respect to input data) and use the results of some computer programs we wrote.

Lemma 6.5.1. *If R is an ordered admissible chain then the equation (*) $e(R) + \alpha/d(R) = 1$ has the following solutions:*

(i) $R = [2, \ldots, 2, 2]$ *for* $\alpha = 1$,

(ii) $R = [2, \ldots, 2, 3]$ *for* $\alpha = 2$,

(iii) $R = [2, \ldots, 2, 3, 2]$ *or* $R = [2, \ldots, 2, 4]$ *for* $\alpha = 3$.

Proof. Using a recurrence formula for a determinant of a chain (cf. 1.1.1(i)) it is easy to check that $R = [2, a_1 \dots, a_k]$ satisfies (*) if and only if $[a_1, \dots, a_k]$ does, so we may assume that $R = [a_1, \dots, a_k]$ with $a_1 \geq 3$. We have $d'(R) + \alpha = d(R) = a_1 d'(R) - d''(R)$, so then $2d'(R) \leq (a_1 - 1)d'(R) = d''(R) + \alpha < d'(R) + \alpha$, hence $d'(R) < \alpha \leq 3$ and $k \leq 2$. For $d'(R) = 2$ we get $R = [3, 2]$, for $d'(R) = 1$ we get $R = [4]$ or $R = [3]$ and for $d'(R) = 0$ we get $R = \emptyset$. □

Lemma 6.5.2. *If $F = [(k), c+1, a_1, \dots, a_n]$ is admissible then $e(F) < \frac{kc-(k-1)}{(k+1)c-k}$.*

Proof. By induction using the fact that for a chain $T = [c, \dots]$ the equality $e(T) = \frac{1}{c - e'(T)}$ holds. □

Lemma 6.5.3.

(i) W is almost minimal and $K + T + \widehat{E} \equiv \lambda \mathcal{P} + \operatorname{Bk} \widehat{E} + \operatorname{Bk}^* T$ (cf. 6.1.3), where $\lambda = 1 - \frac{\widetilde{e} - b}{1 - \delta}$.

(ii) If $\overline{\kappa}(W) \geq 0$ then $\lambda \mathcal{P} \equiv (K + T + \widehat{E})^+$.

(iii) If $\overline{\kappa}(W) \geq 0$ then $b + 1 \geq \widetilde{e} + \delta$, $\delta + \frac{1}{|G|} \geq 1$ and $\epsilon \neq 0$. The inequalities are strict if $\overline{\kappa}(W) = 2$.

(iv) If $\overline{\kappa}(W) \neq 2$ then $\overline{\kappa}(W) \leq 0$, $\widetilde{e} + \delta \geq 2$ and $b = 1$.

Proof. (i) Recall that $\operatorname{Bk}^* T = \operatorname{Bk} T + \operatorname{Bk} T^t$. Using 6.4.4(iii) we have $K + T + \widehat{E} \equiv \mathcal{P} - B + \operatorname{Bk} D + \operatorname{Bk} \widehat{E} = \mathcal{P} - B - \sum_{i=1}^3 \operatorname{Bk} T_i^t + \sum_{i=1}^3 \operatorname{Bk}^* T_i + \operatorname{Bk} \widehat{E} = (1 - \frac{\widetilde{e} - b}{1 - \delta})\mathcal{P} + \operatorname{Bk}^* T + \operatorname{Bk} \widehat{E}$. Suppose W is not almost minimal. Then there exists a (-1)-curve C, such that $C + \operatorname{Bk} \widehat{E} + \operatorname{Bk}^* T$ is negative definite. Since $\operatorname{Supp}(\operatorname{Bk} \widehat{E} + \operatorname{Bk}^* T) = \operatorname{Supp}(\widehat{E} + T)$, $(K + T + \widehat{E})^-$ has at least $\#T + \#\widehat{E} + 1 = b_2(\overline{S})$ numerically independent components, a contradiction with the Hodge index theorem.

(ii) From (i) and from the definition of Bk we see that $\mathcal{P}$ intersects trivially with every component of $T + \widehat{E}$. If $\overline{\kappa}(W) \geq 0$ then by the properties of Zariski decomposition the same is true for $(K + T + \widehat{E})^+$, so $(K + T + \widehat{E})^+ \equiv \lambda P$ (cf. 2.2.3(i)).

(iii) We have $\chi(W) = -1$, so $\delta + \frac{1}{|G|} \geq 1 + \frac{1}{3}\lambda^2 \mathcal{P}^2$ by the Kobayashi inequality (see 1.6.13(ii)). By (ii) and 1.6.8 $\overline{\kappa}(W) > 0$ ($\overline{\kappa}(W) = 0$) if and only if $\lambda > 0$ (respectively $\lambda = 0$), which is equivalent to $b + 1 > \widetilde{e} + \delta$ (respectively $b + 1 = \widetilde{e} + \delta$). Suppose $\epsilon = 0$. Then $\widehat{E} = [|G|]$ by 6.2.3(i), so by 6.1.4(i) $\delta + \frac{1}{|G|} \leq e + \frac{1}{|G|} \leq 1$. Together with the inequality above this implies $e = \delta$, so maximal twigs of D are tips, a contradiction with 6.1.2(vi).

(iv) Suppose $\overline{\kappa}(W) = 1$. Then by (ii) $\lambda^2 \mathcal{P}^2 = 0$, so $\lambda = 0$ and $(K + T + \widehat{E})^+ \equiv 0$, a contradiction. Thus $\overline{\kappa}(W) \leq 0$ and we have $b + 1 \leq \widetilde{e} + \delta$, because $\lambda \leq 0$ in this case. Suppose $b = 2$. Since $\widetilde{e}_i + \frac{1}{d_i} \leq 1$, we get $\widetilde{e}_i + \frac{1}{d_i} = 1$ for each i, so D consist of (-2)-curves by 6.5.1(i). By 6.4.4(iv) $\operatorname{Bk}^2 \widehat{E} = 1 - \epsilon$, so $\epsilon = 2$, $\widehat{E}$ is a chain by 1.3.5(v) and $d'(\widehat{E}) + d'(\widehat{E}^t) + 2 = d(\widehat{E})$. One checks easily that this equation can be satisfied only if Δ is connected, hence by 6.4.1 $\widehat{E} = [3, (k)]$ for some $k \geq 0$. Then $d'(\widehat{E}) + d'(\widehat{E}^t) + 2 > d(\widehat{E})$, a contradiction. □

To make further considerations easier (or even possible) it is crucial to prove that D does not contain small 0-divisors, namely the chains $[2, 1, 2]$ and $[3, 1, 2, 2]$. We prove this under additional assumptions and in the second case we restrict ourselves to proving that if D contains $[3, 1, 2, 2]$ then D and $\widehat{E}$ are special. This will be sufficient for our later arguments to work.

Lemma 6.5.4.

(i) If $KT_i = 0$ for some i then $h^0(2K + T + \widehat{E}) \geq 3 - b - \epsilon$.

(ii) Assume $\overline{\kappa}(W) \leq 0$. Then D does not contain the chain $[2, 1, 2]$ and if D contains a chain $[3, 1, 2, 2]$ then $E = [3]$.

(iii) Assume $\#E = 1$. Then D does not contain the chain $[2, 1, 2]$. If D contains a chain $[3, 1, 2, 2]$ then $\Delta = 0$ and some T_i satisfies $KT_i = 0$ and $\#T_i \geq 5$. The (-3)-curve of $[3, 1, 2, 2]$ is not a tip of D.

Proof. (i) Let T_1 consist of (-2)-curves. Riemann-Roch's theorem gives $h^0(-K-T_2-T_3-\widehat{E})+h^0(2K+T_2+T_3+\widehat{E})\geq\frac{1}{2}(K+T_2+T_3+\widehat{E})(2K+T_2+T_3+\widehat{E})+1=K(K+D+\widehat{E}-T_1-B)-3+1=3-b-\epsilon$. If $-K-T_2-T_3-\widehat{E}\geq 0$ then B, and hence T_1, is in the fixed part, so $-K-D-\widehat{E}\geq 0$, which contradicts $\overline{\kappa}(S_0)=2$. Thus $h^0(2K+T_2+T_3+\widehat{E})\geq 3-b-\epsilon$.

(ii) Suppose D contains a 0-divisor $F_\infty=[2,1,2]$ or $F_\infty=[3,1,2,2]$. Since D is snc-minimal, the (-1)-curve of F_∞ is B, the branching component of D. The divisor F_∞ gives a $\mathbb{P}^1$-ruling $p:\overline{S}\to\mathbb{P}^1$ with F_∞ as a fiber. $\widehat{E}$ is vertical because $F_\infty\widehat{E}=0$, so $\Sigma_{S_0}=h+\nu-2=h-1\leq 2$. Denote the fiber of p containing $\widehat{E}$ by F_E. We have $F_ED\leq 5$ because $\mu(B)\leq 3$.

We first need to prove that all S_0-components are exceptional. For any vertical S_0-component L we have $L(K+T^\#+\widehat{E}^\#)=\lambda\mathcal{P}L$. By 6.4.4 we have also $L\mathcal{P}>0$ because $LD>0$. Suppose $L^2\leq -2$. Then $L(T^\#+\widehat{E}^\#)\leq\lambda L\mathcal{P}$, which is possible only if $\lambda=LT^\#=L\widehat{E}^\#=0$. It follows that $L\widehat{E}=L$, so by 6.1.2(ii) $LD>1$, say $LT_1,LT_2>0$. Then $LT^\#=0$ implies that T_1 and T_2 are (-2)-chains, so by 6.5.3(iii) we get $\widetilde{e}_3+\frac{1}{d_3}=0$. This is a contradiction, so we are done.

Let D_h and D_v be respectively the divisor of horizontal components of D and the divisor of D-components contained in F_E. Let D_1 be the multiple section contained in D_h. Denote the (-1)-curves of F_E by $L_1,L_2,\ldots,L_{\sigma(F_E)}$. Clearly, D_v has at most three connected components and they are chains. We will prove that D_h contains a section and $D_v\neq 0$. Suppose D_h does not contain a section. In this case D_v is connected and D_h is either a 2-section or a 3-section, so $\Sigma_{S_0}=0$ and $\sigma(F_E)=1$. We have $F_ED\leq 3$ and since L_1 is not simple, $|L_1\cap D|\geq 2$, so D_h intersects L_1 in exactly one point and $D_v\neq 0$. This gives $\mu(L_1)+1\leq F_ED_h\leq 3$, so $\mu(L_1)=2$ and we get $K\widehat{E}=0$, a contradiction. Suppose $D_v=0$. Since L_i are not simple, $|L_i\cap D_h|\geq 2$ for each i, so $\sigma(F_E)\leq 2$. Since D_h contains a section, the exceptional component intersecting this section, say L_2, has multiplicity one, hence $\sigma(F_E)=2$. The second exceptional component has also multiplicity one, otherwise it could intersect only D_1, which would imply $D_1F_E\geq\mu(L_2)D_1F_E\geq 4$. This shows that $F_E=[1,(k),1]$ for some $k\geq 0$, a contradiction with $K\widehat{E}\neq 0$. Let α be the number of connected components of D_v. We can assume that L_1 intersects $\widehat{E}$ and D_v. Notice that each L_i meeting $\widehat{E}$ intersects D_h, otherwise it would be simple. We consider two cases.

Suppose $\widehat{E}$ intersects more than one L_i, say $L_2\widehat{E}>0$. We have $5\geq F_ED_h\geq(D_v+\mu(L_1)L_1+\mu(L_2)L_2)D_h$ and $\mu(L_2)L_2D_h\geq 2$, so $\alpha+\mu(L_1)L_1D_h\leq 3$, hence $\alpha=1$ and $\mu(L_1)=2$. This gives $F_ED=5$, so $F_\infty=[3,1,2,2]$ and D contains three horizontal components. In particular, no maximal twig of D is contained in F_∞. We have now $L_2D_v=0$, so some section from D_h intersects L_2, which gives $\mu(L_2)=1$. Moreover, there are no more (-1)-curves in F_E. Defining F'_E as the fiber F_E with L_1 (only L_1) contracted we find that the (-1)-curves, and hence all components of F'_E, have multiplicity one, so $F'_E=[1,(k),1]$ for some $k\geq 0$. It follows that $F_E=[1,(k-1),3,1,2]$, hence $E=[3]$ and we are done.

Now suppose $L_i\widehat{E}=0$ for $i\neq 1$, i.e. L_1 is the only S_0-component intersecting $\widehat{E}$. Consider the contraction of (-1)-curves in F_E different than L_1 (if there are any) until L_1 is the unique exceptional component in the image F'_E of the fiber. This contraction does not touch $\widehat{E}$, so $\widehat{E}$ is one of the connected components of $\underline{F}'_E-L_1$. Since $L_1D_h>0$, we have $\mu(L_1)\leq 3$ because D_h contains no n-sections with $n>3$. It follows that either $F'_E=[2,1,2]$ or $F'_E=[3,1,2,2]$, hence $\widehat{E}=[3]$ because $KE\neq 0$. We have also $\mu(L_1)=3$, so D_h contains a 3-section, which implies $F_\infty=[3,1,2,2]$ and we are done.

(iii) Let p, F_∞ and F_E be as in (ii). Here the argument is tricky. By 6.3.12 there exist a pre-minimal affine ruling of $S\setminus\Delta$, let f be its extension as in 6.3.9. We use the notation of 6.3.9. Notice that in general f is not defined on $\overline{S}$. However, the components of $\underline{F}-Z_1-Z_l$ are not touched by φ_f. In particular, Z_l and the divisor of D-components of the second branch of F (F is the fiber of f, not of p) are maximal twigs of D. We denote them by T_2 and T_1 respectively. Similarly the unique (-1)-curve C contained in F is not touched by φ_f, so it is exceptional on $\overline{S}$ and satisfies $CD=1$, $CB=0$ and $C(\Delta+E)\geq 2$, because it is not simple. Now let us look how does C behave with respect to p. Since $\widehat{E}$ is connected, C is horizontal for p and $F_\infty C=F_EC\geq 2$. We have $CD=1$, so C intersects $F_\infty-B$ in a component $D_0\subseteq T_1$ of multiplicity greater than one, hence $F_\infty=[3,1,2,2]$ and D_0 is the middle (-2)-curve. We now look back at the fiber F of f and we find that after contracting C the component D_0 becomes a (-1)-curve, so $\Delta'=0$ and T_1 consists of (-2)-curves. Notice that if f is almost minimal then applying the above argument to $\widetilde{C}$ instead of C we get that $\widetilde{C}$ intersects D_0, which contradicts the fact that C and $\widetilde{C}$ intersect different maximal twigs of D. Thus f is not almost minimal. The contraction of T_1+C touches Z_1 precisely $x=\#T_1$ times, so $Z_1^2=-x-1$, hence φ_f touches Z_1 precisely k times. The proper transform of $\widetilde{Z}_{lu}$ on $\overline{S}$ is not a (-2)-curve, otherwise D would contain the chain $[2,1,2]$, which was excluded above.

Therefore by 6.3.11(ii) we get $x \geq 5$ and $\Delta = 0$.

We need only to prove that the (-3)-curve of F_∞ is not a tip of D. Suppose it is. If $T_3 = [3]$ then $\widetilde{Z}_l$ is a tip, so $\widetilde{G} + \widetilde{Z}_u + \widetilde{Z}_1$ consists of (-2)-curves, which implies that φ_f touches Z_1 once, contradicting 6.3.11(ii). Thus $T_2 = Z_l = [3]$ and we get $Z_u + G = [2,2]$. Then $\widetilde{G} = [4]$, $\widetilde{Z}_u = [(s)]$ for some $s \geq 0$ and $\widetilde{Z}_l = [2,2,s+2]$. We have $Z_1^2 = -k-1$ and now φ_f touches Z_1 $s+3$ times, so $s = k-3$. Then $\widetilde{Z}_{lu}$ is touched once by φ_f and has self-intersection $-k+1$, hence its image on $\overline{S}$ has self-intersection $-k$. By 6.3.11(ii) we get $-k = 2-k$, a contradiction. □

Lemma 6.5.5. *If $\overline{\kappa}(W) \leq 0$ then $\epsilon = 2$, one of the maximal twigs of D is a (-2)-chain and some other is $[(k),3]$ for some $k \geq 0$. This (-2)-chain is a tip of D, unless D contains the chain* $[3,1,2,2]$.

Proof. Notice that by 6.5.4 if D contains the chain $[3,1,2,2]$ then we can assume that T_1 is a (-2)-chain. We will now prove that if D does not contain the chain $[3,1,2,2]$ then $T_1 = [2]$. We explore intensively the inequality 6.5.3(iv): $\widetilde{e} + \delta \geq 2$. Notice that $\widetilde{e}_i + \frac{1}{d_i} \leq 1$ for each i. Assume that $d_1 \leq d_2 \leq d_3$. We prove successive statements.

(1) $T_1 = [3]$ or T_1 ends with a (-2)-curve.

Proof. Suppose not. If T_1 ends with a (-3)-curve then T_2 and T_3 cannot end with two (-2)-curves by 6.5.4. Moreover, if one of T_2 or T_3, say T_2 ends with a (-2)-curve, then T_3 does not, so using 6.5.2 we get $\widetilde{e}_1 < \frac{1}{2}$, $\widetilde{e}_2 < \frac{2}{3}$ and $\widetilde{e}_3 < \frac{1}{2}$, so $\widetilde{e} < \frac{1}{2} + \frac{2}{3} + \frac{1}{2} = \frac{5}{3}$. We use continuously this type of argument below with less details. If T_1 ends with a $\leq (-4)$-curve then in case some other T_i ends with a (-3)-curve we have $\widetilde{e} < \frac{1}{3} + \frac{1}{2} + \frac{2}{3} = \frac{3}{2}$ and $\widetilde{e} < \frac{1}{3} + \frac{1}{3} + 1 = \frac{5}{3}$ if not. This gives $\frac{3}{d_1} \geq \delta \geq 2 - \widetilde{e} > 2 - \frac{5}{3} = \frac{1}{3}$, so $d_1 \leq 8$. By 1.1.2 we have to exclude the following possibilities for T_1: [4], [5], [6], [7], [8], [2, 3], [2, 4], [2, 2, 3], [3, 3].

Case 1. T_1 is one of $[2,4]$, [5], [6], [7] or [8]. In each case $\widetilde{e}_1 + \frac{1}{d_1} \leq \frac{3}{7}$. If T_3 (or similarly T_2) ends with two (-2)-curves then $\widetilde{e}_2 < \frac{1}{3}$ and we get $\frac{1}{d_2} < 2 - \frac{3}{7} - 1 - \frac{1}{3}$, so $d_2 \leq 4$, a contradiction with $d_2 \geq d_1$. In other case $\widetilde{e} < \frac{3}{7} + \frac{2}{3} + \frac{1}{2}$, so $\frac{2}{d_2} \geq \frac{1}{d_2} + \frac{1}{d_3} > 2 - \widetilde{e} > \frac{17}{42}$ and again $d_2 \leq 4$, a contradiction.

Case 2. T_1 is one of $[2,2,3]$ or $[3,3]$. Then $\widetilde{e}_1 + \frac{1}{d_1} \leq \frac{4}{7}$ and $\widetilde{e}_2 + \widetilde{e}_3 < \frac{1}{2} + \frac{2}{3}$, so $\frac{2}{d_2} \geq 2 - \widetilde{e} - \frac{1}{d_1} > \frac{1}{4}$ and $d_2 \leq 7$. Since $d_1 \leq d_2$ we get $T_1 = [2,2,3]$ and $d_1 = d_2 = 7$. By renaming T_1 with T_2 we can assume that T_2 does not end with a (-2)-curve. In fact we can assume that $T_2 = [2,2,3]$ because other cases ([7] and $[2,4]$) were excluded above, hence $\widetilde{e}_3 + \frac{1}{d_3} \geq \frac{6}{7}$. We have $\widetilde{e}_3 < \frac{2}{3}$ because T_3 does not end with two (-2)-curves, so $\frac{1}{d_3} > \frac{6}{7} - \frac{2}{3}$ and $d_3 \leq 5 < d_1$, a contradiction.

Case 3. $T_1 = [4]$. We have $\widetilde{e}_1 + \frac{1}{d_1} = \frac{1}{2}$, so $\frac{1}{d_2} + \frac{1}{d_3} \geq \frac{3}{2} - \widetilde{e}_2 - \widetilde{e}_3$. If T_2 or similarly T_3 ends with a $\leq (-4)$-curve then $\frac{1}{d_2} \geq \frac{3}{2} - \widetilde{e}_2 - 1 > \frac{1}{6}$, so $d_2 \leq 5$. If T_2 (or similarly T_3) ends with a (-3)-curve, then $\frac{2}{d_2} > \frac{3}{2} - \frac{2}{3} - \frac{1}{2} = \frac{1}{3}$, so again $d_2 \leq 5$. Notice that $T_2 \neq [5]$ (similarly $T_3 \neq [5]$), otherwise $\frac{1}{d_3} + \widetilde{e}_3 \geq \frac{11}{10}$, which is impossible. If T_2 is one of $[2,3]$, $[3,2]$ or $[2,2,2,2]$ then we have respectively $\widetilde{e}_2 + \frac{1}{d_2} = \frac{3}{5}, \frac{4}{5}, 1$ and using 6.5.4 and 6.5.2 we bound $\widetilde{e}_3$ from above respectively by $\frac{2}{3}, \frac{1}{2}$ and $\frac{1}{3}$, which gives $d_3 = 5$. However, we check easily that then the inequality $\frac{1}{d_2} + \widetilde{e}_2 + \frac{1}{d_3} + \widetilde{e}_3 \geq \frac{3}{2}$ cannot be satisfied. Thus $d_2 = 4$. By renaming T_1 and T_2 we can assume that $T_2 \neq [2,2,2]$, so $T_2 = [4]$. Then $\widetilde{e}_3 + \frac{1}{d_3} \geq 1$ so $T_3 = [2,2,2]$ by 6.5.1 and after renaming T_1 and T_3 we get a contradiction.

Case 4. $T_1 = [2,3]$. We have $\widetilde{e}_2 + \widetilde{e}_3 + \frac{1}{d_2} + \frac{1}{d_3} \geq \frac{7}{5}$ and $\widetilde{e}_2 + \widetilde{e}_3 < \frac{2}{3} + \frac{1}{2}$, so $d_2 \leq 8$. Suppose $d_2 = 5$. We can assume that $T_2 = [2,3]$, because the case $T_1 = [5]$, $T_2 = [2,3]$ was considered above and in other cases T_2 ends with a (-2)-curve, so after renaming T_1 and T_2 we get a contradiction. If $d_3 \neq 5$ then $\widetilde{e}_3 \geq \frac{4}{5} - \frac{1}{d_3} > \frac{3}{5}$, hence T_3 ends with two (-2)-curves, a contradiction. Therefore $d_3 = 5$ and again we can assume that $T_3 = [2,3]$, so $\widetilde{e}_2 + \widetilde{e}_3 + \frac{1}{d_2} + \frac{1}{d_3} = \frac{6}{5}$, a contradiction. Thus $6 \leq d_2 \leq 8$. If $T_2 = [d_2]$ then $\frac{1}{d_3} + \widetilde{e}_3 > \frac{7}{5} - \frac{2}{5} = 1$, a contradiction. In particular $d_2 \neq 6$, so T_2 is one of $[2,2,3]$, $[3,2,2]$, $[2,4]$, $[3,3]$, $[4,2]$ or $[2,3,2]$. T_2 and T_3 cannot end two (-2)-curves, so $T_2 = [3,2,2]$ is excluded and $\widetilde{e}_3 < \frac{2}{3}$. If T_2 is $[4,2]$ or $[2,3,2]$ then we have a better upper bound $\widetilde{e}_3 < \frac{1}{2}$, in any case we obtain $\widetilde{e}_3 + \widetilde{e}_2 + \frac{1}{d_2} \leq \frac{5}{4}$, hence $d_3 \leq 6 < d_2$, a contradiction. □

(2) T_1 is a tip.

Proof. Suppose not. We have $\widetilde{e}_2 + \widetilde{e}_3 + \frac{1}{d_2} + \frac{1}{d_3} \geq 1$. By (1) T_1 ends with a (-2)-curve, so T_2 and T_3 do not end with (-2)-curves, hence $\widetilde{e}_2 + \widetilde{e}_3 < \frac{1}{2} + \frac{1}{2} = 1$ and from the inequality $\widetilde{e} + \delta \geq 2$ we get $\widetilde{e}_1 + \frac{3}{d_1} > 1$. This gives $d'(T_1^t) = d(T_1^t) - 1$ or $d'(T_1^t) = d(T_1^t) - 2$, so $T_1 = [(k)]$ or $[3,(k)]$ for some $k > 0$ by 6.5.1.

Suppose $k\geq 2$. In this case T_2 and T_3 end with a $\leq(-4)$-curve, so $\widetilde{e}_2,\widetilde{e}_3<\frac{1}{3}$. Then $\frac{1}{d_2}+\frac{1}{d_3}>\frac{1}{3}$ and we get $d_1\leq d_2\leq 5$, which is possible only if T_2 is a tip and $T_1=[(k)]$ for some $k\in\{2,3,4\}$. Since now $\frac{1}{d_3}\geq 1-\widetilde{e}_3-\frac{2}{d_2}>\frac{2}{3}-\frac{1}{2}$, we see that $d_3\leq 5$, so T_3 is also a tip. Then $\widetilde{e}_2=\frac{1}{d_2}$ and $\widetilde{e}_3=\frac{1}{d_3}$, so $\frac{1}{d_2}+\frac{1}{d_3}\geq\frac{1}{2}$ and we conclude that $T_2=T_3=[4]$ and $T_1=[(k)]$ for some $k\in\{2,3\}$. Using 6.4.4 we compute $\operatorname{Bk}^2\widehat{E}=-\epsilon$. The matrix $Q(\widehat{E})$ is negative definite, so $\epsilon\neq 0$, and in fact $\epsilon=1$ by 1.3.6, hence $\widehat{E}$ is a chain by 1.3.5(v). By 2.1.3(ix) $8(k-1)/d(\widehat{E})$ is a square and by 6.2.2 and 6.1.2(vi) we get $\#\widehat{E}=8+KE-k\geq 10-k$. This implies that $k=3$ and $d(\widehat{E})=16$. However, it is easy to check that no chain $\widehat{E}$ with $d(\widehat{E})=16$ satisfies $\#\widehat{E}-K\widehat{E}=5$, a contradiction.

We are left with the case $T_1=[3,2]$, for which $\widetilde{e}_2+\frac{1}{d_2}+\widetilde{e}_3+\frac{1}{d_3}\geq\frac{6}{5}$. The twigs T_2 and T_3 cannot end with a (-2)-curve, so $\widetilde{e}_2,\widetilde{e}_3<\frac{1}{2}$. Suppose T_2 or T_3 ends with a $\leq(-4)$-curve. Then $\widetilde{e}_2+\widetilde{e}_3<\frac{1}{2}+\frac{1}{3}$, so $\frac{1}{d_1}+\frac{1}{d_2}>\frac{1}{3}$ and we get $d_1=d_2=5$, hence $T_2=[5]$ or $T_2=[2,3]$. If $T_2=[5]$ then $\frac{1}{d_3}>\frac{4}{5}-\frac{1}{2}$. If $T_2=[2,3]$ then by assumption T_3 ends with a $\leq(-4)$-curve, so $\widetilde{e}_3<\frac{1}{3}$ and $\frac{1}{d_3}>\frac{3}{5}-\frac{1}{3}$. In both cases we get $d_2\leq 3$, a contradiction. Thus both T_2 and T_3 end with a (-3)-curve, so $\widetilde{e}_2+\widetilde{e}_3<1$ and we get $d_2\leq 9$. However, admissible chains with $d\leq 9$ ending with (-3)-curve satisfy $\widetilde{e}+\frac{1}{d}\leq\frac{3}{5}$ (cf. 1.1.2), the equality occurs only for $[2,3]$. Hence $\frac{1}{d_3}\geq\frac{3}{5}-\widetilde{e}_3>\frac{1}{10}$, so $d_3\leq 9$ too. This implies $T_2=T_3=[2,3]$. Using 6.4.4 we compute $\operatorname{Bk}^2\widehat{E}=\frac{1}{5}-\epsilon$, hence $\epsilon\neq 0$. We compute $d(D)=-50$, so $d(\widehat{E})\in\{2,50\}$ by 2.1.3(ix). By 6.1.4(i) $|G|\leq 7$ and since $G<GL(2,\mathbb{C})$ is small, it is abelian, hence $\widehat{E}$ is a chain and $d(\widehat{E})=2$, a contradiction with $KE\neq 0$. □

(3) $T_1\neq[3]$.

Proof. Suppose $T_1=[3]$. We have $\widetilde{e}_2+\widetilde{e}_3+\frac{1}{d_2}+\frac{1}{d_3}\geq\frac{4}{3}$, so since $\widetilde{e}_2+\widetilde{e}_3<\frac{2}{3}+\frac{1}{2}$, we get $\frac{1}{d_1}+\frac{1}{d_2}>\frac{1}{6}$, which gives $d_2\leq 11$.

Case 1. Suppose $T_2\neq[3]$ or T_3 does not end with $[3,2]$. We prove that $d_3\leq 42$. For $d_2>6$ the inequality $\frac{1}{d_1}+\frac{1}{d_2}>\frac{1}{6}$ gives $d_3\leq 42$. We can therefore assume that $d_2\leq 6$. If $T_2=[3,2]$ then $\widetilde{e}_2+\frac{1}{d_2}=\frac{4}{5}$ and T_3 does not end with a (-2)-curve, so $\frac{1}{d_3}>\frac{4}{3}-\frac{4}{5}-\frac{1}{2}$ and $d_3<30$. If $T_2=[4]$, $[5]$, $[6]$ or $[2,3]$ then $\widetilde{e}_2+\frac{1}{d_2}\leq\frac{3}{5}$ and since T_3 does not end with two (-2)-curves, $\widetilde{e}_3<\frac{2}{3}$, which gives $d_3\leq 14$. Thus we can assume that $T_2=[3]$, hence $\widetilde{e}_3+\frac{1}{d_3}\geq\frac{2}{3}$. If T_3 ends with a $\leq(-3)$-curve then $\frac{1}{d_3}>\frac{2}{3}-\frac{1}{2}$, so $d_3\leq 5$. If T_3 ends with $[v,2]$ for some $v>3$ then $\frac{1}{d_3}>\frac{2}{3}-\frac{3}{5}$, so $d_3\leq 14$ and we are done. Now notice that whenever d_3 is bounded, by 6.4.5 there are finitely many possibilities for the dual graphs of D and $\widehat{E}$. Using a computer program we checked that the conditions $d_2\leq 11$, 6.1.2(vi), 6.1.4, 6.2.5, 6.4.4 and 6.3.13(ii) (which implies that $-d(D)/d(\widehat{E})$ is a square of an integer) are satisfied only in two cases:

(i) $T_1=[3]$, $T_2=[3]$, $T_3=[3,(6)]$ and $\widehat{E}=[2,3,4]$,

(ii) $T_1=[3]$, $T_2=[4]$, $T_3=[2,2,2]$ and $\widehat{E}$ is a fork with a (-2)-curve as a branching component and maximal twigs $[2]$, $[2]$, $[2,2,3]$.

In both cases D contains the chain $[3,1,2,2]$, a contradiction.

Case 2. Suppose $T_2=[3]$ and $T_3=T_0+[3,2]$. We will determine T_0. Since for a chain beginning with a $(-c)$-curve one has $d=cd'-d''$, we get from $\widetilde{e}+\frac{1}{d_3}\geq\frac{2}{3}$ that $d'(T_0^t)+3\geq d(T_0^t)$, so by 6.5.1 $T_3=[(k),3,2]$, $[3,(k),3,2]$, $[4,(k),3,2]$ or $[2,3,(k),3,2]$ for some $k\geq 0$. We conclude that $KT\leq 5$, hence 6.4.5 again reduces the problem to checking finitely many cases (here Noether formula implies $k\leq 9$, which gives $d_3\leq 102$). We checked that each of them leads to a contradiction. □

To finish the proof we have to show that $\epsilon=2$ and one of T_2 or T_3 is $[(k),3]$ for some $k\geq 0$. Since D cannot contain a chain $[2,1,2]$, T_2 and T_3 end with $\leq(-3)$-curves. We have $\widetilde{e}_2+\frac{1}{d_2}+\widetilde{e}_3+\frac{1}{d_3}\geq 1$ and the inequality is strict for $\overline{\kappa}(W)=-\infty$. Using 6.5.1 it is easy to check that an admissible chain R ending with a $\leq(-3)$-curve and satisfying $\widetilde{e}(R)+\frac{1}{d(R)}\geq\frac{1}{2}$ is either $[4]$ or $[3,(k),3]$ or $[(k),3]$ for some $k\geq 0$. Moreover, the inequality is strict only in the last case, hence if $\overline{\kappa}(W)=-\infty$ then T_2 or T_3 is of type $[(k),3]$ and we are done, because by 6.5.4(i) $\epsilon=2$ then. We can therefore assume $\overline{\kappa}(W)=0$. For convenience we put formally $[3,(-1),3]=[4]$, then we have $d([3,(k-2),3])=4k$ for any $k\geq 1$.

Suppose $\epsilon\leq 1$ or T_2, T_3 are not of type $[(k),3]$. In the second case we can write $T_2=[3,(x-2),3]$, $T_3=[3,(y-2),3]$ with $1\leq x\leq y$. We argue that we can do the same in the first case. Indeed, if $\epsilon\leq 1$ then by 6.5.4 $2(K_{\overline{S}}+T+\widehat{E})\geq 0$, so by 1.6.7(ii) $[2(K_{\overline{S}}+T^{\#}+\widehat{E}^{\#})]\sim U$ for some effective U. Then $K_{\overline{S}}+T^{\#}+\widehat{E}^{\#}\equiv 0$ implies $U+\{2(K_{\overline{S}}+T^{\#}+\widehat{E}^{\#})\}\equiv 0$, hence $2\operatorname{Bk}^*T_i$ and $2\operatorname{Bk}\widehat{E}$ are $\mathbb{Z}$-divisors.

Since T_2, T_3 are not (-2)-chains, we obtain $2\operatorname{Bk}^* T_i = T_i$ for $i = 2,3$. It is easy to see that an admissible chain R satisfying $\operatorname{Bk}^* R = \frac{1}{2}R$ is $[3,(k),3]$ for some $k \geq -1$, so we are done. Using 6.4.4(iv) we compute $\operatorname{Bk}^2 \widehat{E} = -\epsilon$, hence $\epsilon = 1$ and the argument above shows that we can write $\widehat{E} = [3,(z-2),3]$ with $z \geq 1$. By 6.1.2(vi) $x+y+z+\#T_1 = 12$, hence $1 \leq x, y, \#T_1 \leq 9$ and $\frac{1}{\#T_1} + \frac{1}{x} + \frac{1}{y} + \frac{1}{12-x-y-\#T_1} \geq 1$ by 6.5.3(iii). This inequality is satisfied only for $(\#T_1, x, y) = (1,1,9)$, hence $T_2 = [4]$, $T_3 = [3,(7),3]$ and $\widehat{E} = [4]$. By 6.4.2 $(\overline{S}, D)$ is affine ruled and since $b = 1$, B is horizontal and the ruling has three singular fibers. This contradicts 6.4.6. ☐

Proposition 6.5.6. $\overline{\kappa}(W) = 2$.

Proof. Suppose $\overline{\kappa}(W) \leq 0$. Then $b = 1$, by 6.5.5 $\epsilon = 2$ and one of the maximal twigs of D consists of (-2)-curves, so $\#E = 1$. Denote the coefficient of E in $\operatorname{Bk}\widehat{E}$ by w_E. We prove successive statements.

(1) If $w_E > \frac{1}{2}$ then $\widehat{E}$ is a chain and Δ is connected. If $w_E = \frac{1}{2}$ then either $\widehat{E}$ is a fork with maximal twigs [3], [2], [2] or $\widehat{E} = [2,3,2]$.

Proof. Suppose $\widehat{E}$ is a fork. By 6.2.3(iii) we know that Δ does not contain a fork and by 6.4.7 E is not the branching component of $\widehat{E}$, so $\widehat{E}$ is of type (b1) (cf. 6.2.5), hence the maximal twig of $\widehat{E}$ containing E is equal to $[(k),3]$ for some $k \geq 0$. Using 1.3.2(ii) and the definition of a bark of an admissible fork it is a straight computation to check that $w_E \leq \frac{1}{2}$ in each case and the equality occurs only for a fork with maximal twigs [3], [2], [2]. If $\widehat{E}$ is a chain then $\widehat{E} = [(m-1),3,(\widetilde{m}-1)]$ for some $m, \widetilde{m} \geq 1$ and $w_E = \frac{m+\widetilde{m}}{m\widetilde{m}+m+\widetilde{m}} = 1 - 1/(1+\frac{1}{m}+\frac{1}{\widetilde{m}})$, so $w_E \geq \frac{1}{2}$ if and only if $\frac{1}{m} + \frac{1}{\widetilde{m}} \geq 1$, hence (1) follows. ☐

By 6.3.12 we can consider a pre-minimal affine ruling $f : (\overline{S}^\dagger, D^\dagger + \Delta) \to \mathbb{P}^1$ of $S \setminus \Delta$. We have $\Sigma_{S_0} = 0$, so each singular fiber of f has a unique S_0-component, which is exceptional. We use the notation 6.3.9. Since $b = 1$ and $Z_1^2 \leq -2$, $n = 1$ and by 6.3.13 $h + \widetilde{h} = 5$, so either $(h, \widetilde{h}) = (3,2)$ or $(h, \widetilde{h}) = (4,1)$. Write $\Delta' = [(m-1)]$, $\widetilde{\Delta} = [(\widetilde{m}-1)]$ for some $m, \widetilde{m} \geq 1$. The maximal twig of $D^\dagger$ contained in the first branch of F, call it T_2, and the one contained in the second branch of F, call it T_1, are not touched by φ_f, hence they are maximal twigs of D.

Let $\pi : \overline{S} \to U$ be the contraction of $T_1 + C + \Delta'$ to a (smooth) point. Since $b = 1$, the image of B has nonnegative self-intersection, because this contraction touches B at least once. Blow up B on the intersection with T_3 until it decreases to zero. Denote the proper transform of B by $\widetilde{B}$, the resulting surface by $\widetilde{U}$ and the morphism by $\rho : \widetilde{U} \to U$. The center of ρ lies outside $T_1 + C + \Delta'$, so these blowups can be done in different order, i.e. we can first blow up on the intersection of B and T_3 and define a morphism $\widetilde{\rho} : \widetilde{S} \to \overline{S}$ and then contract the proper preimage of $T_1 + C + \Delta'$ by a morphism $\widetilde{\pi} : \widetilde{S} \to \widetilde{U}$.

$$\begin{array}{ccccc} \widetilde{S} & \xrightarrow{\widetilde{\pi}} & \widetilde{U} & \xrightarrow{\eta} & \mathbb{P}^1 \\ {\scriptstyle\widetilde{\rho}}\downarrow & & \downarrow{\scriptstyle\rho} & & \\ \overline{S} & \xrightarrow{\pi} & U & & \end{array}$$

Clearly, $\rho \circ \widetilde{\pi} = \pi \circ \widetilde{\rho}$. Consider the $\mathbb{P}^1$-ruling $\eta : \widetilde{U} \to \mathbb{P}^1$ induced by $\widetilde{B}$. Denote by $\widetilde{T}_3, \widetilde{E} \subseteq \widetilde{U}$ the reduced inverse images of T_3 and E respectively. Put $\widetilde{D} = \widetilde{B} + T_2 + \widetilde{T}_3$. Let $D_2 \subseteq T_2$ and $D_3 \subseteq \widetilde{T}_3$ be the sections of η contained in $\widetilde{D}$ and let F' be the generic fiber. Since $\Sigma_{S_0} = 1$ for the ruling $\eta \circ \widetilde{\pi}$, there exists a unique singular fiber F_1 with $\sigma(F_1) = 2$. Let M_1, M_2 be its S_0-components.

(2) M_1 and M_2 are (-1)-curves. If there exists another singular fiber of η then $F_1 = [1,(\widetilde{m}-1),1]$.

Proof. Suppose there is another singular fiber F_0. Notice that vertical (-1)-curves are S_0-components. We have $\sigma(F_0) = 1$ and F_0 is a column fiber by 5.1.8(ii), hence it contains components of T_2 and $\widetilde{T}_3$. Then F_1 does not contain any $\widetilde{D}$-component. Each M_i intersects D_2 or D_3, so has multiplicity one. It follows that both M_i's are (-1)-curves and $F_1 = [1,(\widetilde{m}-1),1]$, so we are done. We can therefore assume that F_1 is the unique singular fiber of η. Suppose F_1 has only one (-1)-curve. Then D_2 and D_3 intersect tips of F_1 belonging to the first branch, so when we contract F_1 to a smooth fiber we touch $D_2 + D_3$ at most once. This gives two disjoint sections of a $\mathbb{P}^1$-ruling of a Hirzebruch surface, one negative and one non-positive, which is a contradiction. ☐

The morphism $\widetilde{\pi}$ contracts the fiber consisting of $T_1+C+\Delta'$ and the proper transform of B to $\widetilde{B}$, so since $h\leq 4$, we can write $\widetilde{\pi}=p_3\circ p_2\circ\sigma_2\circ p_1\circ\sigma_1$, where p_1,p_2 are sprouting (with respect to the image of the fiber), σ_i are compositions of sequences of subdivisional blowdowns and p_3 is either sprouting if $h=4$ or identity if $h=3$. Notice that $p_1\circ\sigma_1$ is the contraction of $C+\Delta'$. Put $\sigma=\sigma_2\circ p_1\circ\sigma_1$ and let R_i for $i=1,2,3$ be the exceptional divisors of p_i. We now analyze the contraction $\widetilde{\pi}$ and singular fibers of η more closely.

(3) $\widetilde{E}(K_{\widetilde{U}}+\widetilde{D})+E\sigma^*R_2+(h-3)EF'=1.$

Proof. Let us use the common letter E' for the birational transforms of E. We compute how the quantity $E'(K'+D')$, where D' is the reduced inverse image of $\widetilde{D}$ and K' the canonical divisor on a respective intermediate surface between $\widetilde{S}$ and $\widetilde{U}$, changes under subsequent blowdowns. Clearly, it does not change under blowdowns subdivisional for D', hence it does not change under $\widetilde{\rho}$ too. However, if we make a contraction of an exceptional component V which is sprouting for D' then it decreases by $E'V$ (here E' is contained in an intermediate surface between $\widetilde{S}$ and $\widetilde{U}$, for which $V^2=-1$). At the beginning we have $E'(K'+D')=E(K+D+C+\Delta')$. Under σ it decreases by $E'R_1=E\sigma_1^*R_1=E(C+\Delta')$. Under p_i it decreases by $E'R_i$. If $h=4$ then $E'R_3=E'F'=EF'$ because p_3 is preceded by a sprouting blowdown p_2, hence E' intersects the fiber containing R_3 only in R_3. We obtain $\widetilde{E}(K_{\widetilde{U}}+\widetilde{D})=EK-E\sigma^*R_2-(h-3)EF'$. □

(4) There is a unique exceptional S_0-component L, such that $L\widetilde{D}>1$. It satisfies $K_{\widetilde{U}}+\widetilde{D}+L\equiv 0$.

Proof. By Riemann-Roch's theorem $h^0(-K_{\widetilde{U}}-\widetilde{D})+h^0(2K_{\widetilde{U}}+\widetilde{D})\geq K_{\widetilde{U}}(K_{\widetilde{U}}+\widetilde{D})$. The morphism $\rho:\ \widetilde{U}\to U$ is a composition of subdivisional blowdowns in $\widetilde{D}$ and the morphism $\pi:\overline{S}\to U$ is a composition of blowdowns with at least one of them being sprouting for D, hence $K_{\widetilde{U}}(K_{\widetilde{U}}+\widetilde{D})=K_U(K_U+\pi_*D)>K(K+D)=0$. If $2K_{\widetilde{U}}+\widetilde{D}\geq 0$ then $0\leq\kappa(K_{\widetilde{U}}+\widetilde{D})=\kappa(K_U+\pi_*D)=\kappa(K+D+C+\Delta')$, but $C+\Delta'$ contracts to a point using D, so by 5.1.5 this contradicts $\kappa(K+D)=-\infty$. We get $-K_{\widetilde{U}}-\widetilde{D}\geq 0$. Write $-K_{\widetilde{U}}-\widetilde{D}=\sum C_i$ for irreducible C_i's, such that $C_i^2<0$ (cf. 5.1.2(6)). We have $F'(K_{\widetilde{U}}+\widetilde{D})=0$, so C_i's are vertical.

Each S_0-component L intersects $\widetilde{D}$. Suppose each satisfies $L\widetilde{D}=1$. Then F_1 is the only singular fiber of η. Indeed, if $F'\neq F_1$ is a singular fiber then $\sigma(F')=1$ and by 5.1.8(ii) F' is a column fiber, so its exceptional component does not satisfy our assumption. Let $R\subseteq M_1+\widetilde{\Delta}+M_2$ be a chain of components of F_1 connecting two connected components of $F_1\cap\widetilde{D}$ (these components can be points). By our assumption $R\neq M_1$ and $R\neq M_2$ and since $\widetilde{\Delta}\widetilde{D}=0$, we get $M_1+M_2\subseteq R$, hence R contains a 0-divisor. It follows that $F_1=[1,(\widetilde{m}-1),1]$, hence $T_2=D_2$ and $T_3=D_3$. If we now look at the pre-minimal ruling of $S\setminus\Delta$ then we see that $\widetilde{Z}_l$ and Z_l are tips, so $\widetilde{G}$ and G are (-2)-curves, which implies that D contains a component with non-negative self-intersection, a contradiction. Thus there is an exceptional S_0-component L, such that $L\widetilde{D}>1$.

Notice that if for some $i\in\{2,3\}$ the section D_i intersects L then D_i is a maximal twig of $\widetilde{D}$, because $D_iF=1$. It follows that $L\widetilde{D}=2$. Since $(-K_{\widetilde{U}}-\widetilde{D})L=1-\widetilde{D}L<0$, L appears among C_i's. However, $-K_{\widetilde{U}}-\widetilde{D}-L$ is vertical and satisfies $(-K_{\widetilde{U}}-\widetilde{D}-L)^2=K_{\widetilde{U}}(K_{\widetilde{U}}+\widetilde{D})-1\geq 0$, so $-K_{\widetilde{U}}-\widetilde{D}-L\equiv\alpha F$ for some $\alpha\geq 0$. Multiplying by D_i for $i=2,3$ we get $\beta_{\widetilde{D}}(D_i)+LD_i=2-\alpha$. For $\alpha>0$ we would obtain $LD_2=LD_3=0$, which is impossible because $L\widetilde{D}>0$. Thus $K_{\widetilde{U}}+\widetilde{D}+L\equiv 0$. It follows that if another exceptional S_0-component L' has $L'\widetilde{D}>1$ then $L\equiv -K_{\widetilde{U}}-\widetilde{D}\equiv L'$, so $LL'=-1$, hence L is unique. □

(5) $2\leq E\sigma^*R_2=1+EL\leq 3$ and $h=3$.

Proof. Intersecting $K_{\widetilde{U}}+\widetilde{D}+L\equiv 0$ with components of $\widetilde{D}+\widetilde{\Delta}$ we see that $L\widetilde{\Delta}=0$ and L intersects $\widetilde{D}$ only in tips, each tip once. It follows that ρ and π do not touch L. Intersecting $K+T+\widehat{E}\equiv\lambda\mathcal{P}+\mathrm{Bk}^*T+\mathrm{Bk}\,\widehat{E}$ with L we get $EL(1-w_E)\leq(\mathrm{Bk}^*T_2+\mathrm{Bk}^*T_3)L-1$. We have $(\mathrm{Bk}^*T_1+\mathrm{Bk}^*T_3)L<2$, otherwise T_2 and T_3 would be (-2)-chains, which is impossible by 6.5.4(ii). Thus $EL<\frac{1}{1-w_E}$. By (3) we get $E\sigma^*R_2+(h-3)EF'=1-\widetilde{E}(K_{\widetilde{U}}+\widetilde{D})=1+EL<1+\frac{1}{1-w_E}$. By (2) either $w_E\leq\frac{1}{2}$ or $\widehat{E}=[3,(n-1)]$ for some $n\geq 1$ and then $\frac{1}{1-w_E}=2+\frac{1}{n}\leq 3$. In any case $E\sigma^*R_2+(h-3)EF'\leq 3$.

Consider the ruling $\eta\circ\widetilde{\pi}:\widetilde{S}\to\mathbb{P}^1$. Let μ_C and μ_Δ be the coefficients in σ^*R_2 of C and respectively of a component of Δ' intersecting E (put $\mu_\Delta=0$ for $\Delta'=\emptyset$). Clearly, $\widetilde{\rho}$ does not touch $T_1+C+\Delta'+E$. We have $E\sigma^*R_2=\mu_C EC+\mu_\Delta$ and $\mu_\Delta<\mu_C$. Notice that $E\sigma^*R_2\geq 2$, otherwise E is a section of

$\eta \circ \widetilde{\pi}$, which implies $C(E+\Delta) \leq 1$, a contradiction with 6.1.2(ii). Since $E\sigma^* R_2 \leq EF'$, from (3) we get $(h-2)E\sigma^* R_2 \leq 3$, so $h=3$. □

(6) If $T_1 = [(k)]$ then $k=1$.

Proof. Recall that T_1 is contained in the second branch of F (a fiber of a pre-minimal ruling f). Suppose $k>1$. Then by 6.5.5 D contains a chain $[3,1,2,\underline{2}]$. We are now able to eliminate this possibility. As in the proof of 6.5.4 we consider the $\mathbb{P}^1$-ruling p of $\overline{S}$ with $F_\infty = [3,1,2,2]$ as a fiber. Since by 6.5.4(ii) D does not contain a chain $[2,1,2]$, the two (-2)-curves of F_∞ are components of T_1. Consider the curve L given by (4). It is disjoint from $B+T_1$ and intersects the tips of T_2 and T_3. By 6.5.4 we know that the (-3)-curve of F_∞ is not a tip of D, hence $LF_\infty = 0$. By (5) $EL>0$, so L is contained in the fiber of p containing $\widehat{E}$. We have $LT_1 = 0$ and $LE>0$, so the 3-section contained in $\widetilde{D}$ intersects L because L cannot be simple. Hence the 3-section is a maximal twig of D, say it is T_2 (further arguments work for T_3 as well). We can assume that $T_2^2 \neq -3$, otherwise we could take T_2 as a part of new F_∞ and then get a contradiction with 6.5.4(iii). By 6.5.5 $T_3 = [(l),3]$ for some $l \geq 0$. By 6.5.4 we have $\widehat{E} = [3]$ and we can assume that $l \geq 1$. The inequality $\widetilde{e}+\delta \geq 2$ gives $T_2^2 \in \{-4,-5\}$ for $l=1$ and $T_2^2 = -4$ for $l>1$. Noether formula implies $T_2^2 + k + l = 4$. We check that $-\frac{d(D)}{d(\widehat{E})} = \frac{110}{3}$ for $T_2^2 = -5$ and $-\frac{d(D)}{d(\widehat{E})} = 17+13l-2l^2$ for $T_2^2 = -4$ and this is never a square, a contradiction with 6.3.13(ii). □

(7) $T_1 = [(k),3]$ for some $k \geq 1$. $\widehat{E} = [3,2]$.

Proof. Since $h=3$ and $E\sigma^* R_2 = \mu_C EC + \mu_\Delta \leq 3$, we have two possibilities depending on μ_Δ. If $\mu_\Delta > 0$ then $\mu_C > 1$, so $\mu_C = 2$ and $EC = 1$, hence T_1 is $[3,(k)]$ or $[(k),3]$ for some $k \geq 0$. Since one of the maximal twigs of D consists of (-2)-curves, by 6.5.4(ii) the possibility that $T_1 = [3,(k)]$ for some $k>0$ is excluded. If $\mu_\Delta = 0$ then $\Delta' = 0$, so $EC \geq 2$ and $\mu_C = 1$, hence $T_1 = [(k)]$ for some $k \geq 0$. By (6) $T_1 = [(k)]$ is possible only for $k=1$. We only need to prove that T_1 is not a tip. Suppose T_1 is a tip, i.e. $T_1 = [2]$ or $T_1 = [3]$. Then $E\sigma^* R_2 = EF'$ for a generic fiber F'. By (5) we have $2 \leq EL+1 = EF' = \mu_C EC + \mu_\Delta$. Suppose $L \not\subseteq F_1$ (cf. (2)). Then $F_1 = M_1 + \widetilde{\Delta} + M_2$ by (2) because L is vertical. The fiber containing L has $\sigma = 1$, so $\mu(L) \geq 2$ and since $\mu(L)EL \leq EF' \leq 3$, we get $EF' = EL+1 = 2$. This implies that either $\widetilde{\Delta} \neq \emptyset$ and $EM_i = 0$ for some i or $\widetilde{\Delta} = \emptyset$ and $EM_i \leq 1$ for some i. By (4) $M_1\widetilde{D}, M_2\widetilde{D} \leq 1$, so in both cases M_i is simple, which is a contradiction. Therefore $L \subseteq F_1$, say $L = M_1$. We have $E(M_2+\widetilde{\Delta}) \leq E(\underline{F}_1 - L) = 1$ and $M_2\widetilde{D} \leq 1$ by (4). Since $\widetilde{\Delta}M_2 \leq 1$, M_2 is simple, a contradiction. Thus $T_1 = [(k),3]$ for some $k \geq 1$. We conclude that $\Delta' = [2]$ and $E\sigma^* R_2 = 3$, so $EL = 2$. Since $EL < \frac{1}{1-w_E}$ (cf. (5)), we get $\widehat{E} = [3,2]$ because $\widetilde{\Delta} = \emptyset$ by (1). □

(8) $T_2 = [2]$.

Proof. Recall, that T_2 is the maximal twig of D contained in the first branch of F. We have $\Delta' \neq 0$, so by 6.5.4 D does not contain a chain $[3,1,2,2]$. Therefore by 6.5.5 one of T_2 or T_3 is a (-2)-tip. Suppose this is T_3. Clearly, then f is not almost minimal. Thus by 6.3.11 the morphism $\varphi_f : \overline{S}^\dagger \to \overline{S}$ minimalizing $D^\dagger$ contracts precisely $H^\dagger + \widetilde{Z}_1$. Since $T_3 = [2]$, we can write $\widetilde{Z}_l = [l+3]$ for some $l \geq 0$. Since $\widetilde{\Delta} = \emptyset$, $\widetilde{G} + \widetilde{Z}_u + \widetilde{Z}_1 = [(l+3)]$. It follows that φ_f touches Z_1 once. However, $Z_1^2 = -2-k$ because Z_1 becomes a (-1)-curve after contracting $\Delta' + C + T_1$. We get $k=0$, a contradiction with (7). □

From (8) we see that F is produced by the following sequence of characteristic pairs (cf. 6.3.1 and 6.3.4): $\binom{4k}{2k}$, $\binom{2k}{2}$, $\binom{2}{1}$, so the pairs $\binom{c_i}{\underline{p}_i}$ are $\binom{2k}{k}$, $\binom{k}{1}$ and $\tau = 2CE+1 = 3$. The second fiber $\widetilde{F}$ of the pre-minimal ruling is produced by the sequence $\binom{c}{p}$, $\binom{1}{1}$ for some $c,p \geq 1$. We have $\widetilde{\tau}c = d = \tau\underline{c}_1 = 6k$. By (6.8) $3d+1 = \tau(\underline{2}k+k+1)+\widetilde{\tau}(c+p)$, hence $\widetilde{\tau}p = 3k-2$. Then $\widetilde{\tau} = gcd(\widetilde{\tau}c, \widetilde{\tau}p) = gcd(6k, 3k-2) = gcd(3k-2, 4)$, so $\widetilde{\tau} \in \{2,4\}$ ($\widetilde{C}$ would be simple for $\widetilde{\tau} = 1$). Then (6.9) gives $d^2+3 = \tau^2(2k^2+k)+3LE+LE+1+\widetilde{\tau}^2(cp)+\widetilde{\tau}^2$, hence $\widetilde{\tau}^2 = 3k-2$. For $\widetilde{\tau} = 2$ we get $k=2$, so $gcd(c,p) = 2$, a contradiction. Thus $\widetilde{\tau} = 4$ and we get $k=6$ and $(c,p) = (9,4)$, so $\widetilde{G} + \widetilde{Z}_u = [3,2,2,2]$ and $\widetilde{Z}_l = [5,2]$. It follows that Z_1 is touched six times by φ_f, a contradiction with $Z_1^2 = -8$, since $b=1$.

□

Corollary 6.5.7. *$\widehat{E}$ is one of:* $[2,3]$, $[3]$, $[4]$, $[5]$.

Proof. Suppose $|G|\geq 7$. By 6.5.6 $\overline{\kappa}(W)=2$, so by 6.5.3(iii) we have $\epsilon\neq 0$ and $1>\delta>\frac{6}{7}$. For $d_1\geq 3$ we get $d_2=3$ and $d_3\leq 5$. For $d_1=2$ we have $d_2\geq 3$ and the inequality gives $d_2\leq 5$ and $\frac{1}{d_3}\geq\frac{6}{7}-\frac{1}{2}-\frac{1}{3}=\frac{1}{42}$, so $d_3\leq 42$. By 6.4.5 there are only finitely many possibilities for the dual graphs of $\widehat{E}$ and D. Using a computer program we checked that the conditions 6.1.2(vi), 6.2.5, 6.4.4 and 6.3.13(ii) can be satisfied only for $\widehat{E}=[4]$, which contradicts our assumption. We conclude that $\widehat{E}$ is one of: $[2,3]$, $[3]$, $[4]$, $[5]$, $[6]$. However, $[6]$ is excluded, since $\epsilon\neq 0$. □

6.6 Special cases

We have now to deal with the following cases: $\overline{\kappa}(W)=2$ and $\widehat{E}\in\{[2,3],[3],[4],[5]\}$. Let f be a pre-minimal affine ruling of $(\overline{S}^{\dagger},D^{\dagger})$. We use the notation of 6.3.9. Let (x,y,z) with $x\leq y\leq z$ be the ordering of (d_1,d_2,d_3). By 6.5.3 we have $1>\delta>1-\frac{1}{|G|}\geq\frac{2}{3}$, so $x\leq 4$ and $y\leq 11$.

Lemma 6.6.1. *One of the following cases occurs:*

(1) $(x,y)=(3,3)$ *and* $\widehat{E}=[3]$*,*

(2) $(x,y)=(2,3)$*,*

(3) $(x,y)=(2,4)$ *and* $\widehat{E}$ *is either* $[3]$ *or* $[4]$*,*

(4) $(x,y)\in\{(2,5),(2,6)\}$ *and* $\widehat{E}=[3]$*.*

In particular, dual graphs of two maximal twigs of D belong to the list $\mathcal{L}=\{[2],[2,2],[2,2,2],[2,2,2,2],[2,2,2,2,2],[3],[4],[5],[6],[2,3],[3,2]\}$.

Proof. Suppose $z\leq 42$. Given an upper bound for z there is only finite number of possible dual graphs of D. We used a computer program, which showed that for $x\leq 4$, $y\leq 11$, $z\leq 42$ conditions 6.1.2(vi), 6.2.5, 6.4.4, 6.3.13(ii), 6.1.4 and 6.5.3(iii) are satisfied only in three cases:

(i) $b=1$, $T_1=[2]$, $T_2=[4]$, $T_3=[(8),4]$ and $\widehat{E}=[4]$,

(ii) $b=2$, $T_1=[2]$, $T_2=[2,2]$, $T_3=[4,(6)]$ and $\widehat{E}=[4]$,

(iii) $b=2$, $T_1=[2]$, $T_2=[2,2,2]$, $T_3=[3,3,(4)]$ and $\widehat{E}=[4]$,

hence we are done. Now suppose $z>42$. For $x\geq 4$ we get $\frac{1}{z}>1-\frac{1}{|G|}-\frac{1}{2}\geq\frac{1}{6}$, which is impossible. For $x=3$ we have $\frac{1}{y}+\frac{1}{|G|}>\frac{2}{3}-\frac{1}{42}$, which gives $|G|=y=3$. Since $\delta<1$, for $x=2$ we have $y\geq 3$ and $\frac{1}{y}+\frac{1}{|G|}>\frac{1}{2}-\frac{1}{42}$, hence $y\leq 6$ and the bounds on $\widehat{E}$ follow. □

Corollary 6.6.2. *The ruling f has two singular fibers and $\widetilde{h}=2$.*

Proof. By 6.3.7 f has more than one singular fiber and it has at most three because D is a fork. Suppose it has three. Then $D^{\dagger}=D$ and since $x\leq 3$, for one of the singular fibers, say F_1, $F_1\cap D$ has at most two components, hence F_1 is a chain. Moreover, $\widehat{E}=[2,3]$ and $\Delta\subseteq F_1=[2,1,2]$. It follows that the maximal twigs contained in other singular fibers of f have more than two components, a contradiction with 6.6.1.

We have $1\leq\widetilde{h}\leq 2$ because $\widetilde{F}\cap D$ is a chain (cf. 6.3.9). Suppose $\widetilde{h}=1$. Then $\widehat{E}=[2,3]$ and $\widetilde{F}=[2,1,2]$, so $n\geq 2$, otherwise $\widetilde{G}$ would be contracted by φ_f, contradicting the pre-minimality of f. In particular, $\#T_3>2$. By 6.3.13 $h\geq 5$, so the second branch of F contains more than two D-components. Thus at least two maximal twigs of D have more than two components, a contradiction with 6.6.1. □

Let T_1, T_2 be the maximal twigs of D contained respectively in the second and in the first branch of F. (Notice that we did not assume $d_1\leq d_2\leq d_3$, instead we have introduced x,y,z.) Clearly, they are also maximal twigs of $D^{\dagger}$ and φ_f contracts the chain $H^{\dagger}+\widetilde{Z}_1+\widetilde{Z}_u$ to T_3.

We rewrite the equations of 6.3.5 for two fibers. Put $\alpha=n+\epsilon+EK-4$, then $h=3+\alpha$ and $0\leq\alpha\leq n$. Put $\binom{\widetilde{c}_1}{\widetilde{p}_1}=\binom{\widetilde{c}}{\widetilde{p}}$, $\binom{c_1}{p_1}=\binom{c}{p}$ and $\binom{c_{h-1}}{p_{h-1}}=\binom{c'}{p'}$. Since T_1 is a chain, we have $\binom{c_2}{p_2}=\binom{c_3}{p_3}=\ldots=\binom{c_{h-2}}{p_{h-2}}=\binom{c'}{c'}$. Define $u=\tau CE+c_h'CE+c_h'-\tau^2$ and similarly $\widetilde{u}=\widetilde{\tau}\widetilde{C}E+\widetilde{c}_h'\widetilde{C}E+\widetilde{c}_h'-\widetilde{\tau}^2$. We have $u=0$ for $\Delta'=0$ and $u=\frac{1}{2}(1-\tau^2)$ for $\Delta'=[2]$, analogously for $\widetilde{u}$. Now we can write (6.1) as:

$$dn + \gamma - 2 = \tau(p + \alpha c' + p') + \widetilde{\tau}\widetilde{p}. \tag{6.8}$$

We have $d = c\tau = \widetilde{c}\widetilde{\tau}$, hence multiplying the above equation by d and subtracting (6.2) we obtain:

$$d(\gamma - 2) - \gamma = \tau^2((c - c')(\alpha c' + p') - 1) + u - \widetilde{\tau}^2 + \widetilde{u}. \tag{6.9}$$

Remark. Knowing the dual graph of Z_l it is easy to determine c/c' and p/c'. One has $c/c' = d(G + Z_u) = d(Z_l)$ and $p/c' = d(Z_u) = d(Z_l) - d(Z_l - Z_{ll})$ (cf. Appendix of [KR99]).

Remark 6.6.3. For a fixed dual graph of F there is only a finite number of possible dual graphs of $\widetilde{F} + H$.

Proof. If the graph of F is known then we know c, p, c', p', u. The equation (6.8) gives $n(c - c') + \frac{\gamma-2}{\tau} = p + (\epsilon + EK - 4)c' + p' + \frac{\widetilde{\tau}\widetilde{p}}{\tau}$, so $n(c - c') < p + p' + c \leq 2c$, hence $n < 2 + \frac{2c'}{c-c'} \leq 4$. Since now α is bounded, it is enough to bound τ, because then d, and hence $\widetilde{c}, \widetilde{p}$ are bounded. We have $\widetilde{c}\widetilde{\tau} = c\tau$, so $\widetilde{\tau}|c \cdot gcd(\tau, \widetilde{\tau})$. By (6.8) $gcd(\tau, \widetilde{\tau})|\gamma - 2$ and since $\gamma - 2 \in \{1, 2, 3\}$, we get $\widetilde{\tau}|c(\gamma - 2)$ and $\widetilde{\tau} \leq 3c$. Therefore τ and $\widetilde{u}$ are bounded and (6.9) is a nontrivial (the coefficient of τ does not vanish) equation for τ, so we are done. □

Corollary 6.6.4. *$T_3 \in \mathcal{L}$ and $n = 1$.*

Proof. Suppose $T_3 \notin \mathcal{L}$, then $T_1, T_2 \in \mathcal{L}$. Clearly, having the dual graph of T_1, there is only finitely many possibilities for the dual graphs of $T_1 + C + \Delta'$, in each case Z_1^2 is determined. On the other hand, $T_2 = Z_l$) and $G + Z_u$ are adjoint chains (cf. 5.1.7), so the dual graph of $G + Z_u$ is determined by T_2. Then by 6.6.3 there is finitely many possibilities for the dual graphs of $\widetilde{F} + H$. We use a computer program which for given F (in terms of (c, p, c', p')) computes possible $(\gamma, n, \tau, u, \widetilde{\tau}, \widetilde{c}, \widetilde{p}, \widetilde{u})$ using the algorithm sketched in 6.6.3 and checks if (6.8) and (6.9) can be satisfied. In each case (there are many solutions) the maximal twig T_3 is determined and the program returns only these, for which conditions $\delta + \frac{1}{|G|} > 1$, 6.1.2(vi), 6.4.4, 6.3.13(ii) and 6.1.4 hold, these are:

(i) $(n, \gamma, \tau, \widetilde{\tau}) = (1, 4, 4, 2)$, $\binom{c}{p} = \binom{4}{1}$, $\binom{c'}{p'} = \binom{1}{1}$, $\binom{\widetilde{c}}{\widetilde{p}} = \binom{8}{5}$; $b = 2$, $T_1 = [2]$, $T_2 = [(3)]$, $T_3 = [3, 3, (4)]$,

(ii) $(n, \gamma, \tau, \widetilde{\tau}) = (1, 4, 4, 2)$, $\binom{c}{p} = \binom{4}{3}$, $\binom{c'}{p'} = \binom{1}{1}$, $\binom{\widetilde{c}}{\widetilde{p}} = \binom{8}{1}$; $b = 1$, $T_1 = [2]$, $T_2 = [4]$, $T_3 = [(8), 4]$,

(iii) $(n, \gamma, \tau, \widetilde{\tau}) = (2, 4, 4, 2)$, $\binom{c}{p} = \binom{2}{1}$, $\binom{c'}{p'} = \binom{1}{1}$, $\binom{\widetilde{c}}{\widetilde{p}} = \binom{4}{3}$; $b = 2$, $T_1 = [2, 2]$, $T_2 = [2]$, $T_3 = [4, (6)]$.

In cases (i) and (ii) we have $-d(D)/d(\widehat{E}) = 4$ and $gcd(\mu, \widetilde{\mu}) = 4$, in case (iii) $-d(D)/d(\widehat{E}) = 1$ and $gcd(\mu, \widetilde{\mu}) = 2$. By 6.3.13(ii) this is a contradiction.

Suppose now that $n > 1$. Since $D^\dagger = D$, we have $\#T_3 \geq 5$, so $T_1 = [(5)]$ and $\widehat{E} = [3]$. We get $\widetilde{G} + \widetilde{Z}_u = [2]$ and $G + Z_u = [2]$, so $\binom{c}{p} = \binom{2c'}{c'}$ and $\binom{\widetilde{c}}{\widetilde{p}} = \binom{2}{1}$, hence $\widetilde{\tau} = \frac{d}{\widetilde{c}} = c'\tau$. Since $gcd(\tau, \widetilde{\tau})|\gamma - 2$, we get $\tau = 1$, a contradiction. □

We are ready to finish the proof of our main result:

Theorem 6.6.5. *If S' is a normal singular $\mathbb{Q}$-homology plane of negative Kodaira dimension with smooth locus S_0 then $\overline{\kappa}(S_0) < 2$.*

Proof. Suppose $\overline{\kappa}(S_0) = 2$. By 6.6.4 $T_3 \in \mathcal{L}$. We prove successive statements to eliminate all possibilities.

(1) If T_3 is a tip then $T_1 \in \mathcal{L}$ and $\Delta = 0$.
Proof. Write $T_3 = [d_3]$. In this case φ_f contracts $\widetilde{Z}_1$, so f is not almost minimal and we get $\widetilde{u} = 0$ because $\widetilde{\Delta} = 0$ by 6.3.11. We can write $\widetilde{Z}_l = [x + 3]$ for some $x \geq 0$. Since φ_f contracts exactly $H^\dagger$, we obtain $\widetilde{G} + \widetilde{Z}_u = [(x + 2)]$, $G = [x + 5]$, $Z_u = [(x + 1 - d_3)]$ (hence $x \geq d_3 - 1$), $T_2 = [(x + 3), x + 3 - d_3]$ and $Z_1^2 = -b - 1$.

Suppose $T_2 \in \mathcal{L}$. Since $\#T_2 = x + 4 \geq d_3 + 3 \geq 5$, this is possible only for $T_2 = [(5)]$, which implies $x = 1$ and $d_3 = 2$, hence $\binom{\widetilde{c}}{\widetilde{p}} = \binom{4}{3}$ and $\binom{c}{p} = \binom{6c'}{c'}$. Moreover, by 6.6.1 $\widehat{E} = [3]$ and since $\gcd(\tau, \widetilde{\tau})|\gamma - 2$, we see that τ and $\widetilde{\tau}$ are coprime. We get $6c'\tau = d = 4\widetilde{\tau}$, so $\tau|2\widetilde{\tau}$, which implies $\tau = 2$ and $\widetilde{\tau} = 3c'$. Now (6.8) gives $p' = \frac{1}{2}(c' + 1)$ and then by (6.9) $c'^2 - 2c' = 1$, a contradiction. It follows that $T_1 \in \mathcal{L}$.

Suppose $\Delta\neq 0$. Then $\Delta'=[2]$ and $h=3$, so one checks easily that $T_1\in\mathcal{L}$ is possible only for $T_1=[3]$ or $T_1=[2,3]$. We get $\#T_1=b$ by 6.1.2(vi). If $T_1=[2,3]$ then $T_3=[2]$ by 6.6.1 and we compute $d(D)=11x^2+34x-29>0$ ($x\geq d_3-1=1$), which is impossible by 2.2.3(ii). Thus $T_1=[3]$ and $b=1$. If $d_3=3$ then $x\geq 2$, so $-\frac{d(D)}{d(\widehat{E})}=9-\frac{3}{5}x^2$ is not a positive integer, contradicting 6.3.13. Therefore $d_3=2$ and we compute $-\frac{d(D)}{d(\widehat{E})}=\frac{23+2x-x^2}{5}$, which is a square by 6.3.13. This is possible only for $x=3$. We get $\binom{\widetilde{c}}{\widetilde{p}}=\binom{6}{5}$ and $\binom{c}{p}=\binom{22}{3}$, so $22\tau=d=6\widetilde{\tau}$. Since $gcd(\tau,\widetilde{\tau})|\gamma-2$, we get $\tau=3$ and $\widetilde{\tau}=11$. It follows that $u=-4$, so by (6.9) $\tau|\widetilde{\tau}^2+1$, a contradiction. □

(2) T_3 is not a tip.
Proof. Suppose T_3 is a tip. It follows from (1) that $T_1=[(k)]$ for some $1\leq k\leq 5$. By 6.1.2(vi) $k=b+\alpha$, so $k\leq 3$. Suppose $k>1$. If $d_3\neq 2$ then by 6.6.1 $d_3=3$ and $x\geq 2$, so $b=k=2$ and then $d(D)=9(x^2+2x-7)>0$, a contradiction with 2.2.3(ii). If $d_3=2$ then $x\geq 1$, $b=2$ by 6.5.4 and the condition $d(D)=x^2(k+3)+x(2k+10)-(7k+5)<0$ implies $k=3$ and $x=1$. However, $k=3$ implies $\alpha=1$ and then $-\frac{d(D)}{d(\widehat{E})}=\frac{4}{5}$, a contradiction with 6.3.13. Thus $T_1=[2]$ and $d_3\geq 3$.

From $b+\alpha=1$, we get $b=1$ and $\alpha=0$. We have $x\geq d_3-1$ and $d(D)=x^2(d_3-2)-x(d_3^2-6d_3+12)-(4d_3^2-9d_3+18)$. For $5\leq d_3\leq 6$ we get $\widehat{E}=[3]$ by 6.6.1, and then $-\frac{d(D)}{d(\widehat{E})}$ is not a square. Suppose $d_3=4$. We have $\widehat{E}=[3]$ or $\widehat{E}=[4]$, so $-\frac{d(D)}{d(\widehat{E})}$ is a square only for $\widehat{E}=[4]$ and $x=5$. Then $\binom{\widetilde{c}}{\widetilde{p}}=\binom{8}{7}$ and $\binom{c}{p}=\binom{28}{3}$, so $2\widetilde{\tau}=7\tau$ and then $gcd(\tau,\widetilde{\tau})|\gamma-2$ implies $\tau\in\{2,4\}$. For $\tau=2$ (6.8) gives a contradiction, hence $\tau=4$ and $\widetilde{\tau}=14$. We compute $gcd(\mu,\widetilde{\mu})=4$ and $-d(D)/d(\widehat{E})=4$, a contradiction with 6.3.13(ii). Now suppose $d_3=3$. Then $-\frac{d(D)}{d(\widehat{E})}$ is a square only for $\widehat{E}=[3]$ and $x=3$. We get $\binom{\widetilde{c}}{\widetilde{p}}=\binom{6}{5}$ and $\binom{c}{p}=\binom{15}{2}$, so $2\widetilde{\tau}=5\tau/$ and then $gcd(\tau,\widetilde{\tau})|\gamma-2$ implies $\tau=2$ and $\widetilde{\tau}=5$. We compute $\delta+\widetilde{e}=\frac{11}{5}$, a contradiction with 6.5.3(iii). □

(3) If $\#T_3=2$ then $T_2=[2]$ and $T_3=[2,2]$.

Proof. Since $T_3\in\mathcal{L}$, we have $T_3=[2,3]$, $T_3=[3,2]$ or $T_3=[2,2]$. If f is almost minimal then $\#\widetilde{Z}_l=1$, so $\widetilde{G}+\widetilde{Z}_u=0$ consists of (-2)-curves and we see that $\widetilde{Z}_1$ is touched at least twice by φ_f, hence $\widetilde{Z}_1^2\leq -4$, which contradicts $\#\widetilde{\Delta}\leq 1$. Thus f is not almost minimal, so by 6.3.11 $\widetilde{\Delta}=0$ and φ_f contracts $\widetilde{Z}_1+H^{\dagger}$, hence $\#\widetilde{Z}_l=2$. Suppose $T_3=[3,2]$. Then $\widetilde{Z}_l=[3,x]$ for some $x\geq 3$, hence $\widetilde{G}=[2]$. It follows that $G\neq[2]$, hence $T_2\neq[2]$. Since $d_3=5$, by 6.6.1 we get $T_1=[2]$, which implies $Z_1^2=-2$. Since φ_f touches Z_1, we get $b=1$, a contradiction with 6.5.4. Therefore $T_3=[2,k]$ and $\widetilde{Z}_u=[2,x+3]$, where $k\in\{2,3\}$ and $x\geq 0$. We obtain $\widetilde{Z}_u=[(x+1)]$ and $\widetilde{G}=[3]$.

Suppose $Z_u\neq 0$. Then $G+Z_u=[2,x+5,(x-k+1)]$, $Z_1^2=-b-1$ and $Z_u=T_2=[3,(x+2),x-k+3]$. Since $\#T_2>2$, by 6.6.1 we have either $T_1=[2]$ or $d_1=3$ and $\widehat{E}=[3]$. If $d_1=3$ then we have $k=2$, $h=3$ and $\Delta=0$, so $T_1=[2,2]$, $b=2$ and we check that $-d(D)/d(\widehat{E})=-4x^2-8x+17$ is not a square. We infer that $T_1=[2]$, hence $Z_1^2=-2$ and we get $b=1$. By 6.5.4 it follows that $k=3$, hence $x\geq 1$ and $\widehat{E}=[3]$ by 6.6.1. Now we check that $-\frac{d(D)}{d(\widehat{E})}=25+5x-\frac{2}{3}x^2$ is a square only for $x=9$ and then by 6.4.4 $\mathrm{Bk}^2\,\widehat{E}=-\frac{61}{12}<-2$, a contradiction. This proves $Z_u=0$, which gives $G=[2]$ and $T_2=Z_l=[2]$, as required.

We see that φ_f touches $\widetilde{Z}_l$ once, so $x=k-2$. This implies $\binom{\widetilde{c}}{\widetilde{p}}=\binom{2k+1}{k}$ and $\binom{c}{p}=\binom{2c'}{c'}$. We only need to show that $k=2$. Suppose $k=3$. Then $\widehat{E}=[3]$ by 6.6.1, so we have $\tau|d=7\widetilde{\tau}$ and $gcd(\tau,\widetilde{\tau})|\gamma-3$, hence $\tau=7$ and $\widetilde{\tau}=2c'$. However, (6.8) gives $7p'=c'+1$ and then (6.9) implies $3(c')^2-7c'-46=0$, a contradiction. □

(4) $\#T_3=[(k)]$ for some $3\leq k\leq 5$.

Proof. By (2) and (3) we know that $T_3=[(k)]$ for some $k\in\{2,3,4,5\}$. Suppose $k=2$. By (3) $T_2=[2]$ and as in (3) we get $\widetilde{\Delta}=0$ and $\binom{\widetilde{c}}{\widetilde{p}}=\binom{5}{2}$ and $\binom{c}{p}=\binom{2c'}{c'}$. Then $5\widetilde{\tau}=d=2c'\tau$, so (6.8) can be written as $\frac{1}{5}c'\tau(5\alpha-1)=\gamma-2-\tau p'$. It follows that $\alpha=0$, otherwise $\gamma-2-\tau p'\geq 4$, which is impossible. Suppose $\gamma=3$. Then $gcd(\tau,\widetilde{\tau})=1$, so $\tau=5$. We get $c'=5p'-1$ and then (6.9) implies $(c')^2-5c'+u-22=0$. For $\tau=5$ we get $u=0$ or $u=-12$, a contradiction with $c'\in\mathbb{Z}$. Thus $\gamma=4$ and now $gcd(\tau,\widetilde{\tau})|2$, so $\tau\in\{2,5,10\}$. We check that (6.8) and (6.9) lead to a contradiction for $\tau\neq 2$ and for $\tau=2$ give $\binom{c'}{p'}=\binom{25}{6}$. Then $T_1=[(3),7,(6)]$ and $b=2$, hence $d(D)=-25$, a contradiction with 6.3.13(ii). □

(5) f is not almost minimal.

Proof. Notice that by (4) and 6.6.1 $\widehat{E}=[4]$ or $\widehat{E}=[3]$. In particular $\alpha=0$ and $\Delta=0$. Suppose f is almost minimal. Then $\widetilde{Z}_l$ consists of (-2)-curves, so $\widetilde{Z}_u=0$. Let's write $\widetilde{Z}_l=[(s)]$ and $\widetilde{G}=[s+1]$ for some $s\geq 1$. Since $\Delta=0$, we get $\widetilde{Z}_1^2=-2$, hence φ_f does not contract $\widetilde{G}$, otherwise it would contract the whole chain $\widetilde{G}+\widetilde{Z}_1+\widetilde{Z}_l$. This gives $s\geq 2$ because $n=1$ by 6.6.4. If $G\neq[2]$ then $\#T_3\leq 5$ implies $s=2$, $Z_u=0$ and $G=[3]$, so $T_2=Z_l=[2,2]$ and then $T_1=[2]$, a contradiction with 6.4.4(ii). Therefore $G=[2]$, so φ_f touches $\widetilde{G}$ at least twice. Now $\#T_3\leq 5$ implies $s=3$ and $Z_u=0$. By 6.6.1 $\widehat{E}=[3]$. We have $\binom{\widetilde{c}}{\widetilde{p}}=\binom{4}{1}$ and $\binom{c}{p}=\binom{2c'}{c'}$. Then $4\widetilde{\tau}=d=2c'\tau$ and $gcd(\tau,\widetilde{\tau})=1$, so $\tau=2$. Then (6.8) gives $2p'=c'+1$, hence by (6.9) $(c')^2-2c'=1$, a contradiction.

□

Notice that (4) and 6.6.1 imply that $b=2$, otherwise D would contain a chain $[2,1,2]$, which is impossible by 6.5.4. Since f is not almost minimal, φ_f contracts precisely $H^\dagger$, so it touches Z_1, hence $Z_1^2\leq -3$ and $T_1\neq[2]$. We get $Z_l=T_2=[2]$, which implies $G=[2]$ and $\widetilde{G}=[3]$. However, since $T_3=[(k)]$, we can write $\widetilde{Z}_l=[(k-1),x]$ for some $x\geq 3$. Then $\widetilde{G}=[k+1]$ and $k=2$, a contradiction with (4).

□

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