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Hardy–type inequalities and nonlinear eigenvalue problems

PhD dissertation

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Author's declaration:
aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Abstract

Our purpose is to give the new constructive method of derivation of Hardy and Hardy–Sobolev inequalities. We build inequalities knowing solutions u to p and A–harmonic problems, respectively. We derive Caccioppoli inequalities for u. As a consequence we obtain weighted Hardy and Hardy–Sobolev inequalities, respectively, for compactly supported Lipschitz functions.

In the first part we obtain one parameter family of Hardy inequalities of the form

$$\int_\Omega |\xi(x)|^p \mu_{1,\beta}(dx) \leq \int_\Omega |\nabla \xi(x)|^p \mu_{2,\beta}(dx),$$

where $1 < p < \infty$, $\xi : \Omega \to \mathbb{R}$ is compactly supported Lipschitz function, and Ω is an open subset of $\mathbb{R}^n$ not necessarily bounded. The involved measures $\mu_{1,\beta}(dx), \mu_{2,\beta}(dx)$ depend on certain parameter β and u — a nonnegative weak solution to anticoercive PDI:

$$-\Delta_p u \geq \Phi \quad \text{in} \quad \Omega,$$

with locally integrable function Φ (see Theorem 3.3.1). We allow quite a general function Φ that can be negative or sign changing if only there exists

$$\sigma_0 := \inf \{\sigma \in \mathbb{R} : \Phi \cdot u + \sigma |\nabla u|^p \geq 0 \quad a.e.\ in\ \Omega \cap \{u > 0\}\ \} \in \mathbb{R}. \tag{1}$$

The second part is devoted to Hardy–Sobolev inequalities of the form

$$\int_\Omega F_{\bar{A}}(|\xi|)\mu_1(dx) \leq \int_\Omega \bar{A}(|\nabla \xi|)\mu_2(dx),$$

where $\xi : \Omega \to \mathbb{R}$ is compactly supported Lipschitz function, Ω is an open subset of $\mathbb{R}^n$ not necessarily bounded, $\bar{A}(\lambda) = A(|\lambda|)\lambda$ is an N–function satisfying Δ'–condition and $F_{\bar{A}}(\lambda) = 1/(\bar{A}(1/\lambda))$. The involved measures $\mu_1(dx)$,

$\mu_2(dx)$ depend on u — a nonnegative weak solution to the anticoercive partial differential inequality of elliptic type involving A–Laplacian:

$$-\Delta_A u = -\mathrm{div} A(\nabla u) \geq \Phi \quad \text{in} \quad \Omega,$$

with locally integrable function Φ, satisfying the condition corresponding to (1). The results of the second part imply those of the first part with all details. In particular, the constants which we obtain in both attempts are equal.

Our method of construction of the inequalities is a handy tool. Not only is it easy to conduct, but also give deep results such as classical inequalities with the best constants.

Streszczenie

Naszym celem jest wprowadzić nową konstrukcyjną metodę formułowania nierówności typu Hardy'ego i Hardy'ego–Sobolewa. Konstruujemy je znając rozwiązania u zagadnienień p oraz A–harmonicznych, odpowiednio. Wyprowadzamy nierówności typu Caccioppoliego dla u. Jako wniosek z nich otrzymujemy ważone nierwnoci typu Hardy'ego i Hardy'ego–Sobolewa, odpowiednio, dla funkcji Lipschitzowskich o zwartym nośniku.

W pierwszej części otrzymujemy jednoparametrową rodzinę nierówności typu Hardy'ego postaci

$$\int_\Omega |\xi(x)|^p \mu_{1,\beta}(dx) \leq \int_\Omega |\nabla \xi(x)|^p \mu_{2,\beta}(dx),$$

gdzie $1 < p < \infty$, $\xi : \Omega \to \mathbb{R}$ jest funkcją Lipschitzowską o zwartym nośniku, Ω jest otwartym podzbiorem $\mathbb{R}^n$ nie koniecznie ograniczonym. Miary $\mu_{1,\beta}(dx)$, $\mu_{2,\beta}(dx)$ zależą od pewnego parametru β oraz u — nieujemnego rozwiązania antykoercytywnej nierówności różniczkowej:

$$-\Delta_p u \geq \Phi \quad \text{in} \quad \Omega,$$

z lokalnie całkowalną funkcją Φ (patrz Twierdzenie 3.3.1). Dopuszczamy dość ogólną postać Φ, która może być ujemna lub zmieniająca znak jeśli tylko istnieje

$$\sigma_0 := \inf \{ \sigma \in \mathbb{R} : \Phi \cdot u + \sigma |\nabla u|^p \geq 0 \quad a.e.\ in\ \Omega \cap \{u > 0\} \ \} \in \mathbb{R}. \qquad (2)$$

Druga część jest poświęcona nierównościom typu Hardy'ego–Sobolewa postaci

$$\int_\Omega F_{\bar{A}}(|\xi|) \mu_1(dx) \leq \int_\Omega \bar{A}(|\nabla \xi|) \mu_2(dx),$$

gdzie $\xi : \Omega \to \mathbb{R}$ $\xi : \Omega \to \mathbb{R}$ jest funkcją Lipschitzowską o zwartym nośniku, Ω jest otwartym podzbiorem $\mathbb{R}^n$ nie koniecznie ograniczonym, $\bar{A}(\lambda) = A(|\lambda|)\lambda$

jest N–funkcją spełniającą warunek Δ', a $F_{\bar{A}}(\lambda) = 1/(\bar{A}(1/t))$. Miary $\mu_1(dx)$, $\mu_2(dx)$ zależą od u — nieujemnego rozwiązania antykoercytywnej nierówności różniczkowej uwzględniajcacej A–Laplasjan:

$$-\Delta_A u = -\mathrm{div} A(\nabla u) \geq \Phi \quad \text{in} \quad \Omega,$$

with locally integrable function Φ, spełniającej warunek odpowiadający (2). Wyniki drugiej części implikują te z częci pierwszej ze wszystkimi szczegółami. Nawet otrzymane stałe są równe w obu podejściach.

Nasza metoda konstrukcji nierówności jest poręcznym narzędziem. Nie tylko jest łatwa do przeprowadzenia. Pozwala uzyskać głębokie wyniki, jak na przykład klasyczne nierówności z najlepszymi stałymi.

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Contents

Chapter 1

Introduction

The dissertation introduces the new constructive method of derivation of Hardy and Hardy–Sobolev inequalities. We build inequalities knowing weak solutions to p and A–harmonic problems, respectively. The results are based on [92, 93, 94] by the author.

The construction begins with derivation of Caccioppoli inequalities for solutions. As a consequence we obtain Hardy inequalities, involving certain measures, for test functions, i.e. compactly supported Lipschitz functions. This method of construction of the inequalities is a handy tool. Not only is it easy to conduct, but also give deep results such as classical inequalities with the best constants. We present brief explanation of derivation and a sample of main examples.

Our methods are inspired by the techniques from paper [72] where nonexistence of nontrivial nonnegative weak solutions to the A–harmonic problem

$$-\Delta_A u \geq \Phi(u) \quad \text{on} \quad \mathbb{R}^n, \tag{1.1}$$

where Φ is a nonnegative function is investigated. The paper [72] develops the idea from [86] in the following way. The authors derive Caccioppoli–type estimate for nonnegative weak solutions to (1.1). Then, they obtain more specified a priori estimates involving general test functions and finally, choosing appropriate test functions, they obtain nonexistence.

We concentrate on the Caccioppoli–type estimate. Careful analysis enables us to derive this type of estimate violating assumptions from [72] that $\Phi = \Phi(u)$, $\Phi \geq 0$ and that integrals are over whole space. Instead, we assume only that Φ is in some sense bounded from below (see condition in

(3.5), (3.12)). As a next step we notice that certain substitution in the derived Caccioppoli–type inequality for solutions implies Hardy–type inequality for compactly supported Lipschitz functions.

The first part is based on [92] and [93] by the author. These papers concern derivation and application of one parameter family of Hardy inequalities of the form

$$\int_\Omega |\xi(x)|^p \mu_{1,\beta}(dx) \le \int_\Omega |\nabla\xi(x)|^p \mu_{2,\beta}(dx), \tag{1.2}$$

where $1 < p < \infty$, $\xi : \Omega \to \mathbb{R}$ is compactly supported Lipschitz function, and Ω is an open subset of $\mathbb{R}^n$ not necessarily bounded. The involved measures $\mu_{1,\beta}(dx), \mu_{2,\beta}(dx)$ depend on certain parameter β and u — a nonnegative weak solution to anticoercive PDI

$$\Delta_p u \ge \Phi \text{ in } \Omega, \tag{1.3}$$

with locally integrable function Φ (see Theorem 3.3.1). We allow quite a general function Φ that can be negative or sign changing if only there exists

$$\sigma_0 := \inf\{\sigma \in \mathbb{R} : \Phi \cdot u + \sigma|\nabla u|^p \ge 0 \quad a.e.\ in\ \Omega \cap \{u > 0\}\ \} \in \mathbb{R}. \tag{1.4}$$

Let us mention some special cases which we present in Section 3.4. We obtain classical Hardy inequality

$$\int_0^\infty \left(\frac{|\xi(x)|}{x}\right)^p x^\gamma\, dx \le C_{min} \int_0^\infty |\xi'(x)|^p x^\gamma\, dx,$$

with the optimal constant C_{min}, for all admissible range of parameters γ and p. Another special case is a more general result when measures in (1.2) have a form $\mu_i(dx) = \varrho_i(|x|)dx$, with locally integrable radial functions $\varrho_i(|x|)$ and $\Omega = \mathbb{R}^n \setminus \{0\}$. As a direct consequence of this approach we obtain n–dimensional Hardy inequality

$$\int_{\mathbb{R}^n\setminus\{0\}} |\xi(x)|^p |x|^{\gamma-p}\, dx \le C_{min} \int_{\mathbb{R}^n\setminus\{0\}} |\nabla\xi(x)|^p |x|^\gamma\, dx,$$

with the optimal constant C_{min} within certain range of parameters γ and p.

In Subsection 3.4.3 we present Hardy inequalities with exponential weights. In Subsection 3.4.4 we consider p–superharmonic functions. In that case the measures derived in (1.2) have a simpler form. Such inequalities can be constructed for example by using harmonic function u, which satisfies a given boundary value problem. In Subsection 3.4.5 we investigate problems with the negative lower bound of $-p$–Laplacian (i.e. function Φ from (1.3)).

Section 3.5 is devoted to applications of our methods. As a first of them, we illustrate result by Ghoussoub and Moradifam from a recent paper [53], giving the constructive method to obtain Bessel pairs. Our second application is focused on mathematical models in astrophysics. We investigate Hardy and Hardy–Sobolev inequalities resulting from model by Bertin and Ciotti describing dynamics of elliptic galaxies. For this discussion knowing the exact form of a solution is not necessary but the existence is needed. Using existence result by Badiale and Tarantello [7] we derive the related Hardy inequality. Let us mention that the model by Bertin and Ciotti has a similar form to the well known Matukuma's equation [82] and various other models can be used to build Hardy inequalities as well.

In Section 3.6 we present results of [93] where the author derive Hardy–Poincaré inequalities

$$\bar{C}_{\gamma,n,p}\int_{\mathbb{R}^n} |\xi(x)|^p(1+|x|^{\frac{p}{p-1}})^{(p-1)(\gamma-1)}dx \le \int_{\mathbb{R}^n} |\nabla\xi(x)|^p(1+|x|^{\frac{p}{p-1}})^{(p-1)\gamma}\, dx, \tag{1.5}$$

with $\bar{C}_{\gamma,n,p}$ proven to be optimal for sufficiently big γ's. The version of this result, when $p=2$,

$$C\int_{\mathbb{R}^n} |\xi|^2(1+|x|^2)^{\gamma-1}dx \le \int_{\mathbb{R}^n} |\nabla\xi|^2(1+|x|^2)^{\gamma}dx, \tag{1.6}$$

is of special interest in many disciplines of analysis. Let us recall some applications of (1.6) to the theory of nonlinear diffusions — evolution equations of a form $u_t = \Delta u^m$, which are called fast diffusion equation (FDE) if $m<1$ and porous media equation (PME) if $m>1$. In the theory of FDE, Hardy–Poincaré inequalities (1.6) with $\gamma<0$ are the basic tools to investigate the large–time asymptotic of solutions [4, 14, 26, 41]. For example, the best constant in (1.6) is used in [16, 48] to show the fastest rate of convergence of solutions of fast diffusion equation and to bring some information about spectral properties of the elliptic operator $L_{\alpha,d}u := -h_{1-\gamma}\mathrm{div}\,(h_{-\gamma}\nabla u)$, where $h_\alpha = (1+|x|^2)^\alpha$. We refer also to [26, 40, 98] for the related results.

We are interested in (1.5) with $\gamma>1$, and we take into account all $p\in(1,\infty)$, not only $p=2$. This result is obtained when we consider Theorem 3.3.1 and apply $u_\alpha(x) = (1+|x|^{\frac{p}{p-1}})^{-\alpha}$, $\alpha>0$. We prove inequality (1.5) as well as optimality of the obtained constants for a range of parameters. Details are given in the proof of Theorem 3.6.1.

It appears that in some cases we improve the constants obtained by Blanchet, Bonforte, Dolbeault, Grillo and Vázquez in [16], as well as those by Ghoussoub and Moradifam from [53]. In the case $p = 2$, $\gamma = n$, our constant is the same as in [16] and proven there to be optimal. Moreover, we show that our constants are also optimal for $p > 1$, when $\gamma \geq n + 1 - \frac{n}{p}$, but we do not know if they are optimal for wider range of parameters, either in the case $p = 2$, or generally for $p > 1$. At the and of Subsection 3.6.2 we give a summary of the known values of constants, and their optimality, in different cases.

The second part is devoted to Hardy–Sobolev inequalities of the form

$$\int_\Omega F_{\bar{A}}(|\xi|)\mu_1(dx) \leq \int_\Omega \bar{A}(|\nabla \xi|)\mu_2(dx), \tag{1.7}$$

where $\xi : \Omega \to \mathbb{R}$ is compactly supported Lipschitz function, Ω is an open subset of $\mathbb{R}^n$ not necessarily bounded, $\bar{A}(\lambda) = A(|\lambda|)\lambda$ is an N–function satisfying Δ'–condition and $F_{\bar{A}}(\lambda) = 1/(\bar{A}(1/t))$. The involved measures $\mu_1(dx)$, $\mu_2(dx)$ depend on u — a nonnegative weak solution to the anticoercive partial differential inequality of elliptic type involving A–Laplacian $-\Delta_A u = -\mathrm{div} A(\nabla u) \geq \Phi$ in Ω, with locally integrable function Φ, satisfying the condition corresponding to (1.4).

In Section 4 we give examples of inequalities of a type (1.7) with general $\bar{A}(t)$ satisfying Δ'–condition with various measures. In particular we present application of $\bar{A}(t) = t^p \log^\alpha(2 + t)$, $p > 1, \alpha > 0$. This part extends results from Chapter 3 (based on [92] and [93]), where we considered inequality $-\Delta_p u \geq \Phi$, leading to Hardy inequalities with the best constants. In particular, the obtained constants in both attempts are equal.

Chapter 2

Motivation

Hardy–type inequalities are important tools in functional analysis, harmonic analysis, probability theory, and PDEs. In the last three decades huge progress was made to understand them, see e.g. books: [73, 75, 77, 76, 80, 83, 87] and their references. The applied tools are often expressed in the language of functional analysis, harmonic analysis, and probability.

Applications. In theory of PDEs they are used to obtain a priori estimates, existence, and regularity ([8, 19, 20, 50, 54], Section 2.5 in [83]), as well as to study qualitative properties of solutions and their asymptotic behaviour [98]. Hardy inequalities are also applied in derivation of embedding theorems (Theorem 3.1 in [27], [59, 64]), Gagliardo–Nirenberg interpolation inequalities [32, 33, 58, 69] and in the real interpolation theory [47].

Validity of Hardy inequality.

Several necessary and sufficient conditions for the validity of Hardy–type inequalities are present in the literature. Most of them seems to be rather abstract and the conditions for the validity of inequalities are often very hard to verify in practice.

Let us mention one of such results, where conditions for existence of Hardy–type inequalities involving measures have been characterized completely, however they are hard to apply. The example is Theorem 2.4.1 in [83] (in the case $M(t) = |t|$) which characterizes measures satisfying inequality

$$\int_\Omega |\xi|^p \mu(dx) \le C \int_\Omega |\nabla \xi|^p dx, \quad 1 < p < \infty,$$

holding for smooth compactly supported functions u. The conditions, so–called isoperimetric inequalities, are expressed on compact sets and involve capacities.

There are many conditions equivalent to validity of Hardy inequalities. They are usually associated with the name of Muckenhoupt and his work [88]. We give below the famous theorem, which summarizes efforts and ideas in this topic of wide range of great mathematicians such as Artola, Talenti [96], Tomaselli [97], Chisholm–Everitt [31], Muckenhoupt [88], Boyd–Erdos. The proof that we invoke follows [76] where, apart from this formulation, a lot of additional interesting historical information on the investigation of this problems can be found.

Theorem 2.0.1 (Talenti–Tomaselli–Muckenhoupt)**.** *Let $1 \leq p < \infty$. The inequality*

$$\left(\int_0^b \left(\int_0^x f(t)dt\right)^p u(x)dx\right)^{\frac{1}{p}} \leq C\left(\int_0^b f^p(t)v(x)dx\right)^{\frac{1}{p}} \tag{2.1}$$

holds for all measurable functions $f(x) \geq 0$ on $(0,b)$, $0 < b \leq \infty$ if and only if

$$A = \sup_{r>0}\left(\int_r^b u(x)dx\right)^{\frac{1}{p}}\left(\int_0^r v^{1-p'}(x)dx\right)^{\frac{1}{p'}} < \infty.$$

Moreover, the best constant C in (2.1) satifies $A \leq C \leq p^{\frac{1}{p}}p'^{\frac{1}{p'}}$ for $1 < p < \infty$ and $C = A$ for $p = 1$.

Hardy inequalities in PDEs

Generally speaking, linking nonlinear eigenvalue problems of elliptic and parabolic type with Hardy inequalities is common in the literature. We observe this issue also in the articles [3, 5, 7, 18, 24, 25, 68, 71, 85]. For example it is well known that functions achieving best constants in Hardy–Sobolev type inequalities satisfy the nonlinear eigenvalue problems [22, Chapter 5]. Moreover, the best constants are investigated for proving existence of parabolic eigenvalue problems [8, 35, 50, 52]. What is less understood is the converse: that solutions or subsolutions to differential eigenvalue problems are helpful to construct Hardy–Sobolev inequalities.

The best constant and existence. Analysis of the best constants $c_{n,\gamma,p}$ in Classical n–dimensional Hardy inequalities is crucial to decide existence. We refer to seminal paper of P. Baras and J. A. Goldstein [8], where existence, nonexistence of global solutions, and a blow–up for following parabolic problem is considered. For $x \in \mathbb{R}^n,\ n \geq 3$, and $t \in (0,T)$

$$\begin{cases} u_t - \Delta u = \lambda \frac{u}{|x^2|}, & \lambda \in \mathbb{R}, \\ u(x,0) = u_0(x) > 0, & u_0 \in L^2(\mathbb{R}^n), \end{cases} \tag{2.2}$$

has a solution if and only if $\lambda \leq \left(\frac{n-2}{2}\right)^2$. See [8] for details and [54] for related generalized results.

We note additionally that critical $\lambda = \left(\frac{n-2}{2}\right)^2$ is equal to optimal (but not attained in the Sobolev space) constant in the following n–dimensional Classical Hardy inequality

$$\left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n \setminus \{0\}} |\xi|^2 |x|^{-2}\, dx \leq \int_{\mathbb{R}^n \setminus \{0\}} |\nabla \xi|^p |x|^2\, dx, \quad \xi \in C_0^\infty(\mathbb{R}^n \setminus \{0\}).$$

Nevertheless, the authors of [8] neither apply nor recognize Hardy inequality in any version. Connection with critical λ from (2.2) is revealed in [50] by J. P. García–Azorero and I. Peral–Alonso. The authors study links between Hardy inequality and nonlinear critical p–heat equation (and the related stationary p–Laplacian equation)

$$\begin{cases} u_t - \Delta_p u = \lambda \frac{|u|^{p-2}u}{|x|^p}, & x \in \Omega,\ t > 0,\ \lambda \in \mathbb{R}, \\ u(x,0) = f(x) \geq 0, & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{2.3}$$

where $-\Delta_p u \geq 0$, Ω is a bounded domain in $\mathbb{R}^n$, and $1 < p < N$. Qualitative properties of solutions, such as existence and blow–up, depend in general on the relation between λ and the best constant in Hardy inequality.

Asymptotic behaviour. In [98] J. L. Vazquez and E. Zuazua describe the asymptotic behaviour of the heat equation that reads

$$u_t = \Delta u + V(x)u \quad \text{and} \quad \Delta u + V(x)u + \mu u = 0,$$

where $V(x)$ is an inverse–square potential (e.g. $V(x) = \frac{\lambda}{|x|^2}$). The authors consider the Cauchy–Dirichlet problem in a bounded domain and for the Cauchy problem in $\mathbb{R}^n$ as well. The crucial tool is an improved form of Hardy–Poincaré inequality and its new weighted version. The main results show the decay rate of solutions. Well–posedness of the problem and problems with uniqueness are also considered. Furthermore, in [98] the authors explain and generalize the work of P. Baras and J. A. Goldstein [8].

Radiality. Hardy inequality may play the key role to prove existence, nonexistence, as well as radiality of solutions. All the mentioned applications are studied in [51] by M. Garcia–Huidobro, A. Kufner, R. Manásevich, and C. S. Yarur.

The authors establish a critical exponent for the inclusion of a certain weighted Sobolev space into the weighted Lebesgue space. This result is applied in the proof of radiality of solutions for a quasilinear equation

$$\begin{cases} \operatorname{div}(a(|x|)|\nabla u|^{p-2}\nabla u) = b(|x|)|\nabla u|^{q-2}\nabla u \quad \text{in } B \subseteq \mathbb{R}^n, \\ u = 0 \quad \text{on } \partial B, \end{cases} \tag{2.4}$$

where $1 < p < q$, functions a, b are weight functions, and B is a ball.

Links between existence for differential equations and validity of Hardy inequality

Constructing Hardy–type inequalities on the basis of differential problems is an idea present in the literature.

ODEs. In paper [56] Gurka investigated the existence of one–dimensional Hardy–type inequality between L^q and L^p (allowing the case $p = q$) that reads

$$\left(\int_0^a s(x)|u(x)|^q dx\right)^{\frac{1}{q}} \le C\left(\int_0^a r(x)|u'(x)|^p dx\right)^{\frac{1}{p}} \tag{2.5}$$

and found necessary and sufficient conditions for the existence of (2.5) in a certain class of admitted functions. The work [56] generalises previous results by Beesack [9], Kufner and Triebel [79], Muckenhoupt [88], and Tomaselli [97]. The main result of [56] reads

Theorem 2.0.2 ([56], Theorem 1.3)**.** *Assume* $0 < a \le \infty$, $1 < p \le q < \infty$. *Let* $r(x) > 0$, $s(x) \ge 0$ *be functions measurable on* $[0, a]$.

Moreover, let us suppose that the first derivative $r'(x)$ *exists for all* $x \in (0, a)$. *Then the equation*

$$\lambda \frac{d}{dx}\left(r^{\frac{q}{p}}(x)(y'(x))^{\frac{q}{p'}}\right) + s(x)y^{\frac{q}{p'}}(x) = 0 \tag{2.6}$$

(with a certain $\lambda > 0$*) has a solution* $y(x)$ *(with a locally absolutely continuous first derivative) such that*

$$y(x) > 0,\ y'(x) > 0, \quad (x \in (0, a))$$

if and only if there exists a constant $C_0 > 0$ *such that the inequality* (2.5) *holds for every function* $u(x)$ *absolutely continuous on* $[0, a]$ *such that* $u(0) = \lim_{t\to\infty} u(t) = 0$.

In the recent paper by Ghoussoub and Moradifam [53], the authors proved that the validity of inequalities

$$c\int_B |\xi(x)|^2 W(x)dx \leq \int_B |\nabla\xi(x)|^2 V(x)dx \text{ for all } u \in C_0^\infty(B),$$

with radially symmetric functions V and W (so–called Bessel pairs), where B is a ball with center at zero, is equivalent to the existence of solutions to the one–dimensional nonlinear eigenvalue problem

$$y''(r) + \left(\frac{n-1}{r} + \frac{V'(r)}{V(r)}\right) y'(r) + \frac{cW(r)}{V(r)} y(r) = 0, \quad y > 0.$$

This is in the spirit of Gurka's inequality (2.5).

PDEs. We find connections between p–superharmonic problems and Hardy inequalities in papers [10, 11] by Barbatis, Filippas, and Tertikas. The authors assume that the distance $d(x) = dist(x, K)$, for a certain set $K \subseteq \overline{\Omega}$, satisfies in the weak sense the problem

$$-\Delta_p \left(d^{\frac{p-k}{p-1}}\right) \geq 0 \quad \text{in} \quad \Omega \setminus K,$$

where $p \neq k$. The obtained Hardy inequalities with remainder terms involve function d. Furthermore, in the weight functions the exponent of the function d is rigid.

More general approach is presented in several papers by D'Ambrosio [36, 37, 38]. We find there an alternative method of construction of Hardy inequalities from problems of a type $-\Delta_p u \geq 0$ and similar ones described in terms of Heisenberg groups $\mathbb{H}^n$. We find in [36] sufficient criteria for validity of Hardy inequalities involving various weights, among others those with a term with distance from the boundary. The derived inequalities are described not only in Heisenberg setting but also in more general frameworks containing as particular cases the subelliptic setting as well as the usual Euclidean setting. Our result refers to the latter kind of result, namely the inequality

$$\int |\xi(x)|^p W(x)dx \leq C \int |\nabla\xi(x)|^p V(x)dx, \quad \text{for every } \xi \in C_0^1(\Omega),$$

where the weights $V(x)$ and $W(x)$ depend on a function u, that is a nonnegative solution to $-\Delta_p(u^\alpha) \geq 0$, and on the constant α. We generalize this type of reasoning by allowing the lower bound of $-p$–Laplacian (i.e. function Φ from (1.3)) to be negative.

Chapter 3

Hardy inequalities derived from p–harmonic problems

This part is based on [92, 93] by the author. We consider therein the anti–coercive partial differential inequality of elliptic type involving p–Laplacian: $-\Delta_p u \geq \Phi$, where Φ is a given locally integrable function and u is defined on an open subset $\Omega \subseteq \mathbb{R}^n$. We derive Caccioppoli inequalities for u. Knowing solutions, as a direct consequence we obtain Hardy inequalities involving certain measures for compactly supported Lipschitz functions. We present several applications leading to various weighted Hardy inequalities. Our methods allow to retrieve classical Hardy inequalities with optimal constants. Moreover, we give optimal constants for Hardy–Poincaré inequalities with weights of a type $\left(1 + |x|^{\frac{p}{p-1}}\right)^{\alpha}$ for sufficiently big parameter $\alpha > 0$.

3.1 Preliminaries

In the sequel we assume that $p > 1$, $\Omega \subseteq \mathbb{R}^n$ is an open subset not necessarily bounded.

By p–harmonic problems we understand those which involve p–Laplace operator $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$.

If f is defined on Ω by $f\chi_\Omega$ we understand function f extended by 0 outside Ω.

Having an arbitrary $u \in W^{1,1}_{loc}(\mathbb{R}^n)$ we define its value at every point by the formula (see e.g. [15])

$$u(x) := \limsup_{r\to 0} \fint_{B(x,r)} u(y)dy. \tag{3.1}$$

We write $f \sim g$ if function f is comparable with function g, i.e. if there exist positive constants c_1, c_2 such that for every x

$$c_1 g(x) \le f(x) \le c_2 g(x).$$

Definition 3.1.1 (Weighted Sobolev space)**.** *By $W^{1,p}_{v_1,v_2}(\mathbb{R}^n)$, where nonnegative measurable functions v_1, v_2 are given, we mean the completion of the set of functions $u \in C^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} |u|^p\, v_1 dx < \infty$ and $\int_{\mathbb{R}^n} |\nabla u|^p\, v_2 dx < \infty$, under the norm*

$$\|u\|_{W^{1,p}_{v_1,v_2}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |u|^p\, v_1 dx + \int_{\mathbb{R}^n} |\nabla u|^p\, v_2 dx \right)^{\frac{1}{p}}.$$

Differential inequality

Our analysis is based on the following differential inequality.

Definition 3.1.1. *Let Ω be any open subset of $\mathbb{R}^n$ and Φ be the locally integrable function defined in Ω such that for every nonnegative compactly supported $w \in W^{1,p}(\Omega)$*

$$\int_\Omega \Phi w\, dx > -\infty. \tag{3.2}$$

Let $u \in W^{1,p}_{loc}(\Omega)$. We say that

$$-\Delta_p u \ge \Phi, \tag{3.3}$$

if for every nonnegative compactly supported $w \in W^{1,p}(\Omega)$ we have

$$\langle -\Delta_p u, w \rangle := \int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla w \rangle\, dx \ge \int_\Omega \Phi w\, dx. \tag{3.4}$$

Remark 3.1.1. If $p > 1$ and $u \in W^{1,p}_{loc}(\mathbb{R}^n)$ then $|\nabla u|^{p-2} \nabla u \in L^{\frac{p}{p-1}}_{loc}(\mathbb{R}^n, \mathbb{R}^n)$. In particular the second term in (3.4) is finite for every compactly supported function $w \in W^{1,p}(\Omega)$. Therefore $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2} \nabla u)$ is well–defined in the distributional sense.

Remark 3.1.2. Let us consider the case $\Omega = (-1, 1)$, $u(x) = 1 - |x|$, $p = 2$. Then we have: $-u'' = 2\delta_0 \geq 0 \equiv \Phi$ in the sense of Definition 3.1.1. This shows that in our approach $\Delta_p u$ may not be a distribution represented by locally integrable function.

The following condition is crucial in the sequel. Suppose u and Φ are as in Definition 3.1.1 and moreover there exists a real number

$$(\mathbf{\Phi}, \mathbf{p}) \quad \sigma_0 := \inf \{\sigma \in \mathbb{R} : \Phi \cdot u + \sigma |\nabla u|^p \geq 0 \ a.e. \ in \ \Omega \cap \{u > 0\} \ \} \tag{3.5}$$

where we set $\inf \emptyset = \infty$.

Remark 3.1.3.
1. In the case when $\Phi \geq 0$ a.e. on $\{u > 0, \nabla u \neq 0\}$ we have

$$\sigma_0 = -\mathrm{essinf}_{\{u>0,\nabla u \neq 0\}} \left(\frac{\Phi \cdot u}{|\nabla u|^p} \right).$$

2. In the case $\Phi = -\Delta_p u \in L^1_{loc}(\Omega)$, (3.5) reads

$$\sigma_0 \frac{|\nabla u|^p}{u} \geq \Delta_p u \quad \text{a.e. in} \quad \{u > 0\}. \tag{3.6}$$

3.2 Caccioppoli estimates for solutions to problem $-\Delta_p u \geq \Phi$

Our main goal in this chapter is to obtain following result.

Theorem 3.2.1. *Assume that $1 < p < \infty$ and $u \in W^{1,p}_{loc}(\Omega)$ is a nonnegative solution to the PDI $-\Delta_p u \geq \Phi$, in the sense of Definition 3.1.1, with locally integrable function Φ satisfying (3.5) with $\sigma_0 \in \mathbb{R}$. Assume further that β and σ are arbitrary numbers such that $\beta > 0$ and $\beta > \sigma \geq \sigma_0$.*

Then the inequality

$$\int_\Omega (\Phi \cdot u + \sigma |\nabla u|^p) \, \chi_{\{u>0\}} u^{-\beta-1} \phi \, dx \leq$$

$$\leq \frac{(p-1)^{p-1}}{p^p (\beta - \sigma)^{p-1}} \int_\Omega u^{p-\beta-1} \chi_{\{\nabla u \neq 0\}} \cdot |\nabla \phi|^p \phi^{1-p} \, dx, \tag{3.7}$$

holds for every nonnegative Lipschitz function ϕ with compact support in Ω such that the integral $\int_{\operatorname{supp} \phi} |\nabla \phi|^p \phi^{1-p} \, dx$ is finite.

We call (3.7) *Caccioppoli inequality* because it involves ∇u on the left–hand side and only u on the right–hand side (see e.g. [23, 65]).

We note that we do not assume that the right–hand side in (3.7) is finite.

The proof is based on analysis of the proof of Proposition 3.1 from [72] in the case when the considered A–harmonic operator is p–Laplacian. However, here we are not restricted to $\Phi = \Phi(u)$, $\Phi \geq 0$, and integrals over $\mathbb{R}^n$.

Proof of Theorem 3.2.1. The proof follows by three steps.

Step 1. Derivation of a local inequality.

We obtain the following lemma.

Lemma 3.2.1. *Assume that $1 < p < \infty$ and $u \in W^{1,p}_{loc}(\Omega)$ is a nonnegative solution to the PDI $-\Delta_p u \geq \Phi$, in the sense of Definition 3.1.1, with locally integrable function Φ. Assume further that $\beta, \tau > 0$ are arbitrarily taken numbers.*

Then, for every $0 < \delta < R$, the inequality

$$\begin{aligned} &\int_{\{u \leq R-\delta\}} \left(\Phi \cdot u + \left(\beta - \frac{p-1}{p}\tau \right) |\nabla u|^p \chi_{\{\nabla u \neq 0\}} \right) (u+\delta)^{-\beta-1} \phi \, dx \\ \leq \; & \frac{1}{p\tau^{p-1}} \int_{\Omega \cap \{\nabla u \neq 0,\, u \leq R-\delta\}} (u+\delta)^{p-\beta-1} \cdot |\nabla \phi|^p \phi^{1-p} \, dx + \tilde{C}(\delta, R), \end{aligned} \tag{3.8}$$

where

$$\tilde{C}(\delta, R) = R^{-\beta} \left[\int_{\Omega \cap \{\nabla u \neq 0,\, u > R-\delta\}} |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, dx - \int_{\Omega \cap \{u > R-\delta\}} \Phi \phi \, dx \right],$$

holds for every nonnegative Lipschitz function ϕ with compact support in Ω.

Before we prove the lemma let us formulate the following facts.

Fact 3.2.1. *Let $p > 1$, $\tau > 0$ and $s_1, s_2 \geq 0$, then*

$$s_1 s_2^{p-1} \leq \frac{1}{p\tau^{p-1}} \cdot s_1^p + \frac{p-1}{p} \tau \cdot s_2^p.$$

Proof. We apply classical Young inequality $ab \leq \frac{a^p}{p} + \frac{p-1}{p} b^{\frac{p}{p-1}}$ with $a = \frac{s_1}{\delta^{p-1}}$, $b = (s_2\delta)^{p-1}$, where $\delta > 0$, to get

$$s_1 s_2^{p-1} = \left(\frac{s_1}{\delta^{p-1}} \right) (s_2\delta)^{p-1} \leq \frac{1}{p} \left(\frac{s_1}{\delta^{p-1}} \right)^p + \frac{p-1}{p} (s_2\delta)^{(p-1)\frac{p}{p-1}} =$$

$$= \frac{1}{p\delta^{p(p-1)}} \cdot s_1^p + \frac{p-1}{p}\delta^p \cdot s_2^p.$$

Now it suffices to substitute $\tau = \delta^p$. $\square$

Fact 3.2.2 (e.g. [72], Lemma 3.1)**.** *Let $u \in W^{1,1}_{loc}(\Omega)$ be defined everywhere by* (3.1) *and let $t \in \mathbb{R}$. Then*

$$\{x \in \mathbb{R}^n : u(x) = t\} \subseteq \{x \in \mathbb{R}^n : \nabla u(x) = 0\} \cup N,$$

where N is a set of Lebesgue's measure zero.

Fact 3.2.3. *For u, ϕ as in the assumptions of Lemma 3.2.1 we fix $0 < \delta < R$, $\beta > 0$ and denote*

$$u_{\delta,R}(x) := \min\left(u(x) + \delta, R\right), \qquad G(x) := (u_{\delta,R}(x))^{-\beta}\phi(x).$$

Then $u_{\delta,R} \in W^{1,p}_{loc}(\Omega)$ and $G \in W^{1,p}(\Omega)$.

Proof of Lemma 3.2.1. Let us introduce some notation:

$$\begin{aligned}
\tilde{A}_1(\delta, R) &= \int_{\Omega\cap\{\nabla u \neq 0,\, u \leq R-\delta\}} |\nabla u|^p \,(u+\delta)^{-\beta-1}\,\phi\, dx,\\
\tilde{B}(\delta, R) &= \int_{\Omega\cap\{\nabla u \neq 0,\, u \leq R-\delta\}} |\nabla u|^{p-2}\langle \nabla u, \nabla\phi\rangle (u+\delta)^{-\beta}\, dx,\\
\tilde{C}_1(\delta, R) &= R^{-\beta}\int_{\Omega\cap\{u > R-\delta\}} \Phi\cdot\phi\, dx,\\
\tilde{C}_2(\delta, R) &= R^{-\beta}\int_{\Omega\cap\{\nabla u \neq 0,\, u > R-\delta\}} |\nabla u|^{p-2}\langle \nabla u, \nabla\phi\rangle\, dx,\\
\tilde{D}(\delta, R) &= \int_{\mathrm{supp}\,\phi\cap\{\nabla u \neq 0,\, u \leq R-\delta\}} (u+\delta)^{p-\beta-1}\cdot|\nabla\phi|^p\phi^{1-p}\, dx.
\end{aligned}$$

We take $w = G$ in (3.4) and note that

$$\begin{aligned}
I &:= \int_\Omega \Phi\cdot G\, dx = \int_\Omega \Phi\cdot (u_{\delta,R})^{-\beta}\phi\, dx = \\
&= \int_{\Omega\cap\{u\leq R-\delta\}} \Phi\cdot(u+\delta)^{-\beta}\phi\, dx + R^{-\beta}\int_{\Omega\cap\{u > R-\delta\}} \Phi\cdot\phi\, dx = \\
&= \int_{\Omega\cap\{u\leq R-\delta\}} \Phi\cdot(u+\delta)^{-\beta}\phi\, dx + \tilde{C}_1(\delta, R), \qquad (3.9)
\end{aligned}$$

On the other hand, inequality (3.3) implies

$$
\begin{aligned}
I &:= \int_\Omega \Phi \cdot G \, dx \le \langle -\Delta_p u, G\rangle = \int_{\Omega\cap\{\nabla u\neq 0\}} |\nabla u|^{p-2}\langle \nabla u, \nabla G\rangle \, dx = \\
&= -\beta \int_{\Omega\cap\{\nabla u\neq 0,\, u\le R-\delta\}} |\nabla u|^p (u+\delta)^{-\beta-1}\phi \, dx + \\
&\quad + \int_{\Omega\cap\{\nabla u\neq 0,\, u\le R-\delta\}} |\nabla u|^{p-2}\langle \nabla u, \nabla \phi\rangle (u+\delta)^{-\beta} \, dx + \\
&\quad + R^{-\beta} \int_{\Omega\cap\{\nabla u\neq 0,\, u> R-\delta\}} |\nabla u|^{p-2}\langle \nabla u, \nabla \phi\rangle \, dx = \\
&= -\beta \tilde{A}_1(\delta, R) + \tilde{B}(\delta, R) + \tilde{C}_2(\delta, R).
\end{aligned}
$$

Note that all the above integrals above are finite, what follows from Remark 3.1.1 (for $0 \le u \le R-\delta$ we have $\delta \le u+\delta \le R$). Moreover,

$$
\begin{aligned}
\tilde{B}(\delta, R) &\le \int_{\Omega\cap\{\nabla u\neq 0,\, u\le R-\delta\}} |\nabla u|^{p-1}|\nabla \phi|(u+\delta)^{-\beta} \, dx = \\
&= \int_{\mathrm{supp}\,\phi\cap\{\nabla u\neq 0,\, u\le R-\delta\}} \left(\frac{|\nabla \phi|}{\phi}(u+\delta)\right) \cdot |\nabla u|^{p-1}(u+\delta)^{-\beta-1}\, \phi dx.
\end{aligned}
$$

We apply Fact 3.2.1 with $s_1 = \frac{|\nabla\phi|}{\phi}(u+\delta)$, $s_2 = |\nabla u|$ and arbitrary $\tau > 0$, to get

$$
\begin{aligned}
\tilde{B}(\delta, R) \le &\quad \frac{p-1}{p}\tau \int_{\mathrm{supp}\,\phi\cap\{\nabla u\neq 0,\, u\le R-\delta\}} |\nabla u|^p (u+\delta)^{-\beta-1}\phi \, dx + \\
&+ \frac{1}{p\tau^{p-1}} \int_{\mathrm{supp}\,\phi\cap\{\nabla u\neq 0,\, u\le R-\delta\}} \left(\frac{|\nabla \phi|}{\phi}\right)^p (u+\delta)^{p-\beta-1}\phi \, dx. \\
\le &\quad \frac{p-1}{p}\tau \tilde{A}_1(\delta, R) + \frac{1}{p\tau^{p-1}}\tilde{D}(\delta, R).
\end{aligned}
$$

Combining these estimates we deduce that

$$
\begin{aligned}
I &\le -\beta \tilde{A}_1(\delta, R) + \tilde{B}(\delta, R) + \tilde{C}_2(\delta, R) \le \\
&\le \left(-\beta + \frac{p-1}{p}\tau\right) \tilde{A}_1(\delta, R) + \frac{1}{p\tau^{p-1}}\tilde{D}(\delta, R) + \tilde{C}_2(\delta, R).
\end{aligned}
$$

Recall that $\tilde{C}_1(\delta, R)$ and $\tilde{A}_1(\delta, R)$ are finite ($\tilde{D}(\delta, R)$ is finite as well). This and (3.9) imply

$$\int_{\Omega\cap\{u\le R-\delta\}} \Phi(u+\delta)^{-\beta}\phi\,dx + \left(\beta - \frac{p-1}{p}\tau\right)\tilde{A}_1(\delta,R) \quad \le$$
$$\le \frac{1}{p\tau^{p-1}}\tilde{D}(\delta,R) \quad + \quad \tilde{C}(\delta,R),$$

which implies (3.8), because $\tilde{C}(\delta,R) = \tilde{C}_2(\delta,R) - \tilde{C}_1(\delta,R)$. □

Remark 3.2.1. Introduction of parameters δ and R was necessary as we needed to move some quantities in the estimates to opposite sides of inequalities. For this we have to know that they are finite.

Step 2. Passing to the limit with $\delta \searrow 0$.

We show that when $\beta, \tau > 0$ are arbitrary numbers such that $\beta - \frac{p-1}{p}\tau =: \sigma \ge \sigma_0$ then for any $R > 0$

$$\int_{\{u\le R\}} (\Phi\cdot u + \sigma|\nabla u|^p)\, u^{-\beta-1}\chi_{\{u>0\}}\phi\,dx \tag{3.10}$$
$$\le \frac{1}{p\tau^{p-1}}\int_{\{\nabla u\neq 0,\, u\le R\}} u^{p-\beta-1}\cdot|\nabla\phi|^p\phi^{1-p}\,dx + \tilde{C}(R),$$

where

$$\tilde{C}(R) = R^{-\beta}\left[\left|\int_{\Omega\cap\{u\ge\frac{R}{2}\}} |\nabla u|^{p-1}|\nabla u|\cdot|\nabla\phi|\,dx\right| + \int_{\Omega\cap\{u\ge\frac{R}{2}\}}\Phi\phi\,dx\right]$$

holds for every nonnegative Lipschitz function ϕ with compact support in Ω such that the integral $\int_{\{\mathrm{supp}\,\phi\cap\nabla u\neq 0\}}|\nabla\phi|^p\phi^{1-p}\,dx$ is finite. Moreover, all quantities appearing in (3.10) are finite.

We show first that under our assumptions, when $\delta \searrow 0$, we have

$$\int_{\Omega\cap\{\nabla u\neq 0, u+\delta\le R\}} (u+\delta)^{p-\beta-1}|\nabla\phi|^p\phi^{1-p}dx \to \int_{\Omega\cap\{\nabla u\neq 0, u\le R\}} u^{p-\beta-1}|\nabla\phi|^p\phi^{1-p}dx, \tag{3.11}$$

whenever ϕ is a nonnegative Lipschitz function with compact support in Ω such that the integral $\int_{\mathrm{supp}\,\phi\cap\{\nabla u\neq 0\}}|\nabla\phi|^p\phi^{1-p}\,dx$ is finite.

To verify this, we note that $(u+\delta)^{p-\beta-1}\chi_{\{u+\delta\le R\}} \overset{\delta\to 0}{\to} u^{p-\beta-1}\chi_{\{u\le R\}}$ a.e. This follows from Fact 3.2.2 (which gives that the sets $\{u = 0,\ |\nabla u| \neq 0\}$

and $\{u = R,\ |\nabla u| = 0\}$ are of measure zero) and the continuity outside zero of the involved functions.

The function $\Theta(t) := t^{p-\beta-1}$ is decreasing or dominated in the neighbourhood of zero.

Let us start with the case when there exists $\varepsilon > 0$ such that for $t < \varepsilon$ the function $\Theta(t)$ is decreasing. Without loss of generality we may consider $\varepsilon \leq R$.

We divide the domain of integration

$$\int_{\Omega\cap\{\nabla u\neq 0,\, u+\delta\leq R\}} \Theta(u+\delta)\cdot|\nabla\phi|^p\phi^{1-p}\,dx =$$

$$= \int_{E_\varepsilon} \Theta(u+\delta)\cdot|\nabla\phi|^p\phi^{1-p}\,dx + \int_{F_\varepsilon} \Theta(u+\delta)\chi_{\{u+\delta\leq R\}}\cdot|\nabla\phi|^p\phi^{1-p}\,dx,$$

where

$$E_\varepsilon = \left\{u < \frac{\varepsilon}{2}\right\}\cap\operatorname{supp}\phi, \quad F_\varepsilon = \left\{\frac{\varepsilon}{2}\leq u\right\}\cap\operatorname{supp}\phi.$$

Let us begin with integral over E_ε. We consider $\delta \searrow 0$ so we may assume that $\delta < \varepsilon/2$. Then, over E_ε we have $u + \delta < \varepsilon$. As $\Theta(u)$ is decreasing for $u < \varepsilon$, for $\delta \searrow 0$ the function $\Theta(u + \delta)$ is increasing and convergent almost everywhere. Therefore, due to the Lebesgue's Monotone Convergence Theorem

$$\lim_{\delta\to 0}\int_{E_\varepsilon} \Theta(u+\delta)\cdot|\nabla\phi|^p\phi^{1-p}\,dx = \int_{E_\varepsilon} \Theta(u)\cdot|\nabla\phi|^p\phi^{1-p}\,dx.$$

In the case F_ε we have $\varepsilon/2 \leq u+\delta \leq R$. Over this domain Θ is a bounded function so in particular

$$\int_{F_\varepsilon} \Theta(u+\delta)\chi_{\{u+\delta\leq R\}}\cdot|\nabla\phi|^p\phi^{1-p}\,dx = \int_{\{\varepsilon/2\leq u+\delta\leq R\}} \Theta(u+\delta)\cdot|\nabla\phi|^p\phi^{1-p}\,dx \leq$$

$$\leq \left|R - \frac{\varepsilon}{2}\right| \sup_{t\in[\varepsilon/2,R]}\Theta(t)\cdot\int_{\operatorname{supp}\phi\cap\{\nabla u\neq 0\}} |\nabla\phi|^p\phi^{1-p}\,dx < \infty.$$

Taking into account convergence almost everywhere and boundedness of domain of integrating, we apply the Lebesgue's Dominated Convergence Theorem to write

$$\lim_{\delta\to 0}\int_{F_\varepsilon} \Theta(u+\delta)\chi_{\{u+\delta\leq R\}}\cdot|\nabla\phi|^p\phi^{1-p}\,dx = \int_{F_\varepsilon\cap\{u\leq R\}} \Theta(u)\cdot|\nabla\phi|^p\phi^{1-p}\,dx.$$

This completes the case Θ decreasing in the neighbourhood of 0. Let us consider the case of bounded Θ. We carry out the same reasoning as above for F_ε with $\varepsilon = 0$.

To complete this step we note that (3.11) says that, when $\delta \searrow 0$, the first integral on the right–hand side of (3.8) is convergent to the first integral of the right–hand side of (3.10). To deal with the second expression note that we have, for $\delta \leq \frac{R}{2}$:

$$|\tilde{C}(\delta, R)| \quad \leq \quad |\tilde{C}_2(\delta, R)| + |\tilde{C}_1(\delta, R)| \leq \tilde{C}(R).$$

We observe that condition (3.5) implies

$$(\Phi \cdot u + \sigma |\nabla u|^p)\, \chi_{\{u>0\}} \geq 0, \quad \text{a.e.} \quad \text{whenever} \quad \sigma \geq \sigma_0. \tag{3.12}$$

We can pass to the limit with the left–hand side of (3.8) due to The Lebesgue's Monotone Convergence Theorem as the expression in brackets is nonnegative due to (3.12) (for $\sigma = \beta - \frac{p-1}{p}\tau$) and decreasing. Note that $(\Phi \cdot u + \sigma |\nabla u|^p)\, u^{-\beta-1} \chi_{\{u>0\}} \equiv 0$, when $u \equiv 0$ and in particular does not depend on δ.

Step 3. We let $R \to \infty$ and finish the proof.

Without loss of generality we can assume that the integral in the right–hand side of (3.7) is finite, as otherwise the inequality follows trivially. Note that since $|\nabla u|^{p-2}\langle \nabla u, \nabla \phi\rangle$ and $\Phi\phi$ are integrable we have $\lim_{R\to\infty} \widetilde{C}(R) = 0$. Therefore, (3.7) follows from (3.10) by the Lebesgue's Monotone Convergence Theorem. □

3.3 General Hardy inequality

Now we state our main result of the first part of the thesis.

Theorem 3.3.1. *Assume that $1 < p < \infty$ and $u \in W^{1,p}_{loc}(\Omega)$ is a nonnegative solution to PDI $-\Delta_p u \geq \Phi$, in the sense of Definition 3.1.1, where Φ is locally integrable and satisfies $(\mathbf{\Phi}, \mathbf{p})$ with $\sigma_0 \in \mathbb{R}$ given by (3.5). Assume further that β and σ are arbitrary numbers such that $\beta > 0$ and $\beta > \sigma \geq \sigma_0$.*

Then, for every Lipschitz function ξ with compact support in Ω, we have

$$\int_\Omega |\xi|^p \mu_1(dx) \leq \int_\Omega |\nabla \xi|^p \mu_2(dx), \tag{3.13}$$

where

$$\mu_1(dx) \;= \left(\frac{\beta-\sigma}{p-1}\right)^{p-1} [\Phi\cdot u + \sigma|\nabla u|^p]\cdot u^{-\beta-1}\chi_{\{u>0\}}\; dx, \quad (3.14)$$

$$\mu_2(dx) \;= u^{p-\beta-1}\chi_{\{|\nabla u|\neq 0\}}\; dx. \quad (3.15)$$

Proof. We apply (4.15) with $\phi = \xi^p$, where ξ is nonnegative Lipschitz function with compact support. Then ϕ is Lipschitz and

$$|\nabla\xi|^p = \left(\frac{1}{p}\phi^{\frac{1}{p}-1}|\nabla\phi|\right)^p = \frac{1}{p^p}\left(\frac{|\nabla\phi|}{\phi}\right)^p \phi.$$

Therefore (3.7) becomes (3.13). Note that for every nonnegative Lipschitz function ξ with compact support in Ω we have $\int_\Omega |\nabla\xi|^p\,dx < \infty$, equivalently $\int_{\mathrm{supp}\,\phi} |\nabla\phi|^p\phi^{1-p}\,dx < \infty$. As the absolute value of a Lipschitz function is a Lipschitz function as well, we place it on the left–hand side to avoid requiring its nonnegativeness. $\square$

Remark 3.3.1. Note that, by conversing this substitution, we obtain inequality with a structure of (3.7).

Remark 3.3.2. We do not assume that the density of μ_2, the function $u^{p-\beta-1}\chi_{\{|\nabla u|\neq 0\}}$, is locally integrable. However, if it is locally integrable only on some subset $\Omega_1 \subseteq \Omega$, instead of (3.13) we may derive inequality

$$\int_{\Omega_1} |\xi|^p\mu_1(dx) \le \int_{\Omega_1} |\nabla\xi|^p\mu_2(dx),$$

for every Lipschitz function ξ with compact support in Ω_1, where μ_1 and μ_2 are given by (3.14) and (3.15), respectively.

Remark 3.3.3. Note that μ_1 is locally finite provided that additionally $u^{p-\beta-1}\chi_{\{|\nabla u|\neq 0\}}$ is locally integrable. We obtain it by the substitution of a compactly supported ξ such that $0 \le \xi \le 1$ and $\xi \equiv 1$ on a given ball of radius R contained in Ω shows that the condition: $u^{p-\beta-1}\chi_{\{|\nabla u|\neq 0\}}$ is locally integrable. In that case $\left(\frac{\beta-\sigma}{p-1}\right)^{p-1} [\Phi\cdot u + \sigma|\nabla u|^p]\cdot u^{-\beta-1}\chi_{\{u>0\}}$ is also locally integrable.

3.4 Special cases

3.4.1 Classical Hardy inequality

Our goal is to derive classical Hardy inequality with optimal constant (see e.g. [63], [75]) as a consequence of Theorem 3.3.1.

Theorem 3.4.1 (Classical Hardy inequality)**.** *Let* $1 < p < \infty$ *and* $\gamma \neq p-1$. *Suppose that* $\xi = \xi(x)$ *is an absolutely continuous function in* $(0,\infty)$ *such that*

$$\xi^+(0) := \lim_{x\to 0} \xi(x) = 0 \quad \text{for } \gamma < p-1, \tag{3.16}$$

$$\xi(\infty) := \lim_{x\to\infty} \xi(x) = 0 \quad \text{for } \gamma > p-1. \tag{3.17}$$

Then

$$\int_0^\infty \left(\frac{|\xi|}{x}\right)^p x^\gamma \, dx \leq C_{min} \int_0^\infty |\xi'|^p x^\gamma \, dx, \tag{3.18}$$

where the constant $C_{min} = \left(\frac{p}{|p-1-\gamma|}\right)^p$ *is optimal.*

Proof. We may assume that

$$\int_0^\infty |\xi'(x)|^p x^\gamma \, dx < \infty. \tag{3.19}$$

The proof follows by steps. Step 0 gives an explanation that it suffices to prove (3.18) for every compactly supported Lipschitz function ξ, while Steps 1–5 an present application of Theorem 3.3.1 to reach (3.18).

Step 0. By a standard convolution argument, having (3.18) for compactly supported Lipschitz functions, we deduce the inequality for compactly supported functions from weighted Sobolev space $W^{1,p}_{v_1,v_2}(\mathbb{R}_+)$ (defined in Preliminaries) where $v_1 = x^{\gamma-p},\ v_2 = x^\gamma$. This is because on compact subsets of $(0,\infty)$ the considered weights are comparable with constants.

We concentrate on the proof that (3.18) holds for absolutely continuous functions satisfying (3.19) and vanishing condition (3.16) or (3.17). Let ξ be such a function. We construct an approximative sequence $\xi_N \in W^{1,p}_{v_1,v_2}(\mathbb{R}_+)$ with compact support. Let

$$\varphi_N(x) = \begin{cases} Nx - 1, & x \in (\frac{1}{N}, \frac{2}{N}), \\ 1, & x \in (\frac{2}{N}, N), \\ -\frac{1}{N}x + 2, & x \in (N, 2N), \\ 0, & x \in (0, \frac{1}{N}) \cup (2N, \infty). \end{cases}$$

Then each $\xi_N = \varphi_N \cdot \xi$ is a compactly supported function from $W^{1,p}_{v_1,v_2}(\mathbb{R}_+)$. Thus, inequality (3.55) holds for each ξ_N. We pass to the limit with $N \to \infty$ obtaining

$$\int_0^\infty \left(\frac{|\xi\varphi_N|}{x}\right)^p x^\gamma \, dx \longrightarrow \int_0^\infty \left(\frac{|\xi|}{x}\right)^p x^\gamma \, dx \tag{3.20}$$

due to the Lebesgue's Monotone Convergence Theorem. Therefore, the left–hand side of (3.18) is as required. Furthermore, we notice that

$$\begin{aligned} \left(\frac{1}{C_{min}}\right)^{\frac{1}{p}} & \left(\int_0^\infty \left(\frac{|\xi\varphi_N|}{x}\right)^p x^\gamma \, dx\right)^{\frac{1}{p}} \le \left(\int_0^\infty |\xi_N'|^p x^\gamma \, dx\right)^{\frac{1}{p}} \le \\ \le & \left(\int_0^\infty |\varphi_N'\xi|^p x^\gamma \, dx\right)^{\frac{1}{p}} + \left(\int_0^\infty |\varphi_N\xi'|^p x^\gamma \, dx\right)^{\frac{1}{p}} =: a_N + b_N. \end{aligned} \tag{3.21}$$

Then b_N^p tends to the required right–hand side in (3.18) due to the Lebesgue's Monotone Convergence Theorem. By showing that $\{a_N\}_N$ is bounded, we prove

$$\lim_{N\to\infty} \int_0^\infty |\xi_N'|^p x^\gamma \, dx \le \lim_{N\to\infty} \int_0^\infty |\varphi_N\xi'|^p x^\gamma \, dx = \int_0^\infty |\xi'|^p x^\gamma \, dx.$$

We note that if $\int_0^\infty \left(\frac{|\xi|}{x}\right)^p x^\gamma \, dx < \infty$ then we have it. This follows from the fact that $\varphi_N' \sim \frac{1}{x}(\chi_{[\frac{1}{N},\frac{2}{N}]\cup[N,2N]})$. Indeed, we have

$$\begin{aligned} a_N^p &= \int_0^\infty |\varphi_N'\xi|^p x^\gamma \, dx = \int_{\frac{1}{N}}^{\frac{2}{N}} |\varphi_N'\xi|^p x^\gamma \, dx + \int_N^{2N} |\varphi_N'\xi|^p x^\gamma \, dx \\ &\le c \int_{[\frac{1}{N},\frac{2}{N}]\cup[N,2N]} \left|\frac{\xi}{x}\right|^p x^\gamma \, dx \overset{N\to\infty}{\to} 0. \end{aligned}$$

To show that the case $\int_0^\infty \left(\frac{|\xi|}{x}\right)^p x^\gamma \, dx = \infty$ is impossible, we use the following reasoning. It suffices to show that $\sup_N a_N < const$, because then after passing to the limit with $N \to \infty$ in (3.21) we necessarily have $\lim_{N\to\infty} b_N = \infty$, which contradicts with (3.19).

In this step we denote by c positive constants independent of N.

In the case $\gamma > p-1$ and condition (3.17) we define

$$\bar{\xi}(t) = \int_t^\infty |\xi'(\tau)| d\tau.$$

We observe that $|\xi(t)| = \left|\int_t^\infty \xi'(\tau)d\tau\right| \leq |\bar{\xi}(t)|$. We apply Hölder inequality to functions $f(\tau) = |\xi'(\tau)|\tau^{\frac{\gamma}{p}}$, $g(\tau) = \tau^{-\frac{\gamma}{p}}$ and obtain

$$|\bar{\xi}(t)| \leq \left(\frac{p-1}{\gamma-(p-1)}\right)^{\frac{p-1}{p}} \left(\int_t^\infty |\xi'(\tau)|^p \tau^\gamma d\tau\right)^{\frac{1}{p}} t^{-\frac{\gamma-(p-1)}{p}} \leq ct^{-\frac{\gamma-(p-1)}{p}}.$$

It implies

$$|\xi(t)|^p t^{\gamma-(p-1)} \leq c \quad \text{for every} \quad t > 0.$$

Consequently,

$$\begin{aligned}
\int_{[\frac{1}{N},\frac{2}{N}]} |\xi(\tau)|^p \, \tau^{\gamma-p} d\tau \quad &\leq \frac{1}{N} \sup_{\tau\in[\frac{1}{N},\frac{2}{N}]} \left\{ |\xi(\tau)|^p \, \tau^{\gamma-(p-1)} \cdot \frac{1}{\tau} \right\} \leq \frac{1}{N} cN = c, \\
\int_{[N,2N]} |\xi(t)|^p t^{\gamma-p} dt \quad &\leq N \sup_{\tau\in[N,2N]} \left\{ |\xi(\tau)|^p \tau^{\gamma-(p-1)} \cdot \frac{1}{\tau} \right\} \leq Nc\frac{1}{N} = c.
\end{aligned}$$

Therefore, the sequence $\{a_N\}_N$ is bounded in this case.

In the case $\gamma < p-1$ and condition (3.16) we define

$$\bar{\xi}(t) = \int_0^t |\xi'(\tau)| d\tau.$$

We observe that $|\xi(t)| = \left|\int_0^t \xi'(\tau)d\tau\right| \leq |\bar{\xi}(t)|$ and apply Hölder inequality for functions $f(\tau) = |\xi'(\tau)|\tau^{\frac{\gamma}{p}}$, $g(\tau) = \tau^{-\frac{\gamma}{p}}$ to get

$$|\bar{\xi}(t)| \leq \left(\frac{p-1}{(p-1)-\gamma}\right)^{\frac{p-1}{p}} \left(\int_0^t |\xi'(\tau)|^p \tau^\gamma d\tau\right)^{\frac{1}{p}} t^{-\frac{\gamma-(p-1)}{p}} \leq ct^{-\frac{\gamma-(p-1)}{p}}.$$

It implies $|\xi(t)|^p t^{\gamma-(p-1)} \leq |\bar{\xi}(t)|^p t^{\gamma-(p-1)} \leq c$. The remaining arguments are the same as in case $\gamma > p-1$.

This completes the proof of Step 0.

In the following steps we obtain inequality (3.18) for compactly supported Lipschitz functions by application of Theorem 3.3.1.

Step 1. Let us consider the function $u = u_\alpha(x) = x^\alpha$ where $0 \neq \alpha \in \mathbb{R}$. When $p > 1$ the function $u_\alpha(x)$ is nonnegative solution to the PDE

$$-\Delta_p u = -|\alpha|^{p-2}\alpha(p-1)(\alpha-1)x^{\alpha(p-1)-p} =: \Phi \quad \text{a.e. in } \Omega = (0,\infty).$$

Note that we deal with one–dimensional p–Laplacian $\Delta_p u = (|u'|^{p-2}u')'$.

Step 2. Constant σ_0 defined in (3.5) equals $\sigma_0 = \frac{1}{\alpha}(p-1)(\alpha-1)$. To verify this we note that

$$\begin{aligned}
\sigma_0 &= -\inf \frac{\Phi \cdot u}{|\nabla u|^p \chi_{\{u \neq 0\}}} = -\inf \frac{-|\alpha|^{p-2}\alpha(p-1)(\alpha-1)x^{\alpha(p-1)-p+\alpha}}{|\alpha|^p x^{(\alpha-1)p}} = \\
&= -\inf \frac{-\alpha(p-1)(\alpha-1)}{\alpha^2} = \frac{1}{\alpha}(p-1)(\alpha-1).
\end{aligned}$$

Step 3. For $\gamma \in \mathbb{R}$ and $\gamma \neq p-1$, define $\beta = \beta(\alpha,\gamma) := p - 1 - \frac{\gamma}{\alpha}$. Now we apply Theorem 3.3.1. For this we deal with arbitrary numbers β and σ such that $\beta > 0$ and $\beta > \sigma \geq \sigma_0 = \frac{1}{\alpha}(p-1)(\alpha-1)$. In our case β is already defined, we require that $p - 1 > \frac{\gamma}{\alpha}$ and the existence of the admissible σ is equivalent to the condition $\mathrm{sgn}\alpha(p-1-\gamma) > 0$.

Computing measures given by (3.14) and (3.15) directly, we obtain inequality

$$C \int_0^\infty |\xi|^p \, x^{\gamma - p} \, dx \leq \int_0^\infty |\xi'|^p \, x^\gamma \, dx. \tag{3.22}$$

where $C = C(\alpha,\beta,\sigma,p) = \frac{(\beta-\sigma)^{p-1}|\alpha|^p}{(p-1)^{p-1}}(\sigma - \frac{1}{\alpha}(p-1)(\alpha-1))$, holding for every Lipschitz function ξ with compact support in $(0,\infty)$.

Step 4. We observe that when γ is fixed, we can always choose α such that $p - 1 > \frac{\gamma}{\alpha}$ and $\mathrm{sgn}\alpha(p-1-\gamma) > 0$. The choice of

$$\sigma = \frac{1}{p}(\beta(\alpha) + \sigma_0(p-1)) = p - 1 - \frac{(p-1)^2 + \gamma}{p\alpha},$$

gives the inequality (3.22) with the maximal constant with respect to $\sigma \in [\sigma_0, \beta)$. Then, we divide both sides by the constant and obtain inequality (3.18) with the constant $\bar{C} = \left(\frac{p}{\mathrm{sgn}\alpha(p-1-\gamma)}\right)^p = \left(\frac{p}{|p-1-\gamma|}\right)^p$, holding for every Lipschitz function ξ with compact support in $(0,\infty)$. □

Remark 3.4.1. We point out that in the above proof we admit negative function Φ. For example in the case $p = 2, \ \gamma = 0, \ \alpha = 2, \ \beta = 1$ we have $\Phi \equiv -2$.

3.4.2 Inequalities involving measures with radial densities

Analysing radially symmetric solutions to the PDIs in Theorem 3.3.1 we obtain the following result.

Theorem 3.4.2. *Suppose $1 < p < \infty$ and β, σ are arbitrary numbers such that $\beta > 0$ and $\beta > \sigma \geq \sigma_0$. Let $w(x) \in W^{1,p}_{loc}(\mathbb{R}^n \setminus \{0\}) \cap W^{2,1}_{loc}(\mathbb{R}^n \setminus \{0\})$ be an arbitrary nonnegative radial function in the form $w(x) = u(|x|)$ such that*

$$\sigma_0 := -\operatorname{ess\,inf} \frac{u(t)}{u'(t)} \left(-\frac{u''(t)}{u'(t)}(p-1) - \frac{n-1}{t} \right) < \infty \quad \text{a.e. in } \{u(t) > 0\}. \tag{3.23}$$

Assume further that

$$\Phi(x) \ = \ |u'(|x|)|^{p-2} \left[-u''(|x|)(p-1) - u'(|x|) \cdot \frac{n-1}{|x|} \right] \tag{3.24}$$

is a locally integrable function.

Then, for every Lipschitz function ξ with compact support, we have

$$\int_{\mathbb{R}^n \setminus \{0\}} |\xi|^p \mu_1(dx) \leq \int_{\mathbb{R}^n \setminus \{0\}} |\nabla \xi|^p \mu_2(dx), \tag{3.25}$$

where

$$\begin{aligned} \mu_1(dx) \ &= \ \left(\frac{\beta - \sigma}{p-1} \right)^{p-1} \chi_{\{u>0\}} u^{-\beta-1}(|x|) |u'(|x|)|^{p-2} \cdot \\ &\quad \cdot \left[\sigma(u'(|x|))^2 - u''(|x|) u(|x|)(p-1) - u'(|x|) u(|x|) \frac{n-1}{|x|} \right] \ dx, \\ \mu_2(dx) \ &= \ u^{p-\beta-1}(|x|)\, \chi_{\{|\nabla u| \neq 0\}} \ dx. \end{aligned}$$

Proof. We apply Theorem 3.3.1. At first we observe that when $w(x) = u(|x|)$ we have

$$-\Delta_p w = \Phi(x) \quad \text{a.e. in } \mathbb{R}^n \setminus \{0\},$$

with locally integrable right–hand side. Condition $(\mathbf{\Phi}, \mathbf{p})$ is satified. Indeed, due to (3.23), we observe that

$$\sigma_0 (u'(t))^2 - u''(t) u(t)(p-1) - u'(t) u(t) \frac{n-1}{t} \geq 0 \quad \text{ a.e. in } \{u(t) > 0\}$$

and therefore, when $\sigma \geq \sigma_0$, almost everywhere in $\{u(t) > 0\}$ we have

$$\Phi \cdot u + \sigma |\nabla u|^p = |u'|^{p-2} \left[\sigma(u')^2 - u'' u(p-1) - u' u \frac{n-1}{t} \right] \geq 0.$$

Now it suffices to apply Theorem 3.3.1. □

As a direct consequence we retrieve Hardy inequality on $\mathbb{R}^n \setminus \{0\}$ with best constants [77].

Corollary 3.4.1 (Hardy inequality on $\mathbb{R}^n \setminus \{0\}$). *Suppose $p > 1$, $\gamma < p - n$. Then, for every nonnegative Lipschitz function ξ with compact support, we have*

$$\int_{\mathbb{R}^n\setminus\{0\}} |\xi|^p |x|^{\gamma-p}\, dx \le \widetilde{C}_{min} \int_{\mathbb{R}^n\setminus\{0\}} |\nabla\xi|^p |x|^{\gamma}\, dx.$$

where the constant $\widetilde{C}_{min} = \left(\frac{p}{p-n-\gamma}\right)^p$ is optimal.

Proof. Notice that $w(x) = |x| = u(|x|)$ satisfies assumptions of Theorem 3.4.2 with $\Phi(x) = -\frac{n-1}{|x|}$ and $\sigma_0 = -(n-1)$. Let $\beta > 0$ and $\beta > \sigma > -(n-1)$. Substituting it do the formulae describing measures we derive (3.25) with

$$\begin{aligned} \mu_1(dx) &= \left(\frac{\beta-\sigma}{p-1}\right)^{p-1} (\sigma - (n-1))|x|^{-\beta-1}\, dx, \\ \mu_2(dx) &= |x|^{p-\beta-1}\, dx. \end{aligned}$$

The choice of

$$\sigma = \frac{1}{p}(\beta + (n-1)(p-1))$$

gives the inequality (3.22) with the maximal constant with respect to $\sigma \in [-(n-1), \beta)$. The substitution of $\gamma = p - \beta - 1$ and division of both sides by the constant implies the final result. The fact that constant $\widetilde{C}_{min} = \left(\frac{p}{p-n-\gamma}\right)^p$ is the best possible is well known [77]. □

Remark 3.4.2. To ensure that inequality (3.25) has a good interpretation we must assume that function $u^{p-\beta-1}(|x|)\,\chi_{\{|\nabla u|\neq 0\}}$ is locally integrable. Then, also function

$$u^{-\beta-1}(|x|)|u'(|x|)|^{p-2}\chi_{\{u>0\}}\cdot$$
$$\cdot\left[\sigma(u'(|x|))^2 - u''(|x|)u(|x|)(p-1) - u'(|x|)u(|x|)\frac{n-1}{|x|}\right]$$

is locally integrable which follows from the argument from Remark 3.3.3.

Remark 3.4.3. We point out that $w(x) = u(|x|)$ is assumed to be more regular than only $W^{1,p}_{loc}(\mathbb{R}^n \setminus \{0\})$. Namely, by our assumption $u \in W^{2,1}_{loc}((0,\infty))$ (see e. g. Fact 2.1 in [1]). This implies that if $p \ge 2$, the function Φ is always locally integrable.

For qualitative properties of radial solutions to nonlinear eigenvalue problems having the form $-\Delta_p w(x) = \frac{1}{a(|x|)}\varphi(w(x))$, as well as for the nonexistence theorems, we refer to [1] and their references.

3.4.3 Hardy and Hardy–Poincaré inequalities with exponential weights

In this subsection we concentrate on the case when the measures in the derived inequality have exponential terms. We have the following result.

Theorem 3.4.3 (Hardy–Poincaré inequalities with exponential weights). *Assume that $p, be > 1$, $\kappa, q > 0$, $r \geq 0$, $\kappa q b > r(p-1)(b-1)$.*

Then, for every Lipschitz function ξ with compact support in $\mathbb{R}_+$, we have

$$\int_0^\infty |\xi|^p \mu_1(dx) \leq \widetilde{C} \int_0^\infty |\nabla \xi|^p \mu_2(dx), \tag{3.26}$$

where

$$\begin{aligned} \mu_1(dx) &= e^{\kappa x^b} x^{bp-p-b} \left(q + r x^b\right) dx, \\ \mu_2(dx) &= e^{\kappa x^b}\, dx, \end{aligned}$$

and the constant $\widetilde{C} = \left(\frac{p-1}{\kappa q b - r(p-1)(b-1)}\right)^{p-1} \frac{q^p}{(p-1)(b-1)}$.

Proof. Let us consider $a > 0$, $\beta > \sigma \geq p-1$, where those numbers will be stablished later, and the function $u = u_{a,b}(x) = e^{-ax^b}$ where $a > 0$, $b \geq 1$. The proof follows by steps.

Step 1. When $p > 1$ the function u is a nonnegative solution to the PDE

$$-\Delta_p u = |ab|^p (p-1) u^{p-1} x^{(p-1)(b-1)-1} \left(\frac{b-1}{ab} + x^b\right) =: \Phi \quad \text{a.e. in } \Omega = (0, \infty),$$

with locally integrable function Φ.

Indeed, $u'(x) = -abx^{b-1}u(x)$, so

$$\begin{aligned} -\Delta_p u &= -(|u'|^{p-2}u')' = -(-ab|ab|^{p-2} x^{(p-1)(b-1)} u^{p-1})' = \\ &= ab|ab|^{p-2} x^{(p-1)(b-1)} \left((p-1)(b-1)x^{-1}u^{p-1} - ab(p-1)x^{b-1}u^{p-1}\right) \\ &= |ab|^p (p-1) u^{p-1} x^{(p-1)(b-1)-1} \left(\frac{b-1}{ab} - x^b\right). \end{aligned}$$

STEP 2. We recognize that $\sigma_0 = p-1$. Indeed, when we note that $|u'(x)|^p = |ab|^p x^{p(b-1)} u^p(x)$, we obtain

$$\begin{aligned}
[\Phi \cdot u + \sigma|\nabla u|^p] &= |ab|^p u^p x^{(p-1)(b-1)-1} \left((p-1)\left[\frac{b-1}{ab} - x^b\right] + \sigma x^b \right) = \\
&= |ab|^p u^p x^{(p-1)(b-1)-1} \left((p-1)\frac{b-1}{ab} + [\sigma - (p-1)]\, x^b \right),
\end{aligned}$$

which is nonnegative for $\sigma \geq p-1$, as $p>1,\ a>0,\ b\geq 1$.

STEP 3. Computing measures given by (3.14) and (3.15) directly, we obtain inequality (3.13) with

$$\begin{aligned}
\mu_1(dx) &= |ab|^p e^{-a(p-\beta-1)x^b} x^{(p-1)(b-1)-1} \left((p-1)\frac{b-1}{ab} + [\sigma-(p-1)]\, x^b \right) dx, \\
\mu_2(dx) &= e^{-a(p-\beta-1)x^b}\, dx, \\
C &= \left(\frac{p-1}{\beta-\sigma} \right)^{p-1}.
\end{aligned}$$

It suffices to substitute now a, β, σ such that

$$\begin{cases} \kappa = a(\beta - (p-1)), \\ r = \sigma - (p-1), \\ q = \frac{(p-1)(b-1)}{ab}. \end{cases} \tag{3.27}$$

Then

$$\begin{cases} a = \frac{p-1}{q}\frac{b-1}{b}, \\ \sigma = r + (p-1), \\ \beta = p-1+\frac{\kappa q b}{(p-1)(b-1)}. \end{cases} \tag{3.28}$$

The condition $\beta > \sigma \geq p-1$ requires $r \geq 0,\ \kappa, q > 0,\ b>1,\ \kappa q b > r(p-1)(b-1)$. Then, we divide by the constant and obtain (3.26) with $\tilde{C} = \left(\frac{p-1}{\kappa q b - r(p-1)(b-1)} \right)^{p-1} \frac{q^p}{(p-1)(b-1)}$. $\square$

As a consequence we obtain the following theorem, which can also be obtained from Corollary 3.1 from [68]. Two independent arguments are enclosed.

Theorem 3.4.4 (Hardy inequalities with exponential weights)**.** *If $p, b > 1$ and $\kappa > 0$, then for every Lipschitz compactly supported function ξ, we have*

$$\int_0^\infty |\xi(x) x^{b-1}|^p e^{\kappa x^b} dx \leq \left(\frac{p}{\kappa b}\right)^p \int_0^\infty |\xi'(x)|^p e^{\kappa x^b} dx. \tag{3.29}$$

Proof. Method I. We apply Theorem 3.4.3. We omit a positive term in the left–hand side of (3.26): $qe^{\kappa x^b}x^{bp-p-b}$ and we minimize the constant

$$\widetilde{C} = \frac{1}{(p-1)(b-1)^p}\frac{q}{r}\left(\frac{1}{m-\frac{r}{q}}\right)^{p-1} = \frac{1}{m^p(p-1)(b-1)^p}\frac{qm}{r}\left(\frac{1}{1-\frac{r}{qm}}\right)^{p-1},$$

where $m = \frac{\kappa b}{(p-1)(b-1)}$, with respect to arbitrary $q, r > 0$ such that $qm/r > 1$. We reach it by minimizing the function $f(t) = t^{-1}(m-t)^{1-p}$ with respect to $0 < t < 1$. We obtain the constant as required.
Method II. We apply Corollary 3.1 from [68]. In this case we deal with $M(\lambda) = \lambda^p$, $d_M = D_M = p$, $\mu(dx) = e^{-\varphi(x)}dx$, $\varphi(x) = -\kappa x^b$, $\omega(x) = |\varphi'(x)|$, $c(x) = x^p$. In such a case we have

$$\begin{aligned} \varphi'(x) &= -\kappa b x^{b-1}, \\ \varphi''(x) &= -\kappa b(b-1)x^{b-2}, \\ b_1(x, |\varphi'|, \varphi, M) &= 1 + (p-1)\frac{b-1}{\kappa b}\frac{1}{x^b} \end{aligned}$$

Then $b_1 = \inf\{b_1(x, \omega, \varphi, M) : x > 0\} = 1 > 0$. Therefore, Corollary 3.1 from [68] asserts that under this conditions we obtain (3.29). $\square$

3.4.4 Inequalities derived using p–superharmonic functions

In this subsection we analyse the case when nonnegative u is a p–superharmonic function, e.i.

$$-\Delta_p u \geq \Phi \equiv 0 \tag{3.30}$$

in the sense of distributions. These results can be also obtained by the techniques by D'Ambrosio [36]. We present them as a direct consequence of Theorem 3.3.1.

Theorem 3.4.5. *Assume that $1 < p < \infty$, $u \in W^{1,p}_{loc}(\Omega)$ is a nonnegative solution to* (3.30) *in the sense of distributions. Let $\beta > p+1$ be an arbitrary number.*

Then, for every Lipschitz function ξ with compact support in Ω, we have

$$\int_\Omega |\xi|^p \left(\frac{|\nabla u|}{u}\right)^p \, d\mu \leq \frac{1}{\beta - 1 - p}\left(\frac{p-1}{p+1}\right)^{p-1} \int_\Omega |\nabla \xi|^p \, d\mu, \tag{3.31}$$

where $d\mu = u^{-\beta-1+p}dx$.

Proof. It is a consequence of Theorem 3.3.1 when we substitute $\sigma_0 = 0$ and $\sigma = \beta - 1 - p$. $\square$

Remark 3.4.4. If $u^{-\beta-1+p}\chi_{\{|\nabla u|\neq 0\}}$ is locally integrable only on some open subset $\Omega_1 \subseteq \Omega$, we interpret this inequality as in Remark 3.3.2, namely holding for ξ's with compact support in Ω_1.

Since we know superharmonic functions for some domains, we can construct now new Hardy–type inequalities. For example, substituting $p = 2$ in Theorem 3.4.5, we obtain the following corollary.

Corollary 3.4.2. *Assume that $u \in W^{1,2}_{loc}(\Omega)$ is a nonnegative superharmonic function, e.i. $\Delta u \leq 0$ in Ω in the sense of distributions. Assume further that $\beta > 3$ is an arbitrary number.*

Then, for every Lipschitz function ξ with compact support in Ω, we have

$$\int_\Omega |\xi|^2 \ |\nabla u|^2 u^{-\beta-1} \ dx \leq \frac{1}{3(\beta-3)} \int_\Omega |\nabla \xi|^2 u^{-\beta+1} \ dx.$$

Using integral representations of u being a solution to

$$\begin{cases} -\Delta u(x) = f & \text{in } \Omega, \\ u(x) = g & \text{on } \partial\Omega, \end{cases} \tag{3.32}$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded open subset with boundary of class C^1, f, g are sufficiently regular nonnegative functions, we may produce other Hardy inequalities described in terms of f, g and a Green function for a domain.

As an example we state the following theorem dealing with harmonic u, e.i. $f = 0$. Note that in particular, knowing arbitrary $g \in C(\partial\Omega)$, we may construct an inequality inside Ω.

Theorem 3.4.6. *Let $\beta > 3$ be an arbitrary number, $n \geq 2$, $\Omega \subseteq \mathbb{R}^n$ be an open bounded subset with boundary of class C^1, $G(x,y)$ be a Green function for Ω and $g \in C(\partial\Omega)$ is nonnegative and nonzero. We define operator $T : C(\partial\Omega) \to C^\infty(\Omega) \cap C(\bar{\Omega})$ by the formula*

$$Tg(x) = -\int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x,y) g(y) \, \mathcal{H}^{n-1}(dy), \tag{3.33}$$

where ν is the outer normal vector on $\partial\Omega$ and $\mathcal{H}^{n-1}$ is $(n-1)$–dimensional Hausdorff measure on $\partial\Omega$.

Then, for every Lipschitz function ξ with compact support in Ω, we have

$$\int_\Omega |\xi|^2\, \mu_1(dx) \le \int_\Omega |\nabla\xi|^2\, \mu_2(dx),$$

where

$$\begin{aligned} \mu_1(dx) &= |\nabla Tg(x)|^2 \ \ (Tg(x))^{-\beta-1}\ dx, \\ \mu_2(dx) &= \frac{1}{3(\beta-3)}\,(Tg(x))^{-\beta+1}\ dx. \end{aligned}$$

Proof. We apply Corollary 3.4.2. We substitute as u, a solution to a Laplace equation $\Delta u = 0$ in Ω with a boundary condition $u = g$ on $\partial\Omega$. Indeed, $u(x) = Tg(x) = -\int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x,y)g(y)\,\mathcal{H}^{n-1}(dy)$. □

Remark 3.4.5. The above result can be generalised by an application of u being the solution to (3.32) with the nonnegative functions $f \in L^2(\Omega)$, $g \in C(\partial\Omega)$. In such a case the operator Tg in (3.33) should be replaced by

$$T_{f,\ g}(x) = \int_\Omega G(x,y)f(y)\,dy - \int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x,y)g(y)\,\mathcal{H}^{n-1}(dy),$$

where ν is the outer normal vector on $\partial\Omega$ and $\mathcal{H}^{n-1}$ is the $(n-1)$–dimensional Hausdorff measure on $\partial\Omega$.

3.4.5 Hardy inequalities resulting from the PDI $-\Delta_p u \ge \Phi$ with negative Φ

The previous subsection confirms the results of D'Ambrosio [36]. In this subsection we show the example violating one of his assumptions. Namely, we derive here Hardy inequality from the problem $-\Delta_p u \ge \Phi$ when the function Φ is negative. For some choice of parameters we have already admitted nonpositive Φ in the proof of classical Hardy inequality, see Remark 3.4.1. The following theorem deals with the case when $\Phi < 0$ everywhere.

Theorem 3.4.7. *Assume that $1 < p < \infty$ and $\beta > p-1$. Then, there exists a constant $c = c(p,\beta)$ such that for every compactly supported Lipschitz function ξ, we have*

$$\int_0^\infty |\xi|^p \mu_1(dx) \le c\int_0^\infty |\nabla\xi|^p \mu_2(dx), \tag{3.34}$$

where

$$\begin{aligned}\mu_1(dx) &\sim x^{-\beta-1}\ln^{p-\beta-1}(e+x)\ dx, &(3.35)\\ \mu_2(dx) &= x^{p-\beta-1}\ln^{p-\beta-1}(e+x)\ dx. &(3.36)\end{aligned}$$

Proof. We apply Theorem 3.3.1 with a function $u(x) = x\ln(e+x)$. As u is increasing, we have

$$\Phi = -\Delta_p u = -(|u'|^{p-2}u')' = -(|u'|^{p-1})' = -(p-1)|u'|^{p-2}u'',$$

where

$$\begin{aligned}u'(x) &= \ln(e+x) + \frac{x}{e+x} = \frac{(e+x)\ln(e+x)+x}{e+x} = |u'(x)|,\\ u''(x) &= \frac{1}{e+x} + \frac{e}{(e+x)^2} = \frac{x+2e}{(e+x)^2},\\ \Phi &= -(p-1)|u'|^{p-2}\frac{x+2e}{(e+x)^2} < 0.\end{aligned}$$

We have to choose $\sigma_0 \leq \sigma$ ensuring nonnegativeness of

$$\begin{aligned}&\Phi\cdot u + \sigma|u'|^p =\\ =\ & -(p-1)|u'|^{p-2}\frac{x+2e}{(e+x)^2}\cdot u + \sigma|u'|^p =\\ =\ & \left[-(p-1)|u'|^{p-2}\frac{((e+x)\ln(e+x)+x)^2}{(e+x)^2}\frac{(x+2e)x\ln(e+x)}{((e+x)\ln(e+x)+x)^2} + \sigma|u'|^p\right] =\\ =\ & \left[-(p-1)\frac{x(x+2e)\ln(e+x)}{((e+x)\ln(e+x)+x)^2} + \sigma\right]|u'|^p. \qquad (3.37)\end{aligned}$$

We require $x(x+2e)\ln(e+x) < \frac{\sigma}{p-1}((e+x)\ln(e+x)+x)^2$ for $\sigma \geq \sigma_0$. It is easy to compute that $0 \leq \sigma_0 \leq p-1$. This follows from the following arguments:

$$\begin{aligned}0 &\leq h(x) := \frac{x(x+2e)}{(e+x)^2}\cdot\frac{\ln(e+x)}{(\ln(e+x)+x)^2} \leq \frac{(\frac{x+(x+2e))}{2})^2}{(e+x)^2}\frac{1}{\ln(e+x)} = 1\cdot\frac{1}{\ln(e)} = 1,\\ \sigma_0 &= -\inf_{x\in(0,\infty)}(-(p-1)h(x)) = (p-1)\sup(h(x)) \leq p-1.\end{aligned}$$

It is enough to consider $\sigma = p-1$.

We apply Theorem 3.3.1 and obtain (3.34) with the following measures

$$\begin{aligned}\mu_1(dx) &= \frac{(\beta-p+1)^{p-1}}{(p-1)^{p-2}}\left[1-\frac{x(x+2e)\ln(e+x)}{((e+x)\ln(e+x)+x)^2}\right]|u'|^p u^{-\beta-1}dx,\\ \mu_2(dx) &= (x\ln(e+x))^{p-\beta-1}\ dx.\end{aligned}$$

It suffices to estimate the growth rate of the density of $\mu_1(dx)$. We notice that the expression in square brackets in (3.37) is comparable with a constant, moreover

$$|u'|^p u^{-\beta-1} = \left(\frac{(e+x)\ln(e+x)+x}{e+x}\right)^p (x\ln(e+x))^{-\beta-1} \sim x^{-\beta-1}\ln^{p-\beta-1}(e+x).$$

This completes the proof. □

Above result can be compared with the statement of Proposition 5.2 in [70] (expressed in Orlicz setting) stated below. In our case we have $M(\lambda)=\lambda^p$, $\bar\beta=p-\beta-1<0$, $\gamma=\bar\beta$, $\alpha=-1$, $d_M=D_M=p$ and case b) applies. The proofs in both mentioned statements are fairly different.

Proposition 3.4.1. *Let M satisfy the Δ_2–condition, and the weights ω, $\rho:\mathbb{R}_+\to\mathbb{R}_+$ be given by* **(i)** *or* **(ii)** *where:*

(i) $\omega(r)=r^\alpha(\ln(2+r))^\delta$, $\rho(r)=r^{\bar\beta}(\ln(2+r))^\gamma$, $\alpha\in[-1,0)$, $\bar\beta,\delta,\gamma\in\mathbb{R}$,

(ii) $\omega(r)=r^\alpha$, $\rho(r)=r^{\bar\beta}\mathrm{e}^{-cr^\gamma}$, $\alpha\in[-1,0)$, $\bar\beta\in\mathbb{R}$, $\gamma,c>0$.

Then inequalities

$$\int_0^\infty M(\omega(r)|u(r)|)\rho(r)dr \le C_1\int_0^\infty M(|u(r)|)\,\rho(r)dr+C_2\int_0^\infty M(|u'(r)|)\rho(r)dr, \tag{3.38}$$

and

$$\|\omega u\|_{L^M((0,\infty),\rho)} \le \tilde C_1\|u\|_{L^M((0,\infty),\rho)}+\tilde C_2\|u'\|_{L^M((0,\infty),\rho)}, \tag{3.39}$$

hold for every $u\in W$, with positive constants independent of u, where

a) $W=W^{1,M}((0,\infty),\rho)$ *when* $\bar\beta>|\alpha|D_M-1$,

b) $W=\{u\in W^{1,M}((0,\infty),\rho):\liminf_{r\to0^+}M(r^\alpha|u(r)|)r^{\bar\beta+1}=0\}$, *when* $\bar\beta<|\alpha|d_M-1$.

The sets W are maximal subsets of $W^{1,M}((0,\infty),\rho)$ on which (3.38) holds true.

We have more general observation, which implies Theorem 3.4.7.

Theorem 3.4.8. *Suppose that $u : (0,\infty) \to [0,\infty)$ is nondecreasing and convex, $u \not\equiv const$, $u \in W^{2,1}_{loc}((0,\infty))$ and there exists some $a > 0$ such that*

$$a \cdot (u'(x))^2 \geq u''(x)u(x) \quad \text{a.e. in } (0,\infty) \tag{3.40}$$

and $a \cdot (u'(x))^2 - u''(x)u(x) \not\equiv 0$ a.e. Moreover, let $1 < p < \infty$ and $\beta > a(p-1)$. Then there exists a constant $c = c(p,\beta,u)$ such that for every compactly supported Lipschitz function ξ, we have

$$\int_0^\infty |\xi|^p \mu_1(dx) \leq c \int_0^\infty |\nabla \xi|^p \mu_2(dx), \tag{3.41}$$

where

$$\begin{aligned} \mu_1(dx) &= \frac{(\beta - a(p-1))^{p-1}}{(p-1)^{p-2}} |u'|^{p-2} \left\{ -u''(x)u(x) + a(u'(x))^2 \right\} u^{-\beta-1} \chi_{\{u>0\}}\, dx, \\ \mu_2(dx) &= u^{p-\beta-1} \chi_{\{u' \neq 0\}}\, dx. \end{aligned}$$

Proof. An easy verification shows that $-\Delta_p u = -(p-1)|u'|^{p-2}u''$ and condition $(\mathbf{\Phi}, \mathbf{p})$ is satisfied with $0 \leq \sigma_0 \leq (p-1)a$. Therefore we can substitute $\sigma = (p-1)a$ and apply Theorem 3.3.1. $\square$

Remark 3.4.6. Function $u(x) = x \ln(e+x)$ from the proof of Theorem 3.4.7 satisfies condition (3.40) and therefore Theorem 3.4.7 follows from Theorem 3.4.8 as a special case. Below we present some other examples of functions u admitted to Theorem 3.4.8:

i) $u(x) = e^x$ implies (3.34) with $\mu_1(dx) \sim \mu_2(dx) \sim e^{(p-\beta-1)x} dx$,

ii) $u(x) = (x+1)e^x$ implies (3.34) with

$$\mu_1(dx) \sim \mu_2(dx) \sim ((x+1)e^x)^{p-\beta-1}\, dx.$$

Remark 3.4.7. Condition (3.40) can be interpreted as converse pointwise multiplicative inequality dealing with nonnegative functions which states that

$$u'(x)^2 \leq Cu(x)Mu''(x),$$

where $Mv(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |v(y)| dy$ is Hardy–Littlewood maximal function of v and supremum is taken with respect to balls B containing x (see [84], inequality (1.9) on page 93). It is clear that $|v(x)| \leq Mv(x)$ a.e., but converse inequality, even up to a constant, in general does not hold.

3.5 Applications

3.5.1 Analysis of Bessel pairs

Our approach shows constructive way to build Bessel pairs, defined by Ghoussoub and Moradifam in [53] as following.

Definition 3.5.1 (Bessel pair). *Pair (V, U), such that for all $\xi \in C_0^\infty(B_R)$*

$$(H_{V,U}) \qquad \int_{B_R} U(x)\xi^2\,dx \le \int_{B_R} V(x)|\nabla\xi(x)|^2\,dx, \tag{3.42}$$

is called a Bessel pair.

They obtained the following result.

Theorem 3.5.1 ([53], Theorem 2.1). *Let V and U be positive radial C^1–functions on $B_R \setminus \{0\}$, where B_R is a ball centered at zero with radius R ($0 < R \le +\infty$) in $\mathbb{R}^n$ ($n \ge 1$). Assume that $\int_0^a \frac{1}{r^{n-1}V(r)}\,dr = +\infty$ and $\int_0^a r^{n-1}V(r)\,dr < +\infty$ for some $0 < a < R$. Then the following two statements are equivalent:*

1. *The ordinary differential equation*

$$(B_{V,U}) \quad y''(r) + \left(\frac{n-1}{r} + \frac{dV(r)}{dr}\frac{1}{V(r)}\right)y'(r) + \frac{U(r)}{V(r)}y(r) = 0$$

has a positive solution on the interval $(0, R]$ (possibly with $y(R) = 0$).

2. *(V, U) is a Bessel pair.*

As a direct consequence of Theorem 3.4.2 with $p = 2$, we obtain the following corollary related to above theorem.

Corollary 3.5.1. *Suppose B_R is a ball centered at zero with radius R ($0 < R \le +\infty$) in $\mathbb{R}^n$ ($n \ge 1$). Let $w(x) \in C^1(B_R \setminus \{0\})$ be an arbitrary nonnegative radial function in the form $w(x) = u(|x|)$ such that*

$$\sigma_0 := -\inf \frac{u(t)}{u'(t)}\left(\frac{u''(t)}{u'(t)} + \frac{n-1}{t}\right) < \infty \quad \text{a.e.} \quad t \in [0, R],$$

and $W_1(x), W_2(x)$ be positive radial C^1–functions on $B_R \setminus \{0\}$, such that

$$\begin{aligned} W_1(x) &= u^{-\beta+1}(|x|)\, \chi_{\{|\nabla u|\neq 0\}}, \\ W_2(x) &= (\beta-\sigma)\, \chi_{\{u>0\}} u^{-\beta-1}(|x|) \cdot \\ &\cdot \left[\sigma(u'(|x|))^2 - u''(|x|)u(|x|) - u'(|x|)u(|x|)\frac{n-1}{|x|} \right], \end{aligned}$$

where β and σ are arbitrary numbers such that $\beta > 0$ and $\beta > \sigma \geq \sigma_0$. Moreover, assume that $\int_0^a \frac{1}{r^{n-1}W_1(r)}\, dr = +\infty$ and $\int_0^a r^{n-1}W_1(r)\, dr < +\infty$ for some $0 < a < R$.
Then (W_1, W_2) is a Bessel pair.

Proof. We give the proof by applying two methods.
METHOD I. (via Theorem 3.4.2). We apply Theorem 3.4.2 with $p = 2$.
METHOD II (via Theorem 3.5.1). We note that $y(x) = u^{\beta-\sigma}(|x|)$, solves ODE $(B_{V,U})$ with $V = W_1,\ U = W_2$. In particular, the solution to (B_{W_1,W_2}) exists and (W_1, W_2) is a Bessel pair. □

Remark 3.5.1. It would be interesting to obtain generalisation of Corollary 3.5.1, considering the extension of inequality (3.42):

$$(H^p_{V,U}) \qquad \int_{B_R} U(x)|\xi|^p\, dx \leq \int_{B_R} V(x)|\nabla\xi(x)|^p\, dx,$$

to general p.

3.5.2 Inequalities resulting from existence theorems in equations arising in astrophysics

In some cases one can prove existence of solutions to either equation or inequality having the form

$$\begin{cases} -\Delta_p u(x) &= \varphi(x)u^{p-1}(x), \\ u(x) &\geq 0, \end{cases}$$

or more general

$$\begin{cases} -\Delta_p u(x) &\geq \varphi(x)u^{p-1}(x), \\ u(x) &\geq 0 \end{cases} \tag{3.43}$$

under certain general assumptions.

Such problems arise often in astrophysics to model several phenomena. For example, one observes this type of problems in classical models of globular clusters of stars such as Eddington's equation [43]

$$-\Delta u(x) = \frac{1}{1+|x|^2} e^{2u(x)},$$

its improved version — Matukuma's equations [82]

$$-\Delta u(x) = \frac{1}{1+|x|^2} u^p(x),$$

and its generalisations. Qualitative properties of their solutions are also considered from mathematical point of view [42]. Another astrophysical phenomena modelled in this way is the dynamics of elliptic galaxies. The model, which has been proposed by Bertin and Ciotti, has the form

$$\begin{cases} -\Delta u(x) \geq \frac{r^{2\alpha}}{(1+r^2)^{1/2+\alpha}} |u(x)|^{p-2} u(x), & \text{in } \mathbb{R}^3 \\ u(x) > 0 & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} \varphi(r) u^{p-1}(x)\, dx = K < \infty, \end{cases} \tag{3.44}$$

where $x = (x_1, x_2, z) \in \mathbb{R}^3$, $r = \sqrt{x_1^2 + x_2^2}$, $\alpha \geq 0$. For various astrophysical models, their introduction and discussion, we refer to [7, 12, 13, 28].

In paper [7] Badiale and Tarantello consider existence of cylindrically symmetric solutions to a problem of a type (3.43), based on (3.44), having following form

$$\begin{cases} -\Delta u(x) \geq \varphi(r) |u(x)|^{p-2} u(x) & \text{in } \mathbb{R}^3, \\ u(x) > 0 & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} \varphi(r) u^{p-1}(x)\, dx = K < \infty, \end{cases} \tag{3.45}$$

where $p > 1$, $x = (x_1, x_2, z) \in \mathbb{R}^3$, $r = \sqrt{x_1^2 + x_2^2}$, $u(x) = u(r, z)$ is a cylindrically symmetric function and φ is a nonnegative continuous function depending only on r, vanishing both in zero and in infinity, $r\varphi(r) \in L^\infty(\mathbb{R}_+)$. The condition $\int_{\mathbb{R}^3} \varphi(r) u^{p-1}(x)\, dx < \infty$ guarantees that a given solution carries a finite total mass.

Remark 3.5.2. Let us point out that in many cases (see e.g. [7, 53], (4.12) in [8]), existence is an effect of certain Hardy inequalities. Here we obtain possibly new Hardy inequalities as a consequence of existence. It would be interesting to analyse the connections between them.

It appears that even when we do not know u solving (3.45), but we have the information that it exists, we can still deduce some Hardy inequalities for a Lipschitz function ξ with compact support in Ω. We assume existence in a more general setting than (3.45) and derive Hardy and Hardy–Sobolev inequalities. Generalisation admits taking into account p–Laplacian for $p \in (1,\infty)$ instead of Laplacian, possibly other domain (Ω being any open subset of $\mathbb{R}^3$), moreover we do not require cylindrical symmetry of u. For related existence results we refer to [7, 42]. Below we state two results constructing Hardy and Hardy–Sobolev inequalities under the assumption of existence of solutions to the generalized problem, namely (3.46). In the first case we deal with $p = q$ while in the second one we assume $0 < q < p$ and additional information about integrability of the solution. Note that in Theorem 3.5.3 power of integrability of ξ appearing in the left–hand side of derived Hardy–Sobolev inequality is smaller than p.

Theorem 3.5.2. *Suppose* $1 < p < \infty$ *and there exists* $u \in W^{1,p}_{loc}(\Omega)$ *— a solution to*

$$\begin{cases} -\Delta_p u(x) \geq \varphi(x)|u(x)|^{p-2}u(x) & \text{in } \Omega, \\ u(x) \geq 0 & \text{in } \Omega, \end{cases}$$

where $\Phi = \varphi(x)|u(x)|^{p-2}u(x)$ *is locally integrable.*

Then, for every Lipschitz function ξ *with compact support in* Ω*, we have*

$$\int_\Omega |\xi|^p \, \varphi \, dx \leq \int_\Omega |\nabla \xi|^p \, dx.$$

Proof. We note that $\Phi \geq 0$ so we take $\sigma_0 = 0$ and the condition $(\mathbf{\Phi}, \mathbf{p})$ is satisfied with every $\sigma \geq \sigma_0$. We apply Theorem 3.3.1 and we obtain Hardy inequality for every Lipschitz function ξ with compact support in Ω of the following form

$$\int_\Omega |\xi|^p \mu_1(dx) \leq \int_\Omega |\nabla \xi|^p \mu_2(dx),$$

where

$$\begin{aligned} \mu_1(dx) &= \left(\frac{\beta - \sigma}{p-1}\right)^{p-1} \left[\varphi |u|^{p-2}u \cdot u + \sigma |\nabla u|^p\right] \cdot u^{-\beta-1} \chi_{\{u>0\}} \, dx, \\ \mu_2(dx) &= u^{p-\beta-1} \chi_{\{|\nabla u| \neq 0\}} \, dx \end{aligned}$$

where $\beta > \sigma$. We take $\sigma = \sigma_0 \geq 0$ and thus the term $\sigma |\nabla u|^p$ in $\mu_1(dx)$ is cancelled. The choice $\beta = p - 1$ completes the proof. □

Theorem 3.5.3. *Let $1 < p < \infty$. Suppose $1 < q < p$ is such that there exists $u \in W^{1,p}_{loc}(\Omega)$ — a solution to*

$$\begin{cases} -\Delta_p u(x) \geq \varphi(x)|u(x)|^{q-2}u(x) & \text{in } \Omega, \\ u(x) \geq 0 & \text{in } \Omega, \\ \int_\Omega \varphi(x)u^{q-1}(x)\,dx = K < \infty, \end{cases} \tag{3.46}$$

where $\Phi = \varphi(x)|u(x)|^{p-2}u(x)$ is locally integrable.

Then, for every Lipschitz function ξ with compact support in Ω, we have

$$\left(\int_\Omega |\xi|^{\frac{p(q-1)}{p-1}}\varphi(x)\,dx\right)^{\frac{p-1}{p(q-1)}} \leq C\left(\int_\Omega |\nabla\xi|^p\,dx\right)^{\frac{1}{p}} \tag{3.47}$$

with $C = K^{\frac{p-q}{p(q-1)}}\left(\frac{p-1}{p-1-\sigma_0}\right)^{\frac{p-1}{p}}$.

Proof. We note that $\Phi \geq 0$ so we can take $\sigma_0 = 0$ and the condition $(\mathbf{\Phi}, \mathbf{p})$ is satisfied with every $\sigma \geq \sigma_0$. Suppose $\beta > 0$ and ξ is an arbitrary Lipschitz function with compact support in Ω. If $q \in (0,p)$ then Hölder inequality for $f = (\varphi u^{q-1})^{1-\frac{q-1}{p-1}}$ and $g = (|\xi|^p\varphi u^{q-p})^{\frac{q-1}{p-1}}$ with parameter $\frac{p-1}{q-1}$ gives

$$\begin{aligned} \int_\Omega |\xi|^{\frac{p(q-1)}{p-1}}\varphi(x)\,dx &= \int_\Omega (\varphi u^{q-1})^{1-\frac{q-1}{p-1}}(|\xi|^p\varphi u^{q-p})^{\frac{q-1}{p-1}}\,dx \leq \\ &\leq \left(\int_\Omega \varphi(x)u^{q-1}(x)\,dx\right)^{1-\frac{q-1}{p-1}}\left(\int_\Omega |\xi|^p\varphi(x)u^{q-p}\,dx\right)^{\frac{q-1}{p-1}} = \\ &= K^{1-\frac{q-1}{p-1}}\left(\int_\Omega |\xi|^p\varphi(x)u^{q-1-\beta}\,dx\right)^{\frac{q-1}{p-1}}, \end{aligned}$$

where $p = \beta + 1 > 1$. The second inequality comes from the condition from the third line in (3.46). Moreover, Theorem 3.3.1 says that existence of solution to (3.46) implies in particular inequality

$$\int_\Omega |\xi|^p\varphi u^{q-1-\beta}\chi_{\{u>0\}}\,dx \leq \left(\frac{p-1}{\beta-\sigma_0}\right)^{p-1}\int_\Omega |\nabla\xi|^p\,dx.$$

Summing up above observations, we obtain

$$\int_\Omega |\xi|^{\frac{p(q-1)}{p-1}}\varphi(x)\,dx \leq K^{1-\frac{q-1}{p-1}}\left(\frac{p-1}{\beta-\sigma_0}\right)^{q-1}\left(\int_\Omega |\nabla\xi|^p\,dx\right)^{\frac{q-1}{p-1}}.$$

To obtain (3.47) it suffices to put once again $\beta = p - 1$ and rise both sides to power $\frac{p-1}{p(q-1)}$. $\square$

3.6 Hardy–Poincaré inequalities derived from p–harmonic problems

This section is based on [93]. We apply general Hardy type inequalities, obtained in Theorem 3.3.1. As a consequence we obtain a family of Hardy—Poincaré inequalities with certain constants, contributing to the question about precise constants in such inequalities posed in [16]. We confirm optimality of some constants obtained in [16] and [53]. Furthermore, we give constants for generalized inequalities with the proof of their optimality.

3.6.1 The result

In this subsection we show that application of Theorem 3.3.1 with a special function u, namely $u_\alpha(x) = (1 + |x|^{\frac{p}{p-1}})^{-\alpha}$ with $\alpha > 0$, leads to the following theorem.

Theorem 3.6.1. *Suppose $p > 1$ and $\gamma > 1$. Then, for every compactly supported function $\xi \in W^{1,p}_{v_1,v_2}(\mathbb{R}^n)$, where*

$$v_1(x) = \left(1 + |x|^{\frac{p}{p-1}}\right)^{(p-1)(\gamma-1)}, \quad v_2(x) = \left(1 + |x|^{\frac{p}{p-1}}\right)^{(p-1)\gamma},$$

we have

$$\bar{C}_{\gamma,n,p} \int_{\mathbb{R}^n} |\xi|^p \left[(1 + |x|^{\frac{p}{p-1}})^{p-1}\right]^{\gamma-1} dx \leq \int_{\mathbb{R}^n} |\nabla\xi|^p \left[(1 + |x|^{\frac{p}{p-1}})^{p-1}\right]^{\gamma} dx, \tag{3.48}$$

with $\bar{C}_{\gamma,n,p} = n \left(\frac{p(\gamma-1)}{p-1}\right)^{p-1}$. Moreover, for $\gamma > n + 1 - \frac{n}{p}$, the constant $\bar{C}_{\gamma,n,p}$ is optimal and it is achieved by function $\bar{u}(x) = (1 + |x|^{\frac{p}{p-1}})^{1-\gamma}$.

Proof. At first we note that, by standard density argument, it suffices to prove (3.48) for every compactly supported Lipschitz function ξ. Indeed, let $\xi \in W^{1,p}_{v_1,v_2}(\mathbb{R}^n)$ and

$$\phi(x) = \begin{cases} 1, & |x| < 1, \\ -|x| + 2, & 1 \leq |x| \leq 2, \\ 0, & 2 < |x|. \end{cases} \qquad \phi_R(x) = \phi\left(\frac{x}{R}\right), \quad \xi_R(x) = \xi(x)\phi_R(x).$$

An easy verification shows that $\xi_R \to \xi$ in $W^{1,p}_{v_1,v_2}(\mathbb{R}^n)$. Standard convolution argument shows that every compactly supported function $u \in W^{1,p}_{v_1,v_2}(\mathbb{R}^n)$ can be approximated in $W^{1,p}_{v_1,v_2}(\mathbb{R}^n)$ by compactly supported Lipschitz functions.

Let us consider the function $u_\alpha(x) = (1+|x|^{\frac{p}{p-1}})^{-\alpha}$ with $\alpha > 0$. Now the proof follows by steps.

STEP 1. We recognize that the function $u_\alpha \in W^{1,p}_{loc}(\mathbb{R}^n)$ and that it is a nonnegative solution to PDE

$$-\Delta_p(u_\alpha) = d(1+|x|^{\frac{p}{p-1}})^{\alpha-\alpha p-p}(1+\kappa|x|^{\frac{p}{p-1}}) =: \Phi \quad \text{a.e. in } \mathbb{R}^n, \tag{3.49}$$

where

$$d = d(n,\alpha,p) = \left(\frac{\alpha p}{p-1}\right)^{p-1} n \quad \text{and} \quad \kappa = \kappa(n,\alpha,p) = 1 - \frac{\alpha+1}{n}p. \tag{3.50}$$

Moreover, Φ satisfies (3.2). For readers convenience the computations are carried out in the Appendix.

STEP 2. In our case condition $(\mathbf{\Phi}, \mathbf{p})$ becomes

$$\sigma_0 := -\operatorname{ess\,inf}\left(\frac{\Phi \cdot u_\alpha}{|\nabla u_\alpha|^p}\right) = -\frac{p-1}{\alpha p}(n - p(\alpha+1)) \in \mathbb{R}. \tag{3.51}$$

Indeed, by the formulae (3.49) and (3.51), we have

$$\begin{aligned}
\sigma_0 &= -\inf \frac{\left(\frac{\alpha p}{p-1}\right)^{p-1}(1+|x|^{\frac{p}{p-1}})^{-p(\alpha+1)}\left(n+(n-(\alpha+1)p)|x|^{\frac{p}{p-1}}\right)}{\left(\frac{\alpha p}{p-1}\right)^{p}(1+|x|^{\frac{p}{p-1}})^{-p(\alpha+1)}|x|^{\frac{p}{p-1}}} = \\
&= -\inf \frac{n+(n-(\alpha+1)p)|x|^{\frac{p}{p-1}}}{\left(\frac{\alpha p}{p-1}\right)|x|^{\frac{p}{p-1}}} = \\
&= -\left(\frac{p-1}{\alpha p}\right)\left[\inf \frac{n+(n-(\alpha+1)p)|x|^{\frac{p}{p-1}}}{|x|^{\frac{p}{p-1}}}\right] = \\
&= -\frac{(p-1)(n-(\alpha+1)p)}{\alpha p}.
\end{aligned}$$

STEP 3. For given $\alpha > -\gamma$, define $\beta = (p-1)(\frac{\gamma}{\alpha}+1)$. We apply Theorem 3.3.1.

For this we require that $\beta > 0$ and that $\sigma \in \mathbb{R}$ is such that $\beta > \sigma \geq \sigma_0$. This is equivalent to the condition $\gamma > \max\{-\alpha, 1-\frac{n}{p}\}$, which obviously holds for all $\gamma > 1,\ \alpha > 0$.

We are going to compute the measure given by (3.14). Let $b_1 = \left(\frac{\alpha p}{p-1}\right)^p \cdot \sigma$. We note that $\gamma = \alpha\left(\frac{\beta}{p-1}-1\right)$ and $-p(\alpha+1)+\alpha(\beta+1) = (p-1)(\gamma-1)-1$

and recall that d and κ are given in (3.50). Applying these formulae to (3.14), we obtain

$$\begin{aligned}\mu_1(dx) &= \left(\frac{\beta-\sigma}{p-1}\right)^{p-1} [\Phi\cdot u_\alpha + \sigma|\nabla u_\alpha|^p]\, u_\alpha^{-\beta-1}\, dx = \\ &= \left(\frac{\beta-\sigma}{p-1}\right)^{p-1} \left[\frac{d(1+\kappa|x|^{\frac{p}{p-1}})}{(1+|x|^{\frac{p}{p-1}})^{p(\alpha+1)}} + \frac{b_1|x|^{\frac{p}{p-1}}}{(1+|x|^{\frac{p}{p-1}})^{p(\alpha+1)}}\right]\cdot \\ &\quad \cdot(1+|x|^{\frac{p}{p-1}})^{\alpha(\beta+1)}\, dx = \\ &= \left(\frac{(\beta-\sigma)p\alpha}{(p-1)^2}\right)^{p-1} \left\{n + \left[n-(\alpha+1)p + \frac{\sigma\alpha p}{p-1}\right]|x|^{\frac{p}{p-1}}\right\} \\ &\quad \cdot(1+|x|^{\frac{p}{p-1}})^{-1}\left[(1+|x|^{\frac{p}{p-1}})^{p-1}\right]^{\gamma-1} dx,\end{aligned} \tag{3.52}$$

while after substitution of $\beta = \frac{(p-1)(\alpha+\gamma)}{\alpha}$, we obtain from (3.15)

$$\begin{aligned}\mu_2(dx) &= u^{p-\beta-1}\chi_{\{|\nabla u|\neq 0\}}\, dx = \left[(1+|x|^{\frac{p}{p-1}})^{-\alpha}\right]^{p-\beta-1} dx = \\ &= \left[(1+|x|^{\frac{p}{p-1}})^{p-1}\right]^{\gamma} dx.\end{aligned}$$

STEP 4. We choose $\sigma := \frac{(p-1)(\alpha+1)}{\alpha}$ and realize

$$\frac{(p-1)(\alpha+\gamma)}{\alpha} = \beta > \sigma > \sigma_0 = \frac{(p-1)(\alpha+1-\frac{n}{p})}{\alpha},$$

because $\gamma > 1$. Then, in (3.52), the expression in curly brackets equals $n(1+|x|^{\frac{p}{p-1}})$. This leads to the inequality (3.48) with the constant as required.

STEP 5. In this step we prove the optimality of the proposed constant under the assumption $\gamma > n+1-\frac{n}{p}$. It suffices to show that both sides of (3.48), for $u_\alpha := \bar{u}$ defined below, are equal and finite.

We prove first that the function $\bar{u}(x) = v(|x|) = (1+|x|^{\frac{p}{p-1}})^{1-\gamma}$ satisfies

$$-\mathrm{div}(v_2|\nabla\bar{u}|^{p-2}\nabla\bar{u}) = \bar{C}_{\gamma,n,p} v_1 \bar{u}^{p-1}. \tag{3.53}$$

For readers convenience the computations are carried out in the Appendix.

Now we concentrate on (3.48). Simple computations show that $\bar{u} \in W^{1,p}_{v_1,v_2}(\mathbb{R}^n)$. It suffices to prove equality in (3.48) for $\bar{u}$. Due to (3.53), we obtain

$$\bar{C}_{\gamma,n,p}\int_{\mathbb{R}^n}|\bar{u}|^p\,\left(1+|x|^{\frac{p}{p-1}}\right)^{(p-1)(\gamma-1)}dx=\bar{C}_{\gamma,n,p}\int_{\mathbb{R}^n}\bar{u}^p\,v_1\,dx=$$

$$=-\int_{\mathbb{R}^n}\mathrm{div}(v_2|\nabla\bar{u}|^{p-2}\nabla\bar{u})\cdot\bar{u}\,dx=-\lim_{R\to\infty}\int_{|x|<R}\mathrm{div}(v_2|\nabla\bar{u}|^{p-2}\nabla\bar{u})\cdot\bar{u}\,dx=:\mathcal{L}.$$

We apply Gauss–Ostrogradzki Theorem and observe that for an outer normal vector $n_x=\frac{x}{|x|}$ to $\partial B(R)$ we have $\langle\nabla\bar{u},n_x\rangle=|\nabla\bar{u}|$. This implies

$$\mathcal{L}\ =\lim_{R\to\infty}\left(\int_{|x|<R}v_2|\nabla\bar{u}|^p\,dx-\int_{|x|=R}v_2|\nabla\bar{u}|^{p-1}\cdot\bar{u}\,dS\right)=\lim_{R\to\infty}\left(\mathcal{A}-\mathcal{B}\right),$$

where dS denotes the surface measure on the sphere $S^{n-1}(R)$. To deal with the limit we require $\gamma>n+1-\frac{n}{p}$. Let us observe, that $\lim_{R\to\infty}\mathcal{B}=0$, because it is up to a constant equal to $\int_{|x|=R}\bar{u}(x)|x|dS$. Moreover, we notice that finiteness of the limit of $\mathcal{A}$ is ensured by

$$\frac{1}{\bar{C}_{\gamma,n,p}}\mathcal{A}\le\int_{\mathbb{R}^n}\left(1+|x|^{\frac{p}{p-1}}\right)^{-(\gamma-1)}dx\ \ \le\ \int_{\mathbb{R}^n}(1+|x|)^{-\frac{p(\gamma-1)}{p-1}}\,dx,$$

which is finite if the power of $(1+|x|)$ is smaller than $-n$, e.i. for $\gamma>n+1-\frac{n}{p}$. This finishes the proof. □

Remark 3.6.1. *Careful analysis of the quotient*

$$\frac{b(R)}{a(R)}:=\frac{\int_{\mathbb{R}^n}|\nabla u_R|^p(1+|x|^{\frac{p}{p-1}})^{(p-1)\gamma}\ dx}{\bar{C}_{\gamma,n,p}\int_{\mathbb{R}^n}\ |u_R|^p(1+|x|^{\frac{p}{p-1}})^{(p-1)(\gamma-1)}dx},\tag{3.54}$$

where $\bar{u}_R=\phi_R\bar{u}$, leads to optimality result also in the case of $\gamma=n+1-\frac{n}{p}$. We point out that when $\gamma=n+1-\frac{n}{p}$ function $\bar{u}$ does not belong to $W^{1,p}_{v_1,v_2}(\mathbb{R}^n)$. We will prove optimality in this case in another way in Corollary 3.6.1.

3.6.2 Discussion on constants

Comparison with the classical Hardy inequality

We start with showing that constants in Hardy–Poincaré inequalities are not smaller than in the classical Hardy inequalities. At first let us recall the classical results. Partial theorems have been already mentioned (Theorem 3.4.1, Corollary 3.4.1). We refer to [63, 75, 78] for more information on the best constants in various classical Hardy–type inequalities.

Theorem 3.6.2 (Classical Hardy Inequalities)**.** *Let* $1 < p < \infty$.

1. *Assume further that* $\gamma \neq p-1$ *and* ξ *is an arbitrary Lipschitz function with compact support in* $(0,\infty)$. *Then*
$$\int_0^\infty \left(\frac{|\xi|}{x}\right)^p x^\gamma \, dx \leq H_{\gamma,1,p} \int_0^\infty |\xi'|^p x^\gamma \, dx, \tag{3.55}$$
where the constant $H_{\gamma,1,p} = \left(\frac{p}{|p-1-\gamma|}\right)^p$ *is optimal.*

2. *Assume further that* $\gamma \neq p-n$ *and* ξ *is an arbitrary Lipschitz function with compact support in* $\mathbb{R}^n \setminus \{0\}$. *Then*
$$\int_{\mathbb{R}^n\setminus\{0\}} |\xi|^p |x|^{\gamma-p} \, dx \leq H_{\gamma,n,p} \int_{\mathbb{R}^n\setminus\{0\}} |\nabla\xi|^p |x|^\gamma \, dx, \tag{3.56}$$
where the constant $H_{\gamma,n,p} = \left(\frac{p}{|p-n-\gamma|}\right)^p$ *is optimal.*

Remark 3.6.2. The constant $HP_{\gamma,n,p} := 1/\bar{C}_{\gamma,n,p}$, where $\bar{C}_{\gamma,n,p}$ is the constant from Hardy–Poincaré inequality (3.48), is not smaller than the constant $H_{p\gamma,n,p}$ from Hardy inequality (3.56), namely
$$H_{p\gamma,n,p} \leq HP_{\gamma,n,p}.$$

Proof. Let us consider (3.48) with function $\xi_t(y) := \xi(ty)$
$$\begin{aligned}\bar{C}_{\gamma,n,p} \quad & \int_{\mathbb{R}^n} |\xi(ty)|^p \left[(1+|y|^{\frac{p}{p-1}})^{p-1}\right]^{\gamma-1} dy \leq \\ \leq \quad & \int_{\mathbb{R}^n} t^p \, |\nabla\xi(ty)|^p \left[(1+|y|^{\frac{p}{p-1}})^{p-1}\right]^{\gamma} dy,\end{aligned}$$
and realize that it is equivalent to
$$\begin{aligned}\bar{C}_{\gamma,n,p} \quad & \int_{\mathbb{R}^n} |\xi(ty)|^p t^{-p(\gamma-1)} \left[(t^{\frac{p}{p-1}}+|ty|^{\frac{p}{p-1}})^{p-1}\right]^{\gamma-1} dy \leq \\ \leq \quad & \int_{\mathbb{R}^n} t^p |\nabla\xi(ty)|^p t^{-p\gamma} \left[(t^{\frac{p}{p-1}}+|ty|^{\frac{p}{p-1}})^{p-1}\right]^{\gamma} dy.\end{aligned}$$
We multiply both sides by $t^{p(\gamma-1)}$ and substitute $x = ty$, getting
$$\begin{aligned}\bar{C}_{\gamma,n,p} \quad & \int_{\mathbb{R}^n} |\xi(x)|^p \left[(t^{\frac{p}{p-1}}+|x|^{\frac{p}{p-1}})^{p-1}\right]^{\gamma-1} dx \leq \\ \leq \quad & \int_{\mathbb{R}^n} |\nabla\xi(x)|^p \left[(t^{\frac{p}{p-1}}+|x|^{\frac{p}{p-1}})^{p-1}\right]^{\gamma} dy.\end{aligned}$$

It suffices to let $t \to 0$ and divide the inequality by $\bar{C}_{\gamma,n,p}$, to obtain

$$\int_{\mathbb{R}^n} |\xi(x)|^p |x|^{p(\gamma-1)}\, dy \le HP_{\gamma,n,p} \int_{\mathbb{R}^n} |\nabla \xi(x)|^p |x|^{p\gamma}\, dy. \tag{3.57}$$

We already know from Theorem 3.4.1 that the smallest possible constant is $H_{p\gamma,n,p}$. $\square$

Applying this observation, we obtain following result.

Corollary 3.6.1 (Optimal constant)**.** *Suppose that $p > 1, n \ge 1$ and $\gamma = n(1-1/p)+1$. Then, for every nonnegative Lipschitz function ξ with compact support, inequality (3.48) holds with optimal constant $\bar{C}_{\gamma,n,p} = n^p$.*

Proof. We first notice that $HP_{\gamma,n,p} = HP_{n(1-1/p)+1,n,p} = \frac{1}{n}\left(\frac{p-1}{p(\gamma-1)}\right)^{p-1} = n^{-p} = \left(\frac{p\gamma}{|p\gamma-n-\gamma|}\right)^p = H_{p\gamma,n,p}$ (as $p\gamma \neq p-n$), and due to Remark 3.6.2 we recognize the optimality of this constant. $\square$

Hardy—Poincaré Inequalities with improved constants

In this subsection we concentrate on the classical case $p = 2$. We show that, for some values of parameters γ and n, our results improve the previously know constant in the Hardy—Poincaré inequality (1.6).

Links with results by Blanchet, Bonforte, Dolbeault, Grillo and Vázquez in [14, 16]. In [14], the authors apply inequality (1.6) with $\gamma < 0$ to investigate convergence of solutions to fast diffusion equations. In [16], the following constants in (1.6) are established.

Remark 3.6.3 ([16])**.** *For every $v \in W^{1,2}_{v_1,v_2}(\mathbb{R}^n)$ where $v_1(x) = (1+|x|^2)^{\gamma-1}$, $v_2(x) = (1+|x|^2)^{\gamma}$, inequality*

$$\Lambda_{\gamma,n} \int_{\mathbb{R}^n} |v|^2\ (1+|x|^2)^{\gamma-1}\, dx \le \int_{\mathbb{R}^n} |\nabla v|^2\ (1+|x|^2)^{\gamma}\, dx,$$

holds with $\Lambda_{\gamma,n}$ defined below.

1. *For $n = 1$ and $\gamma < 0$ the optimal constant is*

$$\Lambda_{\gamma,1} = \begin{cases} (\gamma - \frac{1}{2})^2 & \text{if} \quad \gamma \in [-\frac{1}{2}, 0), \\ -2\gamma & \text{if} \quad \gamma \in [-\infty, -\frac{1}{2}). \end{cases} \tag{3.58}$$

2. *For $n = 2$ and $\gamma < 0$ the optimal constant is*

$$\Lambda_{\gamma,2} = \begin{cases} \gamma^2 & \text{if} \quad \gamma \in [-2, 0), \\ -2\gamma & \text{if} \quad \gamma \in [-\infty, -2). \end{cases} \tag{3.59}$$

3. *For $n \geq 3$*

   - *and $\gamma < 0$ the optimal constant is*

$$\Lambda_{\gamma,n} = \begin{cases} \frac{(n-2+2\gamma)^2}{4} & \text{if} \quad \gamma \in [-\frac{n+2}{2}, 0) \setminus \{-\frac{n-2}{2}\}, \\ -4\gamma - 2n & \text{if} \quad \gamma \in [-n, -\frac{n+2}{2}), \\ -2\gamma & \text{if} \quad \gamma \in [-\infty, -n). \end{cases} \tag{3.60}$$

   - *and $\gamma = n$ the optimal constant is $\Lambda_{n,n} = 2n(n-1)$,*
   - *and $\gamma \geq n$ the constant is $\Lambda_{\gamma,n} = n(n + \gamma - 2)$,*
   - *and $n \geq \gamma > 0$ the constant is $\Lambda_{\gamma,n} = \gamma(n + \gamma - 2)$.*

Remark 3.6.4. Here we compare our results with the above ones.

1. We preserve the optimal constant if $n \geq 3$ and $\gamma = n$.

2. We extend the above optimality result for $\gamma = n \geq 3$ also to the case $\gamma = n = 2$. Indeed, we recall that Corollary 3.6.1 applied to $p = 2$ gives the optimal constant $\bar{C}_{(n+2)/2,n,2} = n^2$ when $n \geq 1$. In particular, we obtain $\Lambda_{2,2} = 2 \cdot 2(2-1) = \bar{C}_{(2+2)/2,2,2}$.

3. In the case $n \geq 3$, $\gamma > 2$, and $n \neq \gamma$, our constant $\bar{C}_{\gamma,n,2} = 2n(\gamma - 1)$ is better than the constant in [16]:

   - if $\gamma > n$ then $\bar{C}_{\gamma,n,2} > \Lambda_{\gamma,n} = n(n + \gamma - 2)$,
   - if $n > \gamma > 2$ then $\bar{C}_{\gamma,n,2} > \Lambda_{\gamma,n} = \gamma(n + \gamma - 2)$.

4. In the case $n \geq 3,\ 2 > \gamma > 1$ our constant becomes worse than $\Lambda_{\gamma,n}$.

Links with results by Ghoussoub and Moradifam [53]. In a recent paper [53] by Ghoussoub and Moradifam, some improvements to the results of [14] are obtained. In particular, some new estimates for constants from [14] are proven. We can further improve the constants from [53] for some range of parameters.

Among other results, one finds in [53] the following.

Theorem 3.6.3 ([53], Theorem 2.13, part II).
If $a, b, \alpha, \beta > 0$ and $n \geq 2$, then there exists a constant c such that for all $\xi \in C_0^\infty(\mathbb{R}^n)$

$$c \int_{\mathbb{R}^n} (a + b|x|^\alpha)^{\beta - \frac{2}{\alpha}} \xi^2 \, dx \leq \int_{\mathbb{R}^n} (a + b|x|^\alpha)^\beta \, |\nabla \xi|^2 \, dx, \tag{3.61}$$

and moreover $\left(\frac{n-2}{2}\right)^2 =: c_1 \leq c \leq \left(\frac{n+\alpha\beta-2}{2}\right)^2$.

A very special case of the above theorem (when $a = b = 1$, $\alpha = 2$, and $\beta = \gamma$) covers also our case, therefore we present it below and discuss the related constants.

Corollary 3.6.2. *If $\gamma > 0$ and $n \geq 2$, then there exists a constant $\bar{c}_1 > 0$ such that for all $\xi \in C_0^\infty(\mathbb{R}^n)$*

$$\bar{c}_1 \int_{\mathbb{R}^n} |\xi|^2 (1 + |x|^2)^{\gamma - 1} \, dx \leq \int_{\mathbb{R}^n} |\nabla \xi|^2 (1 + |x|^2)^\gamma \, dx, \tag{3.62}$$

and moreover $\left(\frac{n-2}{2}\right)^2 =: c_1 \leq \bar{c}_1 \leq \left(\frac{n+2\gamma-2}{2}\right)^2$.

Note, that we have already pointed out in Remark 3.6.2, that $\bar{c}_1 \leq \left(\frac{n+2\gamma-2}{2}\right)^2$. Therefore, we may concentrate only on the lower bound.

Remark 3.6.5. *Here we compare our results with the above one. The constant $\bar{C}_{\gamma,n,p}$ is the left–hand side constant derived in Theorem 3.6.1 for $\gamma, p > 1$, $n \geq 1$ and it is proven to be optimal for $\gamma \geq n + 1 - \frac{n}{p}$. Let c_1 be the constant from Corollary 3.6.2, where $\gamma > 0$, $p = 2$, $n \geq 2$. We may compare it only when $\gamma > 1$, $p = 2$, $n \geq 2$. We have*

$$C_{\gamma,n,2} = 2n(\gamma - 1) > \left(\frac{n-2}{2}\right)^2 = c_1, \tag{3.63}$$

for every $\gamma > \max\left\{\frac{(n+2)^2}{8n}, 1\right\}$. This shows that for those γs Theorem 3.6.1 gives the inequality (3.62) with the constant better than the one resulting from Corollary 3.6.2. Furthermore, we notice that (3.63) holds also for $\gamma \in \left(\frac{(n+2)^2}{8n}, 1 + \frac{n}{2}\right)$, when we do not have the optimality of $\bar{C}_{\gamma,n,2}$. When $\gamma = \frac{1}{2n}\left(\frac{n+2}{2}\right)^2$, we have $c_1 = \bar{C}_{\gamma,n,2}$, but for such γ we do not prove the optimality of $\bar{C}_{\gamma,n,2}$.

Comparison of the values of the constants $\bar{C}_{\gamma,n,2}$, $\Lambda_{\gamma,n}$, c_1 under common assumptions, in the case when $\bar{C}_{\gamma,n,2}$ is not proven to be optimal, is given in Remark 3.6.6.

Summary of results and open questions

We collect here all the known information about the constants in the Hardy–Poincaré inequality (3.48). We point out that we consider the left–hand side constant, and so the biggest possible one is optimal.

Let us recall that the constants c_1, $\Lambda_{\gamma,n}$ and $\bar{C}_{\gamma,n,p}$.

i) c_1 comes from [53], see Theorem 3.6.3 and Corollary 3.6.2,

ii) $\Lambda_{\gamma,n}$ comes from [16], see Remark 3.6.3,

iii) $\bar{C}_{\gamma,n,p}$ is derived in Theorem 3.6.1 for $p, \gamma > 1$, $n \geq 1$, and proven to be optimal

- for $\gamma > \frac{n}{p}(p-1)+1$ in Theorem 3.6.1,
- for $\gamma = \frac{n}{p}(p-1)+1$ in Corollary 3.6.1.

For $p = 2$, we have $\bar{C}_{\gamma,n,2} = 2n(\gamma-1)$, and moreover

n	γ	constant	optimality	see
$n \geq 1$	$\gamma > 1$	$\bar{C}_{\gamma,n,2}$	for $\gamma > \frac{n+2}{2}$, here	Thm 3.6.1
$n \geq 1$	$\gamma = \frac{n+2}{2}$	$\bar{C}_{\gamma,n,2}$	yes, here	Coro. 3.6.1
$n \geq 1$	$\gamma < 0$	$\Lambda_{\gamma,n}$	yes, [16]	Rem. 3.6.3
$n = 2$	$\gamma = 2$	$\bar{C}_{2,2,2}$	yes, here	Rem. 3.6.4
$n \geq 3$	$\gamma = n$	$\bar{C}_{n,n,2}$	yes, [16]	Rem. 3.6.3
$n \geq 3$	$\gamma > n$	$\bar{C}_{\gamma,n,2} \geq \Lambda_{\gamma,n} > c_1$	yes, here	Rem. 3.6.4
$n = 2$	$0 < \gamma < 1$	c_1	??	Coro. 3.6.2
$n \geq 3$	$\gamma \in (0, \min\{\gamma_c, 1\}]$	$c_1 \geq \Lambda_{\gamma,n}$	??	Coro. 3.6.2
$n \geq 3$	$\gamma_c \leq \gamma \leq 1$	$\Lambda_{\gamma,n} \geq c_1$	??	Coro. 3.6.2
$n \geq 2$	$1 < \gamma \leq \gamma_g$	$c_1 \geq \bar{C}_{\gamma,n,2}$	??	Coro. 3.6.2
$n \geq 2$	$\gamma > \gamma_g$	$\bar{C}_{\gamma,n,2} > c_1$	for $\gamma \geq \frac{n+2}{2}$, here	Rem. 3.6.5

where $\gamma_c = \frac{\sqrt{2}-1}{2}(n-2)$, $\gamma_g = \frac{(n+2)^2}{8n}$.

As we can see above, for sufficiently big values of parameter γ ($\gamma \geq \frac{n+2}{2}$) our constant is optimal, thus $\bar{C}_{\gamma,n,2} \geq \max\{\Lambda_{\gamma,n}, c_1\}$. In the following remark we compare the values of the constants in the case when all three of them are defined (namely $p = 2, n \geq 3, \gamma > 1$) and when $\gamma < \frac{n+2}{2}$.

Remark 3.6.6. *We compare all the mentioned constants under assumptions:* $p = 2$, $n \geq 3$, *and* $1 < \gamma < \frac{n+2}{2}$. *We note* *i)* $c_1 < \Lambda_{\gamma,n}$ *if and only if* $\gamma_c < \gamma$*;* $c_1 > \Lambda_{\gamma,n}$ *if and only if* $\gamma_c > \gamma$*;*

ii) $\bar{C}_{\gamma,n,2} < c_1$ *if and only if* $\gamma < \gamma_g$*;* $\bar{C}_{\gamma,n,2} > c_1$ *if and only if* $\gamma > \gamma_g$*;*

iii) $\bar{C}_{\gamma,n,2} < \Lambda_{\gamma,n}$ *if and only if* $\gamma < 2$*;* $\bar{C}_{\gamma,n,2} > \Lambda_{\gamma,n}$ *if and only if* $\gamma > 2$*.*

Therefore for $p = 2$*,* $n \geq 3$*, and* $n > \gamma > 1$ *we have* $\gamma_c < \frac{n+2}{2}$*,* $1 < \gamma_g < \frac{n+2}{2}$*, moreover*

constants	γ	*such* γ *exists for*
$\bar{C}_{\gamma,n,2} > \Lambda_{\gamma,n} > c_1$	$\gamma \in (\max\{2, \gamma_c\}, \frac{n+2}{2})$	$n \geq 3$
$\bar{C}_{\gamma,n,2} > c_1 > \Lambda_{\gamma,n}$	$\gamma \in (\gamma_g, \gamma_c)$	$n \geq 12$
$\Lambda_{\gamma,n} > \bar{C}_{\gamma,n,2} > c_1$	$\gamma \in (\gamma_g, 2)$	$n \in [3, 11]$
$\Lambda_{\gamma,n} > c_1 > \bar{C}_{\gamma,n,2}$	$\gamma \in (\max\{1, \gamma_c\}, \gamma_g)$	$n \in [3, 11]$
$c_1 > \Lambda_{\gamma,n} > \bar{C}_{\gamma,n,2}$	$\gamma \in (1, \min\{2, \gamma_c\})$	$n \geq 7$
$c_1 > \bar{C}_{\gamma,n,2} > \Lambda_{\gamma,n}$	$\gamma \in (2, \gamma_g)$	$n \geq 12$

For $p > 1,\ n \geq 1$, due to Theorem 3.6.1, we have $\bar{C}_{\gamma,n,p} = n\left(\frac{p(\gamma-1)}{p-1}\right)^{p-1}$, and

γ	constant	optimality
$\gamma \in (1, \frac{n}{p}(p-1)+1)$	$\bar{C}_{\gamma,n,p} = n\left(\frac{p(\gamma-1)}{p-1}\right)^{p-1}$	??
$\gamma = \frac{n}{p}(p-1)+1$	$\bar{C}_{\gamma,n,p} = n^p$	Corollary 3.6.1
$\gamma > \frac{n}{p}(p-1)+1$	$\bar{C}_{\gamma,n,p} = n\left(\frac{p(\gamma-1)}{p-1}\right)^{p-1}$	Theorem 3.6.1

Open questions.

- We do not know what is the optimal constant in (3.62) for $\gamma < \frac{n}{2}+1$.
- We do not know what is the optimal constant in (3.48) for $\gamma < n+1-\frac{n}{p}$ and our methods do not give any estimates for the constant when $\gamma < 1$.

3.6.3 Appendix to Section 3.6

Proof of Step 1 of Proposition 3.6.1. We use the following computations. We recall $u_\alpha(x) = (1+|x|^{\frac{p}{p-1}})^{-\alpha}$ and compute first everything, which is needed to find its p–Laplacian.

$$\begin{aligned}
\nabla u_\alpha(x) &= -\alpha(1+|x|^{\frac{p}{p-1}})^{-\alpha-1}\frac{p}{p-1}|x|^{\frac{p}{p-1}-1}\frac{x}{|x|} = \\
&= \frac{-\alpha p}{p-1}(1+|x|^{\frac{p}{p-1}})^{-\alpha-1}|x|^{\frac{1}{p-1}}\frac{x}{|x|}, \\
|\nabla u_\alpha(x)| &= \left|\frac{\alpha p}{p-1}\right|(1+|x|^{\frac{p}{p-1}})^{-\alpha-1}|x|^{\frac{1}{p-1}}, \\
|\nabla u_\alpha(x)|^{p-2} &= \left|\frac{\alpha p}{p-1}\right|^{p-2}(1+|x|^{\frac{p}{p-1}})^{-(\alpha+1)(p-2)}|x|^{\frac{p-2}{p-1}}, \\
|\nabla u_\alpha(x)|^{p-2}\nabla u_\alpha(x) &= -\frac{\alpha p}{p-1}\left|\frac{\alpha p}{p-1}\right|^{p-2}(1+|x|^{\frac{p}{p-1}})^{-(\alpha+1)(p-1)}x \\
&= \kappa_1 x u_{(\alpha+1)(p-1)}(x),
\end{aligned}$$

where $\kappa_1 = \frac{-\alpha p}{p-1}|\frac{\alpha p}{p-1}|^{p-2}$.

Then (as $\alpha > 0$) we have

$$\begin{aligned}
\Delta_p(u_\alpha(x)) &= \operatorname{div}(|\nabla u_\alpha(x)|^{p-2}\nabla u_\alpha(x)) = \sum_i \frac{\partial(|\nabla u_\alpha(x)|^{p-2}\nabla u_\alpha(x))}{\partial x_i} = \\
&= \kappa_1 \sum_i \frac{\partial(u_{(\alpha+1)(p-1)}(x)x_i)}{\partial x_i} = \\
&= \kappa_1 \left(\sum_i \frac{\partial(u_{(\alpha+1)(p-1)}(x))}{\partial x_i}x_i + u_{(\alpha+1)(p-1)}(x)\sum_i \frac{\partial x_i}{\partial x_i}\right) = \\
&= \kappa_1 \frac{-(\alpha+1)(p-1)p}{p-1}(1+|x|^{\frac{p}{p-1}})^{-(\alpha+1)(p-1)-1}|x|^{\frac{1}{p-1}}\frac{\sum_i x_i^2}{|x|} \\
&\quad +\kappa_1 n u_{(\alpha+1)(p-1)}(x) = \\
&= \kappa_1\left(-(\alpha+1)p(1+|x|^{\frac{p}{p-1}})^{\alpha-\alpha p-p}|x|^{\frac{p}{p-1}} + n u_{(\alpha+1)(p-1)}(x)\right) = \\
&= \left(\frac{\alpha p}{p-1}\right)^{p-1}(1+|x|^{\frac{p}{p-1}})^{\alpha-\alpha p-p}\cdot \\
&\quad \cdot\left((\alpha+1)p|x|^{\frac{p}{p-1}} - n(1+|x|^{\frac{p}{p-1}})\right).
\end{aligned}$$

Therefore, our Φ has a form

$$\begin{aligned}\Phi &= -\mathrm{div}(|\nabla u_\alpha(x)|^{p-2}\nabla u_\alpha(x)) = \\ &= \left(\frac{\alpha p}{p-1}\right)^{p-1}(1+|x|^{\frac{p}{p-1}})^{\alpha-\alpha p-p}\left(n+(n-(\alpha+1)p)|x|^{\frac{p}{p-1}})\right).\end{aligned}$$

$\square$

Proof of (3.53) *in Step 5 of Theorem 3.6.1.* The proof follows from the technical lemmas below (Lemmas 3.6.1, 3.6.2 and 3.6.3). They show that, under assumption of Theorem 3.6.1, $\bar{u}$ satisfies an equation equivalent to equation (3.53). Therefore $\bar{u}$ satisfies (3.53) as well.

Lemma 3.6.1. *Let $\bar{u}(x) = v(|x|) \in C^2((\mathbb{R}\setminus\{0\}))$ be an arbitrary function, $\Phi_p(\lambda) = |\lambda|^{p-2}\lambda$, $v_2(r) = \left(1+r^{\frac{p}{p-1}}\right)^{(p-1)\gamma}$ then*

i) $\nabla\bar{u}(x) = v'(|x|)\frac{x}{|x|}$,

ii) $\Phi_p'(\lambda) = (p-1)|\lambda|^{p-2}$,

iii) $(\Phi_p(\nabla\bar{u}(x))) = \Phi_p(v'(|x|))\cdot\frac{x}{|x|}$,

iv) $\mathrm{div}(\Phi_p(\nabla\bar{u})) = |v'(|x|)|^{p-2}\left((p-1)v''(|x|)+(n-1)\frac{v'(|x|)}{|x|}\right)$

v) $\nabla v_2(|x|) = \gamma p(1+|x|^{\frac{p}{p-1}})^{\gamma(p-1)-1}|x|^{\frac{1}{p-1}}\frac{x}{|x|}$

Proof. We reach the claims *i)–iii)* and *v)* by elementary calculations. Then applying *i)–iii)* we prove the claim *iv)* as follows

$$\begin{aligned}(\Phi_p(\nabla\bar{u})) &= \mathrm{div}\left(\Phi_p(v'(|x|))\frac{x}{|x|}\right) = \\ &= \nabla(\Phi_p(v'(|x|)))\cdot\frac{x}{|x|} + \Phi_p(v'(|x|))\mathrm{div}\left(\frac{x}{|x|}\right) = \\ &= \Phi_p'(v'(|x|))\nabla v'(|x|)\cdot\frac{x}{|x|} + \Phi_p(v'(|x|))\frac{n-1}{|x|} = \\ &= \frac{x}{|x|}\Phi_p'(v'(|x|))v''(|x|)\frac{x}{|x|} + \Phi_p(v'(|x|))\frac{n-1}{|x|} = \\ &= \Phi_p'(v'(|x|))v''(|x|) + \Phi_p(v'(|x|))\frac{n-1}{|x|} = \\ &= (p-1)|v'(|x|)|^{p-2}v''(|x|) + |v'(|x|)|^{p-2}v'(|x|)\frac{n-1}{|x|}.\end{aligned}$$

$\square$

Lemma 3.6.2. *Equation* (3.53)*, where* $\bar{u}(x) = v(|x|) \in C^2(\mathbb{R} \setminus \{0\})$ *is an arbitrary function,* $v_1(r) = \left(1 + r^{\frac{p}{p-1}}\right)^{(p-1)(\gamma-1)}$, $v_2(r) = \left(1 + r^{\frac{p}{p-1}}\right)^{(p-1)\gamma}$ *is equivalent to equation*

$$-A = B, \tag{3.64}$$

where

$$\begin{aligned} A &:= \left((\gamma p + n - 1)|x|^{\frac{1}{p-1}} + \frac{n-1}{|x|}\right) v'(|x|) + (p-1)(1 + |x|^{\frac{p}{p-1}}) v''(|x|) \\ B &:= \bar{C}_{\gamma,n,p}(1 + |x|^{\frac{p}{p-1}})^{-p+2}\ v^{p-1}(|x|)\ (v'(|x|))^{-(p-2)}. \end{aligned}$$

Proof. We concentrate first on the left–hand side of (3.53):

$$\begin{aligned} -LHS &= \operatorname{div}(v_2 \cdot \Phi_p(\nabla \bar{u})) = \nabla v_2 \cdot \Phi_p(\nabla \bar{u}) + v_2 \operatorname{div}(\Phi_p(\nabla \bar{u})) = I + II, \\ I &= \gamma p (1 + |x|^{\frac{p}{p-1}})^{\gamma(p-1)-1} |x|^{\frac{1}{p-1}} \frac{x}{|x|} \cdot \left| v'(|x|) \frac{x}{|x|} \right|^{p-2} v'(|x|) \frac{x}{|x|} = \\ &= \gamma p (1 + |x|^{\frac{p}{p-1}})^{\gamma(p-1)-1} |x|^{\frac{1}{p-1}}\ |v'(|x|)|^{p-2}\ v'(|x|), \\ II &= (1 + |x|^{\frac{p}{p-1}})^{\gamma(p-1)} |v'(|x|)|^{p-2} \left((p-1)v''(|x|) + v'(|x|)\frac{n-1}{|x|}\right). \end{aligned}$$

Therefore,

$$-LHS = (1 + |x|^{\frac{p}{p-1}})^{\gamma(p-1)-1}\ |v'(|x|)|^{p-2} \cdot$$

$$\cdot \left((\gamma p + n - 1)|x|^{\frac{1}{p-1}} v'(|x|) + \frac{n-1}{|x|} v'(|x|) + (p-1)(1 + |x|^{\frac{p}{p-1}}) v''(|x|)\right),$$

while the right–hand side of (3.53) equals

$$RHS = \bar{C}_{\gamma,n,p}(1 + |x|^{\frac{p}{p-1}})^{(\gamma-1)(p-1)} v^{p-1}(|x|).$$

As $LHS = RHS$, by multiplying this equation by $(1 + |x|^{\frac{p}{p-1}})^{-\gamma(p-1)+1}\ |v'(|x|)|^{-(p-2)}$, we obtain (3.64). $\square$

Lemma 3.6.3. *If* $\alpha = 1 - \gamma < 0$*, the function* $v(x) = (1 + |x|^{\frac{p}{p-1}})^{\alpha}$ *satisfies* (3.64)*.*

Proof. We will need the following computations, where we identify $v(x)$ with one variable function $v(r)$

$$
\begin{aligned}
v' &= \frac{\alpha p}{p-1}(1+r^{\frac{p}{p-1}})^{\alpha-1} r^{\frac{1}{p-1}},\\
v'' &= \frac{\alpha p}{p-1}\left(\frac{(\alpha-1)p}{p-1}(1+r^{\frac{p}{p-1}})^{\alpha-2} r^{\frac{2}{p-1}} + \frac{1}{p-1}(1+r^{\frac{p}{p-1}})^{\alpha-1} r^{-\frac{p-2}{p-1}}\right) =\\
&= \frac{\alpha p}{(p-1)^2}(1+r^{\frac{p}{p-1}})^{\alpha-2}\left((\alpha-1)pr^{\frac{2}{p-1}} + (1+r^{\frac{p}{p-1}})r^{-\frac{p-2}{p-1}}\right) =\\
&= \frac{\alpha p}{(p-1)^2}(1+r^{\frac{p}{p-1}})^{\alpha-2}\left(((\alpha-1)p+1)r^{\frac{2}{p-1}} + r^{-\frac{p-2}{p-1}}\right)\\
&= \frac{\alpha p}{(p-1)^2}(1+r^{\frac{p}{p-1}})^{\alpha-2} r^{-\frac{p-2}{p-1}}\left(1+((\alpha-1)p+1)r^{\frac{p}{p-1}}\right),\\
\frac{v^{p-1}}{|v'|^{p-2}} &= \frac{(1+r^{\frac{p}{p-1}})^{\alpha(p-1)}}{\left|\frac{\alpha p}{p-1}\right|^{p-2}(1+r^{\frac{p}{p-1}})^{(\alpha-1)(p-2)} r^{\frac{p-2}{p-1}}} =\\
&= \left|\frac{p-1}{\alpha p}\right|^{p-2} r^{-\frac{p-2}{p-1}} \frac{(1+r^{\frac{p}{p-1}})^{\alpha(p-1)}}{(1+r^{\frac{p}{p-1}})^{(\alpha-1)(p-2)}} =\\
&= \left|\frac{p-1}{\alpha p}\right|^{p-2} r^{-\frac{p-2}{p-1}}(1+r^{\frac{p}{p-1}})^{\alpha+p-2}.
\end{aligned}
$$

When we take into account the above results and substitute $\gamma = -\alpha + 1$, we have in (3.64)

$$
\begin{aligned}
-A &= \left((\gamma p + n - 1)|x|^{\frac{1}{p-1}} + \frac{n-1}{|x|}\right) v'(|x|) + (p-1)(1+|x|^{\frac{p}{p-1}})v''(|x|) =\\
&= \left((\gamma p + n - 1)|x|^{\frac{1}{p-1}} + \frac{n-1}{|x|}\right)\frac{(1-\gamma)p}{p-1}(1+|x|^{\frac{p}{p-1}})^{-\gamma}|x|^{\frac{1}{p-1}} +\\
&\quad +(p-1)(1+|x|^{\frac{p}{p-1}})\frac{(1-\gamma)p}{(p-1)^2}(1+|x|^{\frac{p}{p-1}})^{-\gamma-1}|x|^{-\frac{p-2}{p-1}}\left(1+(-\gamma p+1)|x|^{\frac{p}{p-1}}\right) =\\
&= \frac{(1-\gamma)p}{p-1}(1+|x|^{\frac{p}{p-1}})^{-\gamma}|x|^{-\frac{p-2}{p-1}}\left((n-1)+(\gamma p+n-1)|x|^{\frac{p}{p-1}}\right) +\\
&\quad +\frac{(1-\gamma)p}{p-1}(1+|x|^{\frac{p}{p-1}})^{-\gamma}|x|^{-\frac{p-2}{p-1}}\left(1+(-\gamma p+1)|x|^{\frac{p}{p-1}}\right) =\\
&= n\frac{(1-\gamma)p}{p-1}(1+|x|^{\frac{p}{p-1}})^{-\gamma}|x|^{-\frac{p-2}{p-1}}\left(1+|x|^{\frac{p}{p-1}}\right)
\end{aligned}
$$

and on the the other hand

$$
\begin{aligned}
B &= \bar{C}_{\gamma,n,p}(1+|x|^{\frac{p}{p-1}})^{-p+2}\frac{v^{p-1}(|x|)}{|v'(|x|)|^{p-2}} = \\
&= \bar{C}_{\gamma,n,p}(1+|x|^{\frac{p}{p-1}})^{-p+2}\left(\frac{p-1}{(\gamma-1)p}\right)^{p-2}|x|^{-\frac{p-2}{p-1}}(1+|x|^{\frac{p}{p-1}})^{-\gamma+1+p-2} = \\
&= n\left(\frac{p\,(\gamma-1)}{p-1}\right)^{p-1}\left(\frac{p-1}{(\gamma-1)p}\right)^{p-2}(1+|x|^{\frac{p}{p-1}})^{-\gamma+1}|x|^{-\frac{p-2}{p-1}} = \\
&= n\,(\gamma-1)\,\frac{p}{p-1}(1+|x|^{\frac{p}{p-1}})^{-\gamma+1}|x|^{-\frac{p-2}{p-1}}.
\end{aligned}
$$

We recognize that for all $\gamma > 1$, $n \geq 1$, $p > 1$, we have: $-A = B$. $\square$

Chapter 4

Hardy–Sobolev inequalities derived from A–harmonic problems

This chapter is based on [94] by the author, where the methods of [92, 93] is generalized. The work extends the previous results, described in Chapter 3, where we considered inequality $-\Delta_p u \geq \Phi$, leading among others to Hardy inequalities with the best constants.

we are interested in Hardy–Sobolev type inequalities having a form

$$\int_\Omega f(u)d\mu_1 \leq \int_\Omega g(|\nabla u|)d\mu_2, \tag{4.1}$$

with some functions f, g, $\Omega \subseteq \mathbb{R}^n$, holding for certain class of u's. We consider f, g in the Orlicz class, taking into account the most classical case when $f(t) = g(t) = t^p$.

Multiple authors consider generalized versions of Hardy–Sobolev–type inequalities with remainder terms [2, 6, 39] as well as those expressed in Orlicz setting [21, 68] or combing this both ideas [70].

We consider the anticoercive partial differential inequality of elliptic type involving A–Laplacian: $-\Delta_A u = -\text{div}A(\nabla u) \geq \Phi$, where Φ is a given locally integrable function and u is defined on an open subset $\Omega \subseteq \mathbb{R}^n$. We derive Caccioppoli inequalities for u. Knowing solutions, as a consequence we obtain Hardy inequalities for compactly supported Lipschitz functions involving certain measures, having a form

$$\int_\Omega F_{\bar{A}}(|\xi|)\mu_1(dx) \le \int_\Omega \bar{A}(|\nabla\xi|)\mu_2(dx),$$

where $\bar{A}(\lambda) = A(|\lambda|)\lambda$ is an N–function satisfying Δ'–condition and $F_{\bar{A}}(\lambda) = 1/(\bar{A}(1/t))$. We give several examples starting with $\bar{A}(t) = F_{\bar{A}}(t) = t^p,\ p > 1$ and new various measures, finishing with $\bar{A}(t) = t^p \log^\alpha(2+t),\ p > 1, \alpha > 0$.

4.1 Preliminaries

Notation

In the sequel we assume that $\Omega \subseteq \mathbb{R}^n$ is an open subset not necessarily bounded.

By A–harmonic problems we understand those, which involve A–Laplace operator $\Delta_A u = \mathrm{div}(A(\nabla u))$, understood in the weak sense, where $A : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1–function. Choosing $A(\lambda) = |\lambda|^{p-2}\lambda$ we deal with the usual p–Laplacian.

We restrict ourselves to A's such that $A(\lambda) = B(|\lambda|)\lambda,\ \lambda \in \mathbb{R}^n$, and we set

$$\bar{A}(s) = B(s)s^2, \quad \text{where } s \in [0, \infty). \tag{4.2}$$

We assume that $\bar{A}$ is an N–function, i.e. it is convex and $\lim_{s\to 0} \frac{\bar{A}(s)}{s} = \lim_{s\to\infty} \frac{s}{\bar{A}(s)} = 0$. We refer to the monographs [74, 90] for basic properties of Orlicz spaces. By $\bar{A}^*$ we denote the Legendre transform of $\bar{A}$, e.i. $\bar{A}^* = \sup_{t>0}(st - \bar{A}(t))$.

As usual, $C^k(\Omega)$ (respectively $C_0^k(\Omega)$) denotes functions of class C^k defined on an open set $\Omega \subset \mathbb{R}^n$ (respectively C^k-functions on Ω with compact support). If f is defined on Ω, by $f\chi_\Omega$ we understand function f extended by 0 outside Ω. When $V \subseteq \mathbb{R}^n$, by $|V|$ we denote its Lebesgue's measure.

We deal with Δ_2 and Δ' conditions defined below.

Definition 4.1.1. *We say that the function $F : [0, \infty) \to [0, \infty)$ satisfies the Δ_2–condition (denoted $F \in \Delta_2$), if there exists a constant $\bar{C}_F > 0$ such that for every $s > 0$ we have*

$$F(2s) \le \bar{C}_F F(s). \tag{4.3}$$

Definition 4.1.2. *We say that the function $F : [0, \infty) \to [0, \infty)$ satisfies the Δ'–condition (denoted $F \in \Delta'$), if there exists a constant $C_F > 0$ such that for every $s_1, s_2 > 0$ we have*

$$F(s_1 s_2) \leq C_F F(s_1) F(s_2). \tag{4.4}$$

Remark 4.1.1. Let us note that the Δ'–condition is stronger than the Δ_2–condition.

Typical examples of N–functions satisfying the Δ'–condition can be found among Zygmund–type logarithmic functions. Their construction is based on the following easy observation.

Fact 4.1.1 ([66])**.** *The family of functions satisfying Δ'–condition is invariant under multiplications and compositions.*

Example 4.1.1 ([66])**.** The following N–functions satisfy Δ'–condition:
1. $F(s) = s^p$, $1 < p < \infty$,
2. $M_{p,\alpha}(s) = s^p(\ln(2 + s))^\alpha$, $1 < p < \infty$, $\alpha \geq 0$,
3. $M^1_{p,\alpha}(s) = s^p(\ln(1 + s))^\alpha$, $1 < p < \infty$, $\alpha \geq 0$,
4. $F(s) = M_{p_1,\alpha_1} \circ M_{p_2,\alpha_2} \circ \cdots \circ M_{p_k,\alpha_k}(s)$, $\alpha_1, \ldots, \alpha_k \geq 0$, $p_i > 1$ for $i = 1, \ldots, k$.

Fact 4.1.2. *Let $F_b(s) = s^p \log^\alpha(b + s)$, $b, p > 1$, $\alpha > 0$. Then, the constant from Δ'–condition (see Definition 4.1.2), $C_F \leq \left(\frac{2}{\log b}\right)^\alpha$.*

Proof. Suppose $s_1 \leq s_2$. Then

$$\log(b+s_1s_2) \leq \log(b+s_2^2) \leq \log(b+s_2)^2 = 2\log(b+s_2) \leq 2\log(b+s_2)\cdot\frac{\log(b+s_1)}{\log b},$$

and $F(s_1s_2) = (s_1s_2)^p \log^\alpha(b+s_1s_2) \leq \left(\frac{2}{\log b}\right)^\alpha s_1^p s_2^p \log^\alpha(b+s_1)\log^\alpha(b+s_2) = C_F F(s_1) F(s_2)$. □

Let us state some useful facts and lemmas.

Lemma 4.1.1 ([68], Lemma 4.2)**.** *Suppose that F is a differentiable N–function satisfying Δ_2–condition. Then there exists constants $1 < d_F \leq D_F$, such that for every $r > 0$*

$$d_F \frac{F(r)}{r} \leq F'(r) \leq D_F \frac{F(r)}{r}. \tag{4.5}$$

Moreover, for every $r, s > 0$ the following estimate holds true

$$\frac{F(r)}{r} s \leq \frac{D_F - 1}{d_F} F(r) + \frac{1}{d_F} F(s). \tag{4.6}$$

Remark 4.1.2. Let us comment above lemma.

1. When $F(r) = r^p$, $\frac{1}{p} + \frac{1}{p'} = 1$, we get $r^{p-1}s \leq \frac{1}{p'} r^p + \frac{1}{p} s^p$, equivalent to Young inequality $qs \leq \frac{q^{p'}}{p'} + \frac{s^p}{p}$.
2. For general convex function F the latter inequality in (4.5) with finite constant D_F is equivalent to $F \in \Delta_2$, while the condition $d_F > 1$ is equivalent to $F^* \in \Delta_2$ (see [74], Theorem 4.3 or [67], Proposition 4.1). If d_F and D_F are the best possible in (4.5), they are called Simonenko lower and upper index of F, respectively (see e.g. [17, 49, 57, 91]) for definition and discussion of properties.

Fact 4.1.3. *Let $F(s) = s^p \log^\alpha(b + s)$, $b, p > 1$, $\alpha > 0$. Then, the constants from* (4.5)*, equals $D_F = p + \frac{\alpha}{\log b}$ and $d_F = p$.*

Proof. $F'(s) = (s^p \log^\alpha(b + s))' = ps^{p-1} \log^\alpha(b + s) + \alpha \frac{s^p}{b+s} \log^{\alpha-1}(b + s) = s^{p-1} \log^\alpha(b+s) \left(p + \alpha \frac{s}{(b+s)\log(b+s)}\right) \leq D_F \frac{F(s)}{s}$, with $D_F = \sup \left(p + \alpha \frac{s}{(b+s)\log(b+s)}\right)$.

$F'(s) \geq d_F \frac{F(s)}{s}$, with $d_F = \inf \left(p + \alpha \frac{s}{(b+s)\log(b+s)}\right)$. □

Orlicz—Sobolev spaces

By $W^{1,\bar{A}}(\Omega)$ we mean the completion of the set

$$\{u \in C^\infty(\Omega) : \|u\|_{W^{1,\bar{A}}(\Omega)} := \|u\|_{L^{\bar{A}}(\Omega)} + \|\nabla u\|_{L^{\bar{A}}(\Omega)} < \infty\},$$

under the Luxemburg norm

$$\|f\|_{L^{\bar{A}}(\Omega)} = \inf \left\{K > 0 : \int_\Omega \bar{A}\left(\frac{|f(x)|}{K}\right) dx \leq 1\right\}$$

(in the sequel we assume that $\inf \emptyset = +\infty$). By $W^{1,\bar{A}}_{loc}(\Omega)$ we denote such functions $u : \Omega \to \mathbb{R}$ that $u\phi \in W^{1,\bar{A}}(\Omega)$ for every $\phi \in C^1_0(\Omega)$ (analogous notation is used for local Orlicz spaces $L^{\bar{A}}_{loc}(\Omega)$). Observe that we always have $W^{1,\bar{A}}_{loc}(\Omega) \subseteq W^{1,1}_{loc}(\Omega)$. By $W^{1,\bar{A}}_0(\Omega)$ we denote the completion of smooth compactly supported functions in $W^{1,\bar{A}}(\Omega)$.

The following fact holds true.

Fact 4.1.4 ([72], Fact 2.3)**.** *If $\bar{A}$ is an N–function and $u \in W^{1,\bar{A}}_{loc}(\Omega)$, then*

$$B(|\nabla u|)\nabla u = \frac{\bar{A}(|\nabla u|)}{|\nabla u|}\chi_{\{|\nabla u|\neq 0\}} \in L^{\bar{A}^*}_{loc}(\Omega, \mathbb{R}^n),$$

where B and $\bar{A}$ are the same as in (4.2).

Let $u \in W^{1,\bar{A}}_{loc}(\Omega)$. For $w \in W^{1,\bar{A}}(\Omega)$ with compact support we define

$$\langle \Delta_A u, w\rangle := -\int_\Omega B(|\nabla u|)\langle \nabla u, \nabla w\rangle \, dx. \tag{4.7}$$

According to Fact 4.1.4 the right–hand side in (4.7) is well defined. Obviously when $A(\lambda) = |\lambda|^{p-2}\lambda$, then we retrieve the classical p–Laplacian, $\Delta_p u$.

Differential inequality

The differential inequality we want to analyze is given by the following definition.

Definition 4.1.3. *Let Ω be any open subset of $\mathbb{R}^n$ and Φ be the locally integrable function defined in Ω, such that for every nonnegative compactly supported $w \in W^{1,\bar{A}}(\Omega)$*

$$\left|\int_\Omega \Phi w \, dx\right| < \infty. \tag{4.8}$$

Let $u \in W^{1,\bar{A}}_{loc}(\Omega)$. We will say that

$$-\Delta_A u \geq \Phi \tag{4.9}$$

if for every nonnegative compactly supported $w \in W^{1,\bar{A}}(\Omega)$ we have

$$\langle -\Delta_A u, w\rangle = \int_\Omega B(|\nabla u|)\langle \nabla u, \nabla w\rangle \, dx \geq \int_\Omega \Phi w \, dx. \tag{4.10}$$

Remark 4.1.3. We may choose $\Phi = \Phi(x, u, \nabla u)$.

Set of assumptions. In the sequel we will consider functions satisfying the following assumptions.

($\bar{A}$) $\bar{A}$ is an N–function satisfying Δ'–condition;

(Ψ) there exists a function $\Psi : [0, \infty) \to [0, \infty)$, which is nonnegative and belongs to $C^1((0, \infty))$ and satisfies the following conditions

i) inequality

$$g(t)\Psi^{'}(t) \le -C\Psi(t) \tag{4.11}$$

holds for all $t > 0$ with $C > 0$ independent of t and certain continuous function $g : (0, \infty) \to (0, \infty)$, such that $\Psi(t)/g(t)$ is nonincreasing.

ii) function

$$s \mapsto \Theta(s) := \frac{\bar{A}\left(g(s)\right)\Psi(s)}{g(s)} \tag{4.12}$$

is nonincreasing or bounded in certain neighbourhood of 0.

(u) $u \in W^{1,\bar{A}}_{loc}(\Omega)$ is a given nonnegative solution to (4.9) which is nontrivial, i.e. $u \not\equiv const$, and there exists $\sigma \in \mathbb{R}$ such that

$$\Phi + \sigma\,\frac{\bar{A}(|\nabla u|)}{g(u)}\chi_{\{\nabla u \neq 0\}} \ge 0 \quad \text{a.e.} \tag{4.13}$$

We define

$$\sigma_0 = \inf\{\sigma \in \mathbb{R} : (4.13) \text{ is satisfied}\}, \tag{4.14}$$

where we set $\inf \emptyset = +\infty$.

Remark 4.1.4. Examples when those conditions are satisfied in the case when $\bar{A}(s) = s^p$, $g(s) = s$, $\Psi(s) = s^{-\beta}, \beta > 0$ can be found in [92, 93].

Remark 4.1.5. Let us discuss the assumption (Ψ) **i)**. In particular, it implies that Ψ is decreasing. Elementary calculation leads to following pairs of Ψ and g satisfying condition $g(t)\Psi'(t) \le -C\Psi(t)$ a.e. To ensure that additionally $\Psi(t)/g(t)$ is nonincreasing we have assume that $g'(t) \ge -C$ with th same C. Indeed, Ψ/g is nonincreasing if

$$\begin{aligned}\left(\frac{\Psi(t)}{g(t)}\right)' &= \tfrac{\Psi'(t)g(t)-\Psi(t)g'(t)}{g^2(t)} \le \frac{-C\Psi(t) - \Psi(t)g'(t)}{g^2(t)} = \\ &= -\tfrac{\Psi(t)}{g^2(t)}(C + g'(t)) \le 0\end{aligned}$$

I.e.: when $g'(t) \ge -C$.

The following pairs satisfy assumption (Ψ) (see Table 4.1).

$\Psi(t)$	g(t)	C	remarks
$t^{-\alpha}$	t	α	$\alpha > 0$
e^{-t}	bounded by C, $g' \geq -C$	C	$C > 0$
e^{-t}/t	$t/(1+t)$	1	—
$e^{\frac{1}{2}\log^2(t)}$	$t/\|\log t\|$	1	considered on $(0,1)$

Table 4.1: Good pairs of Ψ and g

4.2 Caccioppoli estimates for solutions to PDI $-\Delta_A u \geq \Phi$

Our main goal in this section is to obtain the following result.

Theorem 4.2.1. *Let $u \in W^{1,\bar{A}}_{loc}(\Omega)$ be a nonnegative solution to PDI: $-\Delta_A u \geq \Phi$, in the sense of Definition 4.1.3, where Φ is locally integrable and assumptions $(\bar{A})$, (Ψ), (u) are satisfied satisfied with $C > 0$ and $\sigma \in [\sigma_0, C)$, where σ_0 is given by* (4.14)*. Let $C_{\bar{A}} > 0$ be a constant coming from Δ'–condition for $\bar{A}$ (see Definition 4.1.2) and $D_{\bar{A}} \geq d_{\bar{A}} > 1$ be constants coming from* (4.5) *applied to $\bar{A}$.*

Then the inequality

$$\int_\Omega \left(\Phi + \sigma \frac{\bar{A}(|\nabla u|)}{g(u)} \chi_{\{\nabla u \neq 0\}} \right) \Psi(u) \phi \, dx \leq \tag{4.15}$$

$$\leq K \int_{\Omega \cap \{\nabla u \neq 0\}} \frac{\bar{A}\left(g(u)\right) \Psi(u)}{g(u)} \cdot \bar{A}\left(\frac{|\nabla \phi|}{\phi} \right) \phi \, dx,$$

holds for every nonnegative Lipschitz function ϕ with compact support in Ω, such that the integral $\int_{\cap\{\nabla u \neq 0\}} \bar{A}\left(\frac{|\nabla\phi|}{\phi}\right) \phi \, dx$ is finite and

$$K = (C - \sigma)\bar{A}\left(\frac{D_{\bar{A}} - 1}{(C - \sigma) d_{\bar{A}}} \right) \frac{C^2_{\bar{A}}}{D_{\bar{A}} - 1}.$$

We call (4.15) Caccioppoli inequality, because it involves ∇u on the left–hand side and only u on the right–hand side (see e.g. [23, 65]).

The proof is based on careful analysis of the proof of Proposition 3.1 from [72]. However, here we are not restricted to $\Phi = \Phi(u)$, $\Phi \geq 0$ and integrals over $\mathbb{R}^n$.

Remark 4.2.1. We do not assume that right–hand side in (4.15) is finite.

Proof of Theorem 4.2.1. The proof follows by three steps.
STEP 1. DERIVATION OF LOCAL INEQUALITY.

We obtain the following lemma.

Lemma 4.2.1. *Let $u \in W^{1,\bar{A}}_{loc}(\Omega)$ be a nonnegative solution to PDI: $-\Delta_A u \geq \Phi$, in the sense of Definition 4.1.3, where Φ is locally integrable and assumptions $(\bar{A})$, (Ψ), (u) are satisfied satisfied with $C > 0$ and $\sigma \in [\sigma_0, C)$, where σ_0 is given by* (4.14)*. Let K be the constant from Theorem 4.2.1.*

Then for every $0 < \delta < R$ and every nonnegative Lipschitz function ϕ with compact support in Ω, the inequality

$$\int_{\{u\leq R-\delta\}} \left(\Phi + \sigma \frac{\bar{A}(|\nabla u|)}{g(u+\delta)} \chi_{\{\nabla u\neq 0\}} \right) \Psi(u+\delta)\phi \; dx$$

$$\leq K \int_{\Omega\cap\{\nabla u\neq 0,\, u\leq R-\delta\}} \Theta(u+\delta)\cdot \bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi \; dx + \tilde{C}(\delta, R), \tag{4.16}$$

holds with $\Theta(u)$ given by (4.12) and

$$\tilde{C}(\delta,R) := \Psi(R)\left[\int_{\Omega\cap\{\nabla u\neq 0,\, u>R-\delta\}} B(|\nabla u|)\langle \nabla u, \nabla\phi\rangle \, dx - \int_{\Omega\cap\{u>R-\delta\}} \Phi\phi \, dx\right]. \tag{4.17}$$

Before we prove the theorem let us formulate the following facts.

Fact 4.2.1 ([72])**.** *For u, ϕ as in the assumptions of Theorem 4.2.1 we fix $0 < \delta < R$ and denote*

$$u_{\delta,R}(x) := \min\left(u(x)+\delta, R\right), \qquad G(x) := \Psi(u_{\delta,R}(x))\phi(x). \tag{4.18}$$

Then $u_{\delta,R} \in W^{1,\bar{A}}_{loc}(\Omega)$ and $G \in W^{1,\bar{A}}_0(\Omega) \subseteq W^{1,\bar{A}}(\Omega)$.

Fact 4.2.2 ([72])**.** *Let $u \in W^{1,1}_{loc}(\Omega)$ be defined everywhere by the formula (3.1) and let $t \in \mathbb{R}$. Then*

$$\{x\in\Omega : u(x) = t\} \subseteq \{x \in \Omega : \nabla u(x) = 0\} \cup N \tag{4.19}$$

where $|N| = 0$.

Proof of Lemma 4.2.1. According to (4.8) integral $\int_\Omega \Phi\phi dx$ is finite. Before we start the proof of (4.16), let us introduce some notation, where $0<\delta<R<\infty$:

$$\begin{aligned}
\tilde{A}(\delta,R) &= \int_{\Omega\cap\{\nabla u\neq 0,\, u\leq R-\delta\}} \bar{A}(|\nabla u|)\Psi^{'}(u+\delta)\phi\, dx,\\
\tilde{A}_1(\delta,R) &= \int_{\Omega\cap\{\nabla u\neq 0,\, u\leq R-\delta\}} \bar{A}(|\nabla u|)\left(\frac{\Psi(u+\delta)}{g(u+\delta)}\right)\phi\, dx,\\
\tilde{B}(\delta,R) &= \int_{\Omega\cap\{\nabla u\neq 0,\, u\leq R-\delta\}} B(|\nabla u|)\langle\nabla u,\nabla\phi\rangle\Psi(u+\delta)\, dx,\\
\tilde{C}_1(\delta,R) &= \Psi(R)\int_{\Omega\cap\{u>R-\delta\}} \Phi\phi\, dx, &(4.20)\\
\tilde{C}_2(\delta,R) &= \Psi(R)\int_{\Omega\cap\{\nabla u\neq 0,\, u>R-\delta\}} B(|\nabla u|)\langle\nabla u,\nabla\phi\rangle\, dx, &(4.21)\\
\tilde{D}(\bar{\epsilon},\delta,R) &= \bar{\epsilon}\bar{A}\left(\frac{1}{\bar{\epsilon}}\right)\frac{C_{\bar{A}}^2}{d_{\bar{A}}}\int_{\mathrm{supp}\,\phi\cap\{\nabla u\neq 0,\, u\leq R-\delta\}} \Theta(u+\delta)\bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\, dx,
\end{aligned}$$

where $\Theta(u)$ is given by (4.12). Let us consider $u_{\delta,R}$ and G defined by (4.18).

We note that

$$\begin{aligned}
I &:= \int_\Omega \Phi G\, dx = \int_\Omega \Phi\Psi(u_{\delta,R})\phi\, dx =\\
&= \int_{\Omega\cap\{u\leq R-\delta\}} \Phi\Psi(u+\delta)\phi\, dx + \Psi(R)\int_{\Omega\cap\{u>R-\delta\}}\Phi\phi\, dx =\\
&= \int_{\Omega\cap\{u\leq R-\delta\}} \Phi\Psi(u+\delta)\phi\, dx + \tilde{C}_1(\delta,R), \qquad (4.22)
\end{aligned}$$

On the other hand, inequality (4.9) implies

$$\begin{aligned}
I &:= \int_\Omega \Phi G\, dx \leq \langle -\Delta_A u, G\rangle = \int_{\Omega\cap\{\nabla u\neq 0\}} B(|\nabla u|)\langle\nabla u,\nabla G\rangle\, dx =\\
&= \int_{\Omega\cap\{\nabla u\neq 0,\, u\leq R-\delta\}} \bar{A}(|\nabla u|)\Psi^{'}(u+\delta)\phi\, dx +\\
&+ \int_{\Omega\cap\{\nabla u\neq 0,\, u\leq R-\delta\}} B(|\nabla u|)\langle\nabla u,\nabla\phi\rangle\Psi(u+\delta)\, dx +\\
&+ \Psi(R)\int_{\Omega\cap\{\nabla u\neq 0,\, u>R-\delta\}} B(|\nabla u|)\langle\nabla u,\nabla\phi\rangle\, dx =\\
&= \tilde{A}(\delta,R)+\tilde{B}(\delta,R)+\tilde{C}_2(\delta,R). \qquad (4.23)
\end{aligned}$$

Note that all integrals above are finite, what follows from Fact 4.1.4 (for $0 \leq u \leq R-\delta$ we have $\delta \leq u+\delta \leq R$). Using assumption (Ψ) we get

$$\begin{aligned}\tilde{A}(\delta,R) &\leq -C\int_{\Omega\cap\{\nabla u\neq 0,\, u\leq R-\delta\}} \bar{A}(|\nabla u|)\left(\frac{\Psi(u+\delta)}{g(u+\delta)}\right)\phi\,dx = \\ &= -C\tilde{A}_1(\delta,R). \end{aligned} \tag{4.24}$$

Moreover, for an arbitrary $\bar{\epsilon}>0$,

$$\begin{aligned}\tilde{B}(\delta,R) &\leq \int_{\Omega\cap\{\nabla u\neq 0,\, u\leq R-\delta\}} B(|\nabla u|)|\nabla u||\nabla\phi|\Psi(u+\delta)\,dx = \\ &= \bar{\epsilon}\int_{\operatorname{supp}\phi\cap\{\nabla u\neq 0,\, u\leq R-\delta\}} B(|\nabla u|)|\nabla u|\left(\frac{|\nabla\phi|}{\phi}\frac{g(u+\delta)}{\bar{\epsilon}}\right)\frac{\Psi(u+\delta)}{g(u+\delta)}\,\phi dx.\end{aligned}$$

As $B(|\nabla u|)|\nabla u| = \frac{\bar{A}(|\nabla u|)}{|\nabla u|}$, we can apply (4.6) for the N–function $\bar{A}$ with $r=|\nabla u|$, $s=\left(\frac{|\nabla\phi|}{\phi}\frac{g(u+\delta)}{\bar{\epsilon}}\right)$ to get

$$\begin{aligned}\tilde{B}(\delta,R) \leq \quad & \bar{\epsilon}\frac{D_{\bar{A}}-1}{d_{\bar{A}}}\int_{\operatorname{supp}\phi\cap\{\nabla u\neq 0,\, u\leq R-\delta\}} \bar{A}(|\nabla u|)\frac{\Psi(u+\delta)}{g(u+\delta)}\phi\,dx + \\ &+ \frac{\bar{\epsilon}}{d_{\bar{A}}}\int_{\operatorname{supp}\phi\cap\{\nabla u\neq 0,\, u\leq R-\delta\}} \bar{A}\left(\frac{|\nabla\phi|}{\phi}\frac{g(u+\delta)}{\bar{\epsilon}}\right)\frac{\Psi(u+\delta)}{g(u+\delta)}\phi\,dx.\end{aligned}$$

Then, applying Δ'–condition for $\bar{A}$ twice in the second expression above, we obtain

$$\tilde{B}(\delta,R) \leq \bar{\epsilon}\frac{D_{\bar{A}}-1}{d_{\bar{A}}}\tilde{A}_1(\delta,R) + \tilde{D}(\bar{\epsilon},\delta,R). \tag{4.25}$$

Combining estimates (4.23), (4.24) and (4.25) we get

$$\begin{aligned} I &\leq -C\tilde{A}_1(\delta,R) + \tilde{B}(\delta,R) + \tilde{C}_2(\delta,R) \leq \\ &\leq \left(-C+\bar{\epsilon}\frac{D_{\bar{A}}-1}{d_{\bar{A}}}\right)\tilde{A}_1(\delta,R) + \tilde{D}(\bar{\epsilon},\delta,R) + \tilde{C}_2(\delta,R).\end{aligned}$$

Moreover, $\tilde{C}_1(\delta,R)$ and $\tilde{A}_1(\delta,R)$ are finite (and $\tilde{D}(\epsilon,\delta,R)$ is finite as well). This and (4.22) imply

$$\begin{aligned}\int_{\Omega\cap\{u\leq R-\delta\}} \Phi\Psi(u+\delta)\phi\,dx + \left(C-\bar{\epsilon}\frac{D_{\bar{A}}-1}{d_{\bar{A}}}\right)\tilde{A}_1(\delta,R) \leq \\ \leq \tilde{D}(\bar{\epsilon},\delta,R) + (\tilde{C}_2(\delta,R)-\tilde{C}_1(\delta,R)).\end{aligned}$$

This is (4.16). Indeed, we have $\tilde{C}(\delta, R) = \tilde{C}_2(\delta, R) - \tilde{C}_1(\delta, R)$. Moreover, when we substitute $\sigma := C - \bar{\epsilon}\frac{D_{\bar{A}}-1}{d_{\bar{A}}}$ we get

$$\begin{aligned}\bar{\epsilon}\bar{A}\left(\frac{1}{\bar{\epsilon}}\right)\frac{C_{\bar{A}}^2}{d_{\bar{A}}} &= \frac{(C-\sigma)d_{\bar{A}}}{D_{\bar{A}}-1}\bar{A}\left(\frac{D_{\bar{A}}-1}{(C-\sigma)d_{\bar{A}}}\right)\frac{C_{\bar{A}}^2}{d_{\bar{A}}} = \\ &= \frac{(C-\sigma)}{D_{\bar{A}}-1}\bar{A}\left(\frac{D_{\bar{A}}-1}{(C-\sigma)d_{\bar{A}}}\right)C_{\bar{A}}^2 = K.\end{aligned}$$

We notice that $\bar{\epsilon} > 0$ is arbitrary and we may always choose $0 < \bar{\epsilon} \leq \frac{(C-\sigma_0)d_{\bar{A}}}{D_{\bar{A}}-1}$, so that $\sigma_0 \leq \sigma < C$. □

We have to introduce parameters δ and R to make sure that some quantities in the estimates, which we move to opposite sides of inequalities, are finite.

STEP 2. PASSING TO THE LIMIT WITH $\delta \searrow 0$.

In this step we show that when assumptions $(\bar{A})$, (Ψ) and (Φ) are satisfied with $\epsilon > 0$, K is the constant from Theorem 4.2.1, then for any $R > 0$ inequality

$$\begin{aligned}&\int_{\{u\leq R\}}\left(\Phi + \sigma\frac{\bar{A}(|\nabla u|)}{g(u)}\chi_{\{\nabla u\neq 0\}}\right)\Psi(u)\phi\,dx \leq \\ &\leq K\int_{\{\nabla u\neq 0,\, u\leq R\}}\frac{\bar{A}\left(g(u)\right)\Psi(u)}{g(u)}\,\bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\,dx + \tilde{C}(R),\end{aligned} \tag{4.26}$$

where

$$\tilde{C}(R) = \Psi(R)\left[\left|\int_{\Omega\cap\{u\geq\frac{R}{2}\}} B(|\nabla u|)|\nabla u|\cdot|\nabla\phi|\,dx\right| + \left|\int_{\Omega\cap\{u\geq\frac{R}{2}\}}\Phi\phi\,dx\right|\right] \tag{4.27}$$

holds for every nonnegative Lipschitz function ϕ with compact support in Ω, such that the integral $\int_{\operatorname{supp}\phi\cap\nabla u\neq 0}\bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\,dx$ is finite. Moreover, all quantities appearing in (4.26) are finite.

For this, we show first that under our assumptions, when $\delta \searrow 0$ we have

$$\int_{\Omega\cap\{\nabla u\neq 0,\, u+\delta\leq R\}}\Theta(u+\delta)\cdot\bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\,dx \to \int_{\Omega\cap\{\nabla u\neq 0,\, u\leq R\}}\Theta(u)\cdot\bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\,dx. \tag{4.28}$$

Note that $\Theta(u+\delta)\chi_{u+\delta\le R} \overset{\delta\to 0}{\to} \Theta(u)\chi_{u\le R}$, a.e. This follows from Lemma 4.2.2 (which gives that the sets $\{u=0,\ |\nabla u|\neq 0\}$ and $\{u=R,\ |\nabla u|=0\}$ are of measure zero) and the continuity outside zero of the involved functions.

We assumed in (Θ) that Θ is nonincreasing or bounded in the neighbourhood of zero. Let we start with the case when there exists $\kappa>0$ such that for $\lambda<\kappa$ the function $\Theta(\lambda)$ is nonincreasing. Without loss of generality we may consider $\kappa\le R$.

We divide the domain of integration

$$\int_{\Omega\cap\{\nabla u\neq 0,\, u+\delta\le R\}} \Theta(u+\delta)\cdot \bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\, dx =$$

$$= \int_{E_\kappa} \Theta(u+\delta)\cdot \bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\, dx + \int_{F_\kappa} \Theta(u+\delta)\chi_{\{u+\delta\le R\}}\cdot \bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\, dx,$$

where

$$E_\kappa = \left\{u<\frac{\kappa}{2},\ \nabla u\neq 0\right\}\cap \operatorname{supp}\phi, \quad F_\kappa = \left\{\frac{\kappa}{2}\le u,\ \nabla u\neq 0\right\}\cap \operatorname{supp}\phi.$$

Let us begin with integral over E_κ. We consider $\delta\to 0$, so we may assume that $\delta<\kappa/2$. Then for $x\in E_\kappa$ we have $u+\delta<\kappa$. As function $\lambda\to\Theta(\lambda)$ is nonincreasing when $\lambda<\kappa$, thus for $\delta\searrow 0$ the function $\delta\to\Theta(u+\delta)$ is nondecreasing and so convergent monotonically almost everywhere to $\Theta(u)$. Therefore, due to The Lebesgue's Monotone Convergence Theorem

$$\lim_{\delta\to 0}\int_{E_\kappa} \Theta(u+\delta)\bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\, dx = \int_{E_\kappa} \Theta(u)\bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\, dx.$$

In the case of F_κ, we have $\kappa/2\le u+\delta\le R$. Over this domain Θ is a bounded function, so in particular on F_κ:

$$\Theta(u+\delta)\chi_{\{u+\delta\le R\}}\bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi \le \sup_{t\in[\kappa/2,R]}\Theta(t)\cdot\bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\in L^1(F_\kappa).$$

We apply The Lebesgue's Dominated Convergence Theorem to deduce that

$$\lim_{\delta\to 0}\int_{F_\kappa} \Theta(u+\delta)\chi_{\{u+\delta\le R\}}\bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\, dx = \int_{F_\kappa\cap\{u\le R\}} \Theta(u)\bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\, dx.$$

This completes the case of Θ nonincreasing in the neighbourhood of 0. In the case when Θ is bounded in the neighbourhood of 0, we note that Θ is bounded on every interval $[0, R]$, where $R > 0$. Hence, we can use previous computations dealing with F_κ in case $\kappa = 0$.

To finish the proof of this step we note that (4.28) says that when $\delta \searrow 0$ the first integral on the right–hand side of (4.16) is convergent to the first integral of right–hand side of (4.26). To deal with the second expression we note that for $\delta \leq \frac{R}{2}$:

$$|\tilde{C}(\delta, R)| \quad \leq \quad |\tilde{C}_2(\delta, R)| + |\tilde{C}_1(\delta, R)| \leq \tilde{C}(R),$$

where $\tilde{C}(\delta, R)$, $\tilde{C}_2(\delta, R)$, $\tilde{C}_1(\delta, R)$, $\tilde{C}(R)$ are given by (4.17), (4.20), (4.21), (4.27), respectively.

We can pass to the limit with $\delta \to 0$ on the left–hand side of (4.16) due to The Lebesgue's Monotone Convergence Theorem as an expression in brackets is nonnegative by (4.13) and the whole integrand therein is nonincreasing by assumption (Ψ).

Step 3. We let $R \to \infty$ and finish the proof.

We are going to let $R \to \infty$ in (4.26). Without loss of generality we can assume that the integral in the right–hand side of (4.15) is finite, as otherwise the inequality follows trivially. Note that as $B(|\nabla u|)\langle \nabla u, \nabla \phi \rangle$ and $\Phi \phi$ are integrable, we have $\lim_{R\to\infty} \widetilde{C}(R) = 0$. Therefore (4.15) follows from (4.26) by the Lebesgue's Monotone Convergence Theorem. □

4.3 Hardy type inequalities

Our most general conclusion resulting from Theorem 4.2.1 reads as follows.

Theorem 4.3.1. *Let $u \in W^{1,\bar{A}}_{loc}(\Omega)$ be a nonnegative solution to PDI: $-\Delta_A u \geq \Phi$, in the sense of Definition 4.1.3, where Φ is locally integrable and assumptions $(\bar{A})$, (Ψ), (u) are satisfied with $C > 0$ and $\sigma \in [\sigma_0, C)$, where σ_0 is given by* (4.14). *Set*

$$F_{\bar{A}}(\lambda) = \frac{1}{\bar{A}\,(1/\lambda)}, \text{ when } \lambda > 0 \text{ and } F_{\bar{A}}(0) = 0. \tag{4.29}$$

Then for every Lipschitz function ξ with compact support in Ω, we have

$$\int_\Omega F_{\bar{A}}(|\xi|)\mu_1(dx) \leq \widetilde{C} \int_\Omega \bar{A}(|\nabla \xi|)\mu_2(dx). \tag{4.30}$$

where

$$\begin{aligned}
\mu_1(dx) &= \Psi(u)\left[\Phi+\sigma\frac{\bar{A}(|\nabla u|)}{g(u)}\right]\chi_{\{u>0\}}\,dx, &(4.31)\\
\mu_2(dx) &= \frac{\bar{A}\left(g(u)\right)\Psi(u)}{g(u)}\chi_{\{\nabla u\neq 0\}}\,dx, &(4.32)\\
\widetilde{C} &= (C-\sigma)\bar{A}\left(\frac{D_{\bar{A}}-1}{(C-\sigma)d_{\bar{A}}}\right)\frac{\bar{A}(D_{\bar{A}})C_{\bar{A}}^4}{D_{\bar{A}}-1}. &(4.33)
\end{aligned}$$

with constants $C_{\bar{A}}>0$ coming from Δ'–condition for $\bar{A}$ (see Definition 4.1.2) and $D_{\bar{A}}>d_{\bar{A}}\geq 1$ coming from (4.5) *applied to $\bar{A}$.*

Proof. Let ξ be a compactly supported Lipschitz function. We define $\phi = F_{\bar{A}}(\xi)$ and apply Theorem 4.2.1. For this we have to verify that ϕ is compactly supported Lipschitz function and $\int_\Omega \bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\,dx<\infty$. We observe that ϕ is compactly supported, because $F_{\bar{A}}(t)$ is continuous at 0. Indeed,

$$\lim_{t\to 0}F_{\bar{A}}(t)=\lim_{t\to 0}\frac{1}{\bar{A}\left(1/t\right)}=\lim_{s\to\infty}\frac{1}{\bar{A}\left(s\right)}=0,$$

which ensures that $\operatorname{supp}\phi=\operatorname{supp}\xi$. Furthermore, $F_{\bar{A}}(t)$ is a locally Lipschitz function. We obtain it from Lemma 4.1.1 which implies

$$F_{\bar{A}}'(t)=\left(\frac{1}{\bar{A}\left(1/t\right)}\right)'\sim\frac{1}{t\bar{A}(1/t)}.$$

Applying the condition $\lim_{s\to\infty}\frac{s}{\bar{A}(s)}=0$ from definition of N–function, we get that $F_{\bar{A}}'(t)$ is a locally bounded function and bounded nearby 0. Therefore, $F_{\bar{A}}(t)$ is locally Lipshitz. The composition of locally Lipshitz function $F_{\bar{A}}(t)$ with Lipschitz and bounded ξ, i.e. $F_{\bar{A}}(\xi)=\phi$, is Lipschitz.

We verify that $\int_\Omega \bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\,dx<\infty$. Note that for every compactly supported Lipschitz function ξ we have $\int_\Omega \bar{A}(|\nabla\xi|)\,dx<\infty$. Therefore, it suffices to prove that

$$\bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\leq C_{\bar{A}}^2\bar{A}(D_{\bar{A}})\bar{A}(|\nabla\xi|). \tag{4.34}$$

As $\bar{A} \in \Delta'$, we note that for each pair of $x, y \geq 0$ we have

$$\begin{aligned}\bar{A}(x)y &= \bar{A}\left(\frac{x}{\bar{A}^{-1}(\frac{1}{y})}\bar{A}^{-1}\left(\frac{1}{y}\right)\right)y \leq \\ &\leq C_{\bar{A}}\bar{A}\left(\frac{x}{\bar{A}^{-1}(\frac{1}{y})}\right)\bar{A}\left(\bar{A}^{-1}\left(\frac{1}{y}\right)\right)y = C_{\bar{A}}\bar{A}\left(\frac{x}{\bar{A}^{-1}(\frac{1}{y})}\right)\end{aligned} \tag{4.35}$$

Hence, taking $x = \frac{|\nabla\phi|}{\phi}$ and $y = \phi$, we obtain from (4.35)

$$\bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi \leq C_{\bar{A}}\bar{A}\left(\frac{|\nabla\phi|}{\phi}\frac{1}{\bar{A}^{-1}(\frac{1}{\phi})}\right), \tag{4.36}$$

for any nonnegative ϕ at every x where $\phi(x) > 0$.

Now we show that at every x, where $\phi(x) > 0$ we have

$$\frac{|\nabla\phi(x)|}{\phi(x)}\frac{1}{\bar{A}^{-1}\left(\frac{1}{\phi(x)}\right)} \leq D_{\bar{A}}|\nabla\xi(x)|. \tag{4.37}$$

Indeed, we have $\phi = \frac{1}{\bar{A}(\frac{1}{\xi})}$, so that

$$\nabla\phi = F'_{\bar{A}}(\xi) = -\frac{1}{\bar{A}^2\left(\frac{1}{\xi}\right)}\bar{A}'\left(\frac{1}{\xi}\right)\left(-\frac{1}{\xi^2}\right)\nabla\xi.$$

Applying (4.5) to $\bar{A} \in \Delta_2$ we have $\bar{A}'(\lambda) \leq D_{\bar{A}}\frac{\bar{A}(\lambda)}{\lambda}$, with the constant $D_{\bar{A}}$. Therefore

$$|\nabla\phi| \leq \frac{1}{\bar{A}^2\left(\frac{1}{\xi}\right)}D_{\bar{A}}\bar{A}\left(\frac{1}{\xi}\right)\frac{|\nabla\xi|}{\xi} = D_{\bar{A}}\phi\frac{|\nabla\xi|}{\xi}.$$

Hence, we have $\frac{|\nabla\phi|}{\phi}\xi \leq D_{\bar{A}}|\nabla\xi|$, which is exactly (4.37).

Summing up the estimates (4.36) and (4.37) we obtain (4.34)

$$\bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi \leq C_{\bar{A}}\bar{A}\left(\frac{|\nabla\phi|}{\phi}\frac{1}{\bar{A}^{-1}(\frac{1}{\phi})}\right) \leq C_{\bar{A}}\bar{A}\left(D_{\bar{A}}|\nabla\xi|\right) \leq C_{\bar{A}}^2\bar{A}(D_{\bar{A}})\bar{A}(|\nabla\xi|).$$

Thus the assumptions of Theorem 4.2.1 are satisfied. We obtain (4.15). The substitution $\phi = F_{\bar{A}}(\xi)$, equivalently taking

$$\xi(x) = \begin{cases} \frac{1}{\bar{A}^{-1}\left(\frac{1}{\phi(x)}\right)}, & \text{when } \phi(x) \neq 0, \\ 0, & \text{when } \phi(x) = 0, \end{cases}$$

where $\bar{A}^{-1}$ is the inverse function of $\bar{A}$, transforms the left–hand side of (4.15) into the left–hand side of (4.30). What remains to show is that the right–hand side in (4.15) is estimated as follows

$$\int_{\{\nabla u \neq 0\}} \frac{\bar{A}\left(g(u)\right)\Psi(u)}{g(u)} \bar{A}\left(\frac{|\nabla\phi|}{\phi}\right)\phi\, dx \leq C_{\bar{A}}^2 \bar{A}(D_{\bar{A}}) \int_{\{\nabla u \neq 0\}} \frac{\bar{A}\left(g(u)\right)\Psi(u)}{g(u)} \bar{A}(|\nabla\xi|)\, dx.$$

This is a direct consequence of (4.34). The proof is complete. □

Examples dealing with various $F_{\bar{A}}$ and g are given in the following sections.

4.4 Retrieving our previous results

When we consider $\Delta_A = \Delta_p$ (i.e. we take $\bar{A}(t) = t^p$), the method becomes much simpler and the obtained inequality (4.30) involves $F_{\bar{A}}(t) = \frac{1}{(1/t)^p} = t^p = \bar{A}(t)$. In this case we have

$$\int_\Omega |\xi|^p \mu_1(dx) \leq \int_\Omega |\nabla\xi|^p \mu_2(dx)$$

with certain measures.

We concentrate on retrieving our previous results from [92, 93] given in Chapter 3. In particular, Theorem 4.3.1 imply Theorem 3.3.1. It leads among others to Hardy and Hardy–Poincaré inequalities with optimal constants (see Chapter 3).

Sketch of the proof of Theorem 3.3.1 via Theorem 4.3.1. We apply Theorem 4.3.1, respectively, with $\bar{A}(t) = t^p = F_{\bar{A}}(t)$, $g(t) = t$, $\Psi(t) = t^{-\beta}$, $C = \beta > 0$ (then $C_{\bar{A}} = 1$, $d_{\bar{A}} = D_{\bar{A}} = p$). We note that the assumption (3.5) matches with the assumption (u). Inequality (3.13) follows from (4.30). Involved measures and constants are the same. □

Remark 4.4.1. Theorem 4.3.1 enables us to derive various measures in (3.13). In the above examples we apply $\Psi(t) = t^{-\beta}$, $g(t) = t$. When we check the other pairs e.g. $\Psi(t) = e^{-t}$, $g(t) \equiv 1$, or $\Psi(t) = \frac{e^{-t}}{t}$, $g(t) = 1/(1+t)$, we obtain comparable inequalities.

4.5 Hardy–Sobolev inequalities dealing with Orlicz functions of power–logarythmic type

Now we deal with the case $\bar{A}(t) = t^p \log^\alpha(2+t)$, $p > 1$, $\alpha > 0$.

Lemma 4.5.1. *Suppose $p > 1$, $\alpha > 0$, $\bar{A}(t) = t^p \log^\alpha(2+t)$ and $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$. Let $u \in W^{1,\bar{A}}_{loc}(\Omega)$ be a nonnegative solution to PDI: $-\Delta_A u \geq \Phi$, in the sense of Definition 4.1.3, where Φ is locally integrable and assumptions (Ψ), (u) are satisfied with $\sigma \in \mathbb{R}$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$.*

Then there exists a constant $\widetilde{C} > 0$, such that for every Lipschitz function ξ with compact support in Ω, we have

$$\int_{\Omega\cap\{\xi\neq 0\}} |\xi|^p \log^{-\alpha}(2 + 1/|\xi|)\mu_1(dx) \leq \widetilde{C} \int_\Omega |\nabla\xi|^p \log^\alpha(2 + |\nabla\xi|)\mu_2(dx),$$

where

$$\mu_1(dx) = \Psi(u)\left(\Phi + \frac{\sigma}{g(u)}|\nabla u|^p \log^\alpha(2 + |\nabla u|)\right)\chi_{\{u>0\}}\,dx, \quad (4.38)$$

$$\mu_2(dx) = g^{p-1}(u)\log^\alpha(2 + g(u))\Psi(u)\chi_{\{\nabla u\neq 0\}}\,dx, \quad (4.39)$$

Proof. We apply Theorem 4.3.1. We remark first that assumption $(\bar{A})$ is satisfied as, according to Example 4.1.1, $\bar{A} \in \Delta'$ if $p > 1$, $\alpha > 0$. We notice, that

$$F_{\bar{A}}(t) = \frac{1}{\bar{A}(1/t)} = \frac{1}{(1/t)^p \log^\alpha(2 + 1/t)} = t^p \log^{-\alpha}(2 + 1/t), \quad F_{\bar{A}}(0) = 0. \quad (4.40)$$

□

As a direct consequence of Lemma 4.5.1 we obtain the following corollary.

Corollary 4.5.1. *Suppose $p > 1$, $\alpha > 0$, $\bar{A}(t) = t^p \log^\alpha(1+t)$ and $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$. Let $u \in W^{1,\bar{A}}_{loc}(\Omega)$ be a nonnegative solution to PDI: $-\Delta_A u \geq \Phi$, in the sense of Definition 4.1.3, where Φ is locally integrable and assumptions (Ψ), (u) are satisfied satisfied with $\sigma \in \mathbb{R}$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$.*

Then there exists $\widetilde{C} > 0$, such that for every Lipschitz function ξ with compact support in Ω, we have

$$\int_\Omega |\xi|^{p+\alpha}\mu_1(dx) \leq \widetilde{C} \int_\Omega \bar{A}(|\nabla\xi|)\mu_2(dx),$$

where $\mu_1(dx), \mu_2(dx), \widetilde{C}$ comes from Theorem 4.5.1.

Proof. We note, that $t^\alpha < \log^{-\alpha}(2+1/t)$. Indeed, $\log(2+1/t) = \log\left(\frac{2t+1}{t}\right) = \frac{\log(2t+1)-\log(t)}{2t+1-t} = \log'(t_1) = \frac{1}{t_1}$, for some $t_1 \in (t, 2t+1)$.

This implies

$$\int_\Omega |\xi|^{p+\alpha}\mu_1(dx) < \int_\Omega |\xi|^p \log^{-\alpha}(1+2/|\xi|)\mu_1(dx)$$

and the result follows from estimate proven in Theorem 4.5.1. □

We give two examples of application Theorem 4.3.1 to power–logarithm function $\bar{A}$ and u being a power function defined on a halfline. We start with a lemma confirming common assumptions.

Lemma 4.5.2. *Suppose $p > 1,\ \alpha > 0,\ \beta \in (0,1)$ and $\Omega \subseteq \mathbb{R}_+$. Assume further that assumption (Ψ) is satisfied with functions Ψ, g and (u) is satisfied with*

$$\sigma > -(1/\beta - 1)(p-1)\inf_{x>0} g(x^\beta)x^{-\beta} =: \sigma_0.$$

Then there exists a constant $\widetilde{C} > 0$, such that for every Lipschitz function ξ with compact support in Ω, we have

$$\int_\Omega |\xi|^p \log^{-\alpha}(2+1/|\xi|)\mu_1(dx) \le \widetilde{C}\int_\Omega |\xi'|^p \log^\alpha(2+|\xi'|)\mu_2(dx),$$

where

$$\mu_1(dx) = \frac{\Psi(x^\beta)}{g(x^\beta)}x^{p(\beta-1)}\log^\alpha\left(2+\beta x^{\beta-1}\right)dx, \tag{4.41}$$

$$\mu_2(dx) = \frac{\Psi(x^\beta)}{g(x^\beta)}g^p(x^\beta)\log^\alpha(2+g(x^\beta))\,dx. \tag{4.42}$$

Moreover

$$\widetilde{C} \le \frac{\beta^{1-p}}{(1-\beta)(p-1)+\sigma\beta}(C-\sigma)\bar{A}\left(\frac{D_{\bar{A}}-1}{(C-\sigma)d_{\bar{A}}}\right)\frac{\bar{A}(D_{\bar{A}})C_{\bar{A}}^4}{D_{\bar{A}}-1}, \tag{4.43}$$

where $C_{\bar{A}} = (\frac{2}{\log 2})^\alpha,\ d_{\bar{A}} = p,\ D_{\bar{A}} = p + \frac{\alpha}{\log 2}$.

Proof. We are to apply Theorem 4.3.1. We consider $\bar{A}(t) = t^p\log^\alpha(1+t)$. The assumption $(\bar{A})$ is satisfied as, according to Example 4.1.1, $\bar{A} \in \Delta'$ for $p > 1,\ \alpha > 0$. We notice, that (as in (4.40)) $F_{\bar{A}}(t) = t^p\log^{-\alpha}(2+1/t)$, when

$t > 0$ and $F_{\bar{A}}(0) = 0$. We note that $u = u_\beta(x) = x^\beta$, with $\beta \in (0,1)$, is the solution to PDI $-\Delta_A u \geq \Phi$, where

$$\Phi = -(\beta-1)\beta^{p-1}(p-1)x^{p\beta-\beta-p}\log^\alpha\left(2+\beta x^{\beta-1}\right). \tag{4.44}$$

Indeed, we have $\nabla u = \beta x^{\beta-1}$, $|\nabla u| = |\beta| x^{\beta-1}$ and we compute the function Φ

$$-\Delta_A u = -\mathrm{div}\left(\frac{\bar{A}(|\nabla u|)}{|\nabla u|^2}\nabla u\right) = -\beta|\beta|^{p-2}\left(x^{(p-1)(\beta-1)}\log^\alpha\left(2+|\beta|x^{\beta-1}\right)\right)' =$$

$$= -\beta|\beta|^{p-2}(\beta-1)x^{(p-1)(\beta-1)-1}\log^{\alpha-1}\left(2+|\beta|x^{\beta-1}\right)\cdot$$

$$\cdot\left((p-1)\log\left(2+|\beta|x^{\beta-1}\right)+\alpha\frac{|\beta|x^{\beta-1}}{2+|\beta|x^{\beta-1}}\right) \geq$$

$$\geq -\beta|\beta|^{p-2}(\beta-1)(p-1)x^{p\beta-p-\beta}\log^\alpha\left(2+|\beta|x^{\beta-1}\right) =$$

$$= |\beta|^p(1/\beta-1)(p-1)x^{p\beta-p-\beta}\log^\alpha\left(2+|\beta|x^{\beta-1}\right) = \Phi,$$

where the inequality holds for $\beta \in (0,1)$, thus we remove the absolute value of β and write (4.44).

Now let us verify assumption (u).
We note first that $\bar{A}(|\nabla u|) = \beta^p x^{p(\beta-1)}\log^\alpha\left(2+\beta x^{\beta-1}\right)$. Therefore

$$g(u)\Phi + \sigma\bar{A}(|\nabla u|) = \beta^p x^{p(\beta-1)}\log^\alpha\left(2+\beta x^{\beta-1}\right)\left[(1/\beta-1)(p-1)g(x^\beta)x^{-\beta}+\sigma\right]$$

is positive for $\sigma > -(1/\beta-1)(p-1)\inf_{x>0} g(x^\beta)x^{-\beta} = \sigma_0$.

We reach the goal by computing the weights according to Theorem 4.3.1 and dividing both sides by the constant.

We notice that, due to the above method, we can estimate the constant $\widetilde{C}$ as in (4.43). For this we have to note that, according to Facts 4.1.2 and 4.1.3, $C_{\bar{A}} = (\frac{2}{\log 2})^\alpha$, $d_{\bar{A}} = p$, $D_{\bar{A}} = p + \frac{\alpha}{\log 2}$. □

4.5.1 Inequalities on $(0, \infty)$

Applying $\Psi(t) = t^{-C}$, $g(t) = t$ in Lemma 4.5.2, we obtain the following result.

Theorem 4.5.1 (Power–logarithm Hardy–Sobolev inequality on $(0,\infty)$)**.** *Let $p>1,\ \alpha>0,\ \beta\in(0,1),\ C>0,\ C>\sigma>-(1/\beta-1)(p-1)$.*

Then there exists $c>0$, such that for every compactly supported Lipschitz function ξ, we have

$$\int_0^\infty |\xi|^p\log^{-\alpha}(2+1/|\xi|)\mu_1(dx)\le c\int_0^\infty |\xi'|^p\log^{\alpha}(2+|\xi'|)\mu_2(dx),$$

where

$$\begin{aligned}\mu_1(dx) &= x^{\gamma-p}\log^{\alpha}\left(2+\beta x^{\beta-1}\right)dx\sim x^{\gamma-p}\log^{\alpha}(2+x)\,dx,\\ \mu_2(dx) &= x^{\gamma}\log(2+x^{\beta})\,dx\sim x^{\gamma}\log(2+x)\,dx,\end{aligned}$$

with $\gamma=-\beta(C+1-p)$ and the constant c depends on $\bar{A},\ p,\ C,\ \beta,\ \sigma$.

Proof. We apply Lemma 4.5.2. It suffices now to check that the pair $\Psi(t)=t^{-C}$, $g(t)=t$ with $C>0$ satisfies the assumption (Ψ) **i)** and **ii)** and finally we compute the weights.

i) The mentioned Ψ,g are positive functions. Ψ is locally Lipschitz, Ψ/g is decreasing, moreover

$$\Psi'(t)g(t)=-Ct^{-C-1}g(t)=-Ct^{-C-1}t=-Ct^{-C-1+1}=-C\Psi(t).$$

ii) The function $\Theta=t^{p-1-C}\log^{\alpha}(2+t)$ (see (4.12)) is bounded in the neighbourhood of 0 when $p-1-C\ge 0$ and decreasing when $p-1-C<0$.

We note that

$$\begin{aligned}\sigma &> -(1/\beta-1)(p-1)\inf_{0<x}g(x^{\beta})x^{-\beta}=\\ &= -(1/\beta-1)(p-1)\inf_{0<x}x^{\beta}x^{-\beta}=-(1/\beta-1)(p-1)=\sigma_0.\end{aligned}$$

Thus there exists $\sigma\in[\sigma_0,C)$ for any $C>0$.

We apply Lemma 4.5.2 and obtain the following measures in inequality (4.41)

$$\begin{aligned}\mu_1(dx) &= (x^{\beta})^{-C-1}\beta^p x^{p(\beta-1)}\log^{\alpha}\left(2+\beta x^{\beta-1}\right)[(1/\beta-1)(p-1)+\sigma]\,dx=\\ &= x^{-\beta(C+1-p)-p}\log^{\alpha}\left(2+\beta x^{\beta-1}\right)\beta^p[(1/\beta-1)(p-1)+\sigma]\,dx,\\ \mu_2(dx) &= \widetilde{C}x^{-\beta(C+1-p)}\log(2+x^{\beta})\,dx.\end{aligned}$$

Now it suffices to take $\gamma=-\beta(C+1-p)$. □

Remark 4.5.1. We may estimate c due to (4.43).

4.5.2 Inequalities on $(0,1)$

We present application with $g(\lambda)$ different from identity. For this, it is convenient to consider the extension of previous results where we consider the restriction of Ψ to the codomain of u. We need Theorems 4.2.1 and 4.3.1, and Lemma 4.5.2, where instead of Assumption (Ψ) we suppose $(\Psi)_2$ (see below). Their proofs in this case are easy modifications of the proofs from previous sections.

$(\Psi)_2$ for a given nonnegative $u \in W^{1,\bar{A}}_{loc}(\Omega)$, there exists a function $\Psi : [0,\infty) \to [0,\infty)$, which is nonnegative and belongs to $C^1(u(\Omega \setminus \{0\})$, where $u(\Omega) = \{u(x) : x \in \Omega\}$. Furthermore, the following conditions are satisfied

i) inequality
$$g(t)\Psi'(t) \le -C\Psi(t),$$
holds for all $t \in u(\Omega)\setminus\{0\}$ with $C > 0$ independent of t and certain continuous function $g : (0,\infty) \to (0,\infty)$, such that $\Psi(t)/g(t)$ is nonincreasing for $t \in u(\Omega)$. Moreover, we set $\Psi(t) \equiv 0 \quad \text{for} \quad t \notin u(\Omega)$.

ii) function $\Theta(t)$ given by (4.12) is nonincreasing or bounded in the neighbourhood of 0.

When we restict ourselves to $(0,1)$ and apply $\Psi(t) = e^{\frac{1}{2}\log^2(t)}$, $g(t) = t/|\log t|$. They do not satisfy assumption (Ψ), but only $(\Psi)_2$. In particular assumption (Ψ) **i)** requires Ψ to be a decreasing function, but it does not hold outside $(0,1)$. This choice in Lemma 4.5.2 leads to the following result.

Theorem 4.5.2 (Hardy–Sobolev inequality on $(0,1)$)**.** *Let $p > 1$, $\alpha > 0$, $\beta \in (0,1)$ and $\bar{A}(t) = t^p \log^\alpha(2+t)$.*

Then there exists a constant $c > 0$, such that for every Lipschitz function ξ with compact support in $(0,1)$, we have
$$\int_0^1 |\xi|^p \log^{-\alpha}(2 + 1/|\xi|)\mu_1(dx) \le c\int_0^1 \bar{A}(|\xi'|)\mu_2(dx),$$
where
$$\mu_1(dx) \;=\; e^{\frac{\beta}{2}\log^2(x)}|\log x|\frac{x^{(p-1)\beta}}{x^p}\log^\alpha\left(2+\beta x^{\beta-1}\right)dx, \tag{4.45}$$
$$\mu_2(dx) \;=\; e^{\frac{\beta}{2}\log^2(x)}|\log x|\frac{x^{(p-1)\beta}}{|\log x|^p}\log^\alpha(2+\frac{x^\beta}{|\log x^\beta|})\,dx. \tag{4.46}$$

Proof. We apply Lemma 4.5.2, where $u = u_\beta(x) = x^\beta$ is considered, with Assumption $(\Psi)_2$ instead of (Ψ). It suffices now to check that the pair $\Psi(t) = e^{\frac{1}{2}\log^2(t)}$, $g(t) = \frac{t}{|\log t|}$, with $C = 1$ (for $t \in (0,1)$) satisfies the assumption $(\Psi)_2$ **i)** and **ii)**.

i) The functions Ψ, g are positive. Ψ is locally Lipschitz. Moreover

$$\begin{aligned}\Psi'(t)g(t) &= -\frac{t}{\log t}\cdot\frac{1}{2}(\log^2 t)' e^{\frac{1}{2}\log^2(t)} = -\frac{t}{\log t}\cdot\frac{1}{2}2\frac{\log t}{t}e^{\frac{1}{2}\log^2(t)} = \\ &= -e^{\frac{1}{2}\log^2(t)} = -\Psi(t).\end{aligned}$$

As $t \in (0,1)$, we have $\log t < 0$. Therefore

$$\begin{aligned}g'(t) &= \left(-\frac{t}{\log t}\right)' = -\frac{t'\log t - t\log' t}{\log^2 t} = -\frac{\log t - 1}{\log^2 t} = \\ &= \frac{1 + |\log t|}{\log^2 t} \;\geq\; 0 \;>\; -1.\end{aligned}$$

According to Remark 4.1.5 it is enough to ensure that Ψ/g is nonincreasing.

ii) The function $\Theta(s) = \frac{\bar{A}(g(s))\Psi(s)}{g(s)} = \left(\frac{s}{|\log s|}\right)^{p-1}\log^\alpha\left(2 + \frac{s}{|\log s|}\right)e^{\frac{1}{2}\log^2(s)}$ is decreasing in the neighbourhood of 0. Indeed, it is easy to show that for sufficiently small positive s we have $\Theta'(s) < 0$.

We note that there exists $\sigma \in [\sigma_0, C) = [0,1)$. Indeed, the only condition for σ is the following

$$\begin{aligned}\sigma > \sigma_0 &= -(1/\beta - 1)(p-1)\inf_{0<x<1} g(x^\beta)x^{-\beta} = -(1/\beta - 1)(p-1)\inf_{0<x<1} x^\beta|\log x^\beta|x^{-\beta} = \\ &= -(1/\beta - 1)(p-1)\inf_{0<x<1}|\log x^\beta| = 0.\end{aligned}$$

We apply Lemma 4.5.2 and obtain the following measures in inequality (4.41)

$$\begin{aligned}\mu_1(dx) &= e^{\frac{1}{2}\log^2(x^\beta)}|\log(x^\beta)|x^{p\beta-\beta-p}\log^\alpha\left(2 + \beta x^{\beta-1}\right)dx, \\ \mu_2(dx) &= e^{\frac{1}{2}\log^2(x^\beta)}|\log(x^\beta)|^{-p+1}x^{p\beta-\beta}\log^\alpha\left(2 + \frac{x^\beta}{|\log x^\beta|}\right)dx.\end{aligned}$$

We compute the final measures by removing unnecessary constants from logarithm terms. $\square$

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