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Collective dynamics of interacting particles

PhD dissertation

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March 2016

Author's declaration:

aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Abstract

The models that describe a collective dynamics of interacting particles belong to a wide class of kinetic models with non-local interaction. A notable example of such model is the classical Vlasov equation. One of the model-defining factors is the kernel of the potential generating the motion. The purpose of this thesis is to analyse one of such models i.e. the Cucker–Smale flocking model with a singular kernel; to approach the problem of well-posedness for this model and to understand the impact of the singularity on its qualitative and quantitative properties.

For the Cucker–Smale particle system with singular kernel $\psi(s) = s^{-\alpha}$ for $\alpha \in (0, 1)$ we prove that the trajectories of the particles can collide and stick together (the latter phenomenon does not occur in case of the model with regular kernel). Moreover we provide the first proof of weak existence and uniqueness of solutions for this range of singularity.

After reducing the range of singularity to $\alpha \in (0, \frac{1}{2})$ we prove existence and uniqueness of strong solutions. Further we apply this result to obtain existence and conditional uniqueness for the singular Cucker–Smale kinetic equation with compactly supported Radon measure as the initial data. This part is achieved by a modified version of the mean–field limit approach resulting in a framework that translates uniform regularity of the particle system to the existence of the measure solution to the kinetic equation. In particular the usual assumption that the solutions of the particle system are stable with respect to the perturbations of the initial data can be omitted.

In the second part we analyse the kinetic Cucker–Smale equation with regular kernel coupled with equations of non-Newtonian shear thickening fluid. We obtain existence and uniqueness of strong solutions in space dimension $d = 3$.

Keywords: alignment, flocking, singular kernel, singular potential, kinetic equation, non-Newtonian fluid.

MSC classes: 82C22, 92C17, 92D50, 92D25, 70B05, 35Q30, 35Q70, 35Q83

Acknowledgements

I would like to thank my parents for their love, care and support from the day I was born up to the day I finish this thesis.

I would like to express my deepest gratitude to my advisor Piotr B. Mucha for his relentless motivation and countless inspiring discussions. I would also like to thank Milan Pokorný from Charles University in Prague for his hospitality and many helpful comments and remarks during my internship in Prague.

During my research I was supported by Foundation for Polish Science as a Ph.D. student in the International Ph.D. Projects operated within the Innovative Economy Operational Programme 2007–2013 (Ph.D. Programme: Mathematical Methods in Natural Sciences). Moreover my research was partially funded by the National Center for Science grant PRELUDIUM 2013/09/N/ST1/04113.

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Chapter 1

Introduction

Flocking, swarming, aggregation - there is a multitude of actual real-life phenomena that from the mathematical point of view can be interpreted as one of these concepts. The mathematical description of collective dynamics of self-propelled agents with nonlocal interaction originates from one of the basic equations of kinetic theory – Vlasov’s equation from 1938. Recently it was noted that such models provide a way to describe a wide range of phenomena that involve interacting agents with a tendency to aggregate their certain qualities. This approach proved to be useful and the language of aggregation now appears not only in the models of groups of animals but also in the description of seemingly unrelated phenomena such as the emergence of common languages in primitive societies, distribution of goods or reaching a consensus among individuals [3, 37, 40, 51]. The general form of equations associated with aggregation models reads as follows:

$$\partial_t f + v \cdot \nabla f + \operatorname{div}_v [(k * f)f] = 0, \quad (1.1)$$

where $f = f(x, v, t)$ is usually interpreted as the density/distribution of particles at the time t with position x and velocity v . Function k is the kernel of the potential generating the motion. It is responsible for the non-local interaction between particles and depending on it the particles may exhibit various tendencies like to flock, aggregate or to disperse. The common properties required from kernel k in most models include Lipschitz continuity and boundedness and it is the case due to the fact that many standard methods work well with such assumptions. For instance if k is Lipschitz continuous and bounded then the particle system associated with (1.1) is well-posed, the characteristic method can be performed for (1.1) and one can usually pass from the particle system to the kinetic equation by mean-field limit. The main goal of this thesis is to consider k that is singular and refine the mean-field limit approach to be applicable in such scenario. We study this problem in a particular case of the Cucker-Smale (C–S) flocking model.

1.1 Cucker–Smale flocking model

In [17] from 2007, Cucker and Smale introduced a model for the flocking of birds associated with the following system of ODEs:

$$\begin{cases} \frac{d}{dt}x_i &= v_i, \\ \frac{d}{dt}v_i &= \sum_{j=1}^N m_j(v_j - v_i)\psi(|x_j - x_i|), \end{cases} \quad (1.2)$$

where N is the number of the particles while $x_i(t)$, $v_i(t)$ and m_i denote the position and velocity of i th particle at the time t and its mass, respectively. Function $\psi : [0, \infty) \rightarrow [0, \infty)$ usually referred to as *the communication weight* is nonnegative and nonincreasing and can be vaguely interpreted as the perception of the particles. The communication weight plays a crucial role in our investigations and we will focus on it more in a while. We refer to system (1.2) as *the C–S particle system* or *the discrete C–S model* (sometimes we omit ‘C–S’).

As $N \rightarrow \infty$ the particle system is replaced by the following Vlasov-type equation:

$$\begin{aligned} \partial_t f + v \cdot \nabla f + \operatorname{div}_v [F(f)f] &= 0, \quad x \in \mathbb{R}^d, \quad v \in \mathbb{R}^d, \\ F(f)(x, v, t) &:= \int_{\mathbb{R}^{2d}} \psi(|y - x|)(w - v)f(y, w, t)dw dy, \end{aligned} \quad (1.3)$$

which can be written as (1.1) with $k(x, v) = v\psi(|x|)$. As mentioned before we are considering (1.3) with a singular kernel

$$\psi(s) = \begin{cases} s^{-\alpha} & \text{for } s > 0, \\ \infty & \text{for } s = 0, \end{cases} \quad \alpha > 0. \quad (1.4)$$

We refer to equation (1.3) as *the kinetic C–S equation*, *the Vlasov-type C–S equation* or *the continuous C–S model* (sometimes we omit ‘C–S’).

Before we proceed with a more detailed statement of our goals let us briefly introduce state of the art for models of flocking and the motivations behind studying such models with singular kernels. The literature on aggregation models associated with Vlasov-type equations of the form (1.1) is very rich thus we will only mention a few examples on some of the more popular branches of the research. Those branches include the analysis of time asymptotics (see e.g. [29]) and pattern formation (see e.g. [28, 50]) or analysis of the models with additional forces that simulate various natural factors (see e.g. [12, 21] - deterministic forces or [16] - stochastic forces). The other variations of the model include forcing particles to avoid collisions (see e.g. [14]) or to aggregate under the leadership of certain individuals (see e.g. [15]). A good example of a paper in which a well rounded analysis of a model that includes effects of attraction, repulsion and alignment is [8]. The story of C–S model should probably begin with [52] by Vicsek *et al.*, where a model of flocking with nonlocal interactions was introduced and it is widely recognized to be up to some degree an inspiration for [17]. Since 2007 the C–S model with a regular communication weight of the form

$$\psi_{cs}(s) = \frac{K}{(1 + s^2)^{\frac{\beta}{2}}}, \quad \beta \geq 0, \quad K > 0 \quad (1.5)$$

was extensively studied in the directions similar to those of more general aggregation models (i.e. collision avoiding, flocking under leadership, asymptotics and pattern formation as well as additional deterministic or stochastic forces - see [2, 11, 27, 30, 42, 47]). Particularly interesting from our point of view is the case of passage from the particle system (1.2) to the kinetic equation (1.3), which in case of the regular communication weight was done for example in [31] or [32]. For a more general overview of the passage from microscopic to mesoscopic and macroscopic descriptions in aggregation models of the form (1.1) we refer to [9, 18, 19].

1.2 C–S model with a singular communication weight

In the paper [31] from 2009 the authors considered C–S model with singular weight (1.4) obtaining asymptotics for the particle system but even the basic question of existence of solutions remained open till later years. The main goal of this thesis is to answer this question. More precisely the goal is to approach the problem of well-posedness for kinetic equations of the type (1.1) with a focus on the C–S model with a singular weight. This problem required a complex approach ranging from quantitative and qualitative analysis of the C–S particle system to applying such analysis in the passage to the kinetic case.

It turned out that system (1.2) possesses drastically different qualitative properties depending on whether $\alpha \in (0, 1)$ or $\alpha \in [1, \infty)$. In [1] the authors observed that for $\alpha \geq 1$ the trajectories of the particles exhibit a tendency to avoid collisions, which they used to prove conditional existence and uniqueness of smooth solutions to the particle system.

On the other hand in [43] (which results are included in this thesis) we proved existence of what we called *piecewise–weak* solutions to the particle system with $\alpha \in (0, 1)$ and gave an example of solution that experienced not only collisions of the trajectories but also sticking (i.e. two different trajectories could start to coincide at some point). This dichotomy is an effect of integrability (or of the lack of thereof) of ψ in a neighbourhood of 0. It is also the reason why the approach to C–S model should vary depending on α . One of the latest contributions to this topic is [10] where the authors showed local in-time well–posedness for the kinetic equation (1.3) with a singular communication weight (1.4) and with an optional non-linear dependence on the velocity in the definition of $F(f)$. They also presented a thorough analysis of the asymptotics for this model. The other more recent addition is [44], where we proved existence and uniqueness of $W^{1,p}$ strong solutions to the particle system (1.2) with a singular weight (1.4) and $\alpha \in (0, \frac{1}{2})$. The paper [44] is a part of this thesis.

There are a few reasons for considering C–S model with a singular weight. Aside from the fact that it appears to be an interesting mathematical problem in itself (as in most models involving singular potentials), there are also some more involved motivations. To understand one of these motivations one needs first to take a closer look at the model with regular weight. In (1.2), the purpose of a regular ψ of the form (1.5) is to suppress the interactions between distant particles. It is only natural to expect that particles that represent members of a flock interact mostly with their closest neighbours. However in some instances such effect can be insufficient. Suppose that we have two groups of particles separated by a very large distance.

The first group (A) consists of two particles while the second (B) consists of the remaining $N - 2$ particles. In such situations it is often reasonable to expect that from the point of view of the model, the interactions between the groups are insignificant compared to the interactions within each group. However if N is large enough, assuming that all particles have the same mass (i.e. $m_i = \frac{1}{N}$) then the factor $m_j = \frac{1}{N}$ in (1.2)₂ makes it so that the interaction between any two particles (including those from group A) is negligible compared to the impact that B has on A. Often such phenomenon is undesirable. There is a couple of ways to deal with this situation (perhaps the most natural can be found in [39]) and one of them is taking ψ of the form (1.4) which not only suppresses the distant interactions but also emphasises the local interactions. The other motivations include the rich dynamics of C-S model with singular communication weight that depending on α allows for the trajectories of the particles to stick together or makes it more difficult for them to even collide. Both of these qualitative phenomena are absent in the case of regular weight. They are on the other hand quite welcome from the modelling point of view. The first indicating a stronger than usual tendency for pattern formation and emergent phenomena, and the second taking into the account that in typical physical scenarios the particles avoid collisions (e.g. birds or fish).

1.3 Models of flocking for fluid-embedded particles

One of other directions of research is the analysis of the motion of agents (described by a kinetic model of the type (1.1)) in their natural habitat. Hence, parallely to the analysis of the kinetic models themselves, research in coupling models of kinetic theory with models of hydrodynamics was performed (see [5–7, 13, 23–25]). From the point of view of this thesis the most important examples of such research is the paper [7] in which the coupling of Navier-Stokes system (N–S) with Vlasov equation is considered and the paper [5] in which the approach of [7] is applied to N–S coupled with C–S (since C–S equation is actually a Vlasov-type equation). The secondary goal of this dissertation is to modify the approach used in [7] and [5] and couple C–S model with models of non-Newtonian fluids, which up to this point was not done.

Our goal is to consider particles embedded in an incompressible, viscous, non-Newtonian shear thickening fluid, i.e. we aim to couple (1.2) with the system

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla \pi - \operatorname{div}(\tau) = f_{ext}, \\ \operatorname{div} u = 0, \end{cases} \quad (1.6)$$

which describes a motion of such fluid in \mathbb{R}^d . The function

$$u = u(t, x) = (u_1(t, x), u_2(t, x), \dots, u_d(t, x))$$

represents velocity of the fluid at the position x and time t . Equation (1.6)₂ expresses the conservation of mass, while (1.6)₁ expresses the conservation of momentum. The term τ in (1.6)₁ denotes a symmetric stress tensor that depends on Du – the symmetric part of the gradient of u i.e. $\tau = \tau(Du)$, where $Du = \frac{1}{2}[\nabla u + (\nabla u)^T]$. In our considerations we assume

that τ is derived from some scalar potential ϑ and through some specified later properties of ϑ we actually impose various other assumptions on τ including polynomial growth or coercivity. Lastly function f_{ext} represents an external force.

The coupling of (1.2) with (1.6) is done via the drag force

$$F_d(t, x, v) := u(t, x) - v,$$

that influence the motion of the particles and the fluid. This way of coupling and such drag force is adopted from [7] and [5] and originally it was used for the modelling of thin spray and fluid (see also [6, 7, 23–25]). Explicitly, the coupled system reads as follows:

$$\begin{cases} \partial_t f + v \nabla f + \operatorname{div}_v [(F(f) + F_d) f] &= 0, \\ \partial_t u + (u \cdot \nabla) u + \nabla \pi - \operatorname{div}(\tau) &= -d \int_{\mathbb{R}^d} F_d f dv, \\ \operatorname{div} u &= 0. \end{cases} \quad (1.7)$$

Let us briefly discuss the difference between coupling of the C–S model with Newtonian and non-Newtonian fluids. In [7] and [5], the authors obtained weak existence for their coupled systems and on top of that in [5], the authors obtained asymptotic flocking. In particular there was little hope to obtain regularity or uniqueness for coupled N-S-Vlasov or N-S-C-S systems without previously obtaining it for N-S system. However in case of coupling with a non-Newtonian fluid, existence, regularity and possibly uniqueness depend on the value of the exponent p and regularity of the external function f_{ext} . For uncoupled non-Newtonian system (1.6) weak existence is known for $p > \frac{2d}{d+2}$ and $f_{ext} \in (W^{1,p})^*$ and it is obtained by Lipschitz truncation method (see [20, 22]). On the other hand if $p \geq \frac{3d+2}{d+2}$ and $f_{ext} \in L^2(0, T; L^2(\mathbb{T}^d))$, not only do we have existence of strong solutions but also uniqueness (see [46]). Therefore, on top of the interesting asymptotics similar to those from the paper [5], we may expect the possibility of better regularity and of uniqueness for the coupled system. However it depends on p and the structure of the external force, which in our case equals

$$f_{ext} = -d \int_{\mathbb{R}^d} (u - v) f dv.$$

Moreover for $p \in (\frac{2d}{d+2}, \frac{3d+2}{d+2})$ in case of the coupled system it appears that a combination of our approach with the Lipschitz truncation method should make obtaining existence of weak solutions possible but it is outside of our scope.

1.4 Main results

The following thesis presents my contribution in the development of the existence theory for the C–S model with a singular communication weight. It should be viewed as a step towards well-posedness for this system. Since 2014 we have managed to make a first successful attempt on proving:

- existence of piecewise-weak solutions to the C–S particle system for the range of singularities $\alpha \in (0, 1)$, published in [43];
- existence and uniqueness of strong solutions to the C–S particle system for the range of singularities $\alpha \in (0, \frac{1}{2})$, published in [44];
- existence and conditional uniqueness of solutions to the C–S kinetic equation for the range of singularities $\alpha \in (0, \frac{1}{2})$, included in the preprint [41];
- possibility of sticking of the trajectories of the particles, published also [43].

To our best knowledge these are the first results on existence for the C–S model with singular weight with $\alpha \in (0, 1)$ and one of the first steps in the direction of well-posedness for this model, which is important from the point of view of applications and numeric analysis. Due to the lack of existing theory we had to develop relatively new (often elementary) techniques or to significantly modify the existing ones. The analysis of the particle system was performed with the methods that originated from elementary techniques of the theory of systems of ODE's. On the other hand the analysis of the kinetic equation was done by passing from the particle to kinetic description. Such passage was performed using much more sophisticated methods: originating from the stochastic analysis of complex many-body systems, mean-field limit method, was used to establish the kinetic equation as a limiting case of the particle system. The topology in which the limiting process was performed was generated by the *Wasserstein W_1 metrics* (sometimes referred to as *Kantorovich-Rubinstein metric* or *bounded-Lipschitz distance*). In order to apply this method in the case of singular communication weight we had to significantly modify it. We explain the methodology more thoroughly at the beginning of Part I.

Additionally we obtained

- existence and uniqueness of strong solutions to the C–S kinetic equation with a regular communication weight coupled with equations of non-Newtonian shear thickening fluids. This work is based on the modified methods from [7] and [5] coupled with results from [46] and [38] and is included in the preprint [45].

This goal is achieved throughout Part II.

The dissertation is organised as follows. It is divided in two major parts: Part I in which we present the results concerning the existence theory for the C–S model with a singular communication weight and Part II in which we present the coupled kinetic-fluid system. These parts are somewhat independent and thus at the beginning of each of them there is a section dedicated to introducing the part-specific notation and presenting a precise statement of the results. The general overview of the strategy of the proofs is also part-specific and is introduced at the beginning of each part. At the end of the dissertation we included Part II with appendices into which we moved the more self-contained and/or tedious proofs.

Part I

Cucker–Smale model with a singular communication weight

Chapter 2

Part I: Introduction

The main goal of the dissertation is realized in the next three chapters. We aim to prove that for any initial Radon measure f_0 and $T > 0$ the Vlasov-type C–S equation

$$\begin{aligned} \partial_t f + v \cdot \nabla f + \operatorname{div}_v [F(f)f] &= 0, \quad x \in \mathbb{R}^d, \quad v \in \mathbb{R}^d, \\ F(f)(x, v, t) &:= \int_{\mathbb{R}^{2d}} \psi(|y - x|)(w - v)f(y, w, t)dw dy, \end{aligned} \quad (2.1)$$

with singular weight

$$\psi(s) = \begin{cases} s^{-\alpha} & \text{for } s > 0, \\ \infty & \text{for } s = 0, \end{cases} \quad \alpha > 0 \quad (2.2)$$

admits solutions in the interval $[0, T]$, provided that the range of singularity of ψ is less than 1 (i.e. $\alpha \in (0, 1)$).

The general strategy is standard and can be summarized as follows. It is natural to expect that the analysis (be it qualitative or quantitative) of the particle system associated with the C–S model

$$\begin{cases} \frac{d}{dt}x_i &= v_i, \\ \frac{d}{dt}v_i &= \sum_{j=1}^N m_j(v_j - v_i)\psi(|x_j - x_i|), \end{cases} \quad (2.3)$$

is more approachable than in the continuous case. Here again N is the number of particles while $x_i(t)$, $v_i(t)$ and $m_i(t)$ are the i th particles' position velocity and mass, respectively. We define the solutions to (2.1) by approximation with the solutions of (2.3) with the number of particles N going to infinity. We use the mean-field limit approach. Given a Radon measure $f_0 = f_0(x, v)$, where $x \in \mathbb{R}^d$ and $v \in \mathbb{R}^d$ as an initial datum, we divide it's support into congruent cubes $Q_{i,\epsilon} \subset \mathbb{R}^d \times \mathbb{R}^d$ of diameter $\epsilon > 0$ (the centres of the cubes, denoted by

$(x_{i,\epsilon}, v_{i,\epsilon})$, form a lattice-shaped ϵ -net on the support of f_0). In the centre of each of the cubes we place a Dirac's delta of a mass $m_{i,\epsilon}$ equal to the total mass of f_0 restricted to the cube i.e.

$$m_{i,\epsilon} := \int_{Q_{i,\epsilon}} f_0(x, v) dx dv.$$

This way, denoting the number of cubes by N_ϵ , we obtain

$$f_{0,\epsilon} := \sum_{i=1}^{N_\epsilon} m_{i,\epsilon} \delta_{x_{i,\epsilon}} \otimes \delta_{v_{i,\epsilon}}$$

which we prove that converges¹ to f_0 as $\epsilon \rightarrow 0$. However, alternatively we may look at the Dirac's deltas $m_{i,\epsilon} \delta_{x_{i,\epsilon}} \otimes \delta_{v_{i,\epsilon}}$ as a description of starting points of the particles in the system (2.3), where $x_{i,\epsilon}$, $v_{i,\epsilon}$ and $m_{i,\epsilon}$ denote the initial position and velocity and the mass of i th particle, respectively. Then the solution of the particle system (denote it by (x_ϵ, v_ϵ)) can be again interpreted as measure valued function f_ϵ from the time interval $[0, T]$. Then we converge with $\epsilon \rightarrow 0$ and hopefully extract a subsequence that converges to some measure valued function f , which serves as a candidate for the solution to (2.1). This general strategy is utilised for example in [31], where the authors prove well-posedness for the kinetic equation (2.1) with a regular communication weight. However they strongly rely on the Lipschitz continuity and boundedness of ψ , which allows them to obtain well-posedness for the particle system, which makes the convergence with the approximate solutions f_ϵ straightforward. On the other hand in case of singular communication weight there is little hope for the well-posedness for the particle system, which in turn makes extraction of the convergent subsequence much more difficult.

We follow the presented above strategy in the next 3 chapters:

- In Chapter 3 we focus on the case of existence for the particle system (2.3) with the range of singularity $\alpha \in (0, 1)$. We prove that for any initial data in the form of finite number of particles there exists piecewise-weak solution with various useful structural properties. We also provide an example of a solution with trajectories that stick together in a finite time (such phenomenon cannot occur in the case of regular communication weight).
- In Chapter 4 we strengthen the results from Chapter 3 proving that by restricting the range of admissible α to $(0, \frac{1}{2})$ we obtain existence and uniqueness of strong solutions to the particle system (2.3). This is the closest result to the well-posedness for the C-S particle system with a singular communication weight.
- In Chapter 5 we use the results from Chapter 4 to obtain existence for the kinetic equation (2.1) with a singular weight with $\alpha \in (0, \frac{1}{2})$. We adopt the mean-field limit method that we sketched at the beginning of this section. The restriction of admissible α to the interval $(0, \frac{1}{2})$ comes directly from Chapter 4 and should be understood in

¹We define the appropriate topology later.

the following way: in order to obtain existence for the kinetic equation by mean-field limit one needs sufficient regularity of solutions to the particle system that grant compactness of the sequences of approximate solutions. In other words, our technique works if only the solutions to the particle system are regular enough.

2.1 Part I: Preliminaries and notation

Throughout Part I $x = (x_1, \dots, x_N) \in \mathbb{R}^{Nd}$, where $x_i = (x_{i,1}, \dots, x_{i,d})$ denotes the position of the particles, $v = \dot{x}$ is their velocity, while N and d are the number of the particles and the dimension of the space respectively. Moreover by $B_i(t)$ we denote the set of all indices j , such that up to the time t , the trajectory of x_j does not coincide with the trajectory of x_i . Assuming that the trajectories, once coinciding cannot separate, we define it as

$$B_i(t) := \{k = 1, \dots, N : x_k(t) \neq x_i(t) \text{ or } v_k(t) \neq v_i(t)\}, \quad (2.4)$$

since any two particles with sufficiently smooth trajectories have the same position and velocity at the time t , if and only if they move on the same trajectory. Further, let $\Omega \subset \mathbb{R}^d$ be an arbitrary domain with $d \in \mathbb{N}$. By $W^{k,p}(\Omega)$ we denote the Sobolev's space of the functions with up to k th weak derivative belonging to the space $L^p(\Omega)$, while by $C(\Omega)$ and $C^1(\Omega)$ we denote the space of continuous and continuously differentiable functions, respectively. Hereinafter, $B((x_0, v_0), R)$ denotes a ball in \mathbb{R}^{2d} centred in (x_0, v_0) with radius R . On the other hand $B_x(x_0, R)$ and $B_v(v_0, R)$ denote balls in \mathbb{R}^d with radius R centred in x_0 and v_0 , respectively. For any positive a , by $aB_v(v_0, R)$ we understand a homothetic transformation of $B_v(v_0, R)$, i.e., $B_v(v_0 a, Ra)$.

Definition 2.1.1. *We say that i th and j th particles collide at the time t if and only if $x_i(t) = x_j(t)$ and we say that they stick together at the time t if and only if $x_i(t) = x_j(t)$ and $v_i(t) = v_j(t)$.*

Throughout the dissertation C denotes a generic positive constant that may change from line to line even in the same inequality.

2.1.1 Piecewise-weak solutions of the particle system

Before we present the definition of the piecewise-weak solutions let us mention one very reasonable characteristic that we would like to make sure that they possess. Namely, that the trajectories of the particles cannot separate if they stick together at some point. This is somewhat related to uniqueness of solutions to the particle system with regular communication weight since (even though in such case sticking of the trajectories is impossible) if two particles have the same position and velocity to begin with, then they cannot separate.

However, since ψ is singular at 0 it may happen that the solutions of the C-S model with ψ are not unique and that the trajectories may split as in the case of the well known example $\dot{y} = cx^{\frac{1}{3}}$. In fact a loss of uniqueness may happen at each time t , such that there exist i and j , such that $x_i(t) = x_j(t)$. It is problematic because such times t include not only each time of collision but also each time at which some particles are stuck together. Thus if for example two particles x_i and x_j start with the same position and velocity, then we may lose uniqueness at an arbitrary time $t > 0$. Therefore we will enforce that the once stuck trajectories cannot separate, by replacing equation (2.3) with (2.5), which does not distinguish trajectories that once stuck together. Therefore in Chapter 3 and to some extent in Chapter 4 we consider C-S model defined by (2.5). For this model we still do not necessarily have uniqueness but the times at which we lose it are restricted only to the times of collisions, which as we will prove occur in some sense rarely.

Definition 2.1.2. Let $0 = T_0 \leq T_1 \leq \dots \leq T_{N_s}$, be the set of all times of sticking and $T_{N_s+1} := T$ be a given positive number. For $n \in \{0, \dots, N_s\}$, on each interval $[T_n, T_{n+1}]$, we consider the problem

$$\begin{cases} \frac{dx_i}{dt} = v_i, \\ \frac{dv_i}{dt} = \frac{1}{N} \sum_{k \in B_i(T_n)} (v_k - v_i) \psi_n(|x_k - x_i|), \\ x_i \equiv x_j \text{ if } j \notin B_i(T_n) \end{cases} \quad (2.5)$$

for $t \in [T_n, T_{n+1}]$, with initial data $x(T_n), v(T_n)$.

We say that (x, v) solves (2.5) on the time interval $[0, T]$, with weight given by (2.2) and arbitrary initial data $x(0) = x_0, v(0) = v_0$ if and only if, for all $n = 0, \dots, N_s$ and arbitrarily small $\epsilon > 0$, the function $x \in (C^1([0, T]))^{N_d}$ is a weak in $(W^{2,1}([T_n, T_{n+1} - \epsilon]))^{N_d}$ solution of (2.5).

Remark 2.1.1. In the above definition and throughout the thesis we sometimes say that a pair such as (x, v) satisfies some equation weakly in $W^{2,1}$. In reality we mean that $x \in W^{2,1}$ satisfies the equation and that $v = \dot{x} \in W^{1,1}$ (and not necessarily does v belong to $W^{2,1}$).

Remark 2.1.2. Definition 2.1.2 may seem not clear at the first glance. Between any two times of sticking T_n and T_{n+1} the solution exists in a $W^{2,1}$ sense. However as we approach T_{n+1} we lose absolute continuity of v and thus we have to separate ourselves from T_{n+1} by some arbitrary $\epsilon > 0$. In general, such separation rises a question of how the piecewise-weak solution on $[T_n, T_{n+1}]$ influences itself on the next interval $[T_{n+1}, T_{n+2}]$. In particular is it really influenced by the initial data beyond the time $t = T_1$ at which the first separation occurs? The answer lies within the continuity of both x and v , which enables us to continuously prolong the solution up to T_{n+1} and in such a way to establish a unique initial data for the solution on $[T_{n+1}, T_{n+2}]$. This way we obtain a continuous function (x, v) defined on $[0, T]$, that truly corresponds to the initial data and solves (2.5) weakly on each interval $[T_n, T_{n+1} - \epsilon]$.

Remark 2.1.3. As mentioned before, in Definition 2.1.2 the purpose of redefining the system (2.5) at each time of sticking T_n (by including the set $B_i(T_n)$) is to ensure that once stuck together trajectories cannot separate. However, in Chapter 4 under the assumption that $\alpha \in (0, \frac{1}{2})$ we prove existence and uniqueness of $W^{2,1}$ solutions to (2.3) and for such solutions trajectories cannot separate anyway and thus there is no need to redefine the system as in Definition 2.1.2. However we prove that in case of $\alpha \in (0, 1)$, the solutions in the sense of Definition 2.1.2 are also unique (whether they belong to $W^{2,1}$ or not – see Theorem 2.2.3) and in this case sets $B_i(T_n)$ are crucial.

2.1.2 Bounded-Lipschitz distance

In Chapter 5 we frequently use the *bounded-Lipschitz distance* (also known as the *flat metric*), which in the sense explained in [48] p. 26 is a version of Kantorovich-Rubinstein distance (or Wasserstein-1 distance).

Definition 2.1.3 (Bounded-Lipschitz distance). *For any probabilistic measures μ and ν we define*

$$d(\mu, \nu) := \sup_g \left| \int_{\Omega} g d\mu - \int_{\Omega} g d\nu \right|,$$

where the supremum is taken over all bounded and Lipschitz continuous functions g , such that $\|g\|_{\infty} \leq 1$ and $Lip(g) \leq 1$.

In the above definition $\|g\|_{\infty}$ and $Lip(g)$ represent L^{∞} norm and Lipschitz constant of g . This also leads to a need of distinction between spaces of measures with different topologies i.e. we denote $\mathcal{M} = \mathcal{M}(\Omega) = (\mathcal{M}, TV)$ as the space of finite Radon measures defined on Ω with *total variation* topology and we denote (\mathcal{M}, d) as the space of finite Radon measures defined on Ω with bounded-Lipschitz distance topology. The importance of the space (\mathcal{M}, d) comes from the prime difference between the bounded-Lipschitz distance and the total variation. Namely for $x_1 \neq x_2$, we have $TV(\delta_{x_1} - \delta_{x_2}) = 2$, while $d(\delta_{x_1}, \delta_{x_2}) \leq C|x_1 - x_2|$. In particular, if $x_n \rightarrow x$ in Ω then $\delta_{x_n} \rightarrow \delta_x$ in d , which is not the case in TV .

In our considerations a crucial role is played by

$$\mathcal{M}_+ := \{\mu \in \mathcal{M} : \mu \text{ is nonnegative}\}$$

both with TV and $d(\cdot, \cdot)$ topology. If Ω is a compact subset of \mathbb{R}^d , then \mathcal{M} is isomorphic to $(C_b(\Omega))^*$ ². There is a very convenient relation between the weak * topology in $(C_b(\Omega))^*$ and the topology generated by bounded-Lipschitz distance on \mathcal{M} that we present below.

²This nice property does not hold in general. For example if $\Omega = \mathbb{R}^d$ then $(C_b(\Omega))^*$ is isomorphic to the space of regular bounded finitely additive measures, while $(C_0(\Omega))^*$ is isomorphic to \mathcal{M} .

Definition 2.1.4. We say that a sequence $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$ converges narrowly to a measure $\mu \in \mathcal{M}$ if

$$\lim_{n \rightarrow \infty} \int_{\Omega} \phi(x) d\mu_n(x) = \int_{\Omega} \phi(x) d\mu(x)$$

for all $\phi \in C_b(\Omega)$.

Remark 2.1.4. The narrow convergence is exactly the weak * convergence in $(C_b(\Omega))^*$ which is equivalent to the weak convergence of measures (which should not be confused with the weak convergence in the sense of functional analysis). It may seem redundant to introduce a new notion for weak * convergence of measures but it is justified if we consider the spaces of measures defined on whole \mathbb{R}^d . Assuming that $\Omega = \mathbb{R}^d$ the space of all Radon measures \mathcal{M} with TV topology is isomorphic to $(C_0(\Omega))^*$ which is by no means isomorphic to $(C_b(\Omega))^*$. Then the weak * convergence in $(C_0(\Omega))^*$ (which plays the role of weak convergence of measures in \mathcal{M}) is different than the narrow convergence.

Remark 2.1.5. In our framework we usually assume that Ω is compact in which case, as mentioned before, the narrow convergence is exactly the weak convergence for measures in \mathcal{M} . Alternatively, since the C–S model has the property of conservation of mass we could consider probabilistic measures and for them the narrow convergence is also equivalent to weak convergence. Either way, throughout Part I, the narrow convergence, the weak * convergence in $(C_b(\Omega))^*$ and the weak convergence in \mathcal{M} are equivalent.

Definition 2.1.5. We say that a sequence $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathcal{M}$ is tight if for all $\epsilon > 0$ there exists a compact $K_\epsilon \subset \subset \Omega$ such that for all $n \in \mathbb{N}$ we have

$$|\mu_n|(\Omega \setminus K_\epsilon) < \epsilon,$$

where $|\mu|$ is the total variation measure of μ .

For compact Ω , the relation between the narrow convergence in \mathcal{M} and the convergence in (\mathcal{M}, d) is as follows.

Proposition 2.1.1. Suppose that Ω is compact and let $\{\mu_n\}_{n \in \mathbb{N}}$ be a tight sequence in \mathcal{M} and let $\mu \in \mathcal{M}$. Then $\mu_n \rightarrow \mu$ narrowly if and only if $d(\mu_n, \mu) \rightarrow 0$ and $\{\mu_n\}_{n \in \mathbb{N}}$ is bounded in \mathcal{M} .

The proof of Proposition 2.1.1 is a modification of the proof of Theorem 2.7 from [26].

Proof of Proposition 2.1.1. (\Rightarrow) Suppose that μ_n converges weakly * to $\mu \in \mathcal{M}$. Then by Banach-Steinhaus theorem we have $\sup_{n \in \mathbb{N}} |\mu_n|(\Omega) < \infty$ i.e. $\{\mu_n\}_{n \in \mathbb{N}}$ is bounded in \mathcal{M} . Using the definition of $d(\cdot, \cdot)$, we obtain

$$d(\mu_n, \mu) \leq \sup \left\{ \int_K g(d\mu_n - d\mu) : \|g\|_\infty \leq 1, \text{Lip}(g) \leq 1 \right\} + \int_{\Omega \setminus K} 1 d(|\mu_n| + |\mu|)$$

for any compact $K \subset \Omega$. Thus by tightness of $\{\mu_n\}_{n \in \mathbb{N}}$ for any $\epsilon > 0$ there exists $K_\epsilon \subset\subset \Omega$ such that

$$\sup_n (|\mu_n| + |\mu|)(\Omega \setminus K_\epsilon) \leq \frac{\epsilon}{3}.$$

Let

$$A_\epsilon := \left\{ g : \|g\|_\infty \leq 1, \text{Lip}(g) \leq 1, \text{supp} g \subset K'_\epsilon \right\},$$

where $K_\epsilon \subset\subset K'_\epsilon \subset\subset \Omega$. Then by Arzela–Ascoli Theorem (Theorem A.4.1) set A_ϵ is compact in $C_b(\Omega)$. Therefore there exists a finite set of Lipschitz functions $\{g_i\}_{i=1}^{k_\epsilon}$ in A_ϵ such that

$$\left(\sup_{n \in \mathbb{N}} |\mu_n|(\Omega) + |\mu|(\Omega) \right) \sup_{g \in A_\epsilon} \left\{ \int_{i=1, \dots, k_\epsilon} \|g - g_i\|_\infty \right\} \leq \frac{\epsilon}{3}$$

and thus

$$d(\mu_n, \mu) \leq \max_{i=1, \dots, k_\epsilon} \int_{K_\epsilon} g_i d(\mu_n - \mu) + \frac{\epsilon}{3} + \frac{\epsilon}{3}.$$

Due to the narrow convergence of $\{\mu_n\}_{n \in \mathbb{N}}$, there exists $m_\epsilon \in \mathbb{N}$ such that

$$\max_{i=1, \dots, k_\epsilon} \left| \int_{K_\epsilon} g_i d(\mu_n - \mu) \right| \leq \frac{\epsilon}{3}$$

for every $n \geq m_\epsilon$. This implies that $d(\mu_n, \mu) \leq \epsilon$ for every $n \geq m_\epsilon$ and thus $d(\mu_n, \mu) \rightarrow 0$ as $n \rightarrow \infty$.

(\Leftarrow) Let $d(\mu_n, \mu) \rightarrow 0$ and let $\sup_{n \in \mathbb{N}} |\mu_n|(\Omega) < \infty$. We will show that $\mu_n \rightarrow \mu$ narrowly. Fix $\phi \in C_b(\Omega)$. Then for any $\epsilon > 0$ there exists a Lipschitz continuous and bounded ϕ_ϵ such that $\|\phi - \phi_\epsilon\|_\infty \leq \epsilon$. Moreover

$$\left| \int_\Omega \phi d\mu_n - \int_\Omega \phi d\mu \right| \leq \left| \int_\Omega (\phi - \phi_\epsilon) d\mu_n - \int_\Omega (\phi - \phi_\epsilon) d\mu \right| + \left| \int_\Omega \phi_\epsilon d\mu_n - \int_\Omega \phi_\epsilon d\mu \right|,$$

where the last term converges to 0 since $d(\mu_n, \mu) \rightarrow 0$. For the first term on the right-hand side of the above inequality we have

$$\left| \int_\Omega (\phi - \phi_\epsilon) d\mu_n - \int_\Omega (\phi - \phi_\epsilon) d\mu \right| \leq \left(\sup_{n \in \mathbb{N}} |\mu_n|(\Omega) + |\mu|(\Omega) \right) \|\phi - \phi_\epsilon\|_\infty \leq C\epsilon,$$

which by arbitrariness of ϵ proves that $\mu_n \rightarrow \mu$ narrowly. \square

In our considerations we will deal only with nonnegative measures, which equipped with the bounded-Lipschitz distance are a complete metric space.

Proposition 2.1.2. *The space (\mathcal{M}_+, d) is a complete metric space.*

The following corollary is the very reason for which we use the bounded-Lipschitz distance. It serves us as a topology with pointwise sequential compactness for measure valued functions. What we mean is that if $f_n : [0, T] \mapsto (\mathcal{M}, d)$ and f_n are uniformly bounded in $L^\infty(0, T; (\mathcal{M}, TV))$ then for each $t \in [0, T]$ the sequence $f_n(t)$ is relatively compact in (\mathcal{M}, d) , which is one of the assumptions of the Arzela-Ascoli theorem (Theorem A.4.1).

Corollary 2.1.1. *Let $\{\mu_n\}_{n \in \mathbb{N}}$ be a sequence bounded in (\mathcal{M}_+, TV) with supports contained in some given ball. Then μ_n has a subsequence convergent in (\mathcal{M}_+, d) .*

Proof. Since the supports of μ_n are uniformly bounded then there exist compact sets $K \subset \subset \Omega \subset \subset \mathbb{R}^d$ such that

$$\bigcup_n \text{supp} \mu_n \subset K.$$

Thus we may treat $\{\mu_n\}_{n \in \mathbb{N}}$ as a sequence of measures defined on a compact Ω . Since then (\mathcal{M}, TV) is isomorphic to $(C_b(\Omega))^*$, which is a separable normed vector space then by Banach-Alaoglu theorem (or Theorem A.4.2) the set $\{\mu_n : n = 1, 2, \dots\}$ is sequentially weakly * compact in $(C_b(\Omega))^*$. Therefore there exists a measure $\mu \in (C_b(\Omega))^*$ such that up to a subsequence μ_n converges to μ weakly * in $(C_b(\Omega))^*$. Proposition 2.1.1 implies that the weak * $(C_b(\Omega))^*$ convergence i.e. the narrow convergence is equivalent to the convergence in $d(\cdot, \cdot)$ if only $\{\mu_n\}_{n \in \mathbb{N}}$ is tight (which is true since μ_n vanish on $\Omega \setminus K$). Thus μ_n converges to μ in $d(\cdot, \cdot)$. Finally since by Proposition 2.1.2 (\mathcal{M}_+, d) is a complete space we conclude that actually $\mu \in (\mathcal{M}_+, d)$ and the proof is finished. \square

Lastly we present a useful lemma related to the bounded-Lipschitz distance.

Lemma 2.1.1. *Let $d(\cdot, \cdot)$ be the bounded-Lipschitz distance. Then for any $\mu, \nu \in \mathcal{M}$ and any bounded and Lipschitz continuous function g , we have*

$$\left| \int_{\Omega} g d\mu - \int_{\Omega} g d\nu \right| \leq \max\{\|g\|_\infty, Lip(g)\} d(\mu, \nu).$$

Proof. The proof of this lemma belongs to the standard theory and can be found, for example, in [31]. \square

2.1.3 Measure solutions of the kinetic equation

We introduce the following weak formulation for (1.3) that will play the major role in Chapter 5:

Definition 2.1.6. Let $T > 0$. We say that f is a weak solution to (1.3) with the initial data $f_0 \in \mathcal{M}_+$, such that $\text{supp} f_0 \subset B(\mathcal{R}_0)$ with $\mathcal{R}_0 > 0$ if

1. $f \in L^\infty(0, T; \mathcal{M}_+)$ and $\partial_t f \in L^p(0, T; (C_b^1(B(\mathcal{R})))^*)$ for some $p > 1$;
2. $\text{supp} f(t) \subset B(\mathcal{R})$ for $t \in (0, T]$ for some positive constant \mathcal{R} ;
3. The following identity holds:

$$\int_0^T \int_{\mathbb{R}^{2d}} f[\partial_t \phi + v \nabla \phi] dx dv dt + \int_0^T \int_{\mathbb{R}^{2d}} F(f) f \nabla_v \phi dx dv dt = \quad (2.6)$$

$$= - \int_{\mathbb{R}^{2d}} f_0 \phi(\cdot, \cdot, 0) dx dv \quad (2.7)$$

for all $\phi \in \mathcal{G}$, where

$$\mathcal{G} := \{\phi \in C^1([0, T] \times \mathbb{R}^{2d}) : \partial_t \phi, \nabla \phi, \nabla_v \phi \text{ are bounded} \\ \text{and Lipschitz continuous and } \phi \text{ has a compact support in } t\};$$

4. The function $g(x, y, v, w, t) := (w - v)\psi(|x - y|)$ is integrable with respect to the measure $f(x, v, t) \otimes f(y, w, t)$, i.e. term $F(f)$ is defined as a measure with respect to the measure f . In particular by Fubini's theorem the integral

$$\int_0^T \int_{\mathbb{R}^{2d}} F(f) f \nabla_v \phi dx dv dt = \int_0^T \int_{\mathbb{R}^{4d}} g \nabla_v \phi f \otimes f dx dv dy dw dt$$

is bounded and the term $\text{div}_v [F(f)f]$ is well defined as a distribution;

5. For each pair of concentric balls $B((x_0, v_0), r) \subset B((x_0, v_0), R)$, the following statement holds: if

$$\text{supp} f_0 \cap B((x_0, v_0), R) \subset B((x_0, v_0), r) \quad (2.8)$$

then there exists $T^* \in [0, T]$, such that

$$\text{supp} f(t) \cap B\left((x_0, v_0), \frac{3R + r}{4}\right) \subset B\left((x_0, v_0), \frac{r + R}{2}\right) \quad (2.9)$$

for all $t \in [0, T^*]$.

Remark 2.1.6. There is a natural question of the correspondence between solutions to (2.1) in the sense of Definition 2.1.6 and solutions to (2.3). The answer to this question is to some

merit positive, which we explain below. Let

$$f_0(x, v) := \sum_{i=1}^N m_i \delta_{x_{i,0}}(x) \otimes \delta_{v_{i,0}}(v) \quad (2.10)$$

with $\sum_{i=1}^N m_i = 1$. Then f_0 defines an initial data $x_0 = (x_{1,0}, \dots, x_{N,0})$, $v_0 = (v_{1,0}, \dots, v_{N,0})$ for the system of ODE's (2.3). For this system let (x, v) be a sufficiently smooth³ solution. Then the function

$$f(x, v, t) := \sum_{i=1}^N m_i \delta_{x_i(t)}(x) \otimes \delta_{v_i(t)}(v) \quad (2.11)$$

is a solution of (2.1) in the sense of Definition 2.1.6 with the initial data f_0 . Indeed, if we plug f defined in (2.11) into (2.6), by a simple use of a chain rule, we obtain

$$\begin{aligned} & \int_0^T \sum_{i=1}^N m_i \left((\partial_t \phi)(x_i, v_i, t) + v_i (\nabla \phi)(x_i, v_i, t) \right) \\ & + \sum_{i,j=1}^N m_i m_j \psi(|x_i - x_j|) (v_j - v_i) (\nabla_v \phi)(x_i, v_i, t) dt \\ = & \int_0^T \sum_{i=1}^N m_i \frac{d}{dt} \phi(x_i, v_i, t) dt = - \sum_{i=1}^N m_i \phi(x_{i,0}, v_{i,0}, 0) = - \int_{\mathbb{R}^{2d}} f_0 \phi(\cdot, \cdot, 0) dx dv \end{aligned}$$

for all $\phi \in \mathcal{G}$.

The converse assertion that a solution to (2.1) in the sense of Definition 2.1.6 corresponds to a solution of (2.3) is also true provided that the initial data are of the form (2.10). However, the proof is much more involved and it is in fact the second part of Chapter 5.

Definition 2.1.7. *We say that f is an atomic solution if it has the form (2.11).*

Remark 2.1.7. Point 5 of Definition 2.1.6 requires some explanation. Its purpose is to establish a local control over the propagation of the support of f . Basically if we can divide the support of f_0 into two parts of distance $R - r$, then in some small time interval $[0, T^*]$ the distances between those parts is no lesser than $\frac{R-r}{4}$.

Remark 2.1.8. In Chapter 5 and especially Section 5.2 we frequently test our weak solution by various test functions that at the first glance may seem not admissible. In particular we test with functions with derivatives in x and v not necessarily Lipschitz continuous. This is however correct since by simple density argument we may test (2.6) with C^1 functions.

³By "sufficiently smooth" we mean for instance that $(x, v) \in W^{1,1}([0, T])$, which is a reasonable assumption in view of Proposition 5.0.3.

Moreover we test (2.6) with functions that are not compactly supported in time. In such case we get a version of (2.6) with both endpoints of the time interval, i.e. by testing in the time interval $[0, t]$ we get

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^{2d}} f[\partial_t \phi + v \nabla \phi] dx dv dt + \int_0^T \int_{\mathbb{R}^{2d}} F(f) f \nabla_v \phi dx dv dt = \\ = \int_{\mathbb{R}^{2d}} f(t) \phi(\cdot, \cdot, t) dx dv - \int_{\mathbb{R}^{2d}} f_0 \phi(\cdot, \cdot, 0) dx dv. \end{aligned}$$

The justification of the above equation is standard but we present it anyway in the proof of Proposition 5.0.3,(v) in Appendix A.

2.2 Part I: Main results

Having all the necessary preliminary definitions and remarks we are finally in position to present the main results of the first part of the thesis. The first one states that the C–S particle system (2.3) with singular weight with $\alpha \in (0, 1)$ admits at least one piecewise-weak solution.

Theorem 2.2.1. *Let $\alpha \in (0, 1)$. For all $T > 0$ there exists a $(C^1([0, T]))^{Nd}$ solution of (2.3) with arbitrary initial data. This solution is in the sense of Definition 2.1.2*

The second result concerns the existence and uniqueness of classical solutions to the C–S particle system with singular communication weight provided that the range of singularity is reduced to $\alpha \in (0, \frac{1}{2})$.

Theorem 2.2.2. *Let $\alpha \in (0, \frac{1}{2})$ be given. Then for all $T > 0$ and arbitrary initial data there exists a unique $x \in W^{2,1}([0, T]) \subset C^1([0, T])$ that solves (2.3) with communication weight given by (2.2) weakly in $W^{2,1}([0, T])$.*

The third result is complementary to Theorem 2.2.1 stating that any piecewise-weak solution to the C–S particle system with singular weight with $\alpha \in (0, 1)$ is unique. Thus it could be expected to appear in Chapter 3, however since the methodology of the proof is developed in Chapter 4, we decided to introduce it separately in that chapter.

Theorem 2.2.3. *Let $\alpha \in (\frac{1}{2}, 1)$ be given. Then the solution in the sense of Definition 2.1.2, which existence is ensured by Theorem 2.2.1 is unique.*

The last result answers the main question of existence and uniqueness of solutions to the C–S kinetic equation with singular communication weight completing at the same time the analysis of the C–S model with singular weight with $\alpha \in (0, 1)$. It states that for any compactly supported initial Radon measure there exists a solution and if the initial measure is atomic then the solution is also atomic and unique.

Theorem 2.2.4. *Let $0 < \alpha < \frac{1}{2}$. For any compactly supported initial data $f_0 \in \mathcal{M}_+$ and any $T > 0$, the Cucker-Smale's kinetic equation (2.1) admits at least one solution in the sense of Definition 2.1.6. Moreover if f_0 is atomic (is of the form (2.10)) then f is atomic (is of the form (2.11)) and it is unique.*

Additionally we present proposition of minor importance from the point of view of this thesis but is an interesting addition to the qualitative analysis of C–S model. It's proof can be found at the end of Chapter 3 (in Section 3.6).

Proposition 2.2.1. *The C–S particle system (2.3) with singular communication weight (2.2) with $\alpha \in (0, 1)$ allows sticking of the trajectories of the particles.*

Remark 2.2.1. The above proposition is important from the point of view of applications as explained in Section 1.2. From the modelling point of view it is good to understand what repertoire of qualitative behaviour does the model exhibit. It may be the lack of sticking of the trajectories typical for the C–S model with regular weight or the lack of collisions whatsoever that is generally expected for the model with singular weight with $\alpha \geq 1$. In case of the C–S model with singular weight and $\alpha \in (0, 1)$ there is a possibility of sticking of the trajectories that we discovered in paper [43] and included in this dissertation.

Chapter 3

Cucker–Smale model with singular weight: piecewise-weak solutions

Our first goal is to obtain existence of piecewise-weak solutions to (2.3) with singular weight (2.2) and $\alpha \in (0, 1)$. Let us briefly present difficulties and ideas how to overcome them. When dealing with the C-S model with bounded weight one makes use of its Lipschitz continuity as well as the structure of the model itself. As an example we present a simple application of the properties of the structure of our model. Namely, we prove that the average velocity of the particles

$$\bar{v}(t) := \frac{1}{N} \sum_{i=1}^N v_i(t)$$

is constant in time. Assuming that $x = (x_1, \dots, x_N)$ and $v = (v_1, \dots, v_N)$ is a sufficiently smooth solution of (2.3), we calculate the derivative of \bar{v} to get

$$\begin{aligned} \frac{d}{dt} \sum_{i=1}^N v_i &= \frac{1}{N} \sum_{i,k=1}^N (v_k - v_i) \psi(|x_i - x_k|) = \\ & \frac{1}{2N} \sum_{i,k=1}^N (v_k - v_i) \psi(|x_i - x_k|) + \frac{1}{2N} \sum_{i,k=1}^N (v_i - v_k) \psi(|x_i - x_k|) = 0, \end{aligned} \tag{3.1}$$

where the latter summand in the second line is obtained by substituting i and k . Clearly each such structure based property of the C-S model will remain true regardless of the communication weight ψ as long as it is nonnegative. This is the first piece of information on which we base our hope to obtain some existence for C-S model with singular weight ψ . The second piece of information is that in case of singular weight given by (2.2) Lipschitz continuity and boundedness of ψ fails only at 0, which means that our main problem will be to prove existence in a neighbourhood of each time t_0 at which some particles collide. However, roughly speaking, in a neighbourhood of each such point we have

$$x_i(t) - x_j(t) \approx t(v_i(t_0) - v_j(t_0)) \approx t(v_i(t) - v_j(t))$$

and since in (2.3)₂ the function $t \mapsto \psi(|x_i(t) - x_j(t)|)$ comes always multiplied by $v_i(t) - v_j(t)$, we have

$$\begin{aligned} \left(\Psi(|x_i(t) - x_j(t)|) \right)' &= \psi(|x_i(t) - x_j(t)|) \frac{(x_i(t) - x_j(t)) \cdot (v_i(t) - v_j(t))}{|x_i(t) - x_j(t)|} \\ &\approx \psi(|x_i(t) - x_j(t)|) |v_i(t) - v_j(t)| \end{aligned}$$

with $\Psi(s) := \frac{1}{1-\alpha} s^{1-\alpha}$ being a primitive of ψ , which is a Hölder continuous function, thus there is hope for some better regularity of v .

These two observations were already used in [31] to obtain asymptotic flocking for C-S model with weight ψ . Occurrence of asymptotic flocking is a further clue that a C-S model with singular weight inherits some nice properties from the model with a smooth weight.

In the following sections we prove existence for the discrete C-S model (2.3) with a singular communication weight given by (2.2) for $\alpha \in (0, 1)$ (Theorem 2.2.1). Our strategy is based on the observation that the function $t \mapsto \psi(|x_i(t) - x_j(t)|)$ is Lipschitz continuous in a neighbourhood of each time t_0 , such that for all i, j , we have $x_i(t_0) \neq x_j(t_0)$, which makes local existence in such points trivial. The idea is that if we prove that the particles collide in some sense rarely, then the only difficulty is to establish existence in a neighbourhood of each point of collision of some particles. Technically we obtain existence of solutions by approximating them with solutions of C-S model with bounded weights.

3.1 Approximate solutions

First, let us define the approximate solutions and present some of their most important properties. For each n let

$$\psi_n(s) = \begin{cases} \psi(s) & \text{if } s \geq (n-1)^{-\frac{1}{\alpha}}, \\ \text{smooth and monotone} & \text{if } n^{-\frac{1}{\alpha}} \leq s \leq (n-1)^{-\frac{1}{\alpha}}, \\ n & \text{if } s \leq n^{-\frac{1}{\alpha}} \end{cases}$$

for all $s \in [0, \infty)$ with ψ given by (2.2). For all n , functions ψ_n are smooth and bounded, thus C-S systems associated with these weights have unique solutions. This can be expressed by the following proposition.

Proposition 3.1.1. *For each positive integer n and for arbitrary initial data, the system*

$$\begin{cases} \dot{x}_i^n &= v_i^n, \\ \dot{v}_i^n &= \frac{1}{N} \sum_{k=1}^N (v_k^n - v_i^n) \psi_n(|x_i^n - x_k^n|) \end{cases} \quad (3.2)$$

has a unique global classical solution x^n belonging to the class $(C^2([0, T]))^{Nd}$.

The proof of this proposition is standard and we omit it. The following properties of the solutions will play an important role throughout the chapter.

Proposition 3.1.2. *Let x^n be a solution of the C-S model associated with weight ψ_n . Then x^n has the following properties:*

1. *It belongs to the class C^∞ in a neighbourhood of every such point t , that*

$$|x_i^n(t) - x_j^n(t)| > 0$$

for all $i, j = 1, \dots, N$.

2. *The average velocity of the particles is constant:*

$$\frac{1}{N} \sum_{i=1}^N v_i^n(t) = \text{const.}$$

3. *Velocity v^n is bounded: there exists a constant $M(n)$ such that for all $i = 1, \dots, N$, we have*

$$\|v_i^n\|_{L^\infty([0, T])} \leq M(n).$$

4. *If the initial data $x^n(0), v^n(0)$ are uniformly bounded, then also v^n is uniformly bounded: there exists a constant M such that for all $i = 1, \dots, N$ and all $n = 1, 2, \dots$, we have*

$$\|v_i^n\|_{L^\infty([0, T])} \leq M.$$

5. *Acceleration \dot{v}^n is bounded:*

$$\|\dot{v}_i^n\|_{L^\infty([0, T])} \leq 2M(n)n.$$

6. *If at some point t we have $x_i^n(t) = x_j^n(t)$ and $v_i^n(t) = v_j^n(t)$ for any $i, j = 1, \dots, N$, then $x_i^n \equiv x_j^n$ on $[t, T]$.*

7. *If at some point t we have $v_i^n(t) = v_j^n(t)$ for all $i, j = 1, \dots, N$, then v^n is constant on $[t, T]$.*

Proof. 1. Since x^n is continuous, if at some point t all the particles have different positions i.e. $|x_i^n(t) - x_j^n(t)| > 0$ for all $i, j = 1, \dots, N$ then it is also true in some neighbourhood of t . Moreover in this neighbourhood of t the right-hand side of (3.2)₂ is differentiable, which by iteration implies that x^n is smooth at t .

2. This part was already done at the beginning of this chapter (in particular in (3.1)).

3. Let $r_n(t) := \sum_{i,j=1}^N (v_i^n(t) - v_j^n(t))^2$. By (3.2)₂, we have

$$\begin{aligned} \dot{r}_n &= 2 \sum_{i,j=1}^N (v_i^n - v_j^n) \left(\frac{1}{N} \sum_{k=1}^N (v_k^n - v_i^n) \psi_n(|x_i^n - x_k^n|) - \frac{1}{N} \sum_{k=1}^N (v_k^n - v_j^n) \psi_n(|x_j^n - x_k^n|) \right) \\ &= \frac{2}{N} \sum_{i,j,k=1}^N (v_i^n - v_j^n)(v_k^n - v_i^n) \psi_n(|x_i^n - x_k^n|) - \frac{2}{N} \sum_{i,j,k=1}^N (v_i^n - v_j^n)(v_k^n - v_j^n) \psi_n(|x_j^n - x_k^n|). \end{aligned}$$

Again, we substitute i and k in the first summand and j and k in the second summand to obtain

$$\begin{aligned} \dot{r}_n &= \frac{1}{N} \sum_{i,j,k=1}^N (v_i^n - v_j^n)(v_k^n - v_i^n) \psi_n(|x_i^n - x_k^n|) + \frac{1}{N} \sum_{i,j,k=1}^N (v_k^n - v_j^n)(v_i^n - v_k^n) \psi_n(|x_i^n - x_k^n|) \\ &\quad - \frac{1}{N} \sum_{i,j,k=1}^N (v_i^n - v_j^n)(v_k^n - v_j^n) \psi_n(|x_j^n - x_k^n|) - \frac{1}{N} \sum_{i,j,k=1}^N (v_i^n - v_k^n)(v_j^n - v_k^n) \psi_n(|x_j^n - x_k^n|) \\ &= -\frac{1}{N} \sum_{i,j,k=1}^N (v_i^n - v_k^n)^2 \psi_n(|x_i^n - x_k^n|) - \frac{1}{N} \sum_{i,j,k=1}^N (v_j^n - v_k^n)^2 \psi_n(|x_j^n - x_k^n|) \\ &= -2 \sum_{i,j=1}^N (v_i^n - v_j^n)^2 \psi_n(|x_i^n - x_j^n|) \leq 0. \end{aligned}$$

Thus for each n , function r_n is nonincreasing with its maximum at 0 i.e. $r_n(t) \leq r_n(0)$. Now let \bar{v}^n be the average velocity, which as we know from property 2 is a constant. We have

$$\begin{aligned} \sum_{i=1}^N (\bar{v}^n - v_i^n)^2 &= \sum_{i=1}^N \left(\frac{1}{N} \sum_{j=1}^N v_j^n - v_i^n \right)^2 = \frac{1}{N^2} \sum_{i=1}^N \left(\sum_{j=1}^N (v_j^n - v_i^n) \right)^2 \\ &\leq \frac{1}{N} \sum_{i,j=1}^N (v_j^n - v_i^n)^2 \leq \frac{1}{N} r_n(0). \end{aligned}$$

Lastly

$$\begin{aligned} |v_i^n| &\leq |v_i^n - \bar{v}^n| + |\bar{v}^n| \leq \sqrt{\sum_{i=1}^N (\bar{v}^n - v_i^n)^2} + |\bar{v}^n| \leq C(N) \sqrt{r_n(0)} + |\bar{v}^n| \\ &\leq C(N) \sqrt{r_n(0)} =: M(n), \end{aligned}$$

where $C(N)$ is a generic constant depending on N .

4. We simply note that if initial velocity is uniformly bounded, then $M(n) \leq M$ for some M independent of n .

Point 5 follows immediately from property 3 and equation (3.2)₂, while points 6 and 7 are obvious consequences of uniqueness of the solutions. \square

Remark 3.1.1. Property 6 from the above proposition implies that the acceleration equation (3.2)₂ can be replaced by:

$$\dot{v}_i^n = \frac{1}{N} \sum_{k \in B_i(t)} (v_k^n - v_i^n) \psi_n(|x_k^n - x_i^n|), \quad (3.3)$$

where $B_i(t)$ is defined by (2.4), with $\dot{v}_i^n = 0$ should set $B_i(t)$ be empty. Indeed, by (3.1.1) we have

$$\dot{v}_i^n(t) = \frac{1}{N} \sum_{k \in B_i(t)} (v_k^n(t) - v_i^n(t)) \psi_n(|x_k^n(t) - x_i^n(t)|) + \frac{1}{N} \sum_{k \notin B_i(t)} (v_k^n(t) - v_i^n(t)) \psi_n(|x_k^n(t) - x_i^n(t)|)$$

and by the definition of $B_i(t)$ the second term above disappears at t (as for all $k \notin B_i(t)$, $v_k^n(t) = v_i^n(t)$). Property 6 from Proposition 3.1.2 implies that this is true also for all $s > t$, thus we may actually ignore the second term altogether, which leaves us with (3.3). This technical observation will be useful later on.

Until the end of the chapter we use $M(n)$ and M in the same roles as in Proposition 3.1.2. We end this section with an important lemma that is in fact our way to deal with existence in a right sided neighbourhood of a point of collision.

Lemma 3.1.1. *Let x^n be a solution of C-S system on the time interval $[0, T]$ with weight ψ_n and initial data $x(0), v(0)$ – independent of n . Then there exists an interval $[0, t]$, such that all velocities v^n are uniformly Hölder continuous on $[0, t]$.*

To prove this lemma we need yet another, technical lemma.

Lemma 3.1.2. *If $x_i(0) = x_j(0)$ and $v_i(0) \neq v_j(0)$, then for all n , there exists an interval $(0, t^n]$, such that*

$$|v_i^n(s) - v_j^n(s)| \leq 4 \frac{|(v_i^n(s) - v_j^n(s)) \cdot (x_i^n(s) - x_j^n(s))|}{|x_i^n(s) - x_j^n(s)|} \quad (3.4)$$

for $s \in (0, t^n]$.

Proof. By property 5 from Proposition 3.1.2, we have

$$v_i^n(s) - v_j^n(s) = v_i^n(0) - v_j^n(0) + \phi_n(s), \quad |\phi_n(s)| \leq 2|s|Mn. \quad (3.5)$$

Moreover as $x_i^n - x_j^n$ is a C^2 function, by Taylor's formula

$$x_i^n(s) - x_j^n(s) = s \cdot (v_i^n(0) - v_j^n(0)) + o_n(s) = s (v_i^n(s) - v_j^n(s) - \phi_n(s)) + o_n(s), \quad (3.6)$$

where

$$o_n(s) := \int_0^s (\dot{v}_i^n - \dot{v}_j^n)(s - \theta) d\theta, \quad |o_n(s)| \leq 2|s|^2 Mn.$$

Thus

$$\begin{aligned}
& |(v_i^n(s) - v_j^n(s))(x_i^n(s) - x_j^n(s))| = \\
& = |s(v_i^n(s) - v_j^n(s))^2 - s(v_i^n(s) - v_j^n(s))\phi_n(s) + (v_i^n(s) - v_j^n(s))o_n(s)| \\
& \geq s(v_i^n(s) - v_j^n(s))^2 - s|(v_i^n(s) - v_j^n(s))\phi_n(s)| - |(v_i^n(s) - v_j^n(s))o_n(s)| \geq \frac{s}{2}(v_i^n(s) - v_j^n(s))^2 \quad (3.7)
\end{aligned}$$

assuming that $s \in (0, t^n]$, where t^n is the supremum of all times s_n , such that for all $s \in (t, s_n]$, we have

$$|\phi_n(s)| \leq \frac{1}{4}|v_i^n(s) - v_j^n(s)|, \quad |o_n(s)| \leq \frac{s}{4}|v_i^n(s) - v_j^n(s)|. \quad (3.8)$$

To check that $t^n > 0$, we notice that for

$$s_n := \frac{|v_i^n(0) - v_j^n(0)|}{10Mn} \quad (3.9)$$

and $s \in [0, s_n]$, we have

$$|\phi_n(s)| \leq \frac{1}{5}|v_i^n(0) - v_j^n(0)|, \quad |o_n(s)| \leq \frac{s}{5}|v_i^n(0) - v_j^n(0)|, \quad (3.10)$$

which together with (3.5) implies that

$$\frac{4}{5}|v_i^n(0) - v_j^n(0)| \leq |v_i^n(s) - v_j^n(s)| \quad (3.11)$$

and condition (3.8) is satisfied. Therefore by taking s_n given by (3.9) we get (3.7). Now by (3.5) and (3.6) on $(0, s_n]$ we also have

$$|x_i^n(s) - x_j^n(s)| \leq 2s|v_i^n(s) - v_j^n(s)|, \quad (3.12)$$

which together with (3.7) proves that there exists $s_n > 0$ such that on $(0, s_n]$ inequality (3.4) holds. Now we define t^n as the supremum of all such times s_n . This finishes the proof. \square

Next we proceed with the proof of Lemma 3.1.1.

Proof of Lemma 3.1.1. The proof will follow by 2 steps. In step 1 we prove that for each n there exists an interval $[0, t^n]$ on which v^n is Hölder continuous with a constant independent of n , while in step 2 we establish a lower bound on t^n that is independent of n .

Step 1. It suffices to show (3.4) separately for all particles, thus let us fix $i = 1, \dots, N$ and consider x_i . By Remark 3.1.1 for all s , we have

$$\begin{aligned}
|v_i^n(s) - v_i^n(0)| &= \left| \int_0^s \dot{v}_i^n(\theta) d\theta \right| \leq \frac{1}{N} \sum_{k \in B_i(0)} \int_0^s |v_k^n - v_i^n| \psi_n(|x_k^n - x_i^n|) d\theta \\
&= \frac{1}{N} \sum_{k \in B_i^0} \int_0^s |v_k^n - v_i^n| \psi_n(|x_k^n - x_i^n|) d\theta + \frac{1}{N} \sum_{k \in B_i^+} \int_0^s |v_k^n - v_i^n| \psi_n(|x_k^n - x_i^n|) d\theta \\
&=: I + II,
\end{aligned}$$

where

$$B_i^0 := \{j \in B_i(0) : |x_j(0) - x_i(0)| = 0\}, \quad B_i^+ := \{j \in B_i(0) : |x_j(0) - x_i(0)| > 0\}$$

and $B_i(0)$ is the defined by (2.4) set of all particles that have different trajectories than x_i (we assume that B_i^0 and B_i^+ are nonempty since otherwise $I = 0$ or $II = 0$ and the estimation is even easier). We estimate I and II separately starting with I . For $j \in B_i^0$, we have $|v_j^n(0) - v_i^n(0)| > 0$ and by it's continuity there exists t^n such that $|x_j^n(s) - x_i^n(s)| > 0$ and consequently $\psi_n(|x_j^n(s) - x_i^n(s)|) \leq \psi(|x_j^n(s) - x_i^n(s)|)$ in $(0, t^n]$. Together with Lemma 3.1.2 it implies that

$$I \leq \frac{4}{N} \sum_{j \in B_i^0} \int_0^s \frac{|(v_j^n - v_i^n) \cdot (x_j^n - x_i^n)|}{|x_j^n - x_i^n|} \psi(|x_j^n - x_i^n|) d\theta.$$

We claim that, since $\Psi(|x_j^n(0) - x_i^n(0)|) = 0$ for all $j \in B_i^0$, then

$$\int_0^s \frac{|(v_j^n - v_i^n) \cdot (x_j^n - x_i^n)|}{|x_j^n - x_i^n|} \psi(|x_j^n - x_i^n|) d\theta = \Psi(|x_j^n(s) - x_i^n(s)|),$$

where $\Psi(s) = \frac{1}{1-\alpha} s^{1-\alpha}$ is a primitive of ψ . Indeed, we have

$$\begin{aligned} \Psi(|x_j^n(s) - x_i^n(s)|) &= \int_0^s \left(\Psi(|x_j^n - x_i^n|) \right)' d\theta \\ &\leq \int_0^s \psi(|x_j^n - x_i^n|) \frac{|(x_j^n - x_i^n)(v_j^n - v_i^n)|}{|x_j^n - x_i^n|} d\theta \end{aligned}$$

and since $\psi \geq 0$ we can substitute the above inequality with an equality provided that on $(0, t^n]$ the function $\xi(s) := (x_j^n(s) - x_i^n(s))(v_j^n(s) - v_i^n(s))$ has a constant sign. It suffices to show that $|\xi| > 0$ in $(0, t^n]$, which is an immediate consequence of Lemma 3.1.2 and (3.11). Thus we proved that

$$\begin{aligned} I &\leq \frac{4}{N} \sum_{j \in B_i^0} \Psi(|x_j^n(s) - x_i^n(s)|) = \frac{4}{N(1-\alpha)} \sum_{j \in B_i^0} |(x_j^n(s) - x_i^n(s))|^{1-\alpha} \\ &\leq \frac{4M^{1-\alpha}}{N(1-\alpha)} \sum_{j \in B_i^0} |s|^{1-\alpha} \leq \frac{4M^{1-\alpha}}{1-\alpha} |s|^{1-\alpha}, \end{aligned}$$

where we use inequality $|x_j^n(s) - x_i^n(s)| \leq M|s|$ that follows from property 4 from Proposition 3.1.2. To estimate II we first notice that since for all $j \in B_i^+$, we have $|x_j^n(0) - x_i^n(0)| > 0$ then there exists $\delta > 0$ such that $|x_j^n(0) - x_i^n(0)| > \delta$ for all $j \in B_i^+$. Then, by property 4 from Proposition 3.1.2 there exists an n independent interval $[0, t_0]$ on which $|x_j^n - x_i^n| > \delta$ for all $j \in B_i^+$. On this interval

$$\psi_n(|x_j^n(s) - x_i^n(s)|) \leq \delta^{-\alpha}.$$

Therefore

$$II \leq \frac{1}{N} \sum_{j \in B_i^+} 2|s| M \delta^{-\alpha} \leq 2t_0^\alpha M \delta^{-\alpha} |s|^{1-\alpha}$$

and adding our estimations of I and II we get

$$|v_i^n(s) - v_i^n(0)| \leq L|s|^{1-\alpha}$$

with $L = \frac{4M^{1-\alpha}}{1-\alpha} + 2t_0^\alpha M \delta^{-\alpha}$ on interval $[0, t^n] \cap [0, t_0]$. For simplicity let us denote $\min\{t^n, t_0\}$ again by t^n . This finishes step 1.

Step 2. In step 1 we proved that for each n there exists an interval $[0, t^n]$ in which v_i^n is Hölder continuous with a constant independent of n . Now we prove that there exists $t > 0$, such that for all n , we have $t \leq t^n$ and thus in $[0, t]$ all functions v_i^n are uniformly Hölder continuous. There are exactly 3 instances, when we bound t^n from the above:

1. In the proof of Lemma 3.1.2.
2. While ensuring that for all $k \in B_i^0$ we have $|v_k^n - v_i^n| > 0$ in $[0, t^n]$.
3. While ensuring that for all $k \in B_i^0$ the function ξ is positive in $(0, t^n]$.

If each of these bounds from above can be bounded from below by a constant independent of n , then so can be t^n for all n .

1. In Lemma 3.1.2, t^n was the supremum of all times s_n , such that for all $s \in (0, t^n]$ conditions (3.8) and (3.12) are satisfied. However from step 1 we may estimate t^n better than we could in the proof of Lemma 3.1.2. We have

$$|\phi_n(s)| \leq 2L|s|^{1-\alpha} \quad \text{and} \quad |o_n(s)| \leq 2L|s|^{2-\alpha},$$

thus by taking

$$\tilde{t} := \left(\frac{1}{10L} |v_k^n(0) - v_i^n(0)| \right)^{\frac{1}{1-\alpha}}, \quad (3.13)$$

we ensure that (3.10) and consequently (3.8) is satisfied. With the same \tilde{t} we obtain also condition (3.12).

2. For $k \in B_i^0$ we have $|v_k^n(0) - v_i^n(0)| > 0$, thus

$$|v_k^n(s) - v_i^n(s)| \geq |v_k^n(0) - v_i^n(0)| - 2L|s|^{1-\alpha},$$

which is positive for $s \leq \tilde{t}$.

3. To prove that ξ has a constant sign in $[0, t^n]$ we applied Lemma 3.1.2 concluding that $|\xi(s)|$ is positive, provided that s belongs to the interval on which the thesis of Lemma 3.1.2 holds and we proved above that this interval includes $(0, \tilde{t}]$.

Therefore all bounds from points 1,2 and 3 are satisfied for t_0 defined by (3.13) and it is clearly n -independent. Thus we proved that there exists an interval $[0, t]$ with $t \geq \tilde{t}$ in which all functions v_i^n are uniformly Hölder continuous. \square

3.2 From approximate solutions to the piecewise-weak solutions

Before we proceed with passing to the limit with the approximate solutions (x^n, v^n) we have to introduce some of the basic notions of piecewise-weak solutions in the language of approximate solutions. In Definition 2.1.2 a crucial role is played by the times at which one or more particles stick together. However, as mentioned in Remark 3.6.1, sticking of the trajectories cannot happen in case of regular weight and the approximate solutions are associated with regularised communication weights. Thus we need to redefine the times of sticking (and for technical reasons also the times of collisions) in terms of the approximate solutions.

Remark 3.2.1. Property 4 from Proposition 3.1.2 implies equicontinuity of x^n , thus by Arzela-Ascoli theorem there exists a $(C([0, T]))^{Nd}$ convergent subsequence x^{n_k} . From this point we pick one of such convergent subsequences and for simplicity of notation assume that $x^n = x^{n_k}$.

Having the above remark in mind we present an alternative definition of the times of collisions by the following recursive formula:

$$t_1 := \inf\{t > 0 : \min_{\substack{i=1,\dots,N \\ j \in B_i(0)}} \lim_{n \rightarrow \infty} |x_i^n - x_j^n| = 0\},$$

$$t_n := \inf\{t > t_{n-1} : \min_{\substack{i=1,\dots,N \\ j \in B_i(t_{n-1})}} \lim_{n \rightarrow \infty} |x_i^n - x_j^n| = 0\} \quad \text{for } n = 2, 3, \dots$$

assuming that $t_n = \infty$ provided that there is no $t > t_{n-1}$, such that

$$\min_{\substack{i=1,\dots,N \\ j \in B_i(t_{n-1})}} \lim_{n \rightarrow \infty} |x_i^n - x_j^n| = 0.$$

Remark 3.2.2. Clearly if $t < t_1$ then there exists $\delta > 0$, such that

$$\min_{\substack{i=1,\dots,N \\ j \in B_i(0)}} \lim_{n \rightarrow \infty} |x_i^n(t) - x_j^n(t)| > \delta,$$

which further implies that for all i, j , there exists n_0 such that for all $n > n_0$, we have $|x_i^n(t) - x_j^n(t)| > \delta$. On the other hand

$$\lim_{n \rightarrow \infty} |x_i^n(t_1) - x_j^n(t_1)| = 0, \quad \text{for some } i = 1, \dots, N \text{ and } j \in B_i(0)$$

and assuming that x is a $(C([0, t_1]))^{Nd}$ limit of x^n , we have

$$x_i(t_1) = x_j(t_1),$$

which means that t_1 is indeed the first time of collision for x . Similarly t_n is the n th time of collision for x .

Remark 3.2.3. The natural question arises whether $t_n \rightarrow \infty$ with $n \rightarrow \infty$, as it is otherwise not clear if $[0, T] \subset [0, t_1] \cup \bigcup_{n=1}^{\infty} [t_n, t_{n+1}]$. The answer to this question is in some sense 'yes', but to specify it and prove it, some careful analysis is required. In fact we will prove it at the very end of this section in the proof of Theorem 2.2.1.

3.3 Existence up to the time of collision

In this section we prove that the approximate solutions converge in every interval

$$[0, t] \subset [0, t_1)$$

, where t_1 is the time of the first collision of the particles. Additionally we will prove that their limit is a weak solution in $(W^{2,1}([0, t]))^{Nd}$. We begin with the following proposition.

Proposition 3.3.1. *For $n = 1, 2, \dots$ let x^n be a solution to the C–S system on interval $[0, T]$ with weight ψ_n and an independent of n initial data $x(0)$ and $v(0)$. There exists an interval $[0, t_1)$ such that for any $[0, t] \in [0, t_1)$ solutions x^n have a subsequence that converges to x in $(C^1([0, t]))^{Nd}$.*

Proof. If for all $i, j = 1, \dots, N$ we have $v_i(0) = v_j(0)$ then by property 7 from Proposition 3.1.2, we have $v^n \equiv v(0)$ for all n and the assertion holds with $t_1 = T$. Thus assuming that there exist $i, j = 1, \dots, N$ such that $v_i(0) \neq v_j(0)$, we have two possibilities:

(A) For all i and $j \in B_i(0)$ we have $x_j(0) \neq x_i(0)$.

(B) There exist i and $j \in B_i(0)$ such that $x_j(0) = x_i(0)$.

(A) In this case for all $t < t_1$, there exists $\delta_t > 0$ such that for all i, j and all n we have $|x_i^n - x_j^n| > \delta_t$ and $\psi(|x_i^n - x_j^n|) \leq \delta_t^{-\alpha}$ on $[0, t]$. Thus all velocities v^n are uniformly Lipschitz continuous on $[0, t]$ and by Arzela-Ascoli theorem there exists a $(C^1([0, t]))^{Nd}$ convergent subsequence of x^n .

(B) In the second case, there exist i and j , such that $x_i(0) = x_j(0)$ and $v_i(0) \neq v_j(0)$ and we may not proceed as in case **(A)**. However for this situation we have prepared Lemma 3.1.1, which implies uniform Hölder continuity of v^n in some neighbourhood of 0. Therefore for sufficiently small s_0 and $j \in B_i(0)$, such that $x_j(0) = x_i(0)$, we have

$$|x_i^n(s) - x_j^n(s)| \geq s \left(|v_i(0) - v_j(0)| - 2Ls^{1-\alpha} \right) \geq s \frac{1}{2} |v_i(0) - v_j(0)| =: \delta_s > 0$$

for $s \in [0, s_0]$. On the other hand, for all $j \in B_i(0)$ such that $x_j(0) \neq x_i(0)$, from property 4 of Proposition 3.1.2, we have

$$|x_i^n(s) - x_j^n(s)| \geq |x_i(0) - x_j(0)| - 2Ms \geq \delta_s$$

for all $n = 1, 2, \dots$ and all $s \in [0, s_1]$ with $0 < s_1 < 1$ possibly smaller than s_0 . Thus in s_1 we end up in a situation from case **(A)** with

$$|x_i^n(s_1) - x_j^n(s_1)| \geq \delta_{s_1}$$

and all velocities v^n are uniformly Hölder continuous on $[0, s_1]$ and uniformly Lipschitz continuous on $[s_1, t]$ for all $t < t_1$. Again by Arzela-Ascoli theorem, there exists a $(C^1([0, t]))^{Nd}$ convergent subsequence of x^n . \square

Remark 3.3.1. As in Remark 3.2.1, even though x from Proposition 3.3.1 is a limit of some subsequence of x^n , we will assume that it is in fact a limit of the whole sequence x^n (by restricting the approximate solutions to only those, which approximate x). Such assumption will pose no threat to our reasoning as long as they will not involve uniqueness of x .

Corollary 3.3.1. *Let x be as in Remark 3.3.1. Then x is a local classical solution to C-S system in the interval $(0, t_1)$. Moreover*

1. *For all $i, j = 1, \dots, N$, we have $|x_j - x_i| > 0$ in $(0, t_1)$.*
2. *The function x is smooth in $(0, t_1)$.*

Proof. By the definition of t_1 we get assertion 1, which on the other hand implies that in a neighbourhood of each $t \in (0, t_1)$ all the derivatives of x^n are uniformly bounded, which by Arzela-Ascoli theorem implies that x is smooth in $(0, t_1)$. With this, to prove that x solves C-S system with weight ψ , it suffices to take a $(C^2([t - \epsilon, t + \epsilon]))^{Nd}$ limit of systems associated with weights ψ_n , with $[t - \epsilon, t + \epsilon] \subset (0, t_1)$. \square

Our next step is to show that the function x actually satisfies our problem in a weak sense in every interval $[0, t] \subset [0, t_1)$ (though to prove that it satisfies Definition 2.1.2 we still need continuity of v at t_1).

Proposition 3.3.2. *For all $t \in [0, t_1]$ the function x is a weak $(W^{2,1}([0, t]))^{Nd}$ solution of (2.5).*

Proof. From Proposition 3.3.1 and Corollary 3.3.1 it follows that $x \in (C^1([0, t_1]))^{Nd}$ and that t_1 is the time of the first collision of the particles. It suffices to show that x satisfies (2.3) weakly in intervals $[0, t]$ for $t \nearrow t_1$. Since x^n satisfy (2.3)₁ and $x^n \rightarrow x$ in $(C^1([0, t]))^{Nd}$, then x satisfies (2.3)₁ with $v = \lim_{n \rightarrow \infty} v^n$. Now for $\phi \in (C_c^\infty([0, t]))^d$, we have

$$\int_0^t v_i^n \dot{\phi} ds = - \int_0^t \dot{v}_i^n \phi ds = - \int_0^t \frac{1}{N} \sum_{k=1}^N (v_k^n - v_i^n) \psi_n (|x_i^n - x_k^n|) \phi ds$$

and the left-hand side converges to $\int_0^t v \dot{\phi} ds$. Thus it remains to show that the right-hand side converges to $[-\int_0^t \dot{v} \phi ds]$, where

$$\dot{v} := \frac{1}{N} \sum_{k=1}^N (v_k - v_i) \psi(|x_i - x_k|). \quad (3.14)$$

To this end we require for example that $v^n \rightarrow v$ in $(L^1([0, t]))^{Nd}$, which follows from Lemma A.1.1 applied to functions $f_n = v_k^n - v_i^n$, $f = v_k - v_i$, $g_n = \psi(|x_i^n - x_k^n|)$, $g = \psi(|x_i - x_k|)$. \square

Remark 3.3.2. In the above proof, we actually have $v^n \rightarrow v$ in $(L^1([0, t]))^{Nd}$, which can be proved using Vitali's convergence theorem.

As our last effort in this section let us make an obvious remark involving properties stated in Proposition 3.1.2.

Corollary 3.3.2. *Properties 1,2,6,7 from Proposition 3.1.2 remain true also for the solution x on $[0, t_1)$. Moreover the following version of properties 3 and 4 holds:*

(4') *For all initial data $x(0)$ and $v(0)$ and all $i = 1, \dots, N$, we have*

$$\|v_i\|_{L^\infty([0, t_1])} \leq M,$$

where M is the constant from property 4 from Proposition 3.1.2.

Proof. Properties 1,2,4' follow by similar argumentation as in the proof of Proposition 3.1.2. Property 6 follows by the definition of our system (namely by substituting equation (2.3)₂ with (2.5)₂) and property 7 follows by calculating the derivative of $r(t) := \sum_{i,j} (v_i - v_j)^2$. \square

Remark 3.3.3. In general, property 5 from Proposition 3.1.2 does not hold on $[0, t_1)$ even though it holds on $[0, t]$ for $t \nearrow t_1$ due to the blowup of $\psi(|x_i - x_j|)$ at the time of collision of i th and j th particles.

3.4 Clustering at the time of collision

In the previous section we established existence of solutions on the interval $[0, t_1)$, where t_1 is time of the first collision of some pair of particles. The solution x belongs to

$$(W^{2,1}([0, t]))^{Nd} \cap (C^1([0, t_1]))^{Nd} \cap (C([0, T_0]))^{Nd}$$

for all $0 < t < t_1$ and satisfies (2.3) in a classical sense in $(0, t_1)$ and weakly in $(W^{2,1}([0, t]))^{Nd}$. Therefore we know that v is a Lipschitz continuous function in each interval $[0, t] \subset [0, t_1)$, however we do not know anything about its behaviour in a neighbourhood of t_1 – with our current knowledge the limit of $v(t)$ as $t \rightarrow t_1$ may even not exist. In this section we provide a proof of continuity of v on whole interval $[0, t_1]$.

Definition 3.4.1. For each $i, j = 1, \dots, N$ we define a relation $i \sim j$ if and only if $j \notin B_i(0)$ or for all $t < t_1$, we have

$$\int_t^{t_1} \psi(|x_i - x_j|) ds = \infty.$$

This relation is clearly symmetric and reflexive but not necessarily transitive. This leads us to another definition.

Definition 3.4.2. For each $i, j = 1, \dots, N$ we define a relation \sim with the following two statements:

1. If $i \sim j$, then $i \sim i$.
2. For $i \not\sim j$, we have $i \sim j$ if and only if there exists k , such that $i \sim k$ and $k \sim j$.

Remark 3.4.1. Relation \sim is an equivalence relation. Since \sim is symmetric and reflexive then so is \sim . Transitivity of \sim follows directly from the definition. Equivalence classes $[i]$ of \sim provide us with a partition of the set of indexes $\{1, \dots, N\}$ with the following property: given $i, j = 1, \dots, N$ if $j \notin [i]$, then $\psi(|x_i - x_j|)$ is integrable in every interval $[t, t_1]$.

Now let us for each $i = 1, \dots, N$ define $w_i = w_i^{t_0}$ by the system of ODE's

$$\begin{aligned} \dot{w}_i &= \frac{1}{N} \sum_{k \in [i]} (w_k - w_i) \psi(|x_i - x_k|), \\ w_i &\equiv w_j \quad \text{if } j \notin B_i(0) \end{aligned}$$

in $[t_0, t_1)$ with the initial data $w_i(t_0) = v_i(t_0)$ for all $i = 1, \dots, N$. All structure based properties 1, 2 and 4' from Corollary 3.3.2 hold also for the functions w_i as in their proof we never make use of the fact that $\dot{x} = v$. We introduce the functions w_i as a tool to study the evolution of v in a neighbourhood of t_1 . First we ensure that w_i and v_i are in some sense close to each other and behave in a similar way.

Lemma 3.4.1. For $t \in [t_0, t_1)$, we have

$$|v_i(t) - w_i(t)| \leq \omega(t_1 - t_0),$$

for some nonnegative continuous function ω with $\omega(0) = 0$.

Proof. Let $r(t) = \sum_{i \in [i]} (v_i(t) - w_i(t))^2$. We have

$$\begin{aligned} \dot{r} &= \frac{2}{N} \sum_{i,j \in [i]} (v_i - w_i) \left((v_j - v_i) - (w_j - w_i) \right) \psi(|x_i - x_j|) \\ &+ \frac{2}{N} \sum_{i,j \notin [i]} (v_i - w_i)(v_j - v_i) \psi(|x_i - x_j|) =: I + II. \end{aligned}$$

By a similar to the proof of property 3 from Proposition 3.1.2 application of the symmetry we conclude that

$$\begin{aligned} I &= \frac{2}{N} \sum_{i,j \in [i]} \left((v_i - w_i)(v_j - w_j) - (v_i - w_i)^2 \right) \psi(|x_i - x_j|) \\ &= -\frac{1}{N} \sum_{i,j \in [i]} \left((v_i - w_i) - (v_j - w_j) \right)^2 \psi(|x_i - x_j|) \leq 0. \end{aligned}$$

On the other hand II is integrable by Remark 3.4.1. Therefore, since $r(t_0) = 0$, for $t \in [t_0, t_1)$, we have

$$r(t) \leq \int_{t_0}^t |II| ds =: \omega^2(t - t_0),$$

where ω is a nonnegative continuous function with $\omega(0) = 0$. □

Our next goal is to prove that if $i \sim j$ then $|w_i(t) - w_j(t)| \rightarrow 0$ as $t \rightarrow t_1$. However before we begin let us make another purely technical assumption that

$$\sum_{i \in [i]} w_i = 0. \tag{3.15}$$

This does not make our reasoning any less general since by the same argumentation as in property 2 from Corollary 3.3.2 this sum is constant in time – thus we may as well assume that it equals 0. Thus our goal can be rewritten in a equivalent form: prove that

$$\lim_{t \rightarrow t_1} w_i(t) = 0 \quad \text{for all } i \in [i]. \tag{3.16}$$

The first step of the proof is to show the following slightly weaker assertion.

Lemma 3.4.2. *If $i \sim j$, then there exists a sequence $s_n \rightarrow t_1$, such that $|w_i(s_n) - w_j(s_n)| \rightarrow 0$.*

Proof. If $j \notin B_i(0)$ then $x_i \equiv x_j$ and $w_i \equiv w_j$ and the assertion holds. If $j \in B_i(0)$ then the proof follows by contradiction. Let us assume that $i \sim j$ and there is no such sequence s_n i.e. there exists $\delta > 0$, such that $|w_i(s) - w_j(s)| > \delta$ for $s \in [t_0, t_1)$. Since $i \sim j$ both i and j belong to $[i]$ and thus for all $s \in [t_0, t_1)$ and for $r(s) := \sum_{k,l \in [i]} (w_k(s) - w_l(s))^2$, we have

$$\dot{r} = \frac{2}{N} \sum_{k,l,m \in [i]} (w_k - w_l) \left((w_m - w_k) \psi(|x_k - x_m|) - (w_m - w_l) \psi(|x_l - x_m|) \right).$$

By the usual symmetry argument

$$\dot{r} = -2 \sum_{k,l \in [i]} (w_k - w_l)^2 \psi(|x_k - x_l|).$$

Now since $|w_i - w_j| > \delta$ and by property 4' from Corollary 3.3.2 also $\delta^2 < r(s) \leq NM^2$ and we have

$$(\ln r)' \leq -2 \frac{(w_i - w_j)^2}{r} \psi(|x_i - x_j|) \leq -\frac{2\delta^2}{NM^2} \psi(|x_i - x_j|)$$

and consequently

$$\delta^2 < r \leq e^{-\frac{2\delta^2}{NM^2} \int_0^s \psi(|x_i - x_j|) d\theta} r(t_0),$$

which is impossible since $\int_0^s \psi(|x_i - x_j|) \rightarrow \infty$ as $s \rightarrow t_1$. Therefore no such δ exists and the proof is complete. \square

Our next step is a technical lemma which is vaguely based on the fact that velocities of the particles only "pull" each other but never push away (which for example means that w_i which is the furthest from 0 may not go any further away from 0 because there is no other velocity to pull it there).

Lemma 3.4.3. *For each $k = 1, \dots, d$ we denote w_i^k – the k th coordinate of w_i and assume that up to permutations $w_1^k(t) \leq \dots \leq w_N^k(t)$. Then the sums*

$$\sum_{i=1}^l w_i^k(t), \quad \text{and} \quad \sum_{i=l}^N w_i^k(t), \quad l = 1, \dots, N$$

are respectively nondecreasing and nonincreasing.

Proof. We prove the assertion only for the first sum as the other differs only by sign. For all $l = 1, \dots, N$, we have

$$\left(\sum_{i=1}^l w_i^k \right)' = \sum_{i,j=1}^l (w_j^k - w_i^k) \psi(|x_i - x_j|) + \sum_{i=1}^l \sum_{j=l+1}^N (w_j^k - w_i^k) \psi(|x_i - x_j|) =: I + II.$$

By symmetry $I = 0$. On the other hand for $j > l$ as long as $w_j^k - w_i^k > 0$, we have $II \geq 0$ and the sum $\sum_{i=1}^l w_i^k$ is nondecreasing. \square

Now we may proceed with our goal which is the following proposition.

Proposition 3.4.1. *If $i \sim j$ then*

$$\lim_{t \rightarrow T_0} |w_i(t) - w_j(t)| = 0. \quad (3.17)$$

Proof. It suffices to show that the assertion holds if we substitute w_i with w_i^k – its k th coordinate, thus let us assume for simplicity of notation that $w_i = w_i^k$. Therefore w_i are real functions and their sum equals to 0 by (3.15). The proof follows by 3 steps.

Step 1. For $t \in [t_0, t_1)$, let $\mathcal{R}(t) := \max_{j \in [l]} w_j(t)$. First we prove that if at some point $t \in [t_0, t_1)$ we have

$$w_i(t) = \mathcal{R}(t) - \delta, \quad (3.18)$$

then

$$\sup_{s \in [t, t_1)} w_i \leq \mathcal{R}(t) - \frac{\delta}{N!}. \quad (3.19)$$

The proof follows by induction with respect to the number of velocities w_j that are bigger than w_i at the time t . For $n = 1$ we are in a situation when there is only one w_j , such that $\mathcal{R}(t) = w_j(t) > w_i(t)$ and (3.18) implies that $\mathcal{R}(t) - w_i(t) = \delta$. Now let

$$p(s) := \max\{w_k(s) : w_k(s) < \mathcal{R}(s)\}, \quad \text{for } s \in [t, T_0)$$

Clearly $p(t) = w_i(t)$ but it is possible that some other velocity may become bigger than w_i at some point in time and this is the only reason to introduce the function p , which will serve us by pointing the right-hand edge of the set of velocities smaller than \mathcal{R} . Clearly $w_i \leq p \leq \mathcal{R}$ in $[t, T_0)$. Moreover Lemma 3.4.3 implies that the sum $p + \mathcal{R}$ is nonincreasing. Therefore

$$\mathcal{R}(t) \geq \mathcal{R}(s) + p(s) - p(t) \geq 2w_i(s) - w_i(t) = 2w_i(s) - \mathcal{R}(t) + \delta,$$

which implies that

$$\sup_{s \in [t, t_1)} w_i \leq \mathcal{R}(t) - \frac{\delta}{2}.$$

Now let us assume that condition (3.18) implies that

$$\sup_{s \in [t, t_1)} w_i \leq \mathcal{R}(t) - \frac{\delta}{(n+1)!} \quad (3.20)$$

in case when at the time t there are exactly n velocities bigger than w_i . We will prove that this implies that if (3.18) holds, then

$$\sup_{s \in [t, t_1)} w_i \leq \mathcal{R}(t) - \frac{\delta}{(n+2)!} \quad (3.21)$$

if only there are exactly $n+1$ velocities bigger than w_i at the time t . In such case we define

$$p(s) := \max\{w_k(s) : k \notin \mathcal{G}\}, \quad \text{for } s \in [t, t_1),$$

where \mathcal{G} is the set of indexes of the $n + 1$ biggest velocities at the time t . Denoting $S(s) := \sum_{k \in \mathcal{G}} w_k(s)$, by Lemma 3.4.3, the function $S + p$ is nonincreasing as long as

$$p(s) < \min_{k \in \mathcal{G}} w_k(s), \quad (3.22)$$

thus

$$(n + 2)p(s) < S(s) + p(s) \leq S(t) + p(t) = S(t) + \mathcal{R}(t) - \delta \leq (n + 2)\mathcal{R}(t) - \delta, \quad (3.23)$$

$$p(s) < \mathcal{R}(t) - \frac{\delta}{n + 2}$$

as long as (3.22) holds. However if at some time s_0 , we have $p(s_0) = \mathcal{R}(t) - \frac{\delta}{n+2}$ then all of the inequalities in (3.23) become an equality which means that also $p(s_0) \geq \min_{k \in \mathcal{G}} w_k(s_0)$. At that point there are at most n velocities bigger than p and the distance between $p(s_0)$ and $\mathcal{R}(s_0)$ is no less than $\delta' := \frac{\delta}{n+2}$. Therefore by (3.20), we have

$$p(s) \leq \mathcal{R}(s_0) - \frac{\delta'}{(n + 1)!} \leq \mathcal{R}(t) - \frac{\delta}{(n + 2)!} \quad \text{for } s \in [s_0, t_1].$$

This proves (3.21). Noticing that $n \leq N - 1$ we get (3.19) and finish step 1.

Step 2. Our next step is the following simple observation.

Lemma 3.4.4. *If Proposition 3.4.1 does not hold, then there exists a class of relation \sim denoted by $[i]$, $\epsilon > 0$ and a sequence $s_n \rightarrow t_1$, such that for all $i \in [i]$ there exists $j \in [i]$, such that*

$$|w_i(s_{n_k}) - w_j(s_{n_k})| \rightarrow 0 \quad \text{and} \quad |w_i(s_{n_l}) - w_j(s_{n_l})| > \epsilon$$

for some subsequences $\{s_{n_k}\}, \{s_{n_l}\} \subset \{s_n\}$.

Proof. Stating that Proposition 3.4.1 does not hold is equivalent to stating that there exists a class $[i]$ that does not 'collapse' at t_1 i.e. it is not true that for all $i, j \in [i]$ we have

$$\lim_{t \rightarrow t_1} |w_i(t) - w_j(t)| = 0.$$

Now suppose that Lemma 3.4.4 also does not hold. Then by Lemma 3.4.2 there exists $i \in [i]$ such that for all $j \in [i]$ we have

$$\lim_{t \rightarrow t_1} |w_i(t) - w_j(t)| = 0$$

and it immediately implies the 'collapse' of $[i]$. □

Step 3. We finish the proof by contradiction assuming that Proposition 3.4.1 does not hold. Let $[i]$ be the 'non-collapsing' class of indices and let us fix $t \in [t_0, t_1)$ and assume that w_i (for

$i \in [i]$) is one of the biggest velocities at the time t i.e. $\mathcal{R}(t) = w_i(t)$. Lemma 3.4.4 ensures existence of $j \in [i]$, such that

$$|w_i - w_j| \rightarrow 0 \quad (3.24)$$

on one subsequence converging to t_1 and

$$|w_i - w_j| > \epsilon \quad (3.25)$$

on some other subsequence converging to t_1 for ϵ independent of i and j . Thus (3.25) implies that at some time $s \in [t, t_1)$ either w_i or w_j (say w_j) is farther from $\mathcal{R}(t)$ than ϵ . Then step 1 implies that

$$\sup_{\theta \in [s, t_1)} w_j \leq \mathcal{R}(t) - \frac{\epsilon}{N!}.$$

Moreover (3.24) implies that at some other time $r \in [s, t_1)$, we have

$$w_i(r) \leq \mathcal{R}(t) - \frac{\epsilon}{2N!}$$

and after that point (again by step 1)

$$\sup_{\theta \in [r, t_1)} w_i \leq \mathcal{R}(t) - \frac{\epsilon}{(2N!)^2}.$$

This procedure can be performed with any velocity w_i that at some time equals to \mathcal{R} as many times as we want. Therefore we may make sure that $\mathcal{R}(t)$ is arbitrarily small at some time $t < t_1$. The same can be done with $\mathcal{L}(t) := \min_{j \in [i]} w_j(t)$ to conclude that the diameter of velocities converges to 0 as $t \rightarrow t_1$ and this contradicts (3.25) and by Lemma 3.4.4 implies that Proposition 3.4.1 holds. This finishes the proof. \square

Remark 3.4.2. In Proposition 3.4.1 we proved that for all $i \in [i]$ we have $w_i \rightarrow 0$. However this was under our assumption (3.15). Now it is time to drop this assumption and conclude that in general there exists a constant $\bar{v} := \sum_{i \in [i]} w_i$, such that for all $i \in [i]$, we have $w_i \rightarrow \bar{v}$.

Our last goal in this subsection is to clarify what does Proposition 3.4.1 imply to the motion of v .

Corollary 3.4.1. *For all $i \in [i]$, we have*

$$\lim_{t \rightarrow t_1^-} v_i(t) = \bar{v}.$$

Proof. From Lemma 3.4.1 and Proposition 3.4.1, given $\epsilon > 0$, we have

$$|v_i(s) - \bar{v}| \leq |v_i(s) - w_i^{t_0}(s)| + |w_i^{t_0}(s) - \bar{v}| < \omega(t_1 - t_0) + |w_i^{t_0}(s) - \bar{v}| < \epsilon$$

for t_0 and s sufficiently close to t_1 . \square

This finally proves that the function v has a limit at t_1^- and we may extend it continuously to $[0, t_1]$.

We finish this section by a corollary that enables us to extend our solutions up to the first time of collision as long as it is not the first time of sticking as well.

Corollary 3.4.2. *Let x be as in Proposition 3.3.2. Then if t_1 is not a time of sticking for x then x is a weak $(W^{2,1}([0, t_1]))^{Nd}$ solution of (2.5).*

Proof. By Corollary 3.4.1 if $i \sim j$ then i th and j th particles stick together at t_1 . Conversely if no particles stick together at t_1 then for all $i, j = 1, \dots, N$ we have $i \not\sim j$ and in particular for all $i, j = 1, \dots, N$ the function $t \mapsto \psi(|x_i(t) - x_j(t)|)$ is integrable in a left-sided neighbourhood of t_1 . Therefore v is absolutely continuous on $[0, t_1]$ and x is a weak $(W^{2,1}([0, t_1]))^{Nd}$ solution of (2.5). \square

3.5 Global existence

In this section we combine our efforts from Sections 3.3 and 3.4 to obtain global existence in the sense of Definition 2.1.2. Proposition 3.3.2 ensures existence of weak solutions in $[0, t] \subset [0, t_1)$ with a continuous velocity v . On top of that Corollary 3.4.2 ensures existence of solutions up to any time of collision as long as none of them is a time of sticking. Assuming that t_1 is a new initial point with initial data equal to $x(t_1)$ and $v(t_1)$ and applying the same reasoning again we conclude that the solution exists on $[0, t_n]$, where t_n is n th time at which some particles collide. Therefore for arbitrary initial data there exists a solution $x \in C^1([0, T^*])^{dN}$ satisfying (2.5) weakly in $W^{2,1}([0, T^*])$, for any T^* satisfying the following conditions:

1. $T^* < T_1$, where T_1 is (as in Definition 2.1.2) the first time of sticking of the particles;
2. $\exists t_n$ such that $T^* \leq t_n$, which ensures that $[0, T^*]$ is included in $[0, t_1] \cup \bigcup_{i=1}^n [t_i, t_{i+1}]$.

The above conditions ensure the existence of $W^{2,1}([0, \tilde{T} - \epsilon])$ weak solutions, where $0 < \epsilon < \tilde{T}$ is arbitrary and \tilde{T} is either the first time of sticking of the particles (and then beyond that point condition 1 fails) or the first density point of the times of collisions (and then beyond that point condition 2 fails), whichever comes first. Thus our remaining goal is to ensure that we may extend the solution beyond \tilde{T} .

Proof of Theorem 2.2.1. It suffices to show that we may extend our solution up to an arbitrary $T > 0$, knowing that it exists in $[0, \tilde{T} - \epsilon] \subset [0, \tilde{T}]$, where

- (a) \tilde{T} is the first density point of the times of collisions (and no particles did stick together before \tilde{T}), or

(b) \tilde{T} is the first time of sticking of the particles (and there was no density points of the times of collisions before \tilde{T}).

Ad. (a). We will show that if \tilde{T} is a density point of the times of collisions then it is a time of sticking, which will reduce case (a) to case (b). Let t_n be a sequence of the points of collision and $t_n \rightarrow \tilde{T}$. We will prove that \tilde{T} is a point of sticking for some particles x_i and x_j . Clearly there exist i and $j \in B_i(0)$ and a subsequence t_{n_k} , such that $x_i(t_{n_k}) - x_j(t_{n_k}) = 0$, which by Lipschitz continuity of x (property 4' from Corollary 3.3.2) implies that

$$x_i(t) - x_j(t) \rightarrow 0 \quad (3.26)$$

as $t \rightarrow \tilde{T}$ and \tilde{T} is a point of collision of x_i and x_j . Now it remains to show that for some i, j satisfying (3.26), we have $v_i(t) - v_j(t) \rightarrow 0$ as $t \rightarrow \tilde{T}$. If there exist i, j , such that

$$\int_t^{\tilde{T}} \psi(|x_i - x_j|) ds = \infty \quad (3.27)$$

for all $t < \tilde{T}$, then by Corollary 3.4.1, we are done. On the other hand if for all i, j the function $t \mapsto \psi(|x_i(t) - x_j(t)|)$ is integrable in a left sided neighbourhood of \tilde{T} then velocity v is in fact uniformly continuous at \tilde{T}^- and in particular has a limit at \tilde{T} . Therefore there exists a limit of $v_i - v_j$ at \tilde{T} . If this limit equals 0 then, again, we are done. If on the other hand it equals to some $\xi \neq 0$, then in a left-sided neighbourhood of \tilde{T} we have $v_i - v_j \in B(\xi, \frac{|\xi|}{2})$, where $B(\xi, \frac{|\xi|}{2})$ is a ball centred at ξ radius $\frac{|\xi|}{2}$. This implies a clearly false statement that

$$0 = x_i(t_{n_{k+1}}) - x_j(t_{n_{k+1}}) \in (t_{n_{k+1}} - t_{n_k})B\left(\xi, \frac{|\xi|}{2}\right)$$

with t_{n_k} and $t_{n_{k+1}}$ sufficiently close to \tilde{T} . This contradicts the assumption that $\xi \neq 0$.

Ad. (b). We have shown that in both cases (a) and (b) we actually know that $\tilde{T} = T_1$, which is the first time of sticking of any particles. Corollary 3.4.1 implies that there exists a left sided limit of v at T_1 . Thus it may be extended continuously up to $[0, T_1]$. Moreover T_1 can be treated as a well defined initial time for (2.5) on $[T_1, T]$ Then we may further extend our solution beyond T_1 . Finally, since there can be at most $N - 1$ times of sticking, then for all $T > 0$ either we find T_n such that $T_n > T$ or all the particles stick together before time T and travel with constant velocity for as long as needed. Such extended solution is not weak in $W^{2,1}([0, T])$ but it satisfies Definition 2.1.2 nevertheless. \square

3.6 On the case of two particles – flocking in a finite time

In this section our goal is to discuss the possibility of a finite in time alignment in case of two particles ($N = 2$). First let us recall that asymptotic flocking was studied before in most papers mentioned in the introduction, see e.g. [31] and we refer to those papers to see general definitions and results. Here, we consider the most strict form of flocking, which is sticking

of the trajectories of the particles in a finite time. By property 2 from Corollary 3.3.2 the average velocity of the particles is constant, which means that

$$v_1 \equiv -v_2 + \bar{v}$$

for some constant \bar{v} . Without a loss of generality we may assume that $\bar{v} = 0$. The above observation implies that

$$x_1(t) = -x_2(t) + t\bar{v} + (x_1(0) + x_2(0))$$

and assuming without a loss of generality that also $x_1(0) = -x_2(0)$, we get $x_1 \equiv -x_2$. Thus both, average velocity and the centre of mass of the particles are equal to 0. Therefore the particles move parallelly to each other, either on two separate parallel lines or on the same line. In the former case, the distance between particles is always no less than the distance of respective lines, thus there is no possibility of a finite in time alignment. In the latter case the distance between particles can be arbitrarily small, thus hypothetically a finite in time alignment may occur.

In order to simplify our calculations, since particles move on the same line, then by a simple change of variables we may assume that $d = 1$. Altogether we have two particles x_1 and x_2 , with $x_1 \equiv -x_2$ and $v_1 \equiv -v_2$. Therefore they are unequivocally defined by the function $\phi(t) := x_2(t) - x_1(t)$. Then the C-S model (2.3) (or (2.5), since in this case they are the same) can be rewritten equivalently as

$$\ddot{\phi}(t) = -2\dot{\phi}(t)\psi(|\phi(t)|), \quad (3.28)$$

with $\phi(0) = x_2(0) - x_1(0) \geq 0$ and $\dot{\phi}(0) = v_2(0) - v_1(0) \in \mathbb{R}$. Moreover Lemma 3.4.3 implies that if at some time t we have $\dot{\phi}(t) = 0$ then it will be constantly equal to 0 from that point in time. This implies that $\dot{\phi}$ may not change sign and this farther implies that there may be at most one collision of the particles. Finally let us notice that by Theorem 2.2.1 there exists a solution to (3.28) with arbitrary initial data and we can easily prove that if $\phi(0) > 0$, then this solutions is unique. Now we are ready to state our main result of this section.

Proposition 3.6.1. *Let ϕ be a solution of (3.28) with $\phi(0) > 0$. Then the following statements are equivalent:*

1. *There exists a time $t_0 < \infty$ such that $\phi(t_0) = \dot{\phi}(t_0) = 0$.*
2. *Initial data satisfy:*

$$\dot{\phi}(0) = -2\Psi(\phi(0)), \quad (3.29)$$

where $\Psi(s) := \frac{1}{1-\alpha} s^{1-\alpha}$ is a primitive of ψ .

Proof. Since there is at most one collision of the particles and we know that they stick together, thus $\phi = |\phi|$ and by simple integration of (3.28) we conclude that the function ϕ satisfies:

$$\dot{\phi}(t) = -2\Psi(\phi(t)) + 2\Psi(\phi(0)) + \dot{\phi}(0) \quad (3.30)$$

Substituting t with t_0 in (3.30) we obtain

$$0 = 2\Psi(\phi(0)) + \dot{\phi}(0),$$

which is exactly condition (3.29). Now let us assume that (3.29) is satisfied. We are going to prove existence of t_0 . First note that in our case (3.30) is satisfied on the set $\{t : \phi(t) \geq 0\}$ and it has the following form:

$$\dot{\phi}(t) = -2\Psi(\phi(t)). \quad (3.31)$$

From (3.31) and by the definition of ψ we obtain

$$\dot{\phi}(t) = -\frac{2}{1-\alpha}\phi(t)\psi(\phi(t))$$

and

$$\phi(t) = e^{-\frac{2}{1-\alpha} \int_0^t \psi(\phi(s)) ds} \phi(0).$$

Thus, since $\max_{t \in [0, t_0]} \phi(t) = \phi(0)$, we have

$$\phi(t) \leq e^{-\frac{2}{1-\alpha} t \psi(\phi(0))} \phi(0),$$

which can become arbitrarily small in a finite time. Now for $n = 2, 3, \dots$ let

$$t_n := \inf\{t > t_{n-1} : \phi(t) \leq 2^{-n}\},$$

with $t_1 := 0$. We have

$$\begin{aligned} \phi(t_n) &= e^{-\frac{2}{1-\alpha} \int_{t_{n-1}}^{t_n} \psi(\phi(s)) ds} \phi(t_{n-1}), \\ 2^{-1} &= e^{-\frac{2}{1-\alpha} \int_{t_{n-1}}^{t_n} \psi(\phi(s)) ds}, \\ \ln 2 &= \frac{2}{1-\alpha} \int_{t_{n-1}}^{t_n} \psi(\phi(s)) ds \geq \frac{2}{1-\alpha} (t_n - t_{n-1}) 2^{\alpha(n-1)}. \end{aligned}$$

Therefore

$$(t_n - t_{n-1}) \leq \frac{(1-\alpha) \ln 2}{2} 2^{\alpha(1-n)}$$

and t_n is a partial sum of a convergent series. Thus t_n converges to a finite limit t_0 such that $\phi(t_0) = \dot{\phi}(t_0) = 0$. \square

Remark 3.6.1. Let us mention that a finite in time alignment may not happen in the case of regular weight ψ_{cs} defined for example by (1.5) since (3.28) implies that

$$|\dot{\phi}(t)| = e^{-2 \int_0^t \psi_{cs}(|\phi(s)|) ds} |\dot{\phi}(0)| \geq e^{-2t \|\psi_{cs}\|_\infty} |\dot{\phi}(0)| > 0,$$

as long as $\dot{\phi}(0) \neq 0$. Similarly we may prove that in case of singular weight ψ a finite in time alignment of velocities is equivalent to sticking of the trajectories of the particles. Finally we may just as easily prove that with unintegrable singular weight ψ , e.g. when $\psi(s) = s^{-\alpha}$ for $\alpha > 1$ in one dimensional setting not only particles cannot stick but they cannot even collide.

Remark 3.6.2. Conditions described in Proposition 3.6.1 refer to the function ϕ and in a simplified case of one dimension. However they can be modified to cover more general cases and refer directly to x_1 and x_2 . Thus the example of Proposition 3.6.1 serves as a proof of Proposition 2.2.1.

Chapter 4

Cucker–Smale model with singular weight: strong solutions

In this chapter we improve the results of Chapter 3 for $\alpha \in (0, \frac{1}{2})$. We show that in such case for any initial data, the piecewise–weak solution has an absolutely continuous velocity component, is unique and satisfies (2.3) in a $W^{2,1}$ weak sense (and in particular a.e.), which is significantly better than what we were able to prove in case of $\alpha \in (0, 1)$. The improvement comes mostly from an inequality originating from [33], that enables us to show a better regularity of the solutions. This inequality is also the reason behind our restriction of the set of admissible α to $(0, \frac{1}{2})$ as for $\alpha \in (\frac{1}{2}, 1)$ it fails to hold and for $\alpha = \frac{1}{2}$ it does not suffice.

First let us rewrite the results of Chapter 3 in a more suitable and compact manner.

Corollary 4.0.1 (Summary). *Let $\alpha \in (0, 1)$. For all initial data x_0, v_0 , there exists at least one solution of Cucker–Smale’s flocking model with a singular communication weight given by (2.2). This solution exists in the sense of Definition 2.1.2. Moreover the following properties hold:*

1. *For all $\epsilon > 0$, the function v is absolutely continuous on each time interval $[T_n, T_{n+1} - \epsilon]$.*
2. *The set of times of collision is at most countable, while the set of times of sticking has at most N elements. Moreover if there exists a point of density of the times of collision¹ then this point itself is a time of sticking. Thus there are at most N points of density of the times of collision.*
3. *Both x and v are uniformly bounded i.e. there exists an N independent constant*

C_1 , such that

$$\begin{aligned} \max_{i=1,\dots,N} \sup_{t \in [0,T]} |x_i(t)| &\leq TC_1, \\ \max_{i=1,\dots,N} \sup_{t \in [0,T]} |v_i(t)| &\leq C_1. \end{aligned}$$

Proof. The proof can be found in Chapter 3. Existence of solutions in the sense of Definition 2.1.2 is the subject of Theorem 2.2.1 while points 1. and 2. were proved along with Theorem 2.2.1. Point 3. is a part of Corollary 3.3.2. \square

Our goal in this chapter is to prove Theorems 2.2.2 and 2.2.3. These theorems come from the natural question of how much worse the piecewise–weak solutions are compared to classical, regular solutions. In the effort to answer this question we found out that if we discriminate two cases of $\alpha \in (0, \frac{1}{2})$ and $\alpha \in [\frac{1}{2}, 1)$, we obtain significantly different results. In the first situation with $\alpha \in (0, \frac{1}{2})$ we are actually able to prove that the piecewise–weak solution has an absolutely continuous velocity component (and thus it is in fact a classical solution) and that it is unique. On the other hand if $\alpha \in [\frac{1}{2}, 1)$ our method is not sufficient to show regularity of the piecewise–weak solutions, however we are still able to obtain uniqueness thanks to the use of the sets $B_i(t)$ in the construction of the solutions (as we mentioned in Remark 2.1.3). In case of $\alpha \in (0, \frac{1}{2})$, by Corollary 4.0.1 it suffices to prove uniqueness and that $v \in W^{1,1}([0, T])$ (i.e. that v is absolutely continuous). In case of $\alpha \in (0, 1)$ we only need to prove uniqueness. We do it in the subsequent sections.

4.1 Absolute continuity of the velocity

In this section we prove the absolute continuity of v . First let us state it in an explicit way.

Proposition 4.1.1. *Let $\alpha \in (0, \frac{1}{2})$ and let (x, v) be a solution of (2.3) in the sense of Definition 2.1.2. Then there exists a constant M , dependent on α, T and the initial data but independent of N , such that*

$$\frac{1}{N} \sum_{i=1}^N \int_0^T |\dot{v}_i(t)| dt \leq M.$$

Thus v belongs to the space $W^{1,1}([0, T])$ and is absolutely continuous.

To prove the above proposition we require the following technical lemmas.

Lemma 4.1.1. *Let $\alpha \in (0, \frac{1}{2})$ and let (x, v) be a solution of (2.3) in the sense of Definition 2.1.2. Then the function*

$$R(t) := \sum_{i,j=1}^N |v_i(t) - v_j(t)|^2 \psi(|x_i(t) - x_j(t)|)$$

is integrable and

$$\int_0^T R(t) dt \leq N^2 C_1^2,$$

where C_1 is the constant from Corollary 4.0.1.3.

Lemma 4.1.2. *Let $\alpha \in (0, \frac{1}{2})$ and let (x, v) be a solution of (2.3) in the sense of Definition 2.1.2. Suppose further that there occurs no sticking in the time interval $[s_1, s_2]$. Then for all $i, j = 1, \dots, N$ and all $\theta \in (0, 1)$, we have*

$$\int_{s_1}^{s_2} |x_j - x_i|^{-\theta} dt < \infty.$$

Lemma 4.1.3. *Let $f = (f_1, \dots, f_d) : [0, T] \rightarrow \mathbb{R}^d$ be a $C^1([0, T]) \cap W_{loc}^{2,1}((0, T))$ vector valued function that is nonzero a.e.. Moreover let $h : [0, \infty) \rightarrow [0, \infty)$ be defined as*

$$h(\lambda) = \lambda^{-\theta},$$

for some $0 < \theta < 1$. Then there exists a constant $C_2 > 0$ depending on $\|f\|_\infty$ and θ , such that we have

$$\int_0^T |f'|^2 h(|f|) dt \leq C_2 \int_0^T |f''| dt + \mathcal{R}(f, T) - \mathcal{R}(f, 0), \quad (4.1)$$

provided that

$$\int_\epsilon^{T-\epsilon} h(|f|) dt < \infty \quad (4.2)$$

for all $\epsilon > 0$. Here, for $H(\lambda) = \frac{1}{1-\theta} \lambda^{1-\theta}$ – a primitive of h , we denote

$$\mathcal{R}(f, t) := \begin{cases} \frac{f(t)f'(t)}{|f(t)|} H(|f(t)|) & \text{for } f(t) \neq 0, \\ 0 & \text{for } f(t) = 0. \end{cases} \quad (4.3)$$

The proofs of Lemmas 4.1.1 and 4.1.2 can be found in Appendix A. On the other hand Lemma 4.1.3 along with its proof comes in almost unchanged form from paper [33]. Equations similar to (4.1) with multiple examples and applications can be found in [33] or [34].

Now we proceed with the proof of Proposition 4.1.1

Proof of Proposition 4.1.1 Let T_k and $N_s \leq N$ be like in Definition 2.1.2. Then, by Corollary 4.0.1.1, velocity v is absolutely continuous on each interval $[T_k, T_{k+1} - \epsilon]$ for arbitrarily small $\epsilon > 0$. Therefore, given $k = 0, \dots, N_s$ by (2.3)₂, we have

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N \int_{T_k}^{T_{k+1}-\epsilon} |\dot{v}_i(t)| dt &= \frac{1}{N} \sum_{i=1}^N \int_{T_k}^{T_{k+1}-\epsilon} \left| \frac{1}{N} \sum_{j=1}^N (v_j - v_i) \psi(|x_j - x_i|) \right| dt \\ &\leq \frac{1}{N^2} \sum_{i,j=1}^N \int_{T_k}^{T_{k+1}-\epsilon} |v_j - v_i| \psi(|x_j - x_i|) dt. \end{aligned}$$

Let us denote

$$\mathcal{L}_{ijk}^\epsilon := \int_{T_k}^{T_{k+1}-\epsilon} |v_j - v_i| \psi(|x_j - x_i|) dt.$$

Then, we have

$$\mathcal{L}_{ijk}^\epsilon = \int_{T_k}^{T_{k+1}-\epsilon} |v_j - v_i|^{2\delta} (\psi(|x_j - x_i|))^\delta \cdot |v_j - v_i|^{1-2\delta} (\psi(|x_j - x_i|))^{1-\delta} dt,$$

where $0 < \delta \ll 1$ is some very small number. We then apply Young's inequality with $\eta > 0$ and exponent $q = \frac{2}{1-2\delta} \in (1, \infty)$ (then it's conjugate $q' = \frac{2}{1+2\delta}$) to get

$$\begin{aligned} \mathcal{L}_{ijk}^\epsilon &\leq C(\eta) \int_{T_k}^{T_{k+1}-\epsilon} |v_j - v_i|^{\frac{4\delta}{1+2\delta}} (\psi(|x_j - x_i|))^{\frac{2\delta}{1+2\delta}} dt \\ &\quad + \eta C \int_{T_k}^{T_{k+1}-\epsilon} |v_j - v_i|^2 (\psi(|x_j - x_i|))^{\frac{2-2\delta}{1-2\delta}} dt =: I_{ijk}^\epsilon + II_{ijk}^\epsilon. \end{aligned}$$

By Hölder's inequality with $q = \frac{1+2\delta}{2\delta}, q' = 1 + 2\delta$, we have

$$I_{ijk}^\epsilon \leq C(\eta) \left(\int_{T_k}^{T_{k+1}-\epsilon} |v_j - v_i|^2 \psi(|x_j - x_i|) dt \right)^{\frac{2\delta}{1+2\delta}} \cdot (T_{k+1} - \epsilon - T_k)^{\frac{1}{1+2\delta}}. \quad (4.4)$$

To deal with the estimation of II_{ijk}^ϵ we use Lemma 4.1.3. First let us check whether, the assumptions are satisfied. However we will check if the assumptions are satisfied on $[T_k, T_{k+1}]$ instead of $[T_k, T_{k+1} - \epsilon]$ since we need estimates to be uniform with respect to ϵ anyway. We take $f = x_j - x_i$, which by Corollary 4.0.1 is a vector valued $C^1([T_k, T_{k+1}]) \cap W_{loc}^{2,1}((T_k, T_{k+1}))$ function that is equal to 0 in at most countable subset of $[T_k, T_{k+1}]$. Moreover we take $h(\lambda) = (\psi(\lambda))^{\frac{2-2\delta}{1-2\delta}} = \lambda^{-\theta}$, for $\theta = \frac{2-2\delta}{1-2\delta} \alpha \in (0, 1)$, provided that δ is sufficiently small. Finally Lemma 4.1.2 implies that assumption (4.2) is also satisfied. Therefore, for \mathcal{R} defined by (4.3) and $C_2 > 0$ (note that C_2 depends on α, δ and C_1), we have

$$\begin{aligned} II_{ijk}^\epsilon &\leq \eta C \left(C_2 \int_{T_k}^{T_{k+1}-\epsilon} |\dot{v}_j - \dot{v}_i| dt + \mathcal{R}(x_j - x_i, T_{k+1} - \epsilon) - \mathcal{R}(x_j - x_i, T_k) \right) \\ &\leq \eta C \left(C_2 \int_{T_k}^{T_{k+1}-\epsilon} |\dot{v}_j| dt + C_2 \int_{T_k}^{T_{k+1}-\epsilon} |\dot{v}_i| dt \right. \\ &\quad \left. + \mathcal{R}(x_j - x_i, T_{k+1} - \epsilon) - \mathcal{R}(x_j - x_i, T_k) \right) \end{aligned} \quad (4.5)$$

and by combining (4.4) with (4.5) we end up with the estimate

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N \int_{T_k}^{T_{k+1}-\epsilon} |\dot{v}_i| dt &\leq \frac{C(\eta)}{N^2} \sum_{i,j=1}^N \left(\int_{T_k}^{T_{k+1}-\epsilon} |v_j - v_i|^2 \psi(|x_j - x_i|) dt \right)^{\frac{2\delta}{1+2\delta}} (T_{k+1} - \epsilon - T_k)^{\frac{1}{1+2\delta}} \\ &\quad + 2\eta C C_2 \frac{1}{N} \sum_{i=1}^N \int_{T_k}^{T_{k+1}-\epsilon} |\dot{v}_i| dt + \frac{\eta C}{N^2} \sum_{i,j=1}^N \left(\mathcal{R}(x_j - x_i, T_{k+1} - \epsilon) - \mathcal{R}(x_j - x_i, T_k) \right), \end{aligned}$$

which with a suitably chosen η leads to the inequality presented below. However before we state the inequality let us mention that at this point for the sake of the simplicity of notation we give up on controlling the constants – we will present them as soon as we finish the prove.

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N \int_{T_k}^{T_{k+1}-\epsilon} |\dot{v}_i| dt &\leq \frac{C}{N^2} \sum_{i,j=1}^N \left(\int_{T_k}^{T_{k+1}-\epsilon} |v_j - v_i|^2 \psi(|x_j - x_i|) dt \right)^{\frac{2\delta}{1+2\delta}} \cdot (T_{k+1} - \epsilon - T_k)^{\frac{1}{1+2\delta}} \\ &\quad + \frac{C(C_2)}{N^2} \sum_{i,j=1}^N \left(\mathcal{R}(x_j - x_i, T_{k+1} - \epsilon) - \mathcal{R}(x_j - x_i, T_k) \right). \end{aligned}$$

By the monotone convergence theorem and continuity of \mathcal{R} (see the end of the proof of Lemma 4.1.3), we may pass with ϵ to 0 obtaining

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N \int_{T_k}^{T_{k+1}} |\dot{v}_i| dt &\leq \frac{C}{N^2} \sum_{i,j=1}^N \left(\int_{T_k}^{T_{k+1}} |v_j - v_i|^2 \psi(|x_j - x_i|) dt \right)^{\frac{2\delta}{1+2\delta}} \cdot (T_{k+1} - T_k)^{\frac{1}{1+2\delta}} \\ &\quad + \frac{C(C_2)}{N^2} \sum_{i,j=1}^N \left(\mathcal{R}(x_j - x_i, T_{k+1}) - \mathcal{R}(x_j - x_i, T_k) \right) \end{aligned}$$

and finally sum over $k = 0, \dots, N_s$ to get

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N \int_0^T |\dot{v}_i| dt &\leq \frac{C}{N^2} \sum_{i,j=1}^N \sum_{k=1}^{N_s} \left(\int_{T_k}^{T_{k+1}} |v_j - v_i|^2 \psi(|x_j - x_i|) dt \right)^{\frac{2\delta}{1+2\delta}} \cdot (T_{k+1} - T_k)^{\frac{1}{1+2\delta}} \\ &\quad + \frac{C(C_2)}{N^2} \sum_{i,j=1}^N \left(\mathcal{R}(x_j - x_i, T) - \mathcal{R}(x_j - x_i, 0) \right) =: I + II. \end{aligned} \quad (4.6)$$

We yet again apply Hölder's inequality (this time for sums) with exponents $q = \frac{1+2\delta}{2\delta}$ and $q' = 1 + 2\delta$ along with Lemma 4.1.1 to get

$$I \leq \frac{C}{N^2} \sum_{i,j=1}^N \left(\int_0^T |v_j - v_i|^2 \psi(|x_j - x_i|) dt \right)^{\frac{2\delta}{1+2\delta}} \cdot T^{\frac{1}{1+2\delta}} \leq C(C_1, T, \delta). \quad (4.7)$$

Moreover by the definition of \mathcal{R} and Corollary 4.0.1.3, we have

$$\begin{aligned} II &\leq \frac{C(C_2)}{N^2} \sum_{i,j=1}^N \left(|v_j(T) - v_i(T)| |H(|x_j(T) - x_i(T)|)| \right. \\ &\quad \left. + |v_j(0) - v_i(0)| |H(|x_j(0) - x_i(0)|)| \right) \\ &\leq C(C_1, C_2, T, \alpha). \end{aligned} \quad (4.8)$$

After combining inequalities (4.6), (4.7) and (4.8), we obtain

$$\frac{1}{N} \sum_{i=1}^N \int_0^T |\dot{v}_i| dt \leq C(C_1, T, \delta) + C(C_1, C_2, T, \alpha) =: M,$$

which finishes the proof. \square

Remark 4.1.1. Careful control over the constants in the above proof leads to the following estimation of M :

$$M \leq 2CC_1^2 \cdot T^{\frac{1}{1+2\delta}} + \frac{2}{(1-\alpha)C_2} C_1^{2-\alpha} T^{1-\alpha}.$$

4.1.1 Uniqueness of solutions

Our goal in this section is to prove uniqueness of solutions to (2.3) for $\alpha \in (0, \frac{1}{2})$ and uniqueness of the piecewise–weak solutions for $\alpha \in (\frac{1}{2}, 1)$.

Proposition 4.1.2. *Let $\alpha \in (0, \frac{1}{2})$. Then the $W^{2,1}$ weak solution of (2.3) is unique.*

Proof. Suppose that (x^1, v^1) and (x^2, v^2) are two $W^{2,1}$ weak solutions of (2.3), with weight ψ given by (2.2) and $\alpha \in (0, \frac{1}{2})$ on the time interval $[0, T]$, subjected to the initial data (x_0, v_0) . We will show that in fact $(x^1, v^1) \equiv (x^2, v^2)$. The proof will follow by four steps. In steps 1-3 we prove uniqueness in a small neighbourhood of the initial time $t = 0$ considering three cases: non-collision initial data, non-sticking initial data and initial data with particles that are stuck together. In step 4 we combine our efforts from previous steps and conclude the proof.

Step 1. If there are no collisions at the initial time, which means that for all $i \neq j$, we have $x_{0,i} \neq x_{0,j}$, then by the fact that $x^1, x^2 \in C^1([0, T])$, there exists $\delta > 0$, such that for all $i \neq j$, we have $|x_i^m(s) - x_j^m(s)| > \delta$ with $m = 1, 2$ for $s \in [0, \delta]$. The communication weight ψ is smooth on the domain $[\delta, +\infty)$ and thus, on the time interval $[0, \delta]$ system (2.3) is a non-linear ODE with a Lipschitz continuous nonlinearity and uniqueness is standard.

Step 2. In the case of non-sticking initial data (which means that for all $i \neq j$ if $x_{0,i} = x_{0,j}$ then $v_{0,i} \neq v_{0,j}$) let us consider

$$r(t) := \sum_{i=1}^N (v_i^1(t) - v_i^2(t))^2.$$

By the assumptions, r is an absolutely continuous function, thus it has a bounded variation and can be represented as a sum of two functions, respectively nonincreasing and nondecreasing. Noting that $r(0) = 0$, let

$$r_{inc}(t) := \int_0^t (\dot{r}(s))_+ ds,$$

where by $(\dot{r})_+$ we denote the positive part of the function \dot{r} . Then if we prove that $r_{inc} \equiv 0$ then we will also know that $r \equiv 0$ and that actually $x^1 \equiv x^2$. By (2.3)₂, we have

$$\frac{d}{dt} r_{inc} = \frac{2}{N} \left[\sum_{i,j=1}^N (v_i^1 - v_i^2) \left((v_j^1 - v_i^1) \psi(|x_j^1 - x_i^1|) - (v_j^2 - v_i^2) \psi(|x_j^2 - x_i^2|) \right) \right]_+.$$

After substituting i and j in the above equation we obtain

$$\begin{aligned} \frac{d}{dt} r_{inc} &= \frac{1}{N} \left[\sum_{i,j=1}^N \left((v_i^1 - v_i^2) - (v_j^1 - v_j^2) \right) \left((v_j^1 - v_i^1) \psi(|x_j^1 - x_i^1|) - (v_j^2 - v_i^2) \psi(|x_j^2 - x_i^2|) \right) \right]_+ \\ &= \frac{1}{N} \left[\sum_{i,j=1}^N \left((v_i^1 - v_i^2) - (v_j^1 - v_j^2) \right) \left((v_j^1 - v_i^1) - (v_j^2 - v_i^2) \right) \psi(|x_j^1 - x_i^1|) \right. \\ &\quad \left. + \sum_{i,j=1}^N \left((v_i^1 - v_i^2) - (v_j^1 - v_j^2) \right) (v_j^2 - v_i^2) \left(\psi(|x_j^1 - x_i^1|) - \psi(|x_j^2 - x_i^2|) \right) \right]_+ \\ &= \frac{1}{N} \left[- \sum_{i,j=1}^N \left((v_i^1 - v_i^2) - (v_j^1 - v_j^2) \right)^2 \psi(|x_j^1 - x_i^1|) \right. \\ &\quad \left. + \sum_{i,j=1}^N \left((v_i^1 - v_i^2) - (v_j^1 - v_j^2) \right) (v_j^2 - v_i^2) \left(\psi(|x_j^1 - x_i^1|) - \psi(|x_j^2 - x_i^2|) \right) \right]_+ \\ &\leq \frac{1}{N} \sum_{i,j=1}^N \left| (v_i^1 - v_i^2) - (v_j^1 - v_j^2) \right| |v_j^2 - v_i^2| \left| \psi(|x_j^1 - x_i^1|) - \psi(|x_j^2 - x_i^2|) \right|. \end{aligned} \quad (4.9)$$

By Corollary 4.0.1.3 the factor $|v_j^2 - v_i^2|$ is bounded uniformly with respect to i, j and t . Next, we fix i and j and consider two cases:

Case 1: $x_i(0) \neq x_j(0)$. This is in fact the situation from step 1, i.e. there exists $\delta > 0$, such that for all i, j with $x_i(0) \neq x_j(0)$, we have

$$|x_i^m - x_j^m| \geq \delta, \quad m = 1, 2$$

on $[0, \delta]$. Then

$$\left| \psi(|x_j^1 - x_i^1|) - \psi(|x_j^2 - x_i^2|) \right| \leq L(\delta) \left| (x_j^1 - x_i^1) - (x_j^2 - x_i^2) \right| \quad (4.10)$$

for some Lipschitz constant $L(\delta)$.

Case 2: $x_i(0) = x_j(0)$. Let us recall that in this step we assume that if $x_i(0) = x_j(0)$ then $v_i(0) \neq v_j(0)$. Therefore for our i and j we have $v_j(0) - v_i(0) =: v_{ji} \neq 0$ and by continuity of v^1 and v^2 there exist $\delta > 0$, such that

$$|v_i^m - v_j^m| \geq \delta, \quad m = 1, 2,$$

which implies that

$$|x_i^m(s) - x_j^m(s)| \geq \frac{1}{2} \delta s$$

on $[0, \delta]$ for all i, j and $m = 1, 2$. Thus, by mean value theorem

$$\begin{aligned}
& \left| \psi(|x_j^1 - x_i^1|) - \psi(|x_j^2 - x_i^2|) \right| \leq \\
& \leq C \left| (x_j^1 - x_i^1) - (x_j^2 - x_i^2) \right| \int_0^1 \left| \theta |x_j^1 - x_i^1| + (1 - \theta) |x_j^2 - x_i^2| \right|^{-1-\alpha} d\theta \\
& \leq C \left| (x_j^1 - x_i^1) - (x_j^2 - x_i^2) \right| \left| \frac{\delta}{2} t \right|^{-1-\alpha} \\
& \leq C(\delta) \frac{\left| (x_j^1 - x_i^1) - (x_j^2 - x_i^2) \right|}{t} |t|^{-\alpha}.
\end{aligned} \tag{4.11}$$

Moreover in either Case 1 or Case 2

$$\begin{aligned}
\left| (x_j^1(t) - x_i^1(t)) - (x_j^2(t) - x_i^2(t)) \right| & \leq t \sup_{s \in [0, t]} \left| (v_j^1(s) - v_i^1(s)) - (v_j^2(s) - v_i^2(s)) \right| \\
& \leq 2t \sup_{s \in [0, t]} \sqrt{r(s)} \leq 2t \sup_{s \in [0, t]} \sqrt{r_{inc}(s)} \\
& \leq 2t \sqrt{r_{inc}(t)}
\end{aligned} \tag{4.12}$$

and thus by combining inequalities (4.9), (4.10), (4.11) and (4.12) with Hölder's inequality one obtains

$$\frac{d}{dt} r_{inc} \leq C r_{inc} \cdot f,$$

where

$$f(t) := \max\{2L(\delta)t, 2C(\delta)|t|^{-\alpha}\},$$

which is an integrable function. Therefore Gronwall's lemma implies that the solution is unique at least on $[0, \delta]$ for a sufficiently small, positive δ .

Step 3. The purpose of this step is to prove uniqueness in case, when at least two particles are stuck together at the initial time, i.e. $x_{0,i} = x_{0,j}$ and $v_{0,i} = v_{0,j}$ for some $i, j = 1, \dots, N$. We present this step as a consequence of the following lemma, which proof, based on reasoning similar to step 2, can be found in Appendix A.

Lemma 4.1.4. *Suppose that at some time $t_0 \in [0, T]$ and some $i, j = 1, \dots, N$, we have $x_i(t_0) = x_j(t_0)$ and $v_i(t_0) = v_j(t_0)$. Then $x_i \equiv x_j$ on $[t_0, t_0 + \delta]$ for some positive δ .*

The above lemma in particular implies that on $[0, \delta]$ any particles that are stuck together can be treated as a single particle. From the point of view of uniqueness it means that we do not have to consider the case, when two or more particles are stuck together, since they cannot separate anyway. Thus if only the trajectory on which they move is unique then their respective trajectories are unique too (since in fact they are the same).

Step 4. In this step we finish the proof of uniqueness by putting together all the information obtained in previous steps. Suppose that we have two distinct solutions (x^1, v^1) and (x^2, v^2) originating in (x_0, v_0) . Then, regardless of the initial data, by all three previous steps, there exists an interval $[0, \delta]$ on which $x^1 \equiv x^2 =: x$. Without a loss of generality we may assume that for $t = \delta$ we have $x_i(t) \neq x_j(t)$ or $x_i \equiv x_j$ on $[0, \delta]$ for all $i, j = 1, \dots, N$. Therefore, by step 1 and step 3 we may prolong the interval

on which $x^1 \equiv x^2$. In fact we may prolong it as long as there is no collision between any particles. Let t_0 be the first time of collision. Then by step 1 and step 3, the uniqueness is ensured up to $t_0 - \epsilon$ for arbitrarily small $\epsilon > 0$. Now, by Corollary 4.0.1, (x, v) is continuous on whole $[0, T]$, thus it has a unique left sided limit at t_0 , which prolongs uniqueness up to t_0 . Finally we may treat t_0 as the new starting point and obtain uniqueness on $[t_0, t_1]$. Therefore the solution is unique between any two times of collision and the (possibly infinite) sum of such intervals include all $[0, T]$. \square

We end this section with the proof of uniqueness of piecewise–weak solutions.

Proof of Theorem 2.2.3. The proof is almost exactly the same as of Proposition 4.1.2. The first difference is that the function r from step 2 was absolutely continuous by the fact that the solutions were $W^{2,1}$ weak on $[0, T]$, while this time they are $W^{2,1}$ weak on each interval $[T_k, T_{k+1} - \epsilon]$ as stated in Corollary 4.0.1.1. This however is of no difference since we need r to be absolutely continuous only on $[0, \delta]$ for some small $\delta > 0$. The second difference is that this time we actually do not need Lemma 4.1.4 since by Definition 2.1.2 and in particular by the use of sets $B_i(t)$ (defined in (2.4)) we already ensured that the trajectories remain stuck together indefinitely. \square

Chapter 5

Cucker–Smale model with singular weight: kinetic equation

We aim to solve the issue of well-posedness for (2.1) with the singular weight (2.2) and initial data from the class of Radon measures i.e. we aim to prove Theorem 2.2.4. The goal is twofold:

– We prove existence and analyse continuous dependence on the initial data. The existence is obtained by approximating measure solutions to (2.1) by solutions to particle system (2.3) using the mean-field limit, similarly to [31]. The key obstacle is the lack of sufficient information about the continuous dependence with respect to perturbations of initial data for solutions to particle system (2.3), thus we are not allowed to apply standard approach. To our best knowledge the most that can be assumed is solvability of (2.3) in the $W^{1,p}$ -class that has been proved in Chapter 4 (or in [44]). Therefore by results from Chapter 4 we restrict our considerations to $\alpha \in (0, \frac{1}{2})$ and modify the mean-field limit procedure to that regularity.

– Concerning the second goal, we prove the property of weak-atomic uniqueness to system (2.1). It means that any weak solution is unique and corresponds to a solution to the particle system (2.3) provided it initiates from a finite sum of Dirac’s deltas $m_i \delta_{x_i(t)} \otimes \delta_{v_i(t)}$. Thus, any atomic solution is preserved by kinetic equation (2.1), and since it is generated by particle system (2.3), by Theorem 2.2.3 it is unique.

The part of the proof concerning the issue of existence follows from analysis of approximation by atomic solutions originating from sums of Dirac’s deltas, which correspond in the sense of Remark 2.1.6 to solutions of (2.3). The main idea behind this approach is twofold. Firstly, there is the very reason why this approach is successful and why only for $\alpha < \frac{1}{2}$, namely the better (and reasonable) regularity of solutions of (2.3) for $\alpha < \frac{1}{2}$. It was in some sense hinted in Chapter 4, where we proved that for $0 < \alpha < \frac{1}{2}$, system (2.3) admits a unique $W^{1,1}([0, T])$ solution (x, v) , which by Remark 2.1.6 corresponds to a solution of (2.1) in the sense of Definition 2.1.6. However, since in fact $\alpha \in (0, \alpha_0)$ for some $\alpha_0 < \frac{1}{2}$, we can push even further and prove that (x, v) is bounded in $W^{1,p}([0, T])$ for some $p > 1$. Such boundedness will provide us with equicontinuity of sequences of solutions of (2.3), which on the other hand will serve us to extract a convergent subsequence. The second idea behind the proof is to change the way we look at the alignment force term

$$\int_0^T \int_{\mathbb{R}^{2d}} F(f_n) f_n \nabla_v \phi dx dv dt, \quad (5.1)$$

where if $f_n \rightharpoonup f$ then it is not clear whether $F(f_n) f_n \rightharpoonup F(f) f$. It happens so, that it is useful to see

(5.1) as

$$\int_0^T \int_{\mathbb{R}^{4d}} \psi(|x-y|)(w-v) \nabla_v \phi d\mu_n dt$$

for $d\mu_n := f_n(t, x, v) \otimes f_n(t, y, w)$.

The uniqueness part of Theorem 2.2.4 is explained and proved in section 5.2.

We begin with an overview of the proof of existence. Suppose that f_0 is a given, compactly supported measure belonging to \mathcal{M}_+ and assume without a loss of generality that

$$\text{supp} f_0 \subset B(R_0), \quad (5.2)$$

$$\int_{\mathbb{R}^{2d}} f_0 dx dv = 1, \quad (5.3)$$

where $B(R)$ is a ball centred at 0 with radius R . For such f_0 we take $f_{0,\epsilon} \in \mathcal{M}_+$ of the form

$$f_{0,\epsilon} = \sum_{i=1}^N m_i \delta_{x_{0,i}^\epsilon} \otimes \delta_{v_{0,i}^\epsilon}, \quad (5.4)$$

which corresponds to the initial data $(x_{0,\epsilon}, v_{0,\epsilon})$ to particle system (2.3). Moreover we assume that

$$d(f_{0,\epsilon}, f_0) \xrightarrow{\epsilon \rightarrow 0} 0$$

and that the support of $f_{0,\epsilon}$ is contained in $B(2R_0)$. The existence of such approximation is standard (we refer for example to the beginning of Section 6.1 in [31] for the details). Now suppose that $(x_\epsilon^n, v_\epsilon^n)$ is a solution to (2.3) with the communication weight

$$\psi_n(s) := \min\{\psi(s), n\}, \quad (5.5)$$

subjected to the initial data $(x_{0,\epsilon}, v_{0,\epsilon})$, which by Remark 2.1.6 means that

$$f_\epsilon^n = \sum_{i=1}^N m_i \delta_{x_{\epsilon,i}^n} \otimes \delta_{v_{\epsilon,i}^n} \quad (5.6)$$

is a solution of (2.1) with the initial data $f_{0,\epsilon}$. Our goal is to converge with ϵ to 0 and with n to ∞ to obtain a solution f of equation (2.1) subjected to the initial data f_0 .

The proof can be summarized in the following steps:

Step 1. Given $T > 0$, for each ϵ and n , we prove existence of a solution f_ϵ^n corresponding to the initial data $f_{0,\epsilon}$ and satisfying various regularity properties.

Step 2. We take a sequence $f_n = f_\epsilon^n$ for $\epsilon = \frac{1}{n}$. Due to the conservation of mass and the regularity proved in step 1 we extract a subsequence f_{n_k} converging in $L^\infty(0, T; (\mathcal{M}_+, d))$ to some $f \in L^\infty(0, T; \mathcal{M}_+)$.

Step 3. We converge with each term in the weak formulation for f_{n_k} to the respective term in the weak formulation for f . This can be easily done for each term except the alignment force term i.e. the term

$$\int_0^T \int_{\mathbb{R}^{2d}} F_{n_k}(f_{n_k}) f_{n_k} \nabla_v \phi dx dv dt.$$

Step 4. In the case of the alignment force term we cannot simply converge. Instead, we replace it with an n_k -independently regular substitute of the form

$$\int_0^T \int_{\mathbb{R}^{2d}} F_m(f_{n_k}) f_{n_k} \nabla_v \phi dx dv dt.$$

We estimate the error between the alignment force term and its substitute proving that it can be controlled in terms of m and uniformly with respect to n_k .

Step 5. For such subsequence we converge with the substitute alignment force term to

$$\int_0^T \int_{\mathbb{R}^{2d}} F_m(f) f \nabla_v \phi dx dv dt.$$

Step 6. We are then left with converging with the substitute alignment force term to the original alignment force term i.e. with $m \rightarrow \infty$. We show that $F(f)$ is an L^1 function with respect to the measure f .

Step 7. We finish the proof by making sure that each and every point of Definition 2.1.6 is satisfied by our candidate for the solution.

Remark 5.0.2. As stated in (5.3) we assume that the total mass of the particles is equal to 1. This assumption is purely for notational simplicity's sake and will hold until the end of the chapter.

Let us state some various properties of the approximate solutions f_ϵ^n . It is in fact the first step of the proof (as presented above) but since it is self-contained, quite lengthy and very similar to the proof of Proposition 4.1.1 from Chapter 4 we will present it in a form of a separate proposition the proof of which can be found in Appendix A.

Proposition 5.0.3. *Given $T > 0$ let $f_{0,\epsilon}$ be of the form (2.10). Then for each $n = 1, 2, \dots$, there exists a unique solution f_ϵ^n to kinetic equation (2.1) that corresponds¹ to a smooth and classical solution (x^n, v^n) of particle system (2.3). Moreover there exists an n and ϵ independent constant $M > 0$ and constants $p, q > 1$, such that the following conditions are satisfied:*

- (i) *For all $t \in [0, T]$ and all n and ϵ the total mass of f_ϵ^n i.e. the value $\int_{\mathbb{R}^{2d}} f_\epsilon^n dx dv$ is equal to 1.*
- (ii) *The support of f_ϵ^n is contained in a ball $B(\mathcal{R})$, where $\mathcal{R} := 2R_0(T + 1)$.*
- (iii) *We have*

$$\int_0^T \sum_{i=1}^{N_n} m_{i,\epsilon}^n |\dot{v}_{i,\epsilon}^n|^p dt + \int_0^T \sum_{i,j=1}^{N_n} m_i m_j \psi_n^p(|x_{i,\epsilon}^n - x_{j,\epsilon}^n|) |v_{i,\epsilon}^n - v_{j,\epsilon}^n|^p dt \leq M(\mathcal{R}).$$

(iv) We have

$$\int_0^T \sum_{i,j=1}^{N_n} m_i m_j \psi_n^q(|x_{i,\epsilon}^n - x_{j,\epsilon}^n|) |v_{i,\epsilon}^n - v_{j,\epsilon}^n| dt \leq M(\mathcal{R}).$$

(v) For each Lipschitz continuous and bounded $g : \mathbb{R}^{2d} \rightarrow \mathbb{R}$, we have

$$\left\| \frac{d}{dt} \int_{\mathbb{R}^{2d}} g f_\epsilon^n dx dv \right\|_{L^p([0,T])} \leq M_g(\text{Lip}(g), \mathcal{R}).$$

Remark 5.0.3. Point (iii) of Proposition 5.0.3 implies in particular that the sequence $(x_\epsilon^n, v_\epsilon^n)$ is uniformly bounded in $W^{1,p}([0, T])$. We mention this to keep the continuity with the idea of the proof presented at the beginning of this section.

Remark 5.0.4. It is worthwhile to note that since by (iii) from Proposition 5.0.3 the derivative of velocity \dot{v} is uniformly integrable, then

$$|v_{i,\epsilon}^n(t) - v_{i,\epsilon}^n(0)| \leq \int_0^t |\dot{v}_{i,\epsilon}^n| ds \leq \omega(t) \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

Moreover the function ω is independent of i, n and ϵ . This remark will be recalled later on.

5.1 Proof of Theorem 2.2.4 (existence)

In this section we follow the steps presented in the previous section and finish the proof of the existence part of Theorem 2.2.4.

Step 1. Proposition 5.0.3 and Remark 2.1.6 ensure the existence of f_ϵ^n with properties (i)-(v) from Proposition 5.0.3. We solve particle system (2.3) with initial data (5.4) in the time interval $[0, T]$ under assumption that the communication weight is in form (3.1.1). By Proposition 5.0.3 we are ensured that

$$\begin{aligned} \|f_\epsilon^n\|_{L^\infty(0,T;\mathcal{M})} &= 1, \\ \|F_n(f_\epsilon^n) f_\epsilon^n\|_{L^p(0,T;\mathcal{M})} &\leq M(T). \end{aligned}$$

Step 2. We take $\epsilon = \frac{1}{n}$ and denote $f_n := f_{\frac{1}{n}}^n$. Since f_n is of the form (5.6) it is clear that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f_n dx dv = \sum_{i=1}^{N_n} m_{i,n} = 1.$$

For each n the function f_n may be treated as a mapping from $[0, T]$ into the metric space (\mathcal{M}_+, d) . For the purpose of showing that f_n has a convergent subsequence we will use Arzela-Ascoli theorem. We have to make sure that f_n is a bounded and equicontinuous sequence of functions with a

relatively compact pointwise sequences $f_n(t)$. Uniform boundedness of f_n is implied by the conservation of mass, while relative compactness of $f_n(t)$ follows from the uniform boundedness of $f_n(t)$ in TV topology and Corollary 2.1.1. Finally in order to prove equicontinuity of f_n we take arbitrary $s, t \in [0, T]$ and arbitrary Lipschitz continuous, bounded function g with $Lip(g) \leq 1$ and $\|g\|_\infty \leq 1$ and use estimation (v) from Proposition 5.0.3 to write

$$\left| \int_{\mathbb{R}^{2d}} g(f_n(s) - f_n(t)) dx dv \right| = \left| \int_t^s \frac{d}{dr} \int_{\mathbb{R}^{2d}} g f_n dx dv dr \right| =: \omega(|s - t|). \quad (5.7)$$

Point (v) of Proposition 5.0.3 states that functions $t \mapsto \frac{d}{dt} \int_{\mathbb{R}^{2d}} g f_n(t) dx dv$ are uniformly bounded in $L^p([0, T])$ for some $p > 1$, which in particular means that they are uniformly integrable. This on the other hand implies that the function ω is a good modulus of uniform continuity for the left-hand side of (5.7). Now since this estimation does not depend on the choice of g (only on the choice of $Lip(g)$), it is also valid for the supremum over all g , which implies that

$$d(f_n(s), f_n(t)) \leq \omega(|s - t|).$$

The above inequality proves that the sequence of functions $t \mapsto f_n(t)$ is equicontinuous as a mapping from $[0, T]$ to (\mathcal{M}, d) (recall the bounded-Lipschitz distance defined in (2.1.3)). Thus the sequence f_n satisfies the assumptions of Arzela-Ascoli theorem. Therefore there exists $f \in L^\infty(0, T; \mathcal{M}_+)$, such that

$$\|d(f_n, f)\|_\infty \rightarrow 0.$$

By (ii) from Proposition 5.0.3 it implies that the support of f is included in $B(\mathcal{R})$. Therefore with $\partial_t f \in L^p(0, T; (C^1(B(\mathcal{R})))^*)$.

Step 3. After a brief look at the weak formulation for f_n i.e. (2.6), we understand that since $f_n \rightarrow f$ in $L^\infty(0, T; (\mathcal{M}_+, d))$, then in particular for $\phi \in \mathcal{G}$, we have

$$\int_0^T \int_{\mathbb{R}^{2d}} f_n [\partial_t \phi + v \nabla \phi] dx dv dt \rightarrow \int_0^T \int_{\mathbb{R}^{2d}} f [\partial_t \phi + v \nabla \phi] dx dv dt$$

and

$$\int_{\mathbb{R}^{2d}} f_{0, \frac{1}{n}} \phi(\cdot, \cdot, 0) dx dv \rightarrow \int_{\mathbb{R}^{2d}} f_0 \phi(\cdot, \cdot, 0) dx dv$$

and the only problem is with the second term on the left-hand side of (2.6) i.e. the alignment force term

$$\int_0^T \int_{\mathbb{R}^{2d}} F_n(f_n) f_n \nabla_v \phi dx dv dt. \quad (5.8)$$

Step 4. To deal with the problem of convergence with the alignment force term we replace it in the following manner

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^{2d}} f_n [\partial_t \phi + v \nabla \phi] dx dv dt + \int_0^T \int_{\mathbb{R}^{2d}} F_m(f_n) f_n \nabla_v \phi dx dv dt \\ = - \int_{\mathbb{R}^{2d}} f_{0, \frac{1}{n}} \phi(\cdot, \cdot, 0) dx dv + \mathcal{J}, \end{aligned}$$

where

$$\mathcal{J} := \int_0^T \int_{\mathbb{R}^{2d}} (F_m(f_n) - F_n(f_n)) f_n \nabla_v \phi dx dv dt$$

for

$$F_m(f_n)(x, v, t) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi_m(|x - y|)(w - v) f_n(y, w, t) dy dw.$$

However, as mentioned at the beginning of this chapter, instead of looking at (5.8) as an integral of a product of $F_n(f_n)$ with f_n , we are going to see it as an integral of

$$g_n(x, y, w, v) := \psi_n(|x - y|)(w - v) \nabla_v \phi(t, x, v) \quad (5.9)$$

with respect to the measure

$$d\mu_n(t, x, y, w, v) := f_n(t, x, v) \otimes f_n(t, y, w).$$

By Fubini's theorem we have

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^{2d}} F_n(f_n) f_n \nabla_v \phi dx dv dt = \\ &= \int_0^T \int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} \psi_n(|x - y|)(w - v) f(t, w, y) dy dw \right) \nabla_v \phi(t, x, v) f(t, x, v) dx dv dt \\ &= \int_0^T \int_{\mathbb{R}^{4d}} g_n d\mu_n dt \end{aligned}$$

and a similar identity holds for $\int_0^T \int_{\mathbb{R}^{2d}} F_m(f_n) f_n \nabla_v \phi dx dv dt$. Therefore

$$\mathcal{J} = \int_0^T \int_{\mathbb{R}^{4d}} (g_m - g_n) d\mu_n dt.$$

Moreover we have

$$g_m - g_n = 0$$

in the set $\{(x, y, w, v) : |x - y| > \max\{m^{-\frac{1}{\alpha}}, n^{-\frac{1}{\alpha}}\}\}$, which provided that² $n > m$ implies that

$$|g_m - g_n| \leq |g_n| \chi_{\{(x, y, w, v) : |x - y| \leq m^{-\frac{1}{\alpha}}\}}. \quad (5.10)$$

Therefore for

$$\begin{aligned} A(m, n) &:= \left\{ t : \int_{B(m, n)} |w - v| d\mu_n > m^{-\frac{1}{2}} \right\}, \\ B(m, n) &:= \left\{ (x, y, w, v) : |x - y| \leq m^{-\frac{1}{\alpha}} \right\} \end{aligned}$$

²Which we may assume since we are going to converge with $n \rightarrow \infty$ for each fixed m .

we have

$$|\mathcal{I}| \leq C \left(\int_{A(m,n)} \int_{B(m,n)} |g_n| d\mu_n dt + \int_{(A(m,n))^c} \int_{B(m,n)} |g_n| d\mu_n dt \right) =: I + II.$$

Now if $|x - y| \leq m^{-\frac{1}{\alpha}}$ then $\psi_n(|x - y|) \geq \min\{m, n\} = m$ and for all $t \in A(m, n)$ we have

$$\begin{aligned} \mathcal{L}_n(t) &:= \int_{\mathbb{R}^{4d}} \psi_n(|x - y|) |w - v| d\mu_n \\ &\geq \int_{B(m,n)} \psi_n(|x - y|) |w - v| d\mu_n \\ &\geq m \cdot \int_{B(m,n)} |w - v| d\mu_n > m^{\frac{1}{2}}. \end{aligned}$$

Furthermore, integrating with respect to $d\mu_n$ reveals that

$$\mathcal{L}_n(t) = \sum_{i,j=1}^N \psi(|x_i^n(t) - x_j^n(t)|) |v_i^n(t) - v_j^n(t)|$$

which by Proposition 5.0.3,(iv) implies that the sequence \mathcal{L}_n is uniformly bounded in $L^p([0, T])$ for some $p > 1$ and thus – it is uniformly integrable which further implies that

$$I \leq C \|\nabla_v \phi\|_\infty \int_{\{t: \mathcal{L}_n(t) > m^{\frac{1}{2}}\}} \mathcal{L}_n(t) dt \leq C(m) \|\nabla_v \phi\|_\infty \xrightarrow{m \rightarrow \infty} 0, \quad (5.11)$$

since $|\mathcal{L}_n(t) > m^{\frac{1}{2}}| \leq \frac{\|\mathcal{L}_n\|_{L^1}}{m^{\frac{1}{2}}} \rightarrow 0$ as $m \rightarrow \infty$.

To estimate II we introduce the set $B_t(m, n)$ of those pairs (i, j) such that $|x_i^n(t) - x_j^n(t)| \leq m^{-\frac{1}{\alpha}}$. Then by Hölder's inequality with exponent $q = \frac{1}{\theta}$, for some arbitrarily small $\theta > 0$, we have

$$\begin{aligned} II &\leq \|\nabla_v \phi\|_\infty \int_{(A(m,n))^c} \sum_{i,j \in B_t(m,n)} m_{i,n} m_{j,n} \psi_n(|x_i^n - x_j^n|) |v_i^n - v_j^n| dt \\ &= \|\nabla_v \phi\|_\infty \int_{(A(m,n))^c} \sum_{i,j \in B_t(m,n)} (m_{i,n} m_{j,n})^{1-\theta} \psi_n(|x_i^n - x_j^n|) |v_i^n - v_j^n|^{1-\theta} \cdot (m_{i,n} m_{j,n})^\theta |v_i^n - v_j^n|^\theta dt \\ &\leq \|\nabla_v \phi\|_\infty \left(\int_{(A(m,n))^c} \sum_{i,j \in B_t(m,n)} m_{i,n} m_{j,n} \psi_n^{\frac{1}{1-\theta}}(|x_i^n - x_j^n|) |v_i^n - v_j^n| dt \right)^{1-\theta} \\ &\quad \cdot \left(\int_{(A(m,n))^c} \sum_{i,j \in B_t(m,n)} m_{i,n} m_{j,n} |v_i^n - v_j^n| dt \right)^\theta \\ &\leq \|\nabla_v \phi\|_\infty \left(\int_0^T \sum_{i,j=1}^{N_n} m_{i,n} m_{j,n} \psi_n^{\frac{1}{1-\theta}}(|x_i^n - x_j^n|) |v_i^n - v_j^n| dt \right)^{1-\theta} \cdot \left(\int_{(A(m,n))^c} \int_{B(m,n)} |w - v| d\mu_n \right)^\theta \\ &\leq \|\nabla_v \phi\|_\infty \left(\int_0^T \sum_{i,j=1}^{N_n} m_{i,n} m_{j,n} \psi_n^{\frac{1}{1-\theta}}(|x_i^n - x_j^n|) |v_i^n - v_j^n| dt \right)^{1-\theta} \cdot \left(T m^{-\frac{1}{2}} \right)^\theta. \end{aligned} \quad (5.12)$$

By Proposition 5.0.3,(iv) the first multiplicand on the right-hand side of (5.12) is uniformly bounded, which implies that

$$II \leq C \|\nabla_v \phi\|_\infty \left(T m^{-\frac{1}{2}} \right)^\theta \xrightarrow{m \rightarrow \infty} 0. \quad (5.13)$$

Estimations (5.11) and (5.13) imply that

$$|\mathcal{J}| \leq C(m) \|\nabla_v \phi\|_\infty$$

for some n -independent positive constant $C(m)$ such that $C(m) \rightarrow 0$ as $m \rightarrow \infty$.

Step 5. Our next goal is to ensure that the convergence

$$\int_0^T \int_{\mathbb{R}^{2d}} F_m(f_n) f_n \nabla_v \phi dx dv dt \rightarrow \int_0^T \int_{\mathbb{R}^{2d}} F_m(f) f \nabla_v \phi dx dv dt \quad (5.14)$$

holds for each m and each $\phi \in \mathcal{G}$. Let us fix $\phi \in \mathcal{G}$ and $m = 1, 2, \dots$. For g_m defined in (5.9), we have

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{R}^{2d}} F_m(f_n) f_n \nabla_v \phi dx dv dt - \int_0^T \int_{\mathbb{R}^{2d}} F_m(f) f \nabla_v \phi dx dv dt \right| = \left| \int_0^T \int_{\mathbb{R}^{4d}} g_m(d\mu_n - d\mu) dt \right| \\ & \leq \left| \int_0^T \int_{\mathbb{R}^{4d}} g_m[d(f_n \otimes f_n) - d(f_n \otimes f)] dt \right| + \left| \int_0^T \int_{\mathbb{R}^{4d}} g_m[d(f_n \otimes f) - d(f \otimes f)] dt \right| =: I + II. \end{aligned} \quad (5.15)$$

Furthermore, again by Fubini's theorem

$$I = \left| \int_0^T \int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} g_m(df_n - df) \right) df_n dt \right|$$

and since for each x, v the function $(y, w) \mapsto g_m(x, y, v, w)$ is Lipschitz continuous and bounded with $Lip(g_m) + \|g_m\|_\infty \leq C_1$ for some $C_1 = C_1(m, \|\nabla_v \phi\|_\infty, Lip(\nabla_v \phi))$ then by Lemma 2.1.1 we have

$$I \leq C_1 \int_0^T \int_{\mathbb{R}^{2d}} d(f_n, f) df_n \leq C_1 T \|d(f_n, f)\|_\infty \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Similarly also $II \rightarrow 0$ with $n \rightarrow \infty$. This concludes the proof of convergence (5.14).

Step 6. At this point after converging with n to infinity we are left with the weak formulation for f that reads as follows:

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^{2d}} f[\partial_t \phi + v \nabla \phi] dx dv dt + \int_0^T \int_{\mathbb{R}^{2d}} F_m(f) f \nabla_v \phi dx dv dt \\ & = - \int_{\mathbb{R}^{2d}} f_0 \phi(\cdot, \cdot, 0) dx dv + \mathcal{J}(m) \end{aligned}$$

for all $m = 1, 2, \dots$ and all $\phi \in \mathcal{G}$ with

$$\mathcal{J}(m) \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Therefore it suffices to show that

$$\int_0^T \int_{\mathbb{R}^{2d}} F_m(f) f \nabla_v \phi dx dv dt \rightarrow \int_0^T \int_{\mathbb{R}^{2d}} F(f) f \nabla_v \phi dx dv dt. \quad (5.16)$$

By Fubini's theorem for

$$\begin{aligned} d\mu &= (f \otimes f)(x, v, y, w, t), \\ g_m &= \psi_m(|x - y|)(w - v) \nabla_v \phi, \\ g &= \psi(|x - y|)(w - v) \nabla_v \phi, \end{aligned}$$

we have

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^{2d}} F_m(f) f \nabla_v \phi dx dv dt &= \int_0^T \int_{\mathbb{R}^{2d}} g_m d\mu dt, \\ \int_0^T \int_{\mathbb{R}^{2d}} F(f) f \nabla_v \phi dx dv dt &= \int_0^T \int_{\mathbb{R}^{2d}} g d\mu dt \end{aligned} \quad (5.17)$$

provided that the integral on the right-hand side of (5.17) is well defined. Therefore to show (5.16) it suffices to prove that

$$g_m \rightarrow g$$

in L^1 with respect to the measure $d\mu$. To prove this we first show that

$$g_m \rightarrow g$$

a.e. with respect to the measure μ . Clearly the convergence holds on

$$A := \{(x, v, y, w, t) : x \neq y\} \cup \{(x, v, y, w, t) : x = y, v = w\}$$

and it suffices to show that the set $A^c = \{(x, v, y, w, t) : x = y, v \neq w\}$ is of measure μ zero. We have $\psi_m \equiv m$ on A^c and thus

$$\begin{aligned} I_m &:= \int_0^T \int_{\mathbb{R}^{4d}} |g_m| d\mu dt = \int_0^T \int_{\mathbb{R}^{4d}} \psi_m(|x - y|) |w - v| |\nabla_v \phi| d\mu dt \\ &\geq \int_{A^c} \psi_m(|x - y|) |w - v| |\nabla_v \phi| d\mu dt = \int_{A^c} m |w - v| |\nabla_v \phi| d\mu dt = m \int_{A^c} |w - v| |\nabla_v \phi| d\mu dt. \end{aligned}$$

Thus either

$$I_m \rightarrow \infty \quad \text{or} \quad \int_{A^c} |w - v| |\nabla_v \phi| d\mu = 0. \quad (5.18)$$

The proofs of Step 4 and Step 5 remain true if we substitute g_m and g_n with $|g_m|$ and $|g_n|$ respectively³. Therefore also the respective convergences hold for $|g_m|$ and $|g_n|$, yielding

$$\left| \int_0^T \int_{\mathbb{R}^{4d}} |g_m| d\mu_n dt - \int_0^T \int_{\mathbb{R}^{4d}} |g_n| d\mu_n dt \right| \leq C(m) \|\nabla_v \phi\|_\infty \xrightarrow{m \rightarrow \infty} 0 \quad (5.19)$$

³Indeed, since $\| |g_m| - |g_n| \| \leq \|g_m - g_n\|$ we may replace in (5.10) g_m and g_n with $|g_m|$ and $|g_n|$ and proceed with the proof in the same way as in Step 4. On the other hand in Step 5, the convergence of I and II from (5.15) was a result of that $\|d(f_n, f)\|_\infty \rightarrow 0$ and that g_m is a Lipschitz continuous function, which remains true for $|g_m|$.

and

$$\left| \int_0^T \int_{\mathbb{R}^{4d}} |g_m| d\mu_n dt - \int_0^T \int_{\mathbb{R}^{4d}} |g_m| d\mu dt \right| \xrightarrow{n \rightarrow \infty} 0. \quad (5.20)$$

Moreover for each m and n , we have

$$\begin{aligned} I_m &\leq \left| \int_0^T \int_{\mathbb{R}^{4d}} |g_m| d\mu dt - \int_0^T \int_{\mathbb{R}^{4d}} |g_m| d\mu_n dt \right| \\ &\quad + \left| \int_0^T \int_{\mathbb{R}^{4d}} |g_m| d\mu_n dt - \int_0^T \int_{\mathbb{R}^{4d}} |g_n| d\mu_n dt \right| + \int_0^T \int_{\mathbb{R}^{4d}} |g_n| d\mu_n dt. \end{aligned}$$

Now, (5.20) implies that for each m we may choose n big enough, so that

$$\left| \int_0^T \int_{\mathbb{R}^{4d}} |g_m| d\mu dt - \int_0^T \int_{\mathbb{R}^{4d}} |g_m| d\mu_n dt \right| \leq 1.$$

Furthermore, by (5.19) for such n we have

$$\left| \int_0^T \int_{\mathbb{R}^{4d}} |g_m| d\mu_n dt - \int_0^T \int_{\mathbb{R}^{4d}} |g_n| d\mu_n dt \right| \leq |\mathcal{J}(m)|$$

and finally by estimation (iii) from Proposition 5.0.3

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^{4d}} |g_n| d\mu_n dt &\leq \|\nabla_v \phi\|_\infty \int_0^T \int_{\mathbb{R}^{4d}} \psi_n(|x-y|) |w-v| d\mu_n dt \\ &= \|\nabla_v \phi\|_\infty \int_0^T \sum_{i,j=1}^{N_n} m_i^n m_j^n \psi_n(|x_i^n - x_j^n|) |v_j^n - v_i^n| dt \leq M \end{aligned}$$

and thus

$$I_m \leq 1 + |\mathcal{J}(m)| + M \leq C_2 \quad (5.21)$$

for some positive constant C_2 . Therefore (5.18) and (5.21) imply that $\int_{A^c} |w-v| \|\nabla_v \phi\| d\mu = 0$ and since the function $|w-v|$ is positive on A^c , then by a standard density argument A^c is of measure μ zero and we have proved that

$$\begin{aligned} \psi_m(|x-y|) (w-v) \nabla_v \phi &\rightarrow \psi(|x-y|) (w-v) \nabla_v \phi, \\ \psi_m(|x-y|) |w-v| \|\nabla_v \phi\| &\rightarrow \psi(|x-y|) |w-v| \|\nabla_v \phi\| \end{aligned}$$

μ -a.e. Moreover by Fatou's lemma

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^{2d}} \psi(|x-y|) |w-v| \|\nabla_v \phi\| d\mu dt &\leq \liminf_{m \rightarrow \infty} \int_0^T \int_{\mathbb{R}^{2d}} \psi_m(|x-y|) |w-v| \|\nabla_v \phi\| d\mu dt \\ &= \liminf_{m \rightarrow \infty} I_m \leq C_2. \end{aligned} \quad (5.22)$$

Therefore the function $(x, y, v, w, t) \mapsto \psi(|x-y|) |w-v| \|\nabla_v \phi\|$ belongs to $L^1(\mu)$. This function is a proper dominating function for $\psi_m(|x-y|) (w-v) \nabla_v \phi$ and by dominated convergence we have (5.16) and the proof of step 6 is finished.

Step 7. Let us now wrap up the proof and compare Definition 2.1.6 with what we were able to prove about f . We took an arbitrary initial data $f_0 \in \mathcal{M}_+$ and proved existence of $f \in L^\infty(0, T; \mathcal{M}_+)$. Moreover in step 2 using estimates (ii) and (v) from Proposition 5.0.3 we proved that actually $\text{supp} f \subset B(\mathcal{R})$ and $\partial_t f \in L^p(0, T; (C^1(B(\mathcal{R})))^*)$ (point 1 of Definition 2.1.6). Point 2 of Definition 2.1.6 is an immediate consequence of (ii) from Proposition 5.0.3, while point 3 was the main focus of all the steps of the proof and it was finally proved in step 6. Point 4 of Definition 2.1.6 follows from (5.22) and Fubini's theorem. We are left with point 5 of Definition 2.1.6. Suppose that $B(R)$ and $B(r)$ are two concentric balls, such that (2.8) is satisfied. Then the construction of $f_{0,n}$ ensures that

$$\text{supp} f_{0,n} \cap B\left(R - \frac{1}{n}\right) \subset B\left(r + \frac{1}{n}\right)$$

and for sufficiently large n we have $r + \frac{1}{n} < r + \frac{R-r}{8} < R - \frac{R-r}{8}$. Translating it according to (5.6) we write that in the set \mathcal{I} of those i that $(x_{0,i}^n, v_{0,i}^n) \in B(R - \frac{R-r}{8})$ we actually have $(x_{0,i}^n, v_{0,i}^n) \in B(r + \frac{R-r}{8})$ and By (ii) and (iii) from Proposition 5.0.3 (and in particular by Remark 5.0.4), for each $i \in \mathcal{I}$ and for each sufficiently big n , we have the n independent bounds:

$$\begin{aligned} |x_i^n(t)| &\leq |x_{0,i}^n| + t\mathcal{R} \xrightarrow{t \rightarrow 0} |x_{0,i}^n|, \\ |v_i^n(t)| &\leq |v_{0,i}^n| + \omega(t) \xrightarrow{t \rightarrow 0} |v_{0,i}^n|. \end{aligned}$$

The above bounds, for sufficiently small t imply that $(x_i^n(t), v_i^n(t)) \in B(r + \frac{R-r}{6})$ as long as $i \in \mathcal{I}$. Similarly for $i \notin \mathcal{I}$ in a sufficiently small neighbourhood of $t = 0$, we have $(x_i^n(t), v_i^n(t)) \notin B(R - \frac{R-r}{6})$. Therefore

$$\text{supp} f_n(t) \cap B\left(R - \frac{R-r}{6}\right) \subset B\left(r + \frac{R-r}{6}\right)$$

for sufficiently large n and sufficiently small t . Thus we may pass to the limit with $n \rightarrow \infty$ to obtain (2.9). This finishes the proof of the existence part of Theorem 2.2.4.

5.2 Proof of Theorem 2.2.4 (weak-atomic uniqueness)

In what follows we aim to prove that if initial configuration f_0 is an atomic measure, i.e. it satisfies (2.10), then solution f in the sense of Definition 2.1.6 is of the form (2.11), and it is unique. We will base the proof on a very careful analysis of the local propagation of the support of f that comes from point 5 of Definition 2.1.6. What, we basically need, is that any amount of the mass f that is separated from the rest of the mass remains separated at least for some time. It is required to refine this property by adding a control over the shape in which the support in the x and v coordinates propagates. The difficulty comes from the fact that in the case of the particle system the position x_i of i th particle changes with its own unique velocity v_i . However in the case of the kinetic equation characteristics are not well defined.

Step 1. By point 1 in Definition 2.1.6 it is sufficient to prove the proposition only in an arbitrarily small neighbourhood of $t = 0$. Let f_0 be of the form (2.10). Our first task is to restrict f_0 to small balls with one particle (say i th particle). Then we will use the local propagation of the support to prove that the mass that initially formed the i th particles remains atomic in some right-sided neighbourhood of

$t = 0$. Since

$$f_0 = \sum_{i=1}^N m_i \delta_{x_{0,i}} \otimes \delta_{v_{0,i}} \quad (5.23)$$

for number of atoms N , we have a finite number of initial positions and velocities of the particles $(x_{0,i}, v_{0,i})$ for $i = 1, \dots, N$, which implies that there exists $R_1 > 0$ such that for all $r_0 < R_1$, we have

$$f_0|_{B_i(r_0)} = m_i \delta_{x_{0,i}} \otimes \delta_{v_{0,i}} \quad (5.24)$$

for $B_i(R) := B_{x,v}((x_{0,i}, v_{0,i}), r_0)$.

At this point let us concentrate on one atom, we fix i . We aim at showing that there exists T^* such that

$$f^D := f(t)|_{B_i(\frac{r_0}{4})} = m_i \delta_{x_i(t)} \otimes \delta_{v_i(t)} \quad (5.25)$$

in $[0, T^*]$ for some \mathbb{R}^d valued functions x_i and v_i . We emphasize that r_0 and $T^*(r_0)$ can be chosen to be arbitrarily small. Identity (5.24) implies that for any $0 < r < r_0$, we have

$$\text{supp} f_0 \cap B_i(r_0) \subset B_i(r)$$

which by point 5 of Definition 2.1.6 ensures that there exists T^* such that

$$\text{dist}\{\text{supp} f^D(t), \text{supp} f^C(t)\} > \frac{r_0}{8} \quad (5.26)$$

for all $t \in [0, T^*]$, where $f^C(t) := f(t) - f^D(t)$. Then one can find a smooth function $\eta : \mathbb{R}^{2d} \times [0, T_*] \rightarrow [0, 1]$ such that $\eta \equiv 1$ over the support of f^D and $\eta \equiv 0$ over the support of f^C . We have then $f^D \eta = f^D$. All these properties allow us to state the following equation satisfied by f^D on $[0, T^*]$:

$$\partial_t f^D + v \cdot \nabla_x f^D + \text{div}_v [(F(f^C) + F(f^D)) f^D] = 0. \quad (5.27)$$

This equation is satisfied in the same sense that (2.6) from Definition 2.1.6. To prove that f^D is indeed of form (5.25) we introduce

$$\begin{cases} \frac{d}{dt} x_a(t) &= v_a(t) \\ \frac{d}{dt} v_a(t) &= \int_{\mathbb{R}^{2d}} \psi(|x_a(t) - y|)(w - v_a(t)) f^C dy dw \end{cases} \quad (5.28)$$

with the initial data $(x_a(0), v_a(0)) = (x_{0,i}, v_{0,i})$. Condition (5.26) ensures that the right-hand side of (5.28)₂ is smooth and thus (5.28) has exactly one smooth solution in $[0, T^*]$. Our goal is to show that f^D is supported on the curve $(x_a(t), v_a(t))$ and that in fact (5.25) holds with $(x_i(t), v_i(t)) \equiv (x_a, v_a)$. Since this feature will hold for all atoms, the whole f will be then atomic.

Step 2. In the next step we characterize possible evolution of the support of the weak solution to (5.27).

Lemma 5.2.1. *Let f be a weak solution to (2.1) in the sense of Definition 2.1.6. Assume further f has the structure of $f = f^D + f^C$ and fulfils the weak formulation of (5.27), and*

$$\text{supp} f_0^D = (x_0, v_0)$$

for some given (x_0, v_0) . Then for any $R > 0$ there exists T^* , such that

$$\text{supp} f^D(t) \subset (x_0, v_0) + (tB_x(v_0, \epsilon)) \times B_v(0, R)$$

for all $t \in [0, T^*]$, with $\epsilon := \sqrt{2R(R + |v_0|)}$, which can be arbitrarily small depending on smallness of R .

To prove Lemma 5.2.1 it is required to show the following result.

Lemma 5.2.2. *Let f^D be a weak solution to (5.27) in the sense of Definition 2.1.6. Assume further that there exists T^* , such that*

$$\text{supp} f^D(t) \subset B((x_0, v_0), R) \tag{5.29}$$

for some given (x_0, v_0) and $R > 0$ and all $t \in [0, T^*]$. Then

$$\text{supp} f^D(t) \subset \text{supp} f_0^D + \bigcup_{s \in (0, t)} (sB_x(v_0, R)) \times B_v(0, R). \tag{5.30}$$

It means that the support in the x -coordinates propagates in a cone defined by the ball $B_x(v_0, R)$ in direction v_0 .

Proof of Lemma 5.2.2. Without a loss of generality we assume that $(x_0, v_0) = (0, 0)$. The boundedness of the support in the v -coordinates is trivial and thus we focus on the support in the x -coordinates. Suppose that $x_1 \in \mathbb{R}^d$ and $\rho > 0$ are such that

$$\text{supp} f_0^D \cap B_x(x_1, \rho) \times \mathbb{R}^d = \emptyset$$

and let

$$\phi(x, t) := ((\rho - Rt)^2 - |x - x_1|^2)_+.$$

Hence

$$\text{supp} \phi(\cdot, t) = \{|x - x_1| \leq |\rho - Rt|\}. \tag{5.31}$$

We test (5.27) by ϕ^2 and integrate over the time interval $[0, T^*]$, obtaining

$$\int_{\mathbb{R}^{2d}} f^D(T^*) \phi(T^*)^2 dx dv + 4 \int_0^{T^*} \int_{\mathbb{R}^{2d}} f^D \phi [(\rho - Rt)R - (x - x_1)v] dx dv dt = \int_{\mathbb{R}^{2d}} f_0^d \phi(0)^2 dx dv = 0.$$

Since the first term on the left-hand side of the above equality is nonnegative, we have

$$\int_0^{T^*} \int_{\mathbb{R}^{2d}} f^D \phi [(\rho - Rt)R - (x - x_1)v] dx dv dt \leq 0.$$

But for the interior of support of ϕ , we have $\rho - Rt > |x - x_1|$ and by (5.29) $R > |v|$. It implies

$$0 < (\rho - Rt)R - (x - x_1)v, \quad \text{hence } f\phi \equiv 0.$$

This way we proved that in the complement of the support in x of $f(t)$ lay all the balls centred outside of $\text{supp} f_0$ and with a radius equals to $\rho - Rt$, which implies (5.30). \square

Proof of Lemma 5.2.1. We base on Lemma 5.2.2. First we establish proper R and T^* . Since f_0^D is concentrated in one point (x_0, v_0) then for arbitrarily small ρ

$$\text{supp } f_0 \subset B((x_0, v_0), \rho).$$

Now, Definition 2.1.6 point 5 ensures that there exist $R(\rho)$ and $T^*(\rho)$ such that

$$\text{supp } f^D(t) \subset B((x_0, v_0), R)$$

in $[0, T^*]$ and R can be chosen arbitrarily small (then also T^* is small but still positive). We fix such R and T^* and note that we may apply Lemma 5.2.2 on $[0, T^*]$. Without a loss of generality we assume that $x_0 = 0$ and test (5.27) with the function ϕ^2 , where

$$\phi(x, t) := ((x - v_0 t)^2 - (t\epsilon)^2)_+$$

and

$$\text{supp } \phi(\cdot, t) = \{x \in \mathbb{R}^d : |x - v_0 t| \geq t\epsilon\}. \quad (5.32)$$

We have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^{2d}} f^D(t) \phi^2(t) dx dv - 4 \int_0^t \int_{\mathbb{R}^{2d}} f^D \phi[-v_0(x - v_0 t) - t\epsilon^2 + v(x - v_0 t)] dx dv dt \\ &\geq 4 \int_0^t \int_{\mathbb{R}^{2d}} f^D \phi[t\epsilon^2 - (v - v_0)(x - v_0 t)] dx dv dt. \end{aligned} \quad (5.33)$$

On the support of f^D , we have $|v - v_0| \leq R$ and by Lemma 5.2.2 it holds

$$|x - v_0 t| \leq |x - \underbrace{x_0}_{=0}| + |v_0 t| \leq t(|v_0| + R) + t|v_0| \leq t(2|v_0| + R).$$

Hence, in view of definition of ϵ , we conclude

$$(v - v_0)(x - v_0 t) \leq (2|v_0| + R)R t < t\epsilon^2.$$

Therefore the integrand on the right-hand side of (5.33) is nonnegative, which means that it has to be equal to 0, which further implies that

$$f^D \phi \equiv 0 \quad \text{in } [0, T^*].$$

By the definition of ϕ it follows that $f^D(t)$ vanishes outside of the cone balls $tB_x(v_0, \epsilon) \times \mathbb{R}^d$. The lemma is proved. \square

Step 3. In this part we show that f initiated by a state of (5.23) stays indeed an atomic solutions for all times.

Proposition 5.2.1. *Let f be a solution to (5.27) in the sense of Definition 2.1.6. Then if f_0 is of the form (2.10) then f is an atomic solution (of the form (2.11)) and it is unique.*

Proof. We show separately for each of atoms that each initial particle generates a mono-atomic solution (at least locally in time). Finiteness of number of atoms allows to conclude that the whole solutions will be atomic. Hence we study (5.27) with a mono-atomic initial data located in (x_0, v_0) .

We test (5.27) by $(v - v_a(t))^2$ getting

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^{2d}} f^D (v - v_a(t))^2 dx dv = -2 \int_{\mathbb{R}^{2d}} f^D (v - v_a(t)) \dot{v}_a(t) dx dv \\ & + 2 \int_{\mathbb{R}^{2d}} F(f^C) f^D (v - v_a(t)) dx dv + 2 \int_{\mathbb{R}^{2d}} F(f^D) f^D (v - v_a(t)) dx dv = -2I + 2II + 2III. \end{aligned} \quad (5.34)$$

First we deal with *III*. By symmetry of $f^D \otimes f^D$ with respect to (x, v) and (y, w) , we have

$$\begin{aligned} III &= \int_{\mathbb{R}^{4d}} \psi(|x - y|) (w - v) f^D f^D (v - v_a(t)) dx dv dy dw \\ &= \int_{\mathbb{R}^{4d}} \psi(|x - y|) (v - w) f^D f^D (w - v_a(t)) dx dv dy dw \\ &= \frac{1}{2} \int_{\mathbb{R}^{4d}} \psi(|x - y|) (w - v) f^D f^D (v - w) dx dv dy dw \\ &= -\frac{1}{2} \int_{\mathbb{R}^{4d}} \psi(|x - y|) (w - v)^2 f^D f^D dx dv dy dw \leq 0. \end{aligned}$$

Next let us take a closer look at *II*. By the definition of $F(f^C)$

$$\begin{aligned} II &= \int_{\mathbb{R}^{4d}} \psi(|x - y|) (w - v) f^D f^C (v - v_a(t)) dx dv dy dw \\ &= \int_{\mathbb{R}^{4d}} \psi(|x - y|) (w - v_a(t) + v_a(t) - v) f^D f^C (v - v_a(t)) dx dv dy dw \\ &= \int_{\mathbb{R}^{4d}} \psi(|x - y|) (w - v_a(t)) f^D f^C (v - v_a(t)) dx dv dy dw \\ &\quad - \underbrace{\int_{\mathbb{R}^{4d}} \psi(|x - y|) f^D f^C (v - v_a(t))^2 dx dv dy dw}_{\leq 0} \\ &\leq \int_{\mathbb{R}^{4d}} \psi(|x - y|) (w - v_a(t)) f^D f^C (v - v_a(t)) dx dv dy dw =: II_2. \end{aligned}$$

Now we compare II_2 with *I*:

$$\begin{aligned} |II_2 - I| &= \left| \int_{\mathbb{R}^{4d}} (\psi(|x_a(t) - y|) - \psi(|x - y|)) (w - v_a(t)) f^D f^C (v - v_a(t)) dx dv dy dw \right| \\ &\leq \int_{\mathbb{R}^{4d}} |\psi(|x_a(t) - y|) - \psi(|x - y|)| |w - v_a(t)| f^D f^C |v - v_a(t)| dx dv dy dw. \end{aligned} \quad (5.35)$$

The main problem with estimating the right-hand side of the above inequality lays in the estimation of

$$|\psi(|x_a(t) - y|) - \psi(|x - y|)|.$$

This is the place where the separation of supports explained by Lemma 5.2.1 comes into play. Both $(x_a(t), v_a(t))$ and (x, v) are in the support of f^D , while (y, w) is in the support of f^C . Thus (5.26) implies that either

$$|x - y| > \frac{r_0}{8} \quad \text{and} \quad |x_a(t) - y| > \frac{r_0}{8} \quad (5.36)$$

or

$$|v - w| > \frac{r_0}{8} \quad \text{and} \quad |v_a(t) - w| > \frac{r_0}{8}. \quad (5.37)$$

We handle above two cases separately.

In case (5.36) it is clear that

$$|\psi(|x_a(t) - y|) - \psi(|x - y|)| \leq L|x - x_a(t)| = Lt^{\frac{1}{2}} \frac{|x - x_a(t)|}{t^{\frac{1}{2}}}. \quad (5.38)$$

for some constant $L = L(r_0) > 0$, since ψ is smooth outside of any neighbourhood of 0.

In case of (5.37) we are actually in a situation when at $t = 0$ multiple particles are situated in the same spot with different velocities i.e. f^C is divided into two parts f^{C_1} and f^{C_2} . The first part submits to the same bounds as (5.36) while for the second, f^{C_2} , we have

$$f^{C_2}(0) = \sum_j m_j \delta_{x_{0,i}} \otimes \delta_{v_{0,j}} =: \sum_j f_j^{C_2}(0).$$

Thus, initially f^{C_2} is concentrated in the same position as f^D but with different velocities. In this case we apply Lemma 5.2.1 multiple times (once for f^D and multiple times for each $f_j^{C_2}$). Even though Lemma 5.2.1 is written for solutions of (2.6) we may still apply it for f^D and each of $f_j^{C_2}$, since the proof does not involve directly the dependence on v . Therefore, by Lemma 5.2.1, we have

$$\text{supp} f^D(t) \subset (x_{0,i}, v_{0,i}) + tB_x(v_{0,i}, \epsilon)$$

and

$$\text{supp} f_j^{C_2}(t) \subset (x_{0,i}, v_{0,j}) + tB_x(v_{0,j}, \epsilon).$$

At this point we fix $R > 0$ and T^* from Lemma 5.2.1, so that ϵ is small enough that

$$B_x(v_{0,i}, \epsilon) \cap B_x(v_{0,j}, \epsilon) = \emptyset \quad \text{for } i \neq j.$$

Moreover

$$\text{dist}(B_x(v_{0,i}, \epsilon), B_x(v_{0,j}, \epsilon)) > C(r_0) > 0.$$

Again, we used that the number of all atoms is finite. If so, then also

$$|x - y| > tC(R) \quad \text{and} \quad |x_a(t) - y| > tC(R)$$

for $x \in \text{supp} f^D$ and $y \in \text{supp} f^{C_2}$. Therefore in such case ($\psi(|s|) = s^{-\alpha}$ and $\psi'(|s|) \sim s^{-1-\alpha}$)

$$|\psi(|x_a(t) - y|) - \psi(|x - y|)| \leq C(R)t^{-1-\alpha}|x - x_a(t)| = C(R)t^{-\frac{1}{2}-\alpha} \frac{|x - x_a(t)|}{t^{\frac{1}{2}}}. \quad (5.39)$$

We combine inequalities (5.35), (5.38)⁴ and (5.39) with the global bounds on the support of f obtaining

$$|II_2 - I| \leq A(t) \int_{\mathbb{R}^{2d}} t^{-\frac{1}{2}} |x - x_a(t)| |v - v_a(t)| f^D dx dv$$

for $A := Lt^{\frac{1}{2}} + C(R)t^{-\frac{1}{2}-\alpha}$, which thanks to the fact that $\alpha < \frac{1}{2}$ is integrable with respect to t over $[0, T^*]$. Taking into the account our estimations of I , II and III we come back to (5.34) and claim that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^{2d}} f^D |v - v_a(t)|^2 dx dv &\leq A(t) \int_{\mathbb{R}^{2d}} t^{-\frac{1}{2}} |x - x_a(t)| |v - v_a(t)| f^D dx dv \\ &\leq A(t) \left(\int_{\mathbb{R}^{2d}} f^D t^{-1} |x - x_a(t)|^2 dx dv + \int_{\mathbb{R}^{2d}} f^D |v - v_a(t)|^2 dx dv \right). \end{aligned} \quad (5.40)$$

To finish the proof there is a need to estimate the second integrand on the right-hand side of (5.40). We test⁵ (5.27) with $|x - x_a(t)|^2 t^{-1}$ getting

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^{2d}} t^{-1} f^D |x - x_a(t)|^2 dx dv &+ \int_{\mathbb{R}^{2d}} t^{-2} f^D |x - x_a(t)|^2 dx dv \\ &+ 2 \int_{\mathbb{R}^{2d}} t^{-1} f^D (x - x_a(t)) \dot{x}_a(t) dx dv - 2 \int_{\mathbb{R}^{2d}} t^{-1} f^D (x - x_a(t)) v dx dv. \end{aligned}$$

and apply Young's inequality with $\delta > 0$ to obtain

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^{2d}} t^{-1} f^D |x - x_a(t)|^2 dx dv &+ \int_{\mathbb{R}^{2d}} t^{-2} f^D |x - x_a(t)|^2 dx dv \\ &\leq 2 \int_{\mathbb{R}^{2d}} t^{-1} f^D |x - x_a(t)| |v - v_a(t)| dx dv \\ &\leq \delta \int_{\mathbb{R}^{2d}} t^{-2} f^D |x - x_a(t)|^2 dx dv + C \int_{\mathbb{R}^{2d}} f^D |v - v_a(t)|^2 dx dv. \end{aligned} \quad (5.41)$$

Finally we fix a suitable $\delta > 0$ and combine inequalities (5.40) and (5.41), which leaves with

$$\begin{aligned} \frac{d}{dt} \left(\int_{\mathbb{R}^{2d}} (t^{-1} f^D |x - x_a(t)| + f^D |v - v_a(t)|^2) dx dv \right) &+ \frac{1}{2} \int_{\mathbb{R}^{2d}} t^{-2} f^D |x - x_a(t)|^2 dx dv \\ &\leq A(t) \int_{\mathbb{R}^{2d}} (t^{-1} f^D |x - x_a(t)|^2 + f^D |v - v_a(t)|^2) dx dv, \end{aligned}$$

which by Gronwall's lemma and the fact that $A(t) \sim t^{-1/2-\alpha}$ is integrable in a neighbourhood of $t = 0$ imply

$$\int_{\mathbb{R}^{2d}} (t^{-1} f^D |x - x_a(t)|^2 + f^D |v - v_a(t)|^2) dx dv \equiv 0 \text{ on } [0, T^*].$$

Thus on $[0, T^*]$ we have $x \equiv x_a$ and $v \equiv v_a$ on the support of f , which is exactly equivalent to (5.25).

We have proved f^D is mono-atomic. Then repeating the procedure for all atoms (the number is finite) we conclude that f is atomic on a time interval $[0, T^*]$ with possibly smaller, but positive $T^* > 0$. As a conclusion, since the solution exists globally it must be atomic all the time. \square

⁴Here is the entire estimation in case (5.36) and the estimation of f^{C_1} in case (5.37).

⁵Even though $|x - x_a(t)|^2 t^{-1}$ is not a good test function for (5.27), we can approximate the singularity at $t = 0$ by modification $(t + l)^{-1}$ and then let $l \rightarrow 0$.

Part II

Flocking particles in a non-Newtonian fluid

Chapter 6

Part II: Introduction

The secondary goal of the dissertation is to obtain existence and uniqueness of solutions to the Cucker–Smale flocking model coupled with equations of motion of non-Newtonian fluids. The notation is slightly different than in Part I. We consider the system

$$\begin{cases} \partial_t f + v \nabla f + \operatorname{div}_v [(F_a(f) + F_d)f] = 0, & x \in \mathbb{T}^d, \quad v \in \mathbb{R}^d, \\ \partial_t u + (u \cdot \nabla)u + \nabla \pi - \operatorname{div}(\tau(Du)) = -d \int_{\mathbb{R}^d} F_d f dv, & x \in \mathbb{T}^d, \\ \operatorname{div} u = 0. \end{cases} \quad (6.1)$$

In (6.1) function $f = f(t, x, v)$ is the density of those particles that at the time $t \in [0, T]$ have position x and velocity v . The alignment force term F_a ¹ is given by

$$F_a(f)(t, x, v) := \int_{\mathbb{T}^d \times \mathbb{R}^d} \psi(|x - y|)(w - v)f(t, y, w)dw dy,$$

where $\psi \in C^1([0, \infty))$ is a given positive, nonincreasing function, called *the communication weight*. This force is responsible for the emergent phenomena that in our case takes the form of flocking of the particles, which we understand as the alignment of particles' velocities. A good example of ψ is ψ_{cs} from (1.5). The drag force F_d is given by

$$F_d(t, x, v) := u(t, x) - v$$

and throughout Part II we will use the notation

$$F(f) := F_a(f) + F_d.$$

The function $u = u(t, x) = (u_1(t, x), u_2(t, x), \dots, u_d(t, x))$ represents velocity of the fluid at the position x and time t . The term τ in (6.1)₂ denotes a symmetric stress tensor that depends on Du – the symmetric part of the gradient of u i.e. $\tau = \tau(Du)$, where $Du = \frac{1}{2}[\nabla u + (\nabla u)^T]$. In our considerations we assume that τ is derived from some scalar potential ϑ and through some specified later properties of ϑ we actually impose various other assumptions on τ including p -growth (where $p > 1$ is given) or coercivity.

The general strategy of the proof is taken from [7]. First we regularise the system, then to obtain existence for the regularised system we introduce an inductive scheme solving C–S and fluid parts of

¹Note that in Part I it was denoted by F .

(6.1) alternating with every step. Thus we solve (6.1)₁ and put it's solution into (6.1)₂ as a given function then solve again obtaining a solution which we again put into (6.1)₁, solve and so on. Then the convergence of the approximations is obtained through a careful technical estimation and analysis.

This strategy is followed in Chapter 7 but before we begin let us introduce the basic notation for Part II and present the main result.

6.1 Part II: Preliminaries and notation

Hereinafter $W^{k,p}(\Omega)$ denotes the Sobolev space of the functions with up to k th weak derivative belonging to the space $L^p(\Omega)$. Moreover \mathcal{D}' denotes the space of distributions and $C^k(\Omega)$ – the space of the functions with up to k th derivative belonging to the space of continuous functions, which itself will be denoted as $C(\Omega)$. Next we present the definition and assumptions on the stress tensor τ , which is a crucial part of the definition of the problem. To establish the set of admissible τ we assume that there exists a scalar potential of τ that we denote $\vartheta \in C^2(\mathbb{R}^{d^2})$, such that for some $p \in (1, \infty)$, $c_1, c_2 > 0$ we have for all $\eta, \xi \in \mathbb{R}_{sym}^{d^2}$ and $i, j, k, l = 1, \dots, d$:

$$\frac{\partial \vartheta(\eta)}{\partial \eta_{ij}} = \tau_{ij}(\eta), \quad (6.2)$$

$$\vartheta(0) = \frac{\partial \vartheta(0)}{\partial \eta_{ij}} = 0, \quad (6.3)$$

$$\frac{\partial^2 \vartheta(\eta)}{\partial \eta_{ij} \partial \eta_{kl}} \xi_{ij} \xi_{kl} \geq c_1 (1 + |\eta|)^{p-2} |\xi|^2, \quad (6.4)$$

$$\left| \frac{\partial \vartheta(\eta)}{\partial \eta_{ij} \partial \eta_{kl}} \right| \leq c_2 (1 + |\eta|)^{p-2}. \quad (6.5)$$

The above assumptions on ϑ impose various properties of τ , that we include in the following lemma.

Lemma 6.1.1. *Let $p \geq 2$ and $\tau : \mathbb{R}_{sym}^{d^2} \rightarrow \mathbb{R}_{sym}^{d^2}$, $\vartheta : \mathbb{R}_{sym}^{d^2} \rightarrow \mathbb{R}$ satisfy (6.2)-(6.5). Then there exist positive constants c_3, c_4, c_5 , such that for all $\xi, \eta \in \mathbb{R}_{sym}^{d^2}$ we have*

$$\tau_{ij}(\xi) \xi_{ij} \geq c_3 (|\xi|^p + |\xi|^2), \quad (6.6)$$

$$|\tau_{ij}(\xi)| \leq c_4 (1 + |\xi|)^{p-1}, \quad (6.7)$$

$$(\tau_{ij}(\xi) - \tau_{ij}(\eta))(\xi - \eta) \geq c_5 (|\xi - \eta|^2 + |\xi - \eta|^p). \quad (6.8)$$

Proof. The proof of the above lemma can be found in [38] page 195. □

Regarding the Cucker-Smale part of the system we assume that ψ is nonnegative, nonincreasing and smooth, with

$$\|\psi\|_{C^1} \leq c_6.$$

Hereinafter we shall denote for nonnegative and integrable functions f :

$$M_\alpha f(t) := \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^\alpha f(t, x, v) dx dv,$$

$$m_\alpha f(t, x) := \int_{\mathbb{R}^d} |v|^\alpha f(t, x, v) dv,$$

with an obvious remark that $M_0 f = \|f\|_{L^1}$ and that for $1 \leq q \leq \infty$,

$$m_\alpha f(t, x) \leq C(R) \|f(t, x, \cdot)\|_q, \quad (6.9)$$

provided that $\text{supp} f(t, x, \cdot) \subset B(R)$, where $B(R)$ is a ball centred at 0 with radius R . Moreover, occasionally we will split $F_a(f)$ into two parts in the following manner:

$$F_a(f)(t, x, v) = a(t, x) - b(t, x)v,$$

where

$$a(t, x) := \int_{\mathbb{T}^3 \times \mathbb{R}^3} \psi(|x-y|) w f(t, y, w) dy dw, \quad \text{with } \|a\|_\infty + \|\nabla a\|_\infty \leq c_6 M_1 f, \quad (6.10)$$

$$b(t, x) := \int_{\mathbb{T}^3 \times \mathbb{R}^3} \psi(|x-y|) f(t, y, w) dy dw, \quad \text{with } \|b\|_\infty + \|\nabla b\|_\infty \leq c_6 M_0 f. \quad (6.11)$$

Thus, we clearly have

$$|F_a(f)(t, x, v)| \leq \|a\|_\infty + |v| \|b\|_\infty \leq c_6 (M_1 f + |v| M_0 f), \quad (6.12)$$

and

$$\text{div}_v F_a(f)(t, x, v) = -db(t, x).$$

Throughout Part II we use the arbitrary 'harmless' constant C that may change it's actual value depending on it's appearances even in the same line. We will also write

$$A \stackrel{H(q)}{\leq} B$$

to emphasise that the estimation $A \leq B$ follows by Hölder's inequality with exponent q . We will also use a similar notation for Young's inequality replacing H with Y .

6.1.1 Weak formulation

We assume that the physical dimension $d = 3$. Let us introduce the basic notation regarding the function spaces.

$$\begin{aligned} L^2_{div,0}(\mathbb{T}^3) &:= \{\omega \in L^2(\mathbb{T}^3) : \text{div} \omega = 0\}, \\ \dot{W}^{1,p}_{div,0}(\mathbb{T}^3) &:= \{\omega \in \mathcal{D}'(\mathbb{T}^3) : \nabla \phi \in L^p(\mathbb{T}^3), \text{div} \omega = 0\}, \\ \mathcal{H} &:= L^\infty(0, T; \dot{W}^{1,p}_{div,0}(\mathbb{T}^3)) \cap C(0, T; L^2_{div,0}(\mathbb{T}^3)) \cap L^2(0, T; W^{2,2}(\mathbb{T}^3)) \cap L^\infty(0, T; W^{1,2}(\mathbb{T}^3)) \\ \mathcal{X} &:= L^\infty([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3) \cap L^\infty(0, T; L^1(\mathbb{T}^3 \times \mathbb{R}^3)) \cap C(0, T; L^2(\mathbb{T}^3 \times \mathbb{R}^3)) \cap \\ &\quad \cap L^\infty(0, T; W^{1,2}(\mathbb{T}^3 \times \mathbb{R}^3)). \end{aligned}$$

We present the definition of a weak solution.

Definition 6.1.1. Let $p \geq \frac{11}{5}$ and $T > 0$. The couple (f, u) is a weak solution of (6.1) on the time interval $[0, T]$ if and only if the following conditions are satisfied:

1. $f \geq 0$, $f \in X$, $\partial_t f \in L^2(0, T; L^2(\mathbb{T}^3 \times \mathbb{R}^3))$, $\nabla_v f \in L^\infty(0, T; L^3(\mathbb{T}^3 \times \mathbb{R}^3))$ and $M_\alpha f \in L^\infty([0, T])$ for $0 \leq \alpha \leq \frac{21}{4}$; the function $v \mapsto f(t, x, \cdot)$ is compactly supported for a.a. $t \in [0, T]$ and $x \in \mathbb{T}^3$.
2. $u \in \mathcal{H}$ and $\partial_t u \in L^2(0, T; L^2(\mathbb{T}^3))$.
3. For all $\phi \in C^1([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$ with compact support in t

$$\int_0^T \int_{\mathbb{T}^3 \times \mathbb{R}^3} f[\partial_t \phi + v \cdot \nabla \phi + F(f) \cdot \nabla_v \phi] dx dv dt = - \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_0 \phi(0, \cdot, \cdot) dx dv.$$

4. For all $\varphi \in W^{1,2}(\mathbb{T}^3) \cap \dot{W}_{div,0}^{1,p}(\mathbb{T}^3)$

$$\int_{\mathbb{T}^3} \left[\frac{\partial u}{\partial t} \varphi + (u \cdot \nabla) u \varphi + \tau(Du)D(\varphi) \right] dx = -3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} (u - v) \cdot \psi f dx dv$$

is satisfied a.e. in $[0, T]$.

Remark 6.1.1. The moment $M_\alpha f$ has various physical interpretations depending on α . For example for $\alpha = 0$ it is the total mass of the particles, for $\alpha = 1$, the total momentum and for $\alpha = 2$, the total kinetic energy of the particles. We realise that the upper bound on α in point 1 of the above definition (i.e. $\frac{21}{4}$) can be both unexpected and difficult to interpret from the physical point of view. The reason we include $\alpha > 2$ is that estimation of $M_5 f$ plays a crucial role in the proof of $W^{2,2}$ regularity of u . As a matter of fact we would be satisfied by taking only $0 \leq \alpha \leq 5$ but our method enables us to estimate moments up to $\frac{21}{4}$.

Remark 6.1.2. In Definition 6.1.1, regularity of f and boundedness of the support of f in v enable us to rewrite point 3. in an equivalent form:

3'. For all $\phi \in L^\infty(\mathbb{T}^3 \times \mathbb{R}^3) \cap L^1(\mathbb{T}^3 \times \mathbb{R}^3)$, we have

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} [\partial_t f + v \nabla f + \operatorname{div}_v [F(f)f]] \phi dx dv = 0.$$

In particular since for a.a. $t \in [0, T]$ we have $f(t) \in L^\infty(\mathbb{T}^3 \times \mathbb{R}^3) \cap L^1(\mathbb{T}^3 \times \mathbb{R}^3)$ then f is an admissible test function for it's weak formulation. Similarly also $|v|^\alpha$ for $\alpha \geq 0$ is a good test function since it belongs to $L^\infty(\mathbb{T}^3 \times \mathbb{R}^3) \cap L^1(\mathbb{T}^3 \times \mathbb{R}^3)$ on the support of f .

6.2 Part II: Main result

We present the main result of Part II which is global strong existence and uniqueness of solutions to (6.1) in the domain $[0, \infty) \times \mathbb{T}^d \times \mathbb{R}^d$ with $d = 3$, with the assumption that $p \geq \frac{11}{5}$. We present the main result in the form of the following theorem.

Theorem 6.2.1. Let $p \geq \frac{11}{5}$ and $T > 0$ and suppose that the initial data (f_0, u_0) satisfy

1. $0 \leq f_0 \in (L^1 \cap L^\infty)(\mathbb{T}^3 \times \mathbb{R}^3)$, $\text{supp} f_0(x, \cdot) \subset B(R)$ for some $R > 0$ and a.a. $x \in \mathbb{T}^3$, where $B(R)$ is a ball centred at 0 with radius R ,
2. $u_0 \in W^{1,2}(\mathbb{T}^3) \cap L^2_{\text{div},0}(\mathbb{T}^3)$,
3. $\nabla_v f_0 \in L^3(\mathbb{T}^3 \times \mathbb{R}^3)$.

Then there exists a unique solution of (6.1) in the sense of Definition (6.1.1) for regular communication weight ψ .

Remark 6.2.1. Assumption 1 in the above theorem immediately implies that

$$M_\alpha f_0 \leq C$$

for some positive constant C and all $\alpha \geq 0$.

Moreover boundedness of the support of f_0 and assumption 3 are needed only for the uniqueness. In order to obtain existence alone we could skip assumption 3 and replace boundedness of the support of f_0 with the assumption that

$$M_\alpha f_0 \leq C$$

for all $0 \leq \alpha \leq \frac{21}{4}$.

Chapter 7

Proof of Theorem 6.2.1

The proof is divided into two parts. The first, much longer part is directed towards proving the existence part of Theorem 6.2.1 while the second towards proving uniqueness.

7.1 Existence of solutions

Our first goal is to prove the existence part of Theorem 6.2.1. The proof follows closely the ideas of [7] and can be sketched as follows:

1. First we regularise the system (6.1). For the particle part we regularise the initial data f_0 as well as F_d in (6.1)₁. It is done to enable the use of the standard method of characteristics, which needs sufficient regularity of the trajectories. For the fluid part, we include a cutoff function in v to the external force in (6.1)₂.
2. Then to solve the regularised system, we introduce an iterative scheme that serves as a way to obtain existence for the coupled system. We alternate between solving (6.1)₁ with F_d taken from previous iterations and (6.1)₂ with the external force defined by previous iterations.
3. For each iteration, we establish existence of solutions through standard techniques originating from [5] and [46] or [38].
4. Next, we converge with the iterations to the solution of the regularised system. The convergence is done by a technique introduced in [7]. Most modifications come from nonlinearity of the stress tensor τ as well as the fact that we aim to prove uniqueness too. Thus we need to uniformly control the support in v of the iterative solutions.
5. We converge with the solutions of the regularised system to a solution of (6.1) locally in time. We estimate the regularised solutions in \mathcal{X} and \mathcal{H} and apply the Aubin–Lions lemma (Theorem B.4.1) to extract a convergent subsequence. The crucial role in these estimations is played by the estimation of $M_5 f_\epsilon$ – the 5th velocity moment of the C–S part of the solution to the regularised system.
6. Lastly, we extend the local solution up to an arbitrary interval $[0, T]$, thus finishing the proof of existence. The extension comes from the global estimate for $M_2 f$ – the 2nd velocity moment

of the C–S part of the solution, which enables us to restart the local-in-time estimates from step 5.

Steps 1-4 of the above sketch are more or less similar to the proof of existence for the coupled Navier-Stokes-Vlasov and Navier-Stokes-Cucker-Smale systems presented in [7] and [5]. Most of the differences and problems with the non-linear viscosity become apparent in steps 5 and 6 and force us to take a different approach.

7.1.1 Step 1: A regularized system

The main difference between our approach and the approach presented in [7] or [5] is that we do not regularise the convective term in (6.1)₂, as we are able to obtain better regularity of u anyway, which is very expected for $p \geq \frac{11}{5}$. Aside from this difference the general idea of the proof is similar to those of [7] and [5]. One of main problems comes from the external force

$$f_{ext}(t, x) = \int_{\mathbb{R}^3} (u(t, x) - v)f(t, x, v)dv,$$

which needs to be controlled by addition of the cutoff function $\gamma_\epsilon : \mathbb{R}^3 \rightarrow \mathbb{R}$ that restricts the support in v of f to a ball of radius $\frac{1}{\epsilon}$, i.e. we replace f_{ext} with

$$f_{ext,\epsilon}(t, x) = \int_{\mathbb{R}^3} (u(t, x) - v)\gamma_\epsilon(v)f(t, x, v)dv,$$

where $\gamma_\epsilon \in C^\infty(\mathbb{R}^3)$:

$$\text{supp}\gamma_\epsilon \subset B\left(\frac{1}{\epsilon}\right), \quad 0 \leq \gamma_\epsilon \leq 1, \quad \gamma_\epsilon = 1 \text{ on } B\left(\frac{1}{2\epsilon}\right), \quad \gamma_\epsilon \rightarrow 1 \text{ as } \epsilon \rightarrow 0.$$

The second problem is caused by the drag force term in the C–S part of the system, which also needs to be regularised. For $\epsilon > 0$ we define θ_ϵ as the standard mollifier i.e.

$$\theta_\epsilon(x) := \frac{1}{\epsilon^3}\theta\left(\frac{x}{\epsilon}\right),$$

for some $0 \leq \theta \in C_0^\infty(\mathbb{T}^3)$ with $\int_{\mathbb{T}^3} \theta dx = 1$. With such θ_ϵ let $F_\epsilon(f_\epsilon)$ be the regularised force given by

$$F_\epsilon(f_\epsilon) := F_a(f_\epsilon) + \theta_\epsilon * u_\epsilon - v.$$

Now we define the regularized system. For $(t, x, v) \in [0, T] \times \mathbb{T}^3 \times \mathbb{R}^3$, let

$$\begin{cases} \partial_t f_\epsilon + v\nabla f_\epsilon + \text{div}_v[F_\epsilon(f_\epsilon)f_\epsilon] &= 0, \\ \partial_t u_\epsilon + (u_\epsilon \cdot \nabla)u_\epsilon + \nabla\pi_\epsilon - \text{div}(\tau(Du_\epsilon)) &= -3 \int_{\mathbb{R}^d} f_\epsilon(u_\epsilon - v)\gamma_\epsilon dv, \\ \text{div}u_\epsilon &= 0, \end{cases} \quad (7.1)$$

with a smooth, compactly supported initial data $f_{0,\epsilon}$, where

1. $0 \leq f_{0,\epsilon} \rightarrow f_0$ strongly in $L^q(\mathbb{T}^3 \times \mathbb{R}^3)$, for all $1 < q < \infty$ and weakly $*$ in $L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$ and has a compact support in v contained in $B(2R)$.
2. $u_{0,\epsilon} = u_0$.

Remark 7.1.1. Note that even though we will refer to the solution of (7.1) as a solution in the sense of Definition 6.1.1, due to it's high regularity f is in fact a classical solution of (7.1)₁.

Step 2: Iterative scheme

To solve the regularized problem we further approximate the solutions using the following iterative scheme. Denoting for notational simplicity

$$f^n := f_\epsilon^n, \quad u^n := u_\epsilon^n,$$

we introduce the following scheme:

Initial step $n = 1$.

We set

$$u^1(t, x) := u_0(x).$$

Next we solve the Cucker-Smale's part of (6.1) with fixed u_1 :

$$\partial_t f^1 + v \cdot \nabla f^1 + \operatorname{div}_v[(F_a(f^1) + F_d)f^1] = 0,$$

with the initial data

$$f^1(0, x, v) = f_{0,\epsilon}(x, v)$$

and

$$F_d = \theta_\epsilon * u^1 - v.$$

Inductive step.

Suppose we have a well defined n th solution (f^n, u^n) . Then we define u^{n+1} as the solution of the system

$$\begin{cases} \partial_t u^{n+1} + (u^{n+1} \cdot \nabla)u^{n+1} + \nabla \pi^{n+1} - \tau(u^{n+1}) &= -3 \int_{\mathbb{R}^d} f^n(u^n - v) \gamma_\epsilon dv, \\ \operatorname{div} u^{n+1} &= 0, \end{cases} \quad (7.2)$$

with the initial data

$$u^{n+1}(0, x) = u_0(x)$$

noting that in such system, the right-hand side depends on f^n and u^n , which are at this point given functions. Thus what we in fact do, is solving (1.6) with a given external force. Then, we define

$$F_d^{n+1} = \theta_\epsilon * u^{n+1} - v,$$

which at this point is a given function. Finally we solve the Vlasov-type equation:

$$\partial_t f^{n+1} + v \cdot \nabla f^{n+1} + \operatorname{div}_v[(F_a(f^{n+1}) + F_d^{n+1})f^{n+1}] = 0, \quad (7.3)$$

with initial data

$$f^{n+1}(0, x, v) = f_{0,\epsilon}(x, v).$$

7.1.2 Step 3: Existence of the iterations

Existence of f^n and u^n is guaranteed by the following propositions.

Proposition 7.1.1. *Let $p \geq \frac{11}{5}$ and $T > 0$. There exists a solution in the sense of Definition 6.1.1 to the problem*

$$\partial_t u + (u \cdot \nabla)u + \nabla \pi - \operatorname{div}(\tau(Du)) = f_{ext},$$

as long as $u_0 \in W^{1,2}(\mathbb{T}^3) \cap L^2_{\operatorname{div},0}(\mathbb{T}^3)$ and the external force f_{ext} belongs to the space $L^2(0, T; L^2(\mathbb{T}^3))$. Moreover

$$\begin{aligned} \|u\|_{\mathcal{H}} &\leq C, \\ \|\partial_t u\|_{L^2(0, T; L^2(\mathbb{T}^3))} &\leq C, \end{aligned}$$

where C is a positive constant depending on $\|u_0\|_{W^{1,2}(\mathbb{T}^3)}$, $\|f_{ext}\|_{L^2(0, T; L^2(\mathbb{T}^3))}$, p and T .

Proof. The proof can be found in [38], page 246 (Theorem 4.5). □

Proposition 7.1.2. *Let $T > 0$. There exists a solution in the sense of Definition 2.1.2 to the problem*

$$\partial_t f + v \cdot \nabla f + \operatorname{div}_v[(F_a(f) + (\theta_\epsilon * u - v))f] = 0, \quad (7.4)$$

as long as $0 \leq f_0 \in C^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$ is compactly supported in v and $u \in L^\infty(0, T; L^2_{\operatorname{div},0}(\mathbb{T}^3))$. This solution f belongs to the space $C^2([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$. Moreover

$$\begin{aligned} \|f\|_{L^\infty(0, T; (L^\infty \cap L^1)(\mathbb{T}^3 \times \mathbb{R}^3))} &\leq C, \\ \|f\|_{C^2} &\leq C(\epsilon), \end{aligned} \quad (7.5)$$

where C is a positive constant depending on $\|f_0\|_{L^1(\mathbb{T}^3 \times \mathbb{R}^3)}$ and $\|f_0\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}$, while $C(\epsilon)$ depends also on ϵ and $\|u\|_{L^\infty(0, T; L^2(\mathbb{T}^3))}$ (both constants depend also on T).

Proof. This proposition along with its proof can be found in [5] Appendix A. However there is one seemingly substantial difference, namely in [5] it is only shown that $f \in C^1$. However it can be obtained by an easy modification of the proof from [5] since both $F_a(f)$ and F_d in (7.4) are smooth. □

Let us return to our sequence of approximate solutions. Assume that u^n and f^n exist in the sense of Definition 6.1.1 (or in case of f^n – with better regularity by Proposition 7.1.2). Then

$u^n \in L^\infty(0, T; L^2_{div,0}(\mathbb{T}^3))$ and $f^n \in C^2([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$. In such case we have

$$\begin{aligned}
\int_0^T \|f_{ext}^{n+1}\|_2^2 &= d^2 \int_0^T \int_{\mathbb{T}^3} \left| \int_{\mathbb{R}^3} (u^n - v) \gamma_\epsilon f^n dv \right|^2 dx dt \\
&= d^2 \int_0^T \int_{\mathbb{T}^3} \left| \int_{B(\frac{1}{\epsilon})} (u^n - v) \gamma_\epsilon f^n dv \right|^2 dx dt \\
&\leq C(T, \epsilon) \int_0^T \int_{\mathbb{T}^3} \int_{B(\frac{1}{\epsilon})} |u^n - v|^2 |f^n|^2 dv dx dt \\
&\leq C(T, \epsilon) \|f^n\|_\infty^2 (\|u^n\|_2^2 + 1).
\end{aligned} \tag{7.6}$$

Therefore f_{ext}^{n+1} belongs to $L^2(0, T; L^2(\mathbb{T}^3))$ with its norm depending on $T, \epsilon, \|f^n\|_\infty$ and $\|u\|_{L^\infty(0, T; L^2(\mathbb{T}^3))}$. Therefore by Proposition 7.1.1, there exists a unique u^{n+1} – a solution to (7.2) in the sense of Definition 6.1.1. Existence of a unique f^{n+1} – a solution to (7.3) belonging additionally to the space $C^2([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$ follows then by Proposition 7.1.2. This argument may be iterated indefinitely and thus, the sequence (f^n, u^n) is well defined.

7.1.3 Step 4: Convergence of the iterations

Our next step is to prove that with $n \rightarrow \infty$, the weak formulations for (f^n, u^n) converge to a weak formulation of (7.1)¹. We begin by estimating u^n and f^n in \mathcal{H} and \mathcal{X} , respectively. First let us denote for simplicity

$$\begin{aligned}
\|f\|_\infty &:= \|f\|_{L^\infty([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)}, \\
\|u\|_q &:= \|u(t)\|_q := \|u(t)\|_{L^q(\mathbb{T}^3)}, \text{ for } 1 \leq q \leq \infty.
\end{aligned}$$

Proposition 7.1.3. *The sequence u^n of the approximate solutions satisfies the following bounds:*

$$\begin{aligned}
(i) \quad & \|u^n\|_{\mathcal{H}} \leq C(\epsilon), \\
(ii) \quad & \|\partial_t u^n\|_{L^2(0, T; L^2(\mathbb{T}^3))} \leq C(\epsilon),
\end{aligned}$$

where $C(\epsilon)$ is independent of n but depends on ϵ .

Proof. By Proposition 7.1.1 and the definition of u^n it is clear that to obtain estimation of u^n in \mathcal{H} it suffices to estimate $\|f_{ext}^{n-1}\|_{L^2(0, T; L^2(\mathbb{T}^3))}$ uniformly with respect to n . By testing the weak formulation for u^n with u^n (which by Proposition 7.1.1 is a suitable test function) applying Korn's inequality B.4.1 and (6.6) we obtain

$$\frac{1}{2} \frac{d}{dt} \|u^n\|_2^2 + c_3 \kappa \|\nabla u^n\|_p^p \leq \|u^n\|_2^2 + \|f_{ext}\|_2^2,$$

¹Keep in mind that we skip the epsilon in $(f^n, u^n) = (f_\epsilon^n, u_\epsilon^n)$.

which by inequality (7.6) and (7.5) implies that

$$\frac{1}{2} \frac{d}{dt} \|u^n\|_2^2 \leq \|u^n\|_2^2 + C(T, \epsilon) \|u^{n-1}\|_2^2.$$

Therefore by Lemma B.1.4 we have

$$\|u^n\|_{L^2(\mathbb{T}^3)} \leq C(T, \epsilon),$$

which together with (7.6) proves (by induction) that f_{ext}^{n-1} is uniformly bounded in $L^2(0, T; L^2(\mathbb{T}^3))$ and this concludes the proof of (i).

The proof of (ii) follows similarly to the proof of (i) by testing the weak formulation for u^n with $\partial_t u^n$ and using the previously proved estimations \square

We follow with the estimation of f^n .

Proposition 7.1.4. *The sequence f^n of the approximate solutions satisfies the following bounds:*

$$\begin{aligned} (iii) \quad & \|\nabla_v f^n\|_{L^\infty(0, T; L^3(\mathbb{T}^3 \times \mathbb{R}^3))} \leq C \\ (iv) \quad & \|f^n\|_X \leq C(\epsilon), \\ (v) \quad & \|\partial_t f^n\|_{L^2(0, T; L^2(\mathbb{T}^3 \times \mathbb{R}^3))} \leq C(\epsilon) \end{aligned}$$

where $C(\epsilon)$ is independent of n but depends on ϵ , while C depends on neither. Moreover there exists a nondecreasing function $\mathcal{R}_\epsilon : [0, T] \rightarrow [0, \infty)$ such that

$$\text{supp } f^n(t, x, \cdot) \subset B(\mathcal{R}_\epsilon(t)), \quad \text{for all } t \text{ and a.a. } x. \quad (7.7)$$

The function \mathcal{R}_ϵ is independent of n but depends on ϵ and R^2 .

Proof. Proof of the propagation of the support.

The estimate of the support of f^n is proved in Lemmas B.3.1 and B.3.2 in Appendix B. Lemma B.3.1 shows that $\mathcal{R}_\epsilon(t)$ depends on $\|u^n\|_{L^2(0, T; W^{2,2}(\mathbb{T}^3))}$ and $\|M_1 f^n\|_\infty$. On the other hand in Lemma B.3.2 we prove that $\|M_1 f^n\|_\infty$ is uniformly bounded by a constant depending on $\|u^n\|_{L^2(0, T; W^{2,2}(\mathbb{T}^3))}$. Therefore by (i) from Proposition 7.1.3 function \mathcal{R}_ϵ is independent of n but depends on ϵ . This observation concludes the proof of (7.7).

Remark 7.1.2. In the proof of (iii) and (iv) we estimate a and b from (6.10) and (6.11) in $W^{1,\infty}$. It follows by the fact $M_1 f^n$ and $M_0 f^n$ are by (7.5) uniformly bounded with respect to n (in fact $M_0 f^n$ is uniformly bounded also with respect to ϵ).

Proof of (iii).

We apply ∇_v to both sides of (7.4), multiply by $|\nabla_v f^n| \nabla_v f^n$ and integrate, obtaining

$$\begin{aligned} -\frac{1}{3} \frac{d}{dt} \|\nabla_v f^n\|_3^3 &= \int_{\mathbb{T}^3 \times \mathbb{R}^3} \nabla_v [\text{div}_v F_a(f^n) f^n + F_a(f^n) \nabla_v f^n - 3f^n + (\theta_\epsilon * u^n - v) \nabla_v f^n] |\nabla_v f^n| \nabla_v f^n dx dv \\ &\stackrel{(6.10), (6.11)}{=} \int_{\mathbb{T}^3 \times \mathbb{R}^3} \nabla_v [-3b f^n + (a - vb) \nabla_v f^n - 3f^n + (\theta_\epsilon * u^n - v) \nabla_v f^n] |\nabla_v f^n| \nabla_v f^n dx dv \\ &= \int_{\mathbb{T}^3 \times \mathbb{R}^3} (-6b - 6) |\nabla_v f^n|^3 dx dv + \int_{\mathbb{T}^3 \times \mathbb{R}^3} [(a - vb) + (\theta_\epsilon * u^n - v)] \nabla_v^{(2)} f^n |\nabla_v f^n| \nabla_v f^n dx dv =: \mathcal{L}. \end{aligned}$$

The second integral appearing in the above definition of \mathcal{L} can be rewritten as

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} [(a - vb) + (\theta_\epsilon * u^n - v)] \frac{1}{3} \nabla_v |\nabla_v f^n|^3 dx dv,$$

which after integrating by parts³ is equal to

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} (b + 1) |\nabla_v f^n|^3 dx dv.$$

Thus

$$\mathcal{L} = -5 \int_{\mathbb{T}^3 \times \mathbb{R}^3} (b + 1) |\nabla_v f^n|^3 dx dv \stackrel{(6.11)}{\leq} (c_6 + 1) \|\nabla_v f^n\|_3^3$$

Altogether we get

$$\frac{1}{3} \frac{d}{dt} \|\nabla_v f^n\|_3^3 \leq (c_6 + 1) \|\nabla_v f^n\|_3^3,$$

thus by Gronwall's lemma for $t \in [0, T]$, we have

$$\frac{1}{3} \cdot \|\nabla_v f^n(t)\|_3^3 \leq C \cdot \|\nabla_v f_0\|_3^3,$$

which together with assumption 3 from Theorem 6.2.1 finishes the proof of estimation (iii).

Proof of (iv).

Taking to account estimation (7.5) and (iii) in order to estimate f^n in \mathcal{X} we only need to estimate ∇f^n in $L^\infty(0, T; L^2(\mathbb{T}^3 \times \mathbb{R}^3))$. Note that in the following estimation we do not use the mollifying effect of θ_ϵ ; we do not need this effect at this point and on top of that ultimately we need ϵ independent estimates. To estimate ∇f^n , we apply ∇ to both sides of (7.4), multiply by ∇f^n and integrate to obtain (similarly to the previous step)

$$\begin{aligned} -\frac{1}{2} \frac{d}{dt} \|\nabla f^n\|_2^2 &= \int_{\mathbb{T}^3 \times \mathbb{R}^3} \nabla [-3bf^n + (a - vb)\nabla_v f^n - 3f^n + (\theta_\epsilon * u^n - v)\nabla_v f^n] \nabla f^n dx dv \\ &= \int_{\mathbb{T}^3 \times \mathbb{R}^3} -3\nabla b f^n \nabla f^n dx dv + \int_{\mathbb{T}^3 \times \mathbb{R}^3} [\nabla a - v\nabla b + \theta_\epsilon * \nabla u^n] \nabla_v f^n \nabla f^n dx dv \\ &+ \int_{\mathbb{T}^3 \times \mathbb{R}^3} [a - vb + \theta_\epsilon * u^n - v] \nabla \nabla_v f^n \nabla f^n dx dv + \int_{\mathbb{T}^3 \times \mathbb{R}^3} (-3 - 3b) |\nabla f^n|^2 dx dv \\ &=: I + II + III + IV. \end{aligned}$$

Then by (6.11), (7.5) and Young's inequality we have

$$|I| \leq C(1 + \|\nabla f^n\|_2^2).$$

To estimate II we use Young's inequality together with (6.10), (6.11) and the estimation of the support in v of f to get

$$|II| \leq C(T, \epsilon) \left(\|\nabla_v f^n\|_2^2 + \|\nabla f^n\|_2^2 + \|\nabla_v f^n\|_3^3 + \|\theta_\epsilon * \nabla u^n \nabla f^n\|_{\frac{3}{2}}^{\frac{3}{2}} \right).$$

³Keeping in mind that $\text{supp} f(x, \cdot, t) \subset B(\mathcal{R}_\epsilon(T))$.

By Hölder's inequality with exponent $\frac{4}{3}$ and Young's inequality for convolution we have

$$|II| \leq C(T, \epsilon) \left(\|\nabla_v f^n\|_2^2 + \|\nabla f^n\|_2^2 + \|\nabla_v f^n\|_3^3 + \|\nabla u^n\|_6^{\frac{3}{2}} \|\nabla f^n\|_2^{\frac{3}{2}} \right)$$

which together with (iii) and the boundedness of the support of f^n implies that

$$|II| \leq C(T, \epsilon) \left(1 + \|\nabla f^n\|_2^2 + \|\nabla u^n\|_6^{\frac{3}{2}} \|\nabla f^n\|_2^{\frac{3}{2}} \right).$$

Finally for III we integrate by parts to obtain

$$III = \frac{1}{2} \int_{\mathbb{T}^3 \times \mathbb{R}^3} (b+1) |\nabla f^n|^2 dx dv \leq c_6 \|\nabla f^n\|_2^2$$

and altogether

$$\frac{1}{2} \frac{d}{dt} \|\nabla f^n\|_2^2 \leq C(T, \epsilon) \left(1 + \|\nabla f^n\|_2^2 + \|\nabla u^n\|_6^{\frac{3}{2}} \left(\|\nabla f^n\|_2^2 \right)^{\frac{3}{4}} \right),$$

which by non-linear Gronwall's lemma (Lemma B.2.1) and estimate (i) proves (iv) (note that $t \mapsto \|\nabla u^n\|_6^{\frac{3}{2}}$ is integrable in $[0, T]$ since $L^2(0, T; W^{2,2}(\mathbb{T}^3)) \hookrightarrow L^1(0, T; W^{1,6}(\mathbb{T}^3))$).

Proof of (v).

We multiply both sides of (7.4) by $\partial_t f^n$ and integrate. Then the proof follows by the use of estimates (i) – (iv) and Young's inequality. \square

With the estimates provided by Propositions 7.1.3 and 7.1.4 we are ready to prove convergence of (f^n, u^n) to a solution of the regularised system (7.1). First let us denote

$$\begin{aligned} \chi^n(t) &:= (x^n(t), v^n(t)), \\ \omega^{n+1} &:= u^{n+1} - u^n, \quad n = 1, 2, \dots \end{aligned}$$

where $(x^n(t), v^n(t))$ is a solution of the following characteristics ODE:

$$\begin{cases} \frac{dx^n}{dt}(t) = v^n(t), & x^n(0) = x, \\ \frac{dv^n}{dt}(t) = F_a(f^n)(x^n(t), v^n(t), t) + (\theta_\epsilon * u^n)(x^n(t), t) - v^n(t), & v^n(0) = v. \end{cases}$$

The following proposition proves the existence of the solution to the regularised system (7.1) and summarises Steps 1–4 of the proof of Theorem 6.2.1.

Proposition 7.1.5. *For each $n = 1, 2, \dots$, there exists a unique solution (f^n, u^n) of (7.2) and (7.3) in the sense of Definition 6.1.1. The sequence (f^n, u^n) converges strongly in $L^\infty([0, T] \times \mathbb{T}^d \times \mathbb{R}^d) \times L^\infty(0, T; W^{1,2}(\mathbb{T}^d))$ towards the weak formulation of (7.1). Such solution of (7.1) satisfies additionally inequality (7.5) and estimates (i) – (v) and (7.7) from Propositions 7.1.3 and 7.1.4.*

Proof. Existence of (f^n, u^n) belonging to appropriate spaces was already explained with the help of Proposition 7.1.1 and Proposition 7.1.2. It remains to show that (f^n, u^n) strongly converges in $L^\infty([0, T] \times \mathbb{T}^d \times \mathbb{R}^d) \times L^\infty(0, T; W^{1,2}(\mathbb{T}^d))$ to the weak formulation of (7.1). To achieve this goal, first let us prove the following lemma.

Lemma 7.1.1. For given positive T and ϵ , let (f^n, u^n) be the n th solution of (7.2) and (7.3). Then for $t \in [0, T)$, we have

1.

$$\|f^n(t) - f^{n-1}(t)\|_\infty + \|\chi^n(t) - \chi^{n-1}(t)\|_\infty \leq C(\epsilon) \int_0^t \|\omega^n(s)\|_2 ds,$$

2.

$$\|\omega^{n+1}(t)\|_2^2 + \int_0^t \|\nabla \omega^{n+1}(s)\|_2^2 ds \leq C(\epsilon) \left(\int_0^t \|\omega^n(s)\|_2^2 ds + \int_0^t \|\omega^{n+1}(s)\|_2^2 ds \right),$$

where $C(\epsilon)$ is a positive constant independent of n .

Proof. Since the Cucker–Smale part of the system is exactly the same as in [5] and the mollifier θ_ϵ makes up for any differences that could appear due to different fluid part of the system, the proof of 1 is the same as in [5], which leaves us with only point 2 to prove. By Proposition 7.1.1, for a.a. t the function ω^{n+1} belongs to $W^{1,2}(\mathbb{T}^3) \cap \dot{W}_{div,0}^{1,p}(\mathbb{T}^3)$, thus it is a good test function for a weak formulation for ω^{n+1} , i.e.

$$\begin{aligned} & \int_{\mathbb{T}^3} \partial_t \omega^{n+1} \cdot \omega^{n+1} dx + \int_{\mathbb{T}^3} (\omega^{n+1} \cdot \nabla) u^{n+1} \cdot \omega^{n+1} dx + \int_{\mathbb{T}^3} (u^n \cdot \nabla) \omega^{n+1} \cdot \omega^{n+1} dx \\ & + \int_{\mathbb{T}^3} [\tau(Du^{n+1}) - \tau(Du^n)] \cdot D\omega^{n+1} dx \\ & = -3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\epsilon f^n(u^n - v) \cdot \omega^{n+1} dx dv - 3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\epsilon f^{n-1}(u^{n-1} - v) \cdot \omega^{n+1} dx dv. \end{aligned}$$

Let us denote the convective term by

$$T_1 := \int_{\mathbb{T}^3} (\omega^{n+1} \cdot \nabla) u^{n+1} \cdot \omega^{n+1} dx + \int_{\mathbb{T}^3} (u^n \cdot \nabla) \omega^{n+1} \cdot \omega^{n+1} dx,$$

the viscosity term by

$$T_2 := \int_{\mathbb{T}^3} [\tau(Du^{n+1}) - \tau(Du^n)] \cdot D\omega^{n+1} dx$$

and the drag force term by

$$T_3 := -3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\epsilon f^n(u^n - v) \cdot \omega^{n+1} dx dv - 3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\epsilon f^{n-1}(u^{n-1} - v) \cdot \omega^{n+1} dx dv.$$

First let us note that by inequality (6.8) and Korn's inequality B.4.1, we have

$$T_2 = \int_{\mathbb{T}^3} [\tau(Du^{n+1}) - \tau(Du^n)] \cdot D\omega^{n+1} dx \geq c_5 \kappa \|\nabla \omega\|_2^2. \quad (7.8)$$

Next, we focus on T_1 . By space periodicity and equation (7.2)₂ the second summand in T_1 equals 0. Hence

$$|T_1| = \left| \int_{\mathbb{T}^d} (\omega^{n+1} \cdot \nabla) u^{n+1} \cdot \omega^{n+1} dx \right| \stackrel{H(2)}{\leq} \|\omega^{n+1}\|_4^2 \|\nabla u^{n+1}\|_2$$

which by interpolation inequality (B.1) and embedding inequality (B.2) implies that

$$|T_1| \leq \|\omega^{n+1}\|_2^{\frac{1}{2}} \|\omega^{n+1}\|_6^{\frac{3}{2}} \|\nabla u^{n+1}\|_2 \stackrel{Y(4)}{\leq} C(\epsilon) \|\omega^{n+1}\|_2^2 + \frac{c_{5K}}{2} \|\nabla \omega^{n+1}\|_2^2. \quad (7.9)$$

Here, we also used the fact that by estimation (ii) from Proposition 7.1.5, $\|\nabla u^{n+1}(t)\|_2 \leq C(\epsilon)$ for all n . Lastly for T_3 , we have

$$T_3 = -3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\epsilon (f^n - f^{n-1})(u^n - v) \omega^{n+1} dx dv - 3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \gamma_\epsilon f^{n-1} (u^n - u^{n-1}) \omega^{n+1} dx dv \\ =: T_{31} + T_{32},$$

where

$$\frac{1}{3} |T_{31}| \stackrel{Y(2)}{\leq} \frac{1}{2} \int_{\mathbb{T}^3 \times B(\frac{1}{\epsilon})} |f^n - f^{n-1}|^2 |u^n - v|^2 dx dv + \frac{1}{2} \int_{\mathbb{T}^3 \times B(\frac{1}{\epsilon})} |\omega^{n+1}|^2 dx dv \\ \leq C(\epsilon) \|f^n - f^{n-1}\|_\infty^2 (\|u^n\|_2^2 + 1) + C(\epsilon) \|\omega^{n+1}\|_2^2, \\ \frac{1}{3} |T_{32}| \stackrel{Y(2)}{\leq} C(\epsilon) \|f^{n-1}\|_\infty (\|u^n\|_2^2 + \|\omega^{n+1}\|_2^2),$$

which together with (7.5) and (i) from Proposition 7.1.3 implies that

$$|T_3| \leq C(\epsilon) (\|f^n - f^{n-1}\|_\infty^2 + \|\omega^{n+1}\|_2^2 + \|\omega^n\|_2^2). \quad (7.10)$$

Combining together estimations (7.8), (7.9), (7.10) and estimation from point 1 we obtain

$$\frac{1}{2} \frac{d}{dt} \|\omega^{n+1}\|_2^2 + \frac{c_{5K}}{2} \|\nabla \omega^{n+1}\|_2^2 \leq C(\epsilon) \left(\int_0^t \|\omega^n\|_2^2(s) ds + \|\omega^{n+1}\|_2^2 + \|\omega^n\|_2^2 \right)$$

and by integration of the previous inequality, having in mind that

$$\int_0^t \int_0^s \|\omega^n(r)\|_2^2 dr ds \leq T \int_0^t \|\omega^n(r)\|_2^2 dr,$$

we obtain point 2. □

Now we are sufficiently equipped to finish the proof of Proposition 7.1.5. By Lemma B.1.4 and Lemma 7.1.1.2 there exists K , such that

$$\|u^n(t) - u^{n-1}(t)\|_2^2 = \|\omega^n(t)\|_2^2 \leq \frac{K^n t^n}{n!},$$

Thus

$$u^n \rightarrow u \quad \text{in } L^\infty(0, T; L^2(\mathbb{T}^3)) \quad (7.11)$$

for some $u \in L^\infty(0, T; L^2(\mathbb{T}^3))$ and thanks to Lemma 7.1.1.2

$$\nabla u^n \rightarrow \nabla u \quad \text{in } L^2(0, T; L^2(\mathbb{T}^3)). \quad (7.12)$$

Moreover by Lemma 7.1.1.1 it follows that

$$f^n \rightarrow f \quad \text{in } L^\infty([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3) \quad (7.13)$$

for some $f \in L^\infty([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$.

To finalise the proof we need to show that (f, u) satisfies (7.1) in the sense of Definition 6.1.1. By (7.11)

$$\int_0^T \int_{\mathbb{T}^3} -u^n \partial_t \phi dx dv dt \rightarrow \int_0^T \int_{\mathbb{T}^3} -u \partial_t \phi dx dv dt, \quad (7.14)$$

for all divergence free $\phi \in C^\infty$ with compact support in t . Thus $\partial_t u^n \rightarrow \partial_t u$ in the distributional sense, where $\partial_t u$ is the distributional derivative of u . However, since $\partial_t u^n$ is bounded in $L^2(0, T; L^2(\mathbb{T}^3))$, by Banach–Alaoglu Theorem A.4.2 it has a weakly * convergent subsequence, which actually implies that $\partial_t u^n \rightarrow \partial_t u$ weakly * in $L^2(0, T; L^2(\mathbb{T}^3))$. Further, by weak * lower semicontinuity of a norm $\partial_t u \in L^2(0, T; L^2(\mathbb{T}^3))$.

By (7.11) and (7.12) $u^n \rightarrow u$ and $\nabla u^n \rightarrow \nabla u$ a.e., which implies that also the convective term $(u^n \cdot \nabla) u^n \rightarrow (u \cdot \nabla) u$ a.e. Moreover for sufficiently small $\epsilon > 0$ we have

$$\|(u^n \cdot \nabla) u^n\|_{L^{1+\epsilon}(0, T; L^{1+\epsilon}(\mathbb{T}^3))} \leq \|u\|_2 \|\nabla u\|_p \leq \|u\|_{\mathcal{H}},$$

which means that $(u^n \cdot \nabla) u^n$ is uniformly bounded in $L^{1+\epsilon}$ and thus it is uniformly integrable. Thus by Vitali theorem B.4.2 $(u^n \cdot \nabla) u^n \rightarrow (u \cdot \nabla) u$ in $L^1(0, T; L^1(\mathbb{T}^3))$ and in particular

$$\int_0^T \int_{\mathbb{T}^3} (u^n \cdot \nabla) u^n \phi dx dv dt \rightarrow \int_0^T \int_{\mathbb{T}^3} (u \cdot \nabla) u \phi dx dv dt, \quad (7.15)$$

for all divergence free $\phi \in C^1$ with compact support in t .

Similarly also $\tau(Du^n) \rightarrow \tau(Du)$ a.e. and by (6.7) it is bounded in $L^{p'}(0, T; L^{p'}(\mathbb{T}^3))$. Vitali's convergence theorem (Theorem B.4.2) implies that $\tau(Du^n) \rightarrow \tau(Du)$ strongly in $L^{p'-\epsilon}(0, T; L^{p'-\epsilon}(\mathbb{T}^3))$ for some $\epsilon > 0$. On the other hand by Banach–Alaoglu Theorem (Theorem A.4.2) by its $L^{p'}(0, T; L^{p'}(\mathbb{T}^3))$ boundedness sequence $\{\tau(Du^n)\}_{n \in \mathbb{N}}$ converges weakly in $L^{p'}(0, T; L^{p'}(\mathbb{T}^3))$ to $\tau(Du)$ and by weak sequential lower semicontinuity of the norm $\tau(Du) \in L^{p'}(0, T; L^{p'}(\mathbb{T}^3))$. We also have

$$\int_0^T \int_{\mathbb{T}^3} \tau(Du^n) D\phi dx dv dt \rightarrow \int_0^T \int_{\mathbb{T}^3} \tau(Du) D\phi dx dv dt \quad (7.16)$$

for all divergence free $\phi \in C^\infty$ with compact support in t . Convergence and boundedness of the external force follows by similar arguments. Altogether (7.14)-(7.16) imply that for all $\phi \in C^\infty$ with compact support in t , we have

$$\int_0^T \int_{\mathbb{T}^3} -u \partial_t \phi + (u \cdot \nabla) u \phi + \tau(Du) D\phi dx dv dt = - \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} (u - v) \cdot \phi f dx dv. \quad (7.17)$$

By previously applied argumentation (using Banach–Alaoglu theorem) we obtain that

$$u \in L^\infty(0, T; \dot{W}^{1,p}(\mathbb{T}^3)) \cap L^\infty(0, T; W^{1,2}(\mathbb{T}^3)) \cap L^2(0, T; W^{2,2}(\mathbb{T}^3))$$

and since $\partial_t u \in L^2(0, T; L^2(\mathbb{T}^3))$, by Lemma B.4.2 we conclude that $u \in C(0, T; L^2(\mathbb{T}^3))$ and thus $u \in \mathcal{H}$.

Finally due to the sufficient regularity of u we may replace (7.17) with equation from point 4 of Definition 6.1.1 and by a density argument extend the class of admissible test functions to $W^{1,2}(\mathbb{T}^3) \cap \dot{W}_{div,0}^{1,p}(\mathbb{T}^3)$.

The proof of the fact that $f \in \mathcal{X}$ and that f satisfies point 3 of Definition 6.1.1 is straightforward since at this point f is still regularised (i.e. $f = f_\epsilon \in C^2([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$).

□

7.1.4 Step 5: Local convergence with the regularised solutions

Until now we have managed to prove existence of solutions to the regularized system (7.1). Our next goal is to converge with ϵ to 0 and obtain local existence for (6.1). We begin with the moment estimates for the regularised system. These estimates are in fact the main difference between the C–S model coupled with N–S and the C–S model coupled with non-Newtonian fluids. This difference originates from the fact that according to Proposition 7.2 in order to obtain higher regularity of the solutions (compared to the solution of NS system) we need L^2 estimates for the external force. We obtain such estimates using some better estimates of the moments of f , which were not needed in [7] and [5].

Proposition 7.1.6. *Let $p \geq \frac{11}{5}$ and (f_ϵ, u_ϵ) be a solution to the system (7.1) constructed as a limit of the approximate solutions as proved in Proposition 7.1.5. Then there exists $T^* \in (0, T]$, such that (f_ϵ, u_ϵ) satisfies the following estimates*

$$\|M_\alpha f_\epsilon\|_{L^\infty[0, T^*]} \leq C, \quad \text{for } 0 \leq \alpha \leq \frac{21}{4} \quad (7.18)$$

$$\|u\|_{L^\infty(0, T^*; L^2_{\text{div}, 0}(\mathbb{T}^3)) \cap L^p(0, T^*; \dot{W}^{1,p}_{\text{div}, 0}(\mathbb{T}^3))} \leq C, \quad (7.19)$$

$$\left\| \int_{\mathbb{R}^3} (u_\epsilon - v) \gamma_\epsilon f_\epsilon dv \right\|_{L^2(0, T^*; L^2(\mathbb{T}^3))} \leq C, \quad (7.20)$$

where C is a positive constant depending only on the initial data.

Proof. For notational simplicity, we assume that $f = f_\epsilon$ and $u = u_\epsilon$. The proof follows by four steps. In step 1 we estimate $\frac{d}{dt} M_\alpha f$, in step 2 we estimate the external force by terms depending on u and $M_\alpha f$ and in step 3 we estimate $\frac{d}{dt} \|u\|_2^2$. Finally in step 4 we put the estimations from previous steps together and use the non-linear version of Gronwall's lemma (Lemma B.2.1).

Step 1. First let us note that if $0 \leq \beta \leq \alpha$ then by Hölder's inequality we can interpolate $M_\beta f$ between $M_\alpha f$ and $M_0 f = \|f\|_1$ and thus by (7.5) it is sufficient to prove (7.18) only for $\alpha \geq 2$. Therefore let us fix $\alpha \geq 2$, multiply (7.1)₁ by $|v|^\alpha$ (which by Remark 6.1.2 is an admissible test function) and integrate to get

$$\begin{aligned} \frac{d}{dt} M_\alpha f &= \int_{\mathbb{T}^6 \times \mathbb{R}^6} |v|^{\alpha-2} v \psi(|x-y|) (w-v) f dy dw dx dv + \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^{\alpha-2} v (\theta_\epsilon * u - v) f dx dv \\ &=: \mathcal{S}_1 + \mathcal{S}_2. \end{aligned}$$

By substituting x with y and w with v we conclude that

$$\begin{aligned} \mathcal{S}_1 &= \int_{\mathbb{T}^6 \times \mathbb{R}^6} (|v|^{\alpha-2} v - |w|^{\alpha-2} w) (w-v) \psi(|x-y|) f dy dw dx dv \\ &\leq - \int_{\mathbb{T}^6 \times \mathbb{R}^6} |w-v|^\alpha \psi(|x-y|) f dy dw dx dv \leq 0 \end{aligned}$$

and this leaves us with

$$\mathcal{S}_2 = - \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^\alpha f dx dv + \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^{\alpha-2} v (\theta_\epsilon * u) f dx dv \leq \int_{\mathbb{T}^3} |\theta_\epsilon * u| m_{\alpha-1} f dx.$$

Now since according to (7.5) function f is bounded, we may apply Lemma B.1.3 and Young's inequality for convolutions to get

$$\begin{aligned} \mathcal{S}_2 &\leq C \int_{\mathbb{T}^3} |\theta_\epsilon * u| (m_\alpha f)^{\frac{\alpha+2}{\alpha+3}} dx \stackrel{H(\frac{\alpha+3}{\alpha+2})}{\leq} C \left(\int_{\mathbb{T}^3} |\theta_\epsilon * u|^{\alpha+3} dx \right)^{\frac{1}{\alpha+3}} (M_\alpha f)^{\frac{\alpha+2}{\alpha+3}} \\ &= C \|\theta_\epsilon * u\|_{\alpha+3} (M_\alpha f)^{\frac{\alpha+2}{\alpha+3}} \stackrel{(B.2)}{\leq} C (\|\nabla u\|_p + \|u\|_2) (M_\alpha f)^{\frac{\alpha+2}{\alpha+3}}, \end{aligned}$$

as long as $\alpha+3 \leq p^* := \frac{3p}{3-p}$, which is the case for $p \geq \frac{11}{5}$ exactly when $\alpha \leq \frac{21}{4}$. Therefore by Young's inequality with exponent p we obtain

$$\begin{aligned} \frac{d}{dt} M_\alpha f &\leq C (\|\nabla u\|_p + \|u\|_2) (M_\alpha f)^{\frac{\alpha+2}{\alpha+3}} \\ &\leq \delta \|\nabla u\|_p^p + \delta \|u\|_2^p + C(\delta) (M_\alpha f)^{\frac{p(\alpha+2)}{(p-1)(\alpha+3)}}, \end{aligned} \tag{7.21}$$

with arbitrary $\delta > 0$ and a δ dependent constant $C(\delta)$.

Step 2. Next we focus on (7.20). We have

$$\|f_{ext}\|_{L^2(\mathbb{T}^3)}^2 \leq C \left\| \int_{\mathbb{R}^3} |u| f dv \right\|_{L^2(\mathbb{T}^3)}^2 + C \left\| \int_{\mathbb{R}^3} |v| f dv \right\|_{L^2(\mathbb{T}^3)}^2 =: A^2 + B^2,$$

where we immediately skipped the cutoff function $\gamma_\epsilon \leq 1$. Furthermore, by Lemma B.1.3

$$\begin{aligned} A^2 &= \int_{\mathbb{T}^3} |u|^2 |m_0 f|^2 dx \leq C \int_{\mathbb{T}^3} |u|^2 |m_5|^{\frac{3}{4}} dx \stackrel{H(4)}{\leq} C \left(\int_{\mathbb{T}^3} |u|^8 dx \right)^{\frac{1}{4}} (M_5 f)^{\frac{3}{4}} \\ &\leq C \|u\|_{L^8(\mathbb{T}^3)}^2 (M_5 f)^{\frac{3}{4}} \end{aligned}$$

and by inequality (B.2) (we easily check that $8 \leq \frac{3p}{3-p}$ for $p \geq \frac{11}{5}$) and again by Young's inequality with exponent $\frac{p}{2}$ we conclude that

$$A^2 \leq C (\|\nabla u\|_p^2 + \|u\|_2^2) (M_5 f)^{\frac{3}{4}} \leq \delta \|\nabla u\|_p^p + \delta \|u\|_2^p + C(\delta) M_5 f^{\frac{3p}{4p-8}}$$

On the other hand, by Lemma B.1.3

$$B^2 = \int_{\mathbb{T}^3} |m_1 f|^2 dx \leq C M_5 f.$$

Altogether, we have

$$\|f_{ext}\|_2^2 \leq \delta \|\nabla u\|_p^p + \delta \|u\|_2^p + C(\delta) (M_5 f)^{\frac{3p}{4p-8}} + C M_5 f. \tag{7.22}$$

Step 3. To obtain estimates of $\|u\|_2^2$, we test the weak formulation for u with u , (which by Proposition 7.2 is a good test function), which with the help of Korn inequality B.4.1 leads to

$$\frac{d}{dt} \|u\|_2^2 + 2c_3 \kappa \|\nabla u\|_p^p \leq \|f_{ext}\|_2^2 + \|u\|_2^2, \tag{7.23}$$

since the convective and pressure terms vanish thanks to the fact that $\operatorname{div} u = 0$.

Step 4. We combine inequalities (7.21)-(7.23), with a suitably chosen δ (actually $\delta = \frac{c_3\kappa}{2}$) to get

$$\frac{d}{dt} (M_\alpha f + \|u\|_2^2) + c_3\kappa \|\nabla u\|_p^p \leq C \left((M_\alpha f)^{\frac{p(\alpha+2)}{(p-1)(\alpha+3)}} + (M_5 f)^{\frac{3p}{4p-8}} + M_5 f + \|u\|_2^2 + \|u\|_2^p \right),$$

which after fixing $\alpha = 5$ leads to

$$\begin{aligned} \frac{d}{dt} (M_5 f + \|u\|_2^2) + c_3\kappa \|\nabla u\|_p^p &\leq C \left((M_5 f)^{\frac{7p}{(8p-8)}} + (M_5 f)^{\frac{3p}{4p-8}} + M_5 f + \|u\|_2^2 + \|u\|_2^p \right) \\ &\leq \tilde{C} \left(1 + (M_5 f + \|u\|_2^2)^{q_1} \right), \end{aligned} \quad (7.24)$$

where \tilde{C} is a positive constant which we fix at this moment and

$$q_1 := \max \left\{ \frac{7p}{(8p-8)}, \frac{3p}{4p-8}, \frac{p}{2} \right\}.$$

We denote

$$r(t) = M_5 f(t) + \|u(t)\|_2^2$$

and integrate inequality (7.24) with respect to time obtaining

$$r(t) \leq \tilde{C}t + r(0) + \tilde{C} \int_0^t (r(s))^{q_1} ds.$$

Since $q_1 > 1$, we use the non-linear Gronwall's lemma i.e. Lemma B.2.1 (with $c = \tilde{C}T + r(0)$, $t_0 = 0$ and $a \equiv 0$, $b \equiv \tilde{C}$), which implies that

$$r(t) \leq (\tilde{C}T + r(0)) \left[1 - \frac{\tilde{C}t(q_1 - 1)}{\tilde{C}T + r(0)} \right]^{\frac{1}{q_1-1}} \leq \tilde{C}T + r(0) \quad (7.25)$$

for all $t \in [0, T^*]$, where T^* is small enough to ensure that

$$\tilde{C}T + r(0) < \left[(q_1 - 1)\tilde{C}T^* \right]^{\frac{1}{1-q_1}}, \quad (7.26)$$

i.e.

$$T^* < \frac{(\tilde{C}T + r(0))^{1-q_1}}{\tilde{C}(q_1 - 1)}.$$

Consequently $M_5 f$ and $\|u\|_2^2$ are uniformly bounded in $L^\infty([0, T^*])$ and by (7.24) we have

$$\int_0^{T^*} \|\nabla u\|_p^p dt \leq C(M_5 f, \|u\|_2, T), \quad (7.27)$$

which in consequence proves (7.18) for $\alpha = 5$ and (7.19). To prove (7.18) for $\alpha \neq 5$ we use inequality (7.21) and Lemma B.2.1 (this time with $q_2 = \frac{\alpha+2}{\alpha+3} < 1$), obtaining

$$M_\alpha f(t) \leq \left[(M_\alpha f(0))^{1-q_2} + (1 - q_2) \int_0^t \|\nabla u(s)\|_p + \|u(s)\|_2 ds \right]^{\frac{1}{1-q_2}}, \quad (7.28)$$

which again is uniformly bounded since $\nabla u \in L^p(0, T^*; L^p(\mathbb{T}^3)) \hookrightarrow L^1(0, T^*; L^p(\mathbb{T}^3))$. This finishes the proof of (7.18). Finally (7.20) follows by inequality (7.22) from (7.18) and (7.19). \square

The following corollary combines all the necessary local in time uniform estimations of u_ϵ and f_ϵ proved throughout this section.

Corollary 7.1.1. *The solution (f_ϵ, u_ϵ) satisfies Propositions 7.1.3 and 7.1.4 uniformly with respect to $\epsilon > 0$ on the time interval $[0, T^*]$. Moreover estimation (iii) holds also on whole $[0, T]$. These estimates depend on T rather than T^* .*

Proof. To prove estimation (i) we notice that by Proposition 7.2 the $\|\cdot\|_{\mathcal{H}}$ norm of u_n^ϵ depends only on $\|u_0\|_{W^{1,2}(\mathbb{T}^3)}$, which is fixed and on $\|f_{\epsilon,xt}\|_{L^2(0,T;L^2(\mathbb{T}^3))}$ which by Proposition 7.1.6 is uniformly bounded with respect to both n and ϵ on the time interval $[0, T^*]$. Therefore also $\|u^\epsilon\|_{\mathcal{H}}$ is uniformly bounded on $[0, T^*]$. The exact same argument is valid for an ϵ independent estimation (ii). Estimation (iii) was already proved in Proposition 7.1.4, while estimations (iv) and (v) were shown to hold with constants depending on the constants from (now proved to be ϵ independent) estimations (i) – (iii) and the estimation of the support of f_ϵ^n from Proposition 7.1.4. Therefore (iv) and (v) hold with ϵ (and T^*) independent constants on the time interval $[0, T^*]$ if only the support of f_ϵ is uniformly bounded.

It remains to show that f_ϵ satisfies (7.7) with ϵ -independent \mathcal{R} . By Lemmas B.3.1 and B.3.2 each iterative solution f_ϵ^n has a support contained in a ball of radius \mathcal{R}_ϵ with $\mathcal{R}_\epsilon(t)$ depending on $\|u_\epsilon^n\|_{L^2(0,T;W^{2,2}(\mathbb{T}^3))}$ and $\|M_1 f_\epsilon^n\|_\infty$ (and R , which depends only on the initial data). However by Proposition 7.1.6 these quantities are uniformly bounded on $[0, T^*]$ thus so is \mathcal{R}_ϵ . Finally f_ϵ inherits the uniform boundedness of the support from f_ϵ^n as an L^∞ limit. \square

Proof of Theorem 6.2.1 – local existence. With the uniform bounds from Corollary 7.1.1, what remains is to converge with ϵ to 0. By virtue of those uniform bounds it follows that u_ϵ is uniformly bounded in $\mathcal{H} \hookrightarrow L^2(0, T; W^{2,2}(\mathbb{T}^3))$ and $\partial_t u^n$ is uniformly bounded in $L^2(0, T; L^2(\mathbb{T}^3))$ and since

$$W^{2,2}(\mathbb{T}^3) \hookrightarrow W^{1,2}(\mathbb{T}^3) \hookrightarrow L^2(\mathbb{T}^3),$$

by Aubin–Lions lemma (or Theorem B.4.1), we may extract from u_ϵ a strongly convergent subsequence in $L^2(0, T^*; W^{1,2}(\mathbb{T}^3))$. Then we converge with every term in Definition 6.1.1,4 (similarly to the proof of Proposition 7.1.5) obtaining the weak formulation for u . The convergence of f_ϵ is done in the same way as in [5]. To prove that $u \in C(0, T; L^2_{div,0}(\mathbb{T}^3))$ and $f \in C(0, T; L^2(\mathbb{T}^3 \times \mathbb{R}^3))$ we use estimates (i) and (ii) from Corollary 7.1.1 together with Lemma B.4.2. This finishes the proof of local existence of solutions in the sense of Definition 6.1.1. \square

7.1.5 Step 6: Global existence

In this section we continue the proof of Theorem 6.2.1 by extending the local solutions up to $[0, T]$.

Proof of Theorem 6.2.1 – global existence. To obtain global existence we need to ensure that we can extend the interval $[0, T^*]$ up to $[0, T]$. First let us note that all bounds presented in Proposition 7.1.6 and Corollary 7.1.1 are not only independent of ϵ but also of T^* (even though they hold only on $[0, T^*]$). Moreover the bounds from Corollary 7.1.1 are a direct consequence of the bounds from Proposition 7.1.6 in the sense that as long as we have (7.20), then we may make all the bounds from Propositions 7.1.3 and 7.1.4 ϵ -independent on $[0, T^*]$. Therefore in order to extend the solution (f, u)

up to $[0, T]$ we only need to prove that the bounds from Proposition 7.1.6 can be extended up to $[0, T]$. We multiply equation (7.1)₁ by v^2 and integrate to obtain

$$\begin{aligned} 0 &= \frac{d}{dt} M_2 f_\epsilon + \int_{\mathbb{T}^3 \times \mathbb{R}^3} v^2 v \cdot \nabla f_\epsilon dx dv + \int_{\mathbb{T}^3 \times \mathbb{R}^3} v^2 \operatorname{div}_v [F_a(f_\epsilon) f_\epsilon + (\theta_\epsilon * u_\epsilon - v) f_\epsilon] dx dv \\ &= \frac{d}{dt} M_2 f_\epsilon - 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} v F_a(f_\epsilon) f_\epsilon dx dv - 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} v (\theta_\epsilon * u_\epsilon - v) f_\epsilon dx dv \end{aligned}$$

and since by substituting x with y and v with w (just like when we were estimating \mathcal{S}_1 in the proof of Proposition 7.1.6) we have

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} v F_a(f_\epsilon) f_\epsilon dx dv = - \int_{\mathbb{T}^3 \times \mathbb{R}^3} |w - v|^2 \psi(|x - y|) f_\epsilon dy dw dx dv \leq 0,$$

which implies the inequality

$$\frac{d}{dt} M_2 f_\epsilon \leq 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} v (u_\epsilon - v) f_\epsilon dx dv + 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} v (\theta_\epsilon * u_\epsilon - u_\epsilon) f_\epsilon dx dv. \quad (7.29)$$

Next we test the weak formulation for u_ϵ with u_ϵ to get

$$\begin{aligned} \frac{1}{3} \frac{d}{dt} \|u_\epsilon\|_2^2 + \frac{2}{3} c_3 \kappa \|\nabla u_\epsilon\|_p^p &\leq -2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} u_\epsilon (u_\epsilon - v) f_\epsilon dx dv \\ &\quad + 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} |u_\epsilon|^2 (1 - \gamma_\epsilon) f_\epsilon dx dv - 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} u_\epsilon \cdot v (1 - \gamma_\epsilon) f_\epsilon dx dv. \end{aligned} \quad (7.30)$$

We add (7.29) and (7.30) obtaining

$$\frac{d}{dt} \left(M_2 f_\epsilon + \frac{1}{3} \|u_\epsilon\|_2^2 \right) + \frac{2}{3} c_3 \kappa \|\nabla u_\epsilon\|_p^p + 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} |u_\epsilon - v|^2 f_\epsilon dx dv \leq \mathcal{J},$$

where

$$\begin{aligned} \mathcal{J} &:= 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} v (\theta_\epsilon * u_\epsilon - u_\epsilon) f_\epsilon dx dv - 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} |u_\epsilon|^2 (1 - \gamma_\epsilon) f_\epsilon dx dv \\ &\quad - 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} u_\epsilon \cdot v (1 - \gamma_\epsilon) f_\epsilon dx dv =: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 \end{aligned}$$

still require estimating. Regarding \mathcal{J}_2 for $h_\epsilon := (1 - \gamma_\epsilon) f_\epsilon$ we have

$$0 \leq \mathcal{J}_2 \stackrel{H(3)}{\leq} \|m_0 h_\epsilon\|_{\frac{3}{2}} \|u_\epsilon\|_6^2.$$

By Lemma B.1.3

$$\|m_0 h_\epsilon\|_{\frac{3}{2}} \leq C (M_{\frac{3}{2}} h_\epsilon)^{\frac{2}{3}},$$

where

$$\begin{aligned} M_{\frac{3}{2}} h_\epsilon &= \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^{\frac{3}{2}} f_\epsilon (1 - \gamma_\epsilon) \\ &\leq \int_{\mathbb{T}^3 \times \{|v| \geq \frac{1}{2\epsilon}\}} |v|^{\frac{3}{2}} f_\epsilon \\ &\leq \sqrt{2\epsilon} M_2 f_\epsilon. \end{aligned}$$

Therefore, since u_ϵ is uniformly bounded in $L^\infty(0, T^*; W^{1,2}(\mathbb{T}^3)) \hookrightarrow L^\infty(0, T^*; L^6(\mathbb{T}^3))$, we have

$$0 \leq \mathcal{J}_2 \stackrel{H(3)}{\leq} (2\epsilon)^{\frac{1}{3}} (M_2 f_\epsilon)^{\frac{2}{3}} \stackrel{Y(3)}{\leq} 2\epsilon + M_2 f_\epsilon. \quad (7.31)$$

Similarly

$$0 \leq |\mathcal{J}_3| \stackrel{H(6)}{\leq} \|m_1 h_\epsilon\|_{\frac{6}{5}} \|u_\epsilon\|_6^2$$

and again by Lemma B.1.3

$$\|m_1 h_\epsilon\|_{\frac{6}{5}} \leq C(M_{\frac{9}{5}} h_\epsilon)^{\frac{5}{6}}.$$

Moreover

$$\begin{aligned} M_{\frac{9}{5}} h_\epsilon &= \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^{\frac{9}{5}} f_\epsilon (1 - \gamma_\epsilon) \\ &\leq \int_{\mathbb{T}^3 \times \{|v| \geq \frac{1}{2\epsilon}\}} |v|^{\frac{9}{5}} f_\epsilon \\ &\leq (2\epsilon)^{\frac{1}{5}} M_2 f_\epsilon \end{aligned}$$

and thus like in the case of \mathcal{J}_2 we obtain

$$0 \leq \mathcal{J}_3 \leq 2\epsilon + M_2 f_\epsilon. \quad (7.32)$$

Finally to estimate \mathcal{J}_1 for all $\mu > 0$ we consider

$$\mathcal{J}_1 = 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} v(\theta_\epsilon * u_\epsilon - u_\epsilon) f_\epsilon (1 - \gamma_\mu) dx dv + 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} v(\theta_\epsilon * u_\epsilon - u_\epsilon) f_\epsilon \gamma_\mu dx dv =: \mathcal{J}_{1,1} + \mathcal{J}_{1,2}.$$

To estimate $\mathcal{J}_{1,1}$ we simply note that for $h_{\mu,\epsilon} := (1 - \gamma_\mu) f_\epsilon$, we have

$$\begin{aligned} \mathcal{J}_{1,1} &= 2 \int_{\mathbb{T}^3} (\theta_\epsilon * u_\epsilon - u_\epsilon) m_1 h_{\mu,\epsilon} dx \\ &\stackrel{H(6)}{\leq} 2 \|\theta_\epsilon * u_\epsilon - u_\epsilon\|_6 \|m_1 h_{\mu,\epsilon}\|_{\frac{6}{5}}. \end{aligned}$$

Lemma B.1.3 implies that

$$\|m_1 h_{\mu,\epsilon}\|_{\frac{6}{5}} \leq C \left(M_{\frac{9}{5}} h_{\mu,\epsilon} \right)^{\frac{5}{6}}$$

and

$$\begin{aligned} M_{\frac{9}{5}} h_{\mu,\epsilon} &= \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^{\frac{9}{5}} f_\epsilon (1 - \gamma_\mu) dx dv \\ &\leq \int_{\mathbb{T}^3 \times \{|v| \geq \frac{1}{2}\mu\}} |v|^{\frac{9}{5}} f_\epsilon dx dv \\ &\leq C \mu^{\frac{1}{5}} M_2 f_\epsilon. \end{aligned}$$

Therefore, as before, since u_ϵ is uniformly bounded in $L^\infty(0, T^*; W^{1,2}(\mathbb{T}^3)) \hookrightarrow L^\infty(0, T^*; L^6(\mathbb{T}^3))$, we have

$$\mathcal{J}_{1,1} \leq C\mu^{\frac{1}{6}} (M_2 f_\epsilon)^{\frac{5}{6}} \stackrel{Y(6)}{\leq} C\mu + CM_2 f_\epsilon. \quad (7.33)$$

On the other hand f_ϵ is uniformly bounded in $L^\infty([0, T^*] \times \mathbb{T}^3 \times \mathbb{R}^3)$, and thus

$$\mathcal{J}_{1,2} = \mathcal{J}_{1,2}(t) \leq C(\mu) \|\theta_\epsilon * u_\epsilon - u_\epsilon\|_2 \xrightarrow{\epsilon \rightarrow 0} 0 \quad (7.34)$$

for a.a $t \in [0, T^*]$. From (7.31)-(7.34) we conclude that after converging⁴ with $\epsilon \rightarrow 0$, we have

$$\frac{d}{dt} \left(M_2 f + \frac{1}{3} \|u\|_2^2 \right) + \frac{2}{3} c_3 \kappa \|\nabla u\|_p^p + 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} |u - v|^2 f dx dv \leq C\mu + CM_2 f_\epsilon$$

for a.a. $t \in [0, T^*]$. Therefore by Gronwall's lemma for a.a $t \in [0, T^*]$ it holds

$$M_2 f(t) + \frac{1}{3} \|u(t)\|_2^2 + \frac{2}{3} c_3 \kappa \int_0^t \|\nabla u(s)\|_p^p ds \leq e^{Ct} \left(M_2 f_0 + \frac{1}{3} \|u_0\|_2^2 + \mu t \right)$$

and by arbitrariness of $\mu > 0$

$$M_2 f(t) + \frac{1}{3} \|u(t)\|_2^2 + \frac{2}{3} c_3 \kappa \int_0^t \|\nabla u(s)\|_p^p ds \leq e^{CT} \left(M_2 f_0 + \frac{1}{3} \|u_0\|_2^2 \right). \quad (7.35)$$

This is indeed a better estimate than the one from Proposition 7.1.6 and it will suffice to extend the estimation to $[0, T]$. Assume that \mathcal{K} is such constant that for r from the proof of Proposition 7.1.6, we have

$$\begin{aligned} \mathcal{L} := & \left[(M_5 f_0)^{\frac{1}{8}} + \frac{T^{\frac{p-1}{p}}}{8} \left(\frac{3e^{CT}}{2c_3 \kappa} M_2 f_0 + \frac{e^{CT}}{2c_3 \kappa} \|u_0\|_2^2 \right)^{\frac{1}{p}} + \frac{T}{8} \left(3e^{CT} M_2 f_0 + e^{CT} \|u_0\|_2^2 \right)^{\frac{1}{2}} \right]^8 \\ & + 6M_2 f_0 + 2r(0) + \tilde{C}T \leq \mathcal{K}. \end{aligned} \quad (7.36)$$

Such constant exists since \mathcal{L} is a fixed number. Assume further that $h \in (0, T^*]$ is such that

$$\mathcal{K} < \left[(q_1 - 1) \tilde{C}h \right]^{\frac{1}{1-q_1}},$$

where q_1 and \tilde{C} are constants defined in step 4 of the proof of Proposition 7.1.6. Note that in order to use Lemma B.2.1 it was necessary (and sufficient) to have inequality (7.26) that served as an upper bound on T^* . However now, since $\tilde{C}T + r(0) \leq \mathcal{K} \leq \left[(q_1 - 1) \tilde{C}h \right]^{\frac{1}{1-q_1}}$, inequality (7.27) and all bounds from Corollary 7.1.1 hold on $[0, h]$ and as a consequence also inequality (7.35) holds on $[0, h]$ and implies that

$$\begin{aligned} \int_0^t \|\nabla u(s)\|_p ds & \leq T^{\frac{p-1}{p}} \left(\int_0^t \|\nabla u(s)\|_p^p ds \right)^{\frac{1}{p}} \\ & \leq T^{\frac{p-1}{p}} \left(\frac{3e^{CT}}{2c_3 \kappa} M_2 f_0 + \frac{e^{CT}}{2c_3 \kappa} \|u_0\|_2^2 \right)^{\frac{1}{p}} \end{aligned}$$

⁴Which we may already do in $[0, T^*]$.

and

$$\|u(t)\|_2^2 \leq 3e^{CT} M_2 f_0 + e^{CT} \|u_0\|_2^2 \quad (7.37)$$

for a.a. $t \in [0, h]$. Applying the above inequalities to (7.28) we get

$$M_5 f(t) \leq \left[(M_5 f_0)^{\frac{1}{8}} + \frac{T^{\frac{p-1}{p}}}{8} \left(\frac{3e^{CT}}{2c_3\kappa} M_2 f_0 + \frac{e^{CT}}{2c_3\kappa} \|u_0\|_2^2 \right)^{\frac{1}{p}} + \frac{T}{8} \left(3e^{CT} M_2 f_0 + e^{CT} \|u_0\|_2^2 \right)^{\frac{1}{2}} \right]^8 \quad (7.38)$$

on $[0, h]$, which together with (7.37), (7.36) and the definition of r implies that

$$\tilde{C}T + r(h) \leq \mathcal{L} \leq \mathcal{K}$$

and we can extend inequality (7.27) and all other estimations from Corollary 7.1.1 up to $[0, 2h]$. Then again (7.35) and (7.28) hold on $[0, 2h]$, which as before implies that (7.38) also holds on $[0, 2h]$ and consequently

$$\tilde{C}T + r(2h) \leq \mathcal{K}$$

and by repeating this procedure we may extend our estimates indefinitely up to $[0, T]$. Therefore the solution (f, u) may be extended to $[0, T]$ and it satisfies all the bounds from Proposition 7.1.6 and Corollary 7.1.1 on $[0, T]$. Hence the proof of existence part of Theorem 7.20 is concluded. \square

7.2 Uniqueness of solutions

In the previous section we proved the existence part of Theorem 6.2.1. In this section we prove the uniqueness of the solutions.

Proof of Theorem 6.2.1 – uniqueness. To prove uniqueness let us denote

$$\begin{aligned} \omega &:= u^1 - u^2, \\ g &:= f^1 - f^2, \end{aligned}$$

where (f_1, u_1) and (f_2, u_2) are two supposedly different solutions to (6.1) subjected to the same initial data (f_0, u_0) .

First we deal with the fluid part of the problem. We subtract weak formulation of u_2 from the weak formulation of u_1 obtaining the weak formulation for ω , which then, we test with $\psi = \omega$ (which is a good test function by Definition 6.1.1), thus obtaining

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\omega\|_2^2 + c_3\kappa \|\nabla\omega\|_2^2 &\leq 3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} |\omega|^2 f^1 dx dv + 3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \omega |u^2 - v| |f^1 - f^2| dx dv \\ &\quad + \int_{\mathbb{T}^3} |\omega|^2 |\nabla u^1| dx =: I_1 + II_1 + III_1, \end{aligned} \quad (7.39)$$

where we used (6.8) and (B.4) to deal with the stress term. We estimate I_1 , II_1 and III_1 separately. By Lemma B.1.3, we have

$$I_1 = \int_{\mathbb{T}^3} |\omega|^2 m_0 f^1 dx \leq \int_{\mathbb{T}^3} |\omega|^2 (m_3 f^1)^{\frac{1}{2}} dx \leq \|\omega\|_4^2 (M_3 f^1)^{\frac{1}{2}}.$$

Since $M_3 f^1 \leq C$, it follows that

$$I_1 \stackrel{(B.1)}{\leq} C \|\omega\|_2^{\frac{1}{2}} \|\omega\|_6^{\frac{3}{2}} \stackrel{(B.2)}{\leq} C \|\omega\|_2^{\frac{1}{2}} \|\nabla \omega\|_2^{\frac{3}{2}} + \|\omega\|_2^2 \stackrel{Y(4)}{\leq} C(\delta) \|\omega\|_2^2 + \delta \|\nabla \omega\|_2^2. \quad (7.40)$$

Furthermore

$$\begin{aligned} II_1 &\leq \int_{\mathbb{T}^3} |\omega| |u^2| |m_0| |g| dx + \int_{\mathbb{T}^3} |\omega| |m_1| |g| dx dv \\ &\leq \frac{1}{2} (\|u^2\|_\infty^2 + 1) \|\omega\|_2^2 + \frac{1}{2} \|m_0|g\|_2^2 + \frac{1}{2} \|m_1|g\|_2^2 \end{aligned}$$

and by the fact that by Definition 6.1.1.1 $\text{supp} f(x, \cdot, t) \subset B(\mathcal{R}(T))$

$$\|m_\alpha|g\|_2^2 = \int_{\mathbb{T}^3} \left(\int_{\mathbb{R}^3} |v|^\alpha |g| dv \right)^2 dx \leq C \mathcal{R}(T)^{2\alpha+6} \|g\|_2^2,$$

which implies that

$$II_1 \leq \frac{1}{2} (\|u^2\|_\infty^2 + 1) \|\omega\|_2^2 + C \|g\|_2^2 \quad (7.41)$$

Finally we estimate III_1 :

$$III_1 \stackrel{H(2)}{\leq} \|\omega\|_4^2 \|\nabla u^1\|_2 \stackrel{(B.1)(B.2)}{\leq} C \|\omega\|_2^{\frac{1}{2}} \|\nabla \omega\|_2^{\frac{3}{2}} + C \|\omega\|_2^2 \stackrel{Y(4)}{\leq} C(\delta) \|\omega\|_2^2 + \delta \|\nabla \omega\|_2^2, \quad (7.42)$$

where we used the fact that $u^1 \in L^\infty(0, T; W^{1,2}(\mathbb{T}^3))$ to get rid of $\|\nabla u^1\|_2$. Inequality (7.39) and estimations (7.40)-(7.42) imply that

$$\frac{1}{2} \frac{d}{dt} \|\omega\|_2^2 + \frac{1}{2} c_3 \kappa \|\nabla \omega\|_2^2 \leq C (\|u^2\|_\infty^2 + 1) \|\omega\|_2^2 + C \|g\|_2^2. \quad (7.43)$$

This finishes the estimations of the fluid part of the system.

Next we estimate the particle part in $L^\infty(0, T; L^2(\mathbb{T}^3 \times \mathbb{R}^3))$. We subtract the weak formulation for f^2 from the weak formulation for f^1 obtaining the weak formulation for g , which then we test with g^5 to get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|g\|_2^2 &= \int_{\mathbb{T}^3 \times \mathbb{R}^3} \left[F_a(g) f^2 + F_a(f^1) g + \omega f^1 + (u^2 - v) g \right] \nabla_v g dx dv \\ &=: I_2 + II_2 + III_2 + IV_2. \end{aligned} \quad (7.44)$$

Next we estimate I_2, II_2, III_2, IV_2 . We have

$$I_2 = \int_{\mathbb{T}^3 \times \mathbb{R}^3} F_a(g) f^2 \nabla_v g dx dv = - \int_{\mathbb{T}^3 \times \mathbb{R}^3} \text{div}_v F_a(g) f^2 g dx dv - \int_{\mathbb{T}^3 \times \mathbb{R}^3} F(g) \nabla_v f^2 g dx dv =: I_{21} + I_{22},$$

for

$$\begin{aligned} |I_{21}| &\leq \|f^2\|_2 \|F(g)\|_\infty \|g\|_2 \stackrel{(6.12)}{\leq} C \|g\|_2^2, \\ |I_{22}| &\leq \|\nabla_v f^2\|_2 \|F(g)\|_\infty \|g\|_2 \stackrel{(6.12)}{\leq} C \|g\|_2^2. \end{aligned}$$

⁵Let us note that thanks to the effort we put into estimations of $\nabla_v f$ and ∇f we made sure that f is a good test function in point 3 of Definition 6.1.1.

Furthermore

$$II_2 = \int_{\mathbb{T}^3 \times \mathbb{R}^3} F_a(f^1) \nabla_v (g^2) dx dv = - \int_{\mathbb{T}^3 \times \mathbb{R}^3} \operatorname{div}_v F_a(f^1) |g|^2 dx dv,$$

thus by (6.11)

$$|II_2| \leq C \|g\|_2^2.$$

Moreover

$$III_2 = \int_{\mathbb{T}^3 \times \mathbb{R}^3} \omega f^1 \nabla_v g dx dv = - \int_{\mathbb{T}^3 \times \mathbb{R}^3} \omega \nabla_v f^1 g dx dv,$$

thus

$$\begin{aligned} |III_2| &\stackrel{H(2)}{\leq} \|g\|_\infty \|\nabla_v f^1\|_2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \omega^2 g dx dv \\ &\leq \|g\|_\infty \|\nabla_v f^1\|_2 \left(\frac{1}{2} \|\omega\|_4^2 + \frac{1}{2} \|m_0 |g|\|_2^2 \right) \end{aligned}$$

and by (B.1), (B.2) and the fact that $f^1 \in \mathcal{X}$, using Young's inequality with arbitrary constant δ , we get

$$III_2 \leq \delta \|\nabla \omega\|_2^2 + C(\delta) \|\omega\|_2^2 + C \|g\|_2^2.$$

For IV_2 , we integrate by parts obtaining

$$IV_2 = \int_{\mathbb{T}^3 \times \mathbb{R}^3} (u^2 - v) \nabla_v (g^2) dx dv = 3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} |g^2| dx dv.$$

Finally we combine (7.43), (7.44) with the estimations of $I_2 - IV_2$ to obtain

$$\frac{1}{2} \frac{d}{dt} (\|\omega\|_2^2 + \|g\|_2^2) + \frac{1}{4} c_3 \kappa \|\nabla \omega\|_2^2 \leq C (\|u^2\|_\infty^2 + 1) \|\omega\|_2^2 + C \|g\|_2^2.$$

We aim to use Gronwall's lemma to conclude that $\omega = 0$ and $g = 0$, which means that the solution is unique. To do it we need to ensure that $t \mapsto \|u^2\|_\infty^2$ is integrable. However by Definition 6.1.1 $u^2 \in \mathcal{H}$ and thus $u \in L^2(0, T; W^{2,2}(\mathbb{T}^3)) \hookrightarrow L^2(0, T; L^\infty(\mathbb{T}^3))$, which implies the integrability of $t \mapsto \|u^2\|_\infty^2$. \square

Part III
Appendices

Appendix A

In Appendix A we present various lemmas and theorems applied throughout Part I. We also present proofs of other lemmas.

A.1 Weak-strong convergence

The following lemma plays a role in Chapter 3.

Lemma A.1.1. *Let $\Omega \subset \mathbb{R}^d$ and $f_n, f, g_n, g : \Omega \rightarrow \mathbb{R}$ be measurable functions. If $f_n \rightarrow f$ a.e. in Ω , f_n is uniformly bounded in $L^\infty(\Omega)$ and $g_n \rightarrow g$ in $L^1(\Omega)$, then*

$$f_n g_n \rightarrow f g \text{ in } L^1(\Omega).$$

A.2 Proofs of technical lemmas from Chapter 4

Proof of Lemma 4.1.1 We have

$$\int_0^T R(t) dt = \sum_{k=0}^{N_s} \int_{T_k}^{T_{k+1}} R(t) dt,$$

with T_k and N_s from Definition 2.1.2. By Corollary 4.0.1.1, the function

$$r(t) := \sum_{i,j=1}^N (v_i(t) - v_j(t))^2$$

is absolutely continuous on each interval $[T_k, T_{k+1} - \epsilon]$ with arbitrarily small $\epsilon > 0$. Then, by (2.3)₂ on each such interval, we have

$$\begin{aligned} \frac{d}{dt} r &= 2 \sum_{i,j=1}^N (v_i - v_j) \left(\frac{1}{N} \sum_{k=1}^N (v_k - v_i) \psi(|x_i - x_k|) - \frac{1}{N} \sum_{k=1}^N (v_k - v_j) \psi(|x_j - x_k|) \right) \\ &= \frac{2}{N} \sum_{i,j,k=1}^N (v_i - v_j)(v_k - v_i) \psi(|x_i - x_k|) - \frac{2}{N} \sum_{i,j,k=1}^N (v_i - v_j)(v_k - v_j) \psi(|x_j - x_k|). \end{aligned}$$

We substitute i and k in the first summand and j and k in the second summand to obtain

$$\begin{aligned}
\frac{d}{dt}r &= \frac{1}{N} \sum_{i,j,k=1}^N (v_i - v_j)(v_k - v_i)\psi(|x_i - x_k|) + \frac{1}{N} \sum_{i,j,k=1}^N (v_k - v_j)(v_i - v_k)\psi(|x_i - x_k|) \\
&\quad - \frac{1}{N} \sum_{i,j,k=1}^N (v_i - v_j)(v_k - v_j)\psi(|x_j - x_k|) - \frac{1}{N} \sum_{i,j,k=1}^N (v_i - v_k)(v_j - v_k)\psi(|x_j - x_k|) \\
&= -\frac{1}{N} \sum_{i,j,k=1}^N (v_i - v_k)^2\psi(|x_i - x_k|) - \frac{1}{N} \sum_{i,j,k=1}^N (v_j - v_k)^2\psi(|x_j - x_k|) \\
&= -2 \sum_{i,j=1}^N (v_i - v_j)^2\psi(|x_i - x_j|) = -2R.
\end{aligned}$$

Therefore

$$\int_{T_k}^{T_{k+1}-\epsilon} R dt = \frac{1}{2} (r(T_k) - r(T_{k+1} - \epsilon))$$

and thus, by the monotone convergence theorem and continuity of r , we pass to the limit with $\epsilon \rightarrow 0$ obtaining

$$\int_{T_k}^{T_{k+1}} R dt = \frac{1}{2} (r(T_k) - r(T_{k+1})). \tag{A.1}$$

Finally, we take a sum over all $k = 0, \dots, N_s$ of the equations of the form (A.1) to get

$$\int_0^T R(t) dt = \frac{1}{2} (r(0) - r(T)) \leq C_1 N^2,$$

where the final estimation is justified by Corollary 4.0.1.3. \square

Proof of Lemma 4.1.2 Given $i, j = 1, \dots, N$, we have

$$\int_{s_1}^{s_2} |x_j - x_i|^{-\theta} dt = \sum_k \int_{t_{k-1}}^{t_k} |x_j - x_i|^{-\theta} dt, \tag{A.2}$$

where t_k denote the times of collision of x_j and x_i that happen in the time interval $[s_1, s_2]$. By Corollary 4.0.1.3, the only density points of the times of collision are times of sticking and since there are no times of sticking in $[s_1, s_2]$ – the sum on the right-hand side of (A.2) is finite. Thus it is sufficient to show that each summand is finite (even if it is arbitrarily large), hence from this point we fix k . Now, if the particles do not stick together in $[s_1, s_2]$, then for $t \in [s_1, s_2]$ either $x_i(t) \neq x_j(t)$ or $v_i(t) \neq v_j(t)$. In particular $v_j(t_{k-1}) - v_i(t_{k-1}) =: v_{k-1} \neq 0$ and $v_j(t_k) - v_i(t_k) =: v_k \neq 0$ and by continuity of v (see Corollary 4.0.1), there exist positive ρ and δ , such that

$$\begin{aligned}
v_j - v_i &\in B(v_{k-1}, \rho) \quad \text{in } [t_{k-1}, t_{k-1} + \delta] \quad \text{and} \\
v_j - v_i &\in B(v_k, \rho) \quad \text{in } [t_k - \delta, t_k]
\end{aligned}$$

and 0 does not belong to neither $B(v_{k-1}, \rho)$ nor $B(v_k, \rho)$. Let us split the integral from the right-hand side of (A.2) in the following manner:

$$\int_{t_{k-1}}^{t_k} |x_j - x_i|^{-\theta} dt = \left(\int_{t_{k-1}}^{t_{k-1}+\delta} + \int_{t_{k-1}+\delta}^{t_{k-1}-\delta} + \int_{t_{k-1}-\delta}^{t_k} \right) |x_j - x_i|^{-\theta} dt =: I + II + III.$$

Then there exists an arbitrarily large constant $C(\delta)$, that bounds II from the above since $|x_j - x_i|$ is continuous and nonzero on $[t_{k-1} + \delta, t_k - \delta]$. To estimate I we notice that for $t \in [t_{k-1}, t_{k-1} + \delta]$ it holds:

$$|x_j(t) - x_i(t)| = \left| \int_{t_{k-1}}^t v_j - v_i ds \right| \geq \inf_{\xi \in B(v_{k-1}, \rho)} |\xi| (t - t_{k-1}) \geq c(t - t_{k-1})$$

for some small constant $c > 0$. Thus

$$\int_{t_{k-1}}^{t_{k-1}+\delta} |x_j - x_i|^{-\theta} dt \leq c^{-\theta} \int_{t_{k-1}}^{t_{k-1}+\delta} (t - t_{k-1})^{-\theta} dt < \infty,$$

since $\theta < 1$. Estimation of III proceeds similarly to the estimation of I . \square

Proof of Lemma 4.1.4 The proof follows similarly to that of step 2. Let

$$r(t) := \sum_{i,j \in [i]} (v_i(t) - v_j(t))^2,$$

where $[i]$ denotes the set of those j that $x_j(t_0) = x_i(t_0)$ and $v_j(t_0) = v_i(t_0)$. Therefore if we show that $r \equiv 0$ then the thesis of Lemma 4.1.4 will be satisfied. We have

$$\begin{aligned} \frac{d}{dt} r_{inc} &\leq \frac{2}{N} \sum_{i,j \in [i]} \sum_{k=1}^N \left[(v_i - v_j) \left((v_k - v_i) \psi(|x_k - x_i|) - (v_k - v_j) \psi(|x_k - x_j|) \right) \right]_+ \\ &\leq \frac{2}{N} \left(\sum_{i,j,k \in [i]} + \sum_{\substack{i,j \in [i] \\ k \notin [i]}} \right) [-as\ above -]_+ \\ &\stackrel{see\ below}{\leq} \frac{2}{N} \sum_{\substack{i,j \in [i] \\ k \notin [i]}} \left[(v_i - v_j) \left((v_k - v_i) \psi(|x_k - x_i|) - (v_k - v_j) \psi(|x_k - x_j|) \right) \right]_+ \quad (A.3) \\ &\leq \frac{2}{N} \sum_{\substack{i,j \in [i] \\ k \notin [i]}} \left[-(v_i - v_j)^2 \psi(|x_k - x_i|) \right]_+ \\ &\quad + \frac{2}{N} \sum_{\substack{i,j \in [i] \\ k \notin [i]}} \left[(v_i - v_j)(v_k - v_j) (\psi(|x_k - x_i|) - \psi(|x_k - x_j|)) \right]_+ \\ &\leq \frac{C}{N} \sum_{\substack{i,j \in [i] \\ k \notin [i]}} |v_i - v_j| |\psi(|x_k - x_i|) - \psi(|x_k - x_j|)|. \end{aligned}$$

Inequality (A.3) follows by the fact that in the triple sum over the set $[i]$ the indexes may be substituted in the same fashion as in the proof of Lemma 4.1.1. Since at this point $k \notin [i]$ and $i, j \in [i]$, the k th

particle cannot stick to neither i th nor j th particle (even though it may collide with them). We are already familiar with this situation and thus, we estimate

$$|\psi(|x_k - x_i|) - \psi(|x_k - x_j|)|$$

similarly to the estimations from Case 1 and Case 2 in the previous step obtaining altogether

$$\frac{d}{dt}r_{inc} \leq Cr_{inc} \cdot f$$

for some integrable function f . Then by Gronwall's lemma $r_{inc} \equiv 0$ on $[0, \delta]$, which means that also $r \equiv 0$ on $[0, \delta]$ and that for all $i, j \in [i]$ we have $x_i \equiv x_j$ on $[0, \delta]$. \square

A.3 Proof of Proposition 5.0.3

We present the proof of Proposition 5.0.3 from Chapter 5. It is similar and yet quite different than the proof of Proposition 4.1.1 from Chapter 4. In Proposition 4.1.1 we prove absolute continuity of solution to the C-S model with singular weight for $\alpha \in (0, \frac{1}{2})$. On the other hand in Proposition 5.0.3 the most important part of the proof revolves around showing the uniform absolute continuity of a sequence of approximate solutions that are supposed to converge to a solution associated with a singular weight with $\alpha \in (0, \frac{1}{2})$. These two conjectures are very similar and in fact the first part of the proof of Proposition 5.0.3 could be almost exactly the same as the proof of Proposition 4.1.1. In other words we could use Lemmas 4.1.1, 4.1.2 and 4.1.3 in a more convoluted proof of Proposition 4.1.1. Alternatively we could also prove Proposition 4.1.1 in a much simpler way using the approach of Proposition 5.0.3. Ultimately we decided to include both proofs to illustrate how the first attempt to solve a mathematical problem can differ from the subsequent more refined attempts.

Proof of Proposition 5.0.3. The existence and uniqueness part as well as points (i) and (ii) are no different than in the case of regular weight and we will not prove them here. Their proofs can be found in the literature (see for instance [31] or [43]). Thus it remains to prove (iii)-(v).

(iii) – (v)

First, assuming for notational simplicity that $(x^n, v^n, N_n, m_i^n) = (x, v, N, m_i)$ let us prove a particularly useful estimate. Let $1 < p < q$ be given numbers satisfying additional conditions that will be specified later. For each $n = 1, 2, \dots$, velocity v^n (denoted by v) is absolutely continuous on $[0, T]$ and thus by (2.3)₂, we have

$$\begin{aligned}
m_i \int_0^T |\dot{v}_i|^p dt &= m_i \int_0^T \left| \sum_{j=1}^N m_j (v_j - v_i) \psi_n(|x_i - x_j|) \right|^p dt \\
&\leq \sum_{j=1}^N m_i m_j \int_0^T |v_j - v_i|^p \psi_n^p(|x_i - x_j|) dt \\
&= \sum_{j=1}^N \int_0^T (m_i m_j)^{\frac{p}{q}} |v_j - v_i|^{\frac{p}{q}} \psi_n^{\frac{p}{q}}(|x_i - x_j|) \cdot (m_i m_j |v_j - v_i|^p)^{(1-\frac{p}{q})} dt \\
&\leq \sum_{j=1}^N m_i m_j \int_0^T |v_j - v_i|^p \psi_n^q(|x_i - x_j|) dt + \sum_{j=1}^N m_i m_j \int_0^T |v_j - v_i|^p dt \quad (\text{A.4}) \\
&\leq \epsilon \sum_{j=1}^N m_i m_j \underbrace{\int_0^T |v_j - v_i|^2 \psi_n^{\frac{2q}{p}}(|x_i - x_j|) dt}_{=:A} + C(\epsilon) T m_i + \sum_{j=1}^N m_i m_j \int_0^T |v_j - v_i|^p dt. \quad (\text{A.5})
\end{aligned}$$

Inequality (A.4) is obtained by Young's inequality with exponent $\frac{q}{p}$ while (A.5) follows by Young's inequality with exponent $\frac{2}{p}$. In both of the above inequalities we also use the assumption that $\sum_{i=1}^N m_i = 1$.

Furthermore recalling that $\psi_n^{\frac{2q}{p}}(s) \leq \psi_n^{\frac{2q}{p}}(s) = |s|^{-\lambda}$, where $\lambda := \frac{2q\alpha}{p}$, integral A can be estimated as follows:

$$\begin{aligned}
A &\leq \sum_{k=1}^d \int_0^T (v_j^k - v_i^k) \cdot (v_j^k - v_i^k) |x_i^k - x_j^k|^{-\lambda} dt = \sum_{k=1}^d \int_0^T (v_j^k - v_i^k) \cdot ((x_j^k - x_i^k) |x_i^k - x_j^k|^{-\lambda})' dt \\
&= - \sum_{k=1}^d \int_0^T (\dot{v}_j^k - \dot{v}_i^k) \cdot (x_j^k - x_i^k) |x_i^k - x_j^k|^{-\lambda} dt + \sum_{k=1}^d (v_j^k - v_i^k) \cdot (x_j^k - x_i^k) |x_i^k - x_j^k|^{-\lambda} \Big|_0^T \\
&\leq C \int_0^T |\dot{v}_i| |x_i - x_j|^{1-\lambda} dt + C \int_0^T |\dot{v}_j| |x_i - x_j|^{1-\lambda} dt + 2C \sup_{t \in [0, T]} |v_j - v_i| |x_i - x_j|^{1-\lambda}.
\end{aligned}$$

However, the above estimation is valid only if $\lambda < 1$, which means that $\frac{q}{p} \cdot 2\alpha < 1$ and such condition can be easily satisfied if $\alpha < \frac{1}{2}$ and $1 < p < q$ are small enough. By point (ii) we have $|v| \leq \mathcal{R}$ and $|x| \leq \mathcal{R}$. This leads to the concluding estimation of A , which reads:

$$A \leq C(\mathcal{R})^{1-\lambda} \int_0^T |\dot{v}_i| dt + C(\mathcal{R})^{1-\lambda} \int_0^T |\dot{v}_j| dt + C(\mathcal{R})^{2-\lambda}. \quad (\text{A.6})$$

Now we will apply the above calculation (particularly estimations (A.5) and (A.6)) in the effort to prove (iii) and (iv). For (iii) let us assume that $p = q = 1^1$. We sum (A.5) over $i = 1, \dots, N$ to get

$$\sum_{i=1}^N m_i \int_0^T |\dot{v}_i| dt \leq \epsilon \sum_{i,j=1}^N m_i m_j A + C(\epsilon) T + 2\mathcal{R} T$$

¹Note that (A.4) remains true also for $p = q = 1$.

and plug in (A.6) to obtain

$$\sum_{i=1}^N m_i \int_0^T |\dot{v}_i| dt \leq 2\epsilon C(\mathcal{R})^{1-\lambda} \sum_{i=1}^N m_i \int_0^T |\dot{v}_i| dt + \epsilon C(\mathcal{R})^{2-\lambda} + C(\epsilon)T + 2\mathcal{R}T,$$

which after fixing sufficiently small ϵ and rearranging yields

$$\sum_{i=1}^N m_i \int_0^T |\dot{v}_i| dt \leq C(\mathcal{R})^{2-\lambda} + CT + C\mathcal{R}T, \quad (\text{A.7})$$

which proves (iii) for $p = 1$. Then for $1 < p = q$ using (A.5), (A.6) and (A.7), we have

$$\sum_{i=1}^N m_i \int_0^T |\dot{v}_i|^p dt \leq 2C(\mathcal{R})^{1-\lambda} \sum_{i=1}^N m_i \int_0^T |\dot{v}_i| dt + C(\mathcal{R})^{2-\lambda} + CT + C\mathcal{R}^p T \leq C(\mathcal{R}, p, T, \lambda) \quad (\text{A.8})$$

and (iii) is proved for some sufficiently small $p > 1$. In order to prove (iv) we take $1 = p < q$ in (A.5), which leads us to a very similar result to (A.8) and to the end of the proof of (iv).

(v)

Let us fix $n = 1, 2, \dots$ and a bounded, Lipschitz continuous function $g = g(x, v)$. Then according to Definition 2.1.6, for $t \in [0, T]$, $\epsilon > 0$ and

$$\chi_{\epsilon,t}(s) := \begin{cases} 1 & \text{for } 0 \leq s \leq t - \epsilon \\ -\frac{1}{2\epsilon}(s - t - \epsilon) & \text{for } t - \epsilon < s \leq t + \epsilon \\ 0 & \text{for } t_\epsilon < s \end{cases}$$

the function $\phi(s, x, v) := \chi_{\epsilon,t}(s)g(x, v) \in \mathcal{G}$ is a good test function in the weak formulation for each f_n . Thus we plug ϕ into (2.6) obtaining

$$\begin{aligned} & \frac{1}{2\epsilon} \int_{t-\epsilon}^{t+\epsilon} \int_{\mathbb{R}^{2d}} f_n g dx dv dt = \\ & = - \int_0^T \int_{\mathbb{R}^{2d}} f_n \chi_{\epsilon,t} v \nabla g dx dv dt - \int_0^T \int_{\mathbb{R}^{2d}} F_n(f_n) f_n \chi_{\epsilon,t} \nabla_v g dx dv dt - \int_{\mathbb{R}^{2d}} f_0 g dx dv. \end{aligned}$$

Since $t \mapsto \int_{\mathbb{R}^{2d}} f_n g dx dv$, $t \mapsto \int_{\mathbb{R}^{2d}} f_n \chi_{\epsilon,t} v \nabla g dx dv$ and $t \mapsto \int_{\mathbb{R}^{2d}} F_n(f_n) f_n \chi_{\epsilon,t} \nabla_v g dx dv$ are integrable functions (for fixed n and g), then converging with $\epsilon \rightarrow 0$ leads to the following equation holding for a.a $t \in [0, T]$:

$$\begin{aligned} \int_{\mathbb{R}^{2d}} f_n(t) g dx dv dt &= \int_0^t \int_{\mathbb{R}^{2d}} f_n v \nabla g dx dv dt + \int_0^t \int_{\mathbb{R}^{2d}} F_n(f_n) f_n \nabla_v g dx dv dt - \int_{\mathbb{R}^{2d}} f_0 g dx dv \\ &= \int_0^t G(t) dt - \int_{\mathbb{R}^{2d}} f_0 g dx dv, \end{aligned}$$

where

$$\begin{aligned} G(t) &:= \int_{\mathbb{R}^{2d}} f_n(t) v \nabla g dx dv + \int_{\mathbb{R}^{2d}} F_n(f_n)(t) f_n(t) \nabla_v g dx dv \\ &= \sum_{i=1}^N m_i v_i^n(t) \nabla g(x_i^n(t), v_i^n(t)) + \sum_{i,j=1}^N m_i m_j (v_j^n(t) - v_i^n(t)) \psi(|x_i^n(t) - x_j^n(t)|) \nabla_v g(x_i^n(t), v_i^n(t)). \end{aligned}$$

By virtue of points (ii) and (iii) of this proposition, we have

$$\begin{aligned} \int_0^T |G(t)|^p dt &\leq \int_0^T \left| \sum_{i=1}^N m_i v_i^n(t) (\nabla g)(x_i^n(t), v_i^n(t)) \right|^p dt \\ &\quad + \int_0^T \left| \sum_{i,j=1}^N m_i m_j \psi_n(|x_i^n(t) - x_j^n(t)|) (v_j^n(t) - v_i^n(t)) (\nabla_v g)(x_i^n(t), v_i^n(t)) \right|^p dt \\ &\leq Lip(g)^p T(\mathcal{R})^p + Lip(g)^p M(\mathcal{R}) =: M_g(Lip(g), \mathcal{R}) \end{aligned}$$

which finishes the proof of (v). \square

A.4 Compactness

The main tools used in Chapter 5 to obtain compactness are Arzela–Ascoli Theorem and Banach–Alaoglu Theorem.

Theorem A.4.1 (Arzela–Ascoli Theorem). *Let $T > 0$ and let Y be a metric space. Assume further that $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of continuous mappings between $[0, T]$ and Y . Then if f_n are equicontinuous and pointwise relatively compact then $\{f_n\}_{n \in \mathbb{N}}$ has a uniformly convergent subsequence.*

Proof. Our version of Arzela–Ascoli theorem is a direct consequence of a more general version from [35], Chapter 7. \square

Remark A.4.1. Pointwise relative compactness means that for all $t \in [0, T]$ the sequence $\{f_n(t)\}_{n \in \mathbb{N}}$ is relatively compact in Y .

Theorem A.4.2 (Banach–Alaoglu Theorem). *Let X be a (separable) normed vector space. Then the closed ball in X^* is (sequentially) weakly $*$ compact.*

Remark A.4.2. In our applications (in the proof of Corollary 2.1.1) we take $X = C_b(\Omega)$ for compact Ω . Then X is a separable normed vector space and any sequence bounded in X^* belongs to some closed ball in X^* , which by Theorem A.4.2 implies that it has a weakly $*$ convergent subsequence.

Appendix B

We present the basic tools that we use throughout Part II.

B.1 Miscellaneous inequalities

Lemma B.1.1. For $u \in L^6(0, T; L^6(\mathbb{T}^3))$, we have

$$\|u\|_4 \leq C(\|u\|_6^3 \|u\|_2)^{\frac{1}{4}}. \quad (\text{B.1})$$

Lemma B.1.2. For $u \in L^p(0, T; \dot{W}_{div,0}^{1,p}(\mathbb{T}^3))$, we have

$$\|u\|_q \leq C(\|\nabla u\|_p + \|u\|_2) \quad (\text{B.2})$$

for all $q \leq p^* := \frac{dp}{d-p}$.

Proof. Sobolev's inequality implies that

$$\|u\|_q \leq C\|u\|_p + \|\nabla u\|_p,$$

while Poincaré's inequality implies that

$$\|u - \int u\|_p \leq C\|\nabla u\|_p$$

and thus

$$\|u\|_p \leq \|\int u\|_p + \|u - \int u\|_p \leq C(\|u\|_2 + \|\nabla u\|_p).$$

□

We present two crucial lemmas from [7].

Lemma B.1.3. Let $\beta > 0$ and g be a nonnegative function in $L^\infty([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$. The following estimate holds for any $\alpha < \beta$:

$$m_\alpha g(t, x) \leq \left(\frac{4}{3} \pi \|g(t, x, \cdot)\|_\infty + 1 \right) m_\beta g(t, x)^{\frac{\alpha+3}{\beta+3}},$$

for a.a. (t, x) .

Proof. The proof can be found in [7], page 9 (Lemma 1). □

Lemma B.1.4. For $T > 0$, let $\{a_n\}$ be a sequence of nonnegative continuous functions defined on $[0, T]$ satisfying the relation:

$$a_{n+1}(t) \leq A + B \int_0^t a_n(s) ds + C \int_0^t a_{n+1}(s) ds, 0 \leq t \leq T,$$

where A, B and C are nonnegative constants, Then there exists a positive constant K , such that for all $n \in \mathbb{N}$

$$a_n(t) \leq \begin{cases} \frac{K^n t^n}{n!}, & A = 0, \\ Ke^{Kt}, & A > 0. \end{cases}$$

Proof. The proof can be found in [7], page 15 (Lemma 3). □

B.2 Nonlinear Gronwall's lemma

We include the formulation of the classical Gronwall's lemma with it's less popular non-linear varieties.

Lemma B.2.1 (Gronwall's lemma). Let f be a nonnegative function satisfying inequality

$$f(t) \leq c + \int_{t_0}^t (a(s)f(s) + b(s)f^q(s)) ds, \quad c \geq 0, \quad q \geq 0,$$

where a and b are nonnegative, integrable functions for $t \geq t_0$. Then we have

for $0 \leq q < 1$

$$f(t) \leq \left[c^{1-q} e^{(1-q) \int_{t_0}^t a(s) ds} + (1-q) \int_{t_0}^t b(s) e^{(1-q) \int_s^t a(r) dr} ds \right]^{\frac{1}{1-q}};$$

for $q = 1$

$$f(t) \leq c e^{\int_{t_0}^t a(s) + b(s) ds};$$

for $q > 1$

$$f(t) \leq c \left[e^{(1-q) \int_{t_0}^t a(s) ds} - c^{-1}(q-1) \int_{t_0}^t b(s) e^{(1-q) \int_s^t a(r) dr} ds \right]^{\frac{1}{q-1}}$$

for $t \in [t_0, h]$ for $h > 0$ provided that

$$c < \left[e^{(1-q) \int_{t_0}^{t_0+h} a(s) ds} \right]^{\frac{1}{q-1}} \left[(q-1) \int_{t_0}^{t_0+h} b(s) ds \right]^{\frac{1}{1-q}}$$

B.3 Propagation of the support

Finally we present the proof of propagation of the support of f .

Lemma B.3.1 (Propagation of the support of velocity). *Let f be a solution to (7.4) subjected to the initial data with the support in v contained in the ball $B(R)$. Then there exists a nondecreasing function $\mathcal{R} : [0, T] \rightarrow [0, \infty)$ such that for all $t \in [0, T]$ and almost all $x \in \mathbb{T}^3$, the support of $f(t, x, \cdot) : \mathbb{R}^3 \rightarrow \mathbb{R}$ is contained in a ball of radius $\mathcal{R}(t)$. Moreover for each $t \in [0, T]$ the value $\mathcal{R}(t)$ depends only on t , $\|u\|_{L^2(0,t;W^{2,2}(\mathbb{T}^3))}$, $\|M_1 f\|_\infty$ and R .*

Proof. Let f be a solution to (7.4). Consider the solution of the system of ODE's:

$$\begin{cases} \frac{dx}{dt}(t) = v(t), & x(0) = x_0, \\ \frac{dv}{dt}(t) = F_a(f)(t, x(t), v(t)) + (\theta_\epsilon * u)(t, x(t)) - v(t), & v(0) = v_0. \end{cases} \quad (\text{B.3})$$

Then the function $\tilde{f}(t, x_0, v_0) := f(t, x(t), v(t))$ satisfies the equation

$$\partial_t \tilde{f} = (-\operatorname{div}_v F_a(f) + 3)\tilde{f} = 3(b+1)\tilde{f}$$

(recall b defined in (6.11)) and thus

$$\tilde{f}(t, x_0, v_0) = e^{3 \int_0^t (b+1) ds} f_0(x_0, v_0).$$

Therefore $\tilde{f}(t, x_0, v_0) = 0$ whenever $f_0(x_0, v_0) = 0$, which implies that $f(t, x, v) = 0$ whenever the characteristic that contains point (x, v) starts at (x_0, v_0) such that $f_0(x_0, v_0) = 0$. We solve (B.3)₂, to get

$$v(t) = e^{-\int_0^t b(s, x(s))+1 ds} \left(v_0 + \int_0^t e^{\int_s^t b(r, x(r))+1 dr} a(s, x(s)) + (\theta_\epsilon * u)(s, x(s)) ds \right),$$

which, since by (6.11) $1 \leq b+1 \leq c_6 M_0 f + 1$ and by (7.5) $M_0 f \leq C$, implies that

$$\begin{aligned} |v(t)| &\leq C e^{Ct} \left(|v_0| + \int_0^t |a(x(s), s)| ds + \int_0^t \|u(s)\|_\infty ds \right) \\ &\stackrel{(6.10)}{\leq} C e^{Ct} \left(|v_0| + t \|M_1 f\|_\infty + \|u\|_{L^2(0,t;W^{2,2}(\mathbb{T}^3))} \right) \\ &\leq C e^{Ct} \left(R + t \|M_1 f\|_\infty + \|u\|_{L^2(0,T;W^{2,2}(\mathbb{T}^3))} \right) =: \mathcal{R}(t), \end{aligned}$$

where we also used the embedding $L^2(0, t; W^{2,2}(\mathbb{T}^3)) \hookrightarrow L^1(0, t; L^\infty(\mathbb{T}^3))$. \square

Lemma B.3.2. *Let f be a solution to (7.4) subjected to the initial data with the support in v contained in the ball $B(R)$. Then*

$$M_1 f \leq C(\epsilon),$$

for some positive ϵ dependent constant $C(\epsilon)$.

Proof. First we integrate (7.4) to see that $M_0f = \text{const.}$ Next we multiply (7.4) by $|v|$ and integrate to get

$$\begin{aligned} \frac{d}{dt}M_1f + \underbrace{\int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|v \nabla f dx dv}_{=0} + \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v| \text{div}_v [(F_a(f) + (\theta_\epsilon * u - v))f] dx dv \\ = \frac{d}{dt}M_1f - \int_{\mathbb{T}^3 \times \mathbb{R}^3} \frac{v}{|v|} (F_a(f) + (\theta_\epsilon * u - v))f dx dv. \end{aligned}$$

Thus

$$\begin{aligned} \frac{d}{dt}M_1f &= \int_{\mathbb{T}^3 \times \mathbb{R}^3} \frac{v}{|v|} F_a(f) f dx dv + \int_{\mathbb{T}^3 \times \mathbb{R}^3} \text{sgn}(v) \theta_\epsilon * u f dx dv - \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v| f dx dv \\ &\leq \int_{\mathbb{T}^3 \times \mathbb{R}^3} |F_a(f)| f dx dv + \int_{\mathbb{T}^3 \times \mathbb{R}^3} |\theta_\epsilon * u| f dx dv \\ &\stackrel{(6.12)}{\leq} 2M_1f M_0f + C \|u\|_{L^2(0,T;W^{2,2}(\mathbb{T}^3))} M_0f. \end{aligned}$$

Since $M_0f = \text{const.}$, Gronwall's lemma implies that M_1f is bounded on $[0, T]$ as long as M_1f_0 is finite, which is the case by Remark 6.2.1. \square

B.4 Korn, Aubin–Lions, Vitali

The following theorems can be found in [46] or [38].

Lemma B.4.1 (Korn's inequality). *Let $1 < p < \infty$. Then for all $u \in W^{1,p}(\mathbb{T}^d)$, we have*

$$\kappa \|\nabla u\|_p \leq \|Du\|_p \tag{B.4}$$

for some positive constant κ .

Theorem B.4.1 (Aubin–Lions lemma). *Let $1 < p, q < \infty$. Let X be a Banach space and let X_0, X_1 be separable and reflexive Banach spaces. Provided that $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$ we have*

$$\{u \in L^p(0, T; X_0); \partial_t u \in L^q(0, T; X_1)\} \hookrightarrow\hookrightarrow L^p(0, T; X).$$

Proof. The proof can be found in [36], Section 1.5. \square

Lemma B.4.2. *Let $V \subset H \subset V^*$ be three Hilbert spaces. Moreover let $u \in L^2(0, T; V)$ and $\partial_t u \in L^2(0, T; V^*)$. Then $u \in C(0, T; H)$.*

Proof. The proof can be found in [49], page 261 (Lemma 1.2). \square

Remark B.4.1. In our case we have $V = H = V^* = L^2(\mathbb{T}^3)$, which of course is a Hilbert space.

Theorem B.4.2 (Vitali Convergence Theorem). *Let Ω be a bounded domain in \mathbb{R}^d and $f^n : \Omega \rightarrow \mathbb{R}$ be integrable for every $n \in \mathbb{N}$. Assume that*

- $\lim_{n \rightarrow \infty} f^n(x)$ exists and is finite for a.a. $x \in \Omega$;

- for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{n \in \mathbb{N}} \int_H |f^n(x)| dx < \epsilon \quad \forall H \subset \Omega, |H| < \delta. \quad (\text{B.5})$$

Then

$$\lim_{n \rightarrow \infty} \int_{\Omega} f^n(x) dx = \int_{\Omega} \lim_{n \rightarrow \infty} f^n(x) dx.$$

Proof. The proof can be found in [4], page 63. □

Remark B.4.2. Condition (B.5) is called *the uniform integrability condition*. In our applications we usually use the fact that if a sequence is bounded in $L^{1+\epsilon}$ for any $\epsilon > 0$ then it is uniformly integrable.

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