University of Warsaw
Faculty of Mathematics, Informatics and Mechanics

Filip Z. Klawe

Mathematical analysis of thermo-visco-elastic models

PhD dissertation

Supervisors

prof. dr hab. Piotr Andrzej Gwiazda

Institute of Applied Mathematics and Mechanics
University of Warsaw

dr hab. Agnieszka Świerczewska-Gwiazda

Institute of Applied Mathematics and Mechanics
University of Warsaw

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aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

January 26 , 2015 . . . .

date *Filip Z. Klawe*

Supervisors' declaration:
the dissertation is ready to be reviewed

January 26 , 2015 . . . .

date *prof. dr hab. Piotr Andrzej Gwiazda*

. . . .

dr hab. Agnieszka Świerczewska-Gwiazda

Abstract

Our research is directed to quasi-static evolution of thermo-visco-elastic models. We assume that the material is subject to two kinds of mechanical deformations: elastic and inelastic. Moreover, our analysis captures the influence of the temperature on visco-elastic properties of the body. The subject of this dissertation is to study a thermodynamically consistent models which describe such type of phenomena related to Mróz model, Norton-Hoff-type model and model with growth conditions in Orlicz spaces.

The proofs base on two level Galerkin approximation. We present the construction of approximate solutions and discuss their existence. Moreover, the problem of low data regularity in parabolic equation appears for considered models. The paper presents two possible ways how to deal with it, i.e. the approach of Boccardo & Gallouët and the approach of renormalized solutions.

We provide proofs regarding existence of solutions to thermo-visco-elastic models in a simplified setting, namely the thermal expansion effects are neglected. Consequently, the coupling between the temperature and the displacement occurs only in the constitutive function for the evolution of the visco-elastic strain.

Keywords

visco-elasticity, thermal effects, Galerkin approximation, monotonicity method, renormalizations, generalized Orlicz space, Young measure

AMS Mathematics Subject Classification

74C10, 35Q74, 74F05

Streszczenie

Nasze badania koncentrują się na analizie modeli termo-lepko-sprężystych opisujących ewolucję quasi-statyczną. Rozważamy modele, które łączą odkształcenia odwracalne (sprężyste) i nieodwracalne (lepko-sprężyste). Dodatkowo, pojawienie się w modelu odkształceń nieodwracalnych związane jest z dysypacją energii mechanicznej i pojawieniem się efektów cieplnych, które również są przedmiotem analizy.

Przedmiotem badań prezentowanych w niniejszej pracy są modele termodynamicznie domknięte opisujące to zjawisko. Dowodzimy istnienia rozwiązań dla modelu Mroza, modelu typu Nortona-Hoffa i modelu z warunkami wzrostu w przestrzeniach Orlicza.

Dowody istnienia rozwiązań oparte są na dwustopniowej aproksymacji Galerkina. Prezentujemy konstrukcję rozwiązań przybliżonych oraz dowodzimy ich istnienia. Ponadto, w rozważanych modelach pojawia się problem niskiej regularności danych w równaniu przewodnictwa cieplnego. Rozważamy dwa sposoby rozwiązania tego problemu, tj. podejście Boccardo & Gallouëta oraz podejście oparte na teorii rozwiązań zrenormalizowanych.

Dodatkowo zakładamy, że rozważane materiały nie ulegają rozszerzalności cieplnej. W związku z tym, zależność przemieszczenia i temperatury jest spowodowana tylko przez funkcję konstytutywną opisującą ewolucję tensora lepko-sprężystego.

Słowa kluczowe

lepko-sprężystość, efekty termiczne, aproksymacja Galerkina, metody monotoniczności, renormalizacje, uogólnione przestrzenie Orlicza, miary Younga

Klasyfikacja tematyczna według AMS

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Contents

1 Introduction **1**
1.1 Derivation of the model . . . 4
1.1.1 Balance of momentum . . . 4
1.1.2 Balance of energy . . . 4
1.1.3 Cauchy stress tensor . . . 6
1.1.4 Evolutionary equation for visco-elastic deformation . . . 9
1.1.5 Full model . . . 10
1.2 Simplifications . . . 11
1.3 Thermodynamical consistency . . . 12
1.4 Main problems . . . 14

2 Construction of approximate solutions **17**
2.1 Definition of bases functions . . . 17
2.2 Definition of approximate solution . . . 22

3 Solution to heat equation **27**
3.1 Boccardo-Gallouët approach . . . 28
3.2 Renormalised approach . . . 32
3.2.1 Uniqueness . . . 45

4 Mróz model **49**
4.1 Young measure tools . . . 49
4.2 Formulation of problem . . . 60
4.3 Proof of Theorem 4.2.1 . . . 60
4.3.1 Transformation to a homogeneous boundary-value problem . . . 61
4.3.2 Potential Energy estimates . . . 62
4.3.3 Uniform boundedness of temperature . . . 64
4.3.4 Limit passage with l going to ∞ . . . 66
4.3.5 Passing to the limit with k going to ∞ . . . 71

5 Norton-Hoff-type models **75**
5.1 Formulation of the problem . . . 76
5.2 Proof of Theorem 5.1.1 . . . 77
5.2.1 Energy estimates . . . 77
5.2.2 Limit passage $l \to \infty$ and uniform estimates. . . . 79
5.2.3 Limit passage $k \to \infty$ . . . 82

6 Models with growth conditions in generalized Orlicz spaces **85**
6.1 Generalized Orlicz spaces 86
6.2 Formulation of the problem 91
6.3 Proof of Theorem 6.2.1 92
6.3.1 Transformation to a homogeneous boundary-value-problem 92
6.3.2 Boundedness of energy 93
6.3.3 Limit passage $l \to \infty$ and uniform estimates 96
6.3.4 Limit passage $k \to \infty$ 105

List of Notations **114**

Chapter 1

Introduction

The objective of this dissertation is to show the existence of solution to special class of models which describes deformation of solid material. We consider the material body which occupies the domain Ω and is treated by external forces and heat flux through the boundary. There are many different types of such phenomena. Our goal is to study the visco-elastic type of deformation. Moreover, we also take into account changes of temperature because what happens during this phenomena is loss of energy. The result of such situation is the naming convention used to describe this class of models, i.e. thermo-visco-elastic models.

Different properties of deformations cause their different naming convention. Let us start with elasticity. If the deformation is reversible and the mechanical energy is not dissipate, i.e. after termination of action of external forces the body returns to its initial state, we say that this is an elastic deformation. Moreover, if the strain is proportional to stress we say that this is a linear elasticity. To classify the inelastic deformation, we use a book of Duvaut and Lions, see [26].

If there exists the region where the deformation is elastic and after some threshold the deformation starts to be inelastic we add a suffix *plastic* to describe it (elasto-visco-plastic, rigid-perfectly plastic, plastic with work hardening, elastic-perfectly-plastic). When the external forces start acting on the material, firstly the elastic deformation appears, however it occurs only till specified threshold, after which deformation ceased to be elastic.

When no such threshold exist and there are two types of deformation from the beginning of action of external forces we say that this deformation is *visco-elastic*. We focus on visco-elastic types of materials in this dissertation.

It is obvious that the properties of the material depend on many different factors, e.g. the temperature. Hence, all models hold only in some specific regimes. The same material in different temperature may be characterized by different properties. Let us take the rubber which is elastic in the room temperature. The same rubber in the temperature of liquid nitrogen is brittle. In this dissertation, we focus on the process in such temperature regimes that they have no influence on material properties.

In the case of inelastic deformations, the relation between stress and strain may be time dependent, e.g. the reaction of material depends on the speed of load. Elastic material does not care how fast the load is applied. However, if the stress rate does take into account this relation we say that material is *viscous*.

For the visco-elastic materials we may observe many different phenomena, e.g. creep, stress relaxation or phase shift in stress response if sinusoidal load is applied. Here, we do not focus on this phenomena. We only want to stress that such phenomena happen. Thanks to that, the visco-elastic materials have many applications, e.g. as energy absorbers (damping the vibrations), noise reducer (in HH-53C rescue helicopter produced by Sikorsky), car bumpers or in computer devices to protect them from mechanical shock, see [47]. Many materials which behave elastically at room temperature attain the visco-elastic properties after heating.

There are many visco-elastic materials, e.g. synthetic polymers, wood, human tissue (ligaments, tendons or disc in human spine) or some metals in specific temperature. It is important to take into account the properties of material in engineering, designing or during physical experiments. In some situations the visco-elastic properties are desirable and in some they are not, e.g. material which is not visco-elastic must be used in the filaments in light bulbs. Many metals in so heigh temperature (greater than $3000°C$) creep, hence the filaments are made from tungsten which is not visco-elastic in this temperature.

Following [11, 18, 22, 26, 45, 46, 60, 70, 71, 72, 78, 79] we study the quasi-static evolution, i.e. the evolution, which is slow and we neglect the acceleration term in the equation for balance of momentum. As mentioned previously, the reactions of visco-elastic materials may be different for different loads speed. Our interest is to examine slow and long-time behaviour of materials.

Furthermore, time dependency between stress and strain is defined by evolutionary equation for visco-elastic strain (flow rule). The difference between symmetric gradient of displacement and visco-elastic strain characterizes the deformation which defines the potential energy of the system, i.e. elastic deformation.

Additionally, considerations regarding solid mechanics with thermal effect included should take into account the thermal expansion of the body, i.e. changes of the volume Ω with changes of temperature. We focus on this problem in [60]. We consider materials which do not change the volume with the changes of temperature.

The most common phenomenon is thermal expansion in a sense that materials expand when temperature increases and contract when temperature decreases. However, there is a group of materials that behave in a different way: materials with negative thermal expansion (denoted by NTE) and zero thermal expansion (denoted by ZTE). ZTE materials prevent or reduce resulting strain or internal stresses in systems subject to large temperature fluctuations. Their behaviours are different than our expectations but they have many technical applications, e.g. they are used in systems that are subject to thermal shock, in functional materials (thermomechanical actuators and space applications, see [66]), in precision engineered parts and microdevices, cf. [23, 51, 66, 67, 68, 88].

ZTE in a single, uncombined material is known only in a few cases, e.g.

- YbGaGe, has negligible volume change between 100 and 400 K, see [66];
- Mn_3AN, where A = Cu/Sn, Zn/Sn, cf. [76];
- $Fe[Co(CN)_6]$, cf. [54];
- $N(CH_3)_4CuZn(CN)_4$, see [62].

There are many different components that contain negative and positive thermal expansion materials such that zero thermal expansion material is obtained. However, the case of components is much more complicated than the case considered in our paper. In components, internal stresses appear, which are not subject of our work.

Negative thermal expansion may be observed in: silicon and germanium in very low temperature (less than $100K$), glasses in the titania–silica family, Kevlar, carbon fibres, anisotropic Invar Fe-Ni alloys and ZrW_2O_8 (see [55]) in room temperature.

Moreover, we consider the model with infinitesimal displacement. In a consequence, the dependence between the Cauchy stress tensor and $\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}^{\mathbf{p}}$ is linear (generalized Hooke's law, for more details see [60] or [63]). Additionally, we assume that thermal expansion of material is linear. Majority of different approaches involve models that are purely mechanical, namely concern the theory of inelastic and infinitesimal deformations with the nonlinear inelastic constitutive relation of monotone type, however they neglect all thermal influences, see [2] and also [18, 19, 20, 21]. On the other hand, the mathematical analysis of linear thermo-elasticity is also classical, well understood topic, cf. [44], contrary to an analysis of thermo-inelastic models. By thermo-inelastic models we mean systems consisting of balance of momentum for inelastic deformation and the equation for evolution of temperature. In the literature there are only some results for special models or for simplified models, see [11, 12, 22].

Furthermore, we assume that the visco-elastic part of deformation is an isochoric process. It means that visco-elastic deformation does not change the volume of Ω.

All of the results presented in this dissertation are motivated by the papers of Hömberg [43], Chełmiński and Racke [22]. In [43] author considered the general physical phenomena which, inter alia, consists of thermo-visco-elastic deformation related to Mróz model. However, the dependency between temperature and visco-elastic strain tensor is defined by a general operator. In [22] authors considered the Norton-Hoff model but in isothermal case (with omission of temperature changes). We have generalize their result to temperature dependent problem. Moreover, they assume that the inelastic deformation is not an isochoric process. Considerations of isochoric process contains additional mathematical difficulties such as the fact that test functions are not regular enough.

Moreover, Norton-Hoff-type models and models with growth conditions in Orlicz spaces are approximations of Prandtl-Reuss model, see [22, 79]. Prandtl-Reuss model describes elastic-perfectly-plastic deformation. It means that there exist a threshold such that before it the deformation is elastic and after it the deformation is perfectly-plastic. Perfectly-plastic deformation is such deformation, for which all mechanical energy of external forces is dissipated and after its termination the body does not change its shape and does not come back to any previous state. Perfect-plasticity is a type of irreversible deformation which occurs without any increase in stress or load.

In the literature many different models are considered. For general information we refer the reader to [2, 26, 60]. A lot of information about different models as well ad many various ways to prove the existence of solution may be found there. All simplifications used in this paper are a standard way to consider such problem. However, thermal dependency of visco-elastic constitutive function causes that many methods used before do not work in this case.

This paper presents the reader with new approaches regarding analysis of thermo-inelastic model. First of all, we use Gallerkin approximation to construct the approximate solution of system, see Chapter 2. This requires us to construct proper bases functions. Then, the proof of approximate solutions existence is not trivial. The second novelty is a consideration with regards to heat equation. Assumptions on the visco-elastic functions provide low regular right-hand side of heat equation. Using two independent approaches we prove the existence of two different solutions. The first one is a solution of Boccardo & Gallouet type, see [15]. The second one is a renormalised solution. Both approaches were introduced for heat equation with Dirichlet boundary conditions. We provide the existence of this solution for Neumann boundary conditions, see Chapter 3. Additionally, we apply these approaches to complicated system of equations and not only for a single equation. The last novelty is the proof of solution regarding three thermo-visco-elastic models: Mróz model, see Chapter 4, Norton-Hoff-type models, see Chapter 5 and models with growth conditions in Orlicz spaces, see Chapter 6. Models with growth conditions in Orlicz spaces allow us to consider the nonhomogeneous materials.

1.1 Derivation of the model

We start considerations regarding thermo-visco-elastic models with the derivation of model. Our goal is to show that considered system of equations really describes the physical phenomena. Furthermore, following calculations we obtain the relations between terms in different equations, e.g. thermal expansion of Ω and existence of nonlinear term in heat equation.

Thermo-visco-elastic system of equations, as a consequence of balance of momentum and balance of energy, cf. [30, 50], see also [33], captures displacement, temperature and visco-elastic strain. Since these two principles do not take into account the material properties of considered body, we may complement it by adding constitutive relations which complete missing information. A standard technique in the visco-elastic deformation is to work with two constitutive relations. First one describes the dependency between stress and strains, i.e. this is an equation for the Cauchy stress tensor. To obtain the equation for Cauchy stress tensor we start from physics. Using Helmholtz free energy we get the necessary relation. The entropy for such models is a consequence of statistical mechanics. The second one is a constitutive equation which is characterized by the evolution of visco-elastic strain tensor, named also the *flow rule*.

1.1.1 Balance of momentum

A linear momentum is a conserved quantity. Hence, changes of a linear momentum correspond to the action of external forces, i.e. volume, where $\boldsymbol{f}$ is a density of external volume force, and surface forces which are defined by normal part of Cauchy stress tenor $\boldsymbol{\sigma n}$ to this surface. Let us consider an open subset $\mathcal{O}$ of Ω. Then the balance of momentum has the following form

$$\frac{d}{dt}\int_{\mathcal{O}} \rho \boldsymbol{u}_t \,\mathrm{d}x = \int_{\mathcal{O}} \boldsymbol{f}\,\mathrm{d}x + \int_{\partial\mathcal{O}} \boldsymbol{\sigma n}\,\mathrm{d}s, \tag{1.1.1}$$

where ρ is the density of the body, $\boldsymbol{\sigma}$ stands for the Cauchy stress tensor and $\boldsymbol{n}$ is an unit outward normal vector to the boundary $\partial\mathcal{O}$ and $\boldsymbol{u}$ is a displacement. Using the Green theorem we obtain

$$\int_{\mathcal{O}} \rho \boldsymbol{u}_{tt}\,\mathrm{d}x - \int_{\mathcal{O}} \operatorname{div}\boldsymbol{\sigma}\,\mathrm{d}x = \int_{\mathcal{O}} \boldsymbol{f}\,\mathrm{d}x. \tag{1.1.2}$$

Equation (1.1.2) is tantamount to the weak formulation of the following equation

$$\rho \boldsymbol{u}_{tt} - \operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f}. \tag{1.1.3}$$

To be exact, we may observe that in the abovementioned equation we use two different systems of coordinates. Displacement is presented in Lagrangian coordinates and Cauchy stress tensor in Eulerian coordinates. This complication will disappear due to the small displacements hypothesis, because we may approximate stress tensor in Eulerian coordinates by stress tensor in Lagrangian coordinates, cf. e.g. [81, pages 203-205].

1.1.2 Balance of energy

The second conservation law used to derive the thermo-visco-elastic system is balance of energy. Let us start with the definition of energy density for this problem. Since we consider the visco-elastic deformation of Ω and heat flow, we take into account three different types of energy: thermal, potential and kinetic:

$$e = c\theta + \frac{1}{2}\boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) + \frac{1}{2}\rho|\boldsymbol{u}_t|^2, \tag{1.1.4}$$

where θ is temperature and constant c stands for the heat capacity of the body. Moreover, by $\boldsymbol{\varepsilon}(\boldsymbol{u})$ we denote a symmetric gradient of displacement $\boldsymbol{u}$: $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u})$ and by $\boldsymbol{\varepsilon}^{\mathbf{p}}$ visco-elastic strain tensor. The operator $\boldsymbol{D}$ is a linear, positively defined and bounded operator from $\mathcal{S}^3$ to $\mathcal{S}^3$, where $\mathcal{S}^3$ is a set of symmetric $3\times3-$matrices with real entries. Symmetry of $\boldsymbol{\varepsilon}^{\mathbf{p}}$ follows from the material objectivity and isotropy of the material. Thus implies symmetry of $\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}$. Tensor $\boldsymbol{D}$ describes the material behaviour, in further coming sections we discuss more precisely operator $\boldsymbol{D}$. Let us define tensor $\boldsymbol{T}$ as follows

$$\boldsymbol{T} := \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}). \tag{1.1.5}$$

Since deformation may be split into elastic and visco-elastic one, i.e.

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \underbrace{\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}}_{\text{elastic}} + \overbrace{\boldsymbol{\varepsilon}^{\mathbf{p}}}^{\text{visco-elastic}}, \tag{1.1.6}$$

tensor $\boldsymbol{T}$ stands for elastic stress. The density of global energy may be reformulated as follows

$$e = c\theta + \frac{1}{2}\boldsymbol{D}^{-1}\boldsymbol{T} : \boldsymbol{T} + \frac{1}{2}\rho|\boldsymbol{u}_t|^2. \tag{1.1.7}$$

The changes of global energy of the closed system are equal to the work done on the system and the heat supplied to the system, namely

$$\frac{d}{dt}\mathcal{E} = P_{external} + \frac{d}{dt}Q, \tag{1.1.8}$$

where $\mathcal{E}$ is the global energy of the system, $P_{external}$ denotes the rate of work of external forces and Q is the heat. Let us again consider an open subset $\mathcal{O}$ of the body Ω. Changes of the global energy in the set $\mathcal{O}$ are then prescribed as follows

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_{\mathcal{O}} &= \frac{d}{dt}\int_{\mathcal{O}} \big(c\theta + \frac{1}{2}\boldsymbol{D}^{-1}\boldsymbol{T} : \boldsymbol{T} + \frac{1}{2}\rho|\boldsymbol{u}_t|^2\big)\,\mathrm{d}x \\
&= \int_{\mathcal{O}} \big(c\theta_t + \boldsymbol{T} : \boldsymbol{\varepsilon}(\boldsymbol{u}_t) - \boldsymbol{T} : \boldsymbol{\varepsilon^p}_t + \frac{1}{2}\rho\frac{d}{dt}|\boldsymbol{u}_t|^2\big)\,\mathrm{d}x. \\
&= \int_{\mathcal{O}} \big(c\theta_t + \boldsymbol{T} : \nabla\boldsymbol{u}_t - \boldsymbol{T} : \boldsymbol{\varepsilon^p}_t + \frac{1}{2}\rho\frac{d}{dt}|\boldsymbol{u}_t|^2\big)\,\mathrm{d}x,
\end{aligned} \tag{1.1.9}$$

where the last equation is caused by symmetry of $\boldsymbol{T}$. The rate of work of external forces acting on the set $\mathcal{O}$ is equal to the rate of work of surface forces and volume forces

$$\begin{aligned}
P_{external} &= \int_{\partial\mathcal{O}} \boldsymbol{\sigma}\boldsymbol{n}\cdot\boldsymbol{u}_t\,\mathrm{d}s + \int_{\mathcal{O}} \boldsymbol{f}\cdot\boldsymbol{u}_t\,\mathrm{d}x \\
&= \int_{\mathcal{O}} \mathrm{div}\,(\boldsymbol{\sigma}\boldsymbol{u}_t)\,\mathrm{d}x + \int_{\mathcal{O}} \boldsymbol{f}\cdot\boldsymbol{u}_t\,\mathrm{d}x \\
&= \int_{\mathcal{O}} \boldsymbol{\sigma} : \nabla\boldsymbol{u}_t\,\mathrm{d}x + \int_{\mathcal{O}} \mathrm{div}\,\boldsymbol{\sigma}\cdot\boldsymbol{u}_t\,\mathrm{d}x + \int_{\mathcal{O}} \boldsymbol{f}\cdot\boldsymbol{u}_t\,\mathrm{d}x \\
&= \int_{\mathcal{O}} \boldsymbol{\sigma} : \nabla\boldsymbol{u}_t\,\mathrm{d}x + \int_{\mathcal{O}} (\mathrm{div}\,\boldsymbol{\sigma} + \boldsymbol{f})\cdot\boldsymbol{u}_t\,\mathrm{d}x.
\end{aligned} \tag{1.1.10}$$

On the basis of (1.1.2) we conclude further

$$\begin{aligned}
P_{external} &= \int_{\mathcal{O}} \boldsymbol{\sigma} : \nabla\boldsymbol{u}_t\,\mathrm{d}x + \int_{\mathcal{O}} \rho\boldsymbol{u}_{tt}\cdot\boldsymbol{u}_t\,\mathrm{d}x \\
&= \int_{\mathcal{O}} \boldsymbol{\sigma} : \nabla\boldsymbol{u}_t\,\mathrm{d}x + \int_{\mathcal{O}} \rho\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}_t|^2\,\mathrm{d}x.
\end{aligned} \tag{1.1.11}$$

The changes of the heat are equal to the heat produced by the heat sources in the body (in our case the density of the heat sources is denoted by r) and the heat flux through the boundary of $\mathcal{O}$

$$\frac{d}{dt}Q = \int_{\mathcal{O}} r \,\mathrm{d}x - \int_{\partial\mathcal{O}} \boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}s = \int_{\mathcal{O}} r\,\mathrm{d}x - \int_{\mathcal{O}} \operatorname{div}\boldsymbol{q}\,\mathrm{d}x. \tag{1.1.12}$$

According to the Fourier's law, the heat flux is proportional to the gradient of the temperature ($\boldsymbol{q} = -\kappa\nabla\theta$). Using this observation, we obtain

$$\frac{d}{dt}Q = \int_{\mathcal{O}} r\,\mathrm{d}x + \int_{\mathcal{O}} \kappa\Delta\theta\,\mathrm{d}x. \tag{1.1.13}$$

Collecting all components of energy, we get the complete form of energy balance

$$\int_{\mathcal{O}} \big(c\theta_t + \boldsymbol{T} : \nabla\boldsymbol{u}_t - \boldsymbol{T} : \boldsymbol{\varepsilon}^{\mathrm{p}}_t + \rho\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}_t|^2\big)\,\mathrm{d}x = \int_{\mathcal{O}} \big(\boldsymbol{\sigma} : \nabla\boldsymbol{u}_t + r + \kappa\Delta\theta + \rho\frac{1}{2}\frac{d}{dt}|\boldsymbol{u}_t|^2\big)\,\mathrm{d}x.$$

This equation holds for arbitrary subset $\mathcal{O}$ of Ω, hence it is equivalent to

$$c\theta_t - \kappa\Delta\theta + \big(\boldsymbol{T} - \boldsymbol{\sigma}\big) : \nabla\boldsymbol{u}_t = \boldsymbol{T} : \boldsymbol{\varepsilon}^{\mathrm{p}}_t + r. \tag{1.1.14}$$

1.1.3 Cauchy stress tensor

Cauchy stress tensor is an equation describing the relation between the stress and the strain. We start with physical properties of Cauchy stress tensor. Let us observe that Cauchy stress tensor, as a physical quantity, is a symmetric tensor and its symmetry follows from the principle of the conservation of angular momentum, cf. [50]. Indeed, let us assume that our body is in the rest. Then, by linear conservation of momentum, see (1.1.1), we get

$$\int_{\mathcal{O}} (\operatorname{div}\boldsymbol{\sigma} + \boldsymbol{f})\,\mathrm{d}x = 0, \tag{1.1.15}$$

for arbitrary subset $\mathcal{O}$ of Ω. Since (1.1.15) is a vector equation, we may consider each of its components independently. Then, for $i = 1,2,3$ it holds

$$\int_{\mathcal{O}} (\sum_{j=1}^{3} \sigma_{ij,j} + f_i)\,\mathrm{d}x = 0, \tag{1.1.16}$$

where $\boldsymbol{\sigma} = \{\sigma_{ij}\}_{i,j=1,2,3}$ and by $\sigma_{ij,j}$ we denote $\frac{\partial\sigma_{ij}}{\partial x_j}$. Moreover, if Ω is in the rest then the angular momentum of $\mathcal{O}$ is equal to zero. As in the case of linear momentum, the angular momentum is split into volume and surface angular momentum

$$\int_{\mathcal{O}} \boldsymbol{r}\times\boldsymbol{f}\,\mathrm{d}x + \int_{\partial\mathcal{O}} \boldsymbol{r}\times(\boldsymbol{\sigma}\boldsymbol{n})\,\mathrm{d}s = \boldsymbol{0}, \tag{1.1.17}$$

where $\boldsymbol{r} = (x_1, x_2, x_3)^T$ is a position vector. To rewrite (1.1.17), we use the Levi-Civita symbol ε_{ijk}[1] and we obtain

$$\sum_{i,j=1}^{3} \left(\int_{\mathcal{O}} \varepsilon_{ijk} x_j f_k \,\mathrm{d}x + \int_{\partial\mathcal{O}} \sum_{m=1}^{3} \varepsilon_{ijk} x_j \sigma_{km} n_m \,\mathrm{d}s \right) = 0, \tag{1.1.18}$$

[1] ε_{ijk} is called the sign of a permutation. $\varepsilon_{ijk} = 1$ if i,j,k is an even permutation of $1,2,3$, $\varepsilon_{ijk} = -1$ if it is an odd permutation. Otherwise $\varepsilon_{ijk} = 0$.

which holds for $k = 1, 2, 3$. Using Green theorem we get

$$\begin{aligned} 0 &= \sum_{i,j=1}^{3} \left(\int_{\mathcal{O}} \varepsilon_{ijk} x_j f_k \,\mathrm{d}x + \int_{\mathcal{O}} \sum_{m=1}^{3} \frac{\partial(\varepsilon_{ijk} x_j \sigma_{km})}{\partial x_m} \,\mathrm{d}x \right) \\ &= \sum_{i,j=1}^{3} \left(\int_{\mathcal{O}} \varepsilon_{ijk} x_j f_k \,\mathrm{d}x + \int_{\mathcal{O}} \varepsilon_{ijk} x_j \sum_{m=1}^{3} \sigma_{km,m} \,\mathrm{d}x + \int_{\mathcal{O}} \sum_{m=1}^{3} \varepsilon_{ijk} \sigma_{km} \frac{\partial x_j}{\partial x_m} \,\mathrm{d}x \right) \\ &= \sum_{i,j=1}^{3} \left(\int_{\mathcal{O}} \varepsilon_{ijk} x_j \left(f_k + \sum_{m=1}^{3} \sigma_{km,m} \right) \mathrm{d}x + \int_{\mathcal{O}} \varepsilon_{ijk} \sigma_{kj} \,\mathrm{d}x \right), \end{aligned} \tag{1.1.19}$$

where the last inequality is a consequence of $\frac{\partial x_j}{\partial x_m} = \delta_{jm}$. By (1.1.16) first term of right-hand side is equal to zero. Then, for $k = 1, 2, 3$, the following inequality holds

$$\sum_{i,j=1}^{3} \left(\int_{\mathcal{O}} \varepsilon_{ijk} \sigma_{kj} \,\mathrm{d}x \right) = 0. \tag{1.1.20}$$

Arbitrary choice of $\mathcal{O}$ and definition of Levi-Civita symbol imply

$$\sigma_{jk} = \sigma_{kj}. \tag{1.1.21}$$

To obtain the equation for Cauchy stress tensor we follow [2] or [41, 47]. We know that there exists a Helmholtz free energy, denoted by $\Psi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta)$, and the following relation holds

$$\boldsymbol{\sigma} = \frac{\partial \Psi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta)}{\partial \boldsymbol{\varepsilon}(\boldsymbol{u})}. \tag{1.1.22}$$

Since we consider the linear case, we may use the Taylor series to represent Cauchy stress tensor

$$\begin{aligned} \boldsymbol{\sigma} = \frac{\partial \Psi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta)}{\partial \boldsymbol{\varepsilon}(\boldsymbol{u})} &\approx \frac{\partial \Psi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta)}{\partial \boldsymbol{\varepsilon}(\boldsymbol{u})}\Big|_{\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}^{\mathbf{p}}, \theta = \theta_R} \\ &+ \frac{\partial^2 \Psi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta)}{\partial \boldsymbol{\varepsilon}(\boldsymbol{u})^2}\Big|_{\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}^{\mathbf{p}}, \theta = \theta_R} (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) + \frac{\partial^2 \Psi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta)}{\partial \boldsymbol{\varepsilon}(\boldsymbol{u}) \partial \theta}\Big|_{\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}^{\mathbf{p}}, \theta = \theta_R} (\theta - \theta_R). \end{aligned} \tag{1.1.23}$$

In the reference temperature and without any external forces Cauchy stress tensor is equal to 0, hence $\frac{\partial \Psi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta)}{\partial \boldsymbol{\varepsilon}(\boldsymbol{u})}\Big|_{\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}^{\mathbf{p}}, \theta = \theta_R} = 0$. Then, we may define

$$\boldsymbol{D} := \frac{\partial^2 \Psi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta)}{\partial \boldsymbol{\varepsilon}(\boldsymbol{u})^2}\Big|_{\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}^{\mathbf{p}}, \theta = \theta_R} \quad \text{and} \quad \boldsymbol{\alpha} := \frac{\partial^2 \Psi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta)}{\partial \boldsymbol{\varepsilon}(\boldsymbol{u}) \partial \theta}\Big|_{\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}^{\mathbf{p}}, \theta = \theta_R}. \tag{1.1.24}$$

And this implies the following assumption.

Assumption 1.1.1. *Cauchy stress tensor*
The Cauchy stress tensor is in the form (Hooke's law) of

$$\boldsymbol{\sigma} = \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) - \alpha(\theta - \theta_R)\boldsymbol{I}, \tag{1.1.25}$$

where θ_R is the reference temperature, α is a positive constant, $\boldsymbol{I}$ is an identity matrix from $\mathcal{S}^3$.

Linear dependence holds for infinitesimal displacements, it should be understood as a relation after neglecting the higher order terms in Taylor expansion. Similar approach was used in [2, 3, 11, 21, 22, 45, 46, 60, 63, 71, 72, 79]. Additionally, we assume that changes of temperature are also infinitesimal.

We assume that the body in the reference temperature and without action of external forces is in the rest, i.e. the Helmholtz free energy is equal to 0, entropy is equal to 0, stress $\boldsymbol{\sigma}$ and strains $\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}^{\mathbf{p}}$ are equal to zero. Thus,

$$\begin{aligned} \Psi(\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}^{\mathbf{p}}, \theta_R) = 0. \\ s(\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}^{\mathbf{p}}, \theta_R) = 0, \end{aligned} \tag{1.1.26}$$

where s is an entropy of the system. Using the thermodynamics relationship we obtain

$$\Psi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta) = e - \theta s + g(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta_R), \tag{1.1.27}$$

where $g(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta_R)$ is a normalization function. Thus,

$$\frac{\partial \Psi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta)}{\partial \boldsymbol{\varepsilon}(\boldsymbol{u})} = \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) - \theta \frac{\partial s(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta)}{\partial \boldsymbol{\varepsilon}(\boldsymbol{u})} + \frac{\partial g(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta_R)}{\partial \boldsymbol{\varepsilon}(\boldsymbol{u})} \tag{1.1.28}$$

This and (1.1.25) implies that

$$\frac{\partial s(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta)}{\partial \boldsymbol{\varepsilon}(\boldsymbol{u})} = \boldsymbol{\alpha} \qquad \text{and} \qquad \frac{\partial g(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta_R)}{\partial \boldsymbol{\varepsilon}(\boldsymbol{u})} = \theta_R \boldsymbol{\alpha} \tag{1.1.29}$$

On the other hand, using the basic thermodynamic we get

$$\mathrm{ds} = \left(\frac{\partial \mathrm{s}}{\partial \theta}\right)_{\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathrm{const}} \mathrm{d}\theta + \left(\frac{\partial \mathrm{s}}{\partial \boldsymbol{\varepsilon}(\boldsymbol{u})}\right)_{\theta = \mathrm{const}} \mathrm{d}\boldsymbol{\varepsilon}(\boldsymbol{u}) \tag{1.1.30}$$

and $\left(\frac{\partial s}{\partial \theta}\right)_{\boldsymbol{\varepsilon}(\boldsymbol{u}) = const} = \frac{c}{\theta}$, where c is a heat capacity. Then

$$s = \boldsymbol{\alpha}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) + c \ln \frac{\theta}{\theta_R}. \tag{1.1.31}$$

And therefore,

$$\Psi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta) = e - \theta \left(\boldsymbol{\alpha}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) + c \ln \frac{\theta}{\theta_R} \right) + g(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta_R), \tag{1.1.32}$$

Let us consider the reference state

$$\begin{aligned} \Psi(\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}^{\mathbf{p}}, \theta = \theta_R) = c\theta_R + g(\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}^{\mathbf{p}}, \theta_R) = 0, \\ s(\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}^{\mathbf{p}}, \theta = \theta_R) = \boldsymbol{\alpha}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) + c \ln \frac{\theta_R}{\theta_R} = 0. \end{aligned} \tag{1.1.33}$$

Thus, it holds that $g(\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}^{\mathbf{p}}, \theta_R) = -c\theta_R$. Finally, we obtain that

$$\begin{aligned} \Psi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta) &= c\theta + \frac{1}{2}\boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) + \frac{1}{2}|\boldsymbol{u}_t|^2 - \theta s - c\theta_R - \theta_R \boldsymbol{\alpha} : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}), \\ s(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta) &= \boldsymbol{\alpha} : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) + c \ln \frac{\theta}{\theta_R}. \end{aligned} \tag{1.1.34}$$

Our interest is devoted to three phenomena: mechanical effects, which can be divided into elastic and visco-elastic deformation and thermal effects. Hence, to describe the problem appropriately we intend to include the dependence on $\boldsymbol{\varepsilon}(\boldsymbol{u})$, $\boldsymbol{\varepsilon}^{\mathbf{p}}$ and θ in the Cauchy stress tensor. Tensor $\boldsymbol{T} = \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}})$ describes the elastic force of the deformation. The second part of Cauchy stress tensor, $\alpha(\theta - \theta_R)\boldsymbol{I}$, is associated with thermal expansion of the body. We may replace $\alpha(\theta - \theta_R)$ by tensor-valued function $\boldsymbol{\alpha}(\theta - \theta_R)$ with some *good* properties, e.g. symmetry. This does not lie in our interest here, however it is worth mentioning. For engineering materials the coefficient α is of the order $10^{-5}K^{-1}$.

Operator $\boldsymbol{D}$ is a four-index matrix, i.e. $\boldsymbol{D} = \{d_{i,j,k,l}\}_{i,j,k,l=1}^{3}$. For general materials, functions $d_{i,j,k,l}$ may depend on the spatial variable x. Additionally, since strains and stress are symmetric, the following equalities hold

$$d_{i,j,k,l} = d_{j,i,k,l}, \quad d_{i,j,k,l} = d_{i,j,l,k} \quad \text{and} \quad d_{i,j,k,l} = d_{k,l,i,j} \quad \forall i,j,k,l = 1,2,3. \tag{1.1.35}$$

If we denote the strain by $\boldsymbol{\varepsilon} = \{\varepsilon_{ij}\}$, then the example of dependency between stress and strain may be defined by

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}, \tag{1.1.36}$$

where λ and μ are Lamé's coordinates, cf. [26, 60, 81]. The specific assumptions on the coefficient of operator $\boldsymbol{D}$ are presented for each model independently.

1.1.4 Evolutionary equation for visco-elastic deformation

In order to complete the system we need to define the evolution equation for the visco-elastic strain tensor. We consider an isochoric visco-elastic flow. We discuss a specific type of constitutive functions in this dissertation, i.e.

$$\boldsymbol{\varepsilon}^{\mathbf{p}}_t = \boldsymbol{G}(\theta, \boldsymbol{T}^d), \tag{1.1.37}$$

for models considered in Chapter 4 and in Chapter 5 or

$$\boldsymbol{\varepsilon}^{\mathbf{p}}_t = \boldsymbol{G}(x, \theta, \boldsymbol{T}^d), \tag{1.1.38}$$

for model in Chapter 6. Moreover, differences of considered models are caused by accurate assumptions on function $\boldsymbol{G}$, which are made on the beginning of Chapters 4–6

Assumption 1.1.2. *Function $\boldsymbol{G} : \mathbb{R}_+ \times \mathcal{S}^3_d \to \mathcal{S}^3_d$ is a function of two variables: temperature θ and deviatoric part of Cauchy stress tensor $\boldsymbol{\sigma}^d$. By $\mathcal{S}^3_d$ we denote a subset of traceless symmetric matrices, $\mathcal{S}^3_d \subset \mathcal{S}^3$. Moreover, let us observe that*

$$\begin{aligned}
\boldsymbol{\sigma}^d &= \boldsymbol{\sigma} - \frac{1}{3}tr(\boldsymbol{\sigma})\boldsymbol{I} \\
&= \boldsymbol{T} - \alpha(\theta - \theta_R)\boldsymbol{I} - \frac{1}{3}tr(\boldsymbol{T})\boldsymbol{I} + \alpha(\theta - \theta_R)\boldsymbol{I} \\
&= \boldsymbol{T} - \frac{1}{3}tr(\boldsymbol{T})\boldsymbol{I} = \boldsymbol{T}^d.
\end{aligned} \tag{1.1.39}$$

Since it is isochoric visco-elastic flow, we assume that it depends only on the deviatoric part of the Cauchy stress tensor and its range is the set of traceless matrices. The last assumption, together with the fact that $\boldsymbol{\varepsilon}^{\mathbf{p}}_0(x)$ is traceless, provides that also $\boldsymbol{\varepsilon}^{\mathbf{p}}$ is traceless.

Vanishing of the deformation tensor's trace corresponds to preservation of the material's volume. Indeed, the volume change is associated only with the elastic response of the material, and the plastic response is essentially incompressible, cf. [31]. The dependence of $\boldsymbol{G}(\theta, \cdot)$ only on

$\boldsymbol{T}^d$ is essential to maintain the coercivity of the model. Once we know that the range of $\boldsymbol{G}(\cdot,\cdot)$ is $\mathcal{S}^3_d$, then even for the isothermal process, namely the case of $\boldsymbol{G}(\theta,\boldsymbol{T}) = \boldsymbol{G}(\boldsymbol{T})$ we observe that $\boldsymbol{G}(\boldsymbol{T}) : \boldsymbol{T} = \boldsymbol{G}(\boldsymbol{T}) : \boldsymbol{T}^d$. Then e.g. taking the identity matrix as $\boldsymbol{T}$ we immediately see that $\boldsymbol{G}(\boldsymbol{I}) : \boldsymbol{I}^d = 0$.

Different models describing the solid body deformation differ in assumptions on the constitutive function $\boldsymbol{G}$. Choice of the function $\boldsymbol{G}$ leads to specific model. There is a broad range of different models considered in the literature, e.g.

- Mróz model [17, 43]:
$$\boldsymbol{G}(\theta,\boldsymbol{T}^d) = g(\theta)\boldsymbol{T}^d, \tag{1.1.40}$$
where $g:\mathbb{R}_+\to\mathbb{R}_+$ is a continuous function.

- Norton-Hoff, model without temperature [3],
$$\boldsymbol{G}(\boldsymbol{T}^d) = |\boldsymbol{T}^d|^{p-2}\frac{\boldsymbol{T}^d}{|\boldsymbol{T}^d|} \tag{1.1.41}$$

- Bodner-Partom model [11, 18, 20]:
$$\begin{aligned} \boldsymbol{G}(\theta,\boldsymbol{T}^d) &= \mathcal{G}\left(\frac{\{|\boldsymbol{T}^d| + \beta(\theta)\}^+}{y}\right)\frac{\boldsymbol{T}^d}{|\boldsymbol{T}^d|},\\ y_t &= \gamma(y)\mathcal{G}\left(\frac{|\boldsymbol{T}^d|}{y}\right)|\boldsymbol{T}^d| - A\delta(y), \end{aligned} \tag{1.1.42}$$
where $y:\Omega\times\mathbb{R}_+\to\mathbb{R}_+$ describes the isotropic hardening of the metal, $\{\cdot\}^+$ stands for the positive part of $\{\cdot\}$, $\gamma:\mathbb{R}_+\supset D(\gamma)\to\mathbb{R}_+$ and $\delta:\mathbb{R}_+\supset D(\delta)\to\mathbb{R}_+$ are given functions and A is a positive constant. Moreover, functions $\mathcal{G}(\cdot)$, $\gamma(\cdot)$, $\delta(\cdot)$ and $\beta(\cdot)$ fulfill some specific properties.

- Prandtl-Reuss model with linear kinematic hardening [22]
$$\boldsymbol{\varepsilon}^{\mathbf{p}}_t \in \partial I_{K(\theta)}(\boldsymbol{T} - \alpha\boldsymbol{\varepsilon}^{\mathbf{p}}), \tag{1.1.43}$$
where $I_{K(\theta)}$ is the indicator function of the closed and convex subset $K(\theta) = \{\boldsymbol{T}\in\mathcal{S}^3 : |\boldsymbol{T}^d| \le k-\theta\}$ and $\alpha, k > 0$ are material parameters. Furthermore, $\partial I_{K(\theta)}$ is a subdifferential of the function $I_{K(\theta)}$.

For further examples of constitutive relations (e.g. classical Maxwell model, models proposed by Chaboche, Hart, Miler, Bruhns and many others) we refer the reader to [2, Chapter 2.2]. In this dissertation we focus on three types of models: Mróz model (Chapter 4), Norton-Hoff-type model (Chapter 5) and models with growth conditions in Orlicz space (Chapter 6).

1.1.5 Full model

Summarizing previous sections we obtain the following system of equations

$$\varrho\boldsymbol{u}_{tt} - \operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f} \qquad \text{in } \Omega\times(0,T), \tag{1.1.44}$$
$$\boldsymbol{\sigma} = \boldsymbol{T} - \alpha(\theta-\theta_R)\boldsymbol{I} \qquad \text{in } \Omega\times(0,T), \tag{1.1.45}$$
$$\boldsymbol{T} = \boldsymbol{D}(\varepsilon(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) \qquad \text{in } \Omega\times(0,T), \tag{1.1.46}$$
$$\boldsymbol{\varepsilon}^{\mathbf{p}}_t = \boldsymbol{G}(\theta,\boldsymbol{T}^d) \qquad \text{in } \Omega\times(0,T), \tag{1.1.47}$$
$$c\theta_t - \kappa\Delta\theta + \alpha(\theta-\theta_R)\operatorname{div}\boldsymbol{u}_{\mathrm{t}} = \boldsymbol{T}^d : \boldsymbol{G}(\theta,\boldsymbol{T}^d) + r \qquad \text{in } \Omega\times(0,T), \tag{1.1.48}$$

which holds in bounded domain Ω. This system of equations should be supplemented by initial $(\boldsymbol{u}_0, \boldsymbol{u}_{0,t}, \boldsymbol{\varepsilon}_0^{\mathbf{p}}, \theta_0)$ and boundary conditions

$$\begin{cases} \boldsymbol{u} = \boldsymbol{g}_{\boldsymbol{u}} \\ \dfrac{\partial \theta}{\partial \boldsymbol{n}} = \boldsymbol{g}_\theta \end{cases} \tag{1.1.49}$$

on $\partial\Omega \times (0, T)$.

The function $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathcal{S}^3$ is the Cauchy stress tensor. The Cauchy stress tensor can be divided into two parts: mechanical and thermal. The mechanical part is $\boldsymbol{T} = \boldsymbol{D}(\varepsilon(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}})$, where the operator $\boldsymbol{D} : \mathcal{S}^3 \to \mathcal{S}^3$ is linear, positively definite and bounded. The operator $\boldsymbol{D}$ is a four-index matrix, i.e. $\boldsymbol{D} = \{d_{i,j,k,l}\}_{i,j,k,l=1}^3$ and the equalities (1.1.35) hold.

The evolution of the visco-elastic strain tensor $\boldsymbol{\varepsilon}^{\mathbf{p}}$ is governed by the constitutive relation $\boldsymbol{G} : \mathbb{R}_+ \times \mathcal{S}_d^3 \to \mathcal{S}_d^3$. The visco-elastic strain tensor $\boldsymbol{\varepsilon}^{\mathbf{p}} = (\boldsymbol{\varepsilon}^{\mathbf{p}})^d$ is traceless if $\boldsymbol{\varepsilon}_0^{\mathbf{p}}$ is traceless. The temperature θ_R is the reference temperature. The function $r : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ describes a given density of heat sources, κ :is a constant material's conductivity, ρ is a constant density of the body and c is a heat capacity. Moreover, α is constant and it describes the thermal expansion of the body. If $\alpha > 0$, then the material expands with the increasing temperature.

1.2 Simplifications

Consideration of full thermo-visco-elastic model (1.1.44)–(1.1.48) with general flow rule is still an open problem. Issues lying on the boundary of mathematics and other science requires special attention and knowledge of both scientific fields, but they will certainly find real-life applications. Better knowledge of processes occurring in the materials and better knowledge of their mathematical properties may help us in modeling and then also to improve the engineering work.

Since the problem of inelastic deformation is a very complex one, we make a list of simplifications which help us in the calculations. Let us explain the character of these simplifications. They are not only mathematical facilitations, but they may be justified from physical point of view.

Following Bartczak [11], Chełmiński [18], Chełmiński and Racke [22], Duvaut and J.L. Lions [26], Johnson [45, 46], Nečas and Hlaváček [60], Suquet [70, 71, 72], Temam [78, 79] we consider the quasi-static case. It means that acting forces cause small and long term displacement.

Assumption 1.2.1. *Consideration of quasi-static problem means that the acceleration term in momentum equation may be neglected, i.e.*

$$\varrho \boldsymbol{u}_{tt} = 0. \tag{1.2.1}$$

The fact of neglecting the acceleration term implies that the system of equations does not have to be supplemented by initial condition to displacement $\boldsymbol{u}_0$ and velocity $\boldsymbol{u}_{0,t}$.

Furthermore, we assume that considered materials do not change the volume with changes of temperature, i.e. the material is characterized by zero thermal expansions (ZTE). There are many different ways to deal with thermal expansion of body. Taking into account the thermal expansion we get also the nonlinear term in heat equation $\alpha(\theta - \theta_R)\mathrm{div}\, \boldsymbol{u}_t$. This term is the biggest troublemaker in inelastic systems. Some authors, cf. Bartczak [11], Chełmiński and Racke [22], try to linearize it in heat equation based on argumentation that the process is close to some temperature, in particular different form the reference temperature. Then the approximation $\alpha(\theta - \theta_R)\mathrm{div}\, \boldsymbol{u}_t \approx \alpha_0 \mathrm{div}\, \boldsymbol{u}_t$ is made. Such approach causes a loss of physical properties of the model, see next section or [32].

Assumption 1.2.2. *We assume that $\alpha = 0$, i.e. the considered material is not subject to thermal expansion.*

Finally, we simplify the calculations by eliminating all constants. Similar results hold even for more general material parameter. Our goal is to focus on the behavior of system for different visco-elastic constitutive functions $\boldsymbol{G}$.

Assumption 1.2.3. *We assume there are no heat sources in the system, hence $r = 0$, material's conductivity κ and capacity c are constant, to simplify $\kappa = 1$ and $c = 1$.*

Taking into account the above assumptions we get the following system of equations:

$$\begin{cases} -\operatorname{div} \boldsymbol{T} &= \boldsymbol{f}, \\ \boldsymbol{T} &= \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}), \\ \boldsymbol{\varepsilon}_t^p &= \boldsymbol{G}(\theta, \boldsymbol{T}^d), \\ \theta_t - \Delta\theta &= \boldsymbol{T}^d : G(\theta, \boldsymbol{T}^d), \end{cases} \tag{1.2.2}$$

with initial and boundary conditions

$$\begin{cases} \theta(x,0) = \theta_0(x) & \text{on } \Omega, \\ \boldsymbol{\varepsilon}^{\mathbf{p}}(x,0) = \boldsymbol{\varepsilon}_0^{\mathbf{p}}(x) & \text{on } \Omega, \\ \boldsymbol{u} = \boldsymbol{g}_{\boldsymbol{u}} & \text{on } \partial\Omega \times (0,T), \\ \frac{\partial\theta}{\partial\boldsymbol{n}} = \boldsymbol{g}_\theta & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{1.2.3}$$

As we may observe there is still an interaction between temperature and displacement. Hence, we cannot split this system into two independent systems of equations, but omission of thermal expansion causes that heat equation is linear.

1.3 Thermodynamical consistency

Simplifications made in previous section may cause a loss of physical properties of model. The purpose of the current section is to show that these simplifications lead to the model which still conserves the energy, the temperature is positive and there exists a function of state, namely the entropy, which has a positive rate of production. We shall say that the system is thermodynamically consistent. In [33] we prove thermodynamical consistency of full thermo-visco-elasticity model, i.e. (1.1.44)–(1.1.48), whereas in [32] we present this result for simplified problem (1.2.2). Through this dissertation we consider only the simplified system of equation, hence we confine here to show the thermodynamic consistency of simplified problem. To derive the whole model we use the physical conservation laws, hence it is not interesting to repeat the same argumentation backward.

Let us consider the isolated system, i.e. the system without external force ($\boldsymbol{f} = 0$), with homogeneous boundary values ($\boldsymbol{g}_{\boldsymbol{u}} = 0$ and $\boldsymbol{g}_\theta = 0$) and without the heat sources ($r = 0$). All of the calculations in this section are formal.

Conservation of total energy

To show that the global energy is preserved we multiply $(1.2.2)_{(1)}$ by $\boldsymbol{u}_t$. After integration over an arbitrary set $\mathcal{O} \subset \Omega$, we obtain

$$-\int_{\mathcal{O}} \operatorname{div} \boldsymbol{T} \cdot \boldsymbol{u}_t \, \mathrm{d}x = 0 \tag{1.3.1}$$

and then

$$\int_{\mathcal{O}} \boldsymbol{T} : \nabla \boldsymbol{u}_t \, \mathrm{d}x - \int_{\partial\mathcal{O}} \boldsymbol{T}\boldsymbol{n} \cdot \boldsymbol{u}_t \, \mathrm{d}s = 0. \tag{1.3.2}$$

Furthermore, let us multiply $(1.2.2)_{(3)}$ by $\boldsymbol{T}$ and integrate over $\mathcal{O}$. Subtracting this equation from (1.3.2) we get

$$\int_{\mathcal{O}} \big(\boldsymbol{T} : \nabla \boldsymbol{u}_t - \boldsymbol{T} : \boldsymbol{\varepsilon}^{\mathbf{p}}_t\big)\,\mathrm{d}x - \int_{\partial\mathcal{O}} \boldsymbol{T}\boldsymbol{n}\cdot\boldsymbol{u}_t\,\mathrm{d}s = -\int_{\mathcal{O}} \boldsymbol{T}^d : \boldsymbol{G}(\theta,\boldsymbol{T}^d)\,\mathrm{d}x. \tag{1.3.3}$$

By symmetry of $\boldsymbol{T}$ we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{O}} \boldsymbol{T} : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}})\,\mathrm{d}x - \int_{\partial\mathcal{O}} \boldsymbol{T}\boldsymbol{n}\cdot\boldsymbol{u}_t\,\mathrm{d}s = -\int_{\mathcal{O}} \boldsymbol{T}^d : \boldsymbol{G}(\theta,\boldsymbol{T}^d)\,\mathrm{d}x. \tag{1.3.4}$$

Considering the quasi-static evolution we omit the acceleration term in $(1.2.2)_{(1)}$. Thus we also omit the kinetic energy in the definition of energy density, i.e.

$$e = \theta + \frac{1}{2}\boldsymbol{D}^{-1}\boldsymbol{T} : \boldsymbol{T}. \tag{1.3.5}$$

Thus we obtain

$$\mathcal{E}_{\mathcal{O}}(t) = \int_{\mathcal{O}} \theta\,\mathrm{d}x + \frac{1}{2}\int_{\mathcal{O}} \boldsymbol{T} : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}})\,\mathrm{d}x. \tag{1.3.6}$$

Consequently, the equation (1.3.4) may be written in the following from

$$\frac{d}{dt}\mathcal{E}_{\mathcal{O}}(t) = \frac{d}{dt}\int_{\mathcal{O}} \theta\,\mathrm{d}x - \int_{\mathcal{O}} \boldsymbol{T}^d : \boldsymbol{G}(\theta,\boldsymbol{T}^d)\,\mathrm{d}x + \int_{\partial\mathcal{O}} \boldsymbol{T}\boldsymbol{n}\cdot\boldsymbol{u}_t\,\mathrm{d}s. \tag{1.3.7}$$

Using $(1.2.2)_4$, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{\mathcal{O}}(t) &= \int_{\mathcal{O}} \theta_t\,\mathrm{d}x - \int_{\mathcal{O}} \theta_t\,\mathrm{d}x + \int_{\mathcal{O}} \Delta\theta\,\mathrm{d}x + \int_{\partial\mathcal{O}} \boldsymbol{T}\boldsymbol{n}\cdot\boldsymbol{u}_t\,\mathrm{d}s \\ &= \int_{\partial\mathcal{O}} \big(\boldsymbol{T}\boldsymbol{u}_t + \nabla\theta\big)\cdot\boldsymbol{n}\,\mathrm{d}s.\end{aligned} \tag{1.3.8}$$

Zero external forces, homogeneous boundary conditions and lack of heat sources imply that $\boldsymbol{u}_t = 0$ and $\nabla\theta\cdot\boldsymbol{n} = 0$ on the boundary $\partial\Omega$. Therefore, the global energy $\mathcal{E}_\Omega$ is constant in time.

Positivity of the temperature

To prove the positivity of temperature we should assume that initial temperature θ_0 is positive. The heat equation after simplifications has a form of

$$\theta_t - \Delta\theta = \boldsymbol{G}(\theta,\boldsymbol{T}^d) : \boldsymbol{T}^d. \tag{1.3.9}$$

For each considered model $\boldsymbol{G}(\theta,\boldsymbol{T}^d) : \boldsymbol{T}^d$ is positive, see Assumption 4.0.1, Assumption 5.0.1 or Assumption 6.0.1. Hence, we consider

$$\theta_t - \Delta\theta \geq 0 \tag{1.3.10}$$

with positive initial condition and homogeneous boundary condition is positive. Positivity of θ is obvious. Moreover, if initial temperature is greater than reference temperature then we obtain that θ is greater then θ_R.

Entropy inequality

Since the temperature is positive we multiply (1.3.9) by $1/\theta$. After integration over an arbitrary set $\mathcal{O}\subset\Omega$, we obtain

$$\frac{d}{dt}\int_{\mathcal{O}} \ln\theta\,\mathrm{d}x - \int_{\mathcal{O}} \mathrm{div}\,\frac{\nabla\theta}{\theta}\,\mathrm{d}x - \int_{\mathcal{O}} \frac{|\nabla\theta|^2}{\theta^2}\,\mathrm{d}x = \int_{\mathcal{O}} \frac{\boldsymbol{G}(\theta,\boldsymbol{T}^d) : \boldsymbol{T}^d}{\theta}\,\mathrm{d}x.$$

Thus

$$\frac{d}{dt}\int_{\mathcal{O}} \ln\theta \,\mathrm{d}x + \int_{\mathcal{O}} \operatorname{div}\left(\frac{\boldsymbol{q}}{\theta}\right) \mathrm{d}x = \int_{\mathcal{O}} \frac{\boldsymbol{G}(\theta,\boldsymbol{T}^d):\boldsymbol{T}^d}{\theta}\,\mathrm{d}x + \int_{\mathcal{O}} \frac{|\nabla\theta|^2}{\theta^2}\,\mathrm{d}x. \tag{1.3.11}$$

By the assumptions on function $\boldsymbol{G}(\cdot,\cdot)$, see Assumption 4.0.1, Assumption 5.0.1 or Assumption 6.0.1, and by positivity of θ, the right-hand side of (1.3.11) is positive. Therefore, an arbitrary choice of the domain $\mathcal{O}$ implies that the inequality holds

$$\Big(\ln\theta\Big)_t + \operatorname{div}\left(\frac{\boldsymbol{q}}{\theta}\right) \geq 0. \tag{1.3.12}$$

The above relation is so-called Clausius-Duhem inequality and it is one of the equivalent formulations of the second principle of thermodynamics. Hence, the homogeneous boundary conditions and the definition of the heat flux ($\boldsymbol{q} = -\nabla\theta$) imply that

$$\frac{d}{dt}\int_{\Omega} \ln\theta \geq 0. \tag{1.3.13}$$

Note that $s(\theta) = \ln\theta$ is one of the admissible entropies for system (1.2.2) which furnishes a formal justification for the thermodynamical consistency of the model. Comparing entropy mentioned in Section 1.1.3 and $s(\theta) = \ln\theta$ we may observe that they differ by a constant. Adding or subtracting the constant to entropy does not change its meaning, hence $s(\theta) = \ln\theta - \ln\theta_R$ is also an admissible entropy. Therefore, $s(\theta) = \ln\theta$ coincides with the results for zero thermal expansion materials.

1.4 Main problems

Finite energy of the system is a starting point of energy estimates. Considering physical phenomena requires conservation of physical properties. In quasi-static case total energy of Ω consists of two kinds of energy, i.e. thermal energy (proportional to temperature) and potential energy, which is defined below.

Definition 1.4.1. *Potential energy*
Let us define potential energy as follows

$$\mathcal{E}(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p) := \frac{1}{2}\int_{\Omega} \boldsymbol{D}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p):(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p)\,\mathrm{d}x.$$

We concentrate on three different models. We start from the simplest one, Mróz model (see Assumption 4.0.1) and then we consider more complicated models, namely Norton-Hoff-type (see Assumption 5.0.1) and model with growth conditions in Orlicz spaces (see Assumption 6.0.1).

We use two level Galerkin approximation. Construction of approximate solutions is presented in Chapter 2 and is the same for each model. Then, we focus on

a) identification of nonlinear term;

b) identification of right-hand side of heat equation;

c) existence of temperature as a solution of low regular data parabolic equation.

We deal with problem *c)* in Chapter 3. We present two possible ways to solve it, i.e. Boccardo Gallouët approach and renormalised solution.

To solve problems *a)* and *b)* we also use two different approaches. For Mróz model (Chapter 4) we use the Young measure tools to identify the limit of nonlinear term and also we get the strong convergence of right-hand side of heat equation. For Norton-Hoff-type model (Chapter 5) and model with growth conditions in Orlicz spaces (Chapter 6) we use three-step method to solve problems *a)* and *b)*. Under three-step method we understand the following steps:

1) showing the inequality for the limit of heat equation's right-hand sides;

2) using Minty-Browder trick to identify the weak limit of nonlinear term;

3) identifying the limit of right-hand side of heat equation.

The first step is similar for both models. A small difference occurs only in the fact that some test functions are not regular enough. The second and the third steps vary to a great extent. Different assumptions on constitutive functions describing the evolution of visco-elastic strain tensor results in the need to use different mathematical tools.

Chapter 2

Construction of approximate solutions

This chapter is dedicated to construction of approximate solution to thermo-visco-elastic model. Throughout the whole dissertation we use Galerkin approximation to prove the existence of solution. Construction of approximate solution is a very important step, which is often neglected in other scientific papers. Omitting this step of reasoning may cause that the rest of our consideration fails. Results presented in this chapter were obtained in [33]. The same method was also used in [32, 33, 48].

In all models considered in this dissertation, problems that appear in the existence proofs are similar. Different assumptions cause use of various analytical tools but data for heat equation are low regular for all models. From physical point of view, it is correct, whereas from mathematical perspective it entails many problems. The existence of solution to parabolic equation with only integrable data is a subject of discussion in Chapter 3 and it requires special attention.

One of crucial steps in approaches presented in Chapter 3 is to test the equation by the solution's truncation. However, this truncation does not need to be a linear combination of basis functions. That is the reason why we use two level approximation. By two level approximation we understand independent parameters of approximation in the displacement and temperature. Moreover, construction of approximate solution to visco-elastic strain also requires a few words of discussion, what we do later.

Furthermore, due to linearity of equations we may split the solutions $(\theta, \boldsymbol{u})$ into two parts. Considering the independent elastostatic and heat equation with non-homogeneous data we get the first set of solutions. The second one is obtained by use of the Galerkin approximation to homogeneous boundary-value problem. Existence of first part of solutions is discussed for each of considered models independently. Different results are obtained for each model. In the rest of this chapter we focus on the second part of solutions, i.e. the part obtained by Galerkin approximation.

2.1 Definition of bases functions

Definition 2.1.1. *Let $k \in \mathbb{N}$ and $\mathcal{T}_k(\cdot)$ be a standard truncation operator*

$$\mathcal{T}_k(x) = \begin{cases} k & x > k, \\ x & |x| \le k, \\ -k & x < -k. \end{cases} \tag{2.1.1}$$

We construct the approximate solution for temperature, displacement and visco-elastic strain, hence we need to construct three independent bases for these physical quantities. We start with bases for displacement. To do this, let us consider the space $L^2(\Omega, \mathcal{S}^3)$ with a scalar product defined as

$$(\boldsymbol{\xi}, \boldsymbol{\eta})_{\boldsymbol{D}} := \int_\Omega \boldsymbol{D}^{\frac{1}{2}}\boldsymbol{\xi} : \boldsymbol{D}^{\frac{1}{2}}\boldsymbol{\eta}\,\mathrm{d}x \qquad \text{for } \boldsymbol{\xi}, \boldsymbol{\eta} \in L^2(\Omega, \mathcal{S}^3), \tag{2.1.2}$$

where $\boldsymbol{D}^{\frac{1}{2}} \circ \boldsymbol{D}^{\frac{1}{2}} = \boldsymbol{D}$. Let $\{\boldsymbol{w}_i\}_{i=1}^\infty$ be the set of eigenfunctions of the elastostatic operator $-\mathrm{div}\,\boldsymbol{D}\boldsymbol{\varepsilon}(\cdot)$ with the domain $W_0^{1,2}(\Omega, \mathbb{R}^3)$ such that $\{\boldsymbol{w}_i\}$ is orthonormal in $L^2(\Omega, \mathbb{R}^3)$ and $\{\boldsymbol{w}_i\}$ orthogonal in $W_0^{1,2}(\Omega, \mathbb{R}^3)$ with the inner product

$$(\boldsymbol{w}, \boldsymbol{v})_{W_0^{1,2}(\Omega)} = (\boldsymbol{\varepsilon}(\boldsymbol{w}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\boldsymbol{D}}. \tag{2.1.3}$$

By $\{\lambda_i\}$ we denote the set of corresponding eigenvalues. Moreover, using the eigenvalue problem for elastostatic operator we obtain

$$\int_\Omega \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_i) : \boldsymbol{\varepsilon}(\boldsymbol{w}_j)\,\mathrm{d}x = \lambda_i \int_\Omega \boldsymbol{w}_i \cdot \boldsymbol{w}_j\,\mathrm{d}x = 0 \tag{2.1.4}$$

Set $\{\boldsymbol{w}_i\}$ is a basis for displacements construction.

Let $\{v_i\}_{i=1}^\infty$ be the subset of $W^{1,2}(\Omega)$ such that

$$\int_\Omega (\nabla v_i \cdot \nabla \phi - \mu_i v_i \phi)\,\mathrm{d}x = 0, \tag{2.1.5}$$

holds for every function $\phi \in C^\infty(\overline{\Omega})$, see [5, 69]. Moreover, we may assume that $\{v_i\}$ is orthogonal in $W^{1,2}(\Omega)$ and orthonormal in $L^2(\Omega)$. Let $\{\mu_i\}$ be the set of corresponding eigenvalues. Set $\{v_i\}$ is to be used to construct approximate solutions to temperature.

To construct the basis for approximating the visco-elastic strain tensor we proceed as follows. For $\frac{3}{2} < s \le 2$ let us denote by $H^s(\Omega, \mathcal{S}^3)$ the fractional Sobolev space with a scalar product $(\!(\cdot,\cdot)\!)_s$. Due to the regularity of eigenfunctions we observe that for each $i \in \mathbb{N}$ tensor $\boldsymbol{\varepsilon}(\boldsymbol{w}_i)$ is an element of $H^s(\Omega, \mathcal{S}^3)$. Let us define

$$V_k := (\mathrm{span}\{\boldsymbol{\varepsilon}(\boldsymbol{w}_1), ..., \boldsymbol{\varepsilon}(\boldsymbol{w}_k)\})^\perp, \tag{2.1.6}$$

where by $^\perp$ we understand the orthogonal complementation in $L^2(\Omega, \mathcal{S}^3)$ taken with respect to the scalar product $(\cdot,\cdot)_{\boldsymbol{D}}$. Then, denote

$$V_k^s := V_k \cap H^s(\Omega, \mathcal{S}^3). \tag{2.1.7}$$

Since the co-dimension of V_k^s is finite, then V_k^s is closed in $H^s(\Omega, \mathcal{S}^3)$ with respect to the $\|\cdot\|_{H^s}$-norm.

Now, the idea is to find the basis of V_k. The following reasoning comes from [53]. We adapt results presented in [53] into our particular case, see also [32]. Let us consider the following problem: find $\boldsymbol{\zeta}_i^k \in V_k^s$ and $\lambda_i \in \mathbb{R}$ such that

$$(\!(\boldsymbol{\zeta}_i^k, \boldsymbol{\Phi})\!)_s = \lambda_i(\boldsymbol{\zeta}_i^k, \boldsymbol{\Phi})_{\boldsymbol{D}} \qquad \forall\ \boldsymbol{\Phi} \in V_k^s. \tag{2.1.8}$$

where $(\!(\cdot,\cdot)\!)_s$ and $(\cdot,\cdot)_{\boldsymbol{D}}$ are previously defined scalar products in $H^s(\Omega, \mathcal{S}^3)$ and in $L^2(\Omega, \mathcal{S}^3)$, respectively.

Theorem 2.1.1 (Theorem 4.11, page 286 from [53])**.** *There exist a countable set of eigenvalues $\{\lambda_i\}_{i=1}^\infty$ and a corresponding family of eigenfunctions $\{\boldsymbol{\zeta}_i\}_{i=1}^\infty$ solving* (2.1.8) *such that*

- $(\boldsymbol{\zeta}_i, \boldsymbol{\zeta}_j)_{\boldsymbol{D}} = \delta_{ij}$ *for all* $i, j \in \mathbb{N}$,

- $1 \le \lambda_1 \le \lambda_2 \le ...$ *and* $\lambda_i \to \infty$ *as* i *tends to* ∞,
- $(\!(\frac{\boldsymbol{\zeta}_i}{\sqrt{\lambda_i}}, \frac{\boldsymbol{\zeta}_j}{\sqrt{\lambda_i}})\!)_s = \delta_{ij}$ *for all* $i, j \in \mathbb{N}$,
- *the set* $\{\boldsymbol{\zeta}_i\}_{i=1}^{\infty}$ *is a basis of* V_k^s.
- *the set* $\{\boldsymbol{\zeta}_i\}_{i=1}^{\infty}$ *is a basis of* V_k.

Moreover, let us define the subspace $H^N \equiv span\{\boldsymbol{\zeta}_1, ..., \boldsymbol{\zeta}_N\}$ *and projection* $P^N : V_k^s \to H^N$ *such that* $P^N \boldsymbol{\varphi} \equiv \sum_{i=1}^N (\boldsymbol{\varphi}, \boldsymbol{\zeta}_i)_{\boldsymbol{D}} \boldsymbol{\zeta}_i$, *then we get*

$$\|P^N \boldsymbol{\varphi}\|_{H^s} \le \|\boldsymbol{\varphi}\|_{H^s}. \tag{2.1.9}$$

Proof. Proof is divided into few steps.
Existence of $\boldsymbol{\zeta}_1$
Let us define

$$\frac{1}{\lambda_1} \equiv \sup_{\substack{\boldsymbol{V} \in V_k^s \\ \|\boldsymbol{v}\|_{H^s} \le 1}} (\boldsymbol{V}, \boldsymbol{V})_{\boldsymbol{D}}. \tag{2.1.10}$$

Consequently, there exists a sequence $\{\boldsymbol{V}_i\}_{i=1}^{\infty}$ such that $(\boldsymbol{V}_i, \boldsymbol{V}_i)_{\boldsymbol{D}} \to \frac{1}{\lambda_1}$ as i tends to ∞ and $\|\boldsymbol{V}_i\|_{H^s(\Omega)} = 1$. Then, there exists a subsequence $\{\boldsymbol{V}_i\}_{i=1}^{\infty}$ (still denoted by i) and $\boldsymbol{\zeta}_1 \in V_k^s$ such that

$$\begin{aligned} \boldsymbol{V}_i &\rightharpoonup \boldsymbol{\zeta}_1 && \text{weakly in } V_k^s, \\ \boldsymbol{V}_i &\to \boldsymbol{\zeta}_1 && \text{in } L^2(\Omega, \mathcal{S}^3). \end{aligned} \tag{2.1.11}$$

If $\|\boldsymbol{\zeta}_1\|_{H^s(\Omega)} < 1$, then let us define $\boldsymbol{\zeta} = \frac{\boldsymbol{\zeta}_1}{\|\boldsymbol{\zeta}_1\|_{H^s(\Omega)}}$ and then

$$\|\boldsymbol{\zeta}\|_{H^s(\Omega)} = 1 \qquad \text{and} \qquad (\boldsymbol{\zeta}, \boldsymbol{\zeta})_{\boldsymbol{D}} = \frac{(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_1)_{\boldsymbol{D}}}{\|\boldsymbol{\zeta}_1\|^2_{H^s(\Omega)}} > \frac{1}{\lambda_1}, \tag{2.1.12}$$

which is contrary with (2.1.10) and it implies that $\|\boldsymbol{\zeta}_1\|_{H^s(\Omega)} = 1$. To finish the first step we show that $\boldsymbol{\zeta}_1$ is an eigenfunction. Let us take arbitrary $\boldsymbol{H} \in V_k^s$ and define the function

$$\Phi(t) = \frac{(\boldsymbol{\zeta}_1 + t\boldsymbol{H}, \boldsymbol{\zeta}_1 + t\boldsymbol{H})_{\boldsymbol{D}}}{(\!(\boldsymbol{\zeta}_1 + t\boldsymbol{H}, \boldsymbol{\zeta}_1 + t\boldsymbol{H})\!)_s}. \tag{2.1.13}$$

Calculating the derivative of function $\Phi(t)$, we obtain

$$\begin{aligned} 0 = \frac{d}{dt}\Phi(t)|_{t=0} &= \frac{2(\boldsymbol{\zeta}_1, \boldsymbol{H})_{\boldsymbol{D}}(\!(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_1)\!)_s - 2(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_1)_{\boldsymbol{D}}(\!(\boldsymbol{\zeta}_1, \boldsymbol{H})\!)_s}{(\!(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_1)\!)_s^2} \\ &= \frac{2(\boldsymbol{\zeta}_1, \boldsymbol{H})_{\boldsymbol{D}} - \frac{2}{\lambda_1}(\!(\boldsymbol{\zeta}_1, \boldsymbol{H})\!)_s}{(\!(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_1)\!)_s^2} \end{aligned} \tag{2.1.14}$$

and then

$$\lambda_1(\boldsymbol{\zeta}_1, \boldsymbol{H})_{\boldsymbol{D}} = (\!(\boldsymbol{\zeta}_1, \boldsymbol{H})\!)_s \qquad \forall\ \boldsymbol{H} \in V_k^s. \tag{2.1.15}$$

Iterative construction
Assume that for $N \ge 1$ there exist a set of eigenvalues $\{\lambda_i\}_{i=1}^N$ and a set of corresponding eigenfunctions $\{\boldsymbol{\zeta}_i\}_{i=1}^N$. Let us define the space

$$W^N \equiv \{\boldsymbol{V} \in V_k^s : (\!(\boldsymbol{V}, \boldsymbol{\zeta}_i)\!)_s = 0, \quad i = 1, ..., N\}. \tag{2.1.16}$$

Using the similar construction as in the previous step, we find the next eigenvalue and eigenfunction

$$(\boldsymbol{\zeta}_{N+1}, \boldsymbol{\zeta}_{N+1})_{\boldsymbol{D}} = \sup_{\substack{\boldsymbol{V}\in W^N \\ \|\boldsymbol{V}\|_{H^s}=1}} (\boldsymbol{V}, \boldsymbol{V})_{\boldsymbol{D}} \equiv \frac{1}{\lambda_{N+1}}. \tag{2.1.17}$$

Finally, we obtain

$$\begin{array}{ll} 1 \leq \lambda_1 \leq \lambda_2 \leq ..., & \\ (\boldsymbol{\zeta}_i, \boldsymbol{\zeta}_j)_{\boldsymbol{D}} = 0 & \text{if } i \neq j, \\ (\!(\boldsymbol{\zeta}_i, \boldsymbol{\zeta}_j)\!)_s = \delta_{ij} & \forall i, j \in \mathbb{N}. \end{array} \tag{2.1.18}$$

Unboundedness of eigenvalues
Let us assume that the set of eigenvalues has a finite limit, i.e. $\lim_{i\to\infty} \lambda_i = \lambda < \infty$. Since $\|\boldsymbol{\zeta}_i\|_{H^s} = 1$, using subsequence if it is necessary, we get $\boldsymbol{\zeta}_{l_i} \to \boldsymbol{\zeta}$ in $L^2(\Omega, \mathcal{S}^3)$ as $i \to \infty$. Hence

$$\begin{aligned} 2 &= (\!(\boldsymbol{\zeta}_{l_i}, \boldsymbol{\zeta}_{l_i})\!)_s + (\!(\boldsymbol{\zeta}_{l_j}, \boldsymbol{\zeta}_{l_j})\!)_s \\ &= (\!(\boldsymbol{\zeta}_{l_i} - \boldsymbol{\zeta}_{l_j}, \boldsymbol{\zeta}_{l_i} - \boldsymbol{\zeta}_{l_j})\!)_s \\ &= (\!(\boldsymbol{\zeta}_{l_i}, \boldsymbol{\zeta}_{l_i} - \boldsymbol{\zeta}_{l_j})\!)_s - (\!(\boldsymbol{\zeta}_{l_j}, \boldsymbol{\zeta}_{l_i} - \boldsymbol{\zeta}_{l_j})\!)_s \\ &= \lambda_{l_i}(\boldsymbol{\zeta}_{l_i}, \boldsymbol{\zeta}_{l_i} - \boldsymbol{\zeta}_{l_j})_{\boldsymbol{D}} - \lambda_{l_j}(\boldsymbol{\zeta}_{l_j}, \boldsymbol{\zeta}_{l_i} - \boldsymbol{\zeta}_{l_j})_{\boldsymbol{D}}. \end{aligned} \tag{2.1.19}$$

Since $\{\boldsymbol{\zeta}_{l_i}\}$ is a Cauchy sequence and sequence $\{\lambda_{l_i}\}$ is bounded, the right-hand side of abovementioned equation may be arbitrary small by letting i, j to ∞. Obviously, this is a contradiction.
The set $\{\lambda_i\}_{i=1}^\infty$ contains all eigenvalues
Let us assume that there exists an eigenvalue λ such that $\lambda \notin \{\lambda_i\}_{i=1}^\infty$. Let $\boldsymbol{\zeta}$ be the corresponding eigenfunction to the eigenvalue λ and

$$(\!(\boldsymbol{\zeta}, \boldsymbol{\Phi})\!)_s = \lambda(\boldsymbol{\zeta}, \boldsymbol{\Phi})_{\boldsymbol{D}} \qquad \boldsymbol{\Phi} \in V_k^s. \tag{2.1.20}$$

Without losing the generality, we assume that $\|\boldsymbol{\zeta}\|_{H^s} = 1$. Moreover, there exists $i \in \mathbb{N}$ such that $\lambda_i < \lambda < \lambda_{i+1}$. Then, by the definition of eigenvalue, for all $k = 1, ..., i$

$$\lambda(\boldsymbol{\zeta}, \boldsymbol{\zeta}_k)_{\boldsymbol{D}} = (\!(\boldsymbol{\zeta}, \boldsymbol{\zeta}_k)\!)_s = (\!(\boldsymbol{\zeta}_k, \boldsymbol{\zeta})\!)_s = \lambda_k(\boldsymbol{\zeta}_k, \boldsymbol{\zeta})_{\boldsymbol{D}}, \tag{2.1.21}$$

which implies that $(\boldsymbol{\zeta}, \boldsymbol{\zeta}_k)_{\boldsymbol{D}} = 0$. Therefore $\boldsymbol{\zeta} \in W^i$ and using the definition of λ_{i+1} we get

$$(\boldsymbol{\zeta}, \boldsymbol{\zeta})_{\boldsymbol{D}} = \frac{1}{\lambda} > \frac{1}{\lambda_i} = \sup_{\substack{\boldsymbol{V}\in W^N \\ \|\boldsymbol{V}\|_{s,2}=1}} (\boldsymbol{V}, \boldsymbol{V})_{\boldsymbol{D}}, \tag{2.1.22}$$

which is contradictory with $\lambda_i < \lambda$.
The set $\{\boldsymbol{\zeta}_i\}_{i=1}^\infty$ is a basis in V_k^s
Let us define $X = \text{span}\{\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, ...\}$ and let us assume that $X \neq V_k^s$. Then, there exists $\boldsymbol{\Phi} \in V_k^s$ such that $\|\boldsymbol{\Phi}\|_{H^s(\Omega)} = 1$ and $(\!(\boldsymbol{\Phi}, \boldsymbol{\zeta}_i)\!)_s = 0$ for all $i \in \mathbb{N}$. Moreover, for all $i \in \mathbb{N}$

$$(\boldsymbol{\Phi}, \boldsymbol{\Phi})_{\boldsymbol{D}} \leq \sup_{\substack{\boldsymbol{V}\in W^i \\ \|\boldsymbol{V}\|_{H^s}=1}} (\boldsymbol{V}, \boldsymbol{V})_{\boldsymbol{D}} = \frac{1}{\lambda_i}, \tag{2.1.23}$$

which implies that $\boldsymbol{\Phi} = \boldsymbol{0}$.
The set $\{\boldsymbol{\zeta}_i\}_{i=1}^\infty$ is a basis in V_k
Let us observe that the space V_k^s is dense in V_k in $L^2(\Omega, \mathcal{S}^3)$ norm. For arbitrary element $\boldsymbol{\xi} \in V_k$ there exist the sequence $\boldsymbol{\xi}_n \in H^s(\Omega, \mathcal{S}^3)$ such that $\boldsymbol{\xi}_n \to \boldsymbol{\xi}$ in $L^2(\Omega, \mathcal{S}^3)$. Let us define the projection $P_k : H^s(\Omega, \mathcal{S}^3) \to \text{lin}\{\boldsymbol{\varepsilon}(\boldsymbol{w}_1), ..., \boldsymbol{\varepsilon}(\boldsymbol{w}_k)\}$ by $P_k\boldsymbol{V} := \sum_{i=1}^k (\boldsymbol{V}, \boldsymbol{\varepsilon}(\boldsymbol{w}_i))_{\boldsymbol{D}}\boldsymbol{\varepsilon}(\boldsymbol{w}_i)$. Since P_k

is the projection on a finite dimensional space, and the dimension of the space is independent of l, there exists a constant, also independent of l such that $\|P_k \boldsymbol{V}\|_{H^s} \leq C\|\boldsymbol{V}\|_{H^s}$. Then, define

$$\overline{\boldsymbol{\xi}}^n := \boldsymbol{\xi}^n - P_k \boldsymbol{\xi}^n. \tag{2.1.24}$$

Hence, we immediately obtain that $\overline{\boldsymbol{\xi}}^n$ is bounded in $H^s(\Omega, \mathcal{S}^3)$ and converges to $\boldsymbol{\xi} \in V_k$. Consequently, $\{\boldsymbol{\zeta}_i\}_{i=1}^{\infty}$ is also a basis in V_k.
Renormalization of basis
To complete the proof we may renormalise the basis

$$\widehat{\boldsymbol{\zeta}}_i \equiv \frac{\boldsymbol{\zeta}_i}{\sqrt{\lambda_i}}. \tag{2.1.25}$$

for all $i \in \mathbb{N}$.
The continuity of P^N
Consider now $\boldsymbol{\varphi} \in V_k^s$. Then

$$\begin{aligned} \|P^N \boldsymbol{\varphi}\|_{H^s}^2 &= \sum_{i=1}^{N} (\boldsymbol{\varphi}, \boldsymbol{\zeta}_i)_{\boldsymbol{D}}^2 (\!(\boldsymbol{\zeta}_i, \boldsymbol{\zeta}_i)\!)_s = \sum_{i=1}^{N} \frac{(\!(\boldsymbol{\varphi}, \boldsymbol{\zeta}_i)\!)_s^2}{\lambda_i^2} (\!(\boldsymbol{\zeta}_i, \boldsymbol{\zeta}_i)\!)_s \\ &\leq \sum_{i=1}^{N} (\!(\boldsymbol{\varphi}, \frac{\boldsymbol{\zeta}_i}{\sqrt{\lambda_i}})\!)_s^2 \leq \|\boldsymbol{\varphi}\|_{H^s}^2. \end{aligned} \tag{2.1.26}$$

Thus (2.1.9) is proved.

□

To construct the basis for approximate solution to visco-elastic strain tensor we use $\{\boldsymbol{w}_i\}$ and $\{\boldsymbol{\zeta}_i^k\}$. For each pair of approximate parameters (k, l) the basis contains two subsets. One of them consist of symmetric gradients of first k functions from the basis for displacement, i.e. set $\{\boldsymbol{\varepsilon}(\boldsymbol{w}_i)\}_{i=1}^k$. The second subset contains first l function from $\{\boldsymbol{\zeta}_i\}_{i=1}^{\infty}$. It is obvious that basis $\{\boldsymbol{\zeta}_j^k\}$ depends on the parameter k. Additionally, for all $k \in \mathbb{N}$ after limit passage with l going to ∞ the set $\{\boldsymbol{\varepsilon}(\boldsymbol{w}_j), \boldsymbol{\zeta}_i^k\}_{j=1,\ldots,k;\ i=1,\ldots,\infty}$ is a basis of whole space $L^2(\Omega, \mathcal{S}^3)$.

At the end of this section we define three projections, which are very important in next chapters.

Definition 2.1.2. *Let us define the following projections:*

- *Let $P^k : H^s(\Omega, \mathcal{S}^3) \to \operatorname{lin}\{\boldsymbol{\varepsilon}(\boldsymbol{w}_1), \ldots, \boldsymbol{\varepsilon}(\boldsymbol{w}_k)\}$ be defined by*

$$P^k(\boldsymbol{v}) := \sum_{n=1}^{k} (\boldsymbol{v}, \boldsymbol{\varepsilon}(\boldsymbol{w}_n))_{\boldsymbol{D}} \boldsymbol{\varepsilon}(\boldsymbol{w}_n). \tag{2.1.27}$$

- *Let $P_{L^2}^{l,k} : L^2(\Omega, \mathcal{S}^3) \to \operatorname{lin}\{\boldsymbol{\zeta}_1^k, \ldots, \boldsymbol{\zeta}_l^k\}$ be defined by*

$$P_{L^2}^{l,k}(\boldsymbol{v}) := \sum_{m=1}^{l} (\boldsymbol{v}, \boldsymbol{\zeta}_m^k)_{\boldsymbol{D}} \boldsymbol{\zeta}_m^k. \tag{2.1.28}$$

- *Let $P_{H^s}^{l,k} : H^s(\Omega, \mathcal{S}^3) \to \operatorname{lin}\{\boldsymbol{\zeta}_1^k, \ldots, \boldsymbol{\zeta}_l^k\}$ be defined by*

$$P_{H^s}^{l,k}(\boldsymbol{v}) := \sum_{m=1}^{l} (\!(\boldsymbol{v}, \frac{\boldsymbol{\zeta}_m^k}{\sqrt{\lambda_m^k}})\!)_s \frac{\boldsymbol{\zeta}_m^k}{\sqrt{\lambda_m^k}}. \tag{2.1.29}$$

As we may observer the projections $P^{l,k}_{H^s}$ and $P^{l,k}_{L^2}$ are equal on V^s_k. Indeed, if $\boldsymbol{\varphi} \in V^s_k$ then

$$P^{l,k}_{L^2}\boldsymbol{\varphi} = \sum_{m=1}^{l} (\boldsymbol{\varphi}, \boldsymbol{\zeta}^k_m)_{\boldsymbol{D}}\boldsymbol{\zeta}^k_m = \sum_{m=1}^{l} ((\boldsymbol{\varphi}, \frac{\boldsymbol{\zeta}^k_m}{\sqrt{\lambda^k_m}}))_s \frac{\boldsymbol{\zeta}^k_m}{\sqrt{\lambda^k_m}} = P^{l,k}_{H^s}\boldsymbol{\varphi}, \tag{2.1.30}$$

where the second equality is condition for eigenvalues. The norms $\|P^{l,k}_{H^s}\|_{\mathcal{L}(H^s)}$ and $\|P^{l,k}_{L^2}\|_{\mathcal{L}(L^2)}$ are equal to 1. Moreover, we may observe that for $\boldsymbol{v} \in H^s(\Omega, \mathcal{S}^3)$ it holds

$$\begin{aligned}(P^{l,k}_{H^s} \circ (Id - P^k))\boldsymbol{v} = \sum_{m=1}^{l} (((Id - P^k)\boldsymbol{v}, \frac{\boldsymbol{\zeta}^k_m}{\sqrt{\lambda^k_m}}))_s \frac{\boldsymbol{\zeta}^k_m}{\sqrt{\lambda^k_m}} &= \sum_{m=1}^{l} ((Id - P^k)\boldsymbol{v}, \boldsymbol{\zeta}^k_m)_{\boldsymbol{D}}\boldsymbol{\zeta}^k_m \\ &= \sum_{m=1}^{l} (\boldsymbol{v}, \boldsymbol{\zeta}^k_m)_{\boldsymbol{D}}\boldsymbol{\zeta}^k_m = P^{l,k}_{L^2}\boldsymbol{v}.\end{aligned} \tag{2.1.31}$$

Since P^k is the projection which does not dependent on l, then there exists $c(k)$ (depending only on k) such that for every $\boldsymbol{\varphi} \in H^s(\Omega, \mathcal{S}^3)$ it holds

$$\max(\|P^k\boldsymbol{\varphi}\|_{H^s}, \|(Id - P^k)\boldsymbol{\varphi}\|_{H^s}) \le c(k)\|\boldsymbol{\varphi}\|_{H^s}. \tag{2.1.32}$$

2.2 Definition of approximate solution

As we have mentioned before it is much easier and more transparent to show the construction of approximate solutions for homogeneous problems. Our idea is to consider displacement, Cauchy stress tensor and temperature as a sum of solutions to three different problems.

Instead of considering thermo-visco-elastic problem

$$\begin{cases} -\operatorname{div} \widehat{\boldsymbol{T}} = \boldsymbol{f}, \\ \qquad \widehat{\boldsymbol{T}} = \boldsymbol{D}(\widehat{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}^p), \\ \qquad \boldsymbol{\varepsilon}^{\mathrm{p}}_t = \boldsymbol{G}(\widehat{\theta}, \widehat{\boldsymbol{T}}^d), \\ \widehat{\theta}_t - \Delta\widehat{\theta} = \widehat{\boldsymbol{T}}^d : \boldsymbol{G}(\widehat{\theta}, \widehat{\boldsymbol{T}}^d). \end{cases} \tag{2.2.1}$$

with initial and boundary conditions (1.2.3) we focus on the following problems:

- elastostatic equation with the same data (volume force and boundary condition) as for the full model
$$\begin{cases} -\operatorname{div} \tilde{\boldsymbol{T}} &= \boldsymbol{f} \quad \text{in } \Omega \times (0, T), \\ \qquad \tilde{\boldsymbol{u}} &= \boldsymbol{g}_{\boldsymbol{u}} \quad \text{on } \partial\Omega \times (0, T), \end{cases} \tag{2.2.2}$$
- heat equation with heat flux through the boundary as in the full model and without any heat sources
$$\begin{cases} \tilde{\theta}_t - \Delta\tilde{\theta} &= 0 \quad \text{in } \Omega \times (0, T), \\ \quad \frac{\partial\tilde{\theta}}{\partial\boldsymbol{n}} &= g_\theta \quad \text{on } \partial\Omega \times (0, T), \\ \quad \tilde{\theta}(x, 0) &= \tilde{\theta}_0 \quad \text{in } \Omega. \end{cases} \tag{2.2.3}$$
- thermo-visco-elastic system of equations with homogeneous boundary data
$$\begin{cases} -\operatorname{div} \boldsymbol{T} = 0, \\ \qquad \boldsymbol{T} = \boldsymbol{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p), \\ \qquad \boldsymbol{\varepsilon}^{\mathrm{p}}_t = \boldsymbol{G}(\theta + \tilde{\theta}, \boldsymbol{T}^d + \tilde{\boldsymbol{T}}^d), \\ \theta_t - \Delta\theta = \left(\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d\right) : \boldsymbol{G}(\tilde{\theta} + \theta, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d). \end{cases} \tag{2.2.4}$$

with initial conditions

$$\begin{cases} \theta(x,0) = \theta_0 - \tilde{\theta}_0, \\ \boldsymbol{\varepsilon}^{\mathbf{p}}(x,0) = \boldsymbol{\varepsilon}^{\mathbf{p}}_0. \end{cases} \tag{2.2.5}$$

To get the same solutions to (2.2.1) and to (2.2.2) – (2.2.4) we introduce the shifts of solution to constitutive functions. Then, by linear character of the system (2.2.1), finding $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{T}}, \tilde{\theta}, \boldsymbol{\varepsilon}^{\mathbf{p}})$ is equivalent to finding $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{T}})$, $\tilde{\theta}$ and $(\boldsymbol{u}, \boldsymbol{T}, \theta, \boldsymbol{\varepsilon}^{\mathbf{p}})$. Moreover, the following relations hold

$$\begin{cases} \tilde{\boldsymbol{u}} = \widehat{\boldsymbol{u}} + \boldsymbol{u}, \\ \tilde{\boldsymbol{T}} = \widehat{\boldsymbol{T}} + \boldsymbol{T}, \\ \tilde{\theta} = \widehat{\theta} + \theta. \end{cases} \tag{2.2.6}$$

Existence proofs to (2.2.2) and (2.2.2) are well known results, see e.g. [29] or [82]. For each model $\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{T}}$ should belong to different functional spaces, hence we discuss their existence for each model separately.

In this section we focus on the construction of approximate solution, i.e. on the construction of $\boldsymbol{u}_{k,l}, \boldsymbol{T}_{k,l}, \theta_{k,l}$ and $\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}$ which are the approximations of $\boldsymbol{u}, \boldsymbol{T}, \theta$ and $\boldsymbol{\varepsilon}^{\mathbf{p}}$, respectively. With the bases from the previous section we may proceed with the definition of approximate solutions. For $k, l \in \mathbb{N}$ let

$$\begin{aligned} \boldsymbol{u}_{k,l} &= \sum_{n=1}^{k} \alpha^n_{k,l}(t)\boldsymbol{w}_n, \\ \theta_{k,l} &= \sum_{m=1}^{l} \beta^m_{k,l}(t)v_m, \\ \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l} &= \sum_{n=1}^{k} \gamma^n_{k,l}(t)\boldsymbol{\varepsilon}(\boldsymbol{w}_n) + \sum_{m=1}^{l} \delta^m_{k,l}(t)\boldsymbol{\zeta}^k_m, \end{aligned} \tag{2.2.7}$$

and $\boldsymbol{u}_{k,l}$, $\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}$ and $\theta_{k,l}$ solve the approximate system of equations

$$\begin{aligned} \int_\Omega \boldsymbol{T}_{k,l} : \boldsymbol{\varepsilon}(\boldsymbol{w}_n)\,\mathrm{d}x &= 0 & n = 1, ..., k, \\ \boldsymbol{T}_{k,l} &= \boldsymbol{D}(\varepsilon(\boldsymbol{u}_{k,l}) - \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}), \\ \int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t : \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_n)\,\mathrm{d}x &= \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_n)\,\mathrm{d}x & n = 1, ..., k, \\ \int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t : \boldsymbol{D}\boldsymbol{\zeta}^k_m\,\mathrm{d}x &= \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{D}\boldsymbol{\zeta}^k_m\,\mathrm{d}x & m = 1, ..., l, \\ \int_\Omega (\theta_{k,l})_t v_m\,\mathrm{d}x + \int_\Omega \nabla\theta_{k,l}\cdot\nabla v_m\,\mathrm{d}x & \\ = \int_\Omega \mathcal{T}_k((\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) :\ & \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}))v_m\,\mathrm{d}x & m = 1, ..., l. \end{aligned} \tag{2.2.8}$$

for a.a. $t \in (0, T)$. For each approximate equation we have the initial conditions in the following form

$$\begin{cases} (\theta_{k,l}(x,0), v_m) = \Big(\mathcal{T}_k(\theta_0 - \tilde{\theta}_0), v_m\Big) & m = 1, .., l, \\ \Big(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}(x,0), \boldsymbol{\varepsilon}(\boldsymbol{w}_n)\Big)_{\boldsymbol{D}} = \big(\boldsymbol{\varepsilon}^{\mathbf{p}}_0, \boldsymbol{\varepsilon}(\boldsymbol{w}_n)\big)_{\boldsymbol{D}} & n = 1, .., k, \\ \Big(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}(x,0), \boldsymbol{\zeta}^k_m\Big)_{\boldsymbol{D}} = \big(\boldsymbol{\varepsilon}^{\mathbf{p}}_0, \boldsymbol{\zeta}^k_m\big)_{\boldsymbol{D}} & m = 1, .., l, \end{cases} \tag{2.2.9}$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$ and $(\cdot,\cdot)_{\boldsymbol{D}}$ the scalar product in $L^2(\Omega,\mathcal{S}^3)$. It is important to mention here that shifts, caused by considering three systems of equations, appear also in initial condition on temperature.

Using $(2.2.8)_{(1)}$ and $(2.2.8)_{(2)}$ we obtain

$$\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}) - \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) : \boldsymbol{\varepsilon}(\boldsymbol{w}_n)\,\mathrm{d}x = 0. \tag{2.2.10}$$

The selection of the Galerkin bases and representation of the approximate solutions (2.2.7) allow us to notice that

$$\alpha^n_{k,l}(t) = \frac{1}{\lambda_n}\gamma^n_{k,l}(t)\int_\Omega \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_n) : \boldsymbol{\varepsilon}(\boldsymbol{w}_n)\,\mathrm{d}x = \gamma^n_{k,l}(t). \tag{2.2.11}$$

Let us define

$$\boldsymbol{\xi}(t) = (\beta^1_{k,l}(t),...,\beta^l_{k,l}(t),\gamma^1_{k,l}(t),...,\gamma^k_{k,l}(t),\delta^1_{k,l}(t),...,\delta^l_{k,l}(t))^T.$$

Then the constitutive function $\boldsymbol{G}$ may be presented as a function $\tilde{\boldsymbol{G}}$ which is a function of x,t and of $\boldsymbol{\xi}$, i.e.

$$\begin{aligned}\tilde{\boldsymbol{G}}(x,t,\boldsymbol{\xi}(t)) &:= \boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\\ &= \boldsymbol{G}\Big(\tilde\theta + \sum_{j=1}^{l}\beta^j_{k,l}(t)v_j(x), \tilde{\boldsymbol{T}}^d - \Big(\boldsymbol{D}\sum_{j=1}^{l}\delta^j_{k,l}(t)\boldsymbol{\zeta}_j\Big)^d\Big).\end{aligned}$$

Hence we obtain

$$\begin{cases}(\gamma^n_{k,l}(t))_t = \dfrac{1}{\lambda_n}\displaystyle\int_\Omega \tilde{\boldsymbol{G}}(x,t,\boldsymbol{\xi}(t)) : \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_n)\,\mathrm{d}x,\\ (\delta^m_{k,l}(t))_t = \displaystyle\int_\Omega \tilde{\boldsymbol{G}}(x,t,\boldsymbol{\xi}(t)) : \boldsymbol{D}\boldsymbol{\zeta}^k_m\,\mathrm{d}x,\\ (\beta^m_{k,l}(t))_t = \displaystyle\int_\Omega \mathcal{T}_k\Big((\tilde{\boldsymbol{T}}^d - (\boldsymbol{D}\sum_{n=1}^{l}\delta^n_{k,l}(t)\boldsymbol{\zeta}_n)^d) : \tilde{\boldsymbol{G}}(x,t,\boldsymbol{\xi}(t))\Big)v_m\,\mathrm{d}x + \mu_m\beta^m_{k,l}(t),\end{cases} \tag{2.2.12}$$

for $n = 1,...,k$ and $m = 1,...,l$. Moreover, the initial conditions (2.2.9) are equivalent to the system

$$\begin{cases}\beta^m_{k,l}(0) = \Big(\mathcal{T}_k(\theta_0 - \tilde\theta_0), v_m\Big),\\ \gamma^n_{k,l}(0) = \dfrac{1}{\lambda_m}\left(\boldsymbol{\varepsilon}^{\mathbf{p}}_0, \boldsymbol{\varepsilon}(\boldsymbol{w}_n)\right)_{\boldsymbol{D}},\\ \delta^m_{k,l}(0) = \Big(\boldsymbol{\varepsilon}^{\mathbf{p}}_0, \boldsymbol{\zeta}^k_m\Big)_{\boldsymbol{D}},\end{cases} \tag{2.2.13}$$

where $n = 1,..,k$ and $m = 1,...,l$, can be equivalently written as the initial value problem

$$\begin{aligned}&\frac{d\boldsymbol{\xi}_2}{dt} = \boldsymbol{F}(\boldsymbol{\xi}(t),t), \qquad t\in[0,T),\\ &\boldsymbol{\xi}(0) = \boldsymbol{\xi}_0,\end{aligned} \tag{2.2.14}$$

where $\boldsymbol{\xi}_0$ is a vector of initial conditions (2.2.13).

Lemma 2.2.1. *(Existence of approximate solution)*
For initial condition satisfying $\boldsymbol{\varepsilon}^{\mathbf{p}}_0 \in L^1(\Omega,\mathcal{S}^3_d)$ and $\theta_0 \in L^1(\Omega)$ there exists an absolutely continuous in time solution to (2.2.14).

Proof. According to Carathéodory Theorem, see [53, Theorem 3.4] or [87, Appendix (61)], there exist unique absolutely continuous functions $\beta_{k,l}^m(t)$, $\gamma_{k,l}^n(t)$ and $\delta_{k,l}^m(t)$ for every $n \leq k$ and $m \leq l$ on some time interval $[0, t^*]$. By (2.2.11) there exists a unique absolutely continuous function $\alpha_{k,l}^n(t)$.

□

Remark. *To get the global existence of approximate solution we need to use the uniform estimates for solutions. For each model considered in this dissertation, we show the estimates independently in the following chapters.*

Chapter 3

Solution to heat equation

The subject of this chapter is to consider the heat equation from thermo-visco-elastic model. In case of all models considered in this dissertation we encounter the same issue, i.e. assumptions on visco-elastic constitutive function cause that the right-hand sides of heat equations are only integrable functions. We deal with low regularity of data in two different ways. The first approach is based on the paper of Boccardo & Gallouët [15] and the second one on papers of Blanchard and Blanchard & Murat [13, 14]. The approach of Boccardo & Gallouët was the first solution regarding parabolic equation with low regular right-hand side. Further result was the renormalised approach. Additionally, renormalised solutions give more information than Boccardo & Gallouët ones. Renormalization methods were used firstly to prove the existence of solution to Boltzmann equation, see [25].

There are two main differences between results presented in [13, 14, 15] and our work. The first one lies in the use of boundary conditions. In [13, 14, 15] authors consider problems with Dirichlet boundary conditions in the contrast to our case, where we use Neumann boundary conditions. The second difference is that the heat equation is a part of system of equations. Thus, we do not have full data information, e.g. we know nothing about the convergence of right-hand side. In [13, 14, 15] authors consider only one equation and they do not have the problem with coupling of equations.

This chapter is divided into two sections which present two different approaches. In both cases, we start with uniform boundedness of right-hand side of heat equation, which is a consequence of uniform boundedness of approximate solutions. In Boccardo-Gallouët's approach this information is sufficient to prove all properties of temperature, whereas it is not enough to prove the existence of renormalised approach. The uniform boundedness of right-hand side of heat equations implies only the almost pointwise convergence of temperature's approximate sequence to a measurable function θ. To prove another properties of θ and also to prove that θ is a renormalised solution (see Definition 3.2.1) we should have some (weak or strong) convergence of right-hand side of heat equations.

For Mróz and Norton-Hoff-type models we use Boccardo-Gallouët's approach, see Chapter 4 and Chapter 5. For models with growth conditions in Orlicz spaces we use renormalised approach, see Chapter 6. This is our arbitrary choice and it is obvious that making some improvements of proofs presented in Chapter 4–6 we may use then conversely. Thus, in the renormalised approach we present results for strong convergence of right-hand sides of heat equations, which takes place in case of Mróz model.

Let $\mathcal{T}_k(\cdot)$ be a standard truncation operator defined in 2.1.1. Then, we focus on the following problem

$$\begin{cases} (\theta_k)_t - \Delta\theta_k = f_k & \text{in } Q, \\ \frac{\partial\theta_k}{\partial \boldsymbol{n}} = 0 & \text{in } \partial\Omega \times (0,T), \\ \theta_k(\cdot,0) = \mathcal{T}_k(\theta_0) & \text{on } \Omega. \end{cases} \tag{3.0.1}$$

where for every $k \in \mathbb{N}$ function f_k belongs to $L^2(Q)$ and the sequence $\{f_k\}$ is uniformly bounded in $L^1(Q)$, i.e. $\|f_k\|_{L^1(Q)} \leq B$. Additionally, we have $\mathcal{T}_k(\theta_0) \in L^2(\Omega)$, $\|\mathcal{T}_k(\theta_0)\|_{L^1(\Omega)} \leq \|\theta_0\|_{L^1(\Omega)}$ and $\mathcal{T}_k(\theta_0) \to \theta_0$ in $L^1(\Omega)$. In the case of thermo-visco-elastic models considered here we have

$$f_k = \mathcal{T}_k((\tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d) : \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)) \qquad \text{in } Q. \tag{3.0.2}$$

3.1 Boccardo-Gallouët approach

Current section is devoted to prove the existence to the heat equation with Neumann boundary conditions. Two dimensional case was considered in [24], and also used in [43]. Lemmas presented below come from [32], where we consider Norton-Hoff-type models.

Lemma 3.1.1. *The sequence $\{\theta_{k,l}\}$ is uniformly bounded in $L^\infty(0,T;L^1(\Omega))$ with respect to k and l.*

Proof. It can be immediately observed that

$$\sup_{0\leq t\leq T} \|\theta_{k,l}(t)\|_{L^1(\Omega)} \leq B + \|\theta_0\|_{L^1(\Omega)},$$

which completes the proof. □

Lemma 3.1.2. *The sequence of approximate solutions to the heat equation* (3.0.1) *is uniformly bounded in space $L^q(0,T,W^{1,q}(\Omega))$ for $q < \frac{2(N+1)-N}{N+1}$ ($q < \frac{5}{4}$ in three dimensional case $N = 3$).*

Proof. We define special truncation function $\psi_m(\cdot)$ for every $m \in \mathbb{N}$:

$$\psi_m(s) = \begin{cases} 1 & \text{if } s \geq m+1, \\ s-m & \text{if } m+1 \geq s \geq m, \\ 0 & \text{if } |s| \leq m, \\ s+m & \text{if } s \geq m+1, \\ -1 & \text{if } s \leq -m-1. \end{cases} \tag{3.1.1}$$

Using $\psi_m(\theta_k)$ as a test function for (3.0.1) we obtain

$$\int_0^T\int_\Omega (\Psi_m(\theta_k))_t \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \nabla\theta_k\cdot\nabla\psi_m(\theta_k)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega f_k\psi_m(\theta_k)\,\mathrm{d}x\,\mathrm{d}t, \tag{3.1.2}$$

where $\Psi_m(s) = \int_0^s \psi_m(\tau)\,\mathrm{d}\tau$. Thus

$$\int_\Omega \Psi_m(\theta_k)(T)\,\mathrm{d}x + \int_0^T\int_\Omega \nabla\theta_k\cdot\nabla\psi_m(\theta_k)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega f_k\psi_m(\theta_k)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \Psi_m(\mathcal{T}_k(\theta_0))\,\mathrm{d}x.$$

Terms on the right-hand side of the above equation can be estimated as follows

$$\int_0^T\int_\Omega f_k\psi_m(\theta_k)\,\mathrm{d}x\,\mathrm{d}t \leq \|f\|_{L^1(0,T,L^1(\Omega))},$$
$$\int_\Omega \Psi_m(\mathcal{T}_k(\theta_0))\,\mathrm{d}x \leq \|\theta_0\|_{L^1(\Omega)},$$

for every $k, m \in \mathbb{N}$. Additionally, $\int_\Omega \Psi_m(\theta_k)(T)dx$ is nonnegative. Hence,

$$\int_{B_m} |\nabla\theta_k|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \nabla\theta_k \cdot \nabla\psi_m(\theta_k)\,\mathrm{d}x\,\mathrm{d}t \le \|f\|_{L^1(0,T,L^1(\Omega))} + \|\theta_0\|_{L^1(\Omega)},$$

where $B_m := \{(x,t) \in \Omega \times (0,T) : m \le \theta_k(x,t) \le m+1\}$. Now let $q \le \frac{2(N+1)-N}{N+1}$ and $r = \frac{N+1}{N}q$ (in our case $q < \frac{5}{4}$ and $r = \frac{4}{3}q$). Using the Hölder inequality we obtain

$$\begin{aligned}
\int_{B_m} |\nabla\theta_k|^q \,\mathrm{d}x\,\mathrm{d}t &\le \left(\int_{B_m} |\nabla\theta_k|^{q\frac{2}{q}} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{q}{2}} \left(\int_{B_m} 1^{\frac{2}{2-q}} \,\mathrm{d}x\,\mathrm{d}t\right)^{1-\frac{q}{2}} \\
&\le \left(\int_{B_m} |\nabla\theta_k|^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{q}{2}} \left(\int_{B_m} \mathrm{d}x\,\mathrm{d}t\right)^{1-\frac{q}{2}} \\
&\le c_3 \left(\int_{B_m} \frac{|\theta_k|^r}{m^r} \,\mathrm{d}x\,\mathrm{d}t\right)^{1-\frac{q}{2}} \\
&\le c_3 \left(\int_{B_m} |\theta_k|^r \,\mathrm{d}x\,\mathrm{d}t\right)^{1-\frac{q}{2}} \frac{1}{m^{\frac{r(2-q)}{2}}} \\
&\le c_3 \left(\int_{B_m} |\theta_k|^r \,\mathrm{d}x\,\mathrm{d}t\right)^{1-\frac{q}{2}} \left(\frac{1}{m^{\frac{r(2-q)}{q}}}\right)^{\frac{q}{2}}.
\end{aligned}$$

Then

$$\begin{aligned}
\int_Q |\nabla\theta_k|^q \,\mathrm{d}x\,\mathrm{d}t &\le c_4(n_0) + c_3 \sum_{m=n_0}^{\infty} \left(\int_{B_m} |\theta_k|^r \,\mathrm{d}x\,\mathrm{d}t\right)^{1-\frac{q}{2}} \left(\frac{1}{m^{\frac{r(2-q)}{q}}}\right)^{\frac{q}{2}} \\
&\le c_4(n_0) + c_3 \left(\sum_{m=n_0}^{\infty} \int_{B_m} |\theta_k|^r \,\mathrm{d}x\,\mathrm{d}t\right)^{1-\frac{q}{2}} \left(\sum_{m=n_0}^{\infty} \frac{1}{m^{\frac{r(2-q)}{q}}}\right)^{\frac{q}{2}} \qquad (3.1.3) \\
&\le c_4(n_0) + c_3 \left(\int_Q |\theta_k|^r \,\mathrm{d}x\,\mathrm{d}t\right)^{1-\frac{q}{2}} \left(\sum_{m=n_0}^{\infty} \frac{1}{m^{\frac{r(2-q)}{q}}}\right)^{\frac{q}{2}},
\end{aligned}$$

where $c_4(n_0) = \int_{\{(x,t):|\theta_k(x,t)|\le n_0\}} |\nabla\theta_k|^q \,\mathrm{d}x\,\mathrm{d}t$. Using the Hölder inequality we observe that $c_4(n_0)$ is bounded by B, $\|u_0\|_{L^1(\Omega)}$ and the measure of set Q. Furthermore, $\frac{r(2-q)}{q} > 1$ and $\sum_{m=n_0}^{\infty} m^{-\frac{r(2-q)}{q}}$ is summable. Using the interpolation inequality for $\|\theta_k\|_{L^q(\Omega)}$ we obtain

$$\|\theta_k\|_{L^q(\Omega)} \le \|\theta_k\|_{L^1(\Omega)}^s \|\theta_k\|_{L^{q^*}(\Omega)}^{1-s}, \qquad (3.1.4)$$

where $q^* = \frac{Nq}{N-q}$ $(= \frac{3q}{3-q})$ and $\frac{1}{q} = \frac{s}{1} + \frac{1-s}{q^*}$. After simple calculations we get that $1 - s = \frac{1-q}{1-q^*}\frac{q^*}{q}$ (and $0 < s < 1$). In Lemma 3.1.1 we showed that $\|\theta_k\|_{L^1(\Omega)}$ is uniformly bounded, hence

$$\int_0^T \int_\Omega |\theta_k|^q \,\mathrm{d}x\,\mathrm{d}t \le C \int_0^T \|\theta_k\|_{L^{q^*}(\Omega)}^{(1-s)q} \,\mathrm{d}t \le C \int_0^T \|\theta_k\|_{L^{q^*}(\Omega)}^{\frac{1-q}{1-q^*}q^*} \,\mathrm{d}t.$$

Using the Hölder inequality we obtain

$$\begin{aligned}\int_0^T \int_\Omega |\theta_k|^q \,\mathrm{d}x\,\mathrm{d}t &\le C \int_0^T \|\theta_k\|_{L^{q^*}(\Omega)}^{\frac{1-q}{1-q^*}q^*} \,\mathrm{d}t \\ &\le C \left(\int_0^T \|\theta_k\|_{L^{q^*}(\Omega)}^{\frac{1-q}{1-q^*}q^* \frac{q^*-1}{q-1}\frac{q}{q^*}} \,\mathrm{d}t \right)^{\frac{q-1}{q^*-1}\frac{q^*}{q}} \\ &= C \left(\int_0^T \|\theta_k\|_{L^{q^*}(\Omega)}^{q} \,\mathrm{d}t \right)^{\frac{q-1}{q^*-1}\frac{q^*}{q}} .\end{aligned}$$

Let us notice that the exponent $\frac{q-1}{q^*-1}\frac{q^*}{q} = \frac{N(q-1)}{N(q-1)+q} < 1$. Using the interpolation inequality for $\|\theta_k\|_{L^r(\Omega)}$ we get

$$\|\theta_k\|_{L^r(\Omega)} \le \|\theta_k\|_{L^1(\Omega)}^s \|\theta_k\|_{L^{q^*}(\Omega)}^{1-s}, \tag{3.1.5}$$

where $\frac{1}{r} = \frac{s}{1} + \frac{1-s}{q^*}$. The parameters s are different in each interpolation inequality (3.1.4) and (3.1.5). Simple calculations yield that $1 - s = \frac{1-r}{1-q^*}\frac{q^*}{r}$. By Lemma 3.1.1 we conclude that

$$\begin{aligned}\|\theta_k\|_{L^r(0,T,L^r(\Omega))}^r &\le \int_0^T \|\theta_k\|_{L^r(\Omega)}^r \,\mathrm{d}t \\ &\le \int_0^T \|\theta_k\|_{L^1(\Omega)}^{sr} \|\theta_k\|_{L^{q^*}(\Omega)}^{\frac{1-r}{1-q^*}\frac{q^*}{r}r} \,\mathrm{d}t \\ &\le C \int_0^T \|\theta_k\|_{L^{q^*}(\Omega)}^q \,\mathrm{d}t = C\|\theta_k\|_{L^q(0,T,L^{q^*}(\Omega))}^q .\end{aligned} \tag{3.1.6}$$

The Sobolev embedding theorem implies that

$$\|\theta_k\|_{L^q(0,T,L^{q^*}(\Omega))}^q = \int_0^T \left(\int_\Omega |\theta_k|^{q^*} \,\mathrm{d}x \right)^{\frac{q}{q^*}} \mathrm{d}t \le C \left(\int_0^T \int_\Omega |\theta_k|^q \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega |\nabla\theta_k|^q \,\mathrm{d}x\,\mathrm{d}t \right).$$

Using the previous inequalities we obtain

$$\begin{aligned}\|\theta_k\|_{L^q(0,T,L^{q^*}(\Omega))}^q &\le C\|\theta_k\|_{L^q(0,T,L^{q^*}(\Omega))}^{\frac{q-1}{q^*-1}\frac{q^*}{q}} + c_4(n_0) + D\left(\int_Q |\theta_k|^r \,\mathrm{d}x\,\mathrm{d}t \right)^{1-\frac{q}{2}} \\ &\le C\|\theta_k\|_{L^q(0,T,L^{q^*}(\Omega))}^{\frac{q-1}{q^*-1}\frac{q^*}{q}} + c_4(n_0) + D\|\theta_k\|_{L^q(0,T,L^{q^*}(\Omega))}^{q\frac{2-q}{2}}\end{aligned}$$

and $\frac{q-1}{q^*-1}\frac{q^*}{q} < 1$ and $q\frac{2-q}{2} < q$, so we have the uniform boundedness

$$\|\theta_k\|_{L^q(0,T,L^{q^*}(\Omega))}^q \le C,$$

and from previous inequalities we get the uniform boundedness of sequence $\{\theta_k\}$ in the space $L^q(0,T,L^{q^*}(\Omega))$. Using this uniform boundedness, inequalities (3.1.3) and (3.1.6) we get the uniform boundedness of the sequence $\{\theta_k\}$ in the spaces $L^q(0,T,W^{1,q}(\Omega))$, which completes the proof. □

Lemma 3.1.3. *Let $f_k \rightharpoonup f$ weakly in $L^1(Q)$. Then the sequence $\{\nabla\theta_k\}$ converges strongly to $\nabla\theta$ in $L^1(0,T,L^1(\Omega))$.*

Proof. Let us define a test function

$$\varphi(s) = \begin{cases} \varepsilon & s > \varepsilon, \\ s & |s| \le \varepsilon, \\ -\varepsilon & s < -\varepsilon, \end{cases} \tag{3.1.7}$$

for fixed $\varepsilon > 0$. Subtracting equation (3.0.1) with function on right-hand side f_n and f_m, and using the test function $\varphi(\theta_n - \theta_m)$ we obtain

$$\int_\Omega \Phi(\theta_n - \theta_m)(T)\,\mathrm{d}x + \int_{D_{n,m,\varepsilon}} |\nabla(\theta_n - \theta_m)|^2\,\mathrm{d}x\,\mathrm{d}t =$$
$$\int_0^T \int_\Omega (f_n - f_m)\varphi(\theta_n - \theta_m)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \Phi(\mathcal{T}_n(\theta_0) - \mathcal{T}_m(\theta_0))\,\mathrm{d}x,$$

where $\Phi(s) = \int_0^s \varphi(\tau)d\tau$ and $D_{n,m,\varepsilon} = \{(x,t) \in \Omega \times (0,T) : |\theta_n(x,t) - \theta_m(x,t)| \le \varepsilon\}$. The sequence $\mathcal{T}_k(\theta_0)$ is convergent to θ_0 in $L^1(\Omega)$, hence, we can find n_0 such that for every n, m greater than n_0 we have $\int_\Omega \Phi(\mathcal{T}_n(\theta_0) - \mathcal{T}_m(\theta_0)) < \varepsilon$. The function Φ is nonnegative and the right-hand side of the equation above is bounded ($\|f_n\|_{L^1(0,T,L^1(\Omega))} \le B$), hence

$$\int_{D_{n,m,\varepsilon}} |\nabla(\theta_n - \theta_m)|^2\,\mathrm{d}x\,\mathrm{d}t \le 2\varepsilon B + \varepsilon = (2B+1)\varepsilon.$$

The Hölder inequality yields

$$\int_{D_{n,m,\varepsilon}} |\nabla(\theta_n - \theta_m)|\,\mathrm{d}x\,\mathrm{d}t \le \left(\int_{D_{n,m,\varepsilon}} |\nabla(\theta_n - \theta_m)|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} (meas(D_{n,m,\varepsilon}))^{\frac{1}{2}}$$
$$\le C(2B+1)^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}.$$

Using the decomposition of $Q = D_{n,m,\varepsilon} \cup (Q \setminus D_{n,m,\varepsilon})$ we have to consider the integral over the second set.

$$\int_{Q\setminus D_{n,m,\varepsilon}} |\nabla(\theta_n - \theta_m)|\,\mathrm{d}x\,\mathrm{d}t \le \left(\int_{Q\setminus D_{n,m,\varepsilon}} |\nabla(\theta_n - \theta_m)|^q\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{q}} (meas(Q \setminus D_{n,m,\varepsilon}))^{1-\frac{1}{q}} \tag{3.1.8}$$

The first term on the right-hand side is bounded, since the sequence $\{\theta_n\}$ is uniformly bounded in $L^q(0,T,W^{1,q}(\Omega))$. The sequence $\{\theta_n\}$ is a Cauchy sequence in $L^1(0,T,L^1(\Omega))$, so there exists n_0 such that for all $n, m > n_0$ occur $(meas(Q \setminus D_{n,m,\varepsilon}))^{1-\frac{1}{q}} < \varepsilon$. Then, from the previous inequalities we obtain

$$\begin{aligned}\int_Q |\nabla\theta_n - \nabla\theta_m|\,\mathrm{d}x\,\mathrm{d}t &= \int_{D_{n,m,\varepsilon}} |\nabla(\theta_n - \theta_m)|\,\mathrm{d}x\,\mathrm{d}t + \int_{Q\setminus D_{n,m,\varepsilon}} |\nabla(\theta_n - \theta_m)|\,\mathrm{d}x\,\mathrm{d}t \\ &\le c_1\varepsilon^{\frac{1}{2}} + c_2\varepsilon\end{aligned} \tag{3.1.9}$$

which implies that $\{\nabla\theta_n\}$ is a Cauchy sequence in $L^1(0,T,L^1(\Omega))$.

□

Lemma 3.1.4 (Aubin-Lions, Lemma 7.7 from [65])**.** *Let V_1, V_2 be Banach spaces, and V_3 be a metrizable Hausdorff locally convex space, V_1 be separable and reflexive, $V_1 \subset\subset V_2$ (a compact embedding), $V_2 \subset V_3$ (a continuous embedding), $1 < p < \infty$, $1 \le q \le \infty$. Then $\{u : u \in L^p(0,T,V_1); u_t \in L^q(0,T,V_3)\} \subset\subset L^p(0,T,V_2)$ (a compact embedding).*

From the uniform boundedness of the sequence $\{f_k\}$ in $L^1(0,T,L^1(\Omega))$ and from the uniform boundedness of the sequence $\{\theta_k\}$ in $L^q(0,T,W^{1,q}(\Omega))$ we obtain that $\{(\theta_k)_t\}$ is a bounded sequence in the space $L^1(0,T,W^{-1,q}(\Omega))$. Consequently, the sequence $\{\theta_k\}$ is relatively compact in $L^1(0,T,L^1(\Omega))$. Due to Lemma 3.1.2 and Lemma 3.1.3 we know that the sequence $\{\theta_k\}$ converges strongly to θ in $L^q(0,T,W^{1,q}(\Omega))$. Moreover, $(\theta_k)_t$ converges strongly to θ_t in $L^1(0,T;W^{-2,2}(\Omega))$ by Rellich–Kondrachov's theorem. Thus, θ_k converges strongly to θ in $C([0,T],W^{-2,2}(\Omega))$ and $\theta_k(\cdot,0)$ converges to $\theta(\cdot,0)$ in $W^{-2,2}(\Omega)$.

Additionally, in Chapter 4 and in Chapter 5 existence of temperature θ gives us information about the convergence of right-hand sides of heat equations. Thus, we may pass to the limit in heat equation.

Lemma 3.1.5. *Let $f_k \rightharpoonup f$ weakly in $L^1(Q)$ and θ_0 belongs to $L^1(\Omega)$. Then, for $q < \frac{2(N+1)-N}{N+1}$ ($q < \frac{5}{4}$ when $N = 3$) there exists $\theta \in L^q(0,T,W^{1,q}(\Omega)) \cap C([0,T],W^{-2,2}(\Omega))$ - a solution to the system*

$$\begin{cases} \theta_t - \Delta\theta = f & \text{in } \Omega\times(0,T), \\ \frac{\partial\theta}{\partial \boldsymbol{n}} = 0 & \text{on } \partial\Omega\times(0,T), \\ \theta(x,0) = \theta_0(x) & \text{in } \Omega. \end{cases} \tag{3.1.10}$$

Proof. Choosing for (3.0.1) the test function $\psi \in C_c^\infty([\infty,T),C^\infty(\Omega))$, we get

$$\int_0^T\int_\Omega (\theta_n)_t\psi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \Delta\theta_n\psi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega f_n\psi\,\mathrm{d}x\,\mathrm{d}t.$$

Then

$$-\int_0^T\int_\Omega \theta_n\psi_t\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \theta_n\psi\,\mathrm{d}x\Big|_0^T + \int_0^T\int_\Omega \nabla\theta_n\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_{\partial\Omega}\frac{\partial\theta_n}{\partial\boldsymbol{n}}\psi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega f_n\psi\,\mathrm{d}x\,\mathrm{d}t.$$

And finally

$$-\int_0^T\int_\Omega \theta_n\psi_t\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \nabla\theta_n\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega f_n\psi\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega T_n(\theta_0)\psi\,\mathrm{d}x.$$

Using the convergence of the temperatures' sequence we obtain

$$-\int_0^T\int_\Omega \theta\psi_t + \int_0^T\int_\Omega \nabla\theta\cdot\nabla\psi = \int_0^T\int_\Omega f\psi + \int_\Omega \theta_0\psi.$$

□

Remark. *The solution θ obtained by Boccardo & Gallouët's approach is not unique.*

3.2 Renormalised approach

The second approach is to find the renormalised solution. The notion of renormalised solution for parabolic equation was introduced in [13, 14], but only for Dirichlet boundary conditions. Some proofs from [13, 14] require modification for the case of Neumann boundary conditions. Moreover, since the heat equation is one out of equations from the whole system, we obtain the result in two steps. Firstly, having only the uniform boundedness of right-hand sides we obtain the existence of temperature θ and almost pointwise convergence of approximate temperatures θ_k to θ as k tends to ∞. Again, we consider systems of equations

$$\begin{cases} (\theta_k)_t - \Delta\theta_k = f_k & \text{in } Q, \\ \frac{\partial\theta_k}{\partial \boldsymbol{n}} = 0 & \text{on } \partial\Omega\times(0,T), \\ \theta_k(t=0) = \theta_{k,0}, & \text{in } \Omega \end{cases} \tag{3.2.1}$$

where for every $k\in\mathbb{N}$ function f_k belongs to $L^2(Q)$, the sequence $\{f_k\}$ is uniformly bounded in $L^1(Q)$ and $\theta_{k,0}$ belongs to $L^2(\Omega)$ and strongly converges to θ_0 in $L^1(\Omega)$ as k tends to ∞.

Secondly, usage of pointwise convergence way prove the convergence of right-hand side of heat equation, see Chapter 6. This information is necessary to complete the prove of renormalised solution.

Definition 3.2.1 (Renormalised solution to heat equation)**.** *Let f belong to $L^1(Q)$ and θ_0 belong to $L^1(\Omega)$. A real-valued function θ defined on Q is a renormalised solution of heat equation* (3.2.1) *if*

a) θ is a measurable function such that $\mathcal{T}_K(\theta)$ belongs to $L^2(0,T,W^{1,2}(\Omega))$ for all positive K;

b) for all positive c

$$\mathcal{T}_{K+c}(\theta) - \mathcal{T}_K(\theta) \to 0 \tag{3.2.2}$$

in $L^2(0,T,W^{1,2}(\Omega))$ as K goes to ∞;

c) and $\theta(t=0)=\theta_0$.

Moreover, for all functions $S\in C^\infty(\mathbb{R})$, such that S' belongs to $C_0^\infty(\mathbb{R})$ (S' has a compact support), the following equality holds

$$\begin{aligned} -\int_Q S(\theta)\frac{\partial\phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega S(\theta_0)\phi(x,0)\,\mathrm{d}x + \int_Q S'(\theta)\nabla\theta\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \\ + \int_Q S''(\theta)|\nabla\theta|^2\phi\,\mathrm{d}x\,\mathrm{d}t = \int_Q fS'(\theta)\phi\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3.2.3}$$

where $\phi\in C_c^\infty([-\infty,T),C^\infty(\Omega))$.

We use the notation $\lim_{k,l\to\infty}$ when the order in the passing to the limit is not relevant, i.e.

$$\lim_{k,l\to\infty} F_{k,l} = \lim_{k\to\infty}\lim_{l\to\infty} F_{k,l} = \lim_{l\to\infty}\lim_{k\to\infty} F_{k,l}.$$

Our reasoning is different then the one presented in [13], e.g. we divide Lemma 1 from [13] into two lemmas. Firstly, our goal is to get the existence of temperature and its almost pointwise convergence in Q. Convergence of right-hand sides of heat equations is a consequence of existence of temperature and it requires some calculation, see Chapter 6. For different models we have different convergence, i.e. strong (for Mróz model) or weak (for Norton-Hoff-type model and model with growth conditions in generalized Orlicz spaces). These differences cause different properties of temperature, see Lemma 3.2.2.

Lemma 3.2.1. *Let us assume that the sequence $\{f_k\}$ is uniformly bounded in $L^1(Q)$. Then, there exists a subsequence of the sequence $\{\theta_k\}$ (still denoted by k) and measurable function θ, such that when k tends to ∞ and for any fixed positive real number K the following conditions are satisfied* *a) θ_k converges to θ almost everywhere in Q;*

b) $\mathcal{T}_K(\theta_k)$ converges weakly to $\mathcal{T}_K(\theta)$ in $L^2(0,T,W^{1,2}(\Omega))$.

Proof. Let us take $\mathcal{T}_K(\theta_k)$ as a test function in (3.2.1). Then for $t \in (0,T)$ it holds

$$\int_0^t \int_\Omega \frac{\partial \theta_k}{\partial t} \mathcal{T}_K(\theta_k)\,\mathrm{d}x\,\mathrm{d}t + \int_0^t \int_\Omega |\nabla \mathcal{T}_K(\theta_k)|^2\,\mathrm{d}x\,\mathrm{d}t = \int_0^t \int_\Omega f_k \mathcal{T}_k(\theta_k)\,\mathrm{d}x\,\mathrm{d}t, \tag{3.2.4}$$

and

$$\int_\Omega \tilde{\mathcal{T}}_K(\theta_k)(t)\,\mathrm{d}x + \int_0^t \int_\Omega |\nabla \mathcal{T}_K(\theta_k)|^2\,\mathrm{d}x\,\mathrm{d}t = \int_0^t \int_\Omega f_k \mathcal{T}_k(\theta_k)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \tilde{\mathcal{T}}_K(\theta_{k,0})\,\mathrm{d}x, \tag{3.2.5}$$

where $\tilde{\mathcal{T}}_K(r) = \int_0^r \mathcal{T}_K(z)\,\mathrm{d}z$ is a positive real valued function. Using definition of the truncation and linear growth of function $\tilde{\mathcal{T}}_K(r)$ at infinity, the following estimate holds

$$\int_\Omega \tilde{\mathcal{T}}_K(\theta_k)(t)\,\mathrm{d}x + \int_0^t \int_\Omega |\nabla \mathcal{T}_K(\theta_k)|^2\,\mathrm{d}x\,\mathrm{d}t \le K\|f\|_{L^1(Q)} + C(K)\|\theta_{k,0}\|_{L^1(\Omega)}. \tag{3.2.6}$$

To show that sequence $\{\mathcal{T}_K(\theta_k)\}$ is uniformly bounded in $L^2(0,T,W^{1,2}(\Omega))$, it is enough to estimate $\|\mathcal{T}_K(\theta_k)\|_{L^2(Q)}$ by $\|\tilde{\mathcal{T}}_K(\theta_k)\|_{L^1(Q)}$ and $\|\nabla\mathcal{T}_K(\theta_k)\|_{L^2(Q)}$. By Poincaré inequality we get

$$\begin{aligned} \|\mathcal{T}_K(\theta_k)\|_{L^2(Q)} &\le \|T_K(\theta_k) - (\mathcal{T}_K(\theta_k))_\Omega\|_{L^2(Q)} + \|(\mathcal{T}_K(\theta_k))_\Omega\|_{L^2(Q)} \\ &\le \|\nabla \mathcal{T}_K(\theta_k)\|_{L^2(Q)} + \|(\mathcal{T}_K(\theta_k))_\Omega\|_{L^2(Q)}, \end{aligned} \tag{3.2.7}$$

where by $(\mathcal{T}_K(\theta_k))_\Omega$ we denote the mean value. Using the definition of truncation operator we obtain

$$\tilde{\mathcal{T}}_K(\theta_k) = \begin{cases} \frac{1}{2}(\theta_k)^2 & |\theta_k| \le K, \\ \frac{1}{2}K^2 + K(|\theta_k| - K) & |\theta_k| > K, \end{cases} \tag{3.2.8}$$

and then it remains to show the estimates for $(\mathcal{T}_K(\theta_k))_\Omega$

$$\int_\Omega |\mathcal{T}_K(\theta_k)|^2\,\mathrm{d}x = \int_{\{x\in\Omega:|\theta_k|\le K\}} |\theta_k|^2\,\mathrm{d}x + \int_{\{x\in\Omega:|\theta_k|>K\}} K^2\,\mathrm{d}x \le 2\int_\Omega \tilde{\mathcal{T}}_K(\theta_k)\,\mathrm{d}x. \tag{3.2.9}$$

Finite measure of Q implies that sequence $\{\mathcal{T}_K(\theta_k)\}$ is uniformly bounded in $L^2(0,T,W^{1,2}(\Omega))$, which completes the proof. □

If we have the convergence of right-hand side functions of heat equation we may improve the result from Lemma 3.2.1.

Lemma 3.2.2. *Let us assume that the sequence $\{f_k\}$ converges weakly to f in $L^1(Q)$ with $k \to \infty$. For a subsequence $\{\theta_k\}$ from Lemma 3.2.1 and for any fixed positive real number K there exists the following limit*

$$\lim_{\eta,\varepsilon\to\infty} \int_Q |\nabla \mathcal{T}_K(\theta_k - \theta_l)|^2\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.2.10}$$

Moreover, if $\{f_k\}$ converges strongly to f then θ belongs to $C([0,T],L^1(\Omega))$.

Proof. To prove (3.2.10) we use $\mathcal{T}_K(\theta_k - \theta_l)$ as a test function for difference of two approximate equations (3.2.1), i.e.

$$\frac{\partial}{\partial t}(\theta_k - \theta_l) - \Delta(\theta_k - \theta_l) = f_k - f_l. \tag{3.2.11}$$

Thus, after integration over Ω and time interval $(0, T)$ we obtain

$$\begin{aligned} \int_\Omega \tilde{\mathcal{T}}_K(\theta_k - \theta_l)(T)\,\mathrm{d}x + &\int_Q |\nabla \mathcal{T}_K(\theta_k - \theta_l)|^2\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_Q (f_k - f_l)\mathcal{T}_K(\theta_k - \theta_l)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \tilde{\mathcal{T}}_K(\theta_{k,0} - \theta_{l,0})\,\mathrm{d}x. \end{aligned} \tag{3.2.12}$$

Positivity of the first term on the left-hand side in abovementioned equation, boundedness of $\mathcal{T}_K(\theta_k - \theta_l)$, weak convergences of the sequence $\{f_k\}$ and strong convergence of initial conditions imply that

$$\lim_{k,l\to\infty} \int_Q |\nabla \mathcal{T}_K(\theta_k - \theta_l)|^2\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.2.13}$$

Let us assume that $\{f_k\}$ converges strongly to f. For $\delta > 0$, let us test (3.2.11) by function $\frac{1}{\delta}\mathcal{T}_\delta(\theta_k - \theta_l)$. Then, we get

$$\begin{aligned} \frac{1}{\delta}\int_\Omega \tilde{\mathcal{T}}_\delta(\theta_k - \theta_l)(t)\,\mathrm{d}x + \frac{1}{\delta}&\int_0^t\int_\Omega |\nabla \mathcal{T}_\delta(\theta_k - \theta_l)|^2\,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{1}{\delta}\int_0^t\int_\Omega (f_k - f_l)\mathcal{T}_\delta(\theta_k - \theta_l)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\delta}\int_\Omega \tilde{\mathcal{T}}_\delta(\theta_{k,0} - \theta_{l,0})\,\mathrm{d}x. \end{aligned} \tag{3.2.14}$$

Using the positivity of the second term of the left-hand side we obtain

$$\frac{1}{\delta}\int_\Omega \tilde{\mathcal{T}}_\delta(\theta_k - \theta_l)(t)\,\mathrm{d}x \le \int_0^t\int_\Omega |f_k - f_l|\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\delta}\int_\Omega \tilde{\mathcal{T}}_\delta(\theta_{k,0} - \theta_{l,0})\,\mathrm{d}x. \tag{3.2.15}$$

Passing to the limit with δ tends to 0 we obtain

$$\begin{aligned} \lim_{\delta\to 0}\frac{1}{\delta}\int_\Omega \tilde{\mathcal{T}}_\delta(\theta_k - \theta_l)(t)\,\mathrm{d}x &= \int_\Omega (\theta_k - \theta_l)(t)\,\mathrm{d}x, \\ \lim_{\delta\to 0}\frac{1}{\delta}\int_\Omega \tilde{\mathcal{T}}_\delta(\theta_{k,0} - \theta_{l,0})\,\mathrm{d}x &= \int_\Omega (\theta_{k,0} - \theta_{l,0})\,\mathrm{d}x. \end{aligned} \tag{3.2.16}$$

Therefore,

$$\int_\Omega (\theta_k - \theta_l)(t)\,\mathrm{d}x \le \int_0^t\int_\Omega |f_k - f_l|\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega (\theta_{k,0} - \theta_{l,0})\,\mathrm{d}x, \tag{3.2.17}$$

and we conclude that the sequence $\{\theta_k\}$ is a Cauchy sequence in $C([0,T], L^1(\Omega))$, hence there exist $\theta \in C([0,T], L^1(\Omega))$, such that $\theta_k \to \theta$ in $C([0,T], L^1(\Omega))$ as k tends to ∞.

□

Let us take any $T' > T$ and let us extend f_k by 0 on $\Omega \times (0, T')$. Then we denote $Q' = \Omega \times (T, T')$ and we consider the following problem

$$\begin{cases} \frac{\partial \theta_k}{\partial t} - \Delta\theta_l = f_k & \text{in } Q', \\ \frac{\partial \theta_k}{\partial \boldsymbol{n}} = 0 & \text{on } \partial\Omega \times (0, T'), \\ \theta_k(t = 0) = \theta_{k,0} & \text{in } \Omega. \end{cases} \tag{3.2.18}$$

Thus, we know that there exists an unique solution $\widehat{\theta_k}$ and $\widehat{\theta_k} = \theta_k$ a.e. on Q. Hence, later we do not distinguish these two solutions.

Now, our goal is to prove that the convergence of a sequence $\{\mathcal{T}_K(\theta_k)\}$ is strong. For this purpose we start with auxiliary Lemma.

Lemma 3.2.3 (Lemma 2 from [13]). *Let H and Z be two real valued functions which belong to $W^{2,\infty}(\mathbb{R})$ such that H' and Z' have compact supports with $Z(0) = Z'(0) = 0$. Then*

$$\lim_{k,l\to\infty} \int_{Q'} (T' - t)H''(\theta_k)Z(\theta_k - \theta_l)|\nabla\theta_k|^2 \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.2.19}$$

Proof. Let θ_k, θ_l be two different solutions to problem (3.2.18). Then taking the difference of these equations we get

$$(\theta_k - \theta_l)_t - \Delta(\theta_k - \theta_l) = f_k - f_l. \tag{3.2.20}$$

Since Z' and H are Lipschitz bounded functions with $Z'(0) = 0$, $Z'(\theta_k - \theta_l)H(\theta_k)$ may be used as a test function in (3.2.20). Thus,

$$\begin{aligned}\int_0^{T'}\int_0^{\tau}\int_\Omega (\theta_k - \theta_l)_t Z'(\theta_k - \theta_l)H(\theta_k)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau& \\ + \int_0^{T'}\int_0^{\tau}\int_\Omega (\nabla\theta_k - \nabla\theta_l)\cdot\nabla\left(Z'(\theta_k - \theta_l)H(\theta_k)\right)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau& \\ = \int_0^{T'}\int_0^{\tau}\int_\Omega (f_k - f_l)\, Z'(\theta_k - \theta_l)H(\theta_k)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau.&\end{aligned} \tag{3.2.21}$$

Choosing now $Z(\theta_k - \theta_l)H'(\theta_k)$ as a test function in the equation $(3.2.18)_{(1)}$ for θ_k gives

$$\begin{aligned}\int_0^{T'}\int_0^{\tau}\int_\Omega (\theta_k)_t Z(\theta_k - \theta_l)H'(\theta_k)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau + \int_0^{T'}\int_0^{\tau}\int_\Omega \nabla\theta_k\cdot\nabla\left(Z(\theta_k - \theta_l)H'(\theta_k)\right)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau& \\ = \int_0^{T'}\int_0^{\tau}\int_\Omega f_k Z(\theta_k - \theta_l)H'(\theta_k)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau.&\end{aligned} \tag{3.2.22}$$

Since $\theta_k \to \theta$ almost pointwise in Q' and Z, H, Z', H' are Lipschitz bounded functions, $Z(\theta_k - \theta_l)H'(\theta_k)$ and $Z'(\theta_k - \theta_l)H(\theta_k)$ both converge to 0 almost pointwise as k and l go to ∞. Moreover, using the weak convergence of $\{f_k\}$ in $L^1(Q)$, we conclude that both right-hand sides of (3.2.21) and (3.2.22) tend to 0 as k and l go to ∞.

Integration by parts results in

$$\begin{aligned}\int_0^{T'}\int_0^{\tau}\int_\Omega (\theta_k)_t Z(\theta_k - \theta_l)H'(\theta_k)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau &= \int_0^{T'}\int_0^{\tau}\int_\Omega (H(\theta_k))_t Z(\theta_k - \theta_l)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau \\ &= -\int_0^{T'}\int_0^{\tau}\int_\Omega H(\theta_k)(Z(\theta_k - \theta_l))_t\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau \\ &\quad + \int_0^{T'}\left[\int_\Omega Z(\theta_k - \theta_l)H(\theta_k)\,\mathrm{d}x\right]_0^{\tau}\mathrm{d}\tau \\ &= -\int_0^{T'}\int_0^{\tau}\int_\Omega (\theta_k - \theta_l)_t Z'(\theta_k - \theta_l)H(\theta_k)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau \\ &\quad + \int_{Q'} Z(\theta_k - \theta_l)H(\theta_k)\,\mathrm{d}x\,\mathrm{d}\tau - T'\int_\Omega Z(\theta_{k,0} - \theta_{l,0})H(\theta_{k,0})\,\mathrm{d}x.\end{aligned} \tag{3.2.23}$$

We know that $\theta_k \to \theta$ a.e. in Q' and functions Z, H belong to $W^{2,\infty}(\mathbb{R})$. Thus, it provides to

$$\begin{aligned}\lim_{k,l\to\infty}\int_{Q'} Z(\theta_k - \theta_l)H(\theta_k)\,\mathrm{d}x\,\mathrm{d}\tau = 0,& \\ \lim_{k,l\to\infty} T\int_\Omega Z(\theta_{k,0} - \theta_{l,0})H(\theta_{k,0})\,\mathrm{d}x = 0.&\end{aligned} \tag{3.2.24}$$

And then we obtain

$$\lim_{k,l\to\infty}\int_0^{T'}\int_0^{\tau}\int_\Omega(\theta_k)_tZ(\theta_k-\theta_l)H'(\theta_k)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau=-\lim_{k,l\to\infty}\int_0^{T'}\int_0^{\tau}\int_\Omega(\theta_k-\theta_l)_tZ'(\theta_k-\theta_l)H(\theta_k)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\tau. \tag{3.2.25}$$

Vanishing of right-hand sides of (3.2.21), (3.2.22) and equality (3.2.25) provides that

$$\begin{aligned}\lim_{k,l\to\infty}\int_0^{T'}\int_\Omega\int_0^{\tau}&((\nabla\theta_k-\nabla\theta_l)\cdot\nabla\left(Z'(\theta_k-\theta_l)H(\theta_k)\right)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\tau\\&=-\lim_{k,l\to\infty}\int_0^{T'}\int_\Omega\int_0^{\tau}\nabla\theta_k\cdot\nabla\left(Z(\theta_k-\theta_l)H'(\theta_k)\right)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned} \tag{3.2.26}$$

In the abovementioned equation we have the following situation. We consider limits in the form of $\int_0^{T'}\int_\Omega\int_0^\tau a(x,t)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\tau$. Using Fubini theorem we may change the order of integration and we obtain

$$\int_0^{T'}\int_\Omega\int_0^{\tau}a(x,t)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\tau=\int_0^{T'}\int_\Omega\int_t^{T'}\mathrm{d}\tau\;a(x,t)\,\mathrm{d}x\,\mathrm{d}t=\int_0^{T'}\int_\Omega(T'-t)a(x,t)\,\mathrm{d}x\,\mathrm{d}t \tag{3.2.27}$$

Thus, two terms in (3.2.26) may be rewritten in the form of

$$\begin{aligned}&\int_0^{T'}\int_\Omega\int_0^{\tau}(\nabla\theta_k-\nabla\theta_l)\cdot\nabla\left(Z'(\theta_k-\theta_l)H(\theta_k)\right)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\tau\\&\quad=\underbrace{\int_{Q'}(T'-t)Z''(\theta_k-\theta_l)H(\theta_k)|\nabla(\theta_k-\theta_k)|^2\,\mathrm{d}x\,\mathrm{d}t}_{=E_K^{k,l}}+\underbrace{\int_{Q'}(T'-t)Z'(\theta_k-\theta_l)H'(\theta_k)\nabla(\theta_k-\theta_k)\cdot\nabla\theta_k\,\mathrm{d}x\,\mathrm{d}t}_{=F_K^{k,l}},\end{aligned} \tag{3.2.28}$$

and

$$\begin{aligned}&\int_0^{T'}\int_\Omega\int_0^{\tau}\nabla\theta_k\cdot\nabla\left(Z(\theta_k-\theta_l)H'(\theta_k)\right)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\tau\\&\quad=\underbrace{\int_{Q'}(T'-t)Z'(\theta_k-\theta_l)H'(\theta_k)\nabla(\theta_k-\theta_k)\cdot\nabla\theta_k\,\mathrm{d}x\,\mathrm{d}t}_{=F_K^{k,l}}+\underbrace{\int_{Q'}(T'-t)Z(\theta_k-\theta_l)H''(\theta_k)|\nabla\theta_k|^2\,\mathrm{d}x\,\mathrm{d}t}_{=G_K^{k,l}}.\end{aligned} \tag{3.2.29}$$

The next step of the proof is to show that $E_K^{k,l}$ and $F_K^{k,l}$ converge to zero as k and l go to ∞. Since (3.2.26) holds, it will imply that $G_K^{k,l}$ converges to zero as k and l go to ∞. Let us take a positive real number M such that $\mathrm{supp}(H')\subset[-M,+M]$ and $\mathrm{supp}(Z')\subset[-M,+M]$. Then, the following estimate holds

$$E_K^{k,l}\le T\|Z''\|_{L^\infty(\mathbb{R})}\|H\|_{L^\infty(\mathbb{R})}\int_{Q'}|\nabla\mathcal{T}_M(\theta_k-\theta_l)|^2\,\mathrm{d}x\,\mathrm{d}t \tag{3.2.30}$$

Using Lemma 3.2.2 we obtain

$$\lim_{k,l\to\infty}E_K^{k,l}=0. \tag{3.2.31}$$

Similarly, $F_K^{\varepsilon;\eta}$ is estimated as follows

$$F_K^{k,l}\le T\|Z'\|_{L^\infty(\mathbb{R})}\|H'\|_{L^\infty(\mathbb{R})}\int_{Q'}|\nabla\mathcal{T}_M(\theta_k-\theta_k)||\nabla\mathcal{T}_M(\theta_k)|\,\mathrm{d}x\,\mathrm{d}t. \tag{3.2.32}$$

Using Hölder's inequality and again Lemma 3.2.2 we get

$$\lim_{k,l\to\infty} F_K^{k,l} = 0, \tag{3.2.33}$$

which completes the proof. □

Now, we pass to the main theorem of this section.

Theorem 3.2.1. *Let K be a fixed positive real number. The sequence $\{\mathcal{T}_K(\theta_k)\}$ strongly converges to $\mathcal{T}_K(\theta)$ in $L^2(0,T,W^{1,2}(\Omega))$.*

Proof. The main point of the following proof is to show that

$$\lim_{k\to\infty}\int_{Q'}(T'-t)|\nabla\mathcal{T}_K(\theta_k)-\nabla\mathcal{T}_K(\theta)|^2\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.2.34}$$

Since θ_k is the unique solution of the problem (3.2.18) it is obvious that

$$\lim_{k\to\infty}\int_{Q'}(T'-t)|\nabla\mathcal{T}_K(\theta_k)-\nabla\mathcal{T}_K(\theta)|^2 \geq (T'-T)\lim_{k\to\infty}\int_{Q}|\nabla\mathcal{T}_K(\theta_k)-\nabla\mathcal{T}_K(\theta)|^2 \geq 0. \tag{3.2.35}$$

Let us start with decomposition of set Q' into four subsets:

- $\{(x,t)\in Q':\ |\theta_k|<K\}\cap\{(x,t)\in Q':\ |\theta_l|<K\}$;
- $\{(x,t)\in Q':\ |\theta_k|<K\}\cap\{(x,t)\in Q':\ |\theta_l|\geq K\}$;
- $\{(x,t)\in Q':\ |\theta_k|\geq K\}\cap\{(x,t)\in Q':\ |\theta_l|<K\}$;
- and $\{(x,t)\in Q':\ |\theta_k|\geq K\}\cap\{(x,t)\in Q':\ |\theta_l|\geq K\}$.

Applying this decomposition and using the truncation operator we obtain

$$\begin{aligned}
&\int_{Q'}(T'-t)|\nabla\mathcal{T}_K(\theta_k)-\nabla\mathcal{T}_K(\theta_l)|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\underbrace{\int_{\{(x,t)\in Q':\ |\theta_k|<K\}\cap\{(x,t)\in Q':\ |\theta_l|<K\}}(T'-t)|\nabla\theta_k-\nabla\theta_l|^2\,\mathrm{d}x\,\mathrm{d}t}_{=A_K^{k,l}}\\
&\qquad+\underbrace{\int_{\{(x,t)\in Q':\ |\theta_k|<K\}\cap\{(x,t)\in Q':\ |\theta_l|\geq K\}}(T'-t)|\nabla\theta_k|^2\,\mathrm{d}x\,\mathrm{d}t}_{=B_K^{k,l}}\\
&\qquad+\underbrace{\int_{\{(x,t)\in Q':\ |\theta_k|\geq K\}\cap\{(x,t)\in Q':\ |\theta_l|<K\}}(T'-t)|\nabla\theta_l|^2\,\mathrm{d}x\,\mathrm{d}t}_{=B_K^{k,l}}\\
&\qquad+\underbrace{\int_{\{(x,t)\in Q':\ |\theta_k|\geq K\}\cap\{(x,t)\in Q':\ |\theta_l|\geq K\}}(T'-t)0\,\mathrm{d}x\,\mathrm{d}t}_{=0}\,.
\end{aligned} \tag{3.2.36}$$

We may observe that $B_K^{k,l}$ and $B_K^{k,l}$ are symmetric with respect to k and l. Our goal is to show that $A_K^{k,l}$ and $B_K^{k,l}$ converge to 0 with $k,l\to\infty$.

The term $A_K^{k,l}$ is easily estimated since

$$0 \le A_K^{k,l} \le \int_{\{(x,t)\in Q':\ |\theta_k-\theta_l|<2K\}} (T'-t)|\nabla(\theta_k-\theta_l)|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_{Q'} (T'-t)|\nabla \mathcal{T}_{2K}(\theta_k-\theta_l)|^2 \,\mathrm{d}x\,\mathrm{d}t \le T' \int_{Q'} |\nabla \mathcal{T}_{2K}(\theta_k-\theta_l)|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{3.2.37}$$

Using Lemma 3.2.2 we conclude that

$$\lim_{k,l\to\infty} A_K^{k,l} = 0. \tag{3.2.38}$$

The next step of the proof is to show the estimates for $B_K^{k,l}$. Let K' be any positive real number. Then we split the set $\{(x,t)\in Q':\ |\theta_k|<K\}\cap\{(x,t)\in Q':\ |\theta_l|\ge K\}$ into two subsets $\{(x,t)\in Q':\ |\theta_k|<K\}\cap\{(x,t)\in Q':\ |\theta_l|\ge K\}\cap\{(x,t)\in Q':\ |\theta_k-\theta_l|\le K'\}$ and $\{(x,t)\in Q':\ |\theta_k|<K\}\cap\{(x,t)\in Q':\ |\theta_l|\ge K\}\cap\{(x,t)\in Q':\ |\theta_k-\theta_l|> K'\}$. Thus

$$B_K^{k,l} = \overbrace{\int_{\{(x,t)\in Q':\ |\theta_k|<K\}\cap\{(x,t)\in Q':\ |\theta_l|\ge K\}\cap\{(x,t)\in Q':\ |\theta_k-\theta_l|\le K'\}} (T'-t)|\nabla\theta_k|^2 \,\mathrm{d}x\,\mathrm{d}t}^{=B_1^{k,l}} + \underbrace{\int_{\{(x,t)\in Q':\ |\theta_k|<K\}\cap\{(x,t)\in Q':\ |\theta_l|\ge K\}\cap\{(x,t)\in Q':\ |\theta_k-\theta_l|> K'\}} (T'-t)|\nabla\theta_k|^2 \,\mathrm{d}x\,\mathrm{d}t}_{=B_2^{k,l}}. \tag{3.2.39}$$

Moreover, the following inequality holds

$$\begin{aligned}
0 \le B_1^{k,l} &= \int_{\{(x,t)\in Q':\ |\theta_k|<K\}\cap\{(x,t)\in Q':\ |\theta_l|\ge K\}\cap\{(x,t)\in Q':\ |\theta_k-\theta_l|\le K'\}} (T'-t)|\nabla\theta_k|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\{|\theta_k|<K\}\cap\{|\theta_l|\ge K\}\cap\{|\theta_k-\theta_l|\le K'\}} (T'-t)\big(|\nabla(\theta_k-\theta_l)|^2 + 2\nabla\theta_k\cdot\nabla\theta_l - |\nabla\theta_l|^2\big) \,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_{\{|\theta_k|<K\}\cap\{K\le|\theta_l|<K+K'\}} (T'-t)\big(|\nabla(\theta_k-\theta_l)|^2 + 2\nabla\theta_k\cdot\nabla\theta_l - |\nabla\theta_l|^2\big) \,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_{\{(x,t)\in Q':\ |\theta_k|<K\}\cap\{(x,t)\in Q':\ K\le|\theta_l|<K+K'\}} (T'-t)|\nabla(\theta_k-\theta_l)|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + 2\int_{\{(x,t)\in Q':\ |\theta_k|<K\}\cap\{(x,t)\in Q':\ K\le|\theta_l|<K+K'\}} (T'-t)\nabla\theta_k\cdot\nabla\theta_l \,\mathrm{d}x\,\mathrm{d}t \\
&\le T'\int_{Q'} |\nabla\mathcal{T}_{2K+K'}(\theta_k-\theta_l)|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + 2\int_{\{(x,t)\in Q':\ |\theta_k|<K\}\cap\{(x,t)\in Q':\ K\le|\theta_l|<K+K'\}} (T'-t)\nabla\theta_k\cdot\nabla\theta_l \,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{3.2.40}$$

By Lemma 3.2.2 the first term in the right-hand side of (3.2.40) converges to 0 when k and l go to ∞. To deal with the second one let us define the following function

$$\Theta_K^{K'}(s) = \begin{cases} 0 & |s|\le K, \\ s - K\mathrm{sgn}(s) & K\le|s|\le K', \\ K'\mathrm{sgn}(s) & K'\le|s|. \end{cases} \tag{3.2.41}$$

Thus, the second term of right-hand side of (3.2.40) is equal to

$$\int_{\{(x,t)\in Q':\ |\theta_k|<K\}\cap\{(x,t)\in Q':\ K\le|\theta_l|<K+K'\}} (T'-t)\nabla\theta_k\cdot\nabla\theta_l \,\mathrm{d}x\,\mathrm{d}t = \int_{Q'} (T'-t)\nabla\mathcal{T}_K(\theta_k)\nabla\Theta_K^{K'}(\theta_l)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.2.42}$$

Through the use of $\Theta_K^{K'}(\theta_l)$ as a test function in the equation (3.2.18) for θ_l we obtain

$$\int_\Omega \tilde{\Theta}_K^{K'}(\theta_l)(t)\,\mathrm{d}x + \int_0^t\int_\Omega |\nabla\Theta_K^{K'}(\theta_l)|^2\,\mathrm{d}x\,\mathrm{d}t = \int_0^t\int_\Omega f^\varepsilon\Theta_K^{K'}(\theta_l)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \tilde{\Theta}_K^{K'}(\theta_{k,0})\,\mathrm{d}x, \tag{3.2.43}$$

where $\tilde{\Theta}_K^{K'}(s) = \int_0^s \Theta_K^{K'}(\tau)\,\mathrm{d}\tau$ is a positive real valued function. This provides that the sequence $\{\nabla\Theta_K^{K'}(\theta_l)\}$ is uniformly bounded in $L^2(Q')$ (for any fixed K and K'). Using the same argumentation as in the proof of Lemma 3.2.1 we obtain that the sequence $\{\Theta_K^{K'}(\theta_l)\}$ is uniformly bounded in $L^2(0,T',W^{1,2}(\Omega))$.

Convergence of sequence $\{\theta_l\}$ to θ almost everywhere with $l\to\infty$ implies that sequence $\{\Theta_K^{K'}(\theta_l)\}$ converges weakly to $\Theta_K^{K'}(\theta)$ in $L^2(0,T,W^{1,2}(\Omega))$. Passing to the limit with $l\to\infty$ and then $k\to\infty$ (or reversibly, since k and l are independent) we obtain

$$\lim_{k\to\infty}\lim_{l\to\infty}\int_{Q'}(T'-t)\nabla\mathcal{T}_K(\theta_k)\nabla\Theta_K^{K'}(\theta_l)\,\mathrm{d}x\,\mathrm{d}t = \int_{Q'}(T'-t)\nabla\mathcal{T}_K(\theta)\nabla\Theta_K^{K'}(\theta)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.2.44}$$

Furthermore, it is obvious that $\mathcal{T}_K(s) = \mathcal{T}_K(\mathcal{T}_{K+1}(s))$ and $\Theta_K^{K'}(s) = \Theta_K^{K'}(\mathcal{T}_{K+K'}(s))$. Applying those equalities to (3.2.44) and using the chain rule we get

$$\int_{Q'}\nabla\mathcal{T}_K(\theta)\nabla\Theta_K^{K'}(\theta)\,\mathrm{d}x\,\mathrm{d}t = \int_{Q'}\mathcal{T}_K'(\theta)(\Theta_K^{K'})'(\theta)\nabla\mathcal{T}_{K+1}(\theta)\nabla\mathcal{T}_{K+K'}(\theta)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.2.45}$$

Due to definition of $\mathcal{T}_K$ and $\Theta_K^{K'}$ the function $\mathcal{T}_K'(\theta)(\Theta_K^{K'})'(\theta)\nabla\mathcal{T}_{K+1}(\theta)\nabla\mathcal{T}_{K+K'}(\theta)$ is equal to 0 a.e. in Q. This information, (3.2.40), (3.2.42) and (3.2.44) imply that

$$\lim_{k,l\to\infty} B_1^{k,l} = 0. \tag{3.2.46}$$

To finish the proof it is enough to show that $B_2^{k,l}$ goes to 0 with $k,l\to\infty$. For this purpose we use Lemma 3.2.3. The natural choice of $H''(\theta_k)$ in (3.2.19) is $|\mathcal{T}_K'(\theta_k)|^2$. Unfortunately, this is not the proper choice, because if $H''(\theta_k) = |\mathcal{T}_K'(\theta_k)|^2$ then $H(\theta_k)$ does not belong to $W^{2,\infty}(\mathbb{R})$. Hence, for positive δ let us define the function

$$(H_K^\delta)''(s) = \begin{cases} 1 & |s|<K, \\ -K\delta & K<|s|<K+\frac{1}{\delta}, \\ 0 & K+\frac{1}{\delta}. \end{cases} \tag{3.2.47}$$

Deriving $H_K^\delta(s)$ from (3.2.47), together with the conditions $H_K^\delta(0) = (H_K^\delta)'(0) = 0$ we obtain the function which belongs to $W^{2,\infty}(\mathbb{R})$ and its support $\mathrm{supp}((H_K^\delta)')$ is contained in the interval $[-K-\frac{1}{\delta}, K+\frac{1}{\delta}]$. Moreover, the sequence $\{(H_K^\delta)''\}$ converges to $(\mathcal{T}_K')^2$ as δ goes to zero. Thus, using (3.2.47) in (3.2.19) we get

$$\lim_{k,l\to\infty}\int_{\{(x,t)\in Q':\ |\theta_k|\le K\}}(T'-t)(H_K^\delta(\theta_l))''Z(\theta_k-\theta_l)|\nabla\theta_k|^2 = 0 \tag{3.2.48}$$

and then

$$\lim_{k,l\to\infty}\int_{\{(x,t)\in Q':\ |\theta_k|\le K\}}(T'-t)Z(\theta_k-\theta_l)|\nabla\theta_k|^2\,\mathrm{d}x\,\mathrm{d}t = K\delta\lim_{k,l\to\infty}\int_{\{(x,t)\in Q':\ K\le|\theta_k|\le K+\frac{1}{\delta}\}}(T'-t)Z(\theta_k-\theta_l)|\nabla\theta_k|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{3.2.49}$$

for any δ. If δ tends to zero in (3.2.49) then

$$\lim_{k,l\to\infty}\int_{\{(x,t)\in Q':\ |\theta_k|\le K\}}(T'-t)Z(\theta_k-\theta_l)|\nabla\theta_k|^2\,\mathrm{d}x\,\mathrm{d}t = K\lim_{\delta\to 0}\left(\delta\lim_{k,l\to\infty}\int_{\{(x,t)\in Q':\ K\le|\theta_k|\le K+\frac{1}{\delta}\}}(T'-t)Z(\theta_k-\theta_l)|\nabla\theta_k|^2\,\mathrm{d}x\,\mathrm{d}t\right). \tag{3.2.50}$$

Furthermore, it holds that

$$\left|\delta\lim_{k,l\to\infty}\int_{\{(x,t)\in Q':\ K\le|\theta_k|\le K+\frac{1}{\delta}\}}(T'-t)Z(\theta_k-\theta_l)|\nabla\theta_k|^2\,\mathrm{d}x\,\mathrm{d}t\right| \le \delta T'\|Z\|_{L^\infty(\mathbb{R})}\int_{\{(x,t)\in Q':\ K\le|\theta_k|\le K+\frac{1}{\delta}\}}|\nabla\theta_k|^2\,\mathrm{d}x\,\mathrm{d}t \tag{3.2.51}$$

To estimate the right-hand side of (3.2.51) let us use $\delta\Theta_K^{\frac{1}{\delta}}(\theta_k)$ (where $\Theta_K^{\frac{1}{\delta}}(\theta_k)$ is defined in (3.2.41)) as a test function in (3.2.18). We obtain

$$\delta\int_\Omega\tilde\Theta_K^{\frac{1}{\delta}}(\theta_k)(T')\,\mathrm{d}x+\delta\int_{\{(x,t)\in Q':\ K\le|\theta_k|\le K+\frac{1}{\delta}\}}|\nabla\theta_k|^2\,\mathrm{d}x\,\mathrm{d}t = \delta\int_{Q'}f^\varepsilon\Theta_K^{\frac{1}{\delta}}(\theta_k)\,\mathrm{d}x\,\mathrm{d}t+\delta\int_\Omega\tilde\Theta_K^{\frac{1}{\delta}}(\theta_{k,0})\,\mathrm{d}x \tag{3.2.52}$$

where $\tilde\Theta_K^{\frac{1}{\delta}}(s)=\int_0^s\Theta_K^{\frac{1}{\delta}}(\tau)\,\mathrm{d}\tau$ is a positive real valued function with linear growth at infinity. Moreover, almost pointwise convergence of $\{\theta_k\}$ and weak convergence of $\{f_k\}$ in $L^1(Q')$ imply that

$$\lim_{k\to\infty}\int_{Q'}f_k\Theta_K^{\frac{1}{\delta}}(\theta_k)\,\mathrm{d}x\,\mathrm{d}t=\int_{Q'}f\Theta_K^{\frac{1}{\delta}}(\theta)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.2.53}$$

Using strong convergence of initial condition $\theta_{k,0}$ to θ_0 in $L^1(\Omega)$ and (3.2.53) in (3.2.52) we obtain

$$\delta\lim_{k\to\infty}\int_{\{(x,t)\in Q':\ K\le|\theta_k|\le K+\frac{1}{\delta}\}}|\nabla\theta_k|^2\,\mathrm{d}x\,\mathrm{d}t\le\delta\int_{Q'}f\Theta_K^{\frac{1}{\delta}}(\theta)\,\mathrm{d}x\,\mathrm{d}t+\delta\int_\Omega\tilde\Theta_K^{\frac{1}{\delta}}(\theta_0)\,\mathrm{d}x \tag{3.2.54}$$

Sequences $\{\delta\Theta_K^{\frac{1}{\delta}}(\theta)\}$ and $\{\delta\tilde\Theta_K^{\frac{1}{\delta}}(\theta_0)\}$ converge almost pointwise to 0 as δ goes to zero. Moreover, terms on the right-hand side of (3.2.54) are uniformly bounded, thus we conclude that right-hand side of (3.2.54) convergence to 0. Then

$$\lim_{\delta\to 0}\left[\delta\lim_{k\to\infty}\int_{\{(x,t)\in Q':\ K\le|\theta_k|\le K+\frac{1}{\delta}\}}|\nabla\theta_k|^2\,\mathrm{d}x\,\mathrm{d}t\right]=0. \tag{3.2.55}$$

According to (3.2.50) and (3.2.51) we deduce that

$$\lim_{k,l\to\infty}\int_{\{(x,t)\in Q'\colon\ |\theta_k|\le K\}}(T-t)Z(\theta_k-\theta_l)|\nabla\theta_k|^2\,\mathrm{d}x\,\mathrm{d}t=0 \tag{3.2.56}$$

Now, let us take $Z\in W^{2,\infty}(\mathbb{R})$ such that $Z(0)=Z'(0)=0$, Z is positive and $Z(s)=1$ for $|s|>K'$. This provides to equation

$$\lim_{k,l\to\infty}\int_{\{(x,t)\in Q'\colon\ |\theta_k|\le K\}\cap\{(x,t)\in Q'\colon\ |\theta_k-\theta_l|>K'\}}(T'-t)|\nabla\theta_k|^2\,\mathrm{d}x\,\mathrm{d}t=0, \tag{3.2.57}$$

which immediately leads to

$$\lim_{k,l\to\infty}B_K^{k,l}=0. \tag{3.2.58}$$

We decomposed Q' into a few subsets and showed that each of them converges to 0 with k,l going to ∞. Thus, we conclude that

$$\lim_{k,l\to\infty}\int_{Q'}(T'-t)|\nabla\mathcal{T}_K(\theta_k)-\nabla\mathcal{T}_K(\theta_l)|^2\,\mathrm{d}x\,\mathrm{d}t=0, \tag{3.2.59}$$

and it provides to

$$\begin{aligned}\lim_{k,l\to\infty}&\int_{Q'}(T'-t)|\nabla\mathcal{T}_K(\theta_k)-\nabla\mathcal{T}_K(\theta_l)|^2\,\mathrm{d}x\,\mathrm{d}t\\&=\lim_{k,l\to\infty}\int_{Q'}(T'-t)|\nabla\mathcal{T}_K(\theta_k)|^2\,\mathrm{d}x\,\mathrm{d}t-2\lim_{k,l\to\infty}\int_{Q'}(T'-t)\nabla\mathcal{T}_K(\theta_k)\cdot\nabla\mathcal{T}_K(\theta_l)^2\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\lim_{k,l\to\infty}\int_{Q'}(T'-t)|\nabla\mathcal{T}_K(\theta_l)|^2\,\mathrm{d}x\,\mathrm{d}t=0.\end{aligned} \tag{3.2.60}$$

Since k and l are independent and the sequence $\{\nabla\mathcal{T}_K(\theta_k)\}$ is weakly convergent in $L^2(Q')$, we rewrite the abovementioned equation as follows

$$\lim_{k\to\infty}\int_{Q'}(T'-t)|\nabla\mathcal{T}_K(\theta_k)|^2\,\mathrm{d}x\,\mathrm{d}t=\int_{Q'}(T'-t)|\nabla\mathcal{T}_K(\theta)|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{3.2.61}$$

Through the use of weak convergence of $\{\nabla\mathcal{T}_K(\theta_k)\}$ and (3.2.61), we get

$$\begin{aligned}\lim_{k\to\infty}&\int_{Q'}(T'-t)|\nabla\mathcal{T}_K(\theta_k)-\nabla\mathcal{T}_K(\theta)|^2\,\mathrm{d}x\,\mathrm{d}t\\&=\lim_{k\to\infty}\int_{Q'}(T'-t)|\nabla\mathcal{T}_K(\theta_k)|^2\,\mathrm{d}x\,\mathrm{d}t-2\lim_{\varepsilon\to0}\int_{Q'}(T'-t)\nabla\mathcal{T}_K(\theta_k)\cdot\nabla\mathcal{T}_K(\theta)\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_{Q'}(T'-t)|\nabla\mathcal{T}_K(\theta)|^2\,\mathrm{d}x\,\mathrm{d}t\\&=2\int_{Q'}(T'-t)|\nabla\mathcal{T}_K(\theta)|^2\,\mathrm{d}x\,\mathrm{d}t-2\int_{Q'}(T'-t)|\nabla\mathcal{T}_K(\theta)|^2\,\mathrm{d}x\,\mathrm{d}t=0,\end{aligned} \tag{3.2.62}$$

which completes the proof.

□

To finish the proof of existence of renormalised solution it is enough to prove that (3.2.2) and (3.2.3) hold. Let us start with the proof of (3.2.2).

Lemma 3.2.4. *For all positive c it holds that*

$$\mathcal{T}_{K+c}(\theta) - \mathcal{T}_K(\theta) \to 0 \tag{3.2.63}$$

in $L^2(0,T,W^{1,2}(\Omega))$ as K goes to ∞.

Proof. Let c be a positive number. We use test function $\mathcal{T}_{K+c}(\theta_k) - \mathcal{T}_K(\theta_k)$ as a test function in (3.2.1). Then,

$$\begin{aligned}\int_Q (\theta_k)_t(\mathcal{T}_{K+c}(\theta_k) - \mathcal{T}_K(\theta_k))\,\mathrm{d}x\,\mathrm{d}t + \int_Q \nabla\theta_k\cdot\nabla(\mathcal{T}_{K+c}(\theta_k) - \mathcal{T}_K(\theta_k))\,\mathrm{d}x\,\mathrm{d}t \\ = \int_Q f_k(\mathcal{T}_{K+c}(\theta_k) - \mathcal{T}_K(\theta_k))\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.2.64}$$

Using chain rule we obtain

$$\begin{aligned}\int_\Omega (\tilde{\mathcal{T}}_{K+c}(\theta_k) - \tilde{\mathcal{T}}_K(\theta_k))(t)\,\mathrm{d}x + \int_Q \nabla\theta_k\cdot\nabla(\mathcal{T}_{K+c}(\theta_k) - \mathcal{T}_K(\theta_k))\,\mathrm{d}x\,\mathrm{d}t \\ = \int_Q f_k(\mathcal{T}_{K+c}(\theta_k) - \mathcal{T}_K(\theta_k))\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega(\tilde{\mathcal{T}}_{K+c}(\theta_{k,0}) - \tilde{\mathcal{T}}_K(\theta_{k,0}))\,\mathrm{d}x,\end{aligned} \tag{3.2.65}$$

where $\tilde{\mathcal{T}}_K(r) = \int_0^r \mathcal{T}_K(z)\,\mathrm{d}z$ and $\tilde{\mathcal{T}}_{K+c}(r) = \int_0^r \mathcal{T}_{K+c}(z)\,\mathrm{d}z$. Furthermore, $\tilde{\mathcal{T}}_{K+c}(\theta_k) - \tilde{\mathcal{T}}_K(\theta_k)$ is a positive function (since c is positive). Thus

$$\begin{aligned}\int_Q \nabla\theta_k\cdot\nabla(\mathcal{T}_{K+c}(\theta_k) - \mathcal{T}_K(\theta_k))\,\mathrm{d}x\,\mathrm{d}t \\ \le \int_Q f_k(\mathcal{T}_{K+c}(\theta_k) - \mathcal{T}_K(\theta_k))\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega(\tilde{\mathcal{T}}_{K+c}(\theta_{k,0}) - \tilde{\mathcal{T}}_K(\theta_{k,0}))\,\mathrm{d}x.\end{aligned} \tag{3.2.66}$$

We may observe that $\nabla(\mathcal{T}_{K+c}(\theta_k) - \mathcal{T}_K(\theta_k))$ is equal to 0 when θ_k does not belong to $\{(x,t)\in Q:\ K<|\theta_k|<K+c\}$. Moreover, on $\{(x,t)\in Q:\ K<|\theta_k|<K+c\}$ it holds that $\nabla\theta_k = \nabla(\mathcal{T}_{K+c}(\theta_k) - \mathcal{T}_K(\theta_k))$. Thus,

$$\begin{aligned}\int_{\{(x,t)\in Q:\ K<|\theta_k|<K+C\}} |\nabla(\mathcal{T}_{K+c}(\theta_k) - \mathcal{T}_K(\theta_k))|^2\,\mathrm{d}x\,\mathrm{d}t \\ \le \int_Q f_k(\mathcal{T}_{K+c}(\theta_k) - \mathcal{T}_K(\theta_k))\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega(\tilde{\mathcal{T}}_{K+c}(\theta_{k,0}) - \tilde{\mathcal{T}}_K(\theta_{k,0}))\,\mathrm{d}x.\end{aligned} \tag{3.2.67}$$

Almost pointwise convergence of $\{\theta_k\}$ to θ in Q with $k\to\infty$ and weak convergence of the sequence $\{f_k\}$ to f in $L^1(Q)$ with $k\to\infty$ imply that $\int_Q f_k(\mathcal{T}_{K+c}(\theta_k) - \mathcal{T}_K(\theta_k))\,\mathrm{d}x\,\mathrm{d}t$ tends to $\int_Q f(\mathcal{T}_{K+c}(\theta) - \mathcal{T}_K(\theta))\,\mathrm{d}x\,\mathrm{d}t$ as k goes to ∞. Furthermore, using the strong convergence of initial conditions and by Lemma 3.2.1 we obtain

$$\begin{aligned}\int_{\{(x,t)\in Q:\ K<|\theta_k|<K+c\}} |\nabla(\mathcal{T}_{K+c}(\theta) - \mathcal{T}_K(\theta))|^2\,\mathrm{d}x\,\mathrm{d}t \\ \le \int_Q f(\mathcal{T}_{K+c}(\theta) - \mathcal{T}_K(\theta))\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega(\tilde{\mathcal{T}}_{K+c}(\theta_0) - \tilde{\mathcal{T}}_K(\theta_0))\,\mathrm{d}x.\end{aligned} \tag{3.2.68}$$

$\mathcal{T}_{K+c}(\theta) - \mathcal{T}_K(\theta)$ is bounded by c and it converges to 0 a.e. in Q. Then,

$$\lim_{K\to\infty} \int_Q f(\mathcal{T}_{K+c}(\theta) - \mathcal{T}_K(\theta))\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.2.69}$$

Moreover,

$$\int_\Omega (\tilde{\mathcal{T}}_{K+c}(\theta_0) - \tilde{\mathcal{T}}_K(\theta_0))\,\mathrm{d}x \le C \int_{\{x\in\Omega:\ |\theta_0|>K\}} |\theta_0|\,\mathrm{d}x, \tag{3.2.70}$$

which implies that right-hand side of (3.2.68) converges to 0 with K goes to ∞. This proves that θ satisfies (3.2.63). □

Multiplying (3.2.1) by $S'(\theta_k)\phi$, where $S \in C^\infty(\mathbb{R})$, S' has a compact support and $\phi \in C_c^\infty([-\infty, T), C^\infty(\Omega))$, we get

$$\begin{aligned} -\int_Q S(\theta_k)\frac{\partial\phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega S(\theta_{k,0})\phi(x,0)\,\mathrm{d}x + \int_Q S'(\theta_l)\nabla\theta_k\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \\ + \int_Q S''(\theta_l)|\nabla\theta_k|^2\phi\,\mathrm{d}x\,\mathrm{d}t = \int_Q f_k S'(\theta_k)\phi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.2.71}$$

S' has a compact support, hence there exist $0 < M < \infty$ such that $\operatorname{supp}(S') \subset [-M, M]$. This allows us to enter the truncations operator into equation (3.2.71). Thus

$$\begin{aligned} -\int_Q S(\theta_k)\frac{\partial\phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega S(\theta_{k,0})\phi(x,0)\,\mathrm{d}x + \int_Q S'(\mathcal{T}_M(\theta_k))\nabla\mathcal{T}_M(\theta_k)\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \\ + \int_Q S''(\mathcal{T}_M(\theta_k))|\nabla\mathcal{T}_M(\theta_k)|^2\phi\,\mathrm{d}x\,\mathrm{d}t = \int_Q f_k S'(\mathcal{T}_M(\theta_k))\phi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.2.72}$$

Using pointwise convergence of $\{\theta_k\}$, bounded character of S', S'' and strong convergence of $\{\mathcal{T}_M(\theta_k)\}$ we obtain the following convergences

$$\begin{array}{ll} S'(\mathcal{T}_M(\theta_k))\nabla\mathcal{T}_M(\theta_k) \to S'(\mathcal{T}_M(\theta))\nabla\mathcal{T}_M(\theta) & \text{in } L^2(Q,\mathbb{R}^3), \\ S''(\mathcal{T}_M(\theta_k))|\nabla\mathcal{T}_M(\theta_k)|^2 \to S''(\mathcal{T}_M(\theta))|\nabla\mathcal{T}_M(\theta)|^2 & \text{in } L^1(Q), \\ f_k S'(\mathcal{T}_M(\theta_k)) \rightharpoonup f S'(\mathcal{T}_M(\theta)) & \text{in } L^1(Q). \end{array} \tag{3.2.73}$$

Hence, passing to the limit with k going to ∞ in (3.2.72) we obtain

$$\begin{aligned} -\int_Q S(\theta)\frac{\partial\phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega S(\theta_0)\phi(x,0)\,\mathrm{d}x + \int_Q S'(\mathcal{T}_M(\theta))\nabla\mathcal{T}_M(\theta)\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \\ + \int_Q S''(\mathcal{T}_M(\theta))|\nabla\mathcal{T}_M(\theta)|^2\phi\,\mathrm{d}x\,\mathrm{d}t = \int_Q f S'(\mathcal{T}_M(\theta))\phi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.2.74}$$

And finally, using the compact support of S' we can omit the truncations in (3.2.74) and we obtain

$$\begin{aligned} -\int_Q S(\theta)\frac{\partial\phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega S(\theta_0)\phi(x,0)\,\mathrm{d}x + \int_Q S'(\theta)\nabla\theta\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \\ + \int_Q S''(\theta)|\nabla\theta|^2\phi\,\mathrm{d}x\,\mathrm{d}t = \int_Q f S'(\theta)\phi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.2.75}$$

It completes the proof of renormalised solution's existence to a parabolic equation with Neumann boundary condition.

3.2.1 Uniqueness

Lemma 3.2.5. *Assume that* $\theta_{0,1}, \theta_{0,2}$ *belong to* $L^1(\Omega)$ *and* f_1, f_2 *belong in* $L^1(Q)$ *and that they satisfy*

$$\begin{cases} \theta_{0,1} \le \theta_{0,2}, \\ f_1 \le f_2. \end{cases} \tag{3.2.76}$$

Then if θ_1 *and* θ_2 *are two renormalised solutions for data* $(\theta_{0,1}, f_1)$ *and* $(\theta_{0,2}, f_2)$*, respectively, we have*

$$\theta_1 \le \theta_2, \tag{3.2.77}$$

almost everywhere in Q.

Proof. Multiplying (3.2.1) for θ_1 and θ_2 by test functions $S'(\theta_1)\phi$ and $S'(\theta_2)\phi$ respectively, where S is $C^\infty(\mathbb{R})$-function ans S' has a compact support. Then, after integration over $\Omega \times (0,t)$ and then over $(0,T)$, we obtain

$$\int_0^T\int_0^t\int_\Omega \frac{\partial S(\theta_1)}{\partial t}\phi \,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t + \int_0^T\int_0^t\int_\Omega S'(\theta_1)\nabla\theta_1\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t + \int_0^T\int_0^t\int_\Omega S''(\theta_1)|\nabla\theta_1|^2\phi\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t = \int_0^T\int_0^t\int_\Omega f_1 S'(\theta_1)\phi\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t.$$

$$\int_0^T\int_0^t\int_\Omega \frac{\partial S(\theta_2)}{\partial t}\phi \,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t + \int_0^T\int_0^t\int_\Omega S'(\theta_2)\nabla\theta_2\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t + \int_0^T\int_0^t\int_\Omega S''(\theta_2)|\nabla\theta_2|^2\phi\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t = \int_0^T\int_0^t\int_\Omega f_2 S'(\theta_2)\phi\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t.$$

Subtracting these equations we obtain

$$\begin{aligned}&\int_0^T\int_0^t\int_\Omega \frac{\partial (S(\theta_1)-S(\theta_2))}{\partial t}\phi \,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t + \int_0^T\int_0^t\int_\Omega \left(S'(\theta_1)\nabla\theta_1 - S'(\theta_2)\nabla\theta_2\right)\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t \\ &+ \int_0^T\int_0^t\int_\Omega \left(S''(\theta_1)|\nabla\theta_1|^2 - S''(\theta_2)|\nabla\theta_2|^2\right)\phi\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t \\ &\qquad = \int_0^T\int_0^t\int_\Omega (f_1 S'(\theta_1) - f_2 S'(\theta_2))\phi\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

Now, let h be $C_0^\infty(\mathbb{R})$-function such that:

$$h(s) = \begin{cases} 1 & |s| \le 1, \\ 0 & 2 \le |s|, \end{cases} \tag{3.2.78}$$

and $\|h\|_{L^\infty} \le 1$. Then we may define

$$h_n(s) = \begin{cases} 1 & |s| \le n-1, \\ h(s-(n-1)\mathrm{sg}(s)) & n-1 \ge |s|, \end{cases} \tag{3.2.79}$$

where $\mathrm{sg}(s)$ denotes the sign of s. Moreover, we define

$$S_n(s) = \int_0^s h_n(\tau)\,\mathrm{d}\tau. \tag{3.2.80}$$

For some positive M let us take the test function $\phi = \mathcal{T}_M^+(S_n(\theta_1) - S_n(\theta_2))$, where $\mathcal{T}_M^+(S_n(\theta_1) - S_n(\theta_2))$ denote the positive part of $\mathcal{T}_M(S_n(\theta_1) - S_n(\theta_2))$. It is obvious that $\mathcal{T}_M^+(S_n(\theta_1) - S_n(\theta_2))$ belongs to $L^2(0,T,W^{1,2}(\Omega)) \cap L^\infty(Q)$. Then

$$\begin{aligned}
&\int_0^T \int_0^t \int_\Omega \frac{\partial(S_n(\theta_1) - S_n(\theta_2))}{\partial t} \mathcal{T}_M^+(S_n(\theta_1) - S_n(\theta_2))\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t \\
&+ \int_0^T \int_0^t \int_\Omega \left(S_n'(\theta_1)\nabla\theta_1 - S_n'(\theta_2)\nabla\theta_2\right) \cdot \nabla \mathcal{T}_M^+(S_n(\theta_1) - S_n(\theta_2))\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t \\
&+ \int_0^T \int_0^t \int_\Omega \left(S_n''(\theta_1)|\nabla\theta_1|^2 - S_n''(\theta_2)|\nabla\theta_2|^2\right) \mathcal{T}_M^+(S_n(\theta_1) - S_n(\theta_2))\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t \\
&\qquad = \int_0^T \int_0^t \int_\Omega (f_1 S_n'(\theta_1) - f_2 S_n'(\theta_2)) \mathcal{T}_M^+(S_n(\theta_1) - S_n(\theta_2))\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned} \tag{3.2.81}$$

It is obvious that for the first term on the left-hand side it holds

$$\begin{aligned}
\int_0^t \int_\Omega \frac{\partial(S_n(\theta_1) - S_n(\theta_2))}{\partial t} \mathcal{T}_M^+(S_n(\theta_1) - S_n(\theta_2))\,\mathrm{d}x\,\mathrm{d}s &= \int_0^t \int_\Omega \frac{\partial \tilde{\mathcal{T}}_M([S_n(\theta_1) - S_n(\theta_2)]^+)}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_\Omega \tilde{\mathcal{T}}_M([S_n(\theta_1) - S_n(\theta_2)]^+)(t)\,\mathrm{d}x \\
&\quad - \int_\Omega \tilde{\mathcal{T}}_M([S_n(\theta_{0,1}) - S_n(\theta_{0,2})]^+)\,\mathrm{d}x,
\end{aligned}$$

where $\tilde{\mathcal{T}}_M(s) = \int_0^s \mathcal{T}_M(\tau)\,\mathrm{d}\tau$. Moreover, the choice of sequence of function $\{S_n\}_{n=1}^\infty$ causes that the following convergences

$$\begin{gathered}
S_n(\theta_1) - S_n(\theta_2) \to \theta_1 - \theta_2 \\
(f_1 S_n'(\theta_1) - f_2 S_n'(\theta_2)) \mathcal{T}_M^+(S_n(\theta_1) - S_n(\theta_2)) \to (f_1 - f_2) \mathcal{T}_M^+(\theta_1 - \theta_2) \\
S_n(\theta_{0,1}) - S_n(\theta_{0,2}) \to \theta_{0,1} - \theta_{0,2}
\end{gathered} \tag{3.2.82}$$

hold strongly in $L^1(Q)$ and $L^1(\Omega)$, respectively, as $n \to \infty$. Let us focus on the second and the third term on the left-hand side of (3.2.81). It is obvious that

$$\begin{aligned}
&\int_0^T\int_0^t\int_\Omega \left(S_n'(\theta_1)\nabla\theta_1 - S_n'(\theta_2)\nabla\theta_2\right)\cdot\nabla\mathcal{T}_M^+(S_n(\theta_1)-S_n(\theta_2))\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t\\
&\quad=\int_0^T\int_0^t\int_\Omega (\nabla S_n(\theta_1)-\nabla S_n(\theta_2))\cdot\nabla\mathcal{T}_M^+(S_n(\theta_1)-S_n(\theta_2))\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t\\
&\quad=\int_0^T\int_{\{(x,s)\in\Omega\times(0,t):\ 0\le\nabla S_n(\theta_1)-\nabla S_n(\theta_2)\le M\}}(\nabla S_n(\theta_1)-\nabla S_n(\theta_2))\cdot\nabla(S_n(\theta_1)-S_n(\theta_2))\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t\\
&\quad=\int_0^T\int_{\{(x,s)\in\Omega\times(0,t):\ 0\le\nabla S_n(\theta_1)-\nabla S_n(\theta_2)\le M\}}|\nabla S_n(\theta_1)-\nabla S_n(\theta_2)|^2\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t\ge 0,\\
&\Big|\int_0^T\int_0^t\int_\Omega\left(S_n''(\theta_1)|\nabla\theta_1|^2 - S_n''(\theta_2)|\nabla\theta_2|^2\right)\mathcal{T}_M^+(S_n(\theta_1)-S_n(\theta_2))\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t\Big|\\
&\quad=\Big|\int_Q (T-t)\left(S_n''(\theta_1)|\nabla\theta_1|^2 - S_n''(\theta_2)|\nabla\theta_2|^2\right)\mathcal{T}_M^+(S_n(\theta_1)-S_n(\theta_2))\,\mathrm{d}x\,\mathrm{d}t\Big|\\
&\quad=T\|S_n''\|_{L^\infty(\mathbb{R})}M\int_{\{(x,t)\in Q:\ n\le|\theta_1|\le n+1\}}|\nabla\theta_1|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+T\|S_n''\|_{L^\infty(\mathbb{R})}M\int_{\{(x,t)\in Q:\ n\le|\theta_1|\le n+1\}}|\nabla\theta_2|^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{3.2.83}$$

Using Lemma 3.2.4 we obtain that the right-hand side of abovementioned equation tends to 0 as n goes to ∞. Then we may rewrite (3.2.81) in the following form

$$\int_Q\tilde{\mathcal{T}}_M([\theta_1-\theta_2]^+)\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_0^t\int_\Omega(f_1-f_2)\mathcal{T}_M^+(\theta_1-\theta_2)\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t+T\int_\Omega\tilde{\mathcal{T}}_M([\theta_{0,1}-\theta_{0,2}]^+)\,\mathrm{d}x.\tag{3.2.84}$$

Since $\theta_{0,1}\le\theta_{0,2}$ and $f_1\le f_2$, then the right-hand side of (3.2.84) is nonpositive and thus $\tilde{\mathcal{T}}_M([\theta_1-\theta_2]^+)$ is nonpositive. Since function $\tilde{\mathcal{T}}_M$ is nonegative (see the definition) and nonpositive, we obtain that $[\theta_1-\theta_2]^+$ is equal to 0. And this implies that $\theta_1\le\theta_2$, which completes the proof. □

Chapter 4

Mróz model

The subject of this chapter is to consider a general class of thermo-visco-elastic models, to which belongs the Mróz model, cf. [57]. In 1967 a Polish engineer prof. Z. Mróz formulated the hardening rule, where the dependency between visco-elastic constitutive function and deviatoric part of Cauchy stress tensor was linear, i.e. $\boldsymbol{G}(\theta, \boldsymbol{T}^d) = G_1(\theta)\boldsymbol{T}^d$. Our goal is to present the proof of existence of solution to visco-elastic models of Mróz-type. Let us start with formulation of assumptions on constitutive function $\boldsymbol{G}$.

Assumption 4.0.1. *For the function $\boldsymbol{G}(\cdot,\cdot)$ the following conditions hold*

a) $\boldsymbol{G}(\theta, \boldsymbol{T}^d)$ *is continuous with respect to* θ *and* $\boldsymbol{T}^d$*;*

b) $(\boldsymbol{G}(\theta, \boldsymbol{T}^d_1) - \boldsymbol{G}(\theta, \boldsymbol{T}^d_2)) : (\boldsymbol{T}^d_1 - \boldsymbol{T}^d_2) > 0$*, where* $\boldsymbol{T}^d_1 \neq \boldsymbol{T}^d_2$*,* $\boldsymbol{T}_1, \boldsymbol{T}_2$ *belong to* $\mathcal{S}^3$*;*

c) $|\boldsymbol{G}(\theta, \boldsymbol{T}^d)| \leq C|\boldsymbol{T}^d|$*, where* $\boldsymbol{T}$ *belongs to* $\mathcal{S}^3$ *and* C *is a positive constant;*

d) $\boldsymbol{G}(\theta, \boldsymbol{T}^d) : \boldsymbol{T}^d \geq \beta|\boldsymbol{T}^d|^2$*, where* $\boldsymbol{T}$ *belongs to* $\mathcal{S}^3$ *and* β *is a positive constant;*

Constants C *and* β *are independent of temperature* θ*.*

We have to additionally assume that $\Omega \subset \mathbb{R}^3$ is an open bounded set with a C^2 boundary and moreover, the considered body is homogeneous in space, i.e. function $\boldsymbol{G}$ and operator $\boldsymbol{D}$ do not depend on spatial variable x.

The results included in this chapter are a combination of two papers. In [33], we showed the sketch of the proof for models satisfying Assumption 4.0.1. Here, we present different proof. We modify the proof from [33] by using the results obtained in [32]. The main idea is still the same, i.e. we still use Young measure tools to complete each limit passage. The differences between these proofs lie in transformation of a system into homogeneous boundary-value problem and in the estimates for approximate solutions.

This chapter is divided into three sections. Firstly, we recall well known facts regarding Young measures. We formulate theorems, lemmas ans their proofs. Then, we formulate the main theorem of this chapter. And finally, we present full proof of solutions existence.

4.1 Young measure tools

We begin this section with a few general remarks about Young measures. Let $n, m \in \mathbb{N}$ and E be a measurable subset of $\mathbb{R}^n$. And let us consider the sequence $\{z_j\}$ such that $z_j : E \to \mathbb{R}^m$. The Young measure is a limiting probability distribution as $j \to \infty$ of the value z_j near point $x \in \mathbb{R}^n$. In the case of many sequences, e.g. sequence which converges only weakly, we are not able to predict how the nonlinear function of this sequence will behave. For example, the weak convergence of velocity does not hold any information about the kinetic energy. The first individual who investigated the failure of classical minimization was L.C. Young in the 1930's.

The Young measures may be a very useful tool to get more information about the sequence behaviour. The idea of looking at the limit of the sequence as a probability distribution comes from L.C. Young [86]. Young applied this technique for problems of calculus of variations without the minimizer in a classical sense. Later, application of Young measures for many other problems was shown, e.g. for optimal control, cf. [52, 83, 86], or nonlinear hyperbolic equations, see [77], as well as many others.

Let us consider a measurable set $E \subset \mathbb{R}^n$. By $C_0(\mathbb{R}^m)$ we denote the closure of continuous functions on $\mathbb{R}^m$ with a compact support. $C_0(\mathbb{R}^m)$ with a norm defined by $\|f\|_\infty = \sup_{\lambda\in\mathbb{R}^m} |f(\lambda)|$ is a Banach space. By Riesz representation theorem, see e.g. [42, p.364], the dual space to $C_0(\mathbb{R}^m)$ is a Banach space $\mathcal{M}(\mathbb{R}^m)$ of bounded Radon measures on $\mathbb{R}^m$. The duality pair between $C_0(\mathbb{R}^m)$ and $\mathcal{M}(\mathbb{R}^m)$ is defined by

$$\langle \nu, f\rangle = \int_{\mathbb{R}^m} f(\lambda)\,\mathrm{d}\nu(\lambda). \tag{4.1.1}$$

Let us denote also by $L^\infty_w(E, \mathcal{M}(\mathbb{R}^m))$ the space of weak* measurable maps $\nu : E \to \mathcal{M}(\mathbb{R}^m)$ that are bounded. Moreover, $L^\infty_w(E, \mathcal{M}(\mathbb{R}^m))$ is the dual of the separable space $L^1(E, C_0(\mathbb{R}^m))$, cf. [27, p.588], and the duality pairing is given by

$$\langle \mu, \Psi\rangle = \int_E \langle \mu_x(\cdot), \Psi(x,\cdot)\rangle\,\mathrm{d}x. \tag{4.1.2}$$

In this section, we present a few theorems and lemmas which are used to prove the existence of solutions to thermo-visco-elastic models. We also use this tools in Chapter 6. Theorem 4.1.1 and Lemma 4.1.1 – Lemma 4.1.3 come from [58]. For the fundamental theorem on Young measures we also refer the reader to [9]. Proof of Theorem 4.1.1 come partially from [58] and partially from [9]. Proofs of lemmas presented below come from [58], but with some additional comment to improve its readability. In its application to nonlinear partial differential equation it is very important that in the fundamental theorem on Young measures we prove point (v), which says that this approach may be used for unbounded functions, e.g. potential energy.

Theorem 4.1.1 (Fundamental theorem on Young measures, Theorem 3.1 from [58])**.** *Let $E \subset \mathbb{R}^n$ be a measurable set of finite measures and let $z_j : E \to \mathbb{R}^m$ be a sequence of measurable functions. Then there exist a subsequence (still denote by z_j) and weak* measurable map $\nu_x : E \to \mathcal{M}(\mathbb{R}^m)$ such that the following holds*

(i) $\nu_x \geq 0$, $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} = \int_{\mathbb{R}^m} \mathrm{d}\nu_x \leq 1$, *for a.e.* $x \in E$;

(ii) for all $f \in C_0(\mathbb{R}^m)$

$$f(z_j) \overset{*}{\rightharpoonup} \overline{f} \quad \text{in } L^\infty(E), \tag{4.1.3}$$

where

$$\overline{f}(x) = \langle \nu_x, f\rangle = \int_{\mathbb{R}^m} f(\lambda)\,\mathrm{d}\nu_x(\lambda); \tag{4.1.4}$$

(iii) Let $K \subset \mathbb{R}^m$ *be compact. If* $dist(z_j, K) \to 0$ *in measure then*

$$supp\ \nu_x \subset K; \tag{4.1.5}$$

(iv) Furthermore one has

$$\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} = 1, \tag{4.1.6}$$

for a.a. $x \in E$ if and only if the sequence does not go to infinity, i.e. if

$$\lim_{M\to\infty} \sup_j meas(\{|z_j| \geq M\}) = 0; \tag{4.1.7}$$

(v) if (4.1.6) holds, $A \subset E$ is measurable, $f \in C(\mathbb{R}^m)$ and if

$$\text{the set } \{f(z_j)\} \text{ is relatively weakly compact in } L^1(A), \tag{4.1.8}$$

then

$$f(z_j) \rightharpoonup \overline{f} \text{ in } L^1(A),\ \overline{f}(x) = \langle \nu_x, f\rangle; \tag{4.1.9}$$

(vi) If (4.1.6) holds, then in (iii) one can replace 'if' by 'if and only if'.

Proof. The idea of this proof is not to consider a real valued vector functions $z_j : E \to \mathbb{R}^m$, but passing to maps $\nu^j : E \to \mathcal{M}(\mathbb{R}^m)$. Let us define

$$\nu^j(x) = \delta_{z_j(x)}. \tag{4.1.10}$$

By (4.1.1) we obtain

$$\langle \nu^j(x), f\rangle = f(z_j(x)) \tag{4.1.11}$$

for $f \in C_0(\mathbb{R}^m)$ and $\|\nu^j(x)\|_{\mathcal{M}(\mathbb{R}^n)} = 1$. This implies that ν^j belongs to $L^\infty_w(E, \mathcal{M}(\mathbb{R}^m))$. Hence, by the Banach-Alaoglu theorem (see [16, Theorem 3.16, page 66]) we obtain that there exists a subsequence of $\{\nu^j\}$ (still denoted by $\{\nu^j\}$) such that

$$\nu^j(\cdot) \overset{*}{\rightharpoonup} \nu \qquad \text{in } L^\infty_w(E, \mathcal{M}(\mathbb{R}^m)). \tag{4.1.12}$$

We should remember that by ν_x we denote $\nu(x)$. Lower semicontinuity of the norm implies that $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} \leq 1$ for a.e. $x \in E$. Let us take a function $\Psi \in L^1(E, C_0(\mathbb{R}^m))$. Using (4.1.12) we get

$$\int_E \Psi(x, z_j(x))\,\mathrm{d}x = \int_E \langle \nu^j(\cdot), \Psi(x,\cdot)\rangle\,\mathrm{d}x \to \int_E \langle \nu_x(\cdot), \Psi(x,\cdot)\rangle\,\mathrm{d}x \tag{4.1.13}$$

as $j \to \infty$. Taking $\Psi(x,\lambda) = \psi(x)f(\lambda)$ where $\psi \in L^1(E)$ and $f \in C_0(\mathbb{R}^m)$ are arbitrary functions we obtain

$$\int_E \psi(x) f(z_j(x))\,\mathrm{d}x \to \int_E \psi(x)\langle \nu_x(\cdot), f(\cdot)\rangle\,\mathrm{d}x, \tag{4.1.14}$$

which implies (*ii*). Moreover, considering all functions $f \geq 0$, $\psi \geq 0$ we get that $\nu_x \geq 0$, which finishes the proof of (*i*).

Let us take arbitrary function $f \in C_0(\mathbb{R}^m \setminus K)$. To prove (*iii*) it is suffient to show that

$$\langle \nu_x, f\rangle = 0. \tag{4.1.15}$$

Let us take arbitrary $\epsilon > 0$, then there exists C_ϵ such that the following inequality holds

$$|f(\lambda)| \leq \epsilon + C_\epsilon \mathrm{dist}(\lambda, K), \tag{4.1.16}$$

and then

$$(|f(\lambda)| - \epsilon)^+ \leq C_\epsilon \mathrm{dist}(\lambda, K), \tag{4.1.17}$$

where we denote a positive part of $\cdot$ by $(\cdot)^+$. Since $\operatorname{dist}(z_j, K) \to 0$ in measure as j goes to ∞ then also $(|f(z_j)| - \epsilon)^+ \to 0$ in measure. By (ii) we get

$$\langle \nu_x, (|f| - \epsilon)^+ \rangle = 0 \tag{4.1.18}$$

for a.e. $x \in E$. Since $\epsilon > 0$ is arbitrary constant (4.1.15) follows.

To prove (iv) let us define the *'hat'* function

$$\phi_M(\lambda) = \begin{cases} 1 & |\lambda| \le M, \\ 1 + M - |\lambda| & M \le |\lambda| \le M + 1, \\ 0 & M + 1 \le |\lambda|, \end{cases} \tag{4.1.19}$$

for positive M. Then

$$\lim_{j\to\infty} \int_E \phi_M(z_j(x))\,\mathrm{d}x = \int_E \langle \nu_x, \phi_M \rangle\,\mathrm{d}x \le \int_E \langle \nu_x, 1 \rangle\,\mathrm{d}x = \int_E \|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)}\,\mathrm{d}x. \tag{4.1.20}$$

Moreover

$$\int_E (1 - \phi_M(z_j(x)))\,\mathrm{d}x \le \operatorname{meas}(\{|z_j| \ge M\}), \tag{4.1.21}$$

and thus

$$\operatorname{meas}(E) - \operatorname{meas}(\{|z_j| \ge M\}) \le \int_E \phi_M(z_j(x))\,\mathrm{d}x. \tag{4.1.22}$$

Passing to the limit with $j \to \infty$ we get

$$\operatorname{meas}(E) - \sup_j \operatorname{meas}(\{|z_j| \ge M\}) \le \int_E \|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)}\,\mathrm{d}x, \tag{4.1.23}$$

and then with $M \to \infty$

$$\operatorname{meas}(E) - \lim_{M\to\infty} \sup_j \operatorname{meas}(\{|z_j| \ge M\}) \le \int_E \|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)}\,\mathrm{d}x. \tag{4.1.24}$$

Finally, if $\lim_{M\to\infty} \sup_j \operatorname{meas}(\{|z_j| \ge M\}) = 0$ then $\|\nu_x\|_{\mathcal{M}} = 1$. We prove the second implication by contradiction. Let us assume that there exists $\epsilon > 0$ and sequence of pairs $\{(M_k, j_k)\}$ such that $M_k \to \infty$ as $k \to \infty$ and $\operatorname{meas}(\{|z_{j_k}| \ge M_k\}) > \epsilon$. Let us take some positive M

$$\lim_{k\to\infty} \int_E \phi_M(z_{j_k}(x))\,\mathrm{d}x = \int_E \langle \nu_x, \phi_M \rangle\,\mathrm{d}x. \tag{4.1.25}$$

Then there exists sufficiently large k, such that $M_k \ge M + 1$ and the following inequality holds

$$\int_E \phi_M(z_{j_k}(x))\,\mathrm{d}x \le \operatorname{meas}(E) - \epsilon. \tag{4.1.26}$$

By monotone convergence theorem we may pass in (4.1.25) to the limit with M going to ∞. Then by (4.1.26) we obtain

$$\begin{aligned} \int_E \|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)}\,\mathrm{d}x &= \int_E \langle \nu_x, 1 \rangle\,\mathrm{d}x \\ &= \lim_{M\to\infty} \int_E \langle \nu_x, \phi_M \rangle\,\mathrm{d}x \\ &= \lim_{M\to\infty} \lim_{k\to\infty} \int_E \phi_M(z_{j_k}(x))\,\mathrm{d}x \\ &\le \operatorname{meas}(E) - \epsilon, \end{aligned} \tag{4.1.27}$$

which stays in a contradiction with $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} = 1$.

To prove (v) it is enough to consider only positive functions $f \in C(\mathbb{R}^m)$. Since (4.1.6) holds, then by (iv) we get

$$f(z_j) \rightharpoonup \chi \qquad \text{in } L^1(A). \tag{4.1.28}$$

Let $f_M(\lambda) = \Phi_M(\lambda) f(\lambda)$, where $\Phi_M(\lambda)$ was defined by (4.1.19). Let us take $\Psi \in L^\infty(A)$. Then

$$|\int_A \Psi(f_M(z_j) - f(z_j))\,\mathrm{d}x| \le C \int_{\{x\in A:\ |z_j|\ge M\}} |f(z_j)|\,\mathrm{d}x, \tag{4.1.29}$$

where constant C depends on the choice of Ψ. By Dunford-Petittis theorem [56, Theorem T23] relatively weak compact set is also uniformly integrable, hence for all $\epsilon > 0$ there exists $R > 0$ such that

$$\sup_{j\in\mathbb{N}} \int_{\{x\in A:\ |f(z_j)|\ge R\}} |f(z_j)|\,\mathrm{d}x < \epsilon. \tag{4.1.30}$$

Then we may continue the estimates in (4.1.29) as follows

$$\begin{aligned}
|\int_A \Psi(f_M(z_j) - f(z_j))\,\mathrm{d}x| &\le C \int_{\{x\in A:\ |z_j|\ge M\}} |f(z_j)|\,\mathrm{d}x \\
&= C \int_{\{x\in A:\ |z_j|\ge M\}\cap\{x\in A:\ |f(z_j)|\ge R\}} |f(z_j)|\,\mathrm{d}x \\
&\quad + C \int_{\{x\in A:\ |z_j|\ge M\}\cap\{x\in A:\ |f(z_j)|\le R\}} |f(z_j)|\,\mathrm{d}x \\
&\le C \int_{\{x\in A:\ |f(z_j)|\ge R\}} |f(z_j)|\,\mathrm{d}x \\
&\quad + C \int_{\{x\in A:\ |z_j|\ge M\}\cap\{x\in A:\ |f(z_j)|\le R\}} |f(z_j)|\,\mathrm{d}x \\
&\le C\epsilon + CR\,\mathrm{meas}(\{x\in A:\ |z_j|\ge M\}).
\end{aligned} \tag{4.1.31}$$

For sufficiently large M the right-hand side of (4.1.31) may be bounded by $2\epsilon C$. Since $\epsilon > 0$ is an arbitrary constant then

$$\lim_{M\to\infty} \int_A \Psi f_M(z_j)\,\mathrm{d}x = \int_A \Psi f(z_j)\,\mathrm{d}x \tag{4.1.32}$$

for all $\Psi \in L^\infty(A)$. Moreover

$$\lim_{j\to\infty} \int_A \Psi f_M(z_j) = \int_A \Psi\langle\nu_x, f_M\rangle\,\mathrm{d}x. \tag{4.1.33}$$

Since $\{f_M\}$ is an increasing sequence then we may pass to the limit with $M\to\infty$ in (4.1.33) by using the monotone convergence theorem. Also we may pass to the limit with $j\to\infty$ in (4.1.32). Then we get

$$\int_A \Psi\langle\nu_x, f\rangle\,\mathrm{d}x = \lim_{M\to\infty}\int_A \Psi\langle\nu_x, f_M\rangle\,\mathrm{d}x = \int_A \Psi\chi\,\mathrm{d}x, \tag{4.1.34}$$

which completes the proof of (v).

Finally, let us prove (vi). We define the function $f(\lambda) = \min(\mathrm{dist}(\lambda, K), 1)$. Then the set $\{f(z_j)\}$ is relatively weakly compact in $L^1(E)$ and by applying (v) we obtain

$$\lim_{j\to\infty}\int_E \phi\min(\mathrm{dist}(z_j, K), 1) = \int_E \phi\langle\nu_x(\cdot), \min(\mathrm{dist}(\cdot, K), 1)\rangle = 0. \tag{4.1.35}$$

for every $\phi \in L^\infty(E)$. The last equality holds because supports of function f and measure ν_x are disjoint.

□

Definition 4.1.1. *Let $f, f_n : E \to \mathbb{R}^n$ be measurable functions. We say that the sequence $\{f_n\}$ converges in measure to f, if for every $\varepsilon > 0$ holds*

$$\lim_{n\to\infty} meas(\{x \in E: \ |f(x) - f_n(x)| \geq \varepsilon\}) = 0. \tag{4.1.36}$$

On the basis of fundamental theorem on Young measures we know that Young measure exists. Now, we prove its properties which are used latterly. The following lemmas come from [58], see also [61]. Similar technique was used in [21, 34, 73].

Lemma 4.1.1 (Corollary 3.2 from [58])**.** *Suppose that a sequence z_j of measurable functions from E to $\mathbb{R}^m$ generates the Young measure $\nu : E \to \mathcal{M}(\mathbb{R}^m)$. Then $z_j \to z$ in measure if and only if $\nu_x = \delta_{z(x)}$ a.e..*

Proof. Let us assume that $z_j \to z$ in measure. Then also $f(z_j) \to f(z)$ in measure for all $f \in C_0(\mathbb{R}^m)$. By Theorem 4.1.1 (*ii*) we obtain that

$$\int_E \Psi\langle \nu_x, f\rangle \,\mathrm{d}x = \lim_{j\to\infty} \int_E \Psi f(z_j)\,\mathrm{d}x = \int_E \Psi f(z(x))\,\mathrm{d}x, \tag{4.1.37}$$

for all $f \in C_0(\mathbb{R}^m)$ and $\Psi \in L^1(E)$. Thus $\nu_x = \delta_{z(x)}$ and the proof of first implication is complete.

Let $\nu_x = \delta_{z(x)}$ a.e. in E. Since $\{\nu_x\}_{x\in E}$ is weak* measurable then the function z is measurable. We define a continuous function $\phi : \mathbb{R}_+ \to [0, 1]$ by

$$\phi(x) = \begin{cases} 0 & x \leq \frac{\epsilon}{2}, \\ \text{continuous} & \frac{\epsilon}{2} \leq x \leq \epsilon, \\ 1 & \epsilon \leq x, \end{cases} \tag{4.1.38}$$

for some positive ϵ. Using Theorem 4.1.1 (v) for function $f(\lambda) = \phi(|\lambda - a|)$ for some constant $a \in \mathbb{R}^m$ we obtain

$$\begin{aligned} \lim_{j\to\infty} \int_E f(z_j)\,\mathrm{d}x &= \int_E \langle \nu_x, f\rangle \,\mathrm{d}x \\ &= \int_E f(z)\,\mathrm{d}x = \int_E \phi(|z-a|)\,\mathrm{d}x = \int_{\{x\in E:\ |z(x)-a|\geq \frac{\epsilon}{2}\}} \phi(|z-a|)\,\mathrm{d}x \end{aligned} \tag{4.1.39}$$

Since $\phi(|z-a|) \leq 1$ then the right-hand side of (4.1.39) is bounded by meas$(\{x \in E: \ |z(x)-a| > \frac{\epsilon}{2}\})$. Moreover, the left-hand side of (4.1.39) may be estimated by

$$\begin{aligned} \lim_{j\to\infty} \int_E f(z_j)\,\mathrm{d}x &= \lim_{j\to\infty} \int_E \phi(|z_j - a|)\,\mathrm{d}x \\ &= \lim_{j\to\infty} \left(\int_{\{x\in E:\ \frac{\epsilon}{2}\leq|z_j(x)-a|<\epsilon\}} \phi(|z_j - a|)\,\mathrm{d}x + \int_{\{x\in E:\ |z_j(x)-a|\geq\epsilon\}} \phi(|z_j - a|)\,\mathrm{d}x \right) \\ &\geq \limsup_{j\to\infty} \int_{\{x\in E:\ |z_j(x)-a|\geq\epsilon\}} \phi(|z_j - a|)\,\mathrm{d}x \\ &= \limsup_{j\to\infty} \mathrm{meas}(\{x \in E: \ |z_j(x) - a| \geq \epsilon\}). \end{aligned} \tag{4.1.40}$$

Hence

$$\limsup_{j\to\infty} \text{meas}(\{x \in E : \ |z_j(x) - a| \geq \epsilon\}) \leq \text{meas}(\{x \in E : \ |z(x) - a| \geq \frac{\epsilon}{2}\}). \tag{4.1.41}$$

Furthermore, similar inequality holds if we choose piecewise constant measurable functions $w : E \to \mathbb{R}^m$. Let $w(x) = \sum_i \mathbf{1}_{\chi_i}(x) a_i$ then

$$\begin{aligned}
\limsup_{j\to\infty} \text{meas}(\{x \in E : \ |z_j(x) - w(x)| > \epsilon\}) &= \limsup_{j\to\infty} \text{meas} \sum_i (\{x \in \chi_i : \ |z_j(x) - a_i| > \epsilon\}) \\
&= \sum_i \limsup_{j\to\infty} \text{meas}(\{x \in \chi_i : \ |z_j(x) - a_i| > \epsilon\}) \\
&\leq \sum_i \text{meas}(\{x \in \chi_i : \ |z(x) - a_i| > \frac{\epsilon}{2}\}) \\
&= \text{meas}(\{x \in E : \ |z(x) - w(x)| > \frac{\epsilon}{2}\}).
\end{aligned} \tag{4.1.42}$$

Thus

$$\begin{aligned}
&\limsup_{j\to\infty} \text{meas}(\{x \in E : |z_j(x) - z(x)| > \epsilon\}) \\
&\qquad \leq \limsup_{j\to\infty} \text{meas}(\{x \in E : \ |z_j(x) - w(x)| > \frac{\epsilon}{2}\}) \\
&\qquad\quad + \limsup_{j\to\infty} \text{meas}(\{x \in E : \ |z(x) - w(x)| > \frac{\epsilon}{2}\}) \\
&\qquad \leq 2 \ \text{meas}(\{x \in E : \ |z(x) - w(x)| > \frac{\epsilon}{4}\}).
\end{aligned} \tag{4.1.43}$$

The right-hand side of (4.1.43) may be made arbitrary small, because we may approximate every measurable function by piecewise constant functions. □

Lemma 4.1.2 (Corollary 3.3 from [58])**.** *Suppose that the sequence of maps $z_j : E \to \mathbb{R}^m$ generates Young measure $\nu : E \to \mathcal{M}(\mathbb{R}^m)$. Let $f : E \times \mathbb{R}^m \to \mathbb{R}^m$ be a Carathéodory function (i.e. measurable in the first argument and continuous in the second). Let us also assume that the negative part $f^-(x, z_j(x))$ is weakly relatively compact in $L^1(E)$. Then*

$$\liminf_{j\to\infty} \int_E f(x, z_j(x))\,\mathrm{d}x \geq \int_E \int_{\mathbb{R}^d} f(x,\lambda)\,\mathrm{d}\nu_x(\lambda)\,\mathrm{d}x \tag{4.1.44}$$

If, in addition, the sequence of functions $\{f^+(\cdot, z_j(\cdot)) + f^-(\cdot, z_j(\cdot))\}$ is weakly relatively compact in $L^1(E)$, then

$$f(\cdot, z_j(\cdot)) \rightharpoonup \int_{\mathbb{R}^d} f(\cdot,\lambda)\,\mathrm{d}\nu_x(\lambda) \qquad \text{in } L^1(E). \tag{4.1.45}$$

Proof. The proof of this lemma consist of four steps. Firstly, we prove this lemma for positive function f with support contained in some ball. Then, we remove the condition on supports boundedness (step 2.) and we remove the assumption on positivity of function (step 3.). The last step is to show that (4.1.45) holds.

Step 1. Let us start the proof with considering the case of positive function $f \geq 0$. Moreover, we assume that $f(x,\lambda) = 0$ if $|\lambda| \geq R$ for some positive R. By Scorza-Dragoni theorem (see e.g. [6, Theorem 12.1.3]) there exists an increasing sequence of compact sets E_k such that

$\mathrm{meas}(E \setminus E_k) \to 0$ as $k \to \infty$ and $f|_{E_k \times \mathbb{R}^m}$ is continuous. Let us define $F_k : E \to C_0(\mathbb{R}^m)$ by $F_k(x, \cdot) = \mathbf{1}_{E_k}(x) f(x, \cdot)$. It is obvious that $F_k \in L^1(E; C_0(\mathbb{R}^m))$. Moreover, $\delta_{z_j(x)} \xrightarrow{*} \nu_x$ in $L^\infty_w(E, \mathcal{M}(\mathbb{R}^m))$ as $j \to \infty$. Hence

$$\begin{aligned}\int_E f(x, z_j(x))\, \mathrm{d}x \geq \int_{E_k} f(x, z_j(x))\, \mathrm{d}x &= \int_E F_k(x, z_j(x))\, \mathrm{d}x \\ &= \int_E \langle \delta_{z_j(x)}, F_k(x, \cdot)\rangle\, \mathrm{d}x \to \int_E \langle \nu_x, F_k(x, \cdot)\rangle\, \mathrm{d}x \\ &= \int_E \int_{\mathbb{R}^m} F_k(x, \lambda)\, \mathrm{d}\nu_x(\lambda)\, \mathrm{d}x = \int_{E_k} \int_{\mathbb{R}^m} f(x, \lambda)\, \mathrm{d}\nu_x(\lambda)\, \mathrm{d}x\end{aligned} \tag{4.1.46}$$

as $j \to \infty$. By monotone convergence theorem, as $k \to \infty$, we obtain

$$\int_E f(x, z_j(x))\, \mathrm{d}x \geq \int_E \int_{\mathbb{R}^m} f(x, \lambda)\, \mathrm{d}\nu_x(\lambda)\, \mathrm{d}x. \tag{4.1.47}$$

Passing to the limit with $j \to \infty$ we complete the step 1.

Step 2. To remove the assumption that support of function $f(x, \cdot)$ is contained in a ball, let us consider an increasing sequence $\{\eta_l\} \subset C_0(\mathbb{R}^m)$, such that η_l converges to 1 as l goes to ∞. Then let $f_l(x, \lambda) = f(x, \lambda)\eta_l(\lambda)$ and by (4.1.47)

$$\int_E f_l(x, z_j(x))\, \mathrm{d}x \geq \int_E \int_{\mathbb{R}^m} f_l(x, \lambda)\, \mathrm{d}\nu_x(\lambda)\, \mathrm{d}x. \tag{4.1.48}$$

holds for every $l \in \mathbb{N}$. By applying again the monotone convergence theorem we may pass to the limit with $l \to \infty$ and finish the proof of step 2. Moreover, by using shifts, (4.1.44) holds for every function bounded from below.

Step 3.

Since $\{f^-(x, z_j(x))\}$ is relatively weakly compact in $L^1(E)$, by [56, Theorem T23], then it is uniformly integrable. Hence, for every $\epsilon > 0$ there exists positive M such that

$$\sup_j \int_{\{x \in E:\ |f^-(x, z_j(x))| \geq M\}} f^-(x, z_j(x))\, \mathrm{d}x < \epsilon. \tag{4.1.49}$$

Let us define an auxiliary function

$$f_M(x, \lambda) = \max(f(x, \lambda), -M). \tag{4.1.50}$$

Then by use of conclusion from step 2. we obtain

$$\liminf_{j \to \infty} \int_E f_M(x, z_j(x))\, \mathrm{d}x \geq \int_E \int_{\mathbb{R}^m} f_M(x, \lambda)\, \mathrm{d}\nu_x(\lambda)\, \mathrm{d}x \geq \int_E \int_{\mathbb{R}^m} f(x, \lambda)\, \mathrm{d}\nu_x(\lambda)\, \mathrm{d}x. \tag{4.1.51}$$

Moreover, definition of function f_M and uniform integrability of f^- implies

$$\begin{aligned}\liminf_{j \to \infty} \int_E f_M(x, z_j(x))\, \mathrm{d}x &= \liminf_{j \to \infty} \int_E \left(f_M^+(x, z_j(x)) - f_M^-(x, z_j(x))\right)\, \mathrm{d}x \\ &= \liminf_{j \to \infty} \left(\int_E f^+(x, z_j(x))\, \mathrm{d}x - \int_E f^-(x, z_j(x))\, \mathrm{d}x \right. \\ &\qquad \left. + \int_{\{x \in E:\ |f^-(x, z_j(x))| \geq M\}} f^-(x, z_j(x))\, \mathrm{d}x\right) \\ &= \liminf_{j \to \infty} \left(\int_E f(x, z_j(x))\, \mathrm{d}x + \int_{\{x \in E:\ |f^-(x, z_j(x))| \geq M\}} f^-(x, z_j(x))\, \mathrm{d}x\right) \\ &\leq \liminf_{j \to \infty} \int_E f(x, z_j(x))\, \mathrm{d}x + \epsilon.\end{aligned} \tag{4.1.52}$$

Hence

$$\liminf_{j\to\infty}\int_E f(x,z_j(x))\,\mathrm{d}x+\epsilon\geq\int_E\int_{\mathbb{R}^m} f(x,\lambda)\,\mathrm{d}\nu_x(\lambda)\,\mathrm{d}x, \tag{4.1.53}$$

and since $\epsilon > 0$ is an arbitrary constant, the proof of step 3. is finished.

Step 4.

It is sufficient to observe that (4.1.45) is an application of (4.1.44) to $\tilde{f}(x,\lambda) = \pm\phi(x)f(x,\lambda)$ for all positive $\phi \in L^\infty(E)$. □

Lemma 4.1.3 (Corollary 3.4 from [58]). *Let $E \subset \mathbb{R}^d$ and let $u_j : E \to \mathbb{R}^n$, $v_j : E \to \mathbb{R}^m$ be measurable and suppose that $u_j \to u$ a.e. while v_j generates the Young measure ν. Then the sequence of pairs $(u_j, v_j) : E \to \mathbb{R}^{n+m}$ generates the Young measure $x \to \delta_{u(x)} \otimes \nu_x$.*

Proof. Let us take the following functions $\phi \in C_0(\mathbb{R}^n)$, $\psi \in C_0(\mathbb{R}^n)$ and $\eta \in L^1(E)$. Since $u_j \to u$ a.e. in E then also $\phi(u_j) \to \phi(u)$ a.e. in E. Using Lebesgue's dominated convergence theorem we obtain that $\eta\phi(u_j) \to \eta\phi(u)$ in $L^1(E)$. Furthermore, by Theorem 4.1.1 (ii) we get

$$\psi(v_j) \overset{*}{\rightharpoonup} \overline{\psi} \text{ in } L^\infty(E) \quad \text{and} \quad \overline{\psi}(x) = \langle \nu_x, \psi\rangle. \tag{4.1.54}$$

Let us denote by $\phi \otimes \psi$ the tensor product. Then

$$\begin{aligned}\int_E \eta(\phi\otimes\psi)(u_j,v_j)\,\mathrm{d}x &= \int_E \eta\phi(u_j)\psi(v_j)\,\mathrm{d}x\\ &\to \int_E \eta\phi(u)\langle\nu_x,\psi\rangle\,\mathrm{d}x,\\ &= \int_E \eta\langle\delta_{u(x)}\otimes\nu_x,\phi\otimes\psi\rangle\,\mathrm{d}x,\end{aligned} \tag{4.1.55}$$

as $j \to \infty$. Then, by the density of linear combinations of products $\phi\otimes\psi$ in $C_0(\mathbb{R}^{n+m})$, we obtain

$$(\phi\otimes\psi)(u_j,v_j) \overset{*}{\rightharpoonup} \langle\delta_{u(\cdot)}\otimes\nu_\cdot,\phi\otimes\psi\rangle \text{ in } L^\infty(E), \tag{4.1.56}$$

which completes the proof. □

The following Lemma comes from [34, Theorem 1.2]. Use of this lemma in the proof of existence is a crucial step of the main theorem of this chapter.

Lemma 4.1.4. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set of finite measure, let $Q = \Omega \times (0,T)$ and let an operator $A(x,t,s,\xi) : \Omega \times \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, satisfy the following conditions:*

(i) $A(x,t,s,\xi)$ is a Carathéodory function (measurable w.r.t. (x,t) and continuous w.r.t. (s,ξ));

(ii) For all $x \in \Omega$, $t \in [0,T]$, $s \in \mathbb{R}^m$ and $\xi_1,\xi_2 \in \mathbb{R}^n$, $\xi_1 \neq \xi_2$

$$\Big(A(x,t,s,\xi_1) - A(x,t,s,\xi_2)\Big)\cdot\Big(\xi_1-\xi_2\Big) > 0; \tag{4.1.57}$$

(iii) There exist positive constants c_1, c_2 such that for $p > 1$ it is held that

$$A(x,t,s,\xi)\cdot\xi \geq c_1|\xi|^p, \tag{4.1.58}$$

and

$$|A(x,t,s,\xi)| \leq c_2|\xi|^{p-1}. \tag{4.1.59}$$

Let $y_j : Q \to \mathbb{R}^m$ and $z_j : Q \to \mathbb{R}^n$ be sequences of measurable functions such that *(iv)* $y_j \to y$ *a.e. in* Q*;*

(v) $z_j \rightharpoonup z$ *in* $L^p(Q)$ *and* $A(x,t,y_j,z_j) \rightharpoonup A^*$ *in* $L^{\frac{p}{p-1}}(Q)$*;*

(vi)

$$\limsup_{j\to\infty} \int_Q A(x,t,y_j,z_j)\cdot z_j \,\mathrm{d}x\,\mathrm{d}t \le \int_Q A^*\cdot z\,\mathrm{d}x\,\mathrm{d}t. \tag{4.1.60}$$

Then there exists a subsequence of $\{z_j\}$*, such that* $z_j \to z$ *in* $L^p(Q)$*.*

Proof. We start the proof with applying Lemma 4.1.2 to the function $A(x,t,y_j,z_j)\cdot z_j$. Since condition (iii) is held, $A(x,t,y_j,z_j)\cdot z_j$ is positive for every $j\in\mathbb{N}$, the sequence of negative parts is relatively weakly compact in $L^1(Q)$. Then we obtain

$$\limsup_{j\to\infty} \int_Q A(x,t,y_j,z_j)\cdot z_j\,\mathrm{d}x\,\mathrm{d}t \ge \int_Q\int_{\mathbb{R}^m\times\mathbb{R}^n} A(x,t,s,\lambda)\cdot\lambda\,\mathrm{d}\mu_{x,t}(s,\lambda)\,\mathrm{d}x\,\mathrm{d}t \tag{4.1.61}$$

where μ_x is the Young measure generated by the sequence (y_n, z_n). By Lemma 4.1.3 we may characterize this measure more precisely, i.e

$$\mu_{x,t}(s,\lambda) = \delta_{y(x,t)}(s)\otimes\nu_{x,t}(\lambda), \tag{4.1.62}$$

since y_j converges to y a.e. in Q and sequence $\{z_j\}$ generate the Young measure $\nu_{x,t}$. Then

$$\int_Q\int_{\mathbb{R}^m\times\mathbb{R}^n} A(x,t,s,\lambda)\cdot\lambda\,\mathrm{d}\mu_{x,t}(s,\lambda)\,\mathrm{d}x\,\mathrm{d}t = \int_Q\int_{\mathbb{R}^n} A(x,t,y,\lambda)\cdot\lambda\,\mathrm{d}\nu_{x,t}(\lambda)\,\mathrm{d}x\,\mathrm{d}t. \tag{4.1.63}$$

Moreover, the set $\{z_j\}$ is bounded in $L^p(Q)$ hence it is relatively weakly compact in $L^1(Q)$. Using Theorem 4.1.1 we obtain that $z(x,t) = \int_{\mathbb{R}^n}\lambda\,\mathrm{d}\nu_{x,t}(\lambda)$. Applying the same argument to function $A(\cdot,\cdot,\cdot,\cdot)$ we get $A^* = \int_{\mathbb{R}^n} A(x,t,y,\lambda)\,\mathrm{d}\nu_{x,t}(\lambda)$. By combining these results with condition (vi) and (4.1.61) we get

$$\begin{aligned}\int_Q\int_{\mathbb{R}^n} A(x,t,y,\lambda)\cdot\lambda\,\mathrm{d}\nu_{x,t}(\lambda)\,\mathrm{d}x\,\mathrm{d}t &\le \int_Q A^*\cdot z\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_Q\int_{\mathbb{R}^n} A(x,t,y,\lambda)\,\mathrm{d}\nu_{x,t}(\lambda)\cdot\int_{\mathbb{R}^n}\lambda\,\mathrm{d}\nu_{x,t}(\lambda)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.1.64}$$

Let us define

$$h(x,t,\lambda) := \left(A(x,t,y,\lambda) - A\left(x,t,y,\int_{\mathbb{R}^n}\xi\,\mathrm{d}\nu_{x,t}(\xi)\right)\right)\cdot\left(\lambda - \int_{\mathbb{R}^n}\xi\,\mathrm{d}\nu_{x,t}(\xi)\right). \tag{4.1.65}$$

Monotonicity of A with respect to last variable implies that

$$\int_Q\int_{\mathbb{R}^n} h(x,t,\lambda)\,\mathrm{d}\nu_{x,t}(\lambda)\,\mathrm{d}x\,\mathrm{d}t \ge 0. \tag{4.1.66}$$

Thus

$$\begin{aligned}&\int_Q\int_{\mathbb{R}^n} h(x,t,\lambda)\,\mathrm{d}\nu_{x,t}(\lambda)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad= \int_Q\int_{\mathbb{R}^n} A(x,t,y,\lambda)\cdot\left(\lambda - \int_{\mathbb{R}^n}\xi\,\mathrm{d}\nu_{x,t}(\xi)\right)\,\mathrm{d}\nu_{x,t}(\lambda)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad\quad - \int_Q\int_{\mathbb{R}^n} A\left(x,t,y,\int_{\mathbb{R}^n}\xi\,\mathrm{d}\nu_{x,t}(\xi)\right)\cdot\left(\lambda - \int_{\mathbb{R}^n}\xi\,\mathrm{d}\nu_{x,t}(\xi)\right)\,\mathrm{d}\nu_{x,t}(\lambda)\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{4.1.67}$$

The second term of right-hand side is equal to zero. Indeed, changing the variables we obtain

$$\begin{aligned}
&\int_Q \int_{\mathbb{R}^n} A\left(x,t,y,\int_{\mathbb{R}^n} \xi \,\mathrm{d}\nu_{x,t}(\xi)\right)\cdot\left(\lambda - \int_{\mathbb{R}^n} \xi \,\mathrm{d}\nu_{x,t}(\xi)\right) \mathrm{d}\nu_{x,t}(\lambda)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad= \int_Q \int_{\mathbb{R}^n} A\left(x,t,y,\int_{\mathbb{R}^n} \xi \,\mathrm{d}\nu_{x,t}(\xi)\right)\cdot \lambda \,\mathrm{d}\nu_{x,t}(\lambda)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\quad - \int_Q \int_{\mathbb{R}^n} A\left(x,t,y,\int_{\mathbb{R}^n} \xi \,\mathrm{d}\nu_{x,t}(\xi)\right)\cdot \int_{\mathbb{R}^n} \xi \,\mathrm{d}\nu_{x,t}(\xi)\,\mathrm{d}\nu_{x,t}(\lambda)\,\mathrm{d}x\,\mathrm{d}t \qquad (4.1.68)\\
&\quad= \int_Q A\left(x,t,y,\int_{\mathbb{R}^n} \xi \,\mathrm{d}\nu_{x,t}(\xi)\right)\cdot \int_{\mathbb{R}^n} \xi \,\mathrm{d}\nu_{x,t}(\xi)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\quad - \int_Q A\left(x,t,y,\int_{\mathbb{R}^n} \xi \,\mathrm{d}\nu_{x,t}(\xi)\right)\cdot \int_{\mathbb{R}^n} \xi \,\mathrm{d}\nu_{x,t}(\xi)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

Coming back to (4.1.67) we get

$$\begin{aligned}
\int_Q \int_{\mathbb{R}^n} h(x,t,\lambda)\,\mathrm{d}\nu_{x,t}(\lambda)\,\mathrm{d}x\,\mathrm{d}t = &\int_\Omega \int_{\mathbb{R}^n} A(x,t,y,\lambda)\cdot\lambda\,\mathrm{d}\nu_{x,t}(\lambda)\,\mathrm{d}x\,\mathrm{d}t\\
&- \int_\Omega \int_{\mathbb{R}^n} A(x,t,y,\lambda)\,\mathrm{d}\nu_{x,t}(\lambda)\cdot \int_{\mathbb{R}^n} \lambda\,\mathrm{d}\nu_{x,t}(\lambda)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \qquad (4.1.69)$$

Using (4.1.64) and (4.1.66) we get

$$\int_{\mathbb{R}^n} h(x,t,\lambda)\,\mathrm{d}\nu_{x,t}(\lambda) = 0, \qquad (4.1.70)$$

for a.a. $(x,t)\in Q$. Using (4.1.65) and properties of a measure $\nu_{x,t}$ we obtain that

$$h(x,t,\lambda) = 0 \quad\iff\quad \lambda - \int_{\mathbb{R}^n} \xi\,\mathrm{d}\nu_{x,t}(\xi) = 0. \qquad (4.1.71)$$

Moreover, by (4.1.70) the support of function $h(x,t,\cdot)$ and measure $\nu_{x,t}$ are disjoint for a.a $(x,t)\in Q$. This implies $\nu_{x,t} = \delta_{z(x,t)}$ a.e..

By Lemma 4.1.1 we obtain that z_j converges to z in measure. Then there exists a subsequence of $\{z_j\}$ such that $z_j \to z$ a.e.. Moreover, using Lemma 4.1.3 and assumption (vi) we get

$$\limsup_{j\to\infty} \int_Q A(x,t,y_j,z_j)\cdot z_j\,\mathrm{d}x\,\mathrm{d}t \le \int_Q A(x,t,y,z)\cdot z\,\mathrm{d}x\,\mathrm{d}t \le \liminf_{j\to\infty} \int_Q A(x,t,y_j,z_j)\cdot z_j\,\mathrm{d}x\,\mathrm{d}t.$$

Let us define $g_n = A(x,t,y_n,z_n)\cdot z_n$ and $g = A(x,t,y,z)\cdot z$. We know that

$$g_n \ge 0, \quad g\in L^1(Q), \quad \int_Q g_n\,\mathrm{d}x\,\mathrm{d}t \to \int_Q g\,\mathrm{d}x\,\mathrm{d}t, \quad g_n \to g \text{ a.e. in } Q. \qquad (4.1.72)$$

Furthermore

$$\int_Q |g_n - g|\,\mathrm{d}x\,\mathrm{d}t = \int_Q (g_n - g)\,\mathrm{d}x\,\mathrm{d}t + 2\int_Q \max\{(g-g_n),0\}\,\mathrm{d}x. \qquad (4.1.73)$$

By Lebesgue's dominated convergence theorem we get that the sequence $\{A(x,t,y_j,z_j)\cdot z_j\}$ converges to $A(x,t,y,z)\cdot z$ in $L^1(Q)$ as j goes to ∞. Thus, it is uniformly integrable and by assumption (iii) also the sequence $|z_j|^p$ is uniformly integrable. Using Vitali's Theorem [53, Lemma 2.11] yields that $z_j \to z$ in $L^p(Q)$, which completes the proof.

□

4.2 Formulation of problem

Let us define $W^{1,2}_{\boldsymbol{g_u}}(\Omega,\mathbb{R}^3) := \{u \in W^{1,2}(\Omega,\mathbb{R}^3) : \boldsymbol{u} = \boldsymbol{g_u} \text{ on } \partial\Omega\}$. Now, we are ready to formulate the solution's definition and the main theorem of this chapter.

Definition 4.2.1. *Solution to Mróz model*
Let $q \in (1, \frac{5}{4})$. The triple of functions $(\boldsymbol{u}, \boldsymbol{T}, \theta)$ is a weak solution to the system (1.2.2) *when*

$$\begin{aligned} &\boldsymbol{u} \in L^2(0,T,W^{1,2}_{\boldsymbol{g_u}}(\Omega,\mathbb{R}^3)),\\ &\boldsymbol{T} \in L^2(Q,\mathcal{S}^3),\\ &\theta \in L^q(0,T,W^{1,q}(\Omega)) \cap C([0,T],W^{-2,2}(\Omega)), \end{aligned} \tag{4.2.1}$$

and it satisfies

$$\int_0^T\int_\Omega \boldsymbol{T} : \nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t, \tag{4.2.2}$$

where $\boldsymbol{T}^d$ is deviatoric part of Cauchy stress tensor $\boldsymbol{T} = \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}})$, and

$$\begin{aligned} &-\int_0^T\int_\Omega \theta\phi_t\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \theta_0(x)\phi(0,x)\,\mathrm{d}x\\ &\quad+\int_0^T\int_\Omega \nabla\theta\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_{\partial\Omega} g_\theta\phi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \boldsymbol{T}^d : \boldsymbol{G}(\theta,\boldsymbol{T}^d)\phi\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{4.2.3}$$

for every test function $\boldsymbol{\varphi} \in C^\infty([0,T],C^\infty_c(\Omega,\mathbb{R}^3))$ and $\phi \in C^\infty_c([-\infty,T),C^\infty(\Omega))$. Furthermore, the visco-elastic strain tensor can be recovered from the equation on its evolution, i.e.

$$\boldsymbol{\varepsilon}^p(x,t) = \boldsymbol{\varepsilon}^p_0(x) + \int_0^t \boldsymbol{G}(\theta(x,\tau),\boldsymbol{T}^d(x,\tau))\,\mathrm{d}\tau, \tag{4.2.4}$$

for a.e. $x \in \Omega$ and $t \in [0,T)$. Then $\boldsymbol{\varepsilon}^{\mathbf{p}}$ belongs to $W^{1,2}(0,T,L^2(\Omega,\mathcal{S}^3_d))$.

Theorem 4.2.1. *Let initial conditions satisfy $\theta_0 \in L^1(\Omega)$, $\boldsymbol{\varepsilon}^p_0 \in L^2(\Omega,\mathcal{S}^3_d)$, boundary conditions satisfy $\boldsymbol{g_u} \in L^2(0,T,H^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^3))$ and $g_\theta \in L^2(0,T,L^2(\partial\Omega))$, function $\boldsymbol{f} \in L^2(0,T,W^{-1,2}(\Omega,\mathbb{R}^3))$ and function $\boldsymbol{G}(\cdot,\cdot)$ satisfy the same condition as in Assumptions 4.0.1. Then there exists a weak solution according to Definition 4.2.1, to system* (1.2.2)*.*

4.3 Proof of Theorem 4.2.1

Proof of Theorem 4.2.1 contains a few steps. Some elements of this reasoning can be found in Chapter 2 and in Chapter 3 and for more details we refer the reader to these chapters. This section is organized as follows: in Section 4.3.1 we concentrate on transforming of full thermo-visco-elastic problem into a homogeneous boundary-value problem, which allows us to focus on the homogeneous boundary-value problem. Sections 4.3.2 and 4.3.3 are dedicated to show the uniform boundedness of approximate solutions. Results presented in Section 4.3.3 are valid for other models presented in this dissertation and we refer to it in the next chapters. Finally, due to use of two level approximation, we pass to limit with $l \to \infty$ in Section 4.3.4 and with $k \to \infty$ in Section 4.3.5.

4.3.1 Transformation to a homogeneous boundary-value problem

The idea of transformation to homogeneous boundary-value problems was discuss in Chapter 2. Let us consider two systems of equations with given initial and boundary data.

$$\begin{cases} -\operatorname{div}\tilde{\boldsymbol{T}} &= \boldsymbol{f} \quad \text{in } \Omega\times(0,T),\\ \tilde{\boldsymbol{T}} &= \boldsymbol{D}\tilde{\boldsymbol{\varepsilon}} \quad \text{in } \Omega\times(0,T),\\ \tilde{\boldsymbol{u}} &= \boldsymbol{g}_{\boldsymbol{u}} \quad \text{on } \partial\Omega\times(0,T), \end{cases} \tag{4.3.1}$$

and

$$\begin{cases} \tilde{\theta}_t-\Delta\tilde{\theta} &= 0 \quad \text{in } \Omega\times(0,T),\\ \frac{\partial\tilde{\theta}}{\partial\boldsymbol{n}} &= g_\theta \quad \text{on } \partial\Omega\times(0,T),\\ \tilde{\theta}(x,0) &= \tilde{\theta}_0 \quad \text{in } \Omega. \end{cases} \tag{4.3.2}$$

Lemma 4.3.1. *Let initial condition satisfy $\tilde{\theta}_0\in L^2(\Omega)$, boundary conditions satisfy $\boldsymbol{g}_{\boldsymbol{u}}\in L^p(0,T,W^{1-\frac{1}{p},p}(\partial\Omega))^3$, $g_\theta\in L^2(0,T,L^2(\partial\Omega))$ and also the volume force $\boldsymbol{f}$ belongs to $L^p(0,T,W^{-1,p}(\Omega,\mathbb{R}^3))$. Then, there exists a solution of systems (4.3.1) and (4.3.2). Additionally, the estimate holds:*

$$\|\tilde{\boldsymbol{u}}\|_{L^p(0,T,W^{1,p}(\Omega))}\le C_1(\|\boldsymbol{g}_{\boldsymbol{u}}\|_{L^p(0,T,W^{1-\frac{1}{p},p}(\partial\Omega,\mathbb{R}^3))}+\|\boldsymbol{f}\|_{L^p(0,T,W^{-1,p}(\Omega,\mathbb{R}^3))},$$

$$\|\tilde{\theta}\|_{L^\infty(0,T,L^1(\Omega))}+\|\tilde{\theta}\|_{L^2(0,T,W^{1,2}(\Omega))}\le C_2\left(\|g_\theta\|_{L^2(0,T,L^2(\partial\Omega))}+\|\tilde{\theta}_0\|_{L^2(\Omega)}\right).$$

Moreover, $\tilde{\theta}$ belongs to $C([0,T],L^2(\Omega))$.

Proof. From the trace theorem [82, Chapter II] there exists $\tilde{\boldsymbol{g}}\in L^p(0,T,W^{1,p}(\Omega,\mathbb{R}^3))$ such that $\tilde{\boldsymbol{g}}|_{\partial\Omega}=\boldsymbol{g}_{\boldsymbol{u}}$. Therefore, instead of finding the solution $\tilde{\boldsymbol{u}}$ to (4.3.1) we focus on finding the solution $\tilde{\boldsymbol{u}}_1$ to system

$$\begin{cases} -\operatorname{div}\boldsymbol{D}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_1) &= \boldsymbol{f}+\operatorname{div}\boldsymbol{D}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{g}}) \quad \text{in } \Omega\times(0,T),\\ \tilde{\boldsymbol{u}}_1 &= 0 \quad \text{on } \partial\Omega\times(0,T). \end{cases} \tag{4.3.3}$$

It can be immediately observed that solution to (4.3.1) $\tilde{\boldsymbol{u}}$ is a sum of $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{g}}$. The existence of solution to elastostatic problem (4.3.3) and its estimate is obtained by usage [82, Corollary 4.4]. The proof and estimates for heat equation are standard. This results can be found, for example, in [29]. □

For Mróz model, i.e. $p=2$, the boundary data satisfy the assumptions from Theorem 4.2.1 and we may transform the problem into homogeneous boundary value problem. Then data are coming into the system by shifts of the solutions in nonlinear function $\boldsymbol{G}(\cdot,\cdot)$. Thus, it is sufficient to consider the following system of equations

$$\begin{cases} -\operatorname{div}\boldsymbol{T}=0,\\ \boldsymbol{T}=\boldsymbol{D}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p),\\ \boldsymbol{\varepsilon}^p_t=\boldsymbol{G}(\tilde{\theta}+\theta,\boldsymbol{T}^d+\boldsymbol{T}^d),\\ \theta_t-\Delta\theta=\left(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d\right):\boldsymbol{G}(\tilde{\theta}+\theta,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d), \end{cases} \tag{4.3.4}$$

with initial conditions:

$$\begin{cases} \theta &= \theta_0-\tilde{\theta}_0 \quad \text{in } \Omega,\\ \boldsymbol{\varepsilon}^p &= \boldsymbol{\varepsilon}^p_0 \quad \text{in } \Omega, \end{cases} \tag{4.3.5}$$

and homogeneous boundary condition on displacement and temperature.

We start the proof with the definition of approximate system of equations and approximate solutions, see (2.2.8). Moreover, we know that the approximate solutions are absolutely continuous on some time interval $(0,t^*)$, see Lemma 2.2.1. Our goal is to show that these solutions exist on the whole time interval $(0,T)$ and that limits of these sequences exist and fulfill Definition 4.2.1.

4.3.2 Potential Energy estimates

As we mentioned in the introduction, the uniform estimates for approximate solutions are consequences of finite initial energy. Density of energy in quasi-static case consists of potential and thermal energy. Let us start with the estimates regarding potential energy, see Definition 1.4.1. The estimates for the temperature are the subject of next section.

Lemma 4.3.2. *There exists a constant C, which is independent of k and l, such that*

$$\sup_{t\in[0,T]} \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(t) + c\|\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}\|^2_{L^2(0,T,L^2(\Omega))} \leq C. \tag{4.3.6}$$

Proof. The potential energy of approximate solutions is an absolutely continuous function. Hence, calculating the time derivative of $\mathcal{E}(t)$ for a.a. $t \in [0,T]$ we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) = &\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}) - \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) : (\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}))_t \,\mathrm{d}x \\ &- \int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}) - \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) : (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t \,\mathrm{d}x.\end{aligned} \tag{4.3.7}$$

Using the approximate system of equations we the changes of potential energy. At the beginning let us multiply $(2.2.8)_{(1)}$ by $\{(\alpha^n_{k,l})_t\}$ for each $n \leq k$. After summing obtained equations over $n = 1,...,k$ we get

$$\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}) - \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) : (\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l})_t \,\mathrm{d}x = 0. \tag{4.3.8}$$

Hence, the first term in (4.3.7) vanishes. Consequently, we multiply $(2.2.8)_{(4)}$ by $\delta^m_{k,l}$ for every $m = 1,...,l$. Summing over $m = 1,...,l$, we obtain the identity, which is equivalent to

$$\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t : \boldsymbol{T}_{k,l} \,\mathrm{d}x = \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{T}_{k,l} \,\mathrm{d}x. \tag{4.3.9}$$

And thus

$$\frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) = -\int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{T}^d_{k,l} \,\mathrm{d}x, \tag{4.3.10}$$

where we use the properties of traceless matrices, i.e. if $\boldsymbol{A}^d = \boldsymbol{A}$ then $\boldsymbol{A}^d : \boldsymbol{B} = \boldsymbol{A}^d : \boldsymbol{B}^d$. Using Assumption 4.0.1 and the Young inequality we get

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) = &-\int_\Omega (\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) \,\mathrm{d}x \\ &+ \int_\Omega \tilde{\boldsymbol{T}}^d : \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) \,\mathrm{d}x \\ &\leq -\beta\|\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}\|^2_{L^2(\Omega)} + \|\tilde{\boldsymbol{T}}^d\|_{L^2(\Omega)}\|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^2(\Omega)} \\ &\leq -\beta\|\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}\|^2_{L^2(\Omega)} + c(\epsilon)\|\tilde{\boldsymbol{T}}^d\|^2_{L^2(\Omega)} + \epsilon\|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|^2_{L^2(\Omega)}\end{aligned}$$

where $\epsilon < \frac{\beta}{2C^2}$, with a constant C coming from Assumption 4.0.1. Hence, we estimate the last term as follows

$$\epsilon\|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|^2_{L^2(\Omega)} \leq C^2\epsilon\|\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}\|^2_{L^2(\Omega)}, \tag{4.3.11}$$

and the choice of ϵ implies that $C^2\epsilon = \frac{\beta}{2}$. Finally, integrating over $(0,t)$, with $0 \le t \le T$ we obtain

$$\begin{aligned}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(t) + \frac{\beta}{2}\|\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}\|^2_{L^2(0,T,L^2(\Omega))} \\ \le c(\epsilon)\|\tilde{\boldsymbol{T}}^d\|^2_{L^2(0,T,L^2(\Omega))} + \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(0),\end{aligned} \tag{4.3.12}$$

which completes the proof. □

Remark. *Using Lemma 4.3.2 and Assumption 4.0.1 we immediately observe that the following inequalities hold*

$$\begin{aligned}\|\boldsymbol{T}_{k,l}\|_{L^\infty(0,T,L^2(\Omega,\mathcal{S}^3))} &\le c_1, \\ \|\boldsymbol{T}^d_{k,l}\|_{L^2(0,T,L^2(\Omega,\mathcal{S}^3))} &\le c_2, \\ \|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^2(0,T,L^2(\Omega,\mathcal{S}^3))} &\le c_3, \\ \|(\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^1(0,T,L^1(\Omega,\mathcal{S}^3))} &\le c_4,\end{aligned} \tag{4.3.13}$$

for all $k,l \in \mathbb{N}$. Moreover, constants c_1, c_2, c_3 as well as c_4 are independent of approximation parameters k and l. Uniform boundedness from $(4.3.13)_{(1)-(3)}$ implies existence of weak limits for each of these quantities.

Lemma 4.3.3. *The sequence $\{(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t\}$ is uniformly bounded in $L^2(0,T,(H^s(\Omega,\mathcal{S}^3))')$ with respect to l.*

Proof. Let $\boldsymbol{\varphi} \in L^2(0,T,H^s(\Omega,\mathcal{S}^3))$. Let us notice that $(P^k + P^{l,k}_{L^2})(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t = (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t$. Using the orthogonality of the subspaces $\mathrm{lin}\{\boldsymbol{\varepsilon}(\boldsymbol{w}_1),\dots,\boldsymbol{\varepsilon}(\boldsymbol{w}_k)\}$ and $\mathrm{lin}\{\boldsymbol{\zeta}_1,\dots,\boldsymbol{\zeta}_l\}$ in the sense of scalar product $(\cdot,\cdot)_{\boldsymbol{D}}$ we obtain

$$\begin{aligned}\int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, \boldsymbol{\varphi})_{\boldsymbol{D}}|\,\mathrm{d}t &= \int_0^T |((P^k + P^{l,k}_{L^2})(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, \boldsymbol{\varphi})_{\boldsymbol{D}}|\,\mathrm{d}x \\ &= \int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, (P^k + P^{l,k}_{L^2})\boldsymbol{\varphi})_{\boldsymbol{D}}|\,\mathrm{d}t \\ &\le \int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, P^k\boldsymbol{\varphi})_{\boldsymbol{D}}|\,\mathrm{d}t + \int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, P^{l,k}_{L^2}\boldsymbol{\varphi})_{\boldsymbol{D}}|\,\mathrm{d}t,\end{aligned} \tag{4.3.14}$$

where P^k and $P^{l,k}_{L^2}$ are projections on $\mathrm{lin}\{\boldsymbol{\varepsilon}(\boldsymbol{w}_n)\}_{n=1}^k$ and $\mathrm{lin}\{\boldsymbol{\zeta}^k_m\}_{m=1}^l$, respectively, in the sense of $(\cdot,\cdot)_{\boldsymbol{D}}$, see Definition 2.1.2. Then, we may estimate

$$\begin{aligned}\int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, \varphi)_{\boldsymbol{D}}|\,\mathrm{d}t &\le \int_0^T |\int_\Omega \boldsymbol{D}\boldsymbol{G}(\theta_{k,l} + \tilde{\theta}, \boldsymbol{T}^d_{k,l} + \tilde{\boldsymbol{T}}^d)P^k\boldsymbol{\varphi}\,\mathrm{d}x|\,\mathrm{d}t \\ &\quad + \int_0^T |\int_\Omega \boldsymbol{D}\boldsymbol{G}(\theta_{k,l} + \tilde{\theta}, \boldsymbol{T}^d_{k,l} + \tilde{\boldsymbol{T}}^d)P^{l,k}_{L^2}\boldsymbol{\varphi}\,\mathrm{d}x|\,\mathrm{d}t \\ &\le d\int_0^T |\int_\Omega \boldsymbol{G}(\theta_{k,l} + \tilde{\theta}, \boldsymbol{T}^d_{k,l} + \tilde{\boldsymbol{T}}^d)P^k\boldsymbol{\varphi}\,\mathrm{d}x|\,\mathrm{d}t \\ &\quad + d\int_0^T |\int_\Omega \boldsymbol{G}(\theta_{k,l} + \tilde{\theta}, \boldsymbol{T}^d_{k,l} + \tilde{\boldsymbol{T}}^d)(P^{l,k}_{H^s} \circ (Id - P^k))\boldsymbol{\varphi}\,\mathrm{d}x|\,\mathrm{d}t,\end{aligned} \tag{4.3.15}$$

where we use the identity (2.1.31) in the last inequality. We continue the estimates as follows

$$
\begin{aligned}
\int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, \boldsymbol{\varphi})_{\boldsymbol{D}}| \,\mathrm{d}t &\le d \int_0^T |\int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) P^k \boldsymbol{\varphi} \,\mathrm{d}x| \,\mathrm{d}t \\
&\quad + d \int_0^T |\int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) (P^{l,k}_{H^s} \circ (Id - P^k)) \boldsymbol{\varphi} \,\mathrm{d}x| \,\mathrm{d}t \\
&\le d \int_0^T \|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^2(\Omega)} \|P^k \boldsymbol{\varphi}\|_{L^2(\Omega)} \,\mathrm{d}t \\
&\quad + d \int_0^T \|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^2(\Omega)} \|(P^{l,k}_{H^s} \circ (Id - P^k)) \boldsymbol{\varphi}\|_{L^2(\Omega)} \,\mathrm{d}t \\
&\le d\tilde{c} \int_0^T \|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^2(\Omega)} \|P^k \boldsymbol{\varphi}\|_{H^s(\Omega)} \,\mathrm{d}t \\
&\quad + d\tilde{c} \int_0^T \|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^2(\Omega)} \|(P^{l,k}_{H^s} \circ (Id - P^k)) \boldsymbol{\varphi}\|_{H^s(\Omega)} \,\mathrm{d}t \\
&\le dc(k)\tilde{c} \int_0^T \|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^2(\Omega)} \|\boldsymbol{\varphi}\|_{H^s(\Omega)} \,\mathrm{d}t \\
&\quad + d\tilde{c} \int_0^T \|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^2(\Omega)} \|(Id - P^k) \boldsymbol{\varphi}\|_{H^s(\Omega)} \,\mathrm{d}t \\
&\le dc(k)\tilde{c} \int_0^T \|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^2(\Omega)} \|\boldsymbol{\varphi}\|_{H^s(\Omega)} \,\mathrm{d}t \\
&\quad + dc(k)\tilde{c} \int_0^T \|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^2(\Omega)} \|\boldsymbol{\varphi}\|_{H^s(\Omega)} \,\mathrm{d}t \\
&\le 2dc(k)\tilde{c} \|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^2(0,T,L^2(\Omega))} \|\boldsymbol{\varphi}\|_{L^2(0,T,H^s(\Omega))},
\end{aligned}
\tag{4.3.16}
$$

where $\tilde{c}$ is an optimal embedding constant of $H^s(\Omega, \mathcal{S}^3) \subset L^2(\Omega, \mathcal{S}^3)$. Consequently, there exists $C(k) > 0$ such that

$$
\sup_{\substack{\boldsymbol{\varphi} \in L^2(0,T,H^s(\Omega)) \\ \|\boldsymbol{\varphi}\|_{L^2(0,T,H^s(\Omega))} \le 1}} \int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, \boldsymbol{\varphi})_{\boldsymbol{D}}| \,\mathrm{d}t \le C(k) \tag{4.3.17}
$$

and hence the sequence $\{(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t\}$ is uniformly bounded in $L^2(0, T, (H^s(\Omega, \mathcal{S}^3))')$. □

4.3.3 Uniform boundedness of temperature

Results presented in this section hold for every model considered in this dissertation. Difference between considered models result from assumptions on the function describing the evolution of visco-elastic strain. The heat equation stays the same in all cases. We draw our attention to all small differences which may appear in next chapters.

Lemma 4.3.4. *The sequence $\{\theta_{k,l}\}$ is uniformly bounded in $L^\infty(0, T; L^1(\Omega))$ with respect to k and l.*

Proof. It can be immediately observed that

$$
\sup_{0 \le t \le T} \|\theta_{k,l}(t)\|_{L^1(\Omega)} \le C \|(\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^1(0,T,L^1(\Omega))} + \|\theta_0\|_{L^1(\Omega)}
$$

and Lemma 4.3.2 holds. □

Lemma 4.3.4 implies that the internal energy of Ω is finite at any time $t \in [0,T]$ if the initial internal energy is finite. It is possible to prove better estimates for the temperature, however this estimate is uniform only with respect to l and not with respect to k. We provide the details in the proceeding lemma.

Lemma 4.3.5. *There exists a constant C, depending on the domain Ω and the time interval $(0,T)$, such that for every $k \in \mathbb{N}$*

$$\sup_{0\le t\le T} \|\theta_{k,l}(t)\|^2_{L^2(\Omega)} + \|\theta_{k,l}\|^2_{L^2(0,T,W^{1,2}(\Omega))} + \|(\theta_{k,l})_t\|^2_{L^2(0,T,W^{-1,2}(\Omega))} \\ \le C\Big(\|\mathcal{T}_k\Big((\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\Big)\|^2_{L^2(0,T,L^2(\Omega))} + \|\mathcal{T}_k(\theta_0)\|^2_{L^2(\Omega)}\Big). \tag{4.3.18}$$

The proof follows from the standard tools for parabolic equations, see e.g. Evans [29].

Proof. Using standard tools for parabolic equations we obtain

$$\sup_{0\le t\le T} \|\theta_{k,l}(t)\|^2_{L^2(\Omega)} + \|\nabla\theta_{k,l}\|^2_{L^2(0,T,L^2(\Omega))} + \|(\theta_{k,l})_t\|^2_{L^2(0,T,W^{-1,2}(\Omega))} \\ \le C\Big(\|T_k((\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}))\|^2_{L^2(0,T,L^2(\Omega))} + \|T_k(\theta_0)\|_{L^2(\Omega)}\Big). \tag{4.3.19}$$

Due to the Neumann boundary condition we cannot use Poincaré inequality in a straightforward way. Let us define an operator $(\theta)_\Omega$ as an average of function θ on Ω. Then

$$\|\theta_{k,l}\|_{L^2(0,T,L^2(\Omega))} \le \|\theta_{k,l} - (\theta_{k,l})_\Omega\|_{L^2(0,T,L^2(\Omega))} + \|(\theta_{k,l})_\Omega\|_{L^2(0,T,L^2(\Omega))}.$$

Applying the Poincaré inequality for functions from $W^{1,2}(\Omega)$ we get boundedness of the first term on the left-hand side by $C\|\nabla\theta_{k,l}\|_{L^2(0,T,L^2(\Omega))}$. For the second one we use simple calculations (Hölder inequality)

$$\begin{aligned} \|(\theta_{k,l})_\Omega\|^2_{L^2(0,T,L^2(\Omega))} &= \int_0^T \int_\Omega (\theta_{k,l})^2_\Omega \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_\Omega \left(\frac{1}{meas(\Omega)} \int_\Omega \theta_{k,l}\,\mathrm{d}y\right)^2 \mathrm{d}x\,\mathrm{d}t \\ &\le \frac{1}{meas(\Omega)^2} \int_0^T \int_\Omega \left((\int_\Omega |\theta_{k,l}|^2\,\mathrm{d}y)^{\frac12} meas(\Omega)^{\frac12}\right)^2 \mathrm{d}x\,\mathrm{d}t \\ &= \frac{1}{meas(\Omega)} \int_0^T \int_\Omega\int_\Omega |\theta_{k,l}|^2\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

From (4.3.19) we know that $\int_\Omega |\theta_{k,l}(t)|^2\,\mathrm{d}y$ is bounded for every $t \in (0,T)$, what completes the proof. □

All terms on the right-hand side of (4.3.18) are bounded and the boundary is independent of l, hence, for every $k \in \mathbb{N}$ we choose the subsequence $\{\theta_{k,l}\}$ (with respect to l) which converges weakly to θ_k in $L^2(0,T,W^{1,2}(\Omega))$ and the subsequence $\{\theta'_{k,l}\}$ which converges weakly to θ'_k in $L^2(0,T,W^{-1,2}(\Omega))$. Let us denote this subsequence by $\{\theta_{k,l}\}$.

Remark. *The uniform boundedness of solutions implies the global existence of approximate solutions, i.e. existence of solutions $\{\alpha^n_{k,l}(t), \beta^m_{k,l}(t), \gamma^n_{k,l}(t), \delta^m_{k,l}(t)\}$ on the whole time interval $[0,T]$ for each $n = 1,...,k$ and $m = 1,...,l$.*

4.3.4 Limit passage with l going to ∞

Due to low regularity of the right-hand side of heat equation and due to necessity of using Boccardo and Galllouët approach, see Section 3.1. We firstly pass to the limit with approximation parameter for temperature. Let us multiply the system (2.2.8) by smooth time-dependent functions and integrate over $[0,T]$. Then we may rewrite the system as follows. For momentum equation

$$\int_0^T\int_\Omega \boldsymbol{T}_{k,l} : \nabla \boldsymbol{w}_n \varphi_1(t)\,\mathrm{d}x\,\mathrm{d}t = 0, \quad n = 1,\dots,k. \tag{4.3.20}$$

Evolutionary equation for visco-elastic strain can be presented as

$$\begin{aligned}
&\int_0^T\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t : \boldsymbol{D\varepsilon}(\boldsymbol{w}_n)\varphi_2(t)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \int_0^T\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) : \boldsymbol{D\varepsilon}(\boldsymbol{w}_n)\varphi_2(t)\,\mathrm{d}x\,\mathrm{d}t, \quad n=1,...,k,\\
&\int_0^T\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t : \boldsymbol{D\zeta}^k_m\varphi_3(t)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \int_0^T\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) : \boldsymbol{D\zeta}^k_m\varphi_3(t)\,\mathrm{d}x\,\mathrm{d}t, \quad m=1,...,l.
\end{aligned} \tag{4.3.21}$$

And finally, balance of energy may be rewritten as

$$\begin{aligned}
&-\int_0^T\int_\Omega \theta_{k,l}\varphi_4'(t)v_m\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \theta_0(x)\varphi_4(0)v_m\,\mathrm{d}x + \int_0^T\int_\Omega \nabla\theta_{k,l}\cdot\nabla v_m\varphi_4(t)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \int_0^T\int_\Omega \mathcal{T}_k\left((\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) : \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\right)\varphi_4(t)v_m\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{4.3.22}$$

for every $m = 1,...,l$. Furthermore, (4.3.20) – (4.3.22) hold for all test functions $\varphi_1,\varphi_2,\varphi_3 \in C^\infty([0,T])$ and $\varphi_4 \in C_c^\infty([-\infty,T))$.

Using uniform boundedness of approximate solutions sequences obtained in the previous sections, we get (passing to the subsequence if it is necessary) the following convergences

$$\begin{array}{rcl}
\boldsymbol{T}_{k,l} \rightharpoonup \boldsymbol{T}_k & & \text{weakly in } L^2(Q,\mathcal{S}^3),\\
\boldsymbol{T}^d_{k,l} \rightharpoonup \boldsymbol{T}^d_k & & \text{weakly in } L^2(Q,\mathcal{S}^3_d),\\
\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) \rightharpoonup \boldsymbol{\chi}_k & & \text{weakly in } L^2(Q,\mathcal{S}^3_d),\\
\theta_{k,l} \rightharpoonup \theta_k & & \text{weakly in } L^2(0,T,W^{1,2}(\Omega)),\\
\theta_{k,l} \to \theta_k & & \text{a.e. in } Q,\\
(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t \rightharpoonup (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t & & \text{weakly in } L^2(0,T,(H^s(\Omega,\mathcal{S}^3))').
\end{array} \tag{4.3.23}$$

Now passing to the limit in (4.3.20)–(4.3.21) results in

$$\int_0^T\int_\Omega \boldsymbol{T}_k : \nabla\boldsymbol{w}_n\varphi_1(t)\,\mathrm{d}x\,\mathrm{d}t = 0, \quad n=1,\dots,k \tag{4.3.24}$$

and

$$\begin{aligned}
\int_0^T\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t : \boldsymbol{D\varepsilon}(\boldsymbol{w}_n)\varphi_2(t)\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega \boldsymbol{\chi}_k : \boldsymbol{D\varepsilon}(\boldsymbol{w}_n)\varphi_2(t)\,\mathrm{d}x\,\mathrm{d}t, \quad n=1,...,k,\\
\int_0^T\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t : \boldsymbol{D\zeta}^k_m\varphi_3(t)\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega \boldsymbol{\chi}_k : \boldsymbol{D\zeta}^k_m\varphi_3(t)\,\mathrm{d}x\,\mathrm{d}t, \quad m\in\mathbb{N},
\end{aligned} \tag{4.3.25}$$

hold for every test function $\varphi_1, \varphi_2, \varphi_3 \in C^\infty([0,T])$. Density of $\mathrm{lin}\{\boldsymbol{\zeta}_m^k\}_{m=1}^\infty$ in $L^2(\Omega, \mathcal{S}^3)$ implies that

$$\int_0^T \int_\Omega (\boldsymbol{\varepsilon}_k^{\mathbf{p}})_t : \boldsymbol{\varphi} \,\mathrm{d}x \,\mathrm{d}t = \int_0^T \int_\Omega \boldsymbol{\chi}_k : \boldsymbol{\varphi} \,\mathrm{d}x \,\mathrm{d}t \tag{4.3.26}$$

holds for all $\boldsymbol{\varphi} \in C^\infty([0,T], L^2(\Omega, \mathcal{S}^3))$ and then also for all $\boldsymbol{\varphi} \in L^2(0,T; L^2(\Omega, \mathcal{S}^3))$.

It remains to pass to the limit in (4.3.22). To do this, we should identify the limit of the right-hand side term

$$\int_0^T \int_\Omega \mathcal{T}_k \left(\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_{k,l}^d) : (\tilde{\boldsymbol{T}}^d + \boldsymbol{T}_{k,l}^d) \right) \mathrm{d}x \,\mathrm{d}t. \tag{4.3.27}$$

The characterization of the limit is not obvious as we are dealing with a product of two weakly converging sequences. For this purpose we will use Young measures tools which were introduced in Section 4.1. Moreover, we should identify $\boldsymbol{\chi}_k$, which can also be done by the use of Young measures tools.

Lemma 4.3.6. *The following inequality holds for the solution of approximate system*

$$\limsup_{l\to\infty} \int_0^\tau \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_{k,l}^d) : \boldsymbol{T}_{k,l}^d \,\mathrm{d}x \,\mathrm{d}t \le \int_0^t \int_\Omega \boldsymbol{\chi}_k : \boldsymbol{T}_k^d \,\mathrm{d}x \,\mathrm{d}t, \tag{4.3.28}$$

for every $\tau \in (0,T)$.

Proof. For each $\mu > 0, \tau \le T - \mu, s \ge 0$, let $\psi_{\mu,\tau} : \mathbb{R}_+ \to \mathbb{R}_+$ be defined as follows

$$\psi_{\mu,\tau}(s) = \begin{cases} 1 & \text{for} \quad s \in [0,\tau), \\ -\frac{1}{\mu}(s-\tau) + 1 & \text{for} \quad s \in [\tau, \tau+\mu), \\ 0 & \text{for} \quad s \ge \tau + \mu. \end{cases} \tag{4.3.29}$$

Next we shall use (4.3.10) and multiply it by $\psi_{\mu,\tau}(t)$ and integrate over $(0,T)$

$$\int_0^T \frac{d}{d\tau} \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}_{k,l}^{\mathbf{p}}) \,\psi_{\mu,\tau} \,\mathrm{d}t \;= -\int_0^T \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_{k,l}^d) : \boldsymbol{T}_{k,l}^d \,\psi_{\mu,\tau} \,\mathrm{d}x \,\mathrm{d}t. \tag{4.3.30}$$

Let us now integrate by parts the left-hand side of (4.3.30)

$$\int_0^T \frac{d}{d\tau} \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}_{k,l}^{\mathbf{p}}) \,\psi_{\mu,\tau} \,\mathrm{d}t = \frac{1}{\mu} \int_\tau^{\tau+\mu} \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}_{k,l}^{\mathbf{p}}) \,\mathrm{d}t - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}_{k,l}^{\mathbf{p}})(0). \tag{4.3.31}$$

Passing to the limit in (4.3.31) with $l \to \infty$ we obtain

$$\begin{aligned} \liminf_{l\to\infty} \int_0^T & \frac{d}{d\tau} \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}_{k,l}^{\mathbf{p}}) \,\psi_{\mu,\tau} \,\mathrm{d}t \\ &= \liminf_{l\to\infty} \frac{1}{\mu} \int_\tau^{\tau+\mu} \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}_{k,l}^{\mathbf{p}}) \,\mathrm{d}t - \lim_{l\to\infty} \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}_{k,l}^{\mathbf{p}})(0) \\ &\ge \frac{1}{\mu} \int_\tau^{\tau+\mu} \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}_k^{\mathbf{p}}) \,\mathrm{d}t - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}_k^{\mathbf{p}})(0) \end{aligned} \tag{4.3.32}$$

Note that the last inequality holds due to weak lower semicontinuity in $L^2(0,T,L^2(\Omega;\mathcal{S}^3))$. Convergence of the initial potential energy is a consequence of strong convergence of initial condition for visco-elastic strain tensor.

Now the problem is low regularity of $\{(\alpha_k^n)_t\}$. Since α_k^n is a limit of absolutely continuous functions, $(\alpha_k^n)_t$ cannot be used as a test function in (4.3.24). Hence, we use time mollifier to solve this problem. Let us take $\varphi_1(t) = ((\alpha_k^n)_t * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon$, as a test function in (4.3.24), where η_ϵ is a standard mollifier with respect to time and $\epsilon < \min(t_1, T - t_2)$. Then

$$\int_0^T \int_\Omega \boldsymbol{T}_k : \boldsymbol{\varepsilon}(((\alpha_k^n)_t * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon \boldsymbol{w}_n) \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{4.3.33}$$

Summing over $n = 1, ..., k$ and using the properties of convolution we obtain

$$\int_{t_1}^{t_2} \int_\Omega \boldsymbol{D} \left(\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}_k^{\mathbf{p}}\right) * \eta_\epsilon : (\boldsymbol{\varepsilon}(\boldsymbol{u}_k) * \eta_\epsilon)_t \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{4.3.34}$$

Let us take $\boldsymbol{\varphi} = (\boldsymbol{T}_k * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon$, as a test function in (4.3.26). Thus we get

$$\int_0^T \int_\Omega (\boldsymbol{\varepsilon}_k^{\mathbf{p}})_t : (\boldsymbol{T}_k * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon \, \mathrm{d}x = \int_0^T \int_\Omega \boldsymbol{\chi}_k : (\boldsymbol{T}_k * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon \, \mathrm{d}x. \tag{4.3.35}$$

Using the properties of convolution, we rewrite it in the following way

$$\int_{t_1}^{t_2} \int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}_k^{\mathbf{p}}) * \eta_\epsilon : (\boldsymbol{\varepsilon}_k^{\mathbf{p}} * \eta_\epsilon)_t \, \mathrm{d}x \, \mathrm{d}t = \int_{t_1}^{t_2} \int_\Omega \boldsymbol{\chi}_k * \eta_\epsilon : \boldsymbol{T}_k * \eta_\epsilon \, \mathrm{d}x \, \mathrm{d}t. \tag{4.3.36}$$

Subtracting (4.3.34) and (4.3.36) we obtain

$$\int_{t_1}^{t_2} \int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}_k^{\mathbf{p}}) * \eta_\epsilon : ((\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}_k^{\mathbf{p}}) * \eta_\epsilon)_t \, \mathrm{d}x \, \mathrm{d}t = -\int_{t_1}^{t_2} \int_\Omega \boldsymbol{\chi}_k * \eta_\epsilon : \boldsymbol{T}_k * \eta_\epsilon \, \mathrm{d}x \, \mathrm{d}t, \tag{4.3.37}$$

and then

$$\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}_k^{\mathbf{p}}) * \eta_\epsilon : (\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}_k^{\mathbf{p}}) * \eta_\epsilon \, \mathrm{d}x \Big|_{t_1}^{t_2} = -\int_{t_1}^{t_2} \int_\Omega \boldsymbol{\chi}_k * \eta_\epsilon : \boldsymbol{T}_k * \eta_\epsilon \, \mathrm{d}x \, \mathrm{d}t. \tag{4.3.38}$$

Since $\{\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}_k^{\mathbf{p}}\}$ belongs to $L^\infty(0, T, L^2(\Omega))$ and $\boldsymbol{\chi}_k, \boldsymbol{T}_k$ belong to $L^2(0, T, L^2(\Omega))$, we pass to the limit with $\epsilon \to 0$. We obtain

$$\frac{1}{2} \int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}_k^{\mathbf{p}}) : (\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}_k^{\mathbf{p}}) \, \mathrm{d}x \Big|_{t_1}^{t_2} = -\int_{t_1}^{t_2} \int_\Omega \boldsymbol{\chi}_k : \boldsymbol{T}_k^d \, \mathrm{d}x \, \mathrm{d}t. \tag{4.3.39}$$

Since $\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}_k^{\mathbf{p}} \in C_w([0, T], L^2(\Omega, \mathcal{S}^3))$ then we may pass with $t_1 \to 0$, replace t_2 by t and conclude (using the definition of potential energy) that

$$\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}_k^{\mathbf{p}})(t) - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}_k^{\mathbf{p}})(0) = -\int_0^t \int_\Omega \boldsymbol{\chi}_k : \boldsymbol{T}_k^d \, \mathrm{d}x \, \mathrm{d}s. \tag{4.3.40}$$

Multiplying (4.3.40) by $\frac{1}{\mu}$ and integrating over the interval $(\tau, \tau + \mu)$ we get

$$\frac{1}{\mu} \int_\tau^{\tau+\mu} \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}_k^{\mathbf{p}})(t) \, \mathrm{d}t - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}_k^{\mathbf{p}})(0) = -\frac{1}{\mu} \int_\tau^{\tau+\mu} \int_0^t \int_\Omega \boldsymbol{\chi}_k : \boldsymbol{T}_k^d \, \mathrm{d}x \, \mathrm{d}s \, \mathrm{d}t. \tag{4.3.41}$$

For conciseness of further calculations let us define

$$F(s) := \int_\Omega \boldsymbol{\chi}_k(s) : \boldsymbol{T}_k^d(s) \, \mathrm{d}x. \tag{4.3.42}$$

It is obvious that F belongs to $L^1(0,T)$. Applying Fubini theorem we have

$$\begin{aligned}\frac{1}{\mu}\int_{\tau}^{\tau+\mu}\int_0^t F(s)\,\mathrm{d}s\,\mathrm{d}t &= \frac{1}{\mu}\int_{\mathbb{R}^2}\mathbf{1}_{\{0\le s\le t\}}(s)\mathbf{1}_{\{\tau\le t\le\tau+\mu\}}(t)F(s)\,\mathrm{d}s\,\mathrm{d}t \\ &= \int_{\mathbb{R}}\left(\frac{1}{\mu}\int_{\mathbb{R}}\mathbf{1}_{\{0\le s\le t\}}(s)\mathbf{1}_{\{\tau\le t\le\tau+\mu\}}(t)\,\mathrm{d}t\right)F(s)\,\mathrm{d}s.\end{aligned} \tag{4.3.43}$$

The crucial observation is that

$$\psi_{\mu,\tau}(s) = \frac{1}{\mu}\int_{\mathbb{R}}\mathbf{1}_{\{0\le s\le t\}}(s)\mathbf{1}_{\{\tau\le t\le\tau+\mu\}}(t)\,\mathrm{d}t, \tag{4.3.44}$$

and then

$$-\frac{1}{\mu}\int_{\tau}^{\tau+\mu}\int_0^t F(s)\,\mathrm{d}s\,\mathrm{d}t = -\int_{\mathbb{R}}F(s)\psi_{\mu,\tau}(s)\,\mathrm{d}s = \int_{\mathbb{R}}\int_{\Omega}\boldsymbol{\chi}_k : \boldsymbol{T}_k^d\psi_{\mu,\tau}\,\mathrm{d}x\,\mathrm{d}s \tag{4.3.45}$$

Finally, using (4.3.30), (4.3.32) and (4.3.41) we conclude

$$-\int_0^T\int_{\Omega}\boldsymbol{\chi}_k : \boldsymbol{T}_k^d\,\psi_{\mu,\tau}\,\mathrm{d}x\,\mathrm{d}t \le \liminf_{l\to\infty}\left(-\int_0^T\int_{\Omega}\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d) : \boldsymbol{T}_{k,l}^d\,\psi_{\mu,\tau}\,\mathrm{d}x\,\mathrm{d}t\right), \tag{4.3.46}$$

which is nothing else than

$$\limsup_{l\to\infty}\int_0^T\int_{\Omega}\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d) : \boldsymbol{T}_{k,l}^d\ \psi_{\mu,\tau}\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T\int_{\Omega}\boldsymbol{\chi}_k : \boldsymbol{T}_k^d\ \psi_{\mu,\tau}\,\mathrm{d}x\,\mathrm{d}t. \tag{4.3.47}$$

To finish the proof let us observe that

$$\begin{aligned}&\limsup_{l\to\infty}\int_0^{\tau}\int_{\Omega}\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d) : \boldsymbol{T}_{k,l}^d\,\mathrm{d}x\,\mathrm{d}t \\ &\quad= \limsup_{l\to\infty}\int_0^{\tau}\int_{\Omega}\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d) : (\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad- \lim_{l\to\infty}\int_0^{\tau}\int_{\Omega}\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d) : \tilde{\boldsymbol{T}}^d\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\le \limsup_{l\to\infty}\int_0^{\tau+\mu}\int_{\Omega}\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d) : (\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d)\psi_{\mu,\tau}\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad- \lim_{l\to\infty}\int_0^{\tau}\int_{\Omega}\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d) : \tilde{\boldsymbol{T}}^d\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{4.3.48}$$

where the last inequality is caused by definition of $\psi_{\mu,\tau}$ and positivity of $\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d) : (\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d)$. Then the estimate may follow

$$\begin{aligned}&\limsup_{l\to\infty}\int_0^{\tau}\int_{\Omega}\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d) : \boldsymbol{T}_{k,l}^d\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\le \limsup_{l\to\infty}\int_0^{\tau+\mu}\int_{\Omega}\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d) : \boldsymbol{T}_{k,l}^d\ \psi_{\mu,\tau}\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad+ \lim_{l\to\infty}\int_0^{\tau+\mu}\int_{\Omega}\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d) : \tilde{\boldsymbol{T}}^d\ \psi_{\mu,\tau}\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad- \lim_{l\to\infty}\int_0^{\tau}\int_{\Omega}\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d) : \tilde{\boldsymbol{T}}^d\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{4.3.49}$$

Hence, using (4.3.47) to the first term on the right-hand side of abovementioned equation and using the identity $\psi_{\mu,\tau} \equiv 1$ on $[0,\tau]$ to remaining terms we obtain

$$\begin{aligned}
\limsup_{l\to\infty} \int_0^\tau \int_\Omega & \boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{T}^d_{k,l}\,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_0^{\tau+\mu} \int_\Omega \boldsymbol{\chi}_k : \boldsymbol{T}^d_k\ \psi_{\mu,\tau}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \lim_{l\to\infty} \int_\tau^{\tau+\mu} \int_\Omega \boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \tilde{\boldsymbol{T}}^d \psi_{\mu,\tau}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^{\tau+\mu} \int_\Omega \boldsymbol{\chi}_k : \boldsymbol{T}^d_k\ \psi_{\mu,\tau}\,\mathrm{d}x\,\mathrm{d}t + \lim_{l\to\infty} \int_\tau^{\tau+\mu} \int_\Omega \boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \tilde{\boldsymbol{T}}^d_k\ \psi_{\mu,\tau}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.3.50}$$

Passing with $\mu \to 0$ results in (4.3.28). The proof is complete. □

Now, we use Theorem 4.1.4 to finish the limit passage in heat equation. In our case $\boldsymbol{G}(\cdot,\cdot)$ is a nonlinear function. Let us check the assumptions of Theorem 4.1.4. The conditions $(i)-(iii)$ are held, because of Assumption 4.0.1. Due to the energy estimate and uniform boundedness of temperature $\{\theta_{k,l}\}$, condition (iv) is fulfilled. Uniform boundedness of the sequences $\{\boldsymbol{T}^d_{k,l}\}$ and $\{\boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\}$ with respect to l in $L^2(Q,\mathcal{S}^3_d)$ implies that the condition (v) is satisfied. Finally, the last condition (vi) is a result of Lemma 4.3.6. Hence, for every $k \in \mathbb{N}$ there exists the subsequence $\{\boldsymbol{T}^d_{k,l}\}$ which converges to $\boldsymbol{T}^d_k$ in $L^2(Q,\mathcal{S}^3_d)$ with $l \to \infty$. Moreover, using Lebesgue's dominated convergence theorem there exists a subsequence $\{\boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\}$ which converges to $\boldsymbol{G}(\tilde\theta + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)$ in $L^2(Q,\mathcal{S}^3)$ with $l \to \infty$. Consequently, product $\{\boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{T}^d_{k,l}\}$ converges to $\boldsymbol{G}(\tilde\theta + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k) : \boldsymbol{T}^d_k$ in $L^1(Q,\mathcal{S}^3)$ with l going to ∞. Hence, we pass to the limit in (4.3.22) and

$$\begin{aligned}
-\int_0^T \int_\Omega \theta_k \varphi_4'(t)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \theta_0(x)\varphi_4(0)\,\mathrm{d}x + \int_0^T \int_\Omega \nabla\theta_k \cdot \nabla\varphi_4\,\mathrm{d}x\,\mathrm{d}t \\
= \int_0^T \int_\Omega \mathcal{T}_k\left((\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k) : \boldsymbol{G}(\tilde\theta + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\right)\varphi_4\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{4.3.51}$$

holds for every test function $\varphi_4 \in C^\infty_c([-\infty,T))$.

Lemma 4.3.7. *The sequence $\{\boldsymbol{\varepsilon}^{\mathbf{p}}_k\}$ is uniformly bounded in $W^{1,2}(0,T,L^2(\Omega,\mathcal{S}^3))$ with respect to k.*

Proof. By Assumption 4.0.1 and the fact that the constant C is independent of temperature, we get

$$\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,t) = \boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,0) + \int_0^t (\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,s))_s\,\mathrm{d}s.$$

Hence

$$|\boldsymbol{\varepsilon}^{\mathbf{p}}_k|^2(x,t) \le 2|\boldsymbol{\varepsilon}^{\mathbf{p}}_k|^2(x,0) + 2t^{1/2}\int_0^t |(\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_s|^2(x,s)\,\mathrm{d}s$$

and consequently

$$\begin{aligned}\int_0^T\int_\Omega |\boldsymbol{\varepsilon}^{\mathbf{p}}_k|^2(x,t)\,\mathrm{d}x\,\mathrm{d}t &\le 2\int_0^T\int_\Omega |\boldsymbol{\varepsilon}^{\mathbf{p}}_k|^2(x,0)\,\mathrm{d}x\,\mathrm{d}t + 2t^{1/2}\int_0^T\int_\Omega\int_0^t |(\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_s|^2(x,s)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\\ &\le C(T)(1+\int_\Omega\int_0^T |\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)|^2)\,\mathrm{d}s\,\mathrm{d}x\\ &\le C(T)(1+\int_0^T\int_\Omega |\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k|^2)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

It follows from Lemma 4.3.2 that the right-hand side is uniformly bounded. □

Lemma 4.3.8. *The sequence $\{\boldsymbol{u}_k\}$ is uniformly bounded in $L^2(0,T,W^{1,2}_0(\Omega,\mathbb{R}^3))$ with respect to k.*

Proof. Using the triangle inequality and the fact that the operator $\boldsymbol{D}$ is positively definite, we obtain

$$|\boldsymbol{\varepsilon}(\boldsymbol{u}_k)|^2 \le 2|\boldsymbol{\varepsilon}(\boldsymbol{u}_k)-\boldsymbol{\varepsilon}^{\mathbf{p}}_k|^2 + 2|\boldsymbol{\varepsilon}^{\mathbf{p}}_k|^2 \le c|\boldsymbol{T}_k|^2 + 2|\boldsymbol{\varepsilon}^{\mathbf{p}}_k|^2. \tag{4.3.52}$$

Integrating over $\Omega\times(0,T)$ we get

$$\begin{aligned}\int_0^T\int_\Omega |\boldsymbol{\varepsilon}(\boldsymbol{u}_k)|^2\,\mathrm{d}x\,\mathrm{d}t &\le c\int_0^T\int_\Omega |\boldsymbol{T}_k|^2\,\mathrm{d}x\,\mathrm{d}t + 2\int_0^T\int_\Omega |\boldsymbol{\varepsilon}^{\mathbf{p}}_k|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\le c\|\boldsymbol{T}_k\|^2_{L^2(0,T,L^2(\Omega))} + 2\|\boldsymbol{\varepsilon}^{\mathbf{p}}_k\|^2_{L^2(0,T,L^2(\Omega))}.\end{aligned} \tag{4.3.53}$$

Due to Lemma 4.3.2 the sequence $\{\boldsymbol{T}_k\}$ is uniformly bounded in $L^2(0,T,L^2(\Omega,\mathcal{S}^3))$. The tensor $\boldsymbol{\varepsilon}(\boldsymbol{u}_k)$ is the symmetric gradient of displacement, thus using Korn inequality (cf. [53, Theorem 1.10]) we conclude that the sequence $\{\boldsymbol{u}_k\}$ is uniformly bounded in $L^2(0,T,W^{1,2}_0(\Omega,\mathbb{R}^3))$. □

4.3.5 Passing to the limit with k going to ∞

To complete the proof we make the second limit passage in the approximate system of equations. In previous sections we have presented the uniform boundedness of solutions sequences. Hence, we have the following convergences

$$\begin{array}{rl}\boldsymbol{u}_k \rightharpoonup \boldsymbol{u} & \text{weakly in } L^2(0,T,W^{1,2}_0(\Omega,\mathbb{R}^3)),\\ \boldsymbol{T}_k \rightharpoonup \boldsymbol{T} & \text{weakly in } L^2(Q,\mathcal{S}^3),\\ \boldsymbol{T}^d_k \rightharpoonup \boldsymbol{T}^d & \text{weakly in } L^2(Q,\mathcal{S}^3_d),\\ \boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k) \rightharpoonup \boldsymbol{\chi} & \text{weakly in } L^2(Q,\mathcal{S}^3_d),\\ \boldsymbol{\varepsilon}^p_k \rightharpoonup \boldsymbol{\varepsilon}^p & \text{weakly in } L^2(Q,\mathcal{S}^3_d).\end{array} \tag{4.3.54}$$

Using these convergences we make the limit passage in (4.3.24) and (4.3.26) and we get

$$\begin{aligned}\int_0^T\int_\Omega \boldsymbol{T}:\nabla\boldsymbol{\varphi}_1\,\mathrm{d}x\,\mathrm{d}t &= 0\\ \int_0^T\int_\Omega \boldsymbol{\varepsilon}^{\mathbf{p}}_t:\boldsymbol{\varphi}_2\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega \boldsymbol{\chi}:\boldsymbol{\varphi}_2\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{4.3.55}$$

for $\boldsymbol{\varphi}_1\in L^2(0,T,L^2(\Omega,\mathbb{R}^3))$ and $\boldsymbol{\varphi}_2\in L^2(0,T,L^2(\Omega,\mathcal{S}^3))$.

As previously we should carefully consider the right-hand side of heat equation as a product of two weakly converging sequences. We have also problem with identification of $\boldsymbol{\chi}$.

Since the sequences $\{\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\}$ and $\{\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k\}$ are uniformly bounded in $L^2(Q,\mathcal{S}^3_d)$ then also the sequence $\{\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k):(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\}$ is uniformly bounded in $L^1(Q)$. Using Boccardo and Gallouët approach for Neumann boundary condition, see Chapter 2 Section 3.1, there exists a subsequence $\{\theta_k\}$ such that $\theta_k\rightharpoonup\theta$ in $L^q(0,T,W^{1,q}(\Omega))$ for every $q\in(1,\frac{5}{4})$.

Hence, it remains to identify the limit of the right-hand side of the heat equation and to identify $\boldsymbol{\chi}$. As in previous limit passage we use Theorem 4.1.4 and we prove that $\boldsymbol{\chi}=\boldsymbol{G}(\tilde\theta+\theta,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d)$. The only difficulties concentrate on checking that assumption (vi) from Theorem 4.1.4 is fulfilled.

Lemma 4.3.9. *The following inequality holds for the solution of approximate system*

$$\limsup_{l\to\infty}\int_0^\tau\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k):\boldsymbol{T}^d_k\,\mathrm{d}x\,\mathrm{d}t\le\int_0^\tau\int_\Omega\boldsymbol{\chi}:\boldsymbol{T}^d\,\mathrm{d}x\,\mathrm{d}t, \tag{4.3.56}$$

for every $\tau\in(0,T)$.

Proof. For each $\mu>0,\tau\le T-\mu,s\ge0$, let $\psi_{\mu,\tau}:\mathbb{R}_+\to\mathbb{R}_+$ be defined by (4.3.29). Due to (4.3.40) we obtain

$$\int_0^T\frac{d}{d\tau}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k),\boldsymbol{\varepsilon}^{\mathbf{p}}_k)\,\psi_{\mu,\tau}\,\mathrm{d}t\;=-\int_0^T\int_\Omega\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k):\boldsymbol{T}^d_k\,\psi_{\mu,\tau}\,\mathrm{d}x\,\mathrm{d}t. \tag{4.3.57}$$

Let us integrate by parts the left-hand side of (4.3.57)

$$\int_0^T\frac{d}{d\tau}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k),\boldsymbol{\varepsilon}^{\mathbf{p}}_k)\,\psi_{\mu,\tau}\,\mathrm{d}t=\frac{1}{\mu}\int_\tau^{\tau+\mu}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k),\boldsymbol{\varepsilon}^{\mathbf{p}}_k)(t)\,\mathrm{d}t-\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k),\boldsymbol{\varepsilon}^{\mathbf{p}}_k)(0). \tag{4.3.58}$$

Passing to the limit in (4.3.58) with $k\to\infty$ we obtain

$$\begin{aligned}\liminf_{k\to\infty}&\int_0^T\frac{d}{d\tau}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k),\boldsymbol{\varepsilon}^{\mathbf{p}}_k)\,\psi_{\mu,\tau}\,\mathrm{d}t\\&=\liminf_{k\to\infty}\frac{1}{\mu}\int_\tau^{\tau+\mu}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k),\boldsymbol{\varepsilon}^{\mathbf{p}}_k)\,\mathrm{d}t-\lim_{k\to\infty}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k),\boldsymbol{\varepsilon}^{\mathbf{p}}_k)(0)\\&\ge\frac{1}{\mu}\int_\tau^{\tau+\mu}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}^{\mathbf{p}})\,\mathrm{d}t-\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}^{\mathbf{p}})(0).\end{aligned} \tag{4.3.59}$$

Note that the last inequality holds due to the weak lower semicontinuity in $L^2(0,T,L^2(\Omega;\mathcal{S}^3))$.

We cannot use $\boldsymbol{u}_t$ as a test function because it is not regular enough with respect to time. Using the time mollifier can help us in dealing with that issue. Then $(\boldsymbol{u}_t*\eta_\epsilon\mathbf{1}_{(t_1,t_2)})*\eta_\epsilon$ is a proper test function to $(4.3.55)_{(1)}$. Further, we use $(\boldsymbol{T}*\eta_\epsilon\mathbf{1}_{(t_1,t_2)})*\eta_\epsilon$ as a test function in $(4.3.55)_{(2)}$. Hence

$$\begin{aligned}&\int_0^T\int_\Omega\boldsymbol{T}:(\boldsymbol{\varepsilon}(\boldsymbol{u})_t*\eta_\epsilon\mathbf{1}_{(t_1,t_2)})*\eta_\epsilon\,\mathrm{d}x\,\mathrm{d}t=0,\\&\int_0^T\int_\Omega\boldsymbol{\varepsilon}^{\mathbf{p}}_t:(\boldsymbol{T}*\eta_\epsilon\mathbf{1}_{(t_1,t_2)})*\eta_\epsilon\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_\Omega\boldsymbol{\chi}:(\boldsymbol{T}*\eta_\epsilon\mathbf{1}_{(t_1,t_2)})*\eta_\epsilon\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{4.3.60}$$

and then

$$\begin{aligned}&\int_{t_1}^{t_2}\int_\Omega\boldsymbol{T}*\eta_\epsilon:\boldsymbol{\varepsilon}(\boldsymbol{u})_t*\eta_\epsilon\,\mathrm{d}x\,\mathrm{d}t=0,\\&\int_{t_1}^{t_2}\int_\Omega\boldsymbol{\varepsilon}^{\mathbf{p}}_t*\eta_\epsilon:\boldsymbol{T}*\eta_\epsilon\,\mathrm{d}x\,\mathrm{d}t=\int_{t_1}^{t_2}\int_\Omega\boldsymbol{\chi}*\eta_\epsilon:\boldsymbol{T}*\eta_\epsilon\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.3.61}$$

Subtracting $(4.3.61)_{(1)}$ and $(4.3.61)_{(2)}$ we get

$$\int_{t_1}^{t_2}\int_\Omega \boldsymbol{T}*\eta_\epsilon : ((\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}^{\mathbf{p}})*\eta_\epsilon)_t \,\mathrm{d}x\,\mathrm{d}t = -\int_{t_1}^{t_2}\int_\Omega \boldsymbol{\chi}*\eta_\epsilon : \boldsymbol{T}*\eta_\epsilon \,\mathrm{d}x\,\mathrm{d}t \tag{4.3.62}$$

and then

$$\int_\Omega \boldsymbol{T}*\eta_\epsilon : (\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}^{\mathbf{p}})*\eta_\epsilon \,\mathrm{d}x\Big|_{t_1}^{t_2} = -\int_{t_1}^{t_2}\int_\Omega \boldsymbol{\chi}*\eta_\epsilon : \boldsymbol{T}*\eta_\epsilon \,\mathrm{d}x\,\mathrm{d}t. \tag{4.3.63}$$

Since $\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}^{\mathbf{p}}$ belongs to $L^\infty(0,T,L^2(\Omega,\mathcal{S}^3))$ and $\boldsymbol{\chi},\boldsymbol{T}$ belongs to $L^2(0,T,L^2(\Omega,\mathcal{S}^3))$ we may pass to limit with ϵ going to 0 for a.a. $t_1,t_2\in(0,T)$

$$\int_\Omega \boldsymbol{T} : (\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}^{\mathbf{p}}) \,\mathrm{d}x\Big|_{t_1}^{t_2} = -\int_{t_1}^{t_2}\int_\Omega \boldsymbol{\chi} : \boldsymbol{T} \,\mathrm{d}x\,\mathrm{d}t. \tag{4.3.64}$$

Using the same argumentation as in Lemma 4.3.6 we may pass to the limit $t_1\to 0$. Finally, repeating the reasoning with function $\psi_{\mu,\tau}$ we complete the proof.

□

By Theorem 4.1.4 we improve convergence from (4.3.54) to

$$\begin{array}{ll} \boldsymbol{u}_k\to\boldsymbol{u} & \text{in } L^2(0,T,W_0^{1,2}(\Omega)),\\ \boldsymbol{T}_k\to\boldsymbol{T} & \text{in } L^2(Q,\mathcal{S}^3),\\ \boldsymbol{T}_k^d\to\boldsymbol{T}^d & \text{in } L^2(Q,\mathcal{S}^3_d),\\ \boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d)\to\boldsymbol{G}(\theta,\boldsymbol{T}) & \text{in } L^2(Q,\mathcal{S}^3_d),\\ \boldsymbol{\varepsilon}_k^{\mathbf{p}}\to\boldsymbol{\varepsilon}^{\mathbf{p}} & \text{in } W^{1,2}(0,T,L^2(\Omega)). \end{array} \tag{4.3.65}$$

Additionally, we have $\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k):(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d)\to\boldsymbol{G}(\tilde\theta+\theta,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d):(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d)$ in $L^1(Q)$. Using these convergences we pass to the limit in (4.3.24), (4.3.26) and (4.3.51) with k going to ∞, we obtain

$$\begin{aligned} \int_0^T\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}^{\mathbf{p}}):\nabla\boldsymbol{\varphi}_1\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}_1\,\mathrm{d}x\,\mathrm{d}t,\\ \int_0^T\int_\Omega \boldsymbol{\varepsilon}_t^{\mathbf{p}}:\boldsymbol{\varphi}_2\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega \boldsymbol{G}(\tilde\theta+\theta,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d):\boldsymbol{\varphi}_2\,\mathrm{d}x\,\mathrm{d}t,\\ -\int_0^T\int_\Omega \theta(\varphi_3)_t\,\mathrm{d}x\,\mathrm{d}t-\int_\Omega\theta_0(x)\varphi_3(0,x)\,\mathrm{d}x &\\ +\int_0^T\int_\Omega\nabla\theta\cdot\nabla\varphi_3\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d):\boldsymbol{G}(\tilde\theta+\theta,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d)\varphi_3\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{4.3.66}$$

for $\boldsymbol{\varphi}_1\in L^2(0,T,L^2(\Omega,\mathbb{R}^3))$, $\boldsymbol{\varphi}_2\in L^2(0,T,L^2(\Omega,\mathcal{S}^3))$ and $\varphi_3\in C_c^\infty(0,T,C^\infty(\Omega))$.

Chapter 5

Norton-Hoff-type models

The subject of this chapter is to present the existence theorem for models of Norton-Hoff-type hardening rule. This class of models is a natural extension of the Mróz model. We generalize the assumption on constitutive function in two ways. Firstly, we allow the p-growth instead of linear growth in Mróz model. Moreover, the visco-elastic constitutive function $\boldsymbol{G}$ is merely monotone and may fail to be strictly monotone. These two changes imply application of other techniques.

Let us assume that $\Omega \subset \mathbb{R}^3$ is an open bounded set with a C^2 boundary and moreover, the body is homogeneous in space. Then we may formulate the assumptions on constitutive function describing the Norton-Hoff-types models.

Assumption 5.0.1. *The function $\boldsymbol{G}(\theta, \boldsymbol{T}^d)$ is continuous with respect to θ and $\boldsymbol{T}^d$ and for $p \geq 2$ satisfies the following conditions:*

a) $(\boldsymbol{G}(\theta, \boldsymbol{T}^d_1) - \boldsymbol{G}(\theta, \boldsymbol{T}^d_2)) : (\boldsymbol{T}^d_1 - \boldsymbol{T}^d_2) \geq 0$*, for all* $\boldsymbol{T}^d_1, \boldsymbol{T}^d_2 \in \mathcal{S}^3_d$ *and* $\theta \in \mathbb{R}_+$*;*

b) $|\boldsymbol{G}(\theta, \boldsymbol{T}^d)| \leq C(1 + |\boldsymbol{T}^d|)^{p-1}$*, where* $\boldsymbol{T}^d \in \mathcal{S}^3_d$*,* $\theta \in \mathbb{R}_+$*;*

c) $\boldsymbol{G}(\theta, \boldsymbol{T}^d) : \boldsymbol{T}^d \geq \beta |\boldsymbol{T}^d|^p$*, where* $\boldsymbol{T}^d \in \mathcal{S}^3_d$*,* $\theta \in \mathbb{R}_+$*,*

where C and β are positive constants, independent of the temperature θ.

Motivation for current considerations were the results of Alber and Chełmiński [3] and of Hömberg [43]. In [3] the authors considered the quasi-static visco-elasticity models with Norton-Hoff constitutive function

$$\boldsymbol{G}(\boldsymbol{T}) = c|\boldsymbol{T}|^{p-2}\boldsymbol{T}, \tag{5.0.1}$$

with $p > 2$. The parameter c was a positive constant.The idea of proof in [3] was to formulate the problem in a way that it fits to the abstract theory of maximal monotone operators, cf. [10].

In the contrast to [3] we include thermal effects of the process. As we mentioned at the beginning of this dissertation we consider materials without thermal expansion. Hence, thermal effects appear only in the form of additional equation on heat conductivity and by taking into account the dependency of the constitutive function $\boldsymbol{G}(\cdot, \cdot)$ on the temperature. This dependence, i.e. the dependence of $\boldsymbol{G}(\cdot, \cdot)$ on temperature, destroys the monotone character of the model and it requires different approach. Moreover, in our case the function $\boldsymbol{G}$ depends only on deviatoric part of Cauchy stress tensor and it has technical consequences. Contrary to the proof of Alber and Chełmiński, where they showed that $\boldsymbol{T}$ belongs to $L^p(0, T, L^p(\Omega, \mathcal{S}^3))$ for $p \geq 2$, the estimates conducted in the present situation provide us only with the fact that $\boldsymbol{T}$ belongs to $L^2(0, T, L^2(\Omega, \mathcal{S}^3))$.

In [43] Hömberg considered more general physical phenomena. Besides deformations and temperature Hömberg was interested in electro-magnetic effects and concentrations of different phases of material. Changes of temperature have got the influence on concentrations. This dependency was prescribed by a general operator $P[\cdot]$, which has got *good* properties. Visco-elastic constitutive functions depend on these concentrations instead of temperature. This assumption and linear dependency between function $\boldsymbol{G}$ and deviatoric part of Cauchy stress tensor (Mróz model) implied that there are no problems with nonlinearities. The similarities between Hömberg's paper and our result lie in the construction of approximated problem (truncation of the terms that are only integrable) and in the approach used in order to deal with right-hand side of heat equation. Nevertheless, because of the different structure of the problem, Hömberg could show strong convergence of approximated sequence of the Cauchy stress tensor. For the concept of showing this strong convergence let us observe that in the case of linear Mróz relation, and in fact also in the case of Norton-Hoff relation (5.0.1), the stronger condition than monotonicity holds, namely the uniform monotonicity condition

$$\boldsymbol{G}(\theta,\boldsymbol{T}_1^d)-\boldsymbol{G}(\theta,\boldsymbol{T}_2^d)):(\boldsymbol{T}_1^d-\boldsymbol{T}_2^d)\geq c|\boldsymbol{T}_1^d-\boldsymbol{T}_2^d|^p, \tag{5.0.2}$$

for all $\boldsymbol{T}_1^d,\boldsymbol{T}_2^d\in\mathcal{S}^3_d$ and $\theta\in\mathbb{R}_+$. For the proof see e.g. [53].

Results presented in this Chapter come from [32]. Here, we skip some details which are analogous to the one used in Chapter 4. The following chapter is organized as follows: Section 5.1 is dedicated to definition of solution and formulation of main Theorem. In Section 5.2 we show the proof of this theorem. In a few places the proof goes in the same way as for Mróz model. Therefore, minor details are omitted.

5.1 Formulation of the problem

Definition 5.1.1 (Solution to Norton-Hoff-type model). *Let $p\geq 2$, $q<\frac{5}{4}$ and $p'=p/(p-1)$. The triple of functions*

$$\boldsymbol{u}\in L^{p'}(0,T,W^{1,p'}_{\boldsymbol{g}}(\Omega,\mathbb{R}^3))$$
$$\boldsymbol{T}\in L^2(0,T,L^2(\Omega,\mathcal{S}^3))$$

and

$$\theta\in L^q(0,T,W^{1,q}(\Omega))\cap C([0,T],W^{-2,2}(\Omega))$$

is a weak solution to the system (1.2.2) *if*

$$\int_0^T\int_\Omega \boldsymbol{T}:\nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t, \tag{5.1.1}$$

where

$$\boldsymbol{T}=\boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\varepsilon}^{\mathbf{p}}), \tag{5.1.2}$$

and

$$\begin{aligned}-\int_0^T\int_\Omega\theta\phi_t\,\mathrm{d}x\,\mathrm{d}t-\int_\Omega\theta_0(x)\phi(0,x)\,\mathrm{d}x\\+\int_0^T\int_\Omega\nabla\theta\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_{\partial\Omega}g_\theta\phi\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_\Omega\boldsymbol{T}^d:\boldsymbol{G}(\theta,\boldsymbol{T}^d)\phi\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{5.1.3}$$

holds for every test function $\boldsymbol{\varphi}\in C^\infty([0,T],C_c^\infty(\Omega,\mathbb{R}^3))$ and $\phi\in C_c^\infty([-\infty,T),C^\infty(\Omega))$. Furthermore, the visco-elastic strain tensor can be recovered from the equation on its evolution, i.e.

$$\boldsymbol{\varepsilon}^{\mathbf{p}}(x,t)=\boldsymbol{\varepsilon}^{\mathbf{p}}_0(x)+\int_0^t\boldsymbol{G}(\theta(x,\tau),\boldsymbol{T}^d(x,\tau))\,\mathrm{d}\tau, \tag{5.1.4}$$

for a.e. $x\in\Omega$ and $t\in[0,T)$. Moreover, $\boldsymbol{\varepsilon}^{\mathbf{p}}\in W^{1,p'}(0,T,L^{p'}(\Omega,\mathcal{S}^3_d))$.

Theorem 5.1.1. *Let $p \geq 2$ and let initial conditions satisfy $\theta_0 \in L^1(\Omega)$, $\boldsymbol{\varepsilon}_0^{\mathbf{p}} \in L^2(\Omega, \mathcal{S}^3_d)$, boundary conditions satisfy $\boldsymbol{g} \in L^p(0,T,W^{1-\frac{1}{p},p}(\partial\Omega,\mathbb{R}^3))$, $g_\theta \in L^2(0,T,L^2(\partial\Omega))$ and volume force $\boldsymbol{f} \in L^p(0,T,W^{-1,p}(\Omega,\mathbb{R}^3))$ and function $\boldsymbol{G}(\cdot,\cdot)$ satisfy the Assumption 5.0.1. Then there exists a weak solution to system* (1.2.2).

5.2 Proof of Theorem 5.1.1

The proof of Theorem 5.1.1 is similar to the proof of Theorem 4.2.1. We point out the differences of these proofs. Differences between assumptions on Norton-Hoff-type models and Mróz models appear in: non-strictly monotone condition with respect to the second variable and p-growth condition of function with respect to second variable. We do not use Young measures tools in the proof, since non-strictly monotone condition with respect to second variable is not sufficient to show that the Young measure reduces to a Dirac measure. Thus, we use Minty-Browder trick to deal with this issue.

In Section 5.2.1 we present transformation into homoguouens boundary-value problem and the energy estimates of approximate solutions. Section 5.2.2 and Section 5.2.3 are dedicated to limit passage with approximation parameters.

5.2.1 Energy estimates

Using the same argumentation as in Chapter 4, see Lemma 4.3.1, we can transform the system into homogenoues boundary value problem and then using results discussed in Chapter 2 we construct the approximate systems of equations.

Now, we show the uniform boundedness of approximate solutions. As in the case of Mróz model, the uniform estimates are the consequences of finite energy of the system. Estimates for potential energy are similar to previous ones.

Lemma 5.2.1. *There exists a constant C which is uniform with respect to k and l such that*

$$\sup_{t\in[0,T]} \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(t) + c\|\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}\|^p_{L^p(0,T,L^p(\Omega))} \leq C. \tag{5.2.1}$$

Proof. Let us start the proof in the same way as proof of Lemma 4.3.2. Since the potential energy is absolutely continuous function, we calculate time derivative of $\mathcal{E}(t)$ and then after simple calculation we obtain

$$\frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) = -\int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{T}^d_{k,l}\,\mathrm{d}x. \tag{5.2.2}$$

Using growth condition of function $\boldsymbol{G}$, Hölder inequality and Young inequality we get

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) = &-\int_\Omega (\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\,\mathrm{d}x \\ &+ \int_\Omega \tilde{\boldsymbol{T}}^d : \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\,\mathrm{d}x \\ \leq &-\beta\|\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}\|^p_{L^p(\Omega)} + \|\tilde{\boldsymbol{T}}^d\|_{L^p(\Omega)}\|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^{p'}(\Omega)} \\ \leq &-\beta\|\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}\|^p_{L^p(\Omega)} + c(\epsilon)\|\tilde{\boldsymbol{T}}^d\|^p_{L^p(\Omega)} + \epsilon\|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|^{p'}_{L^{p'}(\Omega)}\end{aligned} \tag{5.2.3}$$

where $\epsilon = \frac{\beta}{2^p C^{p'}}$. A constant C comes from Assumption 5.0.1. Hence, by the Jensen inequality, the last term may be estimated as follows

$$
\begin{aligned}
\epsilon\|\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\boldsymbol{T}^d_{k,l}+\tilde{\boldsymbol{T}}^d)\|^{p'}_{L^{p'}(\Omega)} &\le C^{p'}\epsilon\int_\Omega (1+|\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}|)^{p'(p-1)}\,\mathrm{d}x \\
&\le C^{p'}\epsilon\int_\Omega (1+|\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}|)^{p}\,\mathrm{d}x \\
&\le C^{p'}2^{p-1}\epsilon\left(|\Omega|+\|\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}\|^p_{L^p(\Omega)}\right) \\
&\le \frac{\beta}{2}|\Omega|+\frac{\beta}{2}\|\boldsymbol{T}^d_{k,l}+\tilde{\boldsymbol{T}}^d\|^p_{L^p(\Omega)}.
\end{aligned}
\tag{5.2.4}
$$

To finish the proof we integrate (5.2.3) over time interval $(0,t)$, with $0 \le t \le T$ and we get

$$
\begin{aligned}
&\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathrm{p}}_{k,l})(t)+\frac{\beta}{2}\|\boldsymbol{T}^d_{k,l}+\tilde{\boldsymbol{T}}^d\|^p_{L^p(0,T,L^p(\Omega))} \\
&\qquad \le c(\epsilon)\|\tilde{\boldsymbol{T}}^d\|^p_{L^p(0,T,L^p(\Omega))}+\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathrm{p}}_{k,l})(0)+\frac{\beta}{2}t|\Omega|.
\end{aligned}
\tag{5.2.5}
$$

□

Remark. *From Lemma 5.2.1 we immediately notice that the sequence* $\{\boldsymbol{T}^d_{k,l}\}$ *is uniformly bounded in the space* $L^p(Q,\mathcal{S}^3_d)$ *with respect to* k *and* l. *Additionally, combining Assumption 5.0.1 with Lemma 5.2.1 we conclude the uniform boundedness of the sequence* $\{\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\}$ *in the space* $L^{p'}(Q,\mathcal{S}^3_d)$. *Summing up, we obtain the uniform boundedness of the sequence* $\{(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\}$ *in* $L^1(Q)$.

Remark. *The uniform boundedness of the potential energy implies that the sequence* $\{\boldsymbol{T}_{k,l}\}$ *is uniformly bounded in* $L^\infty(0,T,L^2(\Omega,\mathcal{S}^3))$ *and in particular in* $L^2(Q,\mathcal{S}^3_d)$.

As a consequence of Lemma 5.2.1 we may observe that the regularity with respect to spatial variable of Cauchy stress tensor and its deviatoric part are significantly different. Deviatoric part of Cauchy stress tensor has higher integrability. We point this out, because it causes main changes in the proof of Theorem 5.1.1 in comparison with proof of Theorem 4.2.1.

Lemma 5.2.2. *The sequence* $\{(\boldsymbol{\varepsilon}^{\mathrm{p}}_{k,l})_t\}$ *is uniformly bounded in* $L^{p'}(0,T,(H^s(\Omega,\mathcal{S}^3))')$ *with respect to* l.

The proof of this lemma is similar to the proof of Lemma 4.3.3. The only difference is that we should use the Hölder inequality with p and p' instead of 2.

Next lemmas are the same as in previous chapter. The regularity of product $\{(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\}$ is the same as for Mróz model. Different assumptions on function $\boldsymbol{G}(\cdot,\cdot)$ in Mróz and Norton-Hoff-type models do not affect temperature results. To see these proofs we refer the reader to Chapter 4.

Lemma 5.2.3. *The sequence* $\{\theta_{k,l}\}$ *is uniformly bounded in* $L^\infty(0,T;L^1(\Omega))$ *with respect to* k *and* l.

Lemma 5.2.4. *There exists a constant* C, *depending on the domain* Ω *and the time interval* $(0,T)$, *such that for every* $k \in \mathbb{N}$

$$
\begin{aligned}
&\sup_{0\le t\le T}\|\theta_{k,l}(t)\|^2_{L^2(\Omega)}+\|\theta_{k,l}\|^2_{L^2(0,T,W^{1,2}(\Omega))}+\|(\theta_{k,l})_t\|^2_{L^2(0,T,W^{-1,2}(\Omega))} \\
&\qquad \le C\Big(\|\mathcal{T}_k\Big((\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\Big)\|^2_{L^2(0,T,L^2(\Omega))}+\|\mathcal{T}_k(\theta_0)\|^2_{L^2(\Omega)}\Big).
\end{aligned}
\tag{5.2.6}
$$

Remark. *The uniform boundedness of solutions implies the global existence of approximate solutions, i.e. existence of solutions* $\{\alpha^n_{k,l}(t), \beta^m_{k,l}(t), \gamma^n_{k,l}(t), \delta^m_{k,l}(t)\}$ *on the whole time interval* $[0,T]$ *for each* $n = 1, ..., k$ *and* $m = 1, ..., l$.

5.2.2 Limit passage $l \to \infty$ and uniform estimates.

Let us multiply the system (2.2.8) by smooth time-dependent functions, integrate over $[0,T]$ and then rewrite the system as follows

$$\int_0^T \int_\Omega \boldsymbol{T}_{k,l} : \nabla \boldsymbol{w}_n \varphi_1(t)\,\mathrm{d}x\,\mathrm{d}t = 0, \quad n = 1, \ldots, k, \tag{5.2.7}$$

and

$$\begin{aligned}
&\int_0^T \int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t : \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_n)\varphi_2(t)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \int_0^T \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_n)\varphi_2(t)\,\mathrm{d}x\,\mathrm{d}t, \quad n = 1, ..., k, \\
&\int_0^T \int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t : \boldsymbol{D}\boldsymbol{\zeta}^k_m\varphi_3(t)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \int_0^T \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{D}\boldsymbol{\zeta}^k_m\varphi_3(t)\,\mathrm{d}x\,\mathrm{d}t, \quad m = 1, ..., l,
\end{aligned} \tag{5.2.8}$$

and for $m = 1, \ldots, l$

$$\begin{aligned}
-\int_0^T \int_\Omega \theta_{k,l}\varphi_4'(t) v_m\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \theta_0(x)\varphi_4(0)v_m\,\mathrm{d}x + \int_0^T \int_\Omega \nabla\theta_{k,l} \cdot \varphi_4(t)\nabla v_m\,\mathrm{d}x\,\mathrm{d}t \\
= \int_0^T \int_\Omega \mathcal{T}_k\left((\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\right)\varphi_4(t)v_m\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{5.2.9}$$

holds for every test function $\varphi_1, \varphi_2, \varphi_3 \in C^\infty([0,T])$ and $\varphi_4 \in C^\infty_c([-\infty, T))$.

Firstly, we pass to the limit with approximation parameter for temperature. In previous section we proved uniform boundedness with respect to l for appropriate sequences. Then, at least for a subsequence, but still denoted by the index l, we get the following convergences

$$\begin{aligned}
\boldsymbol{T}_{k,l} &\rightharpoonup \boldsymbol{T}_k && \text{weakly in } L^2(Q, \mathcal{S}^3), \\
\boldsymbol{T}^d_{k,l} &\rightharpoonup \boldsymbol{T}^d_k && \text{weakly in } L^p(Q, \mathcal{S}^3_d), \\
\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) &\rightharpoonup \boldsymbol{\chi}_k && \text{weakly in } L^{p'}(Q, \mathcal{S}^3_d), \\
\theta_{k,l} &\rightharpoonup \theta_k && \text{weakly in } L^2(0,T, W^{1,2}(\Omega)), \\
\theta_{k,l} &\to \theta_k && \text{a.e. in } Q, \\
(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t &\rightharpoonup (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t && \text{weakly in } L^{p'}(0,T, (H^s(\Omega, \mathcal{S}^3))').
\end{aligned} \tag{5.2.10}$$

Passing now to the limit in (5.2.7)-(5.2.8) yields

$$\int_0^T \int_\Omega \boldsymbol{T}_k : \nabla \boldsymbol{w}_n \varphi_1(t)\,\mathrm{d}x\,\mathrm{d}t = 0, \quad n = 1, \ldots, k, \tag{5.2.11}$$

and

$$\begin{aligned}
\int_0^T \int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t : \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_n)\varphi_2(t)\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_\Omega \boldsymbol{\chi}_k : \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}_n)\varphi_2(t)\,\mathrm{d}x\,\mathrm{d}t, \quad n = 1, ..., k, \\
\int_0^T \int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t : \boldsymbol{D}\boldsymbol{\zeta}^k_m\varphi_3(t)\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_\Omega \boldsymbol{\chi}_k : \boldsymbol{D}\boldsymbol{\zeta}^k_m\varphi_3(t)\,\mathrm{d}x\,\mathrm{d}t, \quad m \in \mathbb{N},
\end{aligned} \tag{5.2.12}$$

holds for every test function $\varphi_1, \varphi_2, \varphi_3 \in C^\infty([0,T])$. Basis functions $\mathrm{lin}\{\boldsymbol{\zeta}_m^k\}_{m=1}^\infty$ are dense in $L^p(\Omega, \mathcal{S}^3)$, hence we conclude that

$$\int_0^T \int_\Omega (\boldsymbol{\varepsilon}_k^{\mathbf{p}})_t : \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega \boldsymbol{\chi}_k : \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t \tag{5.2.13}$$

holds for all $\boldsymbol{\varphi} \in C^\infty([0,T], L^p(\Omega, \mathcal{S}^3))$ and then also for all $\boldsymbol{\varphi} \in L^p(Q, \mathcal{S}^3)$.

The remaining part of this section is dedicated to identification of weak limit of the nonlinear term $\boldsymbol{\chi}_k$ and showing the convergence of right-hand side of heat equation (5.2.9). As previously $(\tilde{\boldsymbol{T}}^d + \boldsymbol{T}_{k,l}^d) : \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_{k,l}^d)$ is a product of two weakly convergent sequences. The same problem was considered for Mróz model. Since we have different monotone condition on function $\boldsymbol{G}(\cdot,\cdot)$, we do not use Young measures to identify the nonlinear term.

Our idea is to solve this problem in three steps method. The first step is to show the limiting inequality as in Lemma 4.3.6. The second one is to identify the weak limit $\boldsymbol{\chi}_k$ by usage of Minty-Browder trick. And finally, we prove the weak convergence of the product $(\tilde{\boldsymbol{T}}^d + \boldsymbol{T}_{k,l}^d)$: $\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_{k,l}^d)$.

Step 1. *Limiting inequality.*

Lemma 5.2.5. *The following inequality holds for the solution of approximate system*

$$\limsup_{l\to\infty} \int_0^t \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_{k,l}^d) : \boldsymbol{T}_{k,l}^d \, \mathrm{d}x \, \mathrm{d}t \le \int_0^t \int_\Omega \boldsymbol{\chi}_k : \boldsymbol{T}_k^d \, \mathrm{d}x \, \mathrm{d}t. \tag{5.2.14}$$

Proof of this lemma is the same as of Lemma 4.3.6. The only difference in the proof is to pass to the limit with convolution in L^p space instead of L^2 space. Hence, we can omit this proof and refer the reader to the proof of 4.3.6.

Step 2. *Minty-Browder trick*

From the monotonicity condition of the function $\boldsymbol{G}(\cdot,\cdot)$ we obtain

$$\int_\Omega \Big(\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_{k,l}^d) - \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d)\Big) : (\boldsymbol{T}_{k,l}^d - \boldsymbol{W}^d) \, \mathrm{d}x \ge 0 \qquad \forall\ \boldsymbol{W}^d \in L^p(Q, \mathcal{S}_d^3). \tag{5.2.15}$$

Hence

$$\int_0^T \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_{k,l}^d) : \boldsymbol{T}_{k,l}^d \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_{k,l}^d) : \boldsymbol{W}^d \, \mathrm{d}x \, \mathrm{d}t \\ - \int_0^T \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d) : \Big(\boldsymbol{T}_{k,l}^d - \boldsymbol{W}^d\Big) \, \mathrm{d}x \, \mathrm{d}t \ge 0. \tag{5.2.16}$$

Our goal is to pass to the limit with $l \to \infty$ in (5.2.16). Limit of the first term comes from Lemma 5.2.5. There is no problem with limit passage in the second term since $\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_{k,l}^d) \rightharpoonup \boldsymbol{\chi}_k$ in $L^{p'}(Q, \mathcal{S}_d^3)$. It remains to consider the last term from (5.2.16).

The pointwise convergence of $\{\theta_{k,l}\}$ implies the pointwise convergence of $\{\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d)\}$ to $\{\boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d)\}$. The function $|\tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d|^{p-1}$ belongs to $L^{p'}(Q)$, hence the sequence $\{\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d)\}$ is uniformly bounded in $L^{p'}(Q, \mathcal{S}_d^3)$. By Lebesgue's dominated convergence theorem we obtain that $\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d) \to \boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)$ in $L^{p'}(Q,\mathcal{S}^3_d)$ for every $\boldsymbol{W}^d \in L^p(Q,\mathcal{S}^3_d)$. Letting $l\to\infty$ in (5.2.16), we get

$$\int_0^T\int_\Omega \Big(\boldsymbol{\chi}_k - \boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{W}^d)\Big) : (\boldsymbol{T}^d_k-\boldsymbol{W}^d)\,\mathrm{d}x\,\mathrm{d}t \geq 0 \qquad \forall\, \boldsymbol{W}^d \in L^p(Q,\mathcal{S}^3_d). \quad (5.2.17)$$

Let us take $\boldsymbol{W}^d = \boldsymbol{T}^d_k - \lambda\boldsymbol{U}^d$, where $\boldsymbol{U}^d\in L^p(Q,\mathcal{S}^3_d)$ and $\lambda>0$. Then we obtain

$$\int_0^T\int_\Omega \Big(\boldsymbol{\chi}_k - \boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k-\lambda\boldsymbol{U}^d)\Big) : (\lambda\boldsymbol{U}^d)\,\mathrm{d}x\,\mathrm{d}t \geq 0 \quad \forall\, \boldsymbol{U}^d \in L^p(Q,\mathcal{S}^3_d). \quad (5.2.18)$$

Hence, dividing by λ, we get

$$\int_0^T\int_\Omega \Big(\boldsymbol{\chi}_k - \boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k-\lambda\boldsymbol{U}^d)\Big) : \boldsymbol{U}^d\,\mathrm{d}x\,\mathrm{d}t \geq 0 \quad \forall\, \boldsymbol{U}^d \in L^p(Q,\mathcal{S}^3_d). \quad (5.2.19)$$

Letting $\lambda\to 0$ we obtain

$$\int_0^T\int_\Omega \Big(\boldsymbol{\chi}_k - \boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\Big) : \boldsymbol{U}^d\,\mathrm{d}x\,\mathrm{d}t \geq 0 \qquad \forall\, \boldsymbol{U}^d \in L^p(0,T,L^p(Q,\mathcal{S}^3_d). \quad (5.2.20)$$

Repeating the reasoning with negative λ we obtain the opposite inequality. Hence

$$\int_0^T\int_\Omega \Big(\boldsymbol{\chi}_k - \boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\Big) : \boldsymbol{U}^d\,\mathrm{d}x\,\mathrm{d}t = 0 \qquad \forall\, \boldsymbol{U}^d \in L^p(Q,\mathcal{S}^3_d). \quad (5.2.21)$$

Thus implies

$$\boldsymbol{\chi}_k = \boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k) \quad \text{a.e. in } Q. \quad (5.2.22)$$

Consequently for every $k\in\mathbb{N}$

$$\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) \rightharpoonup \boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k),$$

in $L^{p'}(Q,\mathcal{S}^3_d)$ as $l\to\infty$.

Step 3. *Limit of the right-hand side of heat equation.*

Lemma 5.2.6. *For each $k\in\mathbb{N}$ it holds*

$$\begin{aligned}\lim_{l\to\infty}\int_0^T\int_\Omega &\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) : (\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T\int_\Omega \boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k) : (\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \quad (5.2.23)$$

Proof. Using monotonicity of the function $\boldsymbol{G}(\cdot,\cdot)$

$$\begin{aligned}0 &\leq \int_0^T\int_\Omega \Big(\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) - \boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\Big) : (\boldsymbol{T}^d_{k,l}-\boldsymbol{T}^d_k)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T\int_\Omega \Big(\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) : (\boldsymbol{T}^d_{k,l}-\boldsymbol{T}^d_k) - \boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k) : (\boldsymbol{T}^d_{k,l}-\boldsymbol{T}^d_k)\Big)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \quad (5.2.24)$$

Passing with l to ∞ we get that the second term from right-hand side of (5.2.24) converges to zero. Furthermore, using Lemma 5.2.5 we obtain

$$\begin{aligned} 0 &\le \limsup_{l\to\infty} \int_0^T \int_\Omega \boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : (\boldsymbol{T}^d_{k,l} + \tilde{\boldsymbol{T}}^d - \tilde{\boldsymbol{T}}^d - \boldsymbol{T}^d_k)\,\mathrm{d}x\,\mathrm{d}t \\ &= \limsup_{l\to\infty} \int_0^T \int_\Omega \boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : (\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \lim_{l\to\infty} \int_0^T \int_\Omega \boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : (\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)\,\mathrm{d}x\,\mathrm{d}t \le 0. \end{aligned} \tag{5.2.25}$$

Hence

$$\begin{aligned} \lim_{l\to\infty} \int_0^T \int_\Omega & \boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : (\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\,\mathrm{d}x\,\mathrm{d}t \\ &= \lim_{l\to\infty} \int_0^T \int_\Omega \boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k) : (\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_\Omega \boldsymbol{G}(\tilde\theta + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k) : (\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)\,\mathrm{d}x\,\mathrm{d}t, \end{aligned}$$

which completes the proof. □

After three steps method we may pass to the limit in the heat equation, namely for all $\varphi_4 \in C^\infty([-\infty, T] \times \Omega)$ we obtain

$$\begin{aligned} -\int_0^T \int_\Omega \theta_k (\varphi_4)_t\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \theta_k(x,0)\varphi_4(x,0)\,\mathrm{d}x + \int_0^T \int_\Omega \nabla\theta_k \cdot \nabla\varphi_4\,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^T \int_\Omega \mathcal{T}_k \left((\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k) : \boldsymbol{G}(\tilde\theta + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k) \right) \varphi_4\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{5.2.26}$$

Next two lemmas are similar to Lemmas 4.3.7 and 4.3.8. We omit its proofs but we should mention that it is crucial that $p \ge 2$.

Lemma 5.2.7. *The sequence $\{\boldsymbol{\varepsilon}^{\mathbf{p}}_k\}$ is uniformly bounded in $W^{1,p'}(0,T,L^{p'}(\Omega,\mathcal{S}^3))$ with respect to k.*

Lemma 5.2.8. *The sequence $\{\boldsymbol{u}_k\}$ is uniformly bounded in $L^{p'}(0,T,W^{1,p'}_0(\Omega,\mathbb{R}^3))$ with respect to k.*

5.2.3 Limit passage $k \to \infty$

We start the second limit passage with considerations on the temperature sequence. Uniform boundedness of sequence $\{(\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k) : \boldsymbol{G}(\tilde\theta + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)\}$ in $L^1(Q)$ allows us to use the Boccardo and Gallouët approach, see Chapter 2 Section 3.1. We obtain that there exists a subsequence $\{\theta_k\}$ such that for each $1 < q < \frac{5}{4}$:

$$\theta_k \rightharpoonup \theta \text{ weakly in } L^q(0,T,W^{1,q}(\Omega)). \tag{5.2.27}$$

Furthermore, uniform estimates from the previous sections imply that the following convergences hold

$$\begin{aligned} \theta_k &\to \theta && \text{a.e. in } Q, \\ \boldsymbol{u}_k &\rightharpoonup \boldsymbol{u} && \text{weakly in } L^{p'}(0,T,W^{1,p'}_0(\Omega,\mathbb{R}^3)), \\ \boldsymbol{T}_k &\rightharpoonup \boldsymbol{T} && \text{weakly in } L^2(Q,\mathcal{S}^3), \\ \boldsymbol{T}^d_k &\rightharpoonup \boldsymbol{T}^d && \text{weakly in } L^p(Q,\mathcal{S}^3_d), \\ \boldsymbol{G}(\tilde\theta + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k) &\rightharpoonup \boldsymbol{\chi} && \text{weakly in } L^{p'}(Q,\mathcal{S}^3_d), \\ (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t &\rightharpoonup (\boldsymbol{\varepsilon}^{\mathbf{p}})_t && \text{weakly in } L^{p'}(Q,\mathcal{S}^3_d). \end{aligned} \tag{5.2.28}$$

Consequently, passing to the limit with $k \to \infty$ in (5.2.11), (5.2.13) we obtain

$$\int_0^T \int_\Omega \boldsymbol{T} : \nabla \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t = 0, \tag{5.2.29}$$

for all $\boldsymbol{\varphi} \in C^\infty([0,T], L^2(\Omega, \mathcal{S}^3))$ and then also for all $\boldsymbol{\varphi} \in L^2(Q, \mathcal{S}^3)$, and

$$\int_0^T \int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}})_t : \boldsymbol{\psi} \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega \boldsymbol{\chi} : \boldsymbol{\psi} \, \mathrm{d}x \, \mathrm{d}t \tag{5.2.30}$$

for all $\boldsymbol{\psi} \in L^p(Q, \mathcal{S}^3)$.

We use three steps method to characterize $\boldsymbol{\chi}$ and to identify weak limit of right-hand side of (5.2.26). Now, it is very important to carefully consider the first step. There appear some difficulties in the limiting inequality and we cannot obtain it by the same argumentation as in Lemma 4.3.6.

Lemma 5.2.9. *The following inequality holds for the solution of approximate systems*

$$\limsup_{k\to\infty} \int_0^{t_2} \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k) : \boldsymbol{T}^d_k \, \mathrm{d}x \, \mathrm{d}t \le \int_0^{t_2} \int_\Omega \boldsymbol{\chi} : \boldsymbol{T}^d \, \mathrm{d}x \, \mathrm{d}t. \tag{5.2.31}$$

Proof. Due to (5.2.22) we can rewrite (4.3.40) as follows

$$\frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}^{\mathbf{p}}_k) = -\int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k) : \boldsymbol{T}^d_k \, \mathrm{d}x. \tag{5.2.32}$$

We multiply the above identity by $\psi_{\mu,\tau}$ given by formula (4.3.29) and integrate over $(0,T)$. Passing to the limit $k \to \infty$ we proceed in the same manner as in the proof of Lemma 4.3.9 and we obtain

$$\begin{aligned}
\liminf_{k\to\infty} \int_0^T & \frac{d}{d\tau}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}^{\mathbf{p}}_k) \, \psi_{\mu,\tau} \, \mathrm{d}t \\
&= \liminf_{k\to\infty} \frac{1}{\mu} \int_\tau^{\tau+\mu} \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}^{\mathbf{p}}_k) \, \mathrm{d}t - \lim_{k\to\infty} \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}^{\mathbf{p}}_k)(0) \\
&\ge \frac{1}{\mu} \int_\tau^{\tau+\mu} \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}^{\mathbf{p}}_k)(t) \, \mathrm{d}t - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}^{\mathbf{p}})(0).
\end{aligned} \tag{5.2.33}$$

For the final step of the proof we need to show that the energy equality holds. Proceeding similarly as in previous limit passage, we shall use as a test function $\boldsymbol{u}_t$ mollified with respect to time. Now, we cannot do this, because of low regularity of $\boldsymbol{u}$ with respect to space ($p' < 2$). Therefore, we proceed differently. We use an approximate sequence as a test function in the limit identity (5.2.29). For $\boldsymbol{u}_k$ we do not have a problem with spatial regularity, because it is a finite combination of basis functions. Hence, we take in (5.2.29) the test function $\boldsymbol{\varphi} = (\boldsymbol{u}_k * \eta_\epsilon)_t \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon$, where again η_ϵ is a standard mollifier and we mollify with respect to time

$$\int_{t_1}^{t_2} \int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) * \eta_\epsilon : (\boldsymbol{\varepsilon}(\boldsymbol{u}_k) * \eta_\epsilon)_t \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{5.2.34}$$

To complete the calculations, we use $\boldsymbol{\psi} = ((\boldsymbol{T}^d * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon$ as a test function in (5.2.13). Having (5.2.22) in mind, we obtain

$$\int_{t_1}^{t_2} \int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_k * \eta_\epsilon)_t : \boldsymbol{T} * \eta_\epsilon \, \mathrm{d}x \, \mathrm{d}t = \int_{t_1}^{t_2} \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k) * \eta_\epsilon : \boldsymbol{T} * \eta_\epsilon \, \mathrm{d}x \, \mathrm{d}t. \tag{5.2.35}$$

Products in (5.2.35) are well defined, since for the matrices $\boldsymbol{A} \in \mathcal{S}^3_d$ and $\boldsymbol{B} \in \mathcal{S}^3$ the equivalence $\boldsymbol{A} : \boldsymbol{B}^d = \boldsymbol{A} : \boldsymbol{B}$ holds and tensor $\boldsymbol{T}^d$ belongs to $L^p(Q, \mathcal{S}^3_d)$. Subtracting (5.2.35) from (5.2.34) we get

$$\int_{t_1}^{t_2} \int_\Omega \boldsymbol{T} * \eta_\epsilon : (\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t * \eta_\epsilon \, \mathrm{d}x \, \mathrm{d}t = -\int_{t_1}^{t_2} \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k) * \eta_\epsilon : \boldsymbol{T}^d * \eta_\epsilon \, \mathrm{d}x \, \mathrm{d}t. \quad (5.2.36)$$

For every $\epsilon > 0$ the sequence $\{(\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t * \eta_\epsilon\}$ belongs to $L^2(Q, \mathcal{S}^3_d)$ and is uniformly bounded in $L^2(Q, \mathcal{S}^3_d)$. Moreover, $\{\boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k) * \eta_\epsilon\}$ belongs to $L^{p'}(Q, \mathcal{S}^3_d)$ and is uniformly bounded in this space. Hence, we pass to the limit with $k \to \infty$

$$\int_{t_1}^{t_2} \int_\Omega \boldsymbol{T} * \eta_\epsilon : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}})_t * \eta_\epsilon \, \mathrm{d}x \, \mathrm{d}t = -\int_{t_1}^{t_2} \int_\Omega \boldsymbol{\chi} * \eta_\epsilon : \boldsymbol{T}^d * \eta_\epsilon \, \mathrm{d}x \, \mathrm{d}t.$$

Using the properties of convolution we get

$$\int_\Omega \boldsymbol{T} * \eta_\epsilon : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) * \eta_\epsilon \, \mathrm{d}x \Big|_{t_1}^{t_2} = -\int_{t_1}^{t_2} \int_\Omega \boldsymbol{\chi} * \eta_\epsilon : \boldsymbol{T}^d * \eta_\epsilon \, \mathrm{d}x \, \mathrm{d}t,$$

and finally by passing to the limit with $\epsilon \to 0$ and then with $t_1 \to 0$

$$\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) \, \mathrm{d}x \Big|_0^{t_2} = -\int_0^{t_2} \int_\Omega \boldsymbol{\chi} : \boldsymbol{T}^d \, \mathrm{d}x \, \mathrm{d}t. \quad (5.2.37)$$

We multiply (5.2.37) by $\frac{1}{\mu}$ and integrate over $(\tau, \tau + \mu)$ and proceed now in the same manner as in the proof of Lemma 4.3.6.

□

Again we identify the weak limit $\boldsymbol{\chi}$ by using the Minty-Browder trick and get $\boldsymbol{\chi} = \boldsymbol{G}(\tilde{\theta} + \theta, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d)$. Moreover, we obtain

$$\boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k) : (\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k) \rightharpoonup \boldsymbol{G}(\tilde{\theta} + \theta, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d) : (\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d), \quad (5.2.38)$$

weakly in $L^1(Q)$. Furthermore,

$$\mathcal{T}_k\Big(\boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k) : (\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)\Big) \rightharpoonup \boldsymbol{G}(\tilde{\theta} + \theta, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d) : (\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d), \quad (5.2.39)$$

weakly in $L^1(Q)$. Here, coming back to removed boundary value problems we conclude that

$$\int_0^T \int_\Omega \Big(\tilde{\boldsymbol{T}} + \boldsymbol{T}\Big) : \nabla \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t, \quad (5.2.40)$$

where

$$\boldsymbol{T} = \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}), \qquad \tilde{\boldsymbol{T}} = \boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}), \quad (5.2.41)$$

and $\boldsymbol{\varphi} \in C^\infty([0,T], C^\infty_c(\Omega, \mathbb{R}^3))$. Moreover,

$$-\int_0^T \int_\Omega (\tilde{\theta} + \theta)\phi_t \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega (\tilde{\theta}_0(x) + \theta_0(x))\phi(x,0) \, \mathrm{d}x + \int_0^T \int_\Omega \nabla(\tilde{\theta} + \theta) \cdot \nabla \phi \, \mathrm{d}x \, \mathrm{d}t$$
$$- \int_0^T \int_{\partial\Omega} g_\theta \phi \, \mathrm{d}s \, \mathrm{d}t = \int_0^T \int_\Omega (\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d) : \boldsymbol{G}(\tilde{\theta} + \theta, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d)\phi \, \mathrm{d}x \, \mathrm{d}t, \quad (5.2.42)$$

for $\phi \in C^\infty_c([-\infty, T], C^\infty(\Omega))$ and

$$\boldsymbol{\varepsilon}^{\mathbf{p}}(x,t) = \boldsymbol{\varepsilon}^{\mathbf{p}}_0(x) + \int_0^t \boldsymbol{G}(\tilde{\theta} + \theta, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d) \, \mathrm{d}\tau. \quad (5.2.43)$$

That completes the proof of Theorem 5.1.1.

Chapter 6

Models with growth conditions in generalized Orlicz spaces

In this chapter we extend the results presented previously. The growth conditions in the generalized Orlicz spaces for visco-elastic constitutive function are natural extensions of Norton-Hoff-type model. First of all, the use of generalized Orlicz spaces takes into consideration more rapid growth than determined by polynomial growth condition (providing solution in Lebesgue spaces), hence it is a better approximation of Prandtl-Reuss model. Secondly, the choice of generalized Orlicz space allows us to consider non-homogeneous materials. Results presented in this chapter are based on [48].

We assume that the body $\Omega \subset \mathbb{R}^3$ is an open bounded set with a C^2 boundary. In previous chapters we considered only homogeneous materials. Here, we omit this assumption. In the case of generalized Orlicz spaces, the N-function depends on spatial variable x. Thus, in different regions of Ω we may have different growth conditions. Furthermore, operator $\boldsymbol{D}$ and function $\boldsymbol{G}$ may also depend on the spatial variable.

Assumption 6.0.1. *The function $\boldsymbol{G}(x,\theta,\boldsymbol{T}^d)$ is a Carethéodory function, i.e. is measurable with respect to x and continuous with respect to θ and $\boldsymbol{T}^d$, and what is more, it satisfies the following conditions:*

a) $(\boldsymbol{G}(x,\theta,\boldsymbol{T}_1^d) - \boldsymbol{G}(x,\theta,\boldsymbol{T}_2^d)) : (\boldsymbol{T}_1^d - \boldsymbol{T}_2^d) \geq 0$, *for all* $\boldsymbol{T}_1^d, \boldsymbol{T}_2^d \in \mathcal{S}_d^3$ *and* $\theta \in \mathbb{R}_+$;

b) $\boldsymbol{G}(x,\theta,\boldsymbol{T}^d) : \boldsymbol{T}^d \geq c\left(M(x,\boldsymbol{T}^d) + M^*(x,\boldsymbol{G}(x,\theta,\boldsymbol{T}^d))\right)$ *for a.a.* $x \in \Omega$, *where* $\boldsymbol{T}^d \in \mathcal{S}_d^3$, $\theta \in \mathbb{R}_+$ *and* c *is a positive constant independent of temperature* θ;

c) $\boldsymbol{G}(x,\theta,\boldsymbol{0}) = \boldsymbol{0}$ *for a.a.* $x \in \Omega$.

Moreover, M is an N-function and M^* is an N-function complementary to M. The class of N-functions is restricted as follows:

1) the inequality holds

$$\int_Q M^*(x,\boldsymbol{A}(x,t))\,\mathrm{d}x\,\mathrm{d}t \leq \int_Q |\boldsymbol{A}|^2\,\mathrm{d}x\,\mathrm{d}t; \tag{6.0.1}$$

2) M^* *satisfies the* Δ_2*-condition.*

Further, we write $\boldsymbol{G}(\theta,\boldsymbol{T}^d)$ instead of $\boldsymbol{G}(x,\theta,\boldsymbol{T}^d)$. We keep in mind that one of variables of function $\boldsymbol{G}$ is x but we omit repetitions in order to make the content more clear for the reader.

Studying mechanical problems in Orlicz spaces is not an isolated issue. The problem of visco-elastic deformation involving Orlicz spaces was considered in [21], but only in the case of N-function independent of spatial variable x. In the case of N-function which depends on the spatial variable x some accurate assumptions must be done. There are two possible ways to do it. Firstly, we may assume the regularity with respect to x, e.g. log-Hölder continuity in [74, 75]. And secondly, upper or lower growth condition of an N-function with respect to the last variable can be considered, e.g. see [35, 36, 37, 84]. There are no results for thermo-visco-elastic problems without any upper and lower growth condition on N-function with respect to the last variable.

In the contrast to results presented in Chapter 4 and Chapter 5 we use another approach to heat equation. In Chapter 3 we presented two different approaches which may be used to solve heat equation. As previously, by Assumption 6.0.1 we know that the right-hand side function $\boldsymbol{G}(\theta,\boldsymbol{T}^d):\boldsymbol{T}^d$ is only an integrable function. Here, we prove existence of renormalised solution to heat equation, see Section 3.2. The use of different approach than one used in previous models causes another definition of solution to heat equation, see Definition 6.2.2. Similarly as for previous model we approximate the difference $\widehat{\theta}-\tilde{\theta}$, where $\tilde{\theta}$ is a solution to *cutting off* problem, see (6.3.2), and $\widehat{\theta}$ is a solution to whole system of equations. In the contrast to Boccardo and Gallouët's solution the difference $\widehat{\theta}-\tilde{\theta}$ appears in the definition of renormalised solution, see Definition 6.2.2.

This chapter is organised as follows. In Section 6.1 we make some general remarks about generalised Orlicz space. We quote main definitions and prove important lemmas. In Section 6.2 we present the statement of the main theorem of this chapter. Finally, Section 6.3 is dedicated to the proof.

6.1 Generalized Orlicz spaces

We recall important definitions which will be used to formulate the statement of this chapter. Let us start with the repetition of generalized Orlicz spaces. For general concept of Orlicz spaces we refer the reader to [1, 49, 59, 64]. Let us start with the definition of N-function.

Definition 6.1.1. *Let Ω be a bounded open domain in $\mathbb{R}^3$. A function $M:\Omega\times\mathcal{S}^3\to\mathbb{R}_+$ is said to be N-function if it satisfies the following conditions:*

1) *M is a Carathéodory function such that $M(x,\boldsymbol{\xi})=0$ if and only if $\boldsymbol{\xi}=\boldsymbol{0}$;*

2) *$M(x,\boldsymbol{\xi})=M(x,-\boldsymbol{\xi})$ a.e. in Ω;*

3) *$M(x,\boldsymbol{\xi})$ is a convex function with respect to $\boldsymbol{\xi}$;*

4) *$\lim_{|\boldsymbol{\xi}|\to 0} M(x,\boldsymbol{\xi})/|\xi|=0$ for a. a. $x\in\Omega$;*

5) *$\lim_{|\boldsymbol{\xi}|\to\infty} M(x,\boldsymbol{\xi})/|\xi|=\infty$ for a. a. $x\in\Omega$.*

Definition 6.1.2. *The complementary function M^* to a function M is defined by*

$$M^*(x,\boldsymbol{\eta})=\sup_{\boldsymbol{\eta}\in\mathcal{S}^3}(\boldsymbol{\xi}:\boldsymbol{\eta}-M(x,\boldsymbol{\xi})),\tag{6.1.1}$$

for $\boldsymbol{\eta}\in\mathcal{S}^3, x\in\Omega$.

Remark. *A complementary function M^* to N-function M is also an N-function.*

The generalized Orlicz class $\mathcal{L}_M(Q,\mathcal{S}^3)$ is the set of all measurable functions $\boldsymbol{\xi}:Q\to\mathcal{S}^3$ such that

$$\int_Q M(x,\boldsymbol{\xi}(x,t))\,\mathrm{d}x\,\mathrm{d}t<\infty. \tag{6.1.2}$$

The generalized Orlicz space $L_M(Q,\mathcal{S}^3)$ can be defined as the smallest linear space containing $\mathcal{L}_M(Q,\mathcal{S}^3)$. It is a Banach space with respect to the Orlicz norm

$$\|\boldsymbol{\xi}\|_{O,M}=\sup\left\{\int_Q\boldsymbol{\xi}:\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t:\ \boldsymbol{\eta}\in L_{M^*}(Q,\mathcal{S}^3),\ \int_Q M^*(x,\boldsymbol{\eta})\,\mathrm{d}x\,\mathrm{d}t\le 1\right\} \tag{6.1.3}$$

or equivalently with respect to Luxemburg norm

$$\|\boldsymbol{\xi}\|_{L,M}=\inf\left\{\lambda>0:\int_Q M\left(x,\frac{\boldsymbol{\xi}(x,t)}{\lambda}\right)\,\mathrm{d}x\,\mathrm{d}t<1\right\}. \tag{6.1.4}$$

By $E_M(Q,\mathcal{S}^3)$ we denote the closure of the set of bounded functions in $\|\cdot\|_O$-norm.

Definition 6.1.3. *We say that an N-function M satisfies Δ_2-condition if for almost all $x\in\Omega$ and for all $\boldsymbol{\xi}\in\mathcal{S}^3$, there exist a constant c and nonnegative integrable function $h:\Omega\to\mathbb{R}$ such that*

$$M(x,2\boldsymbol{\xi})\le cM(x,\boldsymbol{\xi})+h(x). \tag{6.1.5}$$

If this condition fails, we lose numerous properties of the space $L_M(Q,\mathcal{S}^3)$ like separability, reflexivity, cf. [1, 59] and many others. In particular, if (6.1.5) holds, then $\mathcal{L}_M(Q,\mathcal{S}^3)=L_M(Q,\mathcal{S}^3)$.

Remark. *For every M the following inclusion holds*

$$E_M(Q,\mathcal{S}^3)\subseteq\mathcal{L}_M(Q,\mathcal{S}^3)\subseteq L_M(Q,\mathcal{S}^3). \tag{6.1.6}$$

If M satisfies the Δ_2-condition, then $E_M(Q,\mathcal{S}^3)=L_M(Q,\mathcal{S}^3)$.

The proof of abovementioned remark comes from [37, Proposition A.2].

Proof. Inclusions in (6.1.6) are obvious. We show that if M satisfies Δ_2-condition, $\mathcal{L}_M(Q,\mathcal{S}^3)$ is a vector space. Then, by definition of Orlicz spaces $L_M(Q,\mathcal{S}^3)$ and $E_M(Q,\mathcal{S}^3)$ the proof is complete.

Let M satisfy Δ_2-condition. Below we prove that pointwise addition and scalar multiplication are invariant in $\mathcal{L}_M(Q,\mathcal{S}^3)$. $M(x,\cdot)$ is a convex function with respect to second variable, thus for $\boldsymbol{\xi},\boldsymbol{\zeta}\in\mathcal{L}_M(Q,\mathcal{S}^3)$ it holds

$$\begin{aligned}\int_Q M(x,\boldsymbol{\xi}(t,x)+\boldsymbol{\zeta}(t,x))\,\mathrm{d}x\,\mathrm{d}t&=\int_Q M\Big(x,2\frac{\boldsymbol{\xi}(t,x)+\boldsymbol{\zeta}(t,x)}{2}\Big)\,\mathrm{d}x\,\mathrm{d}t\\&\le c\Big(\int_Q M(x,\boldsymbol{\xi}(t,x))\,\mathrm{d}x\,\mathrm{d}t+\int_Q M(x,\boldsymbol{\zeta}(t,x))\,\mathrm{d}x\,\mathrm{d}t\Big)+\int_Q h(x)\,\mathrm{d}x\,\mathrm{d}t<\infty,\end{aligned} \tag{6.1.7}$$

where constant c and function h come from Definition 6.1.3. Let $n\in\mathbb{N}$ such that $|\lambda|\le 2^n$ then

$$\int_Q M(x,\lambda\boldsymbol{\xi})\,\mathrm{d}x\,\mathrm{d}t=\int_Q M\big(x,(\operatorname{sgn}\lambda)2^n\boldsymbol{\xi}\big)\,\mathrm{d}x\,\mathrm{d}t\le c^n\int_Q M(x,\boldsymbol{\xi})\,\mathrm{d}x\,\mathrm{d}t+n\int_Q h(x)\,\mathrm{d}x\,\mathrm{d}t<\infty \tag{6.1.8}$$

which completes the proof.

□

The space $L_{M^*}(Q, \mathcal{S}^3)$ is the dual space of $E_M(Q, \mathcal{S}^3)$. The functional

$$\rho(\boldsymbol{\xi}) = \int_Q M(x, \boldsymbol{\xi})\,\mathrm{d}x\,\mathrm{d}t \tag{6.1.9}$$

is a modular.

Definition 6.1.4. *We say that a sequence $\{\boldsymbol{\xi}_i\}_{i=1}^{\infty}$ convergences modularly to $\boldsymbol{\xi}$ in $L_M(Q, \mathcal{S}^3)$ if there exists $\lambda > 0$ such that*

$$\int_Q M\Big(x, \frac{\boldsymbol{\xi}_i - \boldsymbol{\xi}}{\lambda}\Big)\,\mathrm{d}x\,\mathrm{d}t \to 0 \tag{6.1.10}$$

We will use the notation $\boldsymbol{\xi}_i \xrightarrow{M} \boldsymbol{\xi}$ for modular convergence in $L_M(Q, \mathcal{S}^3)$.

Assumption 6.0.1 requires the usage of basic tools regarding generalized Orlicz spaces. Here we present some basic lemmas, which are used in the proof regarding existence of solution to thermo-visco-elastic models. The following lemmas and their proofs come from [37]. They may be also found in [28, 35, 39, 85] and many other publications.

Lemma 6.1.1 (Fenchel-Young inequality)**.** *Let M be an N-function and M^* be complementary to M. Then following inequality is satisfied*

$$|\boldsymbol{\xi} : \boldsymbol{\eta}| \le M(x, \boldsymbol{\xi}) + M^*(x, \boldsymbol{\eta}) \tag{6.1.11}$$

for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathcal{S}^3$ and for almost all $x \in \Omega$.

Fenchel-Young inequality is a consequence of definition of complementary N-function, see Definition 6.1.2

Lemma 6.1.2 (Hölder inequality)**.** *Let M be an N-function and M^* be complementary to M. Then the following inequality is satisfied*

$$\Big|\int_Q \boldsymbol{\xi} : \boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t\Big| \le 2\|\boldsymbol{\xi}\|_{L,M}\|\boldsymbol{\eta}\|_{L,M^*}. \tag{6.1.12}$$

Proof. By applying the Young inequality to the product of $\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|_{L,M}}$ and $\frac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|_{L,M^*}}$ we obtain

$$\int_Q \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|_{L,M}} : \frac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|_{L,M^*}}\,\mathrm{d}x\,\mathrm{d}t \le \int_Q M\Big(x, \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|_{L,M}}\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_Q M^*\Big(x, \frac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|_{L,M^*}}\Big)\,\mathrm{d}x\,\mathrm{d}t = 2, \tag{6.1.13}$$

where the last equality is the consequence of Luxemburg norms definition. □

Lemma 6.1.3 (Lemma A.3 from [37])**.** *Let $\boldsymbol{\xi}_i : Q \to \mathbb{R}^d$ be a measurable sequence. Then $\boldsymbol{\xi}_i \xrightarrow{M} \boldsymbol{\xi}$ in $L_M(Q, \mathcal{S}^3)$ modularly if and only if $\boldsymbol{\xi}_i \to \boldsymbol{\xi}$ in measure and there exists some $\lambda > 0$ such that the sequence $\{M(\cdot, \lambda\boldsymbol{\xi}_i)\}$ is uniformly integrable, i.e.*

$$\lim_{R\to\infty}\left(\sup_{i\in\mathbb{N}} \int_{\{(t,x):\ |M(x,\lambda\boldsymbol{\xi}_i)|\ge R\}} M(x, \lambda\boldsymbol{\xi}_i)\,\mathrm{d}x\,\mathrm{d}t\right) = 0. \tag{6.1.14}$$

Proof. Note that $\boldsymbol{\xi}_j \to \boldsymbol{\xi}$ in measure if and only if $M(\cdot, (\boldsymbol{\xi}_j - \boldsymbol{\xi})/\lambda) \to 0$ in measure for all $\lambda > 0$. Moreover, the convergence $\boldsymbol{\xi}_j \to \boldsymbol{\xi}$ in measure implies that for all measurable sets $A \subset Q$ it holds

$$\liminf_{j\to\infty} \int_A M(x, \boldsymbol{\xi}_j)\,\mathrm{d}x\,\mathrm{d}t \geq \int_A M(x, \boldsymbol{\xi})\,\mathrm{d}x\,\mathrm{d}t. \tag{6.1.15}$$

Note also that the convexity of M implies

$$\int_A M\Big(x, \frac{\boldsymbol{\xi}_j - \boldsymbol{\xi}}{\lambda}\Big)\,\mathrm{d}x\,\mathrm{d}t \leq \int_A M\Big(x, \frac{\boldsymbol{\xi}_j}{2\lambda}\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_A M\Big(x, \frac{\boldsymbol{\xi}}{2\lambda}\Big)\,\mathrm{d}x\,\mathrm{d}t \tag{6.1.16}$$

Hence by the classical Vitali's lemma for $f_j(x) = M(x, (\boldsymbol{\xi}_j - \boldsymbol{\xi})/\lambda)$ we obtain that $f_j \to 0$ strongly in $L^1(Q)$. □

Lemma 6.1.4 (Lemma A.4 from [37])**.** *Let M be an N-function and for all $i \in \mathbb{N}$, let $\int_Q M(x, \boldsymbol{\xi}_i)\,\mathrm{d}x\,\mathrm{d}t \leq C$. Then the sequence $\{\boldsymbol{\xi}_i\}$ is uniformly integrable.*

Proof. Let us define $\delta(R) = \min_{|\boldsymbol{\xi}|=R} M(x, \boldsymbol{\xi})/|\boldsymbol{\xi}|$. Then, for all $j \in \mathbb{N}$, it holds

$$\begin{aligned}\int_{\{(x,t)\in Q:\ |\boldsymbol{\xi}_j(x,t)|\geq R\}} M(x, \boldsymbol{\xi}_j(t,x))\,\mathrm{d}x\,\mathrm{d}t &= \int_{\{(x,t)\in Q:\ |\boldsymbol{\xi}_j(x,t)|\geq R\}} \frac{M(x,\boldsymbol{\xi}_j)}{|\boldsymbol{\xi}_j|}|\boldsymbol{\xi}_j|\,\mathrm{d}x\,\mathrm{d}t \\ &\geq \delta(R) \int_{\{(x,t)\in Q:\ |\boldsymbol{\xi}_j(x,t)|\geq R\}} |\boldsymbol{\xi}_j|\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{6.1.17}$$

Since the left-hand side is bounded, we obtain

$$\sup_{j\in\mathbb{N}} \int_{\{(x,t)\in Q:\ |\boldsymbol{\xi}_j(x,t)|\geq R\}} |\boldsymbol{\xi}_j(t,x)|\,\mathrm{d}x\,\mathrm{d}t \leq \frac{C}{\delta(R)}. \tag{6.1.18}$$

Function $\delta(R)$ is increasing, hence the proof is complete. □

Lemma 6.1.5 (Lemma A.5 from [37])**.** *Let M be an N-function and M^* its complementary function. Suppose that the sequences $\boldsymbol{\Phi}_i : Q \to \mathcal{S}^3$ and $\boldsymbol{\Psi}_i : Q \to \mathcal{S}^3$ are uniformly bounded in $L_M(Q, \mathcal{S}^3)$ and $L_{M^*}(Q, \mathcal{S}^3)$, respectively. Moreover, $\boldsymbol{\Phi}_i \xrightarrow{M} \boldsymbol{\Phi}$ modularly in $L_M(Q, \mathcal{S}^3)$ and $\boldsymbol{\Phi}_i \xrightarrow{M^*} \boldsymbol{\Phi}$ modularly in $L_{M^*}(Q, \mathcal{S}^3)$. Then, $\boldsymbol{\Phi}_i : \boldsymbol{\Psi}_i \to \boldsymbol{\Phi} : \boldsymbol{\Psi}$ strongly in $L^1(Q)$.*

Proof. Owing to Lemma 6.1.3 the modular convergence of $\{\boldsymbol{\Psi}_j\}$ and $\{\boldsymbol{\Phi}_j\}$ implies the convergence in measure of these sequences and consequently also the convergence in measure of the product. Hence it is sufficient to show the uniform integrability of $\{\boldsymbol{\Psi}_j : \boldsymbol{\Phi}_j\}$. Notice that it is equivalent with the uniform integrability of the term $\{\frac{\boldsymbol{\Psi}_j}{\lambda_1} : \frac{\boldsymbol{\Phi}_j}{\lambda_2}\}$ for any $\lambda_1, \lambda_2 > 0$. The assumptions of the proposition provide that there exist some $\lambda_1, \lambda_2 > 0$ such that the sequences

$$\left\{M\left(x, \frac{\boldsymbol{\Psi}_j}{\lambda_1}\right)\right\} \qquad \text{and} \qquad \left\{M^*\left(x, \frac{\boldsymbol{\Phi}_j}{\lambda_2}\right)\right\} \tag{6.1.19}$$

are uniformly integrable. Hence, let us use the same constants and estimate with the help of Fenchel–Young inequality

$$\Big|\frac{\boldsymbol{\Psi}_j}{\lambda_1} : \frac{\boldsymbol{\Phi}_j}{\lambda_2}\Big| \leq M\Big(x, \frac{\boldsymbol{\Psi}_j}{\lambda_1}\Big) + M\Big(x, \frac{\boldsymbol{\Phi}_j}{\lambda_2}\Big). \tag{6.1.20}$$

Obviously, the uniform integrability of the right-hand side provides the uniform integrability of the left-hand side and this yields the assertion. □

Lemma 6.1.6 (Lemma A.6 from [37]). *Let ρ_i be a standard mollifier, i.e. $\rho \in C^\infty(\mathbb{R})$, ρ has a compact support and $\int_{\mathbb{R}} \rho(\tau)\,\mathrm{d}\tau = 1$, $\rho(\tau) = \rho(-\tau)$. We define $\rho_i(\tau) = i\rho(i\tau)$. Moreover, let $*$ denote a convolution in the variable τ. Then for any function $\mathbf{\Phi} : Q \to \mathcal{S}^3$, such that $\mathbf{\Phi} \in L^1(Q, \mathcal{S}^3)$, it holds*

$$\rho_i * \mathbf{\Phi} \to \mathbf{\Phi} \qquad \text{in measure as } i \to \infty. \tag{6.1.21}$$

Proof. For a.a. $x \in \Omega$ the function $\mathbf{\Psi}(\cdot, x) \in L^1(0,T)$ and $\rho_i * \mathbf{\Psi}(\cdot, x) \to \mathbf{\Psi}(\cdot, x)$ in $L^1(0,T)$ and hence $\rho_i * \mathbf{\Psi} \to \mathbf{\Psi}$ in measure on the set $[0,T] \times \Omega$. □

Lemma 6.1.7 (Lemma A.7 from [37]). *Let ρ_i be a standard mollifier. Given an N-function M and a function $\mathbf{\Phi} : Q \to \mathcal{S}^3$ such that $\mathbf{\Phi} \in \mathcal{L}_M(Q)$, the sequence $\{M(x, \rho_i * \mathbf{\Phi})\}$ is uniformly integrable.*

Proof. We start with an abstract fact concerning the uniform integrability. Namely, the following two conditions are equivalent for any measurable sequence $\{\boldsymbol{\xi}_j\}$

(a) $\forall \epsilon > 0\ \exists \delta > 0 : \sup_{j\in\mathbb{N}} \sup_{|A|\le\delta} \int_A |\boldsymbol{\xi}_j|\,\mathrm{d}x\,\mathrm{d}t \le \epsilon$

(b) $\forall \epsilon > 0\ \exists \delta > 0 : \sup_{j\in\mathbb{N}} \int_Q \left(|\boldsymbol{\xi}_j| - \frac{1}{\sqrt{\delta}}\right)^+ \le \epsilon$ where we use the same notation as in previous chapters, i.e. $z^+ = \max\{0, z\}$.

The implication $(a) \Rightarrow (b)$ is obvious. There exists $\gamma > 0$ such that there exists $A' = \{(x,t) \in Q : |\boldsymbol{\xi}_j| > \frac{1}{\sqrt{\gamma}}\}$ with $\mathrm{meas}(A') \le \delta$. Then

$$\sup_{j\in\mathbb{N}} \int_{A'} |\boldsymbol{\xi}_j|\,\mathrm{d}x\,\mathrm{d}t \le \epsilon, \tag{6.1.22}$$

and

$$\sup_{j\in\mathbb{N}} \int_{A'} \left(|\boldsymbol{\xi}_j| - \frac{1}{\sqrt{\gamma}}\right) \mathrm{d}x\,\mathrm{d}t \le \epsilon + \frac{\delta}{\sqrt{\gamma}}, \tag{6.1.23}$$

By the proper choice of A' we obtain $\sup_{j\in\mathbb{N}} \int_Q \left(|\boldsymbol{\xi}_j| - \frac{1}{\sqrt{\gamma}}\right)^+ \mathrm{d}x\,\mathrm{d}t \le \epsilon + \frac{\delta}{\sqrt{\gamma}}$, which completes this implication. To show that also $(b) \Rightarrow (a)$ holds, let us estimate

$$\begin{aligned}
\sup_{j\in\mathbb{N}} \sup_{\mathrm{meas}(A)\le\delta} \int_A |\boldsymbol{\xi}_j|\,\mathrm{d}x\,\mathrm{d}t &= \sup_{j\in\mathbb{N}} \sup_{\mathrm{meas}(A)\le\delta} \int_A \Big|\boldsymbol{\xi}_j - \frac{1}{\sqrt{\delta}} + \frac{1}{\sqrt{\delta}}\Big|\,\mathrm{d}x\,\mathrm{d}t \\
&\le \sup_{j\in\mathbb{N}} \sup_{\mathrm{meas}(A)\le\delta} \int_A \left(|\boldsymbol{\xi}_j| - \frac{1}{\sqrt{\delta}}\right)^+ \mathrm{d}x\,\mathrm{d}t + \sup_{\mathrm{meas}(A)\le\delta} \mathrm{meas}(A)\frac{1}{\sqrt{\delta}} \\
&\le \sqrt{\delta} + \sup_{j\in\mathbb{N}} \int_Q \left(|\boldsymbol{\xi}_j| - \frac{1}{\sqrt{\delta}}\right)^+ \mathrm{d}x\,\mathrm{d}t \\
&\le \sqrt{\delta} + \epsilon.
\end{aligned} \tag{6.1.24}$$

Notice that since M is a convex function, the following inequality holds for all $\delta > 0$:

$$\int_Q \Big|M(x, \mathbf{\Psi}) - \frac{1}{\delta}\Big|^+ \mathrm{d}x\,\mathrm{d}t \ge \int_Q \Big|M(x, \rho_j * \mathbf{\Psi}) - \frac{1}{\delta}\Big|^+ \mathrm{d}x\,\mathrm{d}t. \tag{6.1.25}$$

Finally, since $\mathbf{\Psi} \in \mathcal{L}_M(Q)$, also $\int_Q |M(x, \mathbf{\Psi}) - (1/\sqrt{\delta})|^+ \,\mathrm{d}x\,\mathrm{d}t$ is finite and hence taking supremum over $j \in \mathbb{N}$ in abovementioned equation we prove the assertion.

□

6.2 Formulation of the problem

Heterogeneity of domain Ω implies the need of making the assumptions on regularity of component in regards of operator $\boldsymbol{D}$.

Assumption 6.2.1. *Let the operator $\boldsymbol{D} : \mathcal{S}^3 \to \mathcal{S}^3$ be a four-index matrix, i.e.*

$$\boldsymbol{D} = \boldsymbol{D}(x) = \{d_{i,j,k,l}(x)\}_{i,j,k,l=1}^3 \tag{6.2.1}$$

and the following equalities hold

$$d_{i,j,k,l}(x) = d_{j,i,k,l}(x), \quad d_{i,j,k,l}(x) = d_{i,j,l,k}(x) \quad \text{and} \quad d_{i,j,k,l}(x) = d_{k,l,i,j}(x), \tag{6.2.2}$$

where function $d_{i,j,k,l}$ belongs to $W^{1,p}(\Omega)$ for each $i, j, k, l = 1, 2, 3$ and for some $p > 3$. Moreover, let $\boldsymbol{D}$ be linear, positively definite and bounded.

Assumption 6.0.1 on function $\boldsymbol{G}(\cdot, \cdot)$ causes use of space for displacement.

Definition 6.2.1. *Let us define the space $BD_{M^*}(Q, \mathbb{R}^3)$ by formula*

$$BD_{M^*}(Q, \mathbb{R}^3) = \left\{\boldsymbol{u} \in L^1(\Omega, \mathbb{R}^3) : \quad \boldsymbol{\varepsilon}(\boldsymbol{u}) \in L_{M^*}(\Omega, \mathcal{S}^3)\right\}. \tag{6.2.3}$$

The space $BD_{M^*}(Q, \mathbb{R}^3)$ is a Banach space with a norm

$$\|\boldsymbol{u}\|_{BD_M(Q)} = \|\boldsymbol{u}\|_{L^1(Q)} + \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{M^*}. \tag{6.2.4}$$

Space $BD_{M^*}(Q, \mathbb{R}^3)$ is a subspace of the space of bounded deformations $BD(Q, \mathbb{R}^3)$

$$BD(Q, \mathbb{R}^3) = \left\{\boldsymbol{u} \in L^1(\Omega, \mathbb{R}^3) : \quad [\boldsymbol{\varepsilon}(\boldsymbol{u})]_{i,j} \in \mathcal{M}(Q)\right\}, \tag{6.2.5}$$

where $\mathcal{M}(Q)$ is a space of bounded measures on Q and $[\boldsymbol{\varepsilon}(\boldsymbol{u})]_{i,j} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, cf. [38]. According to [80, Theorem 1.1] there exist a unique continuous operator γ_0 from $BD_{M^*}(Q)$ onto $L^1((0,T) \times \partial\Omega)$ such that the generalized Green formula

$$2\int_Q \phi \boldsymbol{\varepsilon}_{i,j}(\boldsymbol{u})\,\mathrm{d}x\,\mathrm{d}t = -\int_Q (u_i \frac{\partial \phi}{\partial x_i} + u_j \frac{\partial \phi}{\partial x_j}) + \int_0^T \int_{\partial\Omega} \phi(\gamma_0(u_i)n_j + \gamma_0(u_j)n_i)\mathrm{d}\mathcal{H}^2\,\mathrm{d}t \tag{6.2.6}$$

hold for every $\phi \in C^1(\overline{Q})$ and where $\boldsymbol{n} = (n_1, n_2, n_3)^T$ is an unite outward normal vector on $\partial\Omega$ and H^2 is the 2-Hausdorff measure. Moreover, $BD(Q, \mathbb{R}^3)$ is compactly embedded in $L^q(Q, \mathbb{R}^3)$ for every $1 \leq q < \frac{3}{2}$, see [80, Remark 2.3].

Furthermore, we understand $v \in BD_{M^*}(Q, \mathbb{R}^3) + L^\infty(0, T, W^{2,p}(\Omega, \mathbb{R}^3))$ in the following way: There exists a decomposition $v = v_1 + v_2$, where $v_1 \in BD_{M^*}(Q, \mathbb{R}^3)$ and $v_2 \in L^\infty(0, T, W^{2,p}(\Omega, \mathbb{R}^3))$.

Definition 6.2.2 (Weak-renormalised solution of the system (1.2.2))**.** *The triple of functions $\boldsymbol{u} \in BD_{M^*}(Q, \mathbb{R}^3) + L^\infty(0, T, W^{2,p}(\Omega, \mathbb{R}^3))$, $\boldsymbol{T} \in L^2(Q, \mathcal{S}^3)$ and a measurable function θ such that for every $K \in \mathbb{N}$, $\mathcal{T}_K(\theta) \in L^2(0, T, W^{1,2}(\Omega))$ is a weak-renormalised solution of the system (1.2.2) when*

$$\int_0^T \int_\Omega \boldsymbol{T} : \nabla \boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t, \tag{6.2.7}$$

where

$$\boldsymbol{T} = \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}), \tag{6.2.8}$$

holds for every test function $\boldsymbol{\varphi} \in C^\infty([0,T], C_c^\infty(\Omega,\mathbb{R}^3))$ *and*

$$-\int_Q S(\theta-\tilde\theta)\frac{\partial\phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega S(\theta_0-\tilde\theta_0)\phi(x,0)\,\mathrm{d}x + \int_Q S'(\theta-\tilde\theta)\nabla(\theta-\tilde\theta)\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \\ + \int_Q S''(\theta-\tilde\theta)|\nabla(\theta-\tilde\theta)|^2\phi\,\mathrm{d}x\,\mathrm{d}t = \int_Q \boldsymbol{G}(\theta,\boldsymbol{T}^d):\boldsymbol{T}^d S'(\theta-\tilde\theta)\phi\,\mathrm{d}x\,\mathrm{d}t \tag{6.2.9}$$

holds for every test function $\phi \in C_c^\infty([-\infty,T), C^\infty(\Omega))$*, for every function* $S\in C^\infty(\mathbb{R})$ *such that* $S'\in C_0^\infty(\mathbb{R})$ *and for* $\tilde\theta$ *which is a solution of the problem*

$$\begin{cases} \tilde\theta_t - \Delta\tilde\theta &= 0 \quad \text{in } \Omega\times(0,T),\\ \frac{\partial\tilde\theta}{\partial\boldsymbol{n}} &= g_\theta \quad \text{on } \partial\Omega\times(0,T),\\ \tilde\theta(x,0) &= \tilde\theta_0 \quad \text{in } \Omega, \end{cases} \tag{6.2.10}$$

where $\tilde\theta_0\in L^2(\Omega)$*. Furthermore, the visco-elastic strain tensor can be recovered from the equation on its evolution, i.e.*

$$\boldsymbol{\varepsilon}^{\mathbf{p}}(x,t) = \boldsymbol{\varepsilon}_0^{\mathbf{p}}(x) + \int_0^t \boldsymbol{G}(\theta(x,\tau),\boldsymbol{T}^d(x,\tau))\,\mathrm{d}\tau, \tag{6.2.11}$$

for a.e. $x\in\Omega$ *and* $t\in[0,T)$ *also* $\boldsymbol{\varepsilon}^{\mathbf{p}}, \boldsymbol{\varepsilon}_t^{\mathbf{p}} \in L_{M^*}(Q)$.

Theorem 6.2.1. *Let initial conditions satisfy* $\theta_0\in L^1(\Omega)$*,* $\boldsymbol{\varepsilon}_0^{\mathbf{p}}\in L_{M^*}(\Omega,\mathcal{S}^3_d)$*, boundary conditions satisfy* $g_\theta\in L^2(0,T,L^2(\partial\Omega))$*, for* $p>3$ *function* $\boldsymbol{g}\in L^\infty(0,T,W^{2,p}(\Omega,\mathbb{R}^3))$ *and volume force* $\boldsymbol{f}\in L^\infty(0,T,L^p(\Omega,\mathbb{R}^3))$*, also function* $\boldsymbol{G}(\cdot,\cdot)$ *satisfy the same condition as in Assumptions 6.0.1 and let operator* $\boldsymbol{D}$ *satisfy Assumption 6.2.1. Then there exists a weak solution to system* (1.2.2).

6.3 Proof of Theorem 6.2.1

The idea of the proof is similar to proofs presented in previous chapters. We use two level Galerkin approximations to construct the approximate system of equations. Use of growth condition in Orlicz spaces instead of growth condition in Lebesgue spaces implies usage of different analytic tools. For example, we use Minty-Browder trick in nonreflexive spaces, see [84], to identify the weak limit of nonlinear term. Moreover, to prove the convergence of right-hand side term of heat equation we apply the biting limit, cf. [8], and Young measures tools, which were described in Chapter 4.

Construction of the proof is similar to previous cases. We start with *cutting off* the boundary conditions, Section 6.3.1. Then, we construct the approximate solutions and show their uniform boundedness, see Section 6.3.2. Finally, we pass to the limit independently with approximation parameter for temperature (Section 6.3.3) and with approximation parameter for displacement (Section 6.3.4).

6.3.1 Transformation to a homogeneous boundary-value-problem

As we mentioned in Chapter 2, our idea is to consider three systems of equations, i.e. two systems which take the boundary conditions for displacement and heat flux. i.e.

$$\begin{cases} -\operatorname{div}\tilde{\boldsymbol{T}} &= \boldsymbol{f} \quad \text{in } \Omega\times(0,T),\\ \tilde{\boldsymbol{T}} &= \boldsymbol{D}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}) \quad \text{in } \Omega\times(0,T),\\ \tilde{\boldsymbol{u}} &= \boldsymbol{g} \quad \text{on } \partial\Omega\times(0,T), \end{cases} \tag{6.3.1}$$

and

$$\begin{cases} \tilde{\theta}_t - \Delta\tilde{\theta} &= 0 \quad \text{in } \Omega\times(0,T), \\ \frac{\partial\tilde{\theta}}{\partial \boldsymbol{n}} &= g_\theta \quad \text{on } \partial\Omega\times(0,T), \\ \tilde{\theta}(x,0) &= \tilde{\theta}_0 \quad \text{in } \Omega, \end{cases} \tag{6.3.2}$$

and one with homogeneous boundary conditions

$$\begin{cases} -\operatorname{div}\boldsymbol{T} = 0, \\ \boldsymbol{T} = \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}), \\ \boldsymbol{\varepsilon}^{\mathbf{p}}_t = \boldsymbol{G}(\tilde{\theta}+\theta, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d), \\ \theta_t - \Delta\theta = \left(\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d\right) : \boldsymbol{G}(\tilde{\theta}+\theta, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d). \end{cases} \tag{6.3.3}$$

By the following lemma we show the existence of solutions to (6.3.1) and (6.3.2). In the rest of this chapter we prove that solution to the last system exists.

Lemma 6.3.1. *For $p > 3$, let $\tilde{\theta}_0 \in L^2(\Omega)$, $\boldsymbol{g} \in L^\infty(0,T,W^{2,p}(\Omega,\mathbb{R}^3))$, $g_\theta \in L^2(0,T,L^2(\partial\Omega))$ and $\boldsymbol{f} \in L^\infty(0,T,L^p(\Omega,\mathbb{R}^3))$. Then there exists a solution to systems* (6.3.1) *and* (6.3.2)*. Additionally, the following estimates hold:*

$$\|\tilde{\boldsymbol{u}}\|_{L^\infty(0,T,W^{2,p}(\Omega))} \le C_1 \left((\|\boldsymbol{g}\|_{L^\infty(0,T,W^{2,p}(\Omega))} + \|\boldsymbol{f}\|_{L^\infty(0,T,L^p(\Omega))}\right),$$
$$\|\tilde{\theta}\|_{L^\infty(0,T,L^1(\Omega))} + \|\tilde{\theta}\|_{L^2(0,T,W^{1,2}(\Omega))} \le C_2 \left(\|g_\theta\|_{L^2(0,T,L^2(\partial\Omega))} + \|\tilde{\theta}_0\|_{L^2(\Omega)}\right).$$

Moreover, $\tilde{\theta}$ belongs to $C([0,T],L^2(\Omega))$ and the following estimate for Cauchy stress tensor holds

$$\|\tilde{\boldsymbol{T}}\|_{L^\infty(Q)} \le C_3 \left(\|\boldsymbol{g}\|_{L^\infty(0,T,W^{2,p}(\Omega))} + \|\boldsymbol{f}\|_{L^\infty(0,T,L^{,p}(\Omega))}\right). \tag{6.3.4}$$

Proof. The results for temperature were discussed in previous chapters, hence let us focus on existence of the elastostatic problem. The idea of proof is the same as in the proof of Lemma 4.3.1. The main difference between is that we require more regularity of Cauchy stress tensor $\tilde{\boldsymbol{T}}$. It is caused by usage of Minty-Browder trick in nonreflexive spaces and estimate 6.3.4 is crucial in the next steps of main Theorems proof.

Rewriting the solution in the form $\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}_1 + \boldsymbol{g}$, we may replace finding $\tilde{\boldsymbol{u}}$ by finding $\tilde{\boldsymbol{u}}_1$, where it is a solution to system

$$\begin{cases} -\operatorname{div}\boldsymbol{D}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{u}}_1) &= \boldsymbol{f} + \operatorname{div}\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{g}) \quad \text{in } \Omega\times(0,T), \\ \tilde{\boldsymbol{u}}_1 &= 0 \quad \text{on } \partial\Omega\times(0,T). \end{cases} \tag{6.3.5}$$

Function $\boldsymbol{f} + \operatorname{div}\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{g})$ belongs to $L^\infty(0,T,L^p(\Omega,\mathbb{R}^3))$. By [82, Theorem 7.1] we know that there exists an unique solution $\tilde{\boldsymbol{u}}_1 \in L^\infty(0,T,W^{2,p}(\Omega,\mathbb{R}^3))$. For $p > 3$, using the general Sobolev inequalities [29, Theorem 6, p. 270] we obtain the inequality (6.3.4). □

6.3.2 Boundedness of energy

Following the procedure presented in Chapter 2 we construct the approximate system of equations. Finite initial energy of the system implies the boundedness of approximate solutions. Let us start with estimates for potential energy, see Definition 1.4.1.

Lemma 6.3.2. *There exists a constant C (uniform with respect to k and l) such that*

$$\begin{aligned}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(t)+\frac{2c-d}{2}\int_Q M^*(x,\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}))\,\mathrm{d}x\,\mathrm{d}t\\ +c\int_Q M(x,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\,\mathrm{d}x\,\mathrm{d}t\le C,\end{aligned}\tag{6.3.6}$$

where c is a constant Assumption 6.0.1 and $d=\min(1,c)$. Moreover, constant C depends on solution of additional problem (6.3.1) *and potential energy in the initial time*

$$C=\int_Q M(x,\frac{2}{d}\tilde{\boldsymbol{T}}^d)\,\mathrm{d}x\,\mathrm{d}t+\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(0).\tag{6.3.7}$$

Since $\tilde{\boldsymbol{T}}$ belongs to $L^\infty(Q)$, the constant C in (6.3.2) is finite.

Proof. Let us start with calculating the time derivative of the potential energy $\mathcal{E}(t)$. For a.a. $t\in[0,T]$ we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})=&\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l})-\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}):(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}))_t\,\mathrm{d}x\\ &-\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l})-\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}):(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t\,\mathrm{d}x\end{aligned}$$

Terms on the right-hand side of abovementioned equation may be rewritten with application of approximate system of equations (2.2.8). Firstly, for each $n\le k$ let us multiply $(2.2.8)_1$ by $(\alpha^n_{k,l})_t$. After summing over $n=1,...,k$ we get

$$\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l})-\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}):(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}))_t\,\mathrm{d}x=0.\tag{6.3.8}$$

Then for each $n\le k$ let us multiply $(2.2.8)_3$ by $\gamma^n_{k,l}$ and for each $m\le l$ let us multiply $(2.2.8)_4$ by $\delta^n_{k,l}$. Summing over $n=1,..,k$ and $m=1,...,l$ we obtain

$$\int_\Omega(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t:\boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l})-\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})\,\mathrm{d}x=\int_\Omega \boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{T}_{k,l}\,\mathrm{d}x.\tag{6.3.9}$$

Hence

$$\frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})=-\int_\Omega \boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\boldsymbol{T}_{k,l}\,\mathrm{d}x,\tag{6.3.10}$$

and then

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})=&-\int_\Omega \boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\,\mathrm{d}x\\ &+\int_\Omega \boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):\tilde{\boldsymbol{T}}^d\,\mathrm{d}x.\end{aligned}$$

Thus, using Assumption 6.0.1 and Fenchel-Young inequality we estimate the changes of potential energy by

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}),\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})\le&-c\left(\int_\Omega M(x,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\,\mathrm{d}x+\int_\Omega M^*(x,\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}))\,\mathrm{d}x\right)\\ &+\int_\Omega M(x,\frac{2}{d}\tilde{\boldsymbol{T}}^d)\,\mathrm{d}x+\int_\Omega M^*(x,\frac{d}{2}\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}))\,\mathrm{d}x,\end{aligned}$$

where $d = \min(1, c)$. Then, by convexity of N-function, we obtain

$$\frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l}) \leq -c\left(\int_\Omega M(x, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\,\mathrm{d}x + \int_\Omega M^*(x, \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}))\,\mathrm{d}x\right) + \int_\Omega M(x, \frac{2}{d}\tilde{\boldsymbol{T}}^d)\,\mathrm{d}x + \frac{d}{2}\int_\Omega M^*(x, \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}))\,\mathrm{d}x$$

Finally, integrating over time interval $(0, t)$, with $0 \leq t \leq T$ we obtain

$$\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(t) + c\int_Q M(x, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\,\mathrm{d}x\,\mathrm{d}t + \frac{2c-d}{2}\int_Q M^*(x, \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}))\,\mathrm{d}x\,\mathrm{d}t \leq \int_Q M(x, \frac{2}{d}\tilde{\boldsymbol{T}}^d)\,\mathrm{d}x\,\mathrm{d}t + \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}), \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})(0).$$

which completes the proof. □

Remark. *From Lemma 6.3.2 we know that the sequence $\{\boldsymbol{T}^d_{k,l}\}$ is uniformly bounded in $L_M(Q, \mathcal{S}^3)$ with respect to k and l, as well as the sequence $\{\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\}$ is uniformly bounded in the space $L_{M^*}(Q, \mathcal{S}^3)$ with respect to k and l. Hence, using the Fenchel-Young inequality, the sequence $\{(\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\}$ is uniformly bounded in $L^1(Q)$.*

Remark. *On basis of Lemma 6.3.2 the sequence $\{\boldsymbol{T}_{k,l}\}$ is uniformly bounded in $L^\infty(0, T, L^2(\Omega, \mathcal{S}^3))$ and in particular in $L^2(0, T, L^2(\Omega, \mathcal{S}^3))$.*

The following lemma is similar to Lemma 4.3.3 and Lemma 5.2.2. In the proof we use the projections, see Definition 2.1.2. The differences in the estimates are caused by looking for the solutions in another functional spaces.

Lemma 6.3.3. *The sequence $\{(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t\}$ is uniformly bounded in $L^1(0, T, (H^s(\Omega, \mathcal{S}^3))')$ with respect to l.*

Proof. Let $\boldsymbol{\varphi} \in L^\infty(0, T, H^s(\Omega, \mathcal{S}^3))$. Since $(P^k + P^{l,k}_{L^2})(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t = (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t$ we may estimate as follows

$$\begin{aligned}\int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, \boldsymbol{\varphi})_{\boldsymbol{D}}|\,\mathrm{d}t &= \int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, (P^k + P^{l,k}_{L^2})\boldsymbol{\varphi})_{\boldsymbol{D}}|\,\mathrm{d}t \\ &\leq \int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, P^k\boldsymbol{\varphi})_{\boldsymbol{D}}|\,\mathrm{d}t + \int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, P^{l,k}_{L^2}\boldsymbol{\varphi})_{\boldsymbol{D}}|\,\mathrm{d}t.\end{aligned} \tag{6.3.11}$$

Thus

$$\begin{aligned}\int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, \boldsymbol{\varphi})\boldsymbol{D}|\,\mathrm{d}t &\leq \int_0^T |\int_\Omega \boldsymbol{D}\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})P^k\boldsymbol{\varphi}\,\mathrm{d}x|\,\mathrm{d}t \\ &\quad + \int_0^T |\int_\Omega \boldsymbol{D}\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})P^{l,k}_{L^2}\boldsymbol{\varphi}\,\mathrm{d}x|\,\mathrm{d}t \\ &\leq d\int_0^T \|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^1(\Omega)}\|P^k\boldsymbol{\varphi}\|_{L^\infty(\Omega)}\,\mathrm{d}t \\ &\quad + d\int_0^T \|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^1(\Omega)}\|(P^{l,k}_{H^s} \circ (Id - P^k))\boldsymbol{\varphi}\|_{L^\infty(\Omega)}\,\mathrm{d}t.\end{aligned} \tag{6.3.12}$$

As $s \in (\frac{3}{2}, 2]$, then by Sobolev inequality we get $\|P^{l,k}_{H^s}\boldsymbol{\varphi}\|_{L^\infty(\Omega)} \le \tilde{c}\|P^{l,k}_{H^s}\boldsymbol{\varphi}\|_{H^s(\Omega)}$ and $\|P^k\boldsymbol{\varphi}\|_{L^\infty(\Omega)} \le \tilde{c}\|P^k\boldsymbol{\varphi}\|_{H^s(\Omega)}$, where $\tilde{c}$ is an optimal embedding constant. Then, we may proceed similarly as in the proof of Lemma 4.3.3

$$\begin{aligned}\int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, \boldsymbol{\varphi})_{\boldsymbol{D}}|\,\mathrm{d}t &\le d\tilde{c}\int_0^T \|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^1(\Omega)}\|P^k\boldsymbol{\varphi}\|_{H^s(\Omega)}\,\mathrm{d}t \\ &\quad + d\tilde{c}\int_0^T \|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^1(\Omega)}\|(P^{l,k}_{H^s} \circ (Id - P^k))\boldsymbol{\varphi}\|_{H^s(\Omega)}\,\mathrm{d}t \\ &\le dc(k)\tilde{c}\int_0^T \|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^1(\Omega)}\|\boldsymbol{\varphi}\|_{H^s(\Omega)}\,\mathrm{d}t \\ &\quad + dc(k)\tilde{c}\int_0^T \|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^1(\Omega)}\|\boldsymbol{\varphi}\|_{H^s(\Omega)}\,\mathrm{d}t \\ &\le 2dc\tilde{c}\|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^1(Q)}\|\boldsymbol{\varphi}\|_{L^\infty(0,T,H^s(\Omega))}.\end{aligned} \tag{6.3.13}$$

It is obvious that $\|\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\|_{L^1(Q)}$ is bounded. Hence, there exists $C > 0$ such that

$$\sup_{\substack{\boldsymbol{\varphi} \in L^\infty(0,T,H^s(\Omega)) \\ \|\boldsymbol{\varphi}\|_{L^\infty(0,T,H^s(\Omega))} \le 1}} \int_0^T |((\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t, \boldsymbol{\varphi})_{\boldsymbol{D}}|\,\mathrm{d}t \le C(k), \tag{6.3.14}$$

and hence the sequence $\{(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t\}$ is uniformly bounded in $L^1(0, T, (H^s(\Omega, \mathcal{S}^3))')$. □

Since $\{(\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\}$ is uniformly bounded in $L^1(Q)$ the lemmas for temperature remain the same as in models mentioned previously. Estimates in Lemma 6.3.5 depend on k and it forces us to use two level Galerkin approximation.

Lemma 6.3.4. *The sequence $\{\theta_{k,l}\}$ is uniformly bounded in $L^\infty(0, T; L^1(\Omega))$ with respect to k and l.*

Lemma 6.3.5. *There exists a constant C, depending on the domain Ω and the time interval $(0, T)$, such that for every $k \in \mathbb{N}$*

$$\begin{aligned}\sup_{0\le t\le T}\|\theta_{k,l}(t)\|^2_{L^2(\Omega)} &+ \|\theta_{k,l}\|^2_{L^2(0,T,W^{1,2}(\Omega))} + \|(\theta_{k,l})_t\|^2_{L^2(0,T,W^{-1,2}(\Omega))} \\ &\le C\Big(\|\mathcal{T}_k\Big((\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\Big)\|^2_{L^2(0,T,L^2(\Omega))} + \|\mathcal{T}_k(\theta_0)\|^2_{L^2(\Omega)}\Big).\end{aligned} \tag{6.3.15}$$

We observe that the uniform boundedness of solutions (Lemma 6.3.2 and Lemma 6.3.5) implies the global existence of approximate solutions. For each $n = 1, ..., k$ and $m = 1, ..., l$ the solutions $\{\alpha^n_{k,l}(t), \beta^m_{k,l}(t), \gamma^n_{k,l}(t), \delta^m_{k,l}(t)\}$ exist on the whole time interval $[0, T]$.

6.3.3 Limit passage $l \to \infty$ and uniform estimates

Multiplying (2.2.8) by time dependent test functions $\varphi_1(t), \varphi_2(t), \varphi_3(t) \in C^\infty([0, T])$ and $\varphi_4(t) \in C^\infty_c([-\infty, T))$ and then after integration over time interval $(0, T)$, we obtain the following system of equations

$$\begin{aligned}
&\int_0^T\int_\Omega \boldsymbol{T}_{k,l} : \boldsymbol{\varepsilon}(\boldsymbol{w}_n)\varphi_1(t)\,\mathrm{d}x\,\mathrm{d}t = 0\\
&\int_0^T\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t : \boldsymbol{\varepsilon}(\boldsymbol{w}_n)\varphi_2(t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) : \boldsymbol{\varepsilon}(\boldsymbol{w}_m)\varphi_2(t)\,\mathrm{d}x\,\mathrm{d}t\\
&\int_0^T\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t : \boldsymbol{\zeta}^k_m\varphi_3(t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) : \boldsymbol{\zeta}^k_m\varphi_3(t)\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_0^T\int_\Omega \theta_{k,l}v_m\varphi_4'(t)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \theta_{0,k,l}(x)v_m\varphi_4(0)\,\mathrm{d}x + \int_0^T\int_\Omega \nabla\theta_{k,l}\cdot\nabla v_m\varphi_4(t)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad = \int_0^T\int_\Omega \mathcal{T}_k((\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) : \boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}))v_m\varphi_4(t)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}\tag{6.3.16}$$

where the first and the second equation hold for $n = 1, ..., k$ and the third and the fourth hold for $m = 1, ..., l$. Moreover, it holds $\boldsymbol{T}_{k,l} = \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{k,l}) - \boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})$.

Uniform boundedness proved in previous section implies that the following convergences holds

$$\begin{array}{ll}
\boldsymbol{T}_{k,l} \rightharpoonup \boldsymbol{T}_k & \text{weakly in } L^2(Q,\mathcal{S}^3),\\
\boldsymbol{T}^d_{k,l} \rightharpoonup^* \boldsymbol{T}^d_k & \text{weakly* in } L_M(Q,\mathcal{S}^3_d),\\
\boldsymbol{G}(\tilde\theta+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}) \rightharpoonup^* \boldsymbol{\chi}_k & \text{weakly* in } L_{M^*}(Q,\mathcal{S}^3_d),\\
\theta_{k,l} \rightharpoonup \theta_k & \text{weakly in } L^2(0,T,W^{1,2}(\Omega)),\\
\theta_{k,l} \to \theta_k & \text{a.e. in } \Omega\times(0,T),\\
(\boldsymbol{\varepsilon}^{\mathbf{p}}_{k,l})_t \rightharpoonup (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t & \text{weakly in } L^1(0,T,(H^s(\Omega,\mathcal{S}^3))'),
\end{array}\tag{6.3.17}$$

with $l \to \infty$ and with the use of appropriate subsequences if it is necessary. Using these convergences we pass to the limit in (6.3.16) with $l \to \infty$. Then, for $n = 1, ..., k$ and $m \in \mathbb{N}$, it holds

$$\begin{aligned}
&\int_0^T\int_\Omega \boldsymbol{T}_k : \boldsymbol{\varepsilon}(\boldsymbol{w}_n)\varphi_1(t)\,\mathrm{d}x\,\mathrm{d}t = 0,\\
&\int_0^T\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t : \boldsymbol{\varepsilon}(\boldsymbol{w}_m)\varphi_2(t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \boldsymbol{\chi}_k : \boldsymbol{\varepsilon}(\boldsymbol{w}_m)\varphi_2(t)\,\mathrm{d}x\,\mathrm{d}t,\\
&\int_0^T\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t : \boldsymbol{\zeta}^k_m\varphi_3(t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \boldsymbol{\chi}_k : \boldsymbol{\zeta}^k_m\varphi_3(t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{6.3.18}$$

Moreover, $\{\boldsymbol{\varepsilon}(\boldsymbol{w}_n), \boldsymbol{\zeta}_m\}_{n=1,..,k;m=1,..,\infty}$ is a base of whole space $H^s(\Omega,\mathcal{S}^3)$ and abomentioned equations can be replaced by

$$\int_0^T\int_\Omega (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t : \boldsymbol{\zeta}\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \boldsymbol{\chi}_k : \boldsymbol{\zeta}\,\mathrm{d}x\,\mathrm{d}t \tag{6.3.19}$$

for $\boldsymbol{\zeta} \in L^\infty(0,T,H^s(\Omega,\mathcal{S}^3))$. To show that (6.3.19) holds also for all $\boldsymbol{\zeta} \in L_M(Q,\mathcal{S}^3)$ we proceed similarly as in [35, 37, 48].

It still remains to make the limit passage in $(6.3.16)_{(5)}$ and to identify the weak limit $\boldsymbol{\chi}$. As we know the same problems appeared in models considered previously. For this purpose, we repeat three-step method presented in Chapter 5.

All calculations in three-step method for Norton-Hoff-type model and model with growth conditions in generalized Orlicz spaces are significantly different. Only some parts of proof of

Lemma 6.3.6 are the same as those related to Norton-Hoff-type models. We skip these parts and for more details we refer the reader to Chapter 4. To prove the limiting inequality we use lemmas presented in Section 6.1. In the second step we use Minty-Browder trick for Orlicz space, see [84]. And finally, to show the convergence of $\{(\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\}$ we use biting limit and Young measures tools.

Step 1. *Limiting inequality.*

Lemma 6.3.6. *The following inequality holds for the solution of approximate systems.*

$$\limsup_{l\to\infty} \int_0^\tau \int_\Omega \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{T}^d_{k,l} \,\mathrm{d}x\,\mathrm{d}t \le \int_0^\tau \int_\Omega \boldsymbol{\chi}_k : \boldsymbol{T}^d_k \,\mathrm{d}x\,\mathrm{d}t. \tag{6.3.20}$$

for every $\tau \in (0, T)$.

Proof. At the beginning of the proof procedure is the same as in proof of Lemma 4.3.6. We multiply (6.3.10) by function $\psi_{\mu,\tau}(t)$, see (4.3.29), and then all of the computation proceed the same way as previously. The difference appears when we want to make a limit passage with $\epsilon \to 0$ in the following equation

$$\int_{t_1}^{t_2} \int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}^{\mathbf{p}}_k) * \eta_\epsilon : \left((\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}^{\mathbf{p}}_k) * \eta_\epsilon\right)_t \,\mathrm{d}x\,\mathrm{d}t = -\int_{t_1}^{t_2} \int_\Omega \boldsymbol{\chi}_k * \eta_\epsilon : \boldsymbol{T}^d_k * \eta_\epsilon \,\mathrm{d}x\,\mathrm{d}t. \tag{6.3.21}$$

Since $\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}^{\mathbf{p}}_k$ belongs to $L^2(Q, \mathcal{S}^3)$ we may pass to the limit on the left-hand side of equation (6.3.21), but to make a limit passage on right-hand side we should use lemmas presented in Section 6.1.

From Lemma 6.1.7 sequences $\{M(x, \boldsymbol{T}^d_k * \eta_\epsilon)\}$ and $\{M^*(x, \boldsymbol{\chi}_k * \eta_\epsilon)\}$ are uniformly integrable. Moreover, $\{\boldsymbol{T}^d_k * \eta_\epsilon\}_\epsilon$ converges in measure to $\boldsymbol{T}^d_k$ and $\{\boldsymbol{\chi}_k * \eta_\epsilon\}_\epsilon$ converges in measure to $\boldsymbol{\chi}_k$ (by Lemma 6.1.6) as ϵ goes to 0. Uniform integrability of the sequence and convergence in measure of this sequence implies (by Lemma 6.1.3) that

$$\begin{aligned} \boldsymbol{T}^d_k * \eta_\epsilon &\xrightarrow{M} \boldsymbol{T}^d_k \qquad \text{modularly in } L_M(Q), \\ \boldsymbol{\chi}_k * \eta_\epsilon &\xrightarrow{M^*} \boldsymbol{\chi}_k \qquad \text{modularly in } L_{M^*}(Q), \end{aligned} \tag{6.3.22}$$

as $\epsilon \to 0$. Then, using Lemma 6.1.5 we complete the limit passage in (6.3.21) and we obtain

$$\frac{1}{2} \int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}^{\mathbf{p}}_k) : (\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}^{\mathbf{p}}_k) \,\mathrm{d}x \Big|_{t_1}^{t_2} = -\int_{t_1}^{t_2} \int_\Omega \boldsymbol{\chi}_k : \boldsymbol{T}^d_k \,\mathrm{d}x\,\mathrm{d}t. \tag{6.3.23}$$

The rest part of the proof is similar to proof of Lemma 4.3.6, hence we omit this part.

□

Step 2. *Minty-Browder tick.*

Let us take $s \in (0, T]$ and let us define $Q^s = \Omega \times (0, s)$. By monotonicity condition of function $\boldsymbol{G}(\theta, \cdot)$ we obtain

$$\int_{Q^s} \Big(\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) - \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d)\Big) : (\boldsymbol{T}^d_{k,l} - \boldsymbol{W}^d) \,\mathrm{d}x\,\mathrm{d}t \ge 0 \qquad \forall\ \boldsymbol{W}^d \in L^\infty(Q, \mathcal{S}^3_d). \tag{6.3.24}$$

Lemma 6.3.6 yields

$$\limsup_{l\to\infty} \int_{Q^s} \boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{T}^d_{k,l} \,\mathrm{d}x\,\mathrm{d}t \le \int_{Q^s} \boldsymbol{\chi}_k : \boldsymbol{T}^d_k \,\mathrm{d}x\,\mathrm{d}t. \tag{6.3.25}$$

Since $\boldsymbol{W}^d$ belongs to $L^\infty(Q)$, then it belongs also to $E_M(Q)$. By weak* convergence of $\{\boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l})\}$ in $L_{M^*}(Q)$ we get

$$\lim_{l\to\infty}\int_{Q^s} \boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_{k,l}) : \boldsymbol{W}^d \,\mathrm{d}x\,\mathrm{d}t = \int_{Q^s} \boldsymbol{\chi}_k : \boldsymbol{W}^d \,\mathrm{d}x\,\mathrm{d}t. \tag{6.3.26}$$

Now, we focus on the convergence of the sequence $\{\boldsymbol{G}(\tilde\theta+\theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d)\}$. By pointwise convergence of $\{\theta_{k,l}\}$ we get pointwise convergence of $\{\boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d)\}$. Furthermore, from the Assumption 6.0.1 and non-negativity of N-functions we get

$$|\tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d| \geq c\frac{M^*(x, \boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d))}{|\boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d)|}. \tag{6.3.27}$$

$\tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d$ belongs to $L^\infty(Q, \mathcal{S}^3_d)$ and M^* is an N-function. This implies that the sequence $\{\boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d)\}$ belongs to $L^\infty(Q, \mathcal{S}^3_d)$ and by Lemma 6.1.3 we obtain

$$\boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d) \xrightarrow{M^*} \boldsymbol{G}(\tilde\theta + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d), \tag{6.3.28}$$

modularly in $L_{M^*}(Q)$. Then

$$\begin{aligned}
&\int_{Q^s} |\boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d) : (\boldsymbol{T}^d_{k,l} - \boldsymbol{W}) - \boldsymbol{G}(\tilde\theta + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d) : (\boldsymbol{T}^d_k - \boldsymbol{W}^d)| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leq \int_{Q^s} \left|\Big(\boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d) - \boldsymbol{G}(\tilde\theta + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d)\Big) : (\boldsymbol{T}^d_{k,l} - \boldsymbol{W}^d)\right| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad\quad+ \int_{Q^s} \left|\boldsymbol{G}(\tilde\theta + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d) : (\boldsymbol{T}^d_{k,l} - \boldsymbol{T}^d_k)\right| \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{6.3.29}$$

Finally, using Hölder inequality (Lemma 6.1.2) we get

$$\begin{aligned}
&\int_{Q^s} |\boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d) : (\boldsymbol{T}^d_{k,l} - \boldsymbol{W}) - \boldsymbol{G}(\tilde\theta + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d) : (\boldsymbol{T}^d_k - \boldsymbol{W}^d)| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leq 2\|\boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d) - \boldsymbol{G}(\tilde\theta + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d)\|_{L,M^*} \|\boldsymbol{T}^d_{k,l} - \boldsymbol{W}^d\|_{L,M} \\
&\quad\quad+ \int_{Q^s} \left|\boldsymbol{G}(\tilde\theta + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d) : (\boldsymbol{T}^d_{k,l} - \boldsymbol{T}^d_k)\right| \,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{6.3.30}$$

Since $\|\boldsymbol{T}^d_{k,l} - \boldsymbol{W}^d\|_{L,M}$ is uniformly bounded, $\|\boldsymbol{G}(\tilde\theta+\theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d) - \boldsymbol{G}(\tilde\theta+\theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d)\|_{L,M^*} \to 0$ (M^* satisfies Δ_2-condition and the sequence $\{\boldsymbol{G}(\tilde\theta+\theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d)\}$ convergence to $\boldsymbol{G}(\tilde\theta+\theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d)$ in modular) and $\boldsymbol{T}^d_{k,l} - \boldsymbol{T}^d_k \rightharpoonup 0$ in $L_M(Q, \mathcal{S}^3_d)$ as l goes to ∞, the right-hand side of (6.3.30) goes to 0 as l goes to ∞ and we obtain

$$\lim_{l\to\infty}\int_{Q^s} \boldsymbol{G}(\tilde\theta + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d) : (\boldsymbol{T}^d_{k,l} - \boldsymbol{W}^d) \,\mathrm{d}x\,\mathrm{d}t = \int_{Q^s} \boldsymbol{G}(\tilde\theta + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d) : (\boldsymbol{T}^d_k - \boldsymbol{W}^d) \,\mathrm{d}x\,\mathrm{d}t \tag{6.3.31}$$

Therefore, passing to the limit with $l \to \infty$ in (6.3.24), we get

$$\int_{Q^s} \Big(\boldsymbol{\chi}_k - \boldsymbol{G}(\tilde\theta + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{W}^d)\Big) : (\boldsymbol{T}^d_k - \boldsymbol{W}^d) \,\mathrm{d}x\,\mathrm{d}t \geq 0 \qquad \forall \boldsymbol{W}^d \in L^\infty(Q^s, \mathcal{S}^3). \tag{6.3.32}$$

For $j>0$ let us define the set

$$Q_j = \{(t,x)\in Q^s:\ |\boldsymbol{T}_k^d|\le j \text{ a.e. in } Q^s\}. \tag{6.3.33}$$

Let us use the notation $\mathbf{1}_H$ for characteristic function of set H. Then, for arbitrary $0<j<i$ and $h>0$ we define function

$$\boldsymbol{W}^d = -\tilde{\boldsymbol{T}}^d\mathbf{1}_{Q^s\setminus Q_i} + \boldsymbol{T}_k^d\mathbf{1}_{Q_i} + h\boldsymbol{U}^d\mathbf{1}_{Q_j} \tag{6.3.34}$$

where $\boldsymbol{U}^d\in L^\infty(Q,\mathcal{S}^3)$. We use this function as a test function in (6.3.32) and get

$$\begin{aligned}\int_{Q^s}\Big(\boldsymbol{\chi}_k - \boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d - \tilde{\boldsymbol{T}}^d\mathbf{1}_{Q^s\setminus Q_i} + \boldsymbol{T}_k^d\mathbf{1}_{Q_i} + h\boldsymbol{U}^d\mathbf{1}_{Q_j})\Big):&\\ \Big(\boldsymbol{T}_k^d + \tilde{\boldsymbol{T}}^d\mathbf{1}_{Q^s\setminus Q_i} - \boldsymbol{T}_k^d\mathbf{1}_{Q_i} - h\boldsymbol{U}^d\mathbf{1}_{Q_j}\Big)\,\mathrm{d}x\,\mathrm{d}t &\ge 0\end{aligned} \tag{6.3.35}$$

Since $Q_j\subset Q_i\subset Q^s$ we get

$$\begin{aligned}&-h\int_{Q_j}\Big(\boldsymbol{\chi}_k - \boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d+h\boldsymbol{U}^d)\Big):\boldsymbol{U}^d\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_{Q_i\setminus Q_j}\Big(\boldsymbol{\chi}_k - \boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d)\Big):(\boldsymbol{T}_k^d-\boldsymbol{T}_k^d)\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad+\int_{Q^s\setminus Q_i}\Big(\boldsymbol{\chi}_k - \boldsymbol{G}(\tilde\theta+\theta_k,\mathbf{0})\Big):(\boldsymbol{T}_k^d+\tilde{\boldsymbol{T}}^d)\,\mathrm{d}x\,\mathrm{d}t\ge 0.\end{aligned} \tag{6.3.36}$$

By Assumptions 6.0.1 we know that $\boldsymbol{G}(\tilde\theta+\theta_k,\mathbf{0})=\mathbf{0}$ a.e. in Ω. Hence

$$-h\int_{Q_j}\Big(\boldsymbol{\chi}_k - \boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d+h\boldsymbol{U}^d)\Big):\boldsymbol{U}^d\,\mathrm{d}x\,\mathrm{d}t + \int_{Q^s\setminus Q_i}\boldsymbol{\chi}_k:(\boldsymbol{T}_k^d+\tilde{\boldsymbol{T}}^d)\,\mathrm{d}x\,\mathrm{d}t\ge 0. \tag{6.3.37}$$

Moreover, from the definition of characteristic function

$$\int_{Q^s\setminus Q_i}\boldsymbol{\chi}_k:(\boldsymbol{T}_k^d+\tilde{\boldsymbol{T}}^d)\,\mathrm{d}x\,\mathrm{d}t = \int_Q\Big(\boldsymbol{\chi}_k:(\boldsymbol{T}_k^d+\tilde{\boldsymbol{T}}^d)\Big)\mathbf{1}_{Q^s\setminus Q_i}\,\mathrm{d}x\,\mathrm{d}t. \tag{6.3.38}$$

Since $\int_Q\boldsymbol{\chi}_k:(\boldsymbol{T}_k^d+\tilde{\boldsymbol{T}}^d)<\infty$ and $\Big(\boldsymbol{\chi}_k:(\boldsymbol{T}_k^d+\tilde{\boldsymbol{T}}^d)\Big)\mathbf{1}_{Q^s\setminus Q_i}\to 0$ a.e. in Q as i goes to ∞, Lebesgue's dominated convergence theorem implies that

$$\lim_{i\to\infty}\int_{Q^s\setminus Q_i}\boldsymbol{\chi}_k:(\boldsymbol{T}_k^d+\tilde{\boldsymbol{T}}^d)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{6.3.39}$$

Passing to the limit with i going to ∞ in (6.3.37) and by dividing by h we obtain

$$\int_{Q_j}\Big(\boldsymbol{\chi}_k - \boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d+h\boldsymbol{U}^d)\Big):\boldsymbol{U}^d\,\mathrm{d}x\,\mathrm{d}t\le 0. \tag{6.3.40}$$

Since $\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d+h\boldsymbol{U}^d$ goes to $\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d$ a.e. in Q when $h\to 0^+$, $\{\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d+h\boldsymbol{U}^d)\}_{h>0}$ is uniformly bounded in $L_{M^*}(Q_j,\mathcal{S}^3)$, we conclude that

$$\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d+h\boldsymbol{U}^d)\rightharpoonup^*\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d) \tag{6.3.41}$$

in $L_{M^*}(Q_j, \mathcal{S}^3)$ as h goes to 0^+. Consequently, passing to the limit with h going to 0^+ in (6.3.40) we obtain

$$\int_{Q_j} \left(\boldsymbol{\chi}_k - \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)\right) : \boldsymbol{U}^d \, \mathrm{d}x \, \mathrm{d}t \leq 0, \tag{6.3.42}$$

for all $\boldsymbol{U}^d \in L^\infty(Q, \mathcal{S}^3_d)$, so taking

$$\boldsymbol{U}^d = \begin{cases} \frac{\boldsymbol{\chi}_k - \boldsymbol{G}(\tilde{\theta}+\theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)}{|\boldsymbol{\chi}_k - \boldsymbol{G}(\tilde{\theta}+\theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)|} & \text{when } \boldsymbol{\chi}_k \neq \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d), \\ 0 & \text{when } \boldsymbol{\chi}_k = \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d), \end{cases} \tag{6.3.43}$$

we obtain

$$\int_{Q_j} |\boldsymbol{\chi}_k - \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)| \, \mathrm{d}x \, \mathrm{d}t \leq 0, \tag{6.3.44}$$

i.e. $\boldsymbol{\chi}_k = \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)$ a.e. in Q_j. Arbitrary choice of $j > 0$ and of $0 \leq s \leq T$ implies that $\boldsymbol{\chi}_k = \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)$ a.e. in Q.

Step 3. *Limit of right-hand side of heat equations.*

The idea how to prove the convergence of right-hand side of heat equation came from paper of Gwiazda et al. [40]. Let us denote by $\stackrel{b}{\to}$ the biting limit used, cf. [8].

Definition 6.3.1 (Biting limit)**.** *Let $\{f^\nu\}$ be a bounded sequence in $L^1(Q)$. We say that $f \in L^1(Q)$ is a biting limit of subsequence $\{f^\nu\}$, we denote $f^\nu \stackrel{b}{\to} f$, if there exists nonincreasing sequence $\{E_k\}$ with $E_k \subset Q$ and $\lim_{k\to\infty} |E_k| = 0$, such that f^ν convergence weakly to f in $L^1(Q \setminus E_k)$ for every fixed k.*

The following Lemma and its proof came from [40].

Lemma 6.3.7 (Lemma 4.6 from [40])**.** *Let $a_n \in L^1(Q)$ and let $0 \leq a_0 \in L^1(Q)$ and*

$$a_n \geq -a_0, \quad a_n \stackrel{b}{\to} a \quad \textit{and} \quad \limsup_{n\to\infty} \int_Q a_n \, \mathrm{d}x \, \mathrm{d}t \leq \int_Q a \, \mathrm{d}x \, \mathrm{d}t \tag{6.3.45}$$

then

$$a_n \rightharpoonup a \quad \textit{weakly in } L^1(Q). \tag{6.3.46}$$

Proof. By [4, Theorem 2.5] there exists subsequence $\{a_{n_{sup}}\}$ which converges weakly* in $\mathcal{M}(Q)$ to $\limsup_{n\to\infty} \int_Q a_n \, \mathrm{d}x \, \mathrm{d}t$, and also there exists a nonegative measure $\overline{a}$ such that

$$a_{n_{sup}} \rightharpoonup * \; a + \overline{a} \qquad \text{in } \mathcal{M}(Q). \tag{6.3.47}$$

Then

$$\int_Q a_{n_{sup}} \, \mathrm{d}x \, \mathrm{d}t \to \int_Q a \, \mathrm{d}x \, \mathrm{d}t + \int_Q \overline{a} \, \mathrm{d}x \, \mathrm{d}t, \tag{6.3.48}$$

as $n_{sup} \to \infty$. Since $\limsup_{n\to\infty} \int_Q a_n \, \mathrm{d}x \, \mathrm{d}t \leq \int_Q a \, \mathrm{d}x \, \mathrm{d}t$, we obtain that $\int_Q \overline{a} dx dt = 0$. Therefore, $\overline{a} = 0$ as a measure. Thus, by [4, Theorem 2.9 (ii)] the sequence a_n converges weakly to a in $L^1(Q)$ with $n \to \infty$. □

Lemma 6.3.8. *For each $k \in \mathbb{N}$ sequence $\{\boldsymbol{G}(\tilde{\theta} + \theta_{k,l}, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_{k,l}^d) : (\tilde{\boldsymbol{T}}^d + \boldsymbol{T}_{k,l}^d)\}_{l=1}^\infty$ converges weakly to $\boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d) : (\tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d)$ in $L^1(Q)$.*

Proof. Using the Assumption 6.0.1, Frechet-Young inequality and convexity of N-functions, we get

$$\begin{aligned} c\Big(M(x,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d) &+ M^*(x,\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d))\Big)\\ &\le \boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d)):(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d)\\ &\le M(x,\frac{2}{d}(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d))+M^*(x,\frac{d}{2}\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d))\\ &\le M(x,\frac{2}{d}(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d))+\frac{d}{2}M^*(x,\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d)), \end{aligned} \tag{6.3.49}$$

where $d=\min(c,1)$. And finally

$$cM(x,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d)+\frac{2c-d}{2}M^*(x,\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d))\le M(x,\frac{2}{d}(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d)). \tag{6.3.50}$$

Hence the sequence $\{\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d)\}$ is uniformly bounded in $L_{M^*}(Q)$.Using monotonicity of function $\boldsymbol{G}(\cdot,\cdot)$ with respect to the second variable, we get

$$0\le\Big(\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d)-\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d)\Big):(\boldsymbol{T}_{k,l}^d-\boldsymbol{T}_k^d). \tag{6.3.51}$$

Right-hand side of abovmentioned inequality is uniformly bounded in $L^1(Q)$. Thus, there exists the Young measure denoted by $\mu_{x,t}(\cdot,\cdot)$, see Theorem 4.1.1, such that the following converges hold

$$\begin{aligned} (\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d)&-\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d)):(\boldsymbol{T}_{k,l}^d-\boldsymbol{T}_k^d)\\ &\xrightarrow{b}\int_{\mathbb{R}\times\mathbb{R}^{3\times3}}\Big(\boldsymbol{G}(s,\boldsymbol{\lambda})-\boldsymbol{G}(s,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d)\Big):(\boldsymbol{\lambda}-(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d))d\mu_{x,t}(s,\boldsymbol{\lambda}). \end{aligned} \tag{6.3.52}$$

Using Lemma 4.1.3 we obtain that the measure $\mu_{x,t}(s,\boldsymbol{\lambda})$ can be presented in the form $\delta_{\tilde{\theta}+\theta_k}\otimes\nu_{x,t}(\boldsymbol{\lambda})$. Then

$$\begin{aligned} \int_{\mathbb{R}\times\mathbb{R}^{3\times3}}&\Big(\boldsymbol{G}(s,\boldsymbol{\lambda})-\boldsymbol{G}(s,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d)\Big):(\boldsymbol{\lambda}-(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d))d\mu_{x,t}(s,\boldsymbol{\lambda})\\ &=\int_{\mathbb{R}^{3\times3}}\Big(\boldsymbol{G}(\tilde{\theta}+\theta_k,\boldsymbol{\lambda})-\boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d)\Big):(\boldsymbol{\lambda}-(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d))d\nu_{x,t}(\boldsymbol{\lambda})\\ &=\int_{\mathbb{R}^{3\times3}}\boldsymbol{G}(\tilde{\theta}+\theta_k,\boldsymbol{\lambda}):(\boldsymbol{\lambda}-(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d))d\nu_{x,t}(\boldsymbol{\lambda})\\ &\quad-\int_{\mathbb{R}^{3\times3}}\boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d):(\boldsymbol{\lambda}-(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d))d\nu_{x,t}(\boldsymbol{\lambda}). \end{aligned} \tag{6.3.53}$$

Since sequence $\{\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d\}$ generate the measure $d\nu_{x,t}(\cdot)$ then $\int_{\mathbb{R}^{3\times3}}\boldsymbol{\lambda}d\nu_{x,t}(\boldsymbol{\lambda})=\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d$. The second term in abovementioned equation disappears. Indeed,

$$\begin{aligned} -\int_{\mathbb{R}^{3\times3}}&\boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d):(\boldsymbol{\lambda}-(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d))d\nu_{x,t}(\boldsymbol{\lambda})\\ &=-\boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d):\Big(\int_{\mathbb{R}^{3\times3}}\boldsymbol{\lambda}d\nu_{x,t}(\boldsymbol{\lambda})-(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_k^d)\Big). \end{aligned} \tag{6.3.54}$$

Moreover, uniform boundedness of the sequence $\{\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d):(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d)\}$ in $L^1(Q)$ implies that

$$\begin{aligned} \boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d):(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}_{k,l}^d)&\xrightarrow{b}\int_{\mathbb{R}\times\mathbb{R}^{3\times3}}\boldsymbol{G}(s,\boldsymbol{\lambda}):\boldsymbol{\lambda}d\mu_{x,t}(s,\boldsymbol{\lambda})\\ &=\int_{\mathbb{R}^{3\times3}}\boldsymbol{G}(\tilde{\theta}+\theta_k,\boldsymbol{\lambda}):\boldsymbol{\lambda}d\nu_{x,t}(\boldsymbol{\lambda}). \end{aligned} \tag{6.3.55}$$

Hence, by positivity of $\boldsymbol{G}(\tilde{\theta}+\cdot,\tilde{\boldsymbol{T}}^d+\cdot):(\tilde{\boldsymbol{T}}^d+\cdot)$ and using Lemma 4.1.2 we get

$$\liminf_{l\to\infty}\int_Q \boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})\,\mathrm{d}x\,\mathrm{d}t \ge \int_Q\int_{\mathbb{R}^3}\boldsymbol{G}(\tilde{\theta}+\theta_k,\boldsymbol{\lambda}):\boldsymbol{\lambda}d\nu_{x,t}(\boldsymbol{\lambda})\,\mathrm{d}x\,\mathrm{d}t. \tag{6.3.56}$$

Lemma 6.3.6 and knowledge that $\boldsymbol{\chi}_k=\boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)$ a.e. in Q imply that

$$\int_Q \boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k):(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\,\mathrm{d}x\,\mathrm{d}t \ge \int_Q\int_{\mathbb{R}^3}\boldsymbol{G}(\tilde{\theta}+\theta_k,\boldsymbol{\lambda}):\boldsymbol{\lambda}d\nu_{x,t}(\boldsymbol{\lambda})\,\mathrm{d}x\,\mathrm{d}t. \tag{6.3.57}$$

Since $\boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)=\int_{\mathbb{R}^3}\boldsymbol{G}(\tilde{\theta}+\theta_k,\boldsymbol{\lambda})d\nu_{x,t}(\boldsymbol{\lambda})$ and (6.3.51) holds, we obtain

$$\Big(\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l})-\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k})\Big):(\boldsymbol{T}^d_{k,l}-\boldsymbol{T}^d_k)\xrightarrow{b}0. \tag{6.3.58}$$

Using biting limit once more we get

$$\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k})):(\boldsymbol{T}^d_{k,l}-\boldsymbol{T}^d_k)\xrightarrow{b}0, \tag{6.3.59}$$

with $l\to\infty$. Hence

$$\boldsymbol{G}(\tilde{\theta}+\theta_{k,l},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k,l}):(\boldsymbol{T}^d_{k}+\tilde{\boldsymbol{T}}^d)\xrightarrow{b}\boldsymbol{G}(\tilde{\theta}+\theta_{k},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_{k}):(\boldsymbol{T}^d_{k}+\tilde{\boldsymbol{T}}^d). \tag{6.3.60}$$

with $l\to\infty$. Using Lemma 6.3.7 we complete the proof. □

Thus, we pass to the limit with $l\to\infty$ in $(6.3.16)_{(5)}$

$$\begin{aligned}-\int_0^T\int_\Omega\theta_k v_m(\varphi_4(t))_t\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega\theta_k(x,0)\varphi_4(x,0)\,\mathrm{d}x+\int_0^T\int_\Omega\nabla\theta_k\cdot\nabla v_m\varphi_4(t)\,\mathrm{d}x\,\mathrm{d}t\\ =\int_0^T\int_\Omega\mathcal{T}_k((\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k):\boldsymbol{G}(\theta_k+\tilde{\theta},\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k))v_m\varphi_4(t)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{6.3.61}$$

We finish this section with two lemmas. We prove the uniform boundedness of the sequences $\{\boldsymbol{\varepsilon}^{\mathbf{p}}_k\}$ and $\{\boldsymbol{u}_k\}$ in proper spaces. This allows us to make the limit passage with second parameter in the next section.

Lemma 6.3.9. *The sequence $\{\boldsymbol{\varepsilon}^{\mathbf{p}}_k\}$ is uniformly bounded in $L_{M^*}(Q,\mathcal{S}^3_d)$. Moreover, sequence $\{(\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t\}$ is also uniformly bounded in $L_{M^*}(Q,\mathcal{S}^3_d)$.*

Proof. Let us consider the equation for the evolution of the visco-elastic strain tensor

$$(\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t=\boldsymbol{G}(\tilde{\theta}+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k).$$

Hence

$$\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,t)=\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,0)+\int_0^t(\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,s))_s\,\mathrm{d}s.$$

Integrating the value of $M^*(x,\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,t))$ over cylinder Q and using Δ_2-condition of N-function M^* (6.1.5) we get

$$\begin{aligned}\int_Q M^*(x,\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,t))\,\mathrm{d}x\,\mathrm{d}t&\le c\int_Q M^*\big(x,\tfrac{1}{2}\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,t)\big)\,\mathrm{d}x\,\mathrm{d}t+T\int_\Omega h(x)\,\mathrm{d}x\\ &=c\int_Q M^*\big(x,\tfrac{1}{2}\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,0)+\tfrac{1}{2}\int_0^t(\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,s))_s\,\mathrm{d}s\big)\,\mathrm{d}x\,\mathrm{d}t+T\int_\Omega h(x)\,\mathrm{d}x.\end{aligned}$$

Using the convexity of M^* we obtain

$$\int_Q M^*(x,\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,t))\,\mathrm{d}x\,\mathrm{d}t \le \frac{c}{2}\int_Q M^*(x,\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,0))\,\mathrm{d}x\,\mathrm{d}t \\ + \frac{c}{2}\int_Q M^*\left(x,\int_0^t \boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)(x,s)\,\mathrm{d}s\right)\mathrm{d}x\,\mathrm{d}t + T\int_\Omega h(x)\,\mathrm{d}x. \tag{6.3.62}$$

Let us focus on the middle term on the right-hand side in abovementioned equation. Changing variable $\tau=\frac{t}{T}$ we obtain

$$\int_0^T\int_\Omega M^*\big(x,\int_0^t \boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)(x,s)\,\mathrm{d}s\big)\,\mathrm{d}x\,\mathrm{d}t \\ = T\int_0^1\int_\Omega M^*\left(x,\int_0^{\tau T} \boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)(x,s)\,\mathrm{d}s\right)\mathrm{d}x\,\mathrm{d}\tau.$$

By Jensen inequality we get

$$T\int_0^1\int_\Omega M^*\left(x,\int_0^t \boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)(x,s)\,\mathrm{d}s\right)\mathrm{d}x\,\mathrm{d}t \\ \le T\int_0^1\int_\Omega \frac{1}{\tau T}\int_0^{\tau T} M^*\big(x,\tau T\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\big)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}\tau \\ \le T\int_0^1\int_\Omega \frac{1}{\tau T}\int_0^{\tau T} \tau M^*\big(x,T\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\big)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}\tau \\ = \int_0^1\int_\Omega\int_0^{\tau T} M^*\big(x,T\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\big)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}\tau.$$

There exists $d\in\mathbb{R}$ such that $2^d\ge T$. Then, using the Δ_2-condition, coming back to original variable and using the Fubini theorem we get

$$\int_0^1\int_\Omega\int_0^{\tau T} M^*\big(x,T\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\big)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}\tau \\ \le \int_0^1\int_\Omega\int_0^{\tau T} M^*\big(x,2^d\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\big)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}\tau \\ \le c^d\int_0^1\int_\Omega\int_0^{\tau T} M^*\big(x,\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\big)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}\tau + C(d)\int_\Omega h(x)\,\mathrm{d}x \tag{6.3.63} \\ = \frac{c^d}{T}\int_0^T\int_\Omega\int_0^t M^*\big(x,\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\big)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t + C(d)\int_\Omega h(x)\,\mathrm{d}x \\ \le c^d\int_0^T\int_\Omega M^*\big(x,\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\big)\,\mathrm{d}x\,\mathrm{d}t + C(d)\int_\Omega h(x)\,\mathrm{d}x.$$

Coming back to (6.3.62) we get

$$\int_Q M^*(x,\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,t))\,\mathrm{d}x\,\mathrm{d}t \le \frac{cT}{2}\int_\Omega M^*(x,\boldsymbol{\varepsilon}^{\mathbf{p}}_k(x,0))\,\mathrm{d}x \\ + c^d\int_0^T\int_\Omega M^*\big(x,\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\big)\,\mathrm{d}x\,\mathrm{d}t + C(d)\int_\Omega h(x)\,\mathrm{d}x.$$

Lemma 6.3.2 and initial condition in $L_{M^*}(\Omega,\mathcal{S}^3_d)$ complete the proof. □

Lemma 6.3.10. *The sequence $\{\boldsymbol{u}_k\}$ is uniformly bounded in $BD_{M^*}(\Omega,\mathbb{R}^3)$.*

Proof. Let us start from showing the uniform boundedness of the sequence $\{\boldsymbol{\varepsilon}(\boldsymbol{u}_k)\}$ in the space $L_{M^*}(Q)$. Using Δ_2-condition, convexity of N-function and Assumption 6.0.1 we obtain

$$\begin{aligned}\int_Q M^*(x,\boldsymbol{\varepsilon}(\boldsymbol{u}_k))\,\mathrm{d}x\,\mathrm{d}t &\le c\int_Q M^*\big(x,\frac{1}{2}\boldsymbol{\varepsilon}(\boldsymbol{u}_k)\big)\,\mathrm{d}x\,\mathrm{d}t+\int_Q h(x)\,\mathrm{d}x\,\mathrm{d}t\\ &= c\int_Q M^*\big(x,\frac{1}{2}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k)-\boldsymbol{\varepsilon}^{\mathbf{p}}_k)+\frac{1}{2}\boldsymbol{\varepsilon}^{\mathbf{p}}_k\big)\,\mathrm{d}x\,\mathrm{d}t+T\int_\Omega h(x)\,\mathrm{d}x\\ &\le \frac{c}{2}\int_Q M^*(x,\boldsymbol{\varepsilon}(\boldsymbol{u}_k)-\boldsymbol{\varepsilon}^{\mathbf{p}}_k)\,\mathrm{d}x\,\mathrm{d}t+\frac{c}{2}\int_Q M^*(x,\boldsymbol{\varepsilon}^{\mathbf{p}}_k)\,\mathrm{d}x\,\mathrm{d}t+T\int_\Omega h(x)\,\mathrm{d}x\\ &\le \frac{c}{2}\int_Q |\boldsymbol{\varepsilon}(\boldsymbol{u}_k)-\boldsymbol{\varepsilon}^{\mathbf{p}}_k|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{c}{2}\int_Q M^*(x,\boldsymbol{\varepsilon}^{\mathbf{p}}_k)\,\mathrm{d}x\,\mathrm{d}t+T\int_\Omega h(x)\,\mathrm{d}x\\ &\le \frac{c}{2}\int_Q |\boldsymbol{T}_k|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{c}{2}\int_Q M^*(x,\boldsymbol{\varepsilon}^{\mathbf{p}}_k)\,\mathrm{d}x\,\mathrm{d}t+T\int_\Omega h(x)\,\mathrm{d}x.\end{aligned}\tag{6.3.64}$$

Following Anzellotti and Giaquinta [7, Preposition 1.2 a)] we get the inequality

$$\|\boldsymbol{u}_k\|_{L^1(Q)}\le C\|\boldsymbol{\varepsilon}(\boldsymbol{u}_k)\|_{L^1(Q)},$$

where C is a constant depending on Ω. Finally we get the estimates

$$\|\boldsymbol{u}_k\|_{L^1(Q)}\le C_{Q,M}\int_Q M^*(x,\boldsymbol{\varepsilon}(\boldsymbol{u}_k))\,\mathrm{d}x\,\mathrm{d}t,$$

where constant $C_{Q,M}$ depends on N-function M and space-times cylinder Q. This completes the proof.

□

6.3.4 Limit passage $k\to\infty$

We start the second limit passage with short discussion about solution to heat equation. Since $\{\boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k):(\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\}$ is uniformly bounded in $L^1(Q)$, then there exists a measurable function θ, see Lemma 3.2.1, such that $\theta_k\to\theta$ a.e. in Q.

Furthermore, uniform boundedness of approximate solutions sequences obtained in previous sections imply the following convergences, passing to the subsequence if it is necessary,

$$\begin{array}{ll}\boldsymbol{u}_k\to\boldsymbol{u} & \text{weakly in } L^1(Q,\mathbb{R}^3),\\ \boldsymbol{\varepsilon}(\boldsymbol{u}_k)\rightharpoonup^*\boldsymbol{\varepsilon}(\boldsymbol{u}) & \text{weakly* in } L_{M^*}(Q,\mathbb{R}^3),\\ \boldsymbol{T}_k\rightharpoonup\boldsymbol{T} & \text{weakly in } L^2(Q,\mathcal{S}^3),\\ \boldsymbol{T}^d_k\rightharpoonup^*\boldsymbol{T}^d & \text{weakly* in } L_M(Q,\mathcal{S}^3_d),\\ \boldsymbol{G}(\tilde\theta+\theta_k,\tilde{\boldsymbol{T}}^d+\boldsymbol{T}^d_k)\rightharpoonup^*\boldsymbol{\chi} & \text{weakly* in } L_{M^*}(Q,\mathcal{S}^3_d),\\ (\boldsymbol{\varepsilon}^{\mathbf{p}}_k)_t\rightharpoonup^*(\boldsymbol{\varepsilon}^{\mathbf{p}})_t & \text{weakly* in } L_{M^*}(Q,\mathcal{S}^3_d).\end{array}\tag{6.3.65}$$

Using these convergences in $(6.3.18)_{(1)}$ and (6.3.61), we get

$$\begin{aligned}&\int_Q \boldsymbol{T}:\nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t=0\\ &\int_Q(\boldsymbol{\varepsilon}^{\mathbf{p}})_t:\boldsymbol{\psi}\,\mathrm{d}x\,\mathrm{d}t=\int_Q\boldsymbol{\chi}:\boldsymbol{\psi}\,\mathrm{d}x\,\mathrm{d}t\end{aligned}\tag{6.3.66}$$

for $\boldsymbol{\varphi} \in C^\infty([0,T], L^2(\Omega,\mathbb{R}^3))$ and $\boldsymbol{\psi} \in L_M(Q,\mathcal{S}^3)$. To complete the limit passage in heat equation and to deal with nonlinearities limits we should repeat the three-step method.

Lemma 6.3.11. *The following inequality holds for the solution of approximate systems.*

$$\limsup_{k\to\infty} \int_0^{t_2}\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d) : \boldsymbol{T}_k^d \,\mathrm{d}x\,\mathrm{d}t \le \int_0^{t_2}\int_\Omega \boldsymbol{\chi} : \boldsymbol{T}^d \,\mathrm{d}x\,\mathrm{d}t. \tag{6.3.67}$$

Proof. This proof is similar to proof of Lemma 5.2.9. Using the lower semicontinuity in $L^2(Q)$ we get

$$\begin{aligned}\liminf_{k\to\infty} \int_0^T & \frac{d}{dt}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}_k^{\mathbf{p}})\,\psi_{\mu,\tau}\,\mathrm{d}t \\ &= \liminf_{k\to\infty}\frac{1}{\mu}\int_\tau^{\tau+\mu} \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}_k^{\mathbf{p}})(t)\,\mathrm{d}t - \lim_{k\to\infty}\mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}_k^{\mathbf{p}})(0) \\ &\ge \frac{1}{\mu}\int_\tau^{\tau+\mu} \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}_k), \boldsymbol{\varepsilon}_k^{\mathbf{p}})(t)\,\mathrm{d}t - \mathcal{E}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}^{\mathbf{p}})(0).\end{aligned} \tag{6.3.68}$$

We use $\boldsymbol{\varphi}_1 = ((\boldsymbol{\varepsilon}(\boldsymbol{u}_k) * \eta_\epsilon)_t \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon$ as a test function in (6.3.66), where η_ϵ is a standard mollifier with respect to time, then

$$\int_{t_1}^{t_2}\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) * \eta_\epsilon : (\boldsymbol{\varepsilon}(\boldsymbol{u}_k) * \eta_\epsilon)_t \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{6.3.69}$$

Moreover, we use $\boldsymbol{\psi} = (\boldsymbol{T}^d * \eta_\epsilon \mathbf{1}_{(t_1,t_2)}) * \eta_\epsilon$ as a test function in (6.3.19). Then

$$\int_{t_1}^{t_2}\int_\Omega (\boldsymbol{\varepsilon}_k^{\mathbf{p}} * \eta_\epsilon)_t : \boldsymbol{T} * \eta_\epsilon \,\mathrm{d}x\,\mathrm{d}t = \int_{t_1}^{t_2}\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d) * \eta_\epsilon : \boldsymbol{T} * \eta_\epsilon \,\mathrm{d}x\,\mathrm{d}t. \tag{6.3.70}$$

Products in (6.3.70) are well defined. Subtracting these two equations we get

$$\int_{t_1}^{t_2}\int_\Omega \boldsymbol{T} * \eta_\epsilon : (\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}_k^{\mathbf{p}})_t * \eta_\epsilon \,\mathrm{d}x\,\mathrm{d}t = -\int_{t_1}^{t_2}\int_\Omega \boldsymbol{G}(\tilde\theta+\theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}_k^d) * \eta_\epsilon : \boldsymbol{T}^d * \eta_\epsilon \,\mathrm{d}x\,\mathrm{d}t. \tag{6.3.71}$$

For every $\epsilon > 0$ the sequence $\{(\boldsymbol{\varepsilon}(\boldsymbol{u}_k) - \boldsymbol{\varepsilon}_k^{\mathbf{p}})_t * \eta_\epsilon\}$ belongs to $L^2(Q,\mathcal{S}^3)$ and is uniformly bounded in $L^2(Q,\mathcal{S}^3)$ with respect to k, hence we pass to the limit with $k\to\infty$ and we obtain

$$\int_{t_1}^{t_2}\int_\Omega \boldsymbol{T} * \eta_\epsilon : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}})_t * \eta_\epsilon \,\mathrm{d}x\,\mathrm{d}t = -\int_{t_1}^{t_2}\int_\Omega \boldsymbol{\chi} * \eta_\epsilon : \boldsymbol{T}^d * \eta_\epsilon \,\mathrm{d}x\,\mathrm{d}t.$$

Using the properties of convolution we get

$$\int_\Omega \boldsymbol{T} * \eta_\epsilon : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) * \eta_\epsilon \,\mathrm{d}x\Big|_{t_1}^{t_2} = -\int_{t_1}^{t_2}\int_\Omega \boldsymbol{\chi} * \eta_\epsilon : \boldsymbol{T}^d * \eta_\epsilon \,\mathrm{d}x\,\mathrm{d}t.$$

In the same way as in the previous section we pass to the limit with $\epsilon\to 0$ and then with $t_1 \to 0$

$$\int_\Omega \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}) \,\mathrm{d}x\Big|_0^{t_2} = -\int_0^{t_2}\int_\Omega \boldsymbol{\chi} : \boldsymbol{T}^d \,\mathrm{d}x\,\mathrm{d}t. \tag{6.3.72}$$

We multiply (6.3.72) by $\frac{1}{\mu}$ and integrate over $(\tau, \tau+\mu)$ and proceed now in the same manner as in the proof of Lemma 4.3.6. ∎

The second and the third step of this method are followed in the same way as in the limit passage with $l \to \infty$. Hence, we omit this calculations. Using Minty-Browder trick we obtain

$$\boldsymbol{\chi} = \boldsymbol{G}(\tilde{\theta} + \theta, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d), \tag{6.3.73}$$

a.e. in Q. Moreover, using Young measures tools we may pass to the limit in right-hand side term of heat equation. Repeating the procedure from the previous limit passage we obtain

$$\mathcal{T}_k((\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k) : \boldsymbol{G}(\tilde{\theta} + \theta_k, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d_k)) \rightharpoonup (\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d) : \boldsymbol{G}(\tilde{\theta} + \theta, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d)), \tag{6.3.74}$$

weakly in $L^1(Q)$. Weak convergence of right-hand sides of heat equation and strong convergence of initial conditions imply that there exists a renormalised solution to heat equation θ, see Section 3.2, such that for every $K \in \mathbb{N}$ holds

$$\begin{aligned} \mathcal{T}_K(\theta_k) &\rightharpoonup \mathcal{T}_K(\theta) \quad \text{weakly in } L^2(0,T,W^{1,2}(\Omega)),\\ \theta_k &\to \theta \qquad\quad\ \text{a.e. in } Q. \end{aligned} \tag{6.3.75}$$

Taking $S'(\theta)\phi$ as a test function in (6.3.61), where S is a $C^\infty(\mathbb{R})$ function, such that S' has a compact support we obtain

$$\begin{aligned} &-\int_Q S(\theta)\frac{\partial\phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega S(\theta_0 - \tilde{\theta}_0)\phi(x,0)\,\mathrm{d}x + \int_Q S'(\theta)\nabla\theta\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad + \int_Q S''(\theta)|\nabla\theta|^2\phi\,\mathrm{d}x\,\mathrm{d}t = \int_Q \boldsymbol{G}(\tilde{\theta} + \theta, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d) : (\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d)S'(\theta)\phi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{6.3.76}$$

Since $\theta = \widehat{\theta} - \tilde{\theta}$ we may rewrite (6.3.76) in the following form:

$$\begin{aligned} &-\int_Q S(\widehat{\theta} - \tilde{\theta})\frac{\partial\phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega S(\theta_0 - \tilde{\theta}_0)\phi(x,0)\,\mathrm{d}x + \int_Q S'(\widehat{\theta} - \tilde{\theta})\nabla(\widehat{\theta} - \tilde{\theta})\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_Q S''(\widehat{\theta} - \tilde{\theta})|\nabla(\theta - \tilde{\theta})|^2\phi\,\mathrm{d}x\,\mathrm{d}t = \int_Q \boldsymbol{G}(\tilde{\theta} + \theta, \tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d) : (\tilde{\boldsymbol{T}}^d + \boldsymbol{T}^d)S'(\widehat{\theta} - \tilde{\theta})\phi\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{6.3.77}$$

where $\tilde{\theta}$ is a solution of (6.3.2), $\widehat{\theta}$ is a solution to full thermo-visco-elastic model and (6.3.77) hold for every test function $\phi \in C_c^\infty([-\infty,T), C^\infty(\Omega))$. Moreover, using the solution to problem (6.3.1) we obtain

$$\int_0^T\int_\Omega(\tilde{\boldsymbol{T}} + \boldsymbol{T}) : \nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t, \tag{6.3.78}$$

where

$$\boldsymbol{T} = \boldsymbol{D}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}^{\mathbf{p}}), \tag{6.3.79}$$

and (6.3.78) holds for every test function $\boldsymbol{\varphi} \in C^\infty([0,T], C_c^\infty(\Omega,\mathbb{R}^3))$.

This completes the proof of Theorem 6.2.1.

Bibliography

[1] R. A. Adams. *Sobolev spaces.* Academic Press, 1975.

[2] H.-D. Alber. *Materials with memory.* Lecture Notes in Math. 1682. Springer, Berlin Heidelberg, New York, 1998.

[3] H.-D. Alber and K. Chełmiński. Quasistatic problems in viscoplasticity theory ii: Models with nonlinear hardening. *Mathematical Models and Methods in Applied Sciences*, 17(02):189–213, 2007.

[4] J.J. Aliberi and G. Bouchitté. Non-uniform integrability and generalized young meaesures. *Journal of Convex Analysis*, 4(1):129 – 147, 1997.

[5] H. W. Alt. *Lineare Funktionalanalysis.* Springer-Verlag, 2006.

[6] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient Flows in Metric Spaces and in the Space of Probability Measures.* Birkhäuser, 2005.

[7] G. Anzellotti and M. Giaquinta. Existence of the displacements field for an elasto-plastic body subject to Hencky's law and von mises yield condition. *manuscripta mathematica*, 32(1-2):101–136, 1980.

[8] J. M. Ball and F. Murat. Remarks on Chacon's biting lemma. *Proceedings of the American Mathematical Society*, 107(3), Nov., 1989.

[9] J.M. Ball. A version of the fundamental theorem for young measures. In M. Rascle, D. Serre, and M. Slemrod, editors, *PDEs and Continuum Models of Phase Transitions*, volume 344 of *Lecture Notes in Physics*, pages 207–215. Springer Berlin Heidelberg, 1989.

[10] V. Barbu. *Nonlinear semigroups and differential equations in Banach spaces.* Noordhoff International Publishing, 1976.

[11] L. Bartczak. Mathematical analysis of a thermo-visco-plastic model with Bodner-Partom constitutive equations. *Journal of Mathematical Analysis and Applications*, 385(2):961 – 974, 2012.

[12] S. Bartels and T. Roubíček. Thermo-visco-elasticity with rate-independent plasticity in isotropic materials undergoing thermal expansion. *ESAIM Math. Model. Numer. Anal.*, 45:477–504, 2011.

[13] D. Blanchard. Truncations and monotonicity methods for parabolic equations. *Nonlinear Analysis: Theory, Methods and Applications*, 21(10):725 – 743, 1993.

[14] D. Blanchard and F. Murat. Renormalised solutions of nonlinear parabolic problems with l1 data: existence and uniqueness. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 127:1137–1152, 1997.

[15] L. Boccardo and T. Gallouët. Non-linear elliptic and parabolic equations involving measure data. *Journal of Functional Analysis*, 87(1):149 – 169, 1989.

[16] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Universitext. Springer New York Dordrecht Heidelberg London, 2011.

[17] M. Brokate, P. Krejčí, and D. Rachinskii. Some analytical properties of the multidimensional continuous Mróz model of plasticity. recent advances in structural modelling and optimization. *Control Cybernet.*, 27(2):199 – 215, 1998.

[18] K. Chełmiński. On large solutions for the quasistatic problem in non-linear viscoelasticity with the constitutive equations of Bodner–Partom. *Mathematical Methods in the Applied Sciences*, 19(12):933–942, 1996.

[19] K. Chełmiński and P. Gwiazda. Nonhomogeneous initial-boundary value problems for coercive and self-controlling models of monotone type. *Continuum Mechanics and Thermodynamics*, 12:217–234, 2000.

[20] K. Chełmiński and P. Gwiazda. On the model of Bodner – Partom with nonhomogeneous boundary data. *Mathematische Nachrichten*, 214(1):5–23, 2000.

[21] K. Chełmiński and P. Gwiazda. Convergence of coercive approximations for strictly monotone quasistatic models in inelastic deformation theory. *Math. Methods Appl. Sci.*, 30(12):1357–1374, 2007.

[22] K. Chełmiński and R. Racke. Mathematical analysis of a model from thermoplasticity with kinematic hardening. *J. Appl. Anal.*, 12:37–57, 2006.

[23] J. Chen, X. Xing, C. Sun, P. Hu, R. Yu, X. Wang, and L. Li. Zero thermal expansion in PbTiO3-based perovskites. *J. Am. Chem. Soc.*, 130:1144–1145, 2008.

[24] S. Clain. *Analyse mathématique et numérique d'un modèle de chauffage par induction*. PhD thesis, Lausanne, 1994.

[25] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and sobolev spaces. *Invent. Math.*, 98(3):511 – 547, 1989.

[26] G. Duvaut and J.L. Lions. *Les inéquations en mécanique et en physique*. Dunod, Paris, 1972.

[27] R. E. Edwards. *Functional Analysis: Theory and Applications*. Dover Pub., 1995.

[28] E. Emmrich and A. Wróblewska-Kamińska. Convergence of a full discretization of quasilinear parabolic equations in isotropic and anisotropic Orlicz spaces. *SIAM J. NUMER. ANAL.*, 51(2):1163 – 1184, 2013.

[29] L. C. Evans. *Partial Differential Equations*. American Math Society, 1998.

[30] A. E. Green and P. M. Naghdi. A general theory of an elastic-plastic continuum. *Archive for Rational Mechanics and Analysis*, 18:251–281, 1965.

[31] M. E. Gurtin, E. Fried, and L. Anand. *The Mechanics and Thermodynamics of Continua.* Cambridge, 2013.

[32] P. Gwiazda, F. Z. Klawe, and A. Świerczewska-Gwiazda. Thermo-visco-elasticity for Norton-Hoff-type models. *preprint: http://arxiv.org/abs/1404.1671*, 2014.

[33] P. Gwiazda, F.Z. Klawe, and A. Świerczewska-Gwiazda. Thermo-visco-elasticity for the Mróz model in the framework of thermodynamically complete systems. *Discrete and Continuous Dynamical Systems - Series S*, 7:981 – 991, 2014.

[34] P. Gwiazda and A. Świerczewska. Large eddy simulation turbulence model with Young measures. *Applied Mathematics Letters*, 18(8):923 – 929, 2005.

[35] P. Gwiazda and A. Świerczewska-Gwiazda. On non-newtonian fluids with the property of rapid thickening. *Math. Models Methods Appl. Sci.*, 18(7):1073 – 1092, 2008.

[36] P. Gwiazda and A. Świerczewska-Gwiazda. On steady non-newtonian flows with growth conditions in generalized Orlicz spaces. *Topol. Methods Nonlinear Anal.*, 32(1):103 – 114, 2008.

[37] P. Gwiazda, A. Świerczewska-Gwiazda, and A. Wróblewska. Monotonicity methods in generalized Orlicz spaces for a class of non-Newtonian fluids. *Math. Methods Appl. Sci.*, 33(2):125 – 248, 2010.

[38] P. Gwiazda, A. Świerczewska-Gwiazda, and A. Wróblewska. Generalized Stokes system in Orlicz spaces. *Discrete and continuous dynamical systems*, 32(6), June 2012.

[39] P. Gwiazda, P. Wittbold, A. Wróblewska, and A. Zimmermann. Renormalized solutions of nonlinear elliptic problems in generalized Orlicz spaces. *Journal Differential Equations*, 253:635 – 666, 2012.

[40] P. Gwiazda, P. Wittbold, A. Wróblewska-Kamińska, and A. Zimmermann. Renormalized solutions of nonlinear parabolic problems in generalized Musielak-Orlicz spaces.

[41] R.B. Hetnarski and R.M. Eslami. *Thermal Stresses – Advanced Theory and Applications*, volume 158 of *Solid Mechanics and Its Applications.* Springer, 2009.

[42] E. Hewitt and K. Stromberg. *Real and Abstract Analysis*, volume 25 of *Graduate Texts in Mathematics.* 1975.

[43] D. Hömberg. A mathematical model for induction hardening including mechanical effects. *Nonlinear Analysis: Real World Applications*, 5(1):55 – 90, 2004.

[44] S. Jiang and R. Racke. *Evolution equations in thermoelasticity.* Chapman & Hall/CRC, Boca Raton. 2000.

[45] C. Johnson. Existence theorems for plasticity problem. *J. Math. Pures Appl.*, 55:431–444, 1976.

[46] C. Johnson. On plasticity with hardening. *J. Math. Anal. Appl.*, 62:325–336, 1978.

[47] P. Kelly. *Lecture Notes in Solid Mechanics Part I: An Introduction to Solid Mechanics.* webpage.

[48] F. Z. Klawe. Thermo-visco-elasticity for models with growth conditions in Orlicz spaces. *preprint: http://arxiv.org/abs/1409.2652*, 2014.

[49] M. A. Krasnoselskij and Y. B. Rutitskij. *Convex Functions and Orlicz Spaces.* Noordhoff, Groningen, Netherlands, 1961.

[50] L.D. Landau and E.M. Lifshitz. *Theory of Elasticity.* Pergamon Press, 1970 (7th edition).

[51] J. Li, W. Bi, W. Ki, X. Huang, and S. Reddy. Nanostructured crystals: Unique hybrif semiconduxtors exhibiting nearly zero and tunable uniaxial thermal expansion behavior. *J. Am. Chem. Soc.*, 129:14140–14141, 2007.

[52] E.J. MacShane. Generalized curves. *Duke Math.*, 6:513 – 536, 1940.

[53] J. Málek, J. Nečas, M. Rokyta, and M. Růžička. *Weak and measure-valued solutions to evolutionary PDEs.* Chapman & Hall, London, 1996.

[54] S. Margadonna, K. Prassides, and A. N. Fitch. Zero thermal expansion in a prussian blue analogue. *J. Am. Chem. Soc.*, 126:15390–15391, 2004.

[55] T.A. Mary, J.S. Evans, O. Vogt, and A.W. Sleight. Negative thermal expansion from 0.3 to 1050 kelvin in ZrW_2O_8. *Science*, 272:90–92, 1996.

[56] P.-A. Meyer. *Probability and potentials.* Blaisdell, 1966.

[57] Z. Mróz. On the description of anisotropic workhardening. *J. Mech. Phys. Solids*, 15:163–175, 1967.

[58] S. Müller. *Variational models for microstructure and phase transitions*, volume 1713 of *Lecture Notes in Math.: Calculus of Variations and Geometric Evolution Problems*. Springer-Verlag, Berlin Heidelberg, 1999.

[59] J. Musielak. *Orlicz Spaces and Modular Spaces.* Springer Berlin Heidelberg, 1983.

[60] J. Nečas and I. Hlaváček. *Mathematical theory of elastic and elasto-plastic bodies: an introduction.* Elsevier Scientific Publisher Company, 1980.

[61] P. Pedregal. *Parametrized Measures and Variational Principles*, volume 30 of *Progress in Nonlinear Differential Equations and Their Applications.* Birkhäuser Verlag, 1997.

[62] A. E. Phillips, G. J. Halder, K. W. Chapman, A. L. Goodwin, and C. J. Kepert. Zero thermal expansion in a flexible, stable framework: Tetramethylammonium copper(i) zinc(ii) cyanide. *J. Am. Chem. Soc.*, 132:10–11, 2010.

[63] K. R. Rajagopal. *Implicit constitutive relations.* Encyclopedia of Life Support Systems (EOLSS), Developed under the Auspices of the UNESCO, Eolss Publishers, Oxford, UK.

[64] M. M. Rao and Z. D. Ren. *Theory of Orlicz spaces.* Pure and Applied Math.Dekker, New York, 1991.

[65] T. Roubíček. *Nonlinear Partial Differential Equations with Applications.* Birkhauser Verlag, 2005.

[66] J.R. Salvador, F. Guo, T. Hogan, and M.G. Kanatzidis. Zero thermal expansion in ybgage due to an electronic valence transition. *Nature*, 425:702 –705, 2003.

[67] A. Sleight. Materials science: Zero-expansion plan. *Nature*, 425:674 – 676, 2003.

[68] X. Song, Z. Sun, Q. Huang, M. Rettenmayr, X. Liu, M. Seyring, G. Li, G. Rao, and F. Yin. Adjustable zero thermal expansion in antiperovskite manganese nitride. *Advanced Materials*, 23(40):4690–4694, 2011.

[69] W. Strauss. *Partial differential equations.* Wiley, 2007.

[70] P. Suquet. *Existence and regularity of solution for plasticity problems.* in Variational Methods in Solid Mechanics, Pergamon Press, Oxford, 1980.

[71] P. Suquet. Sur les équations de la plasticité: existence et régularité des solutions. *J. Mécanique*, 20:3–39, 1981.

[72] P. Suquet. *Plasticité et homogénéisation.* PhD thesis, 1982.

[73] A. Świerczewska. A dynamical approach to large eddy simulation of turbulent flows: existence of weak solutions. *Mathematical Methods in the Applied Sciences*, 29(1):99–121, 2006.

[74] A. Świerczewska-Gwiazda. Anisotropic parabolic problems with slowly or rapidly growing terms. *Colloq. Math.*, 134(1):113 – 130, 2014.

[75] A. Świerczewska-Gwiazda. Nonlinear parabolic problems in Musielak-Orlicz spaces. *Nonlinear Analysis: TMA*, 98:48 – 65, 2014.

[76] K. Takenaka and H. Takagi. Zero thermal expansion in a pure-form antiperovskite manganese nitride. *Applied Physics Letters*, 94(13):131904, 2009.

[77] L. Tartar. Compensated compactness and applications to partial differential equations. In *Nonlinear analysis and mechanics: Heriot-Watt Symposium*, volume IV, pages 136–212. Pitman, Boston, Mass.-London, 1979.

[78] R. Temam. *Mathematical Problems in Plasticity.* Gauthier-Villars, Paris-New York, 1984.

[79] R. Temam. A generalized Norton-Hoff model and the Prandtl-Reuss law of plasticity. *Archive for Rational Mechanics and Analysis*, 95(2):137–183, 1986.

[80] R. Temam and G. Strang. Functions of bounded deformation. *Arch. Rational Mech. Anal.*, 75:7–21, 1980/81.

[81] R. M. Temam and A. M. Miranville. *Mathematical modeling in continuum mechanics.* Cambridge University Press, 2005.

[82] T. Valent. *Boundary Value Problems of Finite Elasticity: Local Theorems on Existence, Uniqueness, and Analytic Dependence on Data.* Springer Publishing Company, Incorporated, 1st edition, 2011.

[83] J. Warga. *Optimal Control of Differential and Functional Equations.* Academic Press, 1972.

[84] A. Wróblewska. Steady flow of non-newtonian fluids — monotonicity methods in generalized Orlicz spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 72(11):4136 – 4147, 2010.

[85] A. Wróblewska-Kamińska. *An application of Orlicz spaces in partial differential equations.* PhD thesis, Warsaw, 2012.

[86] L.C. Young. *Lectures on the calculus of variations and optimal control theory.* Saunders, 1969 (reprinted by Chelsea, 1980).

[87] E. Zeidler. *Nonlinear Functional Analysis II/B – Nonlinear Monotone Operators.* Springer–Verlag, Berlin–Heidelberg–New York, 1990.

[88] Y. Zhang, Z. Islam, Y. Ren, P. A. Parilla, S. P. Ahrenkiel, P. L. Lee, A. Mascarenhas, M. J. McNevin, I. Naumov, H. X. Fu, X. Y. Huang, and J. Li. Zero thermal expansion in a nanostructured inorganic-organic hybrid crystal. *Phys. Rev. Lett.*, 99:215901, 2007.

List of Notations

x	point in the reference configuration
t	time moment
$\boldsymbol{u}(x,t)$	displacement
$\theta(x,t)$	temperature
$\boldsymbol{\varepsilon}^{\mathbf{p}}(x,t)$	visco-elastic strain tensor
$\boldsymbol{D}$	linear, positively definite and bounded operator
$\boldsymbol{\varepsilon}(\boldsymbol{u})$	symmetric gradient of displacement $\boldsymbol{u}$
$\boldsymbol{G}(\theta,\boldsymbol{T}^d)$	visco-elastic constitutive function
$\mathcal{S}^3$	space of symmetric matrices 3×3
$\mathcal{S}^3_d$	space of traceless symmetric matrices 3×3
$\boldsymbol{T}^d$	deviatoric part of tensor $\boldsymbol{T}$
$\boldsymbol{I}$	identity matrix form $\mathcal{S}^3$
ρ	density
α	thermal expansion
$\eta(t)$	standard mollifire with respect to the time
$\boldsymbol{\sigma}$	Cauchy stress tensor
r	density of heat sources
θ_R	reference temperature
κ	material's conductivity
c	material's capacity or constant
$\mathcal{O}$	arbitrary subset of Ω
$W^{1,p'}_{\boldsymbol{g}}(\Omega,\mathbb{R}^3)$	space $\left\{\boldsymbol{u}\in W^{1,p'}(\Omega,\mathbb{R}^3):\boldsymbol{u}=\boldsymbol{g} \text{ on } \partial\Omega\right\}$