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Analysis of some systems of partial differential equations describing cellular movement

PhD. Dissertation

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Author's declaration:

aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Abstract.

This PhD Thesis is devoted to various systems of partial differential equations describing cellular movement. First, we consider a model of chemorepulsion. We prove global existence and uniqueness of regular solutions in dimension 2. For dimensions 3 and 4 we prove the global existence of weak solutions. The convergence to steady states is shown in all the cases. Furthermore, in the two dimensional case we provide the explicit rate of convergence to the steady states. Second, we perform an extensive study of existence, uniqueness and asymptotic behavior in 2-dimensional domains for a reduced system of partial differential equations of degenerate type describing tumour invasion. Moreover, we show all the possible positive solutions to the stationary problem associated to such a system. Finally, we study the steady-states, global well-possedness and asymptotic behaviour of some models related to tissue invasion that were proposed in [*A.J. Perumpanani, J. Norbury, J.A. Sherratt and H.M*, *A two parameter family of travelling waves with a singular barrier arising from the modeling of matrix mediated malignant invasion*, Physica D 126 (1999) pp. 145–159] and [*M.A.J. Chaplain, A.R.A. Anderson*, *Mathematical modelling of tissue invasion*, in: L. Preziosi (Ed.), Cancer Modelling and Simulation, Chapman & Hall/CRT, 2003, pp. 269-297].

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Abstrakt.

Rozprawa doktorska poświęcona jest badaniu kilku układów równań różniczkowych cząstkowych (RRCz) opisujących ruch komórek. W rozdziale 2 bada się model chemorepulsji. Dowodzi się globalnego istnienia i jednoznaczności regularnych rozwiązań w wymiarze 2. W wymiarach 3 i 4 dowodzi się globalnego istnienia słabych rozwiązań. Zbieżność do rozwiązań stacjonarnych pokazana jest w każdym przypadku. Następnie, w przypadku 2–wymiarowym podaje się jawnie współczynnik zbieżności. W rozdziale 3 formułuje się twierdzenia o istnieniu, jednoznaczności oraz zachowaniu asymptotycznym, w obszarach 2–wymiarowych, dla zredukowanego układu RRCz, typu zdegenerowanego, opisującego inwazję nowotworu. Ponadto identyfikuje się wszystkie możliwe dodatnie rozwiązania zagadnienia stacjonarnego. W rozdziale 4 bada się rozwiązania stacjonarne, globalne istnienie, jednoznaczność oraz zachowanie asymptotyczne dla pewnych modeli opisujących inwazję nowotworu na otaczającą tkankę. Modele te były zaproponowane w pracach [*A.J. Perumpanani et al, Physica D, (1999)*] oraz [*M.A.J. Chaplain, A.R.A. Anderson, w: L. Preziosi (Ed.), Cancer Modelling and Simulation, Chapman & Hall/CRT, 2003*].

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Contents

Notations

During this work we will use the following basic notations. Also, during this work, we will introduce some additional notations. Sometimes, in order to allow a more fluent reading, we will also remind some of these notations during the work.

- N denotes the spatial dimension.
- C denotes a positive constant that may vary from line to line.
- $\mathbb{N}$ stands for the natural numbers excluding the element zero.
- If A is a set then $\overline{A}$ stands for the closure of A.
- $\Omega \subset \mathbb{R}^N$ denotes a bounded domain with regular boundary, C^3 will be enough during all the text.
- $|\Omega|$ stands for the N-dimensional Lebesgue measure of Ω.
- $Q_T := \Omega \times (0,T)$.
- $C^k(\Omega)$ is the set of functions $k-$ differentiable in Ω.
- $\partial\Omega$ is the boundary of Ω.
- $|\partial\Omega|$ denotes the $N-1$-dimensional Lebesgue measure of $\partial\Omega$.
- n stands for the outward unit normal vector to $\partial\Omega$.
- $W^{k,p}(\Omega)$ is the Sobolev Space of functions $k-$ times differentiable in the distributional sense whose derivatives are $L^p(\Omega)$. The Hilbert space $W^{k,2}(\Omega)$ will be usually denoted by $H^k(\Omega)$.
- $\|\cdot\|_p$ stands for the norm in the space $L^p(\Omega)$, $p \in [1,\infty]$.
- $\|\cdot\|_{k,p}$ denotes the norm in the space $W^{k,p}(\Omega)$.
- If u is a function in Ω, $\overline{u} := \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u$.
- Let X a Banach space then $L^p(0,T;X)$ stands for the space of measurable functions from (0,t) in X such that $\|\cdot\|_X \in L^p(0,T)$.
- Let X is a Banach space, $C([0,T); weak-X)$ denotes the space of measurable functions from $[0,T]$ in X which are continuous respect to time for the weak topology of X.
- $C^{\alpha+k,(\alpha+k)/2}(\overline{Q}_T)$ stands for the Hölder space of exponents $\alpha+k$, $(\alpha+k)/2$ with respect to the spatial and time variables, respectively, in $\overline{Q}_T$ (see, preliminaries of Chapter 3).

Inequalities.

Here we collect various important inequalities that it will be used during this work. We will frequently use the following **Gagliardo-Nirenberg inequality** (see [34, 40])

$$\|w\|_p \leqslant C\|w\|_{1,2}^{\theta}\|w\|_r^{1-\theta} \quad \text{with} \quad \theta = \frac{\frac{N}{r}-\frac{N}{p}}{1-\frac{N}{2}+\frac{N}{r}} \tag{1}$$

which holds true for all $w \in H^1(\Omega)$, $p \in [1, 2N/(N-2))$ and $r \in [1,p]$. Also in various places we will use the **Csiszar-Kullback inequality** (see [14])

$$\frac{1}{2\overline{u}}\|u-\overline{u}\|_1^2 \leqslant \int_\Omega u \ln\left(\frac{u}{\overline{u}}\right) dx \tag{2}$$

Finally, we state the **Poincare-Wintinger inequality** (see [10]). If $u \in W^{1,p}(\Omega)$, then there exists a constant $C > 0$ such that

$$\|u-\overline{u}\|_p \leqslant C\|\nabla u\|_p, \tag{3}$$

for all $p \in [1, +\infty]$.

CHAPTER 1

Introduction

One of the most important features of living systems is that they interact with the environment in which they reside. The way of interaction frequently involves movement toward or away from an external stimulus, and such a response is called a taxis. Taxis can be positive or negative, depending on whether it is toward or away from the external stimulus. Many different types of taxis are known, for example aerotaxis, chemotaxis, haptotaxis, phototaxis... In many cases, taxis can be considered as a survival mechanism (see [61]).

During this Ph.D. Thesis we will assume that the organisms respond to the spatial gradient of the stimulus. Moreover, we will suppose that the organisms diffuse in the environment. In some cases, we will assume a proliferative term, denoting the birth and death of the organisms. To be more precise, throughout this work, the equation that describes the movement of the organisms is given by [1]

$$u_t = \underbrace{\Delta u}_{Diffusion} \pm \underbrace{\nabla \cdot (u \nabla \phi(v))}_{Taxis} + \underbrace{f(u,v)}_{Reaction}$$

u : concentration of the organisms
v : concentration of the stimulus or signal

An extra equation for v is needed. We will consider a diffusive equation for v in Chapter 2 while in Chapters 3,4 we assume that v does not have diffusion.

While the taxis term can lead to aggregation, the diffusive term always has a dispersive effect. Therefore, a natural question can be addressed: which term is the dominant, the taxis term or the dispersive one?

[1] The precise properties of the real functions χ and f will depend on the model, at this stage we skip these properties.

Probably one of the most famous models in mathematical biology is the Keller-Segel system. Such a model was proposed in [44] in order to describe a very particular stage in the life of many species of cellular slime molds, the aggregative stage. At this stage, the amoeba *begin to aggregate in a number of "collecting point" or centers. At each center a slug forms, migrates and eventually forms a multicellular fruiting body* ([44] pg. 399). The aggregation is induced by the presence of a chemical substance produced by some of the amoebas. This particular case of taxis, the one induced by a chemical, is called chemotaxis.

The original model proposed by Keller-Segel consist of four equations, they proved instability results for a reduced system of two equations. A particular case of this reduced system is the following one

$$\begin{cases} u_t &= \nabla \cdot (\nabla u - u \nabla \phi(v)) \\ \epsilon v_t &= \Delta v - v + u \end{cases} \tag{1.1}$$

u : amoeba concentration
v : chemical concentration

It was not till 1992 when the first rigorous results for (1.1) with $\epsilon = 0$ and ϕ a linear function, called chemosensitivity, were given. In their paper [41] Jäger and Luckhaus showed that the solutions to (1.1) may blow-up in finite time in the two-dimensional case. Later on, in a series of papers [59, 60, 35, 6, 29] the different authors studied (1.1) for $\epsilon = 0, 1$ with linear chemosensitivity functions and in [7, 65] for nonlinear chemosensitivity functions, mainly logarithmic functions[2].

In order to avoid the possibility of blow-up for the Keller-Segel system different mechanisms have been proposed. The first one, called volume filling, was proposed by Hillen and Painter in [36, 37], see also [49, 77, 12, 18]. Another way of preventing blow-up was to assume a stronger diffusive effect for u, see for instance [46, 71, 13, 9].

In other words, in the system (1.1) with linear chemosensitivity functions and spatial dimension two the diffusive term and the taxis term are in almost equilibrium i.e. global existence or blow-up is possible and it depends on the initial concentration of u, i.e. u_0.

While most of the authors have focused in the chemoattractive case, that is, the movement is towards regions of high chemical concentration, only few results are known for the chemorepulsive one. In Chapter 2, motivated by [68], we deal with this case. To be more precise, in Chapter 2 we study the following problem

[2] It should be noted that in [65] the author changed the second equation for $v_t = \Delta v - v + \phi'(v)u$.

$$
(1.2) \qquad \begin{cases} u_t = \underbrace{\Delta u}_{Diffusion} + \underbrace{\nabla \cdot (u \nabla v)}_{Chemotaxis} & \text{in} \quad \Omega \times (0,T), \\ \tau \ v_t = \underbrace{D \ \Delta v}_{Diffusion} - \underbrace{\beta v}_{Decay} + \underbrace{u}_{Production} & \text{in} \quad \Omega \times (0,T), \\ \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ (u,v)(x,0) = (u_0,v_0)(x) & \text{in} \quad \Omega, \end{cases}
$$

where Ω is an open, bounded subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, n denotes the outward unit normal vector to $\partial\Omega$ and the parameters τ, D and β are positive real numbers.

If $\tau = 0$ (that is, there is no term v_t in the second equation of (1.2)), it is quite easy to see that no finite time blow-up can take place (at least for $N \leqslant 3$). In fact much more is true and it was proved in [54, 55] that solutions exist globally, are uniformly bounded and converge with an exponential rate to the steady state. A similar result would be expected to be valid for (1.2) but, surprisingly, does not seem to be so easy to prove due to the lack of estimates on v_t. In particular, global existence of solutions is established in [68] under rather artificial conditions. Indeed, for $N = 2$, they require D and $\|u_0\|_1$ to fulfil some conditions (cf. [68, *A1-A3*]). These conditions allow them to construct a Lyapunov functional for (1.2) in the spirit of that constructed in [29] for the chemoattractive case. For $N \geqslant 3$ solutions exist globally only under a smallness condition on the initial data in $L^p(\Omega)$ with $p > N/2 + 1$. To the best of our knowledge, no further result seems to be available for (1.2).

In Chapter 2 we improve the above-mentioned results in the following directions. First, in space dimension $N = 2$ we prove the global existence and uniqueness of uniformly bounded smooth classical solutions without any restriction on the initial data and parameters. In the higher space dimension $N = 3, 4$, we are only able to establish the global existence of weak solutions. In addition, we prove that there exists a unique steady state up to the mass constraint and it is spatially homogeneous. Our approach relies on the observation that there is a natural Lyapunov functional associated to (1.2), from which several estimates can be deduced. However, it does not provide any control on v_t and does not allow us to obtain smooth classical solutions in space dimension $N \geqslant 3$.

Easily can be checked that our results remains true whatever the values of the positive real numbers τ, D and β are, therefore, we set from now on

$$\tau = D = \beta = 1.$$

Since the existence results depend strongly on the space dimension, we separate the statements of the results according to the value of N (see, Theorem 2.15).

Theorem 1.1. *Let $N = 2$. If (u_0, v_0) are non-negative functions in $W^{1,p_0}(\Omega)$ for some $p_0 > 2$ then there exists a unique global in time smooth classical solution to (1.2). Moreover,*

$$\lim_{t \to +\infty} (u,v)(\cdot,t) = (\overline{u}, \overline{v}) \ \ \textit{in} \ \ C^2(\overline{\Omega}; \mathbb{R}^2) \quad \textit{with} \ \ \overline{u} = \overline{v} = \frac{1}{|\Omega|} \int_\Omega u_0 \ dx.$$

Furthermore, the rate of convergence is exponential.

The proof of Theorem 1.1 has various steps. First, based on the abstract theory for quasilinear parabolic systems developed in [3] we prove local well-possedness of solutions. Then, using the Lyapunov functional

$$F(u,v) = \int_\Omega \left(u \ln u + \frac{|\nabla v|^2}{2} \right)$$

we obtain some crucial estimates of solutions to (1.2) independently of time. Those estimates are enough in the two dimensional case to show, based on the regularization of parabolic problems, that the solutions are global and regular in time independently of the size of (u_0, v_0). Moreover, the Lyapunov functional F plays a fundamental role in the convergence to the steady-state.

The estimates provided by the Lyapunov functional F does not seem enough in higher dimensions to obtain global regular solutions (independently of the size of (u_0, v_0)). However, the bounds given by the function F will allow us to prove existence of global weak L^1 solutions. We recall the definition of global weak L^1 solutions here for the reader's convenience (see Definition 2.1).

Definition 1.2. *A global weak solution to (1.2) is a pair of non-negative functions*

$$(u,v) \in C([0,\infty); weak - L^1(\Omega; \mathbb{R}^2))$$

such that

$$\nabla u, \nabla v, u\nabla v \in L^1(\Omega \times (0,T)),$$

and

$$\begin{aligned} \int_\Omega (u(t) - u_0)\ \varphi\ dx + \int_0^t \int_\Omega (\nabla u + u\ \nabla v)\ \cdot \nabla \varphi\ dxds &= 0, \\ \int_\Omega (v(t) - v_0)\ \varphi\ dx + \int_0^t \int_\Omega (\nabla v \cdot \nabla \varphi + (v-u)\ \varphi)\ dxds &= 0, \end{aligned}$$

for each $t \geqslant 0$ and $\varphi \in W^{1,\infty}(\Omega)$ (we recall that $\tau = D = \beta = 1$).

Now, we formulate the precise statements in the higher dimensional case (see Theorems 2.18 and 2.19).

Theorem 1.3. *Let $N = 3$. If (u_0, v_0) are non-negative functions in $W^{1,p_0}(\Omega)$ for some $p_0 > 3$, then there exists a global weak solution (u,v) to (1.2) which satisfies also*

$$(u,v) \in L^{5/4}(0,T; W^{1,5/4}(\Omega; \mathbb{R}^2))$$

for any $T > 0$. Moreover, recalling that $\overline{u}$ and $\overline{v}$ are defined in Theorem 1.1, we have

$$\lim_{t\to+\infty} \left[\int_\Omega (u(t) - \overline{u})\ \ \phi\ dx + \|v(t) - \overline{v}\|_2 \right] = 0$$

for each $\phi \in L^\infty(\Omega)$.

Theorem 1.4. *Let $N = 4$. If (u_0, v_0) are non-negative functions in $W^{1,p_0}(\Omega)$ for some $p_0 > 4$, then there exists a global weak solution (u, v) to (1.2). Moreover,*

$$\lim_{t\to+\infty}\left[\int_\Omega (u(t)-\overline{u})\ \phi\ dx + \|v(t)-\overline{v}\|_2\right] = 0$$

for each $\phi \in L^\infty(\Omega)$.

In the proofs of Theorems 1.3 and 1.4 we will define a regularized problem to the original problem (1.2), see (2.3), and we will show the existence of global solutions to (1.2) by a compactness method.

Finally let us mention that the previous existence results do not seem to extend to space dimension $N \geqslant 5$: this is due to the fact that we can not assure $u\nabla v \in L^1(\Omega\times(0,T))$, for every $T > 0$ if $N \geqslant 5$. In particular, if $N = 4$ then we only have $u\nabla v \in L^1(\Omega\times(0,T))$. Therefore, we will have to apply the Dunford-Pettis Theorem as well as the Vitali convergence Theorem in the compactness method. Since, those results are not standard we will state them in the beginning of Chapter 2.

In Chapter 2 the signal or stimulus diffuses in the environment, however there are cases in which the stimulus is strictly localized, for example the ants, which follow trails left by predecessors, myxobacteria (see, [61]) or in the cancer invasion. The chapters 3 and 4 are devoted to study systems that share this property.

In Chapter 3 we will consider the following problem

$$\text{(1.3)}\qquad \begin{cases} u_t = \nabla\cdot(\nabla u - u\nabla w) + \delta u(1-u) & \text{in } \Omega\times(0,T),\\ w_t = -uw & \text{in } \Omega\times(0,T),\\ \dfrac{\partial u}{\partial n} - u\dfrac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega\times(0,T),\\ (u(x,0), w(x,0)) = (u_0(x), w_0(x)) & \text{in } \Omega, \end{cases}$$

with $\Omega \subset \mathbb{R}^2$ a bounded domain whose boundary is regular and δ is a non-negative constant. For $\delta = 0$ the problem (1.3), as we will see later on, can be consider as a simplified model of invasion. Also (1.3) is a particular case of the models that were proposed in [42] in order to describe bacterial movement. Previously, a version of the problem (1.3) was studied by Rascle in [67] (see also [66]) with the boundary condition replaced by

$$\text{(1.4)}\qquad \frac{\partial u}{\partial n} = 0.$$

Additionally he takes a positive constant as initial condition for the function w, $w_0(x) \equiv w_0 > 0$ and the reaction term $\delta u(1-u)$ is substituted by a function $f(u, w)$ which satisfies the following condition

$$\text{(1.5)}\qquad \exists L > 0,\ \forall u \in \mathbb{R},\ \forall w > 0,\ |f(u,w)| \leqslant L\,|u|\,.$$

Under the previous hypothesis, it has been shown (see [66], [67]) that the problem (1.3) has a unique classical global solution in the one-dimensional case.

In more than one-dimensional space there are some recent results concerning the existence of global weak solutions when $f \equiv 0$, $\Omega = \mathbb{R}^N$ and the initial data satisfy some conditions on the size [20–22]. If the term $-uw$, is replaced by a general function $g(u,w) = \varphi(u,w)h(u,w)$, satisfying the following conditions

$$\begin{aligned} &\varphi(u,w) > 0 \quad \text{if } (u,w) \in H := [u_1,u_2] \times [w_1,w_2], \\ &h(u_1,w_1) = h(u_2,w_2) = 0, \end{aligned} \tag{1.6}$$

for some constants $0 \leqslant u_1 < u_2$, $w_1 < w_2$, furthermore,

$$\begin{aligned} &\varphi, h \in C^1, \\ &\frac{\partial h}{\partial u} > 0\,, \quad \frac{\partial h}{\partial w} + u\frac{\partial h}{\partial u} < 0 \text{ in } H, \end{aligned} \tag{1.7}$$

then, in [28] is shown the existence of a unique global classical solution for all initial data $(u_0, w_0) \in \mathit{int}\,(H)$. Moreover, if additionally $\varphi \equiv 1$ then the solution for large times goes to $(\frac{1}{|\Omega|}\int_\Omega u_0, \frac{1}{|\Omega|}\int_\Omega w_0)$ in L^2 (without rate), where $|\Omega|$ denotes the N-dimensional Lebesgue measure of Ω. Finally, let us mention [32, 33] where problems similar to (1.3) were studied on the real line.

Our aim in Chapter 3 is to prove global existence and uniqueness of classical solution to problem (1.3) in the two-dimensional case and also to present the asymptotic behaviour of the solution. To be more precise we prove the following:

Theorem 1.5. *Let $0 < l < 1$, $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^{l+2} boundary $\partial\Omega$. If $(u_0, w_0) \in \left(C^{l+2}(\overline{\Omega})\right)^2$, $u_0 \geqslant 0$, $w_0 > 0$, $u_0 \neq 0$ and the compatibility condition*

$$\frac{\partial u_0}{\partial n} = u_0 \frac{\partial w_0}{\partial n}$$

is satisfied for every $x \in \partial\Omega$, then the problem (1.3) has a unique global positive solution defined on an interval $[0, +\infty) \subset \mathbb{R}$ and $(u,w) \in \left(C^{l+2,l/2+1}(\overline{\Omega \times (0,t)})\right)^2$, for all $t \in [0, +\infty)$. Moreover, if $\delta = 0$, then

$$\lim_{t\to+\infty} \|u - \overline{u}\|_2 = 0\,, \qquad \lim_{t\to+\infty} \|w\|_2 = 0. \tag{1.8}$$

Furthermore, if $u_0 > 0$ then,

$$\lim_{t\to+\infty} \|u(t) - u_\delta\|_1 \leqslant Ce^{-\theta t}\,, \qquad \lim_{t\to+\infty} \|w\|_\infty \leqslant Ce^{-\theta' t}, \tag{1.9}$$

where θ, θ' are positive constants and

$$u_\delta := \begin{cases} \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u_0 & \text{if } \delta = 0, \\ 1 & \text{if } \delta > 0. \end{cases} \tag{1.10}$$

The proof of Theorem 1.5 is carried out as follows. First, we show local existence of solutions, this is done in a similar way as in [66, 67]. The main difficulty to overcame with respect to those papers is the different boundary condition. Then, in order to obtain the global existence theorem, suitable bounds of the solutions are given. However, by contrast with the chapter 2, the lack of spatial regularization effect in the w-equation demands tedious estimates. In those estimates as in the previous chapter the Lyapunov function

$$F(u,w) = \int_\Omega u(\ln u - 1) + \frac{1}{2}\int_\Omega w^{-1}|\nabla w|^2$$

associated to (1.3) plays an important role. For $\delta = 0$ such a Lyapunov function was introduced in [20] although somehow is hidden in [67]. In the asymptotic behaviour of the solutions since, for $\delta > 0$, the L^1-norm of u is not preserved in time then the proof is more involved than in the previous model. In fact, we are aware about few papers concerning the entropy method in which the L^1 norms of the solutions is not preserved, for example [23, 24]. We will show exponential decay towards a steady-state. Unfortunately we require $u_0 > 0$ and we do not know how to avoid this assumption but we would like to remark that such an assumption has been also encountered in [73][3].

In chapter 3, we also study the steady-state problem. In particular we give explicitly all the solutions to the steady-state problem (see Section 3.4.1). More precisely:

Theorem 1.6. *The positive solutions to the stationary problem associated to (1.3) are given by*

$$\begin{aligned}
(u^*, w^*) &= (0, \widetilde{w}), \quad && \widetilde{w} \in \mathcal{P}_2,\\
(u^*, w^*) &= (1, 0), \quad && \textit{if } \delta > 0,\\
(u^*, w^*) &= (k, 0), \quad && \textit{if } \delta = 0,
\end{aligned}$$

where $k > 0$ is a constant and $\mathcal{P}_2 = \{z \in W^{1,\infty}(\Omega) : \ z \geqslant 0,\ z \neq 0\}$.

Let us stress two facts. First, our concept of solutions to the steady-state problem demands $w \in W^{1,\infty}(\Omega)$. Second, $uw = 0$ does not imply $u = 0$ or $w = 0$. A simple countersample is provided by the the characteristic functions $u = \chi_{[0,1)}$, $w = \chi_{[1,2]}$ on the interval $[0, 2]$.

Finally, the last chapter of the Ph.D. is devoted to a general model of cancer invasion that covers the models presented in [63, 15]. The model reads

$$(1.11)\qquad \begin{cases}
u_t = \underbrace{\rho\Delta u}_{Diffusion} - \underbrace{\nabla\cdot(u\chi(v)\nabla v)}_{Haptotaxis} + \underbrace{\mu u(1-u-v)}_{Proliferation} & \text{in } \ \Omega\times(0,T),\\
v_t = -\underbrace{\gamma m v}_{Degradation} & \text{in } \ \Omega\times(0,T),\\
m_t = \underbrace{\Delta m}_{Diffusion} - \underbrace{\beta m}_{Decay} + \underbrace{\alpha u g(v)}_{Production} & \text{in } \ \Omega\times(0,T),\\
\rho\dfrac{\partial u}{\partial n} - u\chi(v)\dfrac{\partial v}{\partial n} = \dfrac{\partial m}{\partial n} = 0 & \text{on } \ \partial\Omega\times(0,T),\\
(u,v,m)(x,0) = (u_0,v_0,m_0)(x) & \text{in } \ \Omega,
\end{cases}$$

u : concentration of cancer cells
v : concentration of the extracellular matrix (ECM)

[3] The key of such a restriction in [73] is in (5.4)

m : concentration of the matrix degrading enzymes

where $\Omega \subset \mathbb{R}^3$ is a bounded regular domain, $g, \chi \in C^2(\mathbb{R}_+)$ and $\alpha, \beta, \gamma, \rho$ denote positive parameters. The interpretation of the model is as follows, cancer cells on contact with the ECM produce matrix degrading enzymes which degrade the ECM. Then, the cancer cells move towards the gradient of the matrix via a taxis movement called haptotaxis.

Observe that if the production and decay rates of the proteolytic enzymes, denoted by α and β respectively, are much faster than the motility, that is $\alpha \gg 1, \beta \gg 1$, then dividing by β we get

$$\beta^{-1} m_t = \beta^{-1} \Delta m - m + \frac{\alpha}{\beta} u g(v)$$

and, since β is large then β^{-1} is small. Therefore, heuristically, we can claim $m \simeq \frac{\alpha}{\beta} u g(v)$ and in the particular case $g, \chi = 1$ we recover (1.3).

In our knowledge, first models of tissue invasion were proposed in [30] and [62]. In [30] the authors developed a Lotka-Volterra competition model for the cancer and normal cells. They also considered the acidity of the microenvironment through a reaction-diffusion equation for the ion H^+. The paper [62] deals with a complex system of six reaction-diffusion equations with transport terms. Later on, have appeared simpler models in a series of papers [64], [63], [53] in which the authors study travelling waves or they perform some numerical simulations. In the last years, Chaplain and Lolas proposed another models of tissue invasion in [16] and [17]. The main novelty of this papers is the inclusion of a chemotaxis term for the cancer cells and a remodelling term for the ECM. Finally, more recently in [31] the authors consider a model for tissue invasion that take into account the cell-cell adhesion via a nonlocal term in the haptotaxis term.

In most of the papers related to tissue invasion the authors solve the system numerically. We are just aware about few papers that deal with the analytical properties of such systems. In [72] and [57] the authors showed the existence of a local solution of some models of tissue invasion that were proposed in [63] and [15] respectively. The paper [75] contains a global existence and uniqueness result for a simplified version of a model proposed by Anderson in [5] and in [76] Walker has proved the global existence and uniqueness of a model of invasion that also consider the age of the cells. Finally, in [47] the author showed mathematical relationships between different scales of the model proposed in [15].

Concerning to the steady-state problem in chapter 4 we show explicitly, as we did in the previous chapter, all positive solutions to (1.11), under the restriction $v \in W^{1,\infty}(\Omega)$, if $g(s) = s$ or $g(s) = 1$ (see Section 4.3). To be more precise, we have:

Theorem 1.7. *For $g(v) = 1$ the positive steady-states of (1.11) are given by*

$$\begin{aligned}
(u^*, v^*, m^*) &= (0, \widetilde{v}, 0)\,, \quad \widetilde{v} \in \mathcal{P}_2,\\
(u^*, v^*, m^*) &= \left(k, 0, \frac{\beta k}{\alpha}\right)\,, \quad if\ \mu = 0,\\
(u^*, v^*, m^*) &= \left(1, 0, \frac{\beta}{\alpha}\right)\,, \quad if\ \mu > 0,
\end{aligned}$$

where $k > 0$ is a constant and $\mathcal{P}_2 =: \{z \in W^{1,\infty}(\Omega)\ :\ z \geqslant 0,\ z \neq 0\}$.

Theorem 1.8. *Assume $g(v) = v$, then the positive solutions to (1.11) are given by*

$$\begin{aligned}
(u^*, v^*, m^*) &= (0, \widetilde{v}, 0)\,, \quad \widetilde{v} \in \mathcal{P}_2,\\
(u^*, v^*, m^*) &= (k, 0, 0)\,, \quad if\ \mu = 0,\\
(u^*, v^*, m^*) &= (1, 0, 0)\,, \quad if\ \mu > 0.
\end{aligned}$$

where $k > 0$ is a constant.

For the local existence theorem of (1.11) we apply the theory of linear semigroups. We also show the continuity of the solution respect to the initial data (see Corollary 4.7).

Theorem 1.9. *Let $\nu \in \left(\frac{1}{2} + \frac{3}{2p}, 1\right)$, $p \in (3, 6)$ and $X_p^\nu := D((-\Delta + I)^\nu)$. Suppose that the initial data satisfies*

$$\mathbf{x_0} := (u_0, v_0, m_0) \in H^1(\Omega) \times W^{1,\infty}(\Omega) \times X_p^\nu := \mathbf{Y},$$

then there exists $\tau(\|\mathbf{x_0}\|_{\mathbf{Y}})$ such that the problem (1.11) has a unique solution

$$\begin{aligned}
u &\in C\left([0,\tau]; H^1(\Omega)\right) \cap C^1\left((0,\tau); W^{1,\infty}(\Omega)\right)\,,\\
v &\in C\left([0,\tau]; W^{1,\infty}(\Omega)\right) \cap C^1\left((0,\tau); W^{1,\infty}(\Omega)\right)\,,\\
m &\in C\left([0,\tau]; X_p^\nu\right) \cap C^1\left((0,\tau); X_p^\nu\right) \cap C\left((0,\tau); W^{2,p}(\Omega)\right)\,.
\end{aligned} \tag{1.12}$$

Moreover, the solution depends continuously on the initial data. Furthermore, if $u_0(x), w_0(x), m_0(x) \geqslant 0$ then $u(x,t), w(x,t), m(x,t) \geqslant 0$ for all $(x,t) \in \Omega \times (0,\tau]$.

We would like to point out that a similar model of invasion was considered in [75]. However, by contrast with [75, Lemma 2.1], the explicit knowledge of v is not used in the proof of the local existence. Next, in order to show global existence in time of the solutions to (1.11), we use suitable bounds of the solutions. It should be noted that the presence of the m-equation provokes a faster way of improving the regularity than in the previous model. Therefore, by contrast with the previous chapter, we will also show global existence independently of the size of the initial data in the 3-dimensional case. In the asymptotic behaviour the solutions we will focus on the cases $g(s) = 1$ and $g(s) = s$. The absence of a Lyapunov function for (1.11) causes new difficulties. We overcame these difficulties thanks to a careful estimates of the solutions. In particular, for the asymptotic behaviour, (see Theorems 4.23 and 4.25) we have:

Theorem 1.10. *Let $g(v) = 1$, $\tau > 0$, $t \geqslant \tau$, any given initial data $(u_0, v_0, m_0) \geqslant 0$, $v_0 > 0$, $u_0 > 0$ in the class $\mathbf{Y}$ and $v_0 < 1$ if $\mu > 0$. Then the solution to (1.11) (u, v, m) satisfies that,*

- *if* $\mu = 0$,

$$\|u(t) - \overline{u}\|_{X_p^\nu} \leqslant Ce^{-\theta t}, \quad \|v(t)\|_{1,\infty} \leqslant Ce^{-\delta t}, \quad \|m(t) - (\beta/\alpha)\overline{u}\|_{1,\infty} \leqslant Ce^{-\theta' t} \tag{1.13}$$

with $\theta, \delta, \theta' > 0$ *and* $\overline{u} = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u_0$,

- *if* $\mu > 0$,

$$\|u(t) - 1\|_{X_p^\nu} \leqslant Ce^{-\theta t}, \quad \|v(t)\|_\infty \leqslant Ce^{-\delta t}, \quad \|m(t) - \beta/\alpha\|_{\leqslant} Ce^{-\rho' t}, \tag{1.14}$$

with $\rho' > 0$.

Theorem 1.11. *Let* $g(v) = v$, $\tau > 0$, $t \geqslant \tau$ *and any given initial data* $u_0, v_0, m_0 \geqslant 0$, $v_0 > 0$, $u_0 > 0$ *in the class* $\mathbf{Y}$. *Then the solution to (1.11) satisfies,*

- *if* $\mu = 0$,

$$\lim_{t\to+\infty} \|u(t) - \overline{u}\|_2^2 = 0, \quad \lim_{t\to+\infty} \|v(t)\|_2 = 0, \quad \lim_{t\to+\infty} \|m(t)\|_2^2 = 0 \tag{1.15}$$

- *if* $\mu > 0$,

$$\lim_{t\to+\infty} \|u(t) - 1\|_2^2 = 0, \quad \lim_{t\to+\infty} \|v(t)\|_2 = 0, \quad \lim_{t\to+\infty} \|m(t)\|_2^2 = 0, \tag{1.16}$$

under the additional condition $v_0 < 1$.

CHAPTER 2

A chemorepulsion system

Through this chapter we consider a model of chemorepulsion. We prove global existence and uniqueness of regular solutions in dimension 2. For $N = 3, 4$ we prove the global existence of weak solutions. The convergence to steady states is shown in all the cases. Furthermore, in the two dimensional case we provide the explicit rate of convergence to the steady states.

2.1. Preliminaries

In this chapter we consider a chemorepulsion model which is derived in [68] and reads

$$
\begin{cases}
u_t = \underbrace{\Delta u}_{Diffusion} + \underbrace{\nabla \cdot (u \nabla v)}_{Chemotaxis} & \text{in} \quad \Omega \times (0,T), \\
v_t = \underbrace{\Delta v}_{Diffusion} - \underbrace{v}_{Decay} + \underbrace{u}_{Production} & \text{in} \quad \Omega \times (0,T), \\
\\
\dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0,T), \\
(u,v)(x,0) = (u_0, v_0)(x) & \text{in} \quad \Omega,
\end{cases} \tag{2.1}
$$

where Ω is an open and bounded subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geqslant 2$ and n denotes the outward unit normal vector to $\partial\Omega$.

Let us first remind some notations. The norm in the space $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, is denoted by $\|\cdot\|_p$. The classical Sobolev space is denoted by $W^{m,p}(\Omega)$ for $1 \leqslant p \leqslant \infty$ and $m \geqslant 1$ and the associated norm by $\|\cdot\|_{m,p}$. The notation $H^1(\Omega)$ is also used for the Hilbert space $W^{1,2}(\Omega)$. If X is a Banach space, X' denotes its topological dual space. If $k \geqslant 1$, the set of C^k-smooth functions which vanish on the boundary of Ω is denoted by $C_0^k(\Omega)$. Finally, if $T > 0$, $C([0,T]; weak{-}L^1(\Omega))$ denotes the space of functions from $[0,T]$ in $L^1(\Omega)$ which are continuous with respect to time for the weak topology of $L^1(\Omega)$.

Next, we remind the concept of global weak solution for (2.1).

Definition 2.1. *A global weak solution to (2.1) is a pair of non-negative functions*

$$(u, v) \in C([0, \infty); weak - L^1(\Omega; \mathbb{R}^2))$$

such that

$$\nabla u, \nabla v, u\nabla v \in L^1((0, T) \times \Omega),$$

and

$$\begin{aligned} \int_\Omega (u(t) - u_0)\ \varphi\ dx + \int_0^t \int_\Omega (\nabla u + u\ \nabla v)\ \cdot \nabla\varphi\ dxds &= 0, \\ \int_\Omega (v(t) - v_0)\ \varphi\ dx + \int_0^t \int_\Omega (\nabla v \cdot \nabla\varphi + (v - u)\ \varphi)\ dxds &= 0, \end{aligned}$$

for each $t \geqslant 0$ and $\varphi \in W^{1,\infty}(\Omega)$.

From the Definition 2.1 easily follows that a global weak solution (u, v) to (2.1) satisfies

$$\|u(t)\|_1 = \|u_0\|_1 \quad \text{and} \quad \|v(t)\|_1 = e^{-t}\ \|v_0\|_1 + (1 - e^{-t})\ \|u_0\|_1 \quad \text{for} \quad t \geqslant 0. \tag{2.2}$$

Since we will use the compactness method in the L^1-setting. Let us state, for the reader's convenience, Dunford-Pettis Theorem and the Vitali convergence Theorem. In order to do that the following definitions will be needed (see [27, section 2.1.2]).

Definition 2.2. *Let $(X, \mathcal{M}, \mu)$ be a measure space and let $u_n, u : X \to \mathbb{R}$ be measurable functions.*

a) $\{u_n\}$ *is said to* converge to u almost uniformly *if for every $\epsilon > 0$ there exists a set $E \subset \mathcal{M}$ such that $\mu(E) < \epsilon$ and $\{u_n\}$ converges to u uniformly in $X \setminus E$;*

b) $\{u_n\}$ *is said to* converge to u in measure *if for every $\epsilon > 0$,*

$$\lim_{n\to\infty} \mu(\{x \in X : |u_n(x) - u(x)| > \epsilon\}) = 0.$$

Remark 2.3. *If $\{u_n\}$ converges to u almost uniformly, then it converges to u in measure.*

Definition 2.4. *Let $(X, \mathcal{M}, \mu)$ be a measure space. A family $\mathcal{F}$ of measurable functions $u : X \to [-\infty, \infty]$ is said to be*

a) equi-integrable *if for every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\int_E |u| d\mu \leqslant \epsilon$$

for all $u \in \mathcal{F}$ and for every measurable set $E \subset X$ with $\mu(E) \leqslant \delta$.

b) p-equi-integrable*, $p > 0$, if the family of functions $\{|u|^p : u \in \mathcal{F}\}$ is equi-integrable.*

Now, we state the following version of Dunford-Pettis Theorem, see [10], and the Vitali convergence Theorem.

Theorem 2.5. Dunford-Pettis Theorem. *Let $(X, \mathcal{M}, \mu)$ be a measure space with μ finite. A family $\mathcal{F} \subset L^1(X)$ is relatively compact in the weak topology of L^1 if and only if $\mathcal{F}$ is equi-integrable.*

Theorem 2.6. Vitali converge Theorem. *Let $(X, \mathcal{M}, \mu)$ be a measure space, let $1 \leqslant p < \infty$, and let $u_n, u : X \to \mathbb{R}$ be measurable functions. Then $\{u_n\}$ converges to u in $L^p(\Omega)$ if and only if*

a) $\{u_n\}$ converges to u in measure;

b) $\{u_n\}$ is p-equi-integrable;

c) for every $\epsilon > 0$ there exists $E \subset X$ with $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and

$$\int_{X \setminus E} |u_n|^p d\mu \leqslant \epsilon$$

for all n.

Remark 2.7. *Observe that condition $iii)$ is satisfied if X has finite measure.*

Remark 2.8. *In our case we will apply Vitali convergence Theorem for $X = \Omega \times (0, T)$ and μ the Lebesgue measure. Taking into account that $\mu(\Omega \times (0, T)) < \infty$ then if $u_n \to u$ pointwise almost everywhere we have, by the Egoroff Theorem that such a convergence is, in fact, almost uniformly and also in measure. Therefore, in our setting, if we want to show that a sequence u_n of functions in L^1 converges to u in the strong topology of L^1 we have just to show the equi-integrability and the pointwise convergence to u.*

This chapter is organized as follows. The next section is devoted to the local existence and positive steady states. In section 3 we deal with the two dimensional case and finally in section 4 we study the higher dimensional case.

2.2. Local well-posedness and positive steady-states.

First, for each $\epsilon \geqslant 0$, we define the following perturbation of (2.1):

$$\begin{cases} u_t^\epsilon = \Delta u^\epsilon + \nabla \cdot (u^\epsilon(1 - \epsilon u^\epsilon)\nabla v^\epsilon) & \text{in} \quad \Omega \times (0, T), \\ v_t^\epsilon = \Delta v^\epsilon - v^\epsilon + u^\epsilon & \text{in} \quad \Omega \times (0, T), \\ \dfrac{\partial u^\epsilon}{\partial n} = \dfrac{\partial v^\epsilon}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0, T), \\ (u^\epsilon, v^\epsilon)(x, 0) = (u_0, v_0)(x) & \text{in} \quad \Omega. \end{cases} \tag{2.3}$$

Observe that (2.1) is obtained by taking $\epsilon = 0$ in (2.3).

Theorem 2.9. *Let $p_0 > n$ and consider the initial condition $(u_0, v_0) \in W^{1,p_0}(\Omega; \mathbb{R}^2)$ with $u_0, v_0 \geqslant 0$. Then the system (2.3) has a local unique classical solution*

$$(u^\epsilon, v^\epsilon) \in C(\overline{\Omega} \times [0, t_\epsilon^+); \mathbb{R}^2) \cap C^\infty(\overline{\Omega} \times (0, t_\epsilon^+); \mathbb{R}^2)$$

and $u^\epsilon(x, t), v^\epsilon(x, t) \geqslant 0$ for each $(x, t) \in \overline{\Omega} \times [0, t_\epsilon^+)$, t_ϵ^+ denoting the maximal existence time. Moreover, $\|u^\epsilon(t)\|_1$ and $\|v^\epsilon(t)\|_1$ satisfy (2.2) for $t \in [0, t_\epsilon^+)$.

If there is a function $\omega : (0, \infty) \to (0, \infty)$ such that, for each $T > 0$,

$$\|(u^\epsilon(t), v^\epsilon(t))\|_\infty \leqslant \omega(T), \quad 0 < t < \min\{T, t_\epsilon^+\},$$

then $t_\epsilon^+ = +\infty$. In particular, if $\epsilon \in (0, \epsilon_0]$ with $1/\epsilon_0 = \max\{\|u_0\|_\infty, \|v_0\|_\infty\}$ then $0 \leqslant u^\epsilon, v^\epsilon \leqslant 1/\epsilon$ and thus $t_\epsilon^+ = +\infty$.

Note that $1/\epsilon_0 = \max\{\|u_0\|_\infty, \|v_0\|_\infty\}$ is finite thanks to the continuous embedding of $W^{1,p_0}(\Omega)$ in $L^\infty(\Omega)$. Therefore, given $(u_0, v_0) \in W^{1,p_0}(\Omega;\mathbb{R}^2)$ with $u_0, v_0 \geqslant 0$, (2.3) has a global classical solution for $\epsilon > 0$ sufficiently small.

Proof. For $\delta > 0$ we define the set $D_0 := (-\delta, +\infty) \times (-\delta, +\infty)$, $\mathbf{y} = (v^\epsilon, u^\epsilon)$, and $a_{jk} \in C^\infty(D_0, \mathcal{L}(\mathbb{R}^2))$, $1 \leqslant j, k \leqslant n$, by

$$a = a_{jk}(\mathbf{y}) = (a_{jk}^{rs})_{1\leqslant r,s\leqslant 2} := \begin{pmatrix} 1 & 0 \\ u^\epsilon(1-\epsilon u^\epsilon) & 1 \end{pmatrix} \qquad \text{if } j = k,$$

$a_{jk}(\mathbf{y}) = 0$ if $j \neq k$. Next for $\mathbf{z} \in D_0$ we introduce the operators

$$\mathcal{A}(\mathbf{y})\mathbf{z} := \sum_{j,k=1}^{n} -\partial_j(a_{jk}(\mathbf{y})\partial_k \mathbf{z}), \qquad \mathcal{B}(\mathbf{y})\mathbf{z} := \sum_{j,k=1}^{n} \nu_j \cdot a_{jk}(\mathbf{y})\partial_k \mathbf{z},$$

and the function $f \in C^\infty(D_0;\mathbb{R}^2)$

$$f(\mathbf{y}) := \begin{pmatrix} u^\epsilon - v^\epsilon \\ 0 \end{pmatrix}.$$

With these notations (2.3) reads

$$\begin{aligned} \partial_t \mathbf{y} + \mathcal{A}(\mathbf{y})\mathbf{y} &= f(\mathbf{y}), \\ \mathcal{B}(\mathbf{y})\mathbf{y} &= 0, \\ \mathbf{y}(0) &= (v_0, u_0). \end{aligned}$$

Since $(\mathcal{A}, \mathcal{B})$ is of separated divergence form in the sense of [3, Example 4.3 (e)], then the boundary-value operator $(\mathcal{A}, \mathcal{B})$ is normally elliptic. We can therefore apply [3, Theorem 14.4 and Corollary 14.7] to conclude that (2.3) has a unique maximal classical solution

$$\mathbf{y} = (v^\epsilon, u^\epsilon) \in C(\overline{\Omega} \times [0, t_\epsilon^+); \mathbb{R}^2) \cap C^\infty(\overline{\Omega} \times (0, t_\epsilon^+); \mathbb{R}^2).$$

Moreover, since (with the notations of [3, Section 15]) $D_2 = (0, +\infty) \times \{0\}$ and $a_{jj}^{21} = u^\epsilon(1-\epsilon u^\epsilon)$ $1 \leqslant j \leqslant N$, $a_{jk}^{21} = 0, a_{jk}^{12} = 0$ for $j \neq k$, $1 \leqslant j, k \leqslant N$ and all these coefficients vanish on D_2 we can apply [3, Theorem 15.1] to conclude that $u^\epsilon(t) \geqslant 0$ for $[0, t_\epsilon^+)$. Next the non-negativity of v^ϵ follows from the standard maximum principle for parabolic equations. The global existence criterion can be deduced from [3, Theorem 15.5]. Finally, if $\epsilon \in (0, \epsilon_0)$, writing the equation solved by $-u^\epsilon + 1/\epsilon$, we see that we are in a position to apply [3, Theorem 15.1] to establish that $u^\epsilon \leqslant 1/\epsilon$. The similar upper bound for v^ϵ is then a straightforward consequence of the classical comparison principle. ∎

We next turn to the existence of a Lyapunov functional for (2.3) which is the cornerstone of our analysis.

Lemma 2.10. *For $\epsilon \in [0, \epsilon_0]$ and $0 \leqslant s < t < t_\epsilon^+$ the solution (u^ϵ, v^ϵ) to (2.3) satisfies the following equality*

$$F_\epsilon(u^\epsilon(t), v^\epsilon(t)) - F_\epsilon(u^\epsilon(s), v^\epsilon(s)) = -\int_s^t \int_\Omega \left(\frac{|\nabla u^\epsilon|^2}{u^\epsilon(1-\epsilon u^\epsilon)} + |\Delta v^\epsilon|^2 + |\nabla v^\epsilon|^2 \right) dx d\tau \tag{2.4}$$

where F_ϵ is given by

$$F_\epsilon(u,v) = \int_\Omega \left(u\ln u + \frac{1}{\epsilon}(1-\epsilon u)\ln(1-\epsilon u) + \frac{|\nabla v|^2}{2} \right)$$

if $\epsilon > 0$ and

$$F_0(u,v) = \int_\Omega \left(u\ln u + \frac{|\nabla v|^2}{2} \right).$$

Proof. On the one hand, multiplying the first equation of (2.3) by $\ln u^\epsilon - \ln(1-\epsilon u^\epsilon)$ and integrating with respect to space, we obtain

$$\frac{d}{dt}\int_\Omega \left(u^\epsilon \ln u^\epsilon + \frac{1}{\epsilon}(1-\epsilon u^\epsilon)\ln(1-\epsilon u^\epsilon) \right)\, dx = -\int_\Omega \frac{|\nabla u^\epsilon|^2}{u^\epsilon(1-\epsilon u^\epsilon)}\, dx - \int_\Omega \nabla u^\epsilon \cdot \nabla v^\epsilon\, dx. \tag{2.5}$$

On the other hand, multiplying the second equation of (2.3) by $-\Delta v^\epsilon$ and integrating with respect to space, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v^\epsilon|^2\, dx = -\int_\Omega |\Delta v^\epsilon|^2\, dx - \int_\Omega |\nabla v^\epsilon|^2\, dx + \int_\Omega \nabla u^\epsilon \cdot \nabla v^\epsilon\, dx. \tag{2.6}$$

The expected result then follows by adding (2.5), (2.6) and integrating in time. ■

As a consequence of Lemma 2.10 we have the following useful inequality.

Corollary 2.11. *For $\epsilon \in [0,\epsilon_0]$ and $t \in [0, t_\epsilon^+)$, the solution (u^ϵ, v^ϵ) to (2.3) satisfies*

$$\int_\Omega \left(u^\epsilon(t)|\ln u^\epsilon(t)| + \frac{|\nabla v^\epsilon(t)|^2}{2} \right)\, dx + \int_0^t \int_\Omega \left(\frac{|\nabla u^\epsilon|^2}{u^\epsilon} + |\Delta v^\epsilon|^2 + |\nabla v^\epsilon|^2 \right)\, dxds \leqslant C_0,$$

where C_0 depends only on Ω and $F_0(u_0, v_0)$.

Proof. On the one hand, since

$$\begin{aligned} & r + \frac{1}{\epsilon}(1-\epsilon r)\ln(1-\epsilon r) \geqslant 0 \quad \text{for} \quad r \in \left[0, \frac{1}{\epsilon}\right], \\ & 2r\ln r \geqslant -\frac{2}{e} \quad \text{for} \quad r \in [0,1], \end{aligned}$$

we deduce from (2.2) that

$$\begin{aligned} F_\epsilon(u^\epsilon(t), v^\epsilon(t)) &\geqslant \int_\Omega \left(u^\epsilon(t)\ln u^\epsilon(t) - u^\epsilon(t) + \frac{|\nabla v^\epsilon(t)|^2}{2} \right)\, dx \\ &\geqslant \int_\Omega \left(u^\epsilon(t)|\ln u^\epsilon(t)| + \frac{|\nabla v^\epsilon(t)|^2}{2} \right)\, dx - \left(\|u_0\|_1 + \frac{2|\Omega|}{e} \right). \end{aligned}$$

On the other hand,

$$\frac{|\nabla u^\epsilon|^2}{u^\epsilon(1-\epsilon u^\epsilon)} \geqslant \frac{|\nabla u^\epsilon|^2}{u^\epsilon},$$

and Corollary 2.11 follows from Lemma 2.10 and the previous two inequalities.

Next, for $\epsilon = 0$, we may proceed as in [29, Lemma 2.1] to establish a connection between $F_0(u^0, v^0)$ and the right-hand side of (2.4). ■

Lemma 2.12. *If $\epsilon = 0$, the condition*

$$\sup_{t\in[0,t_0^+)} \|u^0\|_{N/2} \leqslant A \tag{2.7}$$

for some $A > 0$ ensures that the functional G given by

$$G(u,v) = \int_\Omega \left(u \ln\left(\frac{u}{\overline{u}}\right) + \frac{1}{2}\int_\Omega |\nabla v|^2 \right)\, dx$$

satisfies the following decay property

$$0 \leqslant G(u^0(t), v^0(t)) \leqslant G(u_0, v_0)e^{-\alpha t} \quad \textit{for} \quad t \in [0, t_0^+),$$

the positive constant α depending only on Ω and A.

Proof. Taking $\varphi = \frac{u^0}{\overline{u^0}}$ and applying Jensen's inequality with the probability measure $d\mu = \frac{\overline{u^0}}{\|\overline{u^0}\|_1} dx$ we get

$$\int_\Omega u^0 \ln\left(\frac{u^0}{\overline{u^0}}\right) dx = \|\overline{u^0}\|_1 \int_\Omega \varphi \ln \varphi \, d\mu \geqslant \left(\int_\Omega \varphi \, d\mu\right) \ln\left(\int_\Omega \varphi \, d\mu\right) = 0. \tag{2.8}$$

Next, recalling that

$$r \ln r - r + 1 \leqslant (r-1)^2 \quad \text{for} \quad r \geqslant 0$$

and taking into account (2.2), we obtain from the Sobolev, Poincaré and Hölder inequalities that

$$\begin{aligned}
\int_\Omega u^0(t) \ln\left(\frac{u^0(t)}{\overline{u^0}}\right) dx &\leqslant \overline{u^0} \int_\Omega \left[\frac{u^0(t)}{\overline{u^0}} - 1 + \left(\frac{u^0(t)}{\overline{u^0}} - 1\right)^2 \right] dx \\
&\leqslant \frac{1}{\overline{u^0}} \left\| u^0(t) - \overline{u^0} \right\|_2^2 \\
&\leqslant C \left\| \nabla \left(u^0(t) - \overline{u^0}\right) \right\|_{2N/(N+2)}^2 \\
&\leqslant C \left\| \sqrt{u^0(t)}\ \nabla \sqrt{u^0(t)} \right\|_{2N/(N+2)}^2 \\
&\leqslant C \left\| u^0(t) \right\|_{N/2} \left\| \nabla \sqrt{u^0(t)} \right\|_2^2 \\
&\leqslant AC \int_\Omega \frac{|\nabla u^0(t)|^2}{u^0(t)}\, dx.
\end{aligned}$$

Consequently

$$G(u^0(t), v^0(t)) \leqslant \frac{1}{\alpha} \int_\Omega \left(\frac{|\nabla u^0(t)|^2}{u^0(t)} + |\nabla v^0(t)|^2 \right) dx, \quad t \in [0, t_0^+).$$

We then obtain from Lemma 2.10 that

$$\frac{d}{dt} G(u^0, v^0) = \frac{d}{dt} F_0(u^0, v^0) \leqslant -\alpha G(u^0, v^0),$$

which completes the proof. ∎

Theorem 2.13. *The only non-negative stationary solutions to (2.1) in $W^{1,p_0}(\Omega)$ for $p_0 > N$ are the pairs (m, m) for $m \in [0, \infty)$.*

Proof. Assume that $(u^0, v^0) \in W^{1,p_0}(\Omega; \mathbb{R}^2)$ for $p_0 > N$ is a stationary solution to (2.1). Then $t_0^+ = +\infty$ by Theorem 2.9 and it follows from Lemma 2.12 that $0 \leqslant G(u^0, v^0) \leqslant G(u^0, v^0)e^{-\alpha t}$ for each $t \geqslant 0$, whence $G(u^0, v^0) = 0$. Consequently,

$$\nabla v^0 = 0 \quad \text{and} \quad u^0 \ln\left(\frac{u^0}{\overline{u}}\right) = 0 \quad \text{a.e. in} \quad \Omega$$

with $\overline{u} = \|u^0\|_1/|\Omega|$, from which we conclude that $u^0 = \overline{u}$ and v^0 is a constant. Taking into account the second equation in (2.1) implies that $(u^0, v^0) = (m, m)$ for some non-negative real number m. ∎

2.3. The two-dimensional case.

In this section, we assume that $N = 2$ and put $(u, v) = (u^0, v^0)$ to simplify the notations, (u^0, v^0) being the solution to (2.3) with $\epsilon = 0$ on $[0, t_0^+)$ given by Theorem 2.9. We recall that, thanks to Theorem 2.9, it is sufficient to establish L^∞-bounds for (u, v). The following lemma is a first step in that direction.

Lemma 2.14. *Let $p \geqslant 2$ and $T > 0$. Then there exists a positive constant $C_1(p, T)$ depending only on Ω, u_0, v_0, p, and T such that*

$$\|u(t)\|_p \leqslant C_1(p, T) \quad \textit{for} \quad t \in [0, t_0^+) \cap [0, T].$$

Proof. We first observe that Lemma 2.10 implies that

$$\int_0^t \int_\Omega |\Delta v|^2 \, dxdt \leqslant F_0(u_0, v_0) + \frac{|\Omega|}{e} \quad \text{for} \quad t \in [0, t_0^+). \tag{2.9}$$

We next multiply the first equation of (2.1) by $(p+1)u^p$, integrate with respect to the space variable and apply the Gagliardo-Nirenberg inequality (1). We thus obtain

$$\begin{aligned}
\frac{d}{dt}\left\|u^{(p+1)/2}\right\|_2^2 &= -\frac{4p}{p+1}\left\|\nabla(u^{(p+1)/2})\right\|_2^2 + p\left\|u^{p+1}\Delta v\right\|_1 \\
&\leqslant -2\left\|\nabla(u^{(p+1)/2})\right\|_2^2 + p\left\|u^{(p+1)/2}\right\|_4^2 \|\Delta v\|_2 \\
&\leqslant -2\left\|\nabla(u^{(p+1)/2})\right\|_2^2 + C\,p\left\|u^{(p+1)/2}\right\|_{1,2}\left\|u^{(p+1)/2}\right\|_2 \|\Delta v\|_2 \\
&\leqslant -\left\|\nabla(u^{(p+1)/2})\right\|_2^2 + \left\|u^{(p+1)/2}\right\|_2^2 + Cp^2\left\|u^{(p+1)/2}\right\|_2^2 \|\Delta v\|_2^2.
\end{aligned}$$

Owing to (2.9) we may apply the Gronwall lemma and complete the proof of Lemma 2.14. ∎

Theorem 2.15. *Let $N = 2$. If (u_0, v_0) are non-negative functions in $W^{1,p_0}(\Omega)$ for some $p_0 > 2$ then there exists a unique global in time smooth classical solution to (2.1). Moreover,*

$$\lim_{t\to+\infty}(u, v)(\cdot, t) = (\overline{u}, \overline{v}) \quad \textit{in} \quad C^2(\overline{\Omega}; \mathbb{R}^2) \quad \textit{with} \quad \overline{u} = \overline{v} = \frac{1}{|\Omega|}\int_\Omega u_0 \, dx.$$

Furthermore, the rate of convergence is exponential.

Outline of the proof. Owing to Lemma 2.14 we may proceed as in [60, Section 4] and use Moser's iteration technique [1] to show that, for every $T > 0$,

$$\|u(t)\|_\infty + \|v(t)\|_\infty \leqslant C(T) \quad \text{for} \quad t \in [0, t_0^+) \cap [0, T].$$

According to the global existence criterion from Theorem 2.9, we have thus shown that $t_0^+ = +\infty$. In addition,

$$(u, v) \in C(\overline{\Omega} \times [0, \infty); \mathbb{R}^2) \cap C^\infty(\overline{\Omega} \times (0, \infty); \mathbb{R}^2),$$

while LaSalle's invariance principle and (2.2) ensure that $(u(t), v(t))$ converges towards $(\overline{u}, \overline{v})$ as $t \to \infty$. For the rate of convergence we may apply Lemma 2.12 thanks to (2.2) and use the Csiszár-Kullback-Pinsker inequality (see, e.g., [14] and the references therein) to obtain

$$\frac{1}{2\overline{u}} \|u - \overline{u}\|_1^2 \leqslant G(u(t), v(t)) \leqslant G(u_0, v_0) e^{-\alpha t},$$

and hence the exponential convergence in $L^1(\Omega)$.

Since $\overline{u}$ is a constant and $\nabla \cdot (u \nabla v)$ is bounded, the exponential convergence in L^∞ may next be proved by Moser's iteration technique [1]. Parabolic estimates then yield the exponential convergence in $W^{2,p}(\Omega)$ for $p > n$. ■

2.4. Global weak solutions in higher space dimensions.

This section is devoted to the proofs of Theorems 1.3 and 1.4. Both are based on a compactness method. Namely, we shall prove that at least a subsequence of the classical solutions (u^ϵ, v^ϵ) to (2.3) converge in suitable topologies towards a (weak) solution to (2.1) as $\epsilon \to 0$. As a first step we deduce some bounds on (u^ϵ, v^ϵ) from Corollary 2.11 and Sobolev embeddings.

Lemma 2.16. *Let $T > 0$. The sequences $(u^\epsilon)_\epsilon$ and $(v^\epsilon)_\epsilon$ enjoy the following properties:*

(2.10) $\quad (u^\epsilon)_\epsilon$ *is bounded in* $L^{(N+2)/(N+1)}(0, T; W^{1,(N+2)/(N+1)}(\Omega))$,

(2.11) $\quad (u_t^\epsilon)_\epsilon$ *is bounded in* $L^1(0, T; C_0^1(\Omega)')$,

(2.12) $\quad (v^\epsilon)_\epsilon$ *is bounded in* $L^{(N+2)/N}(0, T; W^{2,(N+2)/N}(\Omega))$,

(2.13) $\quad (v^\epsilon)_\epsilon$ *is bounded in* $L^\infty(0, T; W^{1,2}(\Omega)) \cap L^2(0, T; W^{2,2}(\Omega))$,

(2.14) $\quad (v_t^\epsilon)_\epsilon$ *is bounded in* $L^{(N+2)/N}(\Omega \times (0, T))$,

(2.15) $\quad (u^\epsilon \nabla v^\epsilon)_\epsilon$ *is bounded in* $L^{(2N+4)/3N}(\Omega \times (0, T))$.

Proof. Consider $\epsilon \in (0, \epsilon_0)$. We first recall that

$$\|u^\epsilon(t)\|_1 \leqslant K \quad \text{for} \quad t \in [0, T] \tag{2.16}$$

by (2.2), where K denotes a constant independently of ϵ and T. Next, since $\nabla \sqrt{u^\epsilon} = \nabla u^\epsilon / (2\sqrt{u^\epsilon})$, we have from (2.16) and Corollary 2.11 that

$$\int_0^T \left\| \sqrt{u^\epsilon}(t) \right\|_{1,2}^2 \, dt \leqslant K.$$

The continuous embedding of $W^{1,2}(\Omega)$ in $L^{2N/(N-2)}(\Omega)$ then entails that

$$\int_0^T \|u^\epsilon(t)\|_{N/(N-2)} \ dt \leqslant K(T). \tag{2.17}$$

Interpolating between (2.16) and (2.17) we obtain

$$\int_0^T \|u^\epsilon(t)\|_{Np/(Np-2)}^p \ dt \leqslant K(T,p) \quad \text{for} \quad p \in [1,\infty]. \tag{2.18}$$

In particular, the choice $p = (N+2)/N$ gives

$$\int_0^T \int_\Omega (u^\epsilon)^{(N+2)/N} \ dxdt \leqslant K(T). \tag{2.19}$$

Next, by Corollary 2.11, (2.19) and the Hölder inequality, we have

$$\begin{aligned}
\int_0^T \int_\Omega |\nabla u^\epsilon|^{(N+2)/(N+1)} &\leqslant \left(\int_0^T \int_\Omega \frac{|\nabla u^\epsilon|^2}{u^\epsilon} \right)^{(N+2)/(2N+2)} \left(\int_0^T \int_\Omega (u^\epsilon)^{(N+2)/N} \right)^{N/(2N+2)} \\
&\leqslant K(T).
\end{aligned}$$

We have thus established (2.10). The bounds (2.12) and (2.14) follow from the second equation of (2.3), (2.19) and classical parabolic regularity results (see, [50]) while Corollary 2.11 and (2.2) ensure that (2.13) holds true. We then deduce from (2.13) and the Sobolev embedding that $(\nabla v^\epsilon)_\epsilon$ is bounded in both $L^\infty(0,T;L^2(\Omega))$ and $L^2(0,T;L^{2N/(N-2)}(\Omega))$, whence

$$\int_0^T \|\nabla v^\epsilon\|_{2Np/(Np-4)}^p \ dt \leqslant K(T,p) \quad \text{for} \quad p \in [2,\infty] \tag{2.20}$$

by interpolation. Combining this estimate for $p = 2(N+2)/N$ with (2.19) yields (2.15).

Consider finally $\phi \in C_0^1(\Omega)$. It follows from the first equation of (2.3) and Corollary 2.11 that

$$\begin{aligned}
\left| \int_\Omega u_t^\epsilon \phi dx \right| &\leqslant \int_\Omega |\nabla u^\epsilon| \ |\nabla \phi| \ dx + \int_\Omega u^\epsilon (1 - \epsilon u^\epsilon) |\nabla v^\epsilon| \ |\nabla \phi| \ dx \\
&\leqslant C \|\nabla u^\epsilon\|_{(N+2)/(N+1)} \|\nabla \phi\|_\infty + \|u^\epsilon\|_2 \|\nabla v^\epsilon\|_2 \|\nabla \phi\|_\infty \\
&\leqslant C(T) \left(\|\nabla u^\epsilon\|_{(N+2)/(N+1)} + \|u^\epsilon\|_2 \right) \|\nabla \phi\|_\infty.
\end{aligned}$$

Therefore,

$$\|u_t^\epsilon\|_{C_0^1(\Omega)'} \leqslant C(T) \left(\|\nabla u^\epsilon\|_{(N+2)/(N+1)} + \|u^\epsilon\|_2 \right),$$

and the right-hand side of the above inequality is bounded in $L^1(0,T)$ by (2.17) since $N/(N-2) \geqslant 2$ for $N = 3, 4$. The proof of Lemma 2.16 is then complete. ∎

We next turn to the relative compactness of the sequences $(u^\epsilon)_\epsilon$ and $(v^\epsilon)_\epsilon$. More specifically, we have the following result:

Lemma 2.17. *There are non-negative functions*

$$\begin{aligned}
u &\in L^{(N+2)/(N+1)}(0,T;W^{1,(N+2)/(N+1)}(\Omega)) \cap C([0,T];C_0^1(\Omega)'), \quad u(0) = u_0, \\
v &\in L^\infty(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega)), \quad v(0) = v_0,
\end{aligned}$$

and a subsequence of $(u^\epsilon)_\epsilon$ *and* $(v^\epsilon)_\epsilon$ *(not relabeled) such that*

$$\begin{aligned} u^\epsilon &\longrightarrow u \quad \textit{in} \quad L^p(\Omega \times (0,T)) \cap C([0,T]; C_0^1(\Omega)') \quad \textit{for} \quad p \in \left[1, \frac{N+2}{N}\right), \\ v^\epsilon &\longrightarrow v \quad \textit{in} \quad L^2(0,T; H^1(\Omega)) \cap C([0,T]; L^2(\Omega)), \end{aligned}$$

and

$$\int_\Omega (v(t) - v_0)\ \varphi\ dx + \int_0^t \int_\Omega (\nabla v \cdot \nabla \varphi + (v-u)\ \varphi)\ dxds = 0$$

for each $t \in [0,T]$ *and* $\varphi \in W^{1,\infty}(\Omega)$.

Proof. In view of (2.10) and (2.11), we see that $(u^\epsilon)_\epsilon$ is relatively compact in $L^{(N+2)/(N+1)}(\Omega \times (0,T))$ by the Aubin-Lions lemma [52, Théorème 5.1]. In fact we can strengthen this claim due to (2.19) and deduce that

$$(2.21) \qquad (u^\epsilon)_\epsilon \ \text{ is relatively compact in } \ L^p(\Omega \times (0,T)) \ \text{ for any } \ p \in \left[1, \frac{N+2}{N}\right).$$

Similarly, it follows from (2.13), (2.14), and [70, Corollary 4] that

$$(2.22) \qquad (v^\epsilon)_\epsilon \ \text{ is relatively compact in } \ C([0,T]; L^2(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega)).$$

Owing to (2.21) and (2.22) we easily obtain the convergences claimed in Lemma 2.17, the convergence of $(u^\epsilon)_\epsilon$ in $C([0,T]; C_0^1(\Omega)')$ being a consequence of (2.11), (2.16), and the Ascoli theorem. It is then straightforward to pass to the limit as $\epsilon \to 0$ in the second equation of (2.3) to deduce the last assertion of Lemma 2.17. ∎

It remains to pass to the limit as $\epsilon \to 0$ in the first equation of (2.3), the main difficulty being the nonlinear term $u^\epsilon(1-\epsilon u^\epsilon)\nabla v^\epsilon$. At this point the difference between $N=3$ and $N=4$ shows up: indeed, though we know that

$$(2.23) \qquad u^\epsilon(1-\epsilon u^\epsilon)\nabla v^\epsilon \longrightarrow u \nabla v \quad \text{a.e. in} \quad \Omega \times (0,T)$$

by Lemma 2.17 (after possibly extracting a further subsequence), we only have an L^1-bound for this term when $N=4$ by (2.15) and this is not sufficient to have strong convergence. Such a difficulty is not encountered when $N=3$ and we now complete the proof of Theorem 2.18.

Theorem 2.18. *Let* $N=3$. *If* (u_0, v_0) *are non-negative functions in* $W^{1,p_0}(\Omega)$ *for some* $p_0 > 3$, *then there exists a global weak solution* (u,v) *to (2.1) which satisfies also*

$$(u,v) \in L^{5/4}(0,T; W^{1,5/4}(\Omega; \mathbb{R}^2))$$

for any $T > 0$. *Moreover, recalling that* $\overline{u}$ *and* $\overline{v}$ *are defined in Theorem 2.15, we have*

$$\lim_{t \to +\infty} \left[\int_\Omega (u(t) - \overline{u})\ \ \phi\ dx + \|v(t) - \overline{v}\|_2 \right] = 0$$

for each $\phi \in L^\infty(\Omega)$.

Proof. According to (2.15), the sequence $(u^\epsilon(1-\epsilon u^\epsilon)\nabla v^\epsilon)_\epsilon$ is bounded in $L^{10/9}(\Omega\times(0,T))$ and thus weakly compact in $L^1(\Omega\times(0,T))$. Since it also converges a.e. in $\Omega\times(0,T)$ by (2.23), we are in a position to apply the Vitali theorem and conclude that

$$u^\epsilon(1-\epsilon u^\epsilon)\nabla v^\epsilon \longrightarrow u\nabla v \quad \text{in} \quad L^1(\Omega\times(0,T)).$$

In view of Lemma 2.17 it is then straightforward to let $\epsilon\to 0$ in the first equation of (2.3) and conclude that (u,v) is a weak solution to (2.1) in the sense of Definition 2.1.

We next turn to the convergence towards steady states. We first recall that the L^1-norms of (u,v) are given by (2.2). It follows from Lemma 2.17 and weak compactness arguments that we may pass to the limit as $\epsilon\to 0$ in the inequality stated in Corollary 2.11 to obtain that

$$\int_\Omega\left(u(t)|\ln u(t)|+\frac{|\nabla v(t)|^2}{2}\right)\,dx+\int_0^t\int_\Omega 4\left(\left|\nabla\sqrt{u}\right|^2+|\Delta v|^2+|\nabla v|^2\right)\,dxds\leqslant C_0. \tag{2.24}$$

Next, we take $0=t_0\leqslant t_1\leqslant ...\leqslant t_k$ with $t_k\to+\infty$ and we define

$$u_k(\cdot,t)=u(\cdot,t+t_k)-<u_0>,\quad t\in(0,1)$$

$$v_k(\cdot,t)=v(\cdot,t+t_k)-<v(t+t_k)>\quad t\in(0,1)$$

where $<\cdot>$ denotes the mean value. Therefore by the Dunford-Pettis Theorem and (2.24) we infer

$$(u_k(0),v_k(0))\to(u_\infty,v_\infty)\ \text{weak-}L^1(\Omega)\times L^2(\Omega). \tag{2.25}$$

Then, taking into account that

$$\left(\int_\Omega|\nabla u|dx\right)^2\leqslant C\int_\Omega\frac{|\nabla u|^2}{u}dx$$

and (2.24) we deduce

$$(\nabla u_k,\nabla v_k)\to(0,0)\ \text{strong in}\ L^2(0,1;L^1(\Omega))\times L^2(0,1;H^1(\Omega)). \tag{2.26}$$

Hence by the Poincare-Wirtinger inequality

$$(u_k,v_k)\to(0,0)\ \text{strong in}\ L^1(0,1;L^1(\Omega)). \tag{2.27}$$

On the other hand, using the definition of a weak solution, the embeddings $W^{1,1}(\Omega)$ in $L^{4/3}(\Omega)$, $W^{1,2}(\Omega)$ in $L^2(\Omega)$ and (2.26) we prove

$$\lim_{k\to+\infty}\sup_{t\in[0,1]}\left|\int_\Omega(u_k(t)-u_k(0))\varphi dx\right|=0\quad\forall\varphi\in W^{1,\infty}(\Omega) \tag{2.28}$$

$$\lim_{k\to+\infty}\sup_{t\in[0,1]}\left|\int_\Omega(v_k(t)-v_k(0))\varphi dx\right|=0\quad\forall\varphi\in W^{1,\infty}(\Omega). \tag{2.29}$$

In view of (2.28), (2.29) and a.e. convergence coming from (2.27) we can identify the limits in (2.25) as zero. ∎

Theorem 2.19. *Let $N=4$. If (u_0,v_0) are non-negative functions in $W^{1,p_0}(\Omega)$ for some $p_0>4$, then there exists a global weak solution (u,v) to (2.1). Moreover,*

$$\lim_{t\to+\infty}\left[\int_\Omega (u(t)-\overline{u})\ \phi\ dx+\|v(t)-\overline{v}\|_2\right]=0$$

for each $\phi\in L^\infty(\Omega)$.

Proof. In that case the weak compactness in $L^1(\Omega\times(0,T))$ of $(u^\epsilon(1-\epsilon u^\epsilon)\nabla v^\epsilon)_\epsilon$ is no longer guaranteed by (2.15) and we thus have to find an alternative way to prove it. To this end, we aim at applying the Dunford-Pettis theorem and first notice that $(u^\epsilon)_\epsilon$ actually enjoys a stronger property than (2.2), namely

$$\sup_{t\in[0,T]}\left\{\int_\Omega u^\epsilon(t)|\ln u^\epsilon(t)|\ dx\right\}\leqslant C_0 \tag{2.30}$$

by Corollary 2.11. Thanks to this property, we can establish the uniform integrability of $(u^\epsilon(1-\epsilon u^\epsilon)\nabla v^\epsilon)_\epsilon$. Indeed, let $E\subset\Omega\times(0,T)$ and $R>1$. We obtain from (2.18) with $p=2$, (2.20) with $p=2$, and (2.30) that

$$\begin{aligned}
&\int\int_E u^\epsilon(1-\epsilon u^\epsilon)\nabla v^\epsilon\ dxdt\leqslant\int\int_E u^\epsilon\nabla v^\epsilon\ dxdt\\
\leqslant\ & R\int\int_E\nabla v^\epsilon\ dxdt+\int\int_E u^\epsilon\mathbf{1}_{(R,\infty)}(u^\epsilon)\nabla v^\epsilon\ dxdt\\
\leqslant\ & CR|E|^{1/2}+\int_0^T\left\|u^\epsilon\mathbf{1}_{(R,\infty)}(u^\epsilon)\right\|_{4/3}\|\nabla v^\epsilon\|_4\ dt\\
\leqslant\ & CR|E|^{1/2}+\sup_{t\in[0,T]}\left\{\left\|u^\epsilon(t)\mathbf{1}_{(R,\infty)}(u^\epsilon(t))\right\|_1\right\}\|u^\epsilon\|_{L^2(0,T;L^{4/3}(\Omega))}\|\nabla v^\epsilon\|_{L^2(0,T;L^4(\Omega))}\\
\leqslant\ & CR|E|^{1/2}+\frac{C}{\ln R}\sup_{t\in[0,T]}\{\|u^\epsilon(t)|\ln u^\epsilon(t)|\|_1\}\\
\leqslant\ & CR|E|^{1/2}+\frac{C}{\ln R},
\end{aligned}$$

where

$$\mathbf{1}_{(R,+\infty)}(u):=\begin{cases}u & \text{if } u>R,\\ 0 & \text{if } u\leqslant R.\end{cases}$$

Letting first $|E|\to0$ and then $R\to\infty$ we end up with

$$\lim_{|E|\to0}\sup_{\epsilon\in(0,\epsilon_0)}\left\{\int\int_E u^\epsilon(1-\epsilon u^\epsilon)\nabla v^\epsilon\ dxdt\right\}=0,$$

which ensures the weak compactness of $(u^\epsilon(1-\epsilon u^\epsilon)\nabla v^\epsilon)_\epsilon$ in $L^1(\Omega\times(0,T))$ by the Dunford-Pettis theorem. Recalling (2.23) we may apply again the Vitali theorem to conclude that

$$u^\epsilon(1-\epsilon u^\epsilon)\nabla v^\epsilon\longrightarrow u\nabla v\quad\text{in}\quad L^1(\Omega\times(0,T)).$$

We then argue as in the proof of Theorem 2.18 to show that (u,v) is a weak solution to (2.1) in the sense of Definition 2.1 and that $(u(t),v(t))$ converges towards $(\overline{u},\overline{v})$ in the expected topologies. ∎

CHAPTER 3

The simplified invasion model

In this chapter we perform an extensive study of existence, uniqueness and asymptotic behavior for (3.1)-(3.5) (see below) in 2-dimensional domains. Moreover, we show, under the assumption $w \in W^{1,\infty}(\Omega)$, all the possible positive solutions to the stationary problem associated to (3.1)-(3.5).

3.1. Preliminaries

Let $\Omega \subset \mathbb{R}^N$ be a domain with smooth boundary $\partial\Omega \in C^{l+2}\left(\mathbb{R}^{N-1}\right)$ and $T > 0$. We consider the cylindrical domain denoted by $Q_T = \Omega \times (0,T)$ with lateral surface $\partial Q_T = \partial\Omega \times (0,T)$.

In this chapter we are going to study a initial-boundary problem for a parabolic-degenerate system of equations which have the general form:

$$\frac{\partial u}{\partial t} = a\Delta u - b\nabla \cdot (uw^{-\alpha}\nabla w) + f(u,w) \qquad x \in \Omega, \quad t \in \mathbb{R}_+ \tag{3.1}$$

$$\frac{\partial w}{\partial t} = -kw^{\beta}u \qquad x \in \Omega, \quad t \in \mathbb{R}_+ \tag{3.2}$$

$$\frac{\partial u}{\partial n} - uw^{-\alpha}\frac{\partial w}{\partial n} = 0 \qquad x \in \partial\Omega, \quad t \in \mathbb{R}_+ \tag{3.3}$$

$$u(x,0) = u_0(x) \geqslant 0 \qquad t \in \mathbb{R}_+ \tag{3.4}$$

$$w(x,0) = w_0(x) > 0 \qquad t \in \mathbb{R}_+ \tag{3.5}$$

where a, b and k are positive constants, $\beta \geqslant 1$, $\beta \geqslant \alpha \geqslant 0$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and n denotes the unit outward normal vector of $\partial\Omega$.

We are using in this chapter the standard notation of functional spaces. $L^p(\Omega)$ and $W^{m,p}(\Omega)$ with $1 \leqslant p \leqslant \infty$, $m \geqslant 1$ are the Lebesgue and Sobolev spaces of functions in Ω, respectively. For a general Banach space X, its norm is denoted by $\|\cdot\|_X$. The space $L^p(0,T;X)$ is the Banach space of all Bochner measurable functions $f : (0,t) \to X$ such that $\|f\|_X \in L^p(0,T)$.

For $0 < l < 1$ we denote by $C^{l+i,l/2+i/2}(\overline{Q}_T)$, $i = 1,2$ the Hölder space of exponents $l+i$ and $l/2+i/2$ with respect to x and t respectively of continuous and bounded functions $\{f(x,t)\}$ defined on $\overline{Q}_T$, provided with continuous and bounded derivatives $\{D_t^r D_x^s f(x,t)\}$ for $2r + |s| \leqslant i$. It is endowed with the norm given by

$$|f|_{Q_T}^{(l+i)} := \sum_{2r+|s|=i} \langle D_t^r D_x^s f\rangle_{x,Q_T}^{(l)} + \sum_{\max\{0,i-1\}\leqslant 2r+|s|\leqslant i} \langle D_t^r D_x^s f\rangle_{t,Q_T}^{((l-2r-|s|)/2+i/2)} + \\ + \sum_{0\leqslant 2r+|s|\leqslant i} \max_{(x,t)\in\overline{Q}_T} |D_t^r D_x^s f|$$

where

$$\langle f\rangle_{x,Q_T}^{(l)} := \sup_{\substack{(x,t),(x',t)\in\overline{Q}_T \\ |x-x'|\leqslant\rho_0}} \frac{|f(x,t)-f(x',t)|}{|x-x'|^l}, \qquad \langle f\rangle_{t,Q_T}^{(l/2)} := \sup_{\substack{(x,t),(x,t')\in\overline{Q}_T \\ |t-t'|\leqslant\rho_0}} \frac{|f(x,t)-f(x,t')|}{|t-t'|^{l/2}}.$$

The norm depends on ρ_0, but changing this constant leads to an equivalent norm.

Throughout this chapter we denote by C, C_i $(i = 1,2,...)$ positive constants which are independent of time, but we shall indicate explicitly on which other parameters they are dependent, if it will be the case. The constants C are not necessarily the same at different occurrences.

Some properties for the norms in the Hölder spaces which will be used often in the next section are given below. Since the proofs are standard, but tedious, we omit the details.

Lemma 3.1. *If $f(x,t) \in C^{l+2,l/2+1}(\overline{Q}_T)$, $0 < l < 1$, then we have:*

(i) $\frac{\partial f}{\partial t} \in C^{l,l/2}(\overline{Q}_T)$,

(ii) $\frac{\partial f}{\partial x_j} \in C^{l+1,l/2+1/2}(\overline{Q}_T)$, $j = 1,...,N$,

(iii) $\Delta f \in C^{l,l/2}(\overline{Q}_T)$,

(iv) $\frac{\partial f}{\partial n} \in C^{l+1,l/2+1/2}(\overline{Q}_T)$, *where n denotes the unit outward normal vector of $\partial\Omega$.*

Lemma 3.2. *If $f(x,t) \in C^{l+2,l/2+1}(\overline{Q}_T)$, $0 < l < 1$, then $F(x,t) = \int_0^t f(x,s)\,ds \in C^{l+2,l/2+1}(\overline{Q}_T)$. Moreover,*

$$|F|_{Q_T}^{(l+2)} \leqslant C \max\left\{T, T^{(1-l)/2}\right\} |f(x,t)|_{Q_T}^{(l+2)} + |f(x,0)|_{Q_T}^{(l)}. \tag{3.6}$$

Lemma 3.3. *If $f, g \in C^{l+i,l/2+i/2}(\overline{Q}_T)$, $0 < l < 1$, then $fg \in C^{l+i,l/2+i/2}(\overline{Q}_T)$ and*

$$|fg|_{Q_T}^{(l+i)} \leqslant C\, |f|_{Q_T}^{(l+i)}\, |g|_{Q_T}^{(l+i)} \tag{3.7}$$

for $i = 0,1,2$.

Lemma 3.4. *([67], Lemma 1) Let $\varphi, \psi : Q_T \to K \subset \mathbb{R}^N$, where K is a compact in $\mathbb{R}^N$, be two functions in $\left(C^{l+2,l/2+1}(\overline{Q}_T)\right)^N$ and let $f \in C^3(K)$. Then $f\circ\varphi$ and $f\circ\psi$ are in $C^{l+2,l/2+1}(\overline{Q}_T)$ and we have*

$$|f\circ\varphi - f\circ\psi|_{Q_T}^{(l+2)} \leqslant \Phi\, \|f\|_{C^3(K)} \left(|\varphi-\psi|_{Q_T}^{(l+2)}\right)^\gamma \tag{3.8}$$

where $\gamma = \min\{l/2, 1-l\}$ and $\Phi = \Phi(|\varphi|_{Q_T}^{(l+2)}, |\psi|_{Q_T}^{(l+2)})$ is an increasing function on both its arguments.

The remaining of this section is devoted to some general results for the existence of solutions for parabolic equations. We consider the problem:

$$\frac{\partial u}{\partial t} - \Delta u + \sum_{i=1}^{N} a_i(x,t)\frac{\partial u}{\partial x_i} + a(x,t)u = F(x,t) \qquad (x,t) \in Q_T \tag{3.9}$$

$$\frac{\partial u}{\partial n} - u\frac{\partial g}{\partial n}(x,t) = G(x,t) \qquad (x,t) \in \partial\Omega_T \tag{3.10}$$

$$u(x,0) = u_0(x) \qquad x \in \Omega \tag{3.11}$$

Let us remark that, if we make the change of variables

$$v(x,t) = u(x,t)e^{-g(x,t)}$$

the system (3.9)-(3.11) becomes:

$$\frac{\partial v}{\partial t} - \Delta v + \sum_{i=1}^{N} b_i(x,t)\frac{\partial v}{\partial x_i} + b(x,t)v = \widetilde{F}(x,t) \qquad (x,t) \in Q_T \tag{3.12}$$

$$\frac{\partial v}{\partial n} = \widetilde{G}(x,t) \qquad (x,t) \in \partial\Omega_T \tag{3.13}$$

$$v(x,0) = v_0(x) \qquad x \in \Omega \tag{3.14}$$

where the coefficients are given by:

$$b_i(x,t) = a_i(x,t) - 2\frac{\partial g}{\partial x_i}(x,t), \qquad 1 \leqslant i \leqslant N \tag{3.15}$$

$$b(x,t) = a(x,t) + \frac{\partial g}{\partial t}(x,t) - \Delta g + \sum_{i=1}^{N} a_i(x,t)\frac{\partial g}{\partial x_i}(x,t) - \sum_{i=1}^{N}\left(\frac{\partial g}{\partial x_i}(x,t)\right)^2 \tag{3.16}$$

$$\widetilde{F}(x,t) = F(x,t)e^{-g(x,t)} \tag{3.17}$$

$$\widetilde{G}(x,t) = G(x,t)e^{-g(x,t)} \tag{3.18}$$

$$v_0(x) = u_0(x)e^{-g(x,0)} \tag{3.19}$$

Theorem 3.5. *([66], Theorem II.2) Let $0 < l < 1$ and $\Omega \subset \mathbb{R}^N$ be a domain with the boundary $\partial\Omega \in C^{l+2}$ and $0 < T < \infty$. We suppose that the following hypothesis are satisfied:*

- *the coefficients $b_i(x,t)$ $(1 \leqslant i \leqslant N)$, $b(x,t)$ belong to the space $C^{l,l/2}(\overline{Q}_T)$;*
- *$\widetilde{F}(x,t) \in C^{l,l/2}(\overline{Q}_T)$, $\widetilde{G}(x,t) \in C^{l+1,l/2+1/2}(\overline{\partial\Omega}_T)$ and $v_0(x) \in C^{l+2}(\overline{\Omega})$;*
- *the compatibility condition $\frac{\partial v}{\partial n}(x,0) = \widetilde{G}(x,0)$ is satisfied for every $x \in \partial\Omega$.*

Then the problem (3.12)-(3.14) has a unique solution $v(x,t) \in C^{l+2,l/2+1}(\overline{Q}_T)$ which verifies

$$|v|^{(l+2)}_{Q_T} \leqslant \Theta\left(\left|\widetilde{F}\right|^{(l)}_{Q_T} + \left|\widetilde{G}\right|^{(l+1)}_{\partial\Omega_T} + |v_0|^{(l+2)}_{\Omega}\right) \tag{3.20}$$

where $\Theta = \Theta(T, \mu(T))$ is an increasing function on T and on the quantity

$$\mu(T) = \sum_{i=1}^{N} |b_i(x,t)|^{(l)}_{Q_T} + |b(x,t)|^{(l)}_{Q_T}\,.$$

Theorem 3.6. *Let $0 < l < 1$ and $\Omega \subset \mathbb{R}^N$ be a domain with the boundary $\partial\Omega \in C^{l+2}$ and $0 < T < \infty$. We suppose that the following hypothesis are satisfied:*

- *the coefficients $a_i(x,t)$ $(1 \leqslant i \leqslant N)$, $a(x,t)$ belong to the space $C^{l,l/2}(\overline{Q}_T)$;*
- *$F(x,t) \in C^{l,l/2}(\overline{Q}_T)$, $G(x,t) \in C^{l+1,l/2+1/2}(\overline{\partial\Omega}_T)$, $g(x,t) \in C^{l+2,l/2+1}(\overline{\partial\Omega}_T)$ and $u_0(x) \in C^{l+2}(\overline{\Omega})$;*
- *the compatibility condition $\frac{\partial u_0}{\partial n} - u_0\frac{\partial g}{\partial n}(x,0) = G(x,0)$ is satisfies for every $x \in \partial\Omega$.*

Then the problem (3.9)-(3.11) has a unique solution $u(x,t) \in C^{l+2,l/2+1}(\overline{Q}_T)$ which satisfies

$$|u|_{Q_T}^{(l+2)} \leqslant \Psi\left(|F|_{Q_T}^{(l)} + |G|_{\partial\Omega_T}^{(l+1)} + |u_0|_{\Omega}^{(l+2)}\right) \tag{3.21}$$

where $\Psi = \Psi\left(T, |g|_{Q_T}^{(l+2)}, \mu(T)\right)$ is an increasing function in T, in $|g|_{Q_T}^{(l+2)}$ and in the quantity

$$\mu(T) = \sum_{i=1}^{N} |b_i(x,t)|_{Q_T}^{(l)} + |b(x,t)|_{Q_T}^{(l)}$$

where $b_i(x,t)$ $(1 \leqslant i \leqslant N)$, $b(x,t)$ are given by (3.15), (3.16).

Proof. The existence and the uniqueness of the solution is proved in [48], Chapter IV, Theorem 5.3. The only thing that we want to point out is the increasing dependence of the function Ψ on its arguments.

From Lemma 3.3 we obtain

$$|u(x,t)|_{Q_T}^{(l+2)} = \left|v(x,t)e^{g(x,t)}\right|_{Q_T}^{(l+2)} \leqslant C\,|v(x,t)|_{Q_T}^{(l+2)}\left|e^{g(x,t)}\right|_{Q_T}^{(l+2)}$$

Now, taking into account (3.20) and Lemma 3.4, we obtain immediately the relation (3.21).

■

This chapter is organized as follows.

In Section 2, the proof of local existence in time and uniqueness of classical solution is accomplished by applying a fixed point argument in a suitable functional space. In order to prove the global existence in time of the classical solutions, in Section 4, we establish a priori bounds of the solutions. The last section is devoted to study the steady-state problem and the asymptotic behaviour of the solutions.

3.2. Local existence in time and uniqueness of classical solutions

In order to simplify the presentation of the results, we consider in what follows the case $\alpha = 0$, $\beta = 1$. The more general case $\beta \geqslant 1$, $\beta \geqslant \alpha > 0$ can be treated similarly, the estimates being more tedious. We consider, without loss of generality, the normalized system (3.1)-(3.5), which means $a = b = k = 1$, with a logistic growing source term, more precisely

$$\frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \nabla w) + \delta u(1-u) \qquad x \in \Omega, \quad t \in \mathbb{R}_+ \tag{3.22}$$

$$\frac{\partial w}{\partial t} = -wu \qquad x \in \Omega, \quad t \in \mathbb{R}_+ \tag{3.23}$$

$$\frac{\partial u}{\partial n} - u\frac{\partial w}{\partial n} = 0 \qquad x \in \partial\Omega, \quad t \in \mathbb{R}_+ \tag{3.24}$$

$$u(x,0) = u_0(x) \geqslant 0 \qquad t \in \mathbb{R}_+ \tag{3.25}$$

$$w(x,0) = w_0(x) > 0 \qquad t \in \mathbb{R}_+ \tag{3.26}$$

where $\delta \geqslant 0$.

In this Section we follow the arguments of Rascle [66], [67], but because in our case the boundary condition is different and the function f does not satisfy the condition (1.5), we briefly give the proof for the local existence for the sake of completeness.

Let us remark that, we can rewrite the initial problem (3.22)-(3.26):

$$\frac{\partial u}{\partial t} = \nabla \cdot \left(\nabla u - u \cdot \nabla \left(w_0 e^{-\varphi}\right)\right) + \delta u \left(1 - \frac{\partial \varphi}{\partial t}\right) \tag{3.27}$$

$$\frac{\partial u}{\partial n} = u \frac{\partial}{\partial n}\left(w_0 e^{-\varphi}\right) \tag{3.28}$$

$$u(x,0) = u_0(x) \tag{3.29}$$

$$\varphi = \int_0^t u(x,s)ds \tag{3.30}$$

We consider now the following linear problem in the variable u

$$\frac{\partial u}{\partial t} = \Delta u - \sum_{i=1}^{N} a_i(x,t)\frac{\partial u}{\partial x_i} - a(x,t)u \qquad x \in \Omega, \quad t \in \mathbb{R}_+ \tag{3.31}$$

$$\frac{\partial u}{\partial n} = u\frac{\partial g}{\partial n} \qquad x \in \partial\Omega, \quad t \in \mathbb{R}_+ \tag{3.32}$$

$$u(x,0) = u_0(x) \geqslant 0 \qquad t \in \mathbb{R}_+ \tag{3.33}$$

where the coefficients are given by

$$g(x,t) = \left(w_0(x)e^{-\varphi(x,t)}\right) \tag{3.34}$$

$$a_i(x,t) = \frac{\partial}{\partial x_i}\left(w_0(x)e^{-\varphi(x,t)}\right) = \frac{\partial g}{\partial x_i} \tag{3.35}$$

$$\begin{aligned} a(x,t) &= \Delta\left(w_0(x)e^{-\varphi(x,t)}\right) - \delta\left(1 - \frac{\partial \varphi}{\partial t}(x,t)\right) = \\ &= \Delta g - \delta\left(1 + g^{-1}\frac{\partial g}{\partial t}\right) \end{aligned} \tag{3.36}$$

Theorem 3.7. *Let $0 < l < 1$, $\Omega \subset \mathbb{R}^N$ be a domain with C^{l+2} boundary $\partial\Omega$ and $0 < T < \infty$. We suppose that the following hypothesis are satisfied:*

- $\phi \in C^{l+2,l/2+1}(\overline{Q}_T)$, $w_0 \in C^{l+2}(\overline{\Omega})$, $u_0 \in C^{l+2}(\overline{\Omega})$, $u_0 \geqslant 0$, $w_0 \geqslant 0$, $u_0 \neq 0$, $w_0 \neq 0$;
- *the compatibility condition* $\frac{\partial u_0}{\partial n}(x) = u_0(x)\frac{\partial g}{\partial n}(x,0)$ *is satisfied for every* $x \in \partial\Omega$.

Then the problem (3.31)-(3.33) has a unique positive solution $u(x,t) \in C^{l+2,l/2+1}(\overline{Q}_T)$ *which verifies*

$$|u|_{Q_T}^{(l+2)} \leqslant \Psi \, |u_0|_{\Omega}^{(l+2)} \tag{3.37}$$

where $\Psi = \Psi\left(T, |g|_{Q_T}^{(l+2)}, \mu(T)\right)$ *is an increasing function in* T*, in* $|g|_{Q_T}^{(l+2)}$ *and in the quantity*

$$\mu(T) = \sum_{i=1}^{N} |a_i|_{Q_T}^{(l)} + \left| \frac{\partial g}{\partial t} - \delta \left(1 + g^{-1}\frac{\partial g}{\partial t}\right)\right|_{Q_T}^{(l)} \tag{3.38}$$

Proof. Taking into account the properties of the norm in Hölder spaces (see Lemma 3.1, Lemma 3.4), we have

$$\begin{aligned} a_i(x,t) &\in C^{l,l/2}(\overline{Q}_T), \qquad i = 1, ..., N \\ a(x,t) &\in C^{l,l/2}(\overline{Q}_T) \\ g(x,t) &\in C^{l+2,l/2+1}(\overline{Q}_T) \end{aligned}$$

so by Theorem 3.6 we obtain that the problem (3.31)-(3.33) has a unique solution $u(x,t) \in C^{l+2,l/2+1}(\overline{Q}_T)$. Moreover, taking into account (3.35), (3.36), this solution verifies (3.37).

The positivity of the solution is a consequence of the maximum principle for linear parabolic equations. ■

We shall prove now the local existence of the solution for the problem (3.1)-(3.5) using a fixed point argument. We consider the set

$$X(T,\sigma) = \left\{\phi \in C^{l+2,l/2+1}(\overline{Q_T});\ \ |\phi|_{QT}^{(l+2)} \leqslant \sigma,\ \ \phi \geqslant 0,\ \ \phi \neq 0,\ \ \phi(\cdot,0) = 0\right\} \tag{3.39}$$

where σ is some positive constant. We define the following operators

$$\begin{aligned} &S : X \to C^{l+2,l/2+1}(\overline{Q}_T) \\ &S(\phi) = u \end{aligned}$$

where u is the unique solution of the problem (3.31)-(3.33), and

$$\begin{aligned} &R : C^{l+2,l/2+1}(\overline{Q_T}) \to C^{l+2,l/2+1}(\overline{Q}_T) \\ &R(u) = \varphi \end{aligned}$$

where φ is given by the relation (3.30).

Let us observe that, in order to find a solution of the problem (3.27)-(3.30), it is enough to find a fixed point for the application

$$\begin{aligned} &R \circ S : X \to C^{l+2,l/2+1}(\overline{Q}_T) \\ &(R \circ S)(\phi) = R(u) = \varphi \end{aligned}$$

Theorem 3.8. *Let* $0 < l < 1$*,* $\Omega \subset \mathbb{R}^N$ *be a domain with* C^{l+2} *boundary* $\partial\Omega$*. We consider satisfied the hypothesis of Theorem 3.7 and, moreover, we pick* $\sigma > 0$ *such that* $|u_0|_{\Omega}^{(l)} < \sigma/2$*. Then for every* $\zeta > 0$ *there exists* $T_0 > 0$ *such that, for all* $\tau \in (0, T_0]$ *the following properties are true:*

(i) the closed set $X(\tau,\sigma)$ is invariant with respect to $R\circ S$;

(ii) the operator $R\circ S$ satisfies the following inequality in $X(\tau,\sigma)$ with respect the norm $|\cdot|_{Q_T}^{(l+2)}$:

$$|(R\circ S)(\phi)-(R\circ S)(\psi)|_{Q_\tau}^{(l+2)} \leqslant \zeta\left(|\phi-\psi|_{Q_\tau}^{(l+2)}\right)^{\gamma} \tag{3.40}$$

where $\gamma=\min\{l/2,1-l\}$. Therefore, $R\circ S$ has a unique fixed point ϕ in $X(\tau,\sigma)$.

Proof. (i) Let $T>0$, taking into account Lemma 3.2, for every $0<\tau\leqslant T$, we have

$$|(R\circ S)(\phi)|_{Q_\tau}^{(l+2)} = |\varphi|_{Q_\tau}^{(l+2)} \leqslant C\max\left\{\tau^{(1-l)/2},\tau\right\}|u|_{Q_\tau}^{(l+2)}+|u_0|_{Q_\tau}^{(l)} \tag{3.41}$$

where C is a constant independent on τ.

Because $u(x,t)$ is the unique solution of problem (3.31)-(3.33) and taking into account Theorem 3.7 and relation (3.37), we obtain from (3.41)

$$\begin{aligned}|(R\circ S)(\phi)|_{Q_\tau}^{(l+2)} \leqslant{}& C\max\left\{\tau^{(1-l)/2},\tau\right\}\Psi\left(\tau,|g(x,t)|_{Q_\tau}^{(l+2)},\mu(\tau)\right)|u_0|_{\Omega}^{(l+2)}+\\ &+|u_0|_{Q_\tau}^{(l)}\end{aligned} \tag{3.42}$$

Now, in order to estimate the function $\Psi\left(\tau,|g|_{Q_\tau}^{(l+2)},\mu(\tau)\right)$ which appears in (3.42), first we estimate $\mu(\tau)$. We obtain, taking into account Lemma 3.1,

$$\mu(\tau)\leqslant C\,|g|_{Q_\tau}^{(l+2)}+\delta\,|\phi|_{Q_\tau}^{(l+2)}+\delta \tag{3.43}$$

where C is a constant independent on τ. The norm $|g|_{Q_\tau}^{(l+2)}$ can be estimated using Lemma 3.4 and finally we obtain

$$|g|_{Q_\tau}^{(l+2)}\leqslant C_1\left(|w_0(x)|_{Q_\tau}^{(l+2)}+\sigma\right)^{\gamma} \tag{3.44}$$

where $C_1=C_1(|w_0(x)|_{Q_\tau}^{(l+2)},\sigma)$ and $\gamma=\min\{l/2,1-l\}$.

We obtain from (3.43) and (3.44)

$$\mu(\tau)\leqslant C_2+\delta\sigma+\delta \tag{3.45}$$

where $C_2=C_2\left(|w_0(x)|_{Q_\tau}^{(l+2)},\sigma\right)$.

From Theorem 3.7 we know that the function Ψ is increasing on τ, $|g|_{Q_\tau}^{(l+2)}$ and $\mu(\tau)$, so we obtain from (3.44) and (3.45) for $0<\tau\leqslant T$

$$\Psi\left(\tau,|g(x,t)|_{Q_\tau}^{(l+2)},\mu(\tau)\right)\leqslant\Psi\left(\tau,C_1\left(|w_0(x)|_{Q_\tau}^{(l+2)}+\sigma\right)^{\gamma},C_2+\delta\sigma+\delta\right)=:\chi(\sigma) \tag{3.46}$$

Finally, from (3.42), we obtain

$$\begin{aligned}|(R\circ S)(\phi)|_{Q_\tau}^{(l+2)}&\leqslant C\max\left\{\tau^{(1-l)/2},\tau\right\}\chi(\sigma)\,|u_0|_{\Omega}^{(l+2)}+|u_0|_{Q_\tau}^{(l)}<\\ &<C\max\left\{\tau^{(1-l)/2},\tau\right\}\chi(\sigma)\,|u_0|_{\Omega}^{(l+2)}+\frac{1}{2}\sigma\end{aligned}$$

It follows that for $\tau>0$ sufficiently small $X(\tau,\sigma)$ is invariant with respect to $R\circ S$. Let $T_1>0$ be sufficiently small, such that, for all $0<\tau\leqslant T_1$, $X(\tau,\sigma)$ is invariant with respect to $R\circ S$.

(*ii*) Let $\phi, \overline{\phi} \in X(T_1, \sigma)$ and

$$\begin{aligned} U &= R(u) = (R \circ S)(\phi) \\ \overline{U} &= R(\overline{u}) = (R \circ S)(\psi) \end{aligned}$$

It is easy to see that the function $z = u - \overline{u}$ satisfies the problem

$$\begin{aligned} &\frac{\partial z}{\partial t} = \Delta z - \sum_{i=1}^{N} a_i(x,t)\frac{\partial z}{\partial x_i} - a(x,t)z + \overline{F}(x,t) \\ &\frac{\partial z}{\partial n} = z\frac{\partial g}{\partial n} + \overline{G}(x,t) \\ &z(x,0) = 0 \end{aligned}$$

where

$$\begin{aligned} \overline{F}(x,t) &= \nabla\overline{u} \cdot \nabla\left(w_0 e^{-\phi} - w_0 e^{-\psi}\right) + \\ &\quad + \overline{u}\Delta\left(w_0 e^{-\phi} - w_0 e^{-\psi}\right)\delta\overline{u}\frac{\partial}{\partial t}(\phi - \psi) \\ \overline{G}(x,t) &= \overline{u}\frac{\partial}{\partial n}\left(w_0 e^{-\phi} - w_0 e^{-\psi}\right) \end{aligned}$$

Let us notice that

$$\overline{G}(x,0) = v(x,0)\frac{\partial}{\partial n}\left(w_0 e^{-\phi(x,0)} - w_0 e^{-\psi(x,0)}\right) = 0$$

so the function $z(x,t) = (u - \overline{u})(x,t)$ satisfies the compatibility condition

$$\frac{\partial z}{\partial n}(x,0) - z(x,0)\frac{\partial g}{\partial n} = G(x,0)$$

We obtain, taking into account the Theorem 3.6

$$|(R \circ S)(\phi) - (R \circ S)(\psi)|_{Q_\tau}^{(l+2)} \leqslant C_3 \max\left\{\tau^{(1-l)/2}, \tau\right\} \Psi(\sigma) |u_0|_{Q_\tau}^{(l+2)} \left(|\phi - \psi|_{Q_\tau}^{(l+2)}\right)^\gamma$$

where $\gamma = \min\{l/2, 1 - l\}$ and $C_3 = C_3(\sigma)$. By taking τ sufficiently small the inequality (3.40) follows. We choose $T_0 < T_1$ such that (*i*) and (*ii*) are fulfilled for all $\tau \in (0, T_0]$.

Next we define the following two sequences

$$\begin{aligned} u_n &= S(\varphi_n) \\ \varphi_{n+1} &= R(u_n) = (R \circ S)(\varphi_n) \end{aligned}$$

where $\varphi_0 \in X(T_0, \sigma)$. Then, from above considerations, the sequence $(\varphi_n)_{n \in \mathbf{N}}$ is a Cauchy sequence, so it converges to an element φ, which is a fixed point of $R \circ S$. The inequality (3.40) implies the uniqueness of this fixed point.

From the continuity of the application S we obtain that the sequence $(u_n)_{n \in \mathbf{N}}$ converges to $u = S(\varphi)$. It is easy to see that (u, φ) is the unique solution of the problem (3.27)-(3.30) on the interval $[0, T_0]$. ■

Theorem 3.9. *Let $0 < l < 1$, $\Omega \subset \mathbb{R}^N$ be a domain with C^{l+2} boundary $\partial\Omega$. If $(u_0, w_0) \in \left(C^{l+2}(\overline{\Omega})\right)^2$, $u_0 \geqslant 0$, $w_0 \geqslant 0$, $u_0 \neq 0$, $w_0 \neq 0$ and the compatibility condition $\frac{\partial u_0}{\partial n} = u_0 \frac{\partial w_0}{\partial n}$ is satisfied for every $x \in \partial\Omega$, then the problem (3.22)-(3.26) has a unique positive solution defined on an interval $[0, T) \subset \mathbb{R}$ and $(u, w) \in \left(C^{l+2, l/2+1}(\overline{Q}_t)\right)^2$, for all $t \in [0, T)$.*

Proof. From Theorem 3.8 we deduce the existence and the uniqueness of the solution of the problem (3.22)-(3.26) on $\overline{\Omega} \times [0, \tau_1]$ with τ_1 sufficiently small. Repeating the previous argument, we can extend this solution on an interval $[\tau_1, \tau_2]$ and so on, each interval having the length given by the fact that $R \circ S$ fulfill the conditions (i) and (ii) in Theorem 3.8. It is obvious that the compatibility condition is satisfied at each step. We obtain in such a way a solution defined in an interval $[0, t) \subset \mathbb{R}$, $0 < t \leqslant \infty$.

In order to prove the uniqueness of the solution, it is enough to notice that each classical solution of the problem (3.22)-(3.26) can be regarded, locally, as a fixed point of a map analogue to $R \circ S$. The uniqueness of such a fixed point implies the uniqueness of the solution. ■

3.3. Global existence in time

In this Section we prove that the solution (u, w) of the problem (3.22)-(3.26) in a two-dimensional bounded domain $\Omega \subset \mathbb{R}^2$ is globally defined in time. For this we start calculating a priori bounds. These bounds will be used for proving that the solution u of the system (3.31)-(3.33) belong to a suitable Hölder space. The regularity is then successively ameliorated until obtaining a bound of $|u(\cdot, t)|_{\Omega}^{(l+2)} \leqslant C(t)$, where $C(t) < +\infty$ if $t < T_{max}$. As the length of the existence interval obtained in Theorem 3.7 depends uniformly on $|u_0|_{\Omega}^{(l+2)}$, this bound will imply that the maximal interval of definition of the solution is $[0, \infty)$.

In what follows, sometimes the function arguments are omitted. Also, the variable t belongs to the maximal time interval of existence of the classical solution (u, v) of the problem (3.22)-(3.26). Moreover, in this section, since we prolong the solution given by Theorem 3.7, we assume the same conditions on the initial data as in Theorem 3.7.

3.3.1. A Lyapunov function for the system

The results obtained in this Subsection do not depend of the dimension of the space, they are valid in a domain $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$.

Proposition 3.10. *The total mass of the solution u is bounded*

$$\int_{\Omega} u(x)dx \leqslant \max\{1, M_0\} \tag{3.47}$$

where $M_0 = \int_{\Omega} u_0(x)dx = \|u_0\|_{L^1(\Omega)}$ represents the initial mass.

Proof. Taking into account the boundary condition (3.24) and integrating the equation (3.22) over Ω, we can easily deduce

$$\int_{\Omega} u_t(x,t)dx = \delta \int_{\Omega} u(x,t)dx - \delta \int_{\Omega} u^2(x,t)dx,$$

and applying Jensen's inequality and Gronwall lemma we obtain the estimate (3.47). ■

Remark 3.11. *1. Since the solution u is nonnegative, a consequence of the property (3.47) is that u satisfies an a priori L^1 estimate uniform in time*

$$\|u\|_{L^\infty(0,t;L^1(\Omega))} = \left(\left\|u^{1/2}\right\|_{L^\infty(0,t;L^2(\Omega))}\right)^2 \leqslant C_4$$

for all $t > 0$, where $C_4 = \max\{1, M_0\} > 0$.

2. Let us observe that, from (3.23)

$$w(x,t) = w_0(x) e^{-\int_0^t u(x,s)ds}$$

In particular, for $w_0(x) > 0$ we obtain

$$0 < w(x,t) \leqslant w_0(x)$$

for all $t > 0$, which implies

$$\|w\|_{L^\infty(0,t;L^\infty(\Omega))} \leqslant \|w_0\|_{L^\infty(\Omega)}.$$

Let (u, v) be the solution of (3.22)-(3.26). We introduce the following two functionals

$$F(u,w) := \int_\Omega u(\ln u - 1)dx + \frac{1}{2}\int_\Omega w^{-1}|\nabla w|^2 dx \tag{3.48}$$

$$D(u,w) := \int_\Omega u^{-1}|\nabla u|^2 dx + \frac{\beta}{2}\int_\Omega u w^{-1}|\nabla w|^2 dxds + \delta\int_\Omega u(u-1)\ln u dx \tag{3.49}$$

and we show that $F(u, w)$ is a Lyapunov function for the system (3.22)-(3.26).

Lemma 3.12. *We have*

$$\frac{d}{dt}F(u,w) = -D(u,w) \leqslant 0 \tag{3.50}$$

and the functional $F(u, w)$ is bounded from below, i.e. there is a constant $C > 0$ such that

$$F(u,w) > -C \tag{3.51}$$

for all $t > 0$.

Proof. We formally differentiate F with respect to t:

$$\frac{d}{dt}F(u,w) = \int_\Omega u_t(\ln u - 1)dx + \int_\Omega u_t dx + \frac{1}{2}\frac{d}{dt}\int_\Omega w^{-1}|\nabla w|^2 dx$$

Multiplying the equation (3.22) by $\ln u$ and formally integrating on Ω (in fact we multiply by $\ln(u + \varepsilon)$, $\varepsilon > 0$ and after integration $\varepsilon \to 0$), we obtain

$$\int_\Omega u_t(\ln u - 1)dx = \int_\Omega \Delta u(\ln u - 1)dx - \int_\Omega \nabla\cdot(u\nabla w)(\ln u - 1)dx + \delta\int_\Omega u(1-u)(\ln u - 1)dx$$

and taking into account the equality

$$\int_\Omega u_t dx = \delta\int_\Omega u(1-u)dx$$

we obtain

$$\int_\Omega u_t(\ln u - 1)dx = -\int_\Omega u^{-1}|\nabla u|^2 dx + \int_\Omega \nabla u\cdot\nabla w dx + \delta\int_\Omega u(1-u)\ln u dx - \int_\Omega u_t dx$$

We estimate now the second term from the right-hand side in the last equality taking into account (3.23):

$$\int_\Omega \nabla u\cdot\nabla w dx = -\frac{1}{2}\frac{d}{dt}\int_\Omega (w^{-1}|\nabla w|^2) - \frac{1}{2}\int_\Omega uw^{-1}|\nabla w|^2$$

so, we have

$$\int_\Omega u_t(\ln u - 1)dx = -\int_\Omega u^{-1}|\nabla u|^2 dx - \frac{1}{2}\frac{d}{dt}\int_\Omega (w^{-1}|\nabla w|^2) - \frac{1}{2}\int_\Omega uw^{-1}|\nabla w|^2 + \\ + \delta\int_\Omega u(1-u)\ln u dx - \int_\Omega u_t dx$$

which means

$$\frac{d}{dt}F(u,w) = -D(u,w) \leqslant 0$$

In order to prove (3.51), let us observe that for all $u \geqslant 0$, $u(\ln u - 1) > -1$ holds and we obtain

$$F(u,w) = \int_\Omega u(\ln u - 1)dx + \frac{1}{2}\int_\Omega w^{-1}|\nabla w|^2 dx \geqslant -|\Omega| \tag{3.52}$$

In fact, all terms in F are bounded from below (it is easy to see that $u\ln u \geqslant -e^{-1}$). ■

From now on we assume the following additional condition for the initial data:

$$(H)\ \ F(u_0, w_0) < +\infty.$$

Proposition 3.13. *If (H) is satisfied then there exists a positive constant C_5 such that*

$$\int_\Omega u\ln u dx < C_5 \tag{3.53}$$

where $C_5 = F(u_0, w_0) + \max\{1, \|u_0\|_{L^1(\Omega)}\}$

Proof. From (3.50) we obtain, after integration in $[0,t]$

$$F(u,w) \leqslant F(u_0, w_0) \tag{3.54}$$

and taking into account (3.47) we have

$$\int_\Omega u\ln u dx \leqslant \int_\Omega u(\ln u - 1)dx + \frac{1}{2}\int_\Omega w^{-1}|\nabla w|^2 dx + \int_\Omega u dx \leqslant \\ \leqslant \int_\Omega u_0(\ln u_0 - 1)dx + \frac{1}{2}\int_\Omega w_0^{-1}|\nabla w_0|^2 dx + \max\left\{1, \|u_0\|_{L^1(\Omega)}\right\}$$

■

Remark 3.14. *We point out that if $\int_\Omega u_0\ln u_0$ is bounded, we also obtain $u_0 \in L^1(\Omega)$ because*

$$\|u_0\|_{L^1(\Omega)} = \int_\Omega u_0 dx \leqslant \int_\Omega (u_0\ln u_0 + 1)\, dx.$$

Corollary 3.15. *If (H) is satisfied then the functional $F(u,w)$ and all its terms are bounded independently of $t>0$.*

Proposition 3.16. *If there exists a positive constant C such that the positive function u satisfy*

$$\int_\Omega u\ln u dx < C$$

then

$$\lim_{k\to\infty} \|u_k\|_{L^1(\Omega)} = 0 \tag{3.55}$$

uniformly with respect to $t>0$, where $u_k=(u-k)_+$.

Proof. Let $k>1$. We define the set

$$\Omega_k = \{x\in\Omega: \ \ u(x)>k\}$$

we have

$$\begin{aligned}\|u_k\|_{L^1} &= \int_{\Omega_k}(u-k)\,dx \leqslant \int_{\Omega_k} udx \leqslant \frac{1}{\ln k}\int_{\Omega_k} u\ln udx \leqslant \\ &\leqslant \frac{1}{\ln k}\left(\int_\Omega u\ln udx - \int_{\Omega\setminus\Omega_1} u\ln udx\right) \leqslant C_6\frac{1}{\ln k}\end{aligned}$$

where $C_6=\left(C+e^{-1}|\Omega|\right)$. The last inequality implies (3.55). ■

3.3.2. Estimates in L^p, $1<p<\infty$

Considering the following change of variables

$$v(x,t) = u(x,t)e^{-w(x,t)}$$

the system (3.22)-(3.26) becomes

$$\frac{\partial v}{\partial t} = \Delta v + \nabla v\cdot\nabla w + e^w v^2 w + \delta v(1-ve^w) \qquad x\in\Omega,\quad t\in\mathbb{R}_+ \tag{3.56}$$

$$\frac{\partial w}{\partial t} = -e^w wv \qquad x\in\Omega,\quad t\in\mathbb{R}_+ \tag{3.57}$$

$$\frac{\partial v}{\partial n} = 0 \qquad x\in\partial\Omega,\quad t\in\mathbb{R}_+ \tag{3.58}$$

$$v(x,0) = u_0(x)e^{-w_0(x)} = v_0(x) \geqslant 0 \qquad t\in\mathbb{R}_+ \tag{3.59}$$

$$w(x,0) = w_0(x) > 0 \qquad t\in\mathbb{R}_+ \tag{3.60}$$

From now on, for simplicity of notation we write v_k instead of $(v-k)_+$, where $k>0$. Moreover, we assume during the rest of the chapter that

$$(H2)\ \ \Omega\subset\mathbb{R}^N, \text{with } N\leqslant 2.$$

Proposition 3.17. *If (H) and $(H2)$ are satisfied then there exists a constant*

$$C_7 = C_7(p, \|v_0\|_{L^p(\Omega)}, \|w_0\|_{L^\infty(\Omega)})$$

independent of time such that the solution of the system (3.56)-(3.60) satisfies

$$\|v\|_{L^p(\Omega)} \leqslant C_7, \qquad 1\leqslant p<+\infty.$$

Proof. Testing the equation (3.56) with $pv_k^{p-1}e^w$, $k>0$, $p>1$, gives

$$\frac{d}{dt}\int_\Omega v_k^p e^w = -p(p-1)\int_\Omega v_k^{p-2}e^w\,|\nabla v_k|^2 + \delta p\int_\Omega v_k^{p-1}\left[ve^w(1-ve^w)\right] + (p-1)\int_\Omega e^{2w}wv_k^{p+1} \tag{3.61}$$

Taking into account the identity

$$\left|\nabla\left(v_k^{p/2}\right)\right|^2 = \frac{p^2}{4}v_k^{p-2}\,|\nabla v_k|^2$$

we obtain from (3.61)

$$\begin{aligned}\frac{d}{dt}\int_\Omega v_k^p e^w = &-\frac{4(p-1)}{p}\int_\Omega e^w\left|\nabla\left(v_k^{p/2}\right)\right|^2 + \\ &+\int_\Omega pke^w\left[ke^w w+\delta\left(1-ke^w\right)\right]v_k^{p-1} + \\ &+\int_\Omega e^w\left[(2p-1)ke^w w+\delta p\left(1-2ke^w\right)\right]v_k^p + \\ &+\int_\Omega e^{2w}\left[(p-1)w-\delta p\right]v_k^{p+1}\end{aligned} \tag{3.62}$$

Since $0<w(x,t)\leqslant w_0(x)$ and $e^{w(x,t)}\geqslant 1$ for all $t>0$, we obtain from (3.62)

$$\begin{aligned}\frac{d}{dt}\int_\Omega v_k^p e^w \leqslant &-\frac{4(p-1)}{p}\left\|\nabla\left(v_k^{p/2}\right)\right\|^2_{L^2(\Omega)} + C_8\int_\Omega v_k^{p-1} + \\ &+C_9\int_\Omega v_k^p + C_{10}\int_\Omega v_k^{p+1}\end{aligned} \tag{3.63}$$

where C_8, C_9 and C_{10} are given by

$$C_8 = C_8(p,k,\delta,\|w_0\|_{L^\infty}) = pk\left[k\left(e^{2\|w_0\|_{L^\infty}}\|w_0\|_{L^\infty}-\delta\frac{p}{p-1}\right)+\delta\left(e^{\|w_0\|_{L^\infty}}+\frac{k}{p-1}\right)\right] \tag{3.64}$$

$$C_9 = C_9(p,k,\delta,\|w_0\|_{L^\infty}) = p\left[\frac{2p-1}{p}k\left(e^{2\|w_0\|_{L^\infty}}\|w_0\|_{L^\infty}-\delta\frac{p}{p-1}\right)+\delta\left(e^{\|w_0\|_{L^\infty}}+\frac{k}{p-1}\right)\right] \tag{3.65}$$

$$C_{10} = C_{10}(p,\delta,\|w_0\|_{L^\infty}) = (p-1)\left(e^{2\|w_0\|_{L^\infty}}\|w_0\|_{L^\infty}-\delta\frac{p}{p-1}\right) \tag{3.66}$$

Adding the term $\sigma\int_\Omega v_k^p$, where $\sigma>0$ is a constant, on both sides of the last inequality, we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega v_k^p e^w + \sigma\int_\Omega v_k^p \leqslant &-\frac{4(p-1)}{p}\left\|\nabla\left(v_k^{p/2}\right)\right\|^2_{L^2(\Omega)} + C_8\int_\Omega v_k^{p-1} + \\ &+\left[C_9+\sigma\right]\int_\Omega v_k^p + C_{10}\int_\Omega v_k^{p+1}\end{aligned} \tag{3.67}$$

We estimate now the last two terms from (3.67) using Gagliardo-Nirenberg's inequality and taking into account the positivity of v. We have

$$\int_\Omega v_k^p = \left\| v_k^{\frac{p}{2}} \right\|_{L^2(\Omega)}^2 \leqslant C_{11}(\Omega) \left\| v_k^{\frac{p}{2}} \right\|_{H^1(\Omega)} \left\| v_k^{\frac{p}{2}} \right\|_{L^1(\Omega)} \tag{3.68}$$

$$\int_\Omega v_k^{p+1} = \left\| v_k^{\frac{p}{2}} \right\|_{L^{\frac{2(p+1)}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \leqslant C_{12}(\Omega) \left\| v_k^{\frac{p}{2}} \right\|_{H^1(\Omega)}^2 \|v_k\|_{L^1(\Omega)} \tag{3.69}$$

We put estimates (3.68), (3.69) on (3.67) and we apply the Cauchy's inequality. We obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega v_k^p e^w + \sigma \int_\Omega v_k^p \leqslant -\frac{4(p-1)}{p} \left\| \nabla \left(v_k^{\frac{p}{2}} \right) \right\|_{L^2(\Omega)}^2 + C_8 \left\| v_k^{p-1} \right\|_{L^1(\Omega)} + \\
&+ [C_9 + \sigma] \left\| v_k^{\frac{p}{2}} \right\|_{L^2(\Omega)}^2 + C_{10} C_{12} \left\| v_k^{\frac{p}{2}} \right\|_{H^1(\Omega)}^2 \|v_k\|_{L^1(\Omega)} \leqslant \\
&\leqslant \left[-\frac{4(p-1)}{p} + C_{10}C_{12} \|v_k\|_{L^1(\Omega)} \right] \left\| \nabla \left(v_k^{\frac{p}{2}} \right) \right\|_{L^2(\Omega)}^2 + C_8 \left\| v_k^{p-1} \right\|_{L^1(\Omega)} + \\
&+ C_{11} \left[C_9 + \sigma + C_{10}C_{12} \|v_k\|_{L^1(\Omega)} \right] \left\| v_k^{\frac{p}{2}} \right\|_{H^1(\Omega)} \left\| v_k^{\frac{p}{2}} \right\|_{L^1(\Omega)} \leqslant \\
&\leqslant \left[-\frac{4(p-1)}{p} + C_{10}C_{12} \|v_k\|_{L^1(\Omega)} + \varepsilon \right] \left\| \nabla \left(v_k^{\frac{p}{2}} \right) \right\|_{L^2(\Omega)}^2 + C_8 \left\| v_k^{p-1} \right\|_{L^1(\Omega)} + \\
&+ \varepsilon \left\| v_k^{\frac{p}{2}} \right\|_{L^2(\Omega)}^2 + \frac{1}{4\varepsilon} \left\{ C_{11} \left[C_9 + \sigma + C_{10}C_{12} \|v_k\|_{L^1(\Omega)} \right] \right\}^2 \left\| v_k^{\frac{p}{2}} \right\|_{L^1(\Omega)}^2
\end{aligned} \tag{3.70}$$

In order to estimate the second term from the right-hand side of (3.70), we apply Young's inequality and we obtain for $\epsilon > 0$:

$$v_k^{p-1} \leqslant \frac{1}{p}\epsilon^{-p} + \frac{p-1}{p}\epsilon^{\frac{p}{p-1}} v_k^p \tag{3.71}$$

Now, choosing ε small enough such that $\varepsilon < \min\{\sigma/2, 2(p-1)/p\}$ and putting (3.71) on (3.70), we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega v_k^p e^w + \frac{\sigma}{2} \int_\Omega v_k^p \leqslant \\
&\leqslant \left[-\frac{4(p-1)}{p} + C_{10}C_{12} \|v_k\|_{L^1(\Omega)} + \varepsilon \right] \left\| \nabla \left(v_k^{\frac{p}{2}} \right) \right\|_{L^2(\Omega)}^2 + \frac{1}{p}\epsilon^{-p} C_8 |\Omega| + \\
&+ \frac{1}{4\varepsilon} \left\{ C_{11} \left[C_9 + \sigma + \frac{p-1}{p}\epsilon^{\frac{p}{p-1}} C_8 + C_{10}C_{12} \|v_k\|_{L^1(\Omega)} \right] \right\}^2 \left\| v_k^{\frac{p}{2}} \right\|_{L^1(\Omega)}^2
\end{aligned}$$

Taking into account the Proposition 3.16, we can choose k sufficiently large such that the coefficient of $\left\| \nabla \left(v_k^{\frac{p}{2}} \right) \right\|_{L^2(\Omega)}^2$ is negative. In this way, the last inequality becomes

$$\frac{d}{dt}\int_\Omega v_k^p e^w + \frac{\sigma}{2} \int_\Omega v_k^p \leqslant C_{13} \left\| v_k^{\frac{p}{2}} \right\|_{L^1(\Omega)}^2 + \frac{1}{p}\epsilon^{-p} C_8 |\Omega| \tag{3.72}$$

where

$$\begin{aligned}
C_{13} &= C_{13}(p, k, \delta, \|w_0\|_{L^\infty}, \sigma, \varepsilon, \epsilon, \Omega) = \\
&= \frac{1}{4\varepsilon} \left\{ C_{11} \left[C_9 + \sigma + \frac{p-1}{p}\epsilon^{\frac{p}{p-1}} C_8 + C_{10}C_{12} \|v_k\|_{L^1(\Omega)} \right] \right\}^2
\end{aligned}$$

Since $\int_\Omega v_k^p e^w \leqslant e^{\|w_0\|_{L^\infty}} \int_\Omega v_k^p$, we obtain from the last inequality

$$\frac{d}{dt}\int_\Omega v_k^p e^w + \frac{\sigma}{2e^{\|w_0\|_{L^\infty}}}\int_\Omega v_k^p e^w \leqslant C_{13}\left\|v_k^{\frac{p}{2}}\right\|^2_{L^1(\Omega)} + \frac{1}{p}\epsilon^{-p}C_8\,|\Omega|$$

and applying Gronwall's inequality

$$\int_\Omega v_k^p \leqslant \int_\Omega v_k^p e^w \leqslant \max\left\{\int_\Omega (v_0-k)_+^p\, e^{w_0}, \frac{2e^{\|w_0\|_{L^\infty}}}{\sigma}\left[C_{13}\left\|v_k^{\frac{p}{2}}\right\|^2_{L^1(\Omega)} + \frac{1}{p}\epsilon^{-p}C_8\,|\Omega|\right]\right\} \tag{3.73}$$

We will show by induction for all $p = 2^j$, with $j \in \mathbb{N}$, that

$$\|v_k(t)\|_{L^p(\Omega)} \leqslant C$$

where C is a constant independent of t.

Let us remark that, taking into account Proposition 3.10, we have

$$\|v_k(t)\|_{L^1(\Omega)} \leqslant \int\limits_\Omega u dx \leqslant \max\{1, M_0\} \tag{3.74}$$

Let $p = 2^j$, and suppose that $\|v_k(t)\|_{L^{2^{j-1}}(\Omega)} = \|v_k(t)\|_{L^{p/2}(\Omega)}$ is uniformly bounded, the bound being independent of $t > 0$. We obtain from (3.73) that $\|v_k(t)\|_{L^{2^j}(\Omega)}$ is bounded, $j \in \mathbb{N}$. We conclude, taking into account the imbedding of $L^p(\Omega)$ spaces, that

$$\|v_k\|_{L^\infty(0,t;L^p(\Omega))} \leqslant C_{14}, \text{ for every } 1 \leqslant p < \infty$$

where $C_{14} = C_{14}(p, \|v_0\|_{L^p(\Omega)}, \|w_0\|_{L^\infty(\Omega)})$ is a positive constant, independent of $t > 0$.

Finally, we obtain

$$\|v\|_{L^p(\Omega)} \leqslant 2\left(\|v_k(t)\|^p_{L^p(\Omega)} + k^p\,|\Omega|\right)^{1/p}$$

and we conclude the proof of Proposition 3.17. ■

Remark 3.18. *The above estimates depends strongly on the dimension of the space.*

The above estimates are done in the case when C_8, C_9, C_{10} are positive. If one or several of these constants are negative (for example, when $\delta > \frac{p-1}{p}e^{2\|w_0\|_{L^\infty}}\|w_0\|_{L^\infty}$, $p > 1$), the result remains true, the upper bound being slightly modified.

3.3.3. Estimates in L^∞

Proposition 3.19. *If (H) and $(H2)$ are satisfied then the solution to the system (3.56)-(3.60) satisfies*

$$\|v\|_{L^\infty(0,\infty;L^\infty(\Omega))} \leqslant C$$

where the constant C will be determined later.

Proof. We introduce the following sets

$$\Omega_k(t) = \{x \in \Omega; \quad v(x,t) > k\}$$

where k is a positive constant. Let us observe that, taking into account (3.74) and choosing $p=2$, the relation (3.70) becomes

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega v_k^2 e^w + \sigma \int_\Omega v_k^2 \leqslant \\
&\leqslant \left[-2 + C_{10}C_{12}\left\|v_k\right\|_{L^1(\Omega)} + \varepsilon\right] \left\|\nabla v_k\right\|^2_{L^2(\Omega)} + \varepsilon \left\|v_k\right\|^2_{L^2(\Omega)} + \\
&+ \left\{ C_8 + \frac{1}{4\varepsilon}\left\|v_k\right\|_{L^1(\Omega)} \left[C_{11}\left(C_9 + \sigma + C_{10}C_{12}\left\|v_k\right\|_{L^1(\Omega)}\right)\right]^2 \right\} \left\|v_k\right\|_{L^1(\Omega)}
\end{aligned} \tag{3.75}$$

We estimate the last term of the right-hand side of the last inequality using Hölder's inequality and the Sobolev embedding

$$\left\|v_k\right\|_{L^1(\Omega)} \leqslant \left\|v_k\right\|_{L^4(\Omega)} \left|\Omega_k\right|^{3/4} \leqslant C_{15} \left\|v_k\right\|_{H^1(\Omega)} \left|\Omega_k\right|^{3/4}$$

where C_{15} is a constant independent of t. Using this inequality and Cauchy's inequality, we obtain from (3.75)

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega v_k^2 e^w + \sigma \int_\Omega v_k^2 \leqslant \\
&\leqslant \left[-2 + C_{10}C_{12}\left\|v_k\right\|_{L^1(\Omega)} + \varepsilon + \varepsilon'\right] \left\|\nabla v_k\right\|^2_{L^2(\Omega)} + \left(\varepsilon + \varepsilon'\right) \left\|v_k\right\|^2_{L^2(\Omega)} + \\
&+ \frac{C_{15}^2}{4\varepsilon'} \left\{ C_8 + \frac{1}{4\varepsilon}\left\|v_k\right\|_{L^1(\Omega)} \left[C_{11}\left(C_9 + \sigma + C_{10}C_{12}\left\|v_k\right\|_{L^1(\Omega)}\right)\right]^2 \right\}^2 \left|\Omega_k\right|^{3/2}
\end{aligned} \tag{3.76}$$

We choose ε and ε' small enough such that $\varepsilon + \varepsilon' < \min\{1, \sigma/2\}$. Taking into account the Proposition 3.16, it follows that there exists a $k_1 > 0$ sufficiently large such that, for every $k > k_1$, the coefficient of $\left\|\nabla\left(v_k^{\frac{p}{2}}\right)\right\|^2_{L^2(\Omega)}$ is negative. Taking into account (3.74) we obtain from (3.76)

$$\frac{d}{dt}\int_\Omega v_k^2 e^w + \frac{\sigma}{2}\int_\Omega v_k^2 \leqslant C_{16}\left|\Omega_k\right|^{3/2} \tag{3.77}$$

for all $k > k_1$, where

$$\begin{aligned}
C_{16} &= C_{16}(k, \delta, \left\|w_0\right\|_{L^\infty}, M_0) = \\
&= \frac{C_{15}^2}{4\varepsilon'} \left\{ C_8 + \frac{1}{4\varepsilon}\max\{1, M_0\} \left[C_{11}\left(C_9 + \sigma + C_{10}C_{12}\max\{1, M_0\}\right)\right]^2 \right\}^2
\end{aligned}$$

Moreover $\int_\Omega v_k^p e^w \leqslant e^{\left\|w_0\right\|_{L^\infty}} \int_\Omega v_k^p$, so we obtain

$$\frac{d}{dt}\int_\Omega v_k^2 e^w + \frac{\sigma}{2e^{\left\|w_0\right\|_{L^\infty}}}\int_\Omega v_k^2 e^w \leqslant C_{16}\left|\Omega_k\right|^{3/2} \tag{3.78}$$

One can notice, using (3.64), (3.65) and (3.66), that C_{16} is a polynomial of degree 4 in k. Let α be the dominant coefficient. It is a constant depending only on the initial data of the system. Now, we try to get rid the dependence on k of C_{16}. Representing the L^q-norm with the use of the level sets, we have (see [25, 69])

$$\int_\Omega v^{q+1} = (q+1)\int_0^\infty s^q \left|\Omega_s\right| ds, \quad q \geqslant 1$$

We obtain, using these facts, a bound for the right-hand side of the inequality (3.78). Namely, taking into account Proposition 3.17, we get

$$(k-1)^q\,|\Omega_k| < \int_{k-1}^{k} s^q\,|\Omega_s|\,ds < \int_0^\infty s^q\,|\Omega_s|\,ds = \frac{1}{q+1}\,\|v\|_{L^{q+1}(\Omega)}^{q+1} < C_{17}$$

where C_{17} is a constant independent of t. From the last inequality, taking $q=16$, we obtain

$$(k-1)^4\,|\Omega_k|^{1/4} < C_{17}^{1/4}$$

It follows that there exists $k_2>0$ such that for every $k>k_2$,

$$C_{16}\,|\Omega_k|^{1/4} < (\alpha+1)C_{17}^{1/4} = C_{18}$$

which implies, from (3.78)

$$\frac{d}{dt}\int_\Omega v_k^2 e^w + \frac{\sigma}{2e^{\|w_0\|_{L^\infty}}}\int_\Omega v_k^2 e^w \leqslant C_{18}\,|\Omega_k|^{5/4} \tag{3.79}$$

Since $v_0\in L^\infty(\Omega)$, there exists $k_3>0$ such that $\|v_k(0)\|=0$ for all $k>k_3$. For $k>\max\{k_1,k_2,k_3\}$, we deduce from (3.79)

$$\|v_k(t)\|_{L^2(\Omega)}^2 \leqslant \left\|e^{w/2}v_k(t)\right\|_{L^2(\Omega)}^2 \leqslant \frac{2C_{18}e^{\|w_0\|_{L^\infty}}}{\sigma}\left(1-e^{-\frac{\sigma}{2e^{\|w_0\|_{L^\infty}}}t}\right)\left(\sup_{t\geqslant0}|\Omega_k(t)|\right)^{5/4} \tag{3.80}$$

On the other hand, taking into account that $\Omega_l\subset\Omega_k$ for $l>k>0$, then

$$\|v_k(t)\|_{L^2(\Omega)}^2 \geqslant \int_{\Omega_l(t)} v_k^2 \geqslant (l-k)^2\,|\Omega_l(t)| \tag{3.81}$$

Taking the supremum on $t\geqslant0$ in the last relation, (3.80) implies

$$(l-k)^2\sup_{t\geqslant0}|\Omega_l(t)| \leqslant \frac{2C_{18}e^{\|w_0\|_{L^\infty}}}{\sigma}\left(\sup_{t\geqslant0}|\Omega_k(t)|\right)^{5/4}$$

for $l>k>\max\{k_1,k_2,k_3\}$. Obviously the function $k\mapsto\sup_{t\geqslant0}|\Omega_k(t)|$ is decreasing, so we can apply [26, Lemma 4.1]. It follows that there exists

$$k_0=\max\{k_1,k_2,k_3\}+\left(\frac{2^{11}C_{18}e^{\|w_0\|_{L^\infty}}}{\sigma}\right)^{1/2}|\Omega|^{1/8}$$

such that

$$\sup_{t\geqslant0}|\Omega_k(t)|=0$$

for all $k\geqslant k_0$. This concludes the proof. ∎

Remark 3.20. *The L^∞ bound can also be proved using the iterative technique of Alikakos [2]. We have chosen the method presented here (inspired by an idea of Gajewski and Zacharias [29]) mainly for aesthetic reasons.*

3.3.4. Estimates for ∇v and Δv

3.3.5. A priori estimates

Lemma 3.21. *If the hypothesis (H) and $(H2)$ are satisfied, then we have*

$$\|v_t\|_{L^2(Q_t)} \leqslant C_{20} \tag{3.82}$$

$$\|\nabla v\|_{L^2(\Omega)} \leqslant C_{20} \qquad \textit{for all } t > 0 \tag{3.83}$$

where C_{20} is a constant independent on t.

Proof. Taking $e^w v_t$ as test function in the equation (3.56) and integrating in space, we obtain

$$\int_\Omega e^w v_t^2 = -\frac{1}{2}\int_\Omega e^w \frac{\partial}{\partial t}\left(|\nabla v|^2\right) + \int_\Omega e^w \left[e^w v^2 w + \delta v\left(1 - v e^w\right)\right] v_t \tag{3.84}$$

We estimate now every term of the right-hand side of (3.84). The first one is

$$-\frac{1}{2}\int_\Omega e^w \frac{\partial}{\partial t}\left(|\nabla v|^2\right) = -\frac{1}{2}\frac{d}{dt}\int_\Omega \left(e^w |\nabla v|^2\right) - \frac{1}{2}\int_\Omega e^{2w} v w |\nabla v|^2 \tag{3.85}$$

In order to estimate the last term we take into account the following inequalities

$$\int_\Omega e^{2w} w v^2 v_t \leqslant \frac{1}{2}\int_\Omega e^w v_t^2 - \frac{1}{2} e^{2\|w_0\|_{L^\infty(\Omega)}} \|w_0\|_{L^\infty(\Omega)} \|v\|^3_{L^\infty(\Omega)} \int_\Omega \frac{\partial w}{\partial t} \tag{3.86}$$

$$\delta \int_\Omega e^w v v_t \leqslant \frac{\delta}{2}\frac{d}{dt}\int_\Omega e^w v^2 - \frac{\delta}{2} e^{\|w_0\|_{L^\infty(\Omega)}} \|v\|^2_{L^\infty(\Omega)} \int_\Omega \frac{\partial w}{\partial t} \tag{3.87}$$

$$-\delta \int_\Omega e^{2w} v^2 v_t \leqslant -\frac{\delta}{3}\frac{d}{dt}\int_\Omega e^{2w} v^3 \tag{3.88}$$

Substituting (3.85), (3.86), (3.87) and (3.88) into (3.84), we have

$$\begin{aligned}\int_\Omega e^w v_t^2 + \frac{d}{dt}\int_\Omega e^w |\nabla v|^2 \leqslant -e^{\|w_0\|_{L^\infty(\Omega)}} \|v\|^2_{L^\infty(\Omega)} \Big[e^{\|w_0\|_{L^\infty(\Omega)}} \|w_0\|_{L^\infty(\Omega)} \|v\|_{L^\infty(\Omega)} + \\ +\delta\Big] \frac{d}{dt}\int_\Omega w + \delta \frac{d}{dt}\int_\Omega e^w v^2 - \frac{2\delta}{3}\frac{d}{dt}\int_\Omega e^{2w} v^3\end{aligned}$$

We integrate the last inequality with respect the time on $[0,t]$ and we obtain

$$\int_0^t \int_\Omega e^w v_s^2 + \int_0^t \frac{d}{ds}\int_\Omega e^w |\nabla v|^2 \leqslant C_{19} \tag{3.89}$$

where

$$\begin{aligned}C_{19} = e^{\|w_0\|_{L^\infty(\Omega)}} \|v\|^2_{L^\infty(0,t;L^\infty(\Omega))} \Big[e^{\|w_0\|_{L^\infty(\Omega)}} \|w_0\|_{L^\infty(\Omega)} \|v\|_{L^\infty(0,t;L^\infty(\Omega))} + \\ + \delta\Big] \int_\Omega w_0 + \delta e^{\|w_0\|_{L^\infty(\Omega)}} \|v\|^2_{L^2(\Omega)} + \frac{2\delta}{3}\int_\Omega e^{2w_0} v_0^3\end{aligned}$$

Finally, from (3.89) we obtain

$$\int_0^t \int_\Omega v_s^2 + \int_\Omega e^w |\nabla v|^2 \leqslant \int_\Omega e^{w_0} |\nabla v_0|^2 + C_{19}$$

The last inequality implies (3.82) and (3.83) where $C_{20} = \left(\int_\Omega e^{w_0} |\nabla v_0|^2 + C_{19}\right)^{1/2}$. ■

Lemma 3.22. *If the hypothesis (H) and (H2) are satisfied, then we have*

$$\|\Delta v\|_{L^1(0,t;L^2(\Omega))} \leqslant e\,(n+1)!k(T_0) \tag{3.90}$$

for all $t \in [0, \min\{(n+1)\,T_0, T\}]$, $n \in \mathbb{N}$, where $k(T_0)$ is a constant independent on t which will be given later.

Proof. From (3.56) we obtain

$$\int_0^t \|\Delta v\|_{L^2(\Omega)} \leqslant \int_0^t \|v_t\|_{L^2(\Omega)} + \int_0^t \|\nabla w \cdot \nabla v\|_{L^2(\Omega)} + \int_0^t \|h(v,w)\|_{L^2(\Omega)} \tag{3.91}$$

where

$$h(v,w) = e^w v^2 w + \delta v(1 - ve^w)$$

We estimate the first term in the right-hand side of (3.91) using (3.82) and the Hölder inequality

$$\int_0^t \|v_t\|_{L^2(\Omega)} \leqslant t^{1/2} \left(\int_0^t \int_\Omega |v_t|^2\right)^{1/2} \leqslant \sqrt{C_{20}}\, t^{1/2} \tag{3.92}$$

We estimate the second term in the right-hand side of (3.91) with the Hölder inequality

$$\|\nabla w \cdot \nabla v\|_{L^2(\Omega)} \leqslant \|\nabla w\|_{L^4(\Omega)} \|\nabla v\|_{L^4(\Omega)} \tag{3.93}$$

In what follows, we are going to obtain an estimate for $\|\nabla w\|_{L^4(\Omega)}$. We deduce from the equation (3.57) the relation

$$\nabla w_t = -e^w wv\nabla w - e^w v\nabla w - e^w w\nabla v$$

Multiplying this last relation by $\nabla w\,|\nabla w|^2$ and after that integrating in Ω, we get

$$\frac{d}{dt}\|\nabla w\|_{L^4(\Omega)} \leqslant C_{21}\|\nabla v\|_{L^4(\Omega)}$$

where $C_{21} = e^{\|w_0\|_{L^\infty(\Omega)}}\|w_0\|_{L^\infty(\Omega)}$. From the last inequality we obtain by integration

$$\|\nabla w\|_{L^4(\Omega)} \leqslant C_{22} + C_{21}\int_0^t \|\nabla v\|_{L^4(\Omega)} \tag{3.94}$$

where $C_{22} = \|\nabla w_0\|_{L^4(\Omega)}$. Taking into account (3.94), the estimate (3.93) becomes

$$\|\nabla w \cdot \nabla v\|_{L^2(\Omega)} \leqslant C_{22}\|\nabla v\|_{L^4(\Omega)} + \frac{C_{21}}{2}\frac{d}{dt}\left[\int_0^t \|\nabla v\|_{L^4(\Omega)}\right]^2 \tag{3.95}$$

We estimate now every term in the last inequality. In order to estimate the first term of (3.95) we use the Cauchy inequality and the Gagliardo-Nirenberg inequality

$$C_{22}\|\nabla v\|_{L^4(\Omega)} \leqslant C_{22}C_{23}\|\Delta v\|^{1/2}_{L^2(\Omega)}\|\nabla v\|^{1/2}_{L^2(\Omega)} \leqslant \varepsilon\|\Delta v\|_{L^2(\Omega)} + \frac{(C_{22}C_{23})^2}{4\varepsilon}\|\nabla v\|_{L^2(\Omega)} \tag{3.96}$$

In order to estimate the second term of (3.95), we obtain from Hölder inequality and (3.83)

$$\begin{aligned}\left[\int_0^t \|\nabla v\|_{L^4(\Omega)}\right]^2 &\leqslant \left[\int_0^t C_{23}\|\Delta v\|^{1/2}_{L^2(\Omega)}\|\nabla v\|^{1/2}_{L^2(\Omega)}\right]^2 \leqslant \\ &\leqslant C_{23}^2 t^{1/2}\left(\int_0^t \|\Delta v\|_{L^2(\Omega)}\right)\left(\int_0^t\int_\Omega |\nabla v|^2\right)^{1/2} \leqslant C_{23}^2\sqrt{C_{20}}\,t\int_0^t \|\Delta v\|_{L^2(\Omega)}\end{aligned} \tag{3.97}$$

Finally, after integration of (3.95) on $[0, t]$ and taking into account (3.96) and (3.97), we obtain

$$\int_0^t \|\nabla w \cdot \nabla v\|_{L^2(\Omega)} \leqslant \varepsilon \int_0^t \|\Delta v\|_{L^2(\Omega)} + \frac{(C_{22}C_{23})^2}{4\varepsilon} \int_0^t \|\nabla v\|_{L^2(\Omega)} + + \frac{C_{21}C_{23}^2\sqrt{C_{20}}}{2} t \int_0^t \|\Delta v\|_{L^2(\Omega)} \tag{3.98}$$

We estimate the last term of (3.91) with the use of Proposition 3.17.

$$\int_0^t \|h(v,w)\|_{L^2(\Omega)} \leqslant e^{\|w_0\|_{L^\infty(\Omega)}} \|w_0\|_{L^\infty(\Omega)} \int_0^t \|v\|^2_{L^4(\Omega)} + \delta \int_0^t \|v\|_{L^2(\Omega)} + \delta e^{\|w_0\|_{L^\infty(\Omega)}} \int_0^t \|v\|^2_{L^4(\Omega)} = = \delta \int_0^t \|v\|_{L^2(\Omega)} + e^{\|w_0\|_{L^\infty(\Omega)}} \left[\delta + \|w_0\|_{L^\infty(\Omega)}\right] \int_0^t \|v\|^2_{L^4(\Omega)} \leqslant C_{24} t \tag{3.99}$$

where $C_{24} = \delta \|v\|_{L^2(\Omega)} + e^{\|w_0\|_{L^\infty(\Omega)}} \left[\delta + \|w_0\|_{L^\infty(\Omega)}\right] \|v\|^2_{L^4(\Omega)}$.

Taking into account (3.92), (3.98) and (3.99) we estimate now $\|\Delta v\|^2_{L^2(\Omega)}$ from (3.91)

$$\int_0^t \|\Delta v\|_{L^2(\Omega)} \leqslant \sqrt{C_{20}} t^{1/2} + \varepsilon \int_0^t \|\Delta v\|_{L^2(\Omega)} + \frac{(C_{22}C_{23})^2}{4\varepsilon} \int_0^t \|\nabla v\|_{L^2(\Omega)} + + \frac{C_{21}C_{23}^2\sqrt{C_{20}}}{2} t \int_0^t \|\Delta v\|_{L^2(\Omega)} + C_{24} t$$

or, thanks to (3.83)

$$\left(1 - \varepsilon - \frac{C_{21}C_{23}^2\sqrt{C_{20}}}{2} t\right) \int_0^t \|\Delta v\|_{L^2(\Omega)} \leqslant \sqrt{C_{20}} t^{1/2} + t \left[\frac{(C_{22}C_{23})^2}{4\varepsilon} \sqrt{C_{20}} + C_{24}\right] \tag{3.100}$$

We take $\varepsilon = \frac{1}{4}$ and t sufficiently small such that

$$1 - \varepsilon - \frac{C_{21}C_{23}^2\sqrt{C_{20}}}{2} t \geqslant \frac{1}{2} \Longrightarrow t \leqslant \frac{1}{2C_{21}C_{23}^2\sqrt{C_{20}}} = T_0$$

which implies from (3.100)

$$\int_0^t \|\Delta v\|_{L^2(\Omega)} \leqslant 2\sqrt{C_{20}} t^{1/2} + 2\left[(C_{22}C_{23})^2 \sqrt{C_{20}} + C_{24}\right] t = k(t) \tag{3.101}$$

In this way we have obtained the boundedness of $\int_0^t \|\Delta v\|_{L^2(\Omega)}$ for all $t \in [0, \min\{T_0, T\}]$. This bound depends on the initial data considered in $\tau_0 = 0$. We can repeat the same procedure taking the initial data on a generic τ. We obtain

$$\left(1 - \varepsilon - \frac{\widetilde{C}_{21}C_{23}^2\sqrt{C_{20}}}{2}(t-\tau)\right) \int_\tau^t \|\Delta v\|_{L^2(\Omega)} \leqslant \sqrt{C_{20}}(t-\tau)^{1/2} + (t-\tau)\left[\frac{\left(\widetilde{C}_{22}C_{23}\right)^2}{4\varepsilon}\sqrt{C_{20}} + \widetilde{C}_{24}\right] \tag{3.102}$$

where

$$\widetilde{C}_{21} = e^{\|w(x,\tau)\|_{L^\infty(\Omega)}} \|w(x,\tau)\|_{L^\infty(\Omega)} \leqslant C_{21} \tag{3.103}$$

$$\widetilde{C}_{22} = \|\nabla w(x,\tau)\|_{L^4(\Omega)} \tag{3.104}$$

$$\widetilde{C}_{24} = \delta \|v\|_{L^2(\Omega)} + e^{\|w(x,\tau)\|_{L^\infty(\Omega)}} \left[\delta + \|w(x,\tau)\|_{L^\infty(\Omega)}\right] \|v\|^2_{L^4(\Omega)} \leqslant C_{24} \tag{3.105}$$

If $T_0 < T$ let us take $\tau = T_0$. Taking into account (3.94) (which is true for all $t \in [0, T_0]$) and (3.97), we have

$$\widetilde{C}_{22}^2 = \|\nabla w(x, T_0)\|_{L^4(\Omega)}^2 \leqslant \left[C_{22} + C_{21} \int_0^{T_0} \|\nabla v\|_{L^4(\Omega)}\right]^2 \leqslant 2C_{22}^2 + C_{21} \int_0^{T_0} \|\Delta v\|_{L^2(\Omega)} \tag{3.106}$$

Using (3.103), (3.105), and(3.106), the inequality (3.102) becomes

$$\begin{aligned}\left(1 - \varepsilon - \frac{C_{21} C_{23}^2 \sqrt{C_{20}}}{2} (t - T_0)\right) \int_{T_0}^t \|\Delta v\|_{L^2(\Omega)} \leqslant \sqrt{C_{20}}\, (t - T_0)^{1/2} + \\ + (t - T_0) \left\{\frac{C_{23}^2 \sqrt{C_{20}}}{4\varepsilon} \left[2C_{22}^2 + C_{21} \int_0^{T_0} \|\Delta v\|_{L^2(\Omega)}\right] + C_{24}\right\}\end{aligned} \tag{3.107}$$

We take $\varepsilon = \frac{1}{4}$ and t sufficiently small such that

$$1 - \varepsilon - \frac{C_{21} C_{23}^2 \sqrt{C_{20}}}{2} (t - T_0) \geqslant \frac{1}{2} \Longrightarrow t \leqslant \frac{1}{2C_{21} C_{23}^2 \sqrt{C_{20}}} + T_0 = 2T_0$$

which implies, from (3.107)

$$\int_{T_0}^t \|\Delta v\|_{L^2(\Omega)} \leqslant k(t - T_0) + \frac{1}{T_0} (t - T_0) \int_0^{T_0} \|\Delta v\|_{L^2(\Omega)}$$

The last relation is true for all $t \in [T_0, \min\{2T_0, T\}]$. More generally, we obtain

$$\int_{nT_0}^t \|\Delta v\|_{L^2(\Omega)} \leqslant k(t - nT_0) + \frac{n}{T_0} (t - nT_0) \int_0^{nT_0} \|\Delta v\|_{L^2(\Omega)} \tag{3.108}$$

for all $t \in [nT_0, \min\{(n+1)\, T_0, T\}]$, if $n \in \mathbb{N}$ is such that $nT_0 < T$.

Let us observe that

$$\frac{n}{T_0} (t - nT_0) \leqslant n$$

and for the function $k(t)$ given by (3.101) we have

$$k(t - nT_0) \leqslant k(T_0)$$

for all $t \in [nT_0, \min\{(n+1)\, T_0, T\}]$. Thus, the inequality (3.108) becomes

$$\int_{nT_0}^t \|\Delta v\|_{L^2(\Omega)} \leqslant k(T_0) + n \int_0^{nT_0} \|\Delta v\|_{L^2(\Omega)}$$

for all $t \in [nT_0, \min\{(n+1)\, T_0, T\}]$.

Finally, for all $t \in [0, \min\{(n+1)\, T_0, T\}]$, taking into account (3.101), we obtain

$$\begin{aligned}\int_0^t \|\Delta v\|_{L^2(\Omega)} &\leqslant (n+1)! k(T_0) \left(\frac{1}{2!} + ... + \frac{1}{(n+1)!}\right) + (n+1)! \int_0^t \|\Delta v\|_{L^2(\Omega)} \leqslant \\ &\leqslant (n+1)! \left(1 + \frac{1}{2!} + ... + \frac{1}{(n+1)!}\right) k(T_0) \leqslant e\, (n+1)! k(T_0)\end{aligned}$$

■

Remark 3.23. *This inequality holds for all $0 < t < T$, and n is maximal with the property $nT_0 \leqslant t$. We emphasize that the bound in terms of n is equivalent with a bound in terms of t, of the same type. Hence we obtain*

$$\int_0^t \|\Delta v\|_{L^2(\Omega)} \leqslant \Psi(t) \tag{3.109}$$

where Ψ is a increasing function of the time t having the properties $\lim\limits_{t\searrow 0}\Psi(t) = 0$, $\lim\limits_{t\nearrow T}\Psi(t) < \infty$ for all T finite.

Henceforth Ψ will stand for a generic function of t having the above properties.

Lemma 3.24. *If the hypothesis (H) and $(H2)$ are satisfied, then we have*

$$\int_0^t \|\nabla v\|_{L^p(\Omega)} \leqslant \Psi(t)$$

for $2 \leqslant p < \infty$.

Proof. Taking into account the Gagliardo-Nirenberg inequality and the Cauchy inequality we obtain the following estimate

$$\int_0^t \|\nabla v\|_{L^{2^j}(\Omega)} \leqslant \int_0^t \|\Delta v\|_{L^2(\Omega)} + \frac{C_{25}^2}{4}\int_0^t \|\nabla v\|_{L^{2^{j-1}}(\Omega)}$$

for $j = 2, 3, ...$ The last inequality implies

$$\int_0^t \|\nabla v\|_{L^{2^j}(\Omega)} \leqslant \frac{1-\left(\frac{C_{25}^2}{4}\right)^{j-1}}{1-\frac{C_{25}^2}{4}}\int_0^t \|\Delta v\|_{L^2(\Omega)} + \left(\frac{C_{25}^2}{4}\right)^{j-1}\int_0^t \|\nabla v\|_{L^2(\Omega)}$$

From (3.83) and (3.90) we obtain for $j = 2, 3, ...$ and for all $t \in [0, \min\{(n+1)\, T_0, T\}]$

$$\int_0^t \|\nabla v\|_{L^p(\Omega)} \leqslant \Psi(t)$$

for $2 \leqslant p < \infty$. ■

Lemma 3.25. *If the hypothesis (H) and $(H2)$ are satisfied, then we have*

$$\int_0^t \|\nabla w\|_{L^p(\Omega)} \leqslant \Psi(t)$$

for $1 < p < \infty$.

Proof. We deduce from the equation (3.57)

$$\nabla w_t = -e^w wv\nabla w - e^w v\nabla w - e^w \nabla v$$

Multiplying this last relation by $\nabla w\,|\nabla w|^{p-2}$ and after that integrating in Ω, we have

$$\frac{d}{dt}\|\nabla w\|_{L^p(\Omega)} \leqslant e^{\|w_0\|_{L^\infty(\Omega)}}\|w_0\|_{L^\infty(\Omega)}\|\nabla v\|_{L^p(\Omega)}$$

From the last inequality we obtain by integration in time

$$\|\nabla w\|_{L^p(\Omega)} \leqslant \|\nabla w_0\|_{L^p(\Omega)} + e^{\|w_0\|_{L^\infty(\Omega)}}\|w_0\|_{L^\infty(\Omega)}\int_0^t \|\nabla v\|_{L^p(\Omega)}$$

which proves the lemma. ■

Lemma 3.26. *If the hypothesis (H) and $(H2)$ are satisfied then*

$$u \in C^{\alpha+1,(\alpha+1)/2}(\overline{Q}_T), \text{ for every } T > 0. \tag{3.110}$$

Proof. We consider the equation (3.56) together with (3.58) and (3.59) as a linear problem in the general form (3.12)-(3.14), considering

$$\begin{aligned} b_i(x,t) &= \frac{\partial w}{\partial x_i}(x,t), \qquad i = 1,2 \\ b(x,t) &= e^w wv + \delta(1 - ve^w) \\ \widetilde{F}(x,t) &= \widetilde{G}(x,t) = 0 \end{aligned}$$

The above estimates show that the hypothesis of , [48, Chapter 4, Theorem 9.1]) are fulfilled, see also [66, Theorem II.3]. This implies that for $q > 2$ we have

$$v \in C^{2-4/q,1-2/q}\left(\overline{Q}_T\right) \tag{3.111}$$

Thanks to the Sobolev embedding, $w \in L^\infty(0,T;C^\alpha(\overline{\Omega}))$. Therefore,

$$\begin{aligned} \|w(t)\|_{C^{1+\alpha}(\overline{\Omega})} &\leqslant \|w_0\|_{C^{1+\alpha}(\overline{\Omega})} + \int_0^t \|e^w w^\beta v\|_{C^{1+\alpha}(\overline{\Omega})} \\ &\leqslant \|w_0\|_{C^{1+\alpha}(\overline{\Omega})} + C\int_0^t \|e^w w^\beta\|_{C^{1+\alpha}(\overline{\Omega})}\|v\|_{C^{1+\alpha}(\overline{\Omega})} \\ &\leqslant \|w_0\|_{C^{1+\alpha}(\overline{\Omega})} + C(t)\int_0^t \|w\|_{C^{1+\alpha}(\overline{\Omega})} \end{aligned} \tag{3.112}$$

Next, Gronwall's Lemma entails $w \in C([0,T];C^{\alpha+1}(\overline{\Omega}))$. This fact together with $w_t \in C([0,T];C^{\alpha+1}(\overline{\Omega}))$, that is a consequence of the equality $w_t = -e^w vw$, assures $w \in C^{\alpha+1,(\alpha+1)/2}(\overline{Q}_T)$. ∎

Now $\frac{\partial w}{\partial x_i} \in C^{\alpha,\alpha/2}(\overline{Q}_T)$ and thanks to [48, Chapter 4, section 5] $v \in C^{\alpha+2,(\alpha+2)/2}(\overline{Q}_T)$. Next we can argue as in (3.112) and get $w \in C^{\alpha+2,(\alpha+2)/2}(\overline{Q}_T)$. Therefore we have reached the desired bound of $|u|_{Q_t}^{(l+2)}$.

3.4. Steady states and asymptotic behaviour of global solutions

3.4.1. Steady states

In this Section we study the steady states of (3.22)-(3.23) with the no-flux boundary condition (3.24). So, we consider the following stationary problem:

$$0 = \Delta u - \nabla\cdot(u\nabla w) + \delta u(1-u) \qquad x \in \Omega, \quad t \in \mathbb{R}_+ \tag{3.113}$$

$$0 = wu \qquad x \in \Omega, \quad t \in \mathbb{R}_+ \tag{3.114}$$

$$\frac{\partial u}{\partial n} = 0 \qquad x \in \partial\Omega, \quad t \in \mathbb{R}_+ \tag{3.115}$$

Observe that, after the change of variables, $v = e^{-w}u$ the system (3.113) - (3.115) reads

$$-\Delta v - \nabla w\cdot\nabla v = \delta v(1 - ve^w) \qquad x \in \Omega, \quad t \in \mathbb{R}_+ \tag{3.116}$$

$$ve^w w = 0 \qquad x \in \Omega, \quad t \in \mathbb{R}_+ \tag{3.117}$$

$$\frac{\partial v}{\partial n} = 0 \qquad x \in \partial\Omega, \quad t \in \mathbb{R}_+ \tag{3.118}$$

Definition 3.27. *We say that (v, w) are positive solutions to (3.116) - (3.118) if (3.116) - (3.118) are satisfied a.e. and*

$$(v, w) \in (\mathcal{P}_1 \cup \{0\}) \times (\mathcal{P}_2 \cup \{0\}), \quad (v, w) \neq (0, 0),$$

with

$$\begin{aligned} \mathcal{P}_1 &:= \left\{ z \in W^{2,p}(\Omega) \ : \ p > N, \ z \geqslant 0, \ z \neq 0, \ \tfrac{\partial z}{\partial n} = 0 \ on \ \partial\Omega \right\} \\ \mathcal{P}_2 &:= \{ z \in W^{1,\infty}(\Omega) \ : \ z \geqslant 0, \ z \neq 0 \}, \end{aligned}$$

for some $p > N$.

Remark 3.28. *Assume $w \neq 0$, $\delta > 0$ then the positive solutions to (3.116)- (3.118) satisfies*

$$L^1(w)z := -\Delta z - \nabla z \cdot \nabla w + e^{-w} v z > 0, \quad \frac{\partial z}{\partial n} = 0. \tag{3.119}$$

Moreover,

$$\sigma_1(L^1(w)) = \sigma_1(L^2(w) + e^{-w} v) > \sigma_1(L^2(w)) = 0, \tag{3.120}$$

where $\sigma_1(L^1(w))$ stands for the principal eigenvalue of $L^1(w)$ with Neumann boundary condition. Thanks to (3.119) and (3.120) the strong maximum principle (Theorem 2.4 in [4]) entails $v \in int(\mathcal{P}_1)$.

Theorem 3.29. *The positive solutions to (3.116)-(3.118) are given by*

$$\begin{aligned} (v^*, w^*) &= (0, \widetilde{w}), \quad \widetilde{w} \in \mathcal{P}_2, \\ (v^*, w^*) &= (1, 0), \quad \text{if } \delta > 0, \\ (v^*, w^*) &= (k, 0), \quad \text{if } \delta = 0. \end{aligned}$$

where k is a positive constant.

Proof. Case 1.- Assume $w = 0$ then

- if $\delta = 0$ taking v as a test function in (3.116) we obtain that $v = k$, $k > 0$ any constant.
- if $\delta > 0$ $v = 1$ is a positive solution to (3.116). Moreover, since the function

  $$f : [0, \infty) \to \mathbb{R}, \qquad f(s) = \delta s(1 - s)$$

  is continuous on $[0, \infty)$ and the function $u \longmapsto f(s)/s$, $\delta > 0$ is decreasing on $(0, \infty)$. Then, thanks to Lemma 1 in [11], is the only possible positive solution.

Case 2.- Assume $w \neq 0$ then

- if $\delta = 0$ taking v as a test function in (3.116) we get $v = k$, $k \geqslant 0$. Therefore, if $k \neq 0$ the condition (3.117) can not be satisfied and if $k = 0$ any $w \in \mathcal{P}_2$ is a positive solution.
- if $\delta > 0$ then either $v = 0$ or, thanks to Remark 3.28, $v \in int(\mathcal{P}_2)$.
  - if $v = 0$ then any $w \in \mathcal{P}_2$ is a positive solution.
  - if $v \in int(\mathcal{P}_2)$ then, from (3.117) we get $w = 0$ a contradiction with the assumption.

■

3.4.2. Asymptotic behaviour

The purpose of this section is to study the large time behaviour of the solutions to (3.22)-(3.23). In particular we will concentrate on the large time behaviour of the trajectories of such a system. The key property for our results is the "energy" equality (3.50).

Remark 3.30. *The equality (3.50) provides us, under the condition $F(u_0, w_0) < +\infty$, with the estimates*

$$\int_0^t \int_\Omega uw^{-1}|\nabla w|^2 \leqslant C\,, \qquad \int_0^t \int_\Omega u^{-1}|\nabla u|^2 \leqslant C\,, \qquad \forall t > 0. \tag{3.121}$$

Thanks to (3.121) and the estimates $w \leqslant w_0$, $u \leqslant C$ we obtain

$$\int_0^t \int_\Omega u|\nabla w|^2 \leqslant C\,, \qquad \int_0^t \int_\Omega |\nabla u|^2 \leqslant C\,, \qquad \forall t > 0. \tag{3.122}$$

Lemma 3.31. *Let $\tau, k > 0$ and $y \in C(\tau, +\infty) \cap L^1(\tau, +\infty)$, $y' \in L^1(\tau, +\infty)$. If*

$$\lim_{t\to+\infty} \int_t^{t+k} (|y(s)| + |y'(s)|)ds = 0$$

then $\lim\limits_{t\to+\infty} |y(t)| = 0$.

Proof. Assume $\lim\limits_{t\to+\infty} |y(t)| \neq 0$, then there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$, $t_n \to +\infty$, such that

$$|y(t_n)| > C > 0\,, \quad \forall n \geqslant n_0.$$

We pick $0 < \theta \leqslant k$, then for all $n \geqslant n_0$ we have

$$||y(t_n + \theta)| - |y(t_n)|| \leqslant |y(t_n + \theta) - y(t_n)| \leqslant \int_{t_n}^{t_n+\theta} |y'(s)|ds \leqslant \int_{t_n}^{t_n+k} |y'(s)|ds$$

Therefore $|y(s)| > C/2$ for all $s \in [t_n, t_n + k]$, $n \geqslant n_0$. The last inequality is a contradiction with

$$\lim_{n\to+\infty} \int_{t_n}^{t_n+k} |y(s)|ds = 0.$$

$\blacksquare$

Theorem 3.32. *Let $\delta = 0$, if $(u_0, w_0) \in \left(\mathcal{C}^{2+\alpha}(\overline{\Omega})\right)^2$, $u_0 \geqslant 0$, $w_0 > 0$ with $F(u_0, w_0) < +\infty$ then,*

$$\lim_{t\to+\infty} \|u - \overline{u}\|_2 = 0\,, \qquad \lim_{t\to+\infty} \|w\|_2 = 0. \tag{3.123}$$

Proof. We are going to apply Lemma 3.31 with $y(t) = \int_\Omega (u - \overline{u})^2(t)dx$. Taking into account the Poincare-Wintinger inequality and Remark 3.30 we get

$$\int_0^\infty \int_\Omega (u - \overline{u})^2 \leqslant C \int_0^\infty \int_\Omega |\nabla u|^2 \leqslant C. \tag{3.124}$$

On the other hand, thanks to the Hölder's inequality, Young's inequality and the uniform bound in L^∞ of u, we have

$$
\begin{aligned}
\left|\frac{d}{2dt}\int_\Omega (u-\overline{u})^2\right| &= \left|\int_\Omega |\nabla u|^2 + \int_\Omega u\nabla u\cdot\nabla w\right| \\
&\leqslant \int_\Omega |\nabla u|^2 + \int_\Omega |u\nabla u\cdot\nabla w| \\
&\leqslant \int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega u|\nabla u|^2 + \frac{1}{2}\int_\Omega u|\nabla w|^2 \\
&\leqslant (1+C)\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega u|\nabla w|^2 ,
\end{aligned}
\tag{3.125}
$$

so, thanks to Lemma 3.31, we infer that for $t\to+\infty$, $u(\cdot,t)\to\overline{u}$ in $L^2(\Omega)$. From (3.23) we have

$$
\begin{aligned}
\frac{d}{2dt}\int_\Omega w^2 + \overline{u}\int_\Omega w^2 &= \int_\Omega (\overline{u}-u)w^2 \\
&\leqslant \left(\int_\Omega (u-\overline{u})^2\right)^{1/2}\left(\int_\Omega w^4\right)^{1/2} \\
&\leqslant w_0\left(\int_\Omega (u-\overline{u})^2\right)^{1/2}\left(\int_\Omega w^2\right)^{1/2} \\
&\leqslant C\int_\Omega (u-\overline{u})^2 + \epsilon\int_\Omega w^2.
\end{aligned}
\tag{3.126}
$$

Therefore, after integrating (3.126) in time, we infer

$$
\int_0^\infty\int_\Omega w^2 \leqslant C,
\tag{3.127}
$$

Moreover, arguing as in (3.126) we get

$$
\int_0^\infty\left|\frac{d}{dt}\int_\Omega w^2\right| \leqslant C.
\tag{3.128}
$$

Hence, for $t\to+\infty$, $w(\cdot,t)\to 0$ in $L^2(\Omega)$. ∎

In what follows we will study also the case $\delta>0$ and we will provide an estimate of the rate of convergence to the equilibrium.

Lemma 3.33. *Let* $(u_0,w_0)\in\left(C^{2+\alpha}(\overline{\Omega})\right)^2$, $w_0>0$. *If* $u_0>0$ *then* $u(t)>0$ *for all* $t>0$.

Proof. We know that

$$
v_t = e^{-w}\nabla\cdot(e^w\nabla v) + \delta v(1-ve^w) + v^2e^w w.
\tag{3.129}
$$

On multiplying (3.129) by $e^w(v-k)_-$ with k a positive constant to be fixed later on, we get

$$
\begin{aligned}
\frac{d}{2dt}\int_\Omega e^w(v-k)_-^2 = &-\int_\Omega e^w|\nabla(v-k)_-|^2 + \delta\int_\Omega e^w(v-k)_-v(1-ve^w)+ \\
&+\int_\Omega v^2e^{2w}w(v-k)_- - \frac{1}{2}\int_\Omega e^{2w}v(v-k)_-^2.
\end{aligned}
\tag{3.130}
$$

In order to show that the right-hand side of (3.130) is non-negative it is enough to prove that

$$
\delta e^w(v-k)_-v(1-ve^w)\leqslant 0.
\tag{3.131}
$$

Observe that for k sufficiently small $ke^w \leqslant 1$, therefore (3.131) is true for such a choice of k. Since the right-hand side of (3.130) we obtain

$$e^{w(t)}(v(t)-k)_-^2 \leqslant e^{w_0}(v_0-k)_-^2. \tag{3.132}$$

Taking into account that $u_0 > 0$ then $v_0 > 0$. Hence $v_0 > \theta > 0$ and for $0 < k < \theta$ we get $v(t) > 0$. ∎

Theorem 3.34. *Assume the same hypothesis as in Theorem 3.32 together with the additional condition $u_0 > 0$ then*

$$\lim_{t\to+\infty} \|u(t)-\overline{u}\|_1 \leqslant Ce^{-\theta t}, \qquad \lim_{t\to+\infty} \|w\|_\infty \leqslant Ce^{-\theta' t}, \tag{3.133}$$

for any $\theta, \theta' > 0$.

Proof. Observe that $w(x,t) = w_0(x)e^{-\int_0^t u}$, thus, applying Lemma 3.33 we get

$$\lim_{t\to+\infty} \|w\|_\infty \leqslant Ce^{-\theta' t}.$$

For the convergence of u we use the following argument. It is not difficult to infer that

$$\frac{d}{dt}G(u(t),w(t)) = -D(u(t),w(t)) \tag{3.134}$$

with

$$G(u(t),w(t)) := \int_\Omega u(t)\ln\left(\frac{u(t)}{\overline{u}}\right)dx + \frac{1}{2}\int_\Omega w(t)^{-1}|\nabla w(t)|^2 dx \tag{3.135}$$

and D was defined in (3.49). We know that for $r \geqslant 0$

$$r\ln r - 1 + r \leqslant (r-1)^2 \tag{3.136}$$

Therefore, putting $r = u/\overline{u}$ and thanks to the Poincare-Wintinger inequality we have

$$\begin{aligned}
\int_\Omega u\ln\left(\frac{u}{\overline{u}}\right) &= \overline{u}\int_\Omega r\ln r + \overline{u}\int_\Omega (r-1) \\
&\leqslant \overline{u}\int_\Omega (r-1)^2 \\
&= \frac{1}{\overline{u}}\int_\Omega (u-\overline{u})^2 \\
&\leqslant \frac{C_{pw}}{\overline{u}}\left(\int_\Omega |\nabla u|\right)^2 \\
&= \frac{4C_{pw}}{\overline{u}}\left(\int_\Omega |u^{1/2}\nabla u^{1/2}|\right)^2 \\
&\leqslant \frac{4C_{pw}}{\overline{u}}\left(\int_\Omega u\right)\left(\int_\Omega |\nabla u^{1/2}|^2\right) \\
&= |\Omega| C_{pw}\int_\Omega u^{-1}|\nabla u|^2
\end{aligned} \tag{3.137}$$

On the other hand, taking $\varphi = u/\overline{u}$ and applying Jensen's inequality with the probability measure $d\mu = \frac{\overline{u}}{\|\overline{u}\|_1}dx$ we get

$$\int_\Omega u\ln\left(\frac{u}{\overline{u}}\right)dx = \|\overline{u}\|_1\int_\Omega \varphi\ln\varphi\, d\mu \geqslant \left(\int_\Omega \varphi\, d\mu\right)\ln\left(\int_\Omega \varphi\, d\mu\right) = 0 \tag{3.138}$$

Hence, thanks to (3.137), (3.138) and Lemma 3.33 we obtain

$$\min\left\{\frac{1}{|\Omega|C_{pw}},\rho\right\}G(u(t),w(t))\leqslant D(u(t),v(t)) \tag{3.139}$$

with $u(t)\geqslant\rho$. Finally the statement of the Theorem follows from the Csiszar-Kullback inequality. ∎

Theorem 3.35. *Assume $\delta>0$, $(u_0,w_0)\in\left(C^{2+\alpha}(\overline{\Omega})\right)^2$, $u_0>0$, $w_0>0$. Then,*

$$\lim_{t\to+\infty}\|u-1\|_2\leqslant Ce^{-\theta''t},\qquad \lim_{t\to+\infty}\|w\|_\infty\leqslant Ce^{-\theta't}, \tag{3.140}$$

for any $\theta',\theta''>0$.

Proof. The convergence for w is exactly as in Theorem 3.34. We know that

$$\frac{d}{dt}H(t)=-D(t) \tag{3.141}$$

with

$$H(u,w):=\int_\Omega u(\ln u-1)+1+\frac{1}{2}\int_\Omega w^{-1}|\nabla w|^2. \tag{3.142}$$

Taking into account Lemma 3.33 we have $u(t)\geqslant\rho$. This fact together with the inequality $\ln s\leqslant s-1$ for $s\geqslant 0$ provide us with the following

$$\begin{aligned}D(u,w)&\geqslant\frac{\rho}{2}\int_\Omega w^{-1}|\nabla w|^2+\delta\rho(u-1)\ln u\\&\geqslant\frac{\rho}{2}\int_\Omega w^{-1}|\nabla w|^2+\delta\rho(u(\ln u-1)+1)\geqslant\min\{\rho,\delta\rho\}H(u(t),w(t))\end{aligned} \tag{3.143}$$

Therefore, $H(t)\leqslant H(0)e^{-\beta t}$, $\beta=\min\{\rho,\delta\rho\}$. In particular, since $u(\ln u-1)+1\geqslant 0$ we have

$$\int_\Omega w^{-1}|\nabla w|^2\leqslant H(0)e^{-\beta t}. \tag{3.144}$$

Let $0<\theta''<\min\{\delta\rho,\beta\}$. On multiplying (3.22) by $u-1$ and integrating in the space variable we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega(u-1)^2&=-2\int_\Omega|\nabla u|^2+2\int_\Omega u\nabla u\cdot\nabla w-2\int_\Omega(u-1)^2\\&\leqslant(\epsilon C-2)\int_\Omega|\nabla u|^2+C\int_\Omega w^{-1}|\nabla w|^2-\theta''\int_\Omega(u-1)^2\\&\leqslant Ce^{-\beta t}-\theta''\int_\Omega(u-1)^2\end{aligned} \tag{3.145}$$

From (3.145) it is not difficult to conclude the Theorem. ∎

CHAPTER 4

The invasion model

The aim of this chapter is to study the steady-states, global well-possedness and asymptotic behaviour of some models related to tissue invasion that were proposed in [63] and [15]. Basically we prove local well-possedness with the use of semigroup theory, we prolong the solutions via a suitable bounds on the solutions and for the asymptotic behaviour we use again suitable estimates on the solutions.

4.1. Introduction

In this chapter we will focus on some models of tissue invasion, more concretively we will study in detail the models proposed in [63] and [15].

Along the chapter $\Omega \subset \mathbb{R}^3$ is a region in which the tumour cells, ECM and proteolytic enzymes lies. We assume that Ω is bounded, connected and has a regular boundary. We denote by u, v and m the concentration of cancer cells, ECM and proteolytic enzymes respectively. The model reads

$$
(4.1)\qquad \begin{cases}
u_t = \underbrace{\Delta u}_{Diffusion} - \underbrace{\nabla \cdot (u\chi(v)\nabla v)}_{Haptotaxis} + \underbrace{\mu u(1-u-v)}_{Proliferation} & \text{in} \quad \Omega \times (0,T), \\
v_t = -\underbrace{mv}_{Degradation} & \text{in} \quad \Omega \times (0,T), \\
m_t = \underbrace{\Delta m}_{Diffusion} - \underbrace{m}_{Decay} + \underbrace{ug(v)}_{Production} & \text{in} \quad \Omega \times (0,T), \\
\dfrac{\partial u}{\partial n} - u\chi(v)\dfrac{\partial v}{\partial n} = \dfrac{\partial m}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0,T), \\
(u,v,m)(x,0) = (u_0,v_0,m_0)(x) & \text{in} \quad \Omega,
\end{cases}
$$

where χ and g are given functions the following conditions

$$
(4.2)\qquad \chi \in C^1(\mathbb{R}),\ \chi, \chi' \text{ is globally Lipchitz},\ \chi \geqslant 0
$$

and

$$g \in C(\mathbb{R}),\ g \text{ is globally Lipchitz},\ g \geqslant 0 \tag{4.3}$$

It should be noted that we are assuming that many of the biological parameters are just 1. The only reason for that is the simplification of the notation. Most of our results does not depend on those parameters, we will just point out the differences, if there any, for different values of the parameters.

The chapter is organized as follows. In section 2 we give some preliminaries and notations. Section 3 is devoted to the stationary problem associated to (4.1). In section 4 we prove local existence and the non-negative of solutions for non-negative initial data. In section 5 we show that the solution constructed in the previous section can be prolonged in time till infinity. In the last section we show the convergence to the steady-states even with explicit rate of convergence in some cases.

4.2. Preliminaries and notations

In this section we collect some tools and notations that will be used along the chapter. We denote by $C^{\nu}(\overline{\Omega})$ the space of Hölder continuous functions. By $L^p(\Omega)$, $W^{k,p}(\Omega)$, $p \geqslant 1$ we denote the Lebesgue space and Sobolev spaces of functions on Ω with the usual norms $\|\cdot\|_p$, $\|\cdot\|_{k,p}$ and $W^{k,2}(\Omega) = H^k(\Omega)$. If X is a Banach space with norm $\|\cdot\|_X$, for $T > 0$ we denotes by $L^p(0,T;X)$ the Banach space of all measurable functions $u : (0,T) \to X$ such that $\|u(\cdot)\|_X \in L^p(0,T)$. If G is an interval of real numbers, the notation $C(G,X)$ stands for the space of continuous functions with values in X. If $f \in L^1(\Omega)$ then $\overline{u}$ denotes the mean value, i.e. $\overline{f} := \frac{1}{|\Omega|}\int_{\Omega} f$. The principal eigenvalue to the problem

$$\begin{cases} -\Delta\phi + a(x)\phi = \lambda\phi & \text{in } \Omega, \\ \frac{\partial\phi}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

is denoted by $\sigma_1(-\Delta + a(x), \mathcal{N})$. Throughout the chapter C will denote a generic constant that may vary from line to line.

Let $p \in (1,\infty)$, we define the operator

$$A_p u := -\Delta u + u$$

with domain

$$D(A_p) := \left\{ u \in W^{2,p}(\Omega) : \ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}$$

Since $Re\ \sigma(A_p) \geqslant 1$, where $\sigma(A_p)$ stands for the spectrum of A_p, then A_p possesses fractional powers, A_p^{β}. Let

$$X_p^{\beta} := D(A_p^{\beta}),$$

then the following embedding properties are known, (see, [34, Theorem 1.6.1])

$$\begin{array}{lll} X_p^{\beta} \hookrightarrow W^{k,q}(\Omega) & \text{for} & k - \frac{d}{q} < 2\beta - \frac{N}{p}, \quad q \geqslant p \\ X_p^{\beta} \hookrightarrow C^{\nu}(\overline{\Omega}) & \text{for} & 0 \leqslant \nu < 2\beta - \frac{N}{p}. \end{array} \tag{4.4}$$

Since A_p is a sectorial operator, then

$$S(t) := e^{-tA_p}$$

defines an analytical semigroup in $L^p(\Omega)$. The operator $S(t)$ has the following properties,

1) In [34, Theorem 1.3.4] it is stated that

$$\|S(t)\|_{\mathcal{L}(L^p,L^p)} \leqslant Ce^{-\delta t}, \tag{4.5}$$

with $\delta \in (0,1)$.

2) We have

$$A_p^\alpha S(t) = S(t)A_p^\alpha \quad \text{on } D(A_p^\alpha), \tag{4.6}$$

Moreover, combining (4.6) with (4.5) we obtain

$$\|S(t)\|_{\mathcal{L}(X_p^\beta,X_p^\beta)} \leqslant Ce^{-\delta t}. \tag{4.7}$$

3) For $u \in L^p(\Omega)$, (see, [34, Theorem 1.4.3]), we have

$$\|S(t)u\|_{X_p^\beta} \leqslant C_\beta t^{-\beta} e^{-\delta t}\|u\|_p, \;\; t > 0, \;\; \delta \in (0,1). \tag{4.8}$$

4.3. Positive steady-states

In this section we will describe the positive solutions to

$$\begin{cases} 0 = \nabla\cdot(\nabla u - u\chi(v)\nabla v) + \mu u(1-u-v) & \text{in} \quad \Omega, \\ 0 = -mv & \text{in} \quad \Omega, \\ 0 = \Delta m - m + ug(v) & \text{in} \quad \Omega, \\ \\ \dfrac{\partial u}{\partial n} - u\chi(v)\dfrac{\partial v}{\partial n} = \dfrac{\partial m}{\partial n} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{4.9}$$

Observe that after the change of variables

$$w := uz, \qquad z := e^{-\int_0^v \chi(s)ds}. \tag{4.10}$$

the system (4.9) reads

$$\begin{cases} 0 = \Delta w + \chi(v)\nabla w\cdot\nabla v + \mu w(1 - z^{-1}w - v) & \text{in} \quad \Omega, \\ 0 = mv & \text{in} \quad \Omega, \\ 0 = \Delta m - m + wz^{-1}g(v) & \text{in} \quad \Omega, \\ \\ \dfrac{\partial w}{\partial n} = \dfrac{\partial m}{\partial n} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{4.11}$$

Since we are interested basically in the models given in [15] and [63] we assume that $g(v) = 1$ and $g(v) = v$ respectively.

Definition 4.1. *We say that (w,v,m) are positive solutions to (4.11) if*

$$(w,v,m) \in (\mathcal{P}_1 \cup \{0\}) \times (\mathcal{P}_2 \cup \{0\}) \times (\mathcal{P}_1 \cup \{0\}), \quad (w,v,m) \neq (0,0,0),$$

with

$$\begin{aligned} \mathcal{P}_1 &:= \left\{ z \in W^{2,p}(\Omega) : \ p > N, \ z \geqslant 0, \ z \neq 0, \ \tfrac{\partial z}{\partial n} = 0 \ on \ \partial\Omega \right\} \\ \mathcal{P}_2 &:= \{ z \in W^{1,\infty}(\Omega) : \ z \geqslant 0, \ z \neq 0 \} \end{aligned}$$

Theorem 4.2. *Assume $g(v) = 1$, then the positive solutions to (4.11) are given by*

$$\begin{aligned} (w^*, v^*, m^*) &= (0, \widetilde{v}, 0), \quad \widetilde{v} \in \mathcal{P}_2, \\ (w^*, v^*, m^*) &= (k, 0, k), \quad if \ \mu = 0, \\ (w^*, v^*, m^*) &= (1, 0, 1), \quad if \ \mu > 0. \end{aligned}$$

where $k > 0$ is any constant.

Proof. Case 1.- Assume $w = 0$. Taking into account that $\sigma_1(-\Delta + 1, \mathcal{N}) > 0$ then the unique solution to $(4.11)_3$ is $m = 0$. Now, for $\mu \geqslant 0$ we can see that $(0, \widetilde{v}, 0)$ with $\widetilde{v} \in \mathcal{P}_2$ are positive solutions to (4.11).
Case 2.- Assume $u \neq 0$. Applying the strong maximum principle to $(4.11)_3$, see for instance [4, Theorem 2.4], we deduce $m \in int(\mathcal{P}_1)$, then from $(4.9)_2$, we get $v = 0$. Now, we distinguish between two cases.

- If $\mu = 0$, then from $(4.11)_1$ we have

$$\left\{ \begin{array}{ll} -\Delta w = 0 & \text{in } \Omega \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega \end{array} \right. \tag{4.12}$$

  Taking w as a test function in (4.12), we obtain that $w = k$ is constant. Finally, from $(4.11)_3$ we get $m = k$. Therefore, $(k, 0, k)$ are positive solutions to (4.11).

- If $\mu > 0$, then it is clear that $w = 1$ is a solution to $(4.9)_3$. The uniqueness of positive solution follows by a standard result, see for example [56, Theorem 2]. Therefore, $(1, 0, 1)$ is a positive solution to (4.11).

■

Theorem 4.3. *Assume $g(v) = v$, then the positive solutions to (4.11) are given by*

$$\begin{aligned} (w^*, v^*, m^*) &= (0, \widetilde{v}, 0), \quad \widetilde{v} \in \mathcal{P}_2, \\ (w^*, v^*, m^*) &= (k, 0, 0), \quad if \ \mu = 0, \\ (w^*, v^*, m^*) &= (1, 0, 0), \quad if \ \mu > 0. \end{aligned}$$

where $k > 0$ is any constant.

Proof. Case 1.- $wz^{-1}v = 0$ then $m = 0$, therefore $mv = 0$ independently of v. Let us observe that w satisfies $(mv = 0)$.

$$-\nabla \cdot (z^{-1} \nabla w) = \mu w z^{-1} (1 - w z^{-1}), \quad \frac{\partial w}{\partial n} = 0. \tag{4.13}$$

Now, we consider two cases,

- $\mu = 0$. Taking w as a test function in the weak formulation of (4.13) we obtain $w = k$, $k \geqslant 0$.
  - if $k = 0$ then $v \in \mathcal{P}_2$. So, $(0, \widetilde{v}, 0)$, $\widetilde{v} \in \mathcal{P}_2$ are solutions to (4.11).
  - if $k > 0$. Then, taking into account that $ke^{\int_0^k \chi(s)ds} v = 0$, then $v = 0$. Therefore, $(k, 0, 0)$ with $k > 0$ are solutions to (4.11).
- $\mu > 0$.
  - If $w = 0$ then $(0, \widetilde{v}, 0)$ with $\widetilde{v} \in \mathcal{P}_2$ are solutions to (4.11).
  - If $w \neq 0$ then from the strong maximum principle $w \in int(\mathcal{P}_1)$. Therefore we have to take $v = 0$ in order to fulfill $wz^{-1}v = 0$. Then reasoning as in Theorem 4.2 we conclude $w = 1$. So, $(1, 0, 0)$ is a solution to (4.11).

Case 2.- $wz^{-1}v \neq 0$ is exactly as in the case 2.- in Theorem 4.2. ∎

Remark 4.4. *From Theorems 4.2, 4.3 we can easily recover the steady-states for the system (4.9).*

Remark 4.5. *We would like to point out that if the m-equation i.e.* $(4.9)_3$ *is of the form* $m_t = \Delta m - \beta m + \alpha u$ *with* $\alpha > 0$, $\beta > 0$ *then the steady-states differs from the case that we have treated in Theorem 4.2* ($\alpha = \beta = 1$). *In this case the positive steady-states of (4.9) are given by*

$$(u^*, v^*, m^*) = (0, \widetilde{v}, 0)\,, \quad \widetilde{v} \in \mathcal{P}_2,$$
$$(u^*, v^*, m^*) = \left(k, 0, \frac{\beta k}{\alpha}\right)\,, \quad if\ \mu = 0,$$
$$(u^*, v^*, m^*) = \left(1, 0, \frac{\beta}{\alpha}\right)\,, \quad if\ \mu > 0.$$

The proof of this fact easily follows from Theorem 4.2.

4.4. Local existence and non-negativity

As we did for the steady-states problem it is convenient to transform the system (4.1) in the following manner

$$w := uz, \qquad z := e^{-\int_0^v \chi(s)ds}. \tag{4.14}$$

Therefore, in terms of the variables w, v, m the system (4.1) becomes

$$\begin{cases} w_t = \Delta w + \chi(v)\nabla w \cdot \nabla v + \mu w(1 - z^{-1}w - v) + w\chi(v)mv & \text{in} \quad \Omega \times (0,T), \\ v_t = -mv & \text{in} \quad \Omega \times (0,T), \\ m_t = \Delta m - m + wz^{-1}g(v) & \text{in} \quad \Omega \times (0,T), \\ \\ \dfrac{\partial w}{\partial n} = \dfrac{\partial m}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ (w, v, m)(x, 0) = \left(u_0 e^{-\int_0^{v_0} \chi}, v_0, m_0\right)(x) & \text{in} \quad \Omega. \end{cases} \tag{4.15}$$

Next, the local existence theorem for (4.15) is formulated.

Theorem 4.6. *Let $\gamma \in \left(\frac{1}{2}+\frac{3}{2p},1\right)$, $p \in (3,6)$. Suppose that the initial data satisfies*

$$\mathbf{y_0} := (w_0, v_0, m_0) \in \mathbf{Y} := X_2^{1/2} \times W^{1,\infty}(\Omega) \times X_p^\gamma,$$

then there exists $\tau(\|\mathbf{y_0}\|_{\mathbf{Y}})$ such that the problem (4.15) has a unique solution

$$\begin{array}{ll} w & \in C\left([0,\tau]; X_2^{1/2}\right) \cap C^1\left((0,\tau); X_2^{1/2}\right) \cap C((0,\tau); W^{2,p}(\Omega))\,, \\ v & \in C\left([0,\tau]; W^{1,\infty}(\Omega)\right) \cap C^1\left((0,\tau); W^{1,\infty}(\Omega)\right)\,, \\ m & \in C\left([0,\tau]; X_p^\gamma\right) \cap C^1\left((0,\tau); X_p^\gamma\right) \cap C\left((0,\tau); W^{2,p}(\Omega)\right)\,. \end{array} \tag{4.16}$$

Moreover, the solution depends continuously on the initial data, i.e. if $\mathbf{u}(\mathbf{u_0})$ and $\mathbf{u}(\overline{\mathbf{u}}_\mathbf{0})$ are the solutions to (4.15) with initial data $\mathbf{u_0}$ and $\overline{\mathbf{u}}_\mathbf{0}$ respectively then

$$\|\mathbf{u}(\mathbf{u_0}) - \mathbf{u}(\overline{\mathbf{u}}_\mathbf{0})\|_{C\left([0,\tau];X_2^{1/2}\right)\times C([0,\tau];W^{1,\infty}(\Omega))\times C\left([0,\tau];X_p^\gamma\right)} \leqslant C\|\mathbf{u_0} - \overline{\mathbf{u}}_\mathbf{0}\|_{\mathbf{Y}}.$$

Proof. The proof of the theorem it is based on a standard contraction argument. Let $T > 0$, $t \leqslant T$, $R > 0$ and

$$\theta(w,v,m) := w + \chi(v)\nabla w \cdot \nabla v + \mu w(1 - z^{-1}w - v) + w\chi(v)mv. \tag{4.17}$$

We consider the spaces

$$\begin{array}{ll} X_T & := C\left([0,T]; X_2^{1/2}\right)\,, \\ Y_T & := C\left([0,T]; W^{1,\infty}(\Omega)\right)\,, \\ Z_T & := C\left([0,T]; X_p^\gamma\right)\,, \end{array}$$

the operator

$$\mathbf{F}(w,v,m) := \begin{pmatrix} F_1(w,v,m) \\ F_2(w,v,m) \\ F_3(w,v,m) \end{pmatrix}$$

with F_1, F_2, F_3 given by

$$\begin{array}{ll} F_1(w,v,m) & := S(t)w_0 + \displaystyle\int_0^t S(t-s)\theta(w,v,m)ds\,, \\ F_2(w,v,m) & := v_0 - \displaystyle\int_0^t mv\, ds\,, \\ F_3(w,v,m) & := S(t)m_0 + \displaystyle\int_0^t S(t-s)\left(wz^{-1}g(v)\right)ds\,. \end{array}$$

and the closed set

$$B_R^T := \{(w,v,m) \in X_T \times Y_T \times Z_T \,:\, \|w-w_0\|_{X_T} + \|v-v_0\|_{Y_T} + \|m-m_0\|_{Z_T} \leqslant R\}.$$

For a fixed R we try to find τ_0 such that $\mathbf{F}(B_R^{\tau_0}) \subset B_R^{\tau_0}$. Let $(w,v,m) \in B_R^T$ and $t \leqslant T$. Having in mind that $w_0 \in X_2^{1/2}$ and (4.8), we obtain

$$\begin{array}{ll} \|F_1(w,v,m) - w_0\|_{X_2^{1/2}} & \leqslant \dfrac{R}{4} + \displaystyle\int_0^t \|S(t-s)\theta(w,v,m)\|_{X_2^{1/2}} \\ & \leqslant \dfrac{R}{4} + \displaystyle\int_0^t (t-s)^{-1/2}e^{-\delta(t-s)}\|\theta(w,v,m)\|_2. \end{array} \tag{4.18}$$

Thanks to (4.2), (4.4) and taking into account that $X_2^{1/2} = H^1(\Omega)$, we estimate

$$\|F_1(w,v,m) - w_0\|_{X_T} \leqslant \frac{R}{4} + C(R, \|w_0\|_{1,2}, \|v_0\|_{1,\infty}, \|m_0\|_{X_p^\gamma}) T^{1/2}. \tag{4.19}$$

Also, we have

$$\|F_2(w,v,m) - v_0\|_{1,\infty} \leqslant \int_0^t \|v\|_{1,\infty}\|m\|_{1,\infty}.$$

Therefore, by (4.4) we deduce

$$\|F_2(w,v,m) - v_0\|_{Y_T} \leqslant C(R, \|v_0\|_{1,\infty}, \|m_0\|_{X_p^\gamma}) T \tag{4.20}$$

In the same manner, by (4.3) and the embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$, we obtain

$$\|F_3(w,v,m) - w_0\|_{Z_T} \leqslant \frac{R}{4} + \frac{C(R, \|w_0\|_{1,2}, \|v_0\|_{1,\infty}, \|m_0\|_{X_p^\gamma})}{1-\gamma} T^{1-\gamma}. \tag{4.21}$$

Finally, thanks to (4.19), (4.20) and (4.21) we can choose $0 < T = \tau_0$ small enough in order to obtain $\mathbf{F}(B_R^{\tau_0}) \subset B_R^{\tau_0}$. Moreover, if $t < \tau_0$ then $\mathbf{F}(B_R^t) \subset B_R^t$.

In what follows we try to have the following estimate

$$\|\mathbf{F}(\mathbf{u}) - \mathbf{F}(\overline{\mathbf{u}})\|_{\mathbf{X}} \leqslant C\|\mathbf{u} - \overline{\mathbf{u}}\|_{\mathbf{X}}, \tag{4.22}$$

where $C < 1$, $\mathbf{X} := X_t \times Y_t \times Z_t$ and $\mathbf{u}, \overline{\mathbf{u}} \in \mathbf{X}$. Basically the proof is as the proof of $F(B_R^t) \subset (B_R^t)$. We have

$$\|F_1(w,v,m) - F_1(\overline{w},\overline{v},\overline{m})\|_{X_t} \leqslant t^{1/2} C(R, \|w_0\|_{1,2}, \|v_0\|_{1,\infty}, \|m_0\|_{X_p^\gamma}) \|\mathbf{u} - \overline{\mathbf{u}}\|_{\mathbf{X}}, \tag{4.23}$$

$$\|F_2(w,v,m) - F_2(\overline{w},\overline{v},\overline{m})\|_{X_t} \leqslant t C(R, \|v_0\|_{1,\infty}, \|m_0\|_{X_p^\gamma}) \|\mathbf{u} - \overline{\mathbf{u}}\|_{\mathbf{X}}, \tag{4.24}$$

$$\|F_3(w,v,m) - F_3(\overline{w},\overline{v},\overline{m})\|_{X_t} \leqslant t^{1-\gamma} \frac{C(R, \|w_0\|_{1,2}, \|v_0\|_{1,\infty}, \|m_0\|_{X_p^\gamma})}{1-\gamma} \|\mathbf{u} - \overline{\mathbf{u}}\|_{\mathbf{X}}. \tag{4.25}$$

Thus, choosing $t = \tau$ small enough we obtain the estimate (4.22).

Let $t_0 \in (0, \tau)$ fixed, then [34, Lemma 3.5.2] entails

$$\frac{d}{dt} w(\cdot, t_0) \in X_2^\alpha, \qquad \frac{d}{dt} m(\cdot, t_0) \in X_p^\alpha \tag{4.26}$$

for any $\alpha < 1$. Therefore, by (4.26) we obtain

$$(w, m) \in C^1((0,\tau); X_2^{1/2}) \times C^1((0,\tau); X_p^\gamma).$$

Next, we rewrite the first equation of (4.15) in the following form

$$-\Delta w - b \cdot \nabla w + w = f - \frac{\partial w}{\partial t} \tag{4.27}$$

with $b \in (L^\infty(\Omega))^N$ and $f \in L^2(\Omega)$. Therefore, from the elliptic regularity, we get $w(\cdot, t_0) \in H^2(\Omega)$. Now, with the estimate $w(\cdot, t_0) \in H^2(\Omega)$, and the Sobolev embedding, we can repeat again the same procedure, but with $f \in L^p(\Omega)$, thus

$$w \in C((0,\tau); W^{2,p}(\Omega)).$$

In the same manner we have

$$m \in C((0,\tau); W^{2,p}(\Omega)).$$

Now, we prove the continuity of the solutions respect to the initial data. Let $R > 0$ large enough to have $\mathbf{u}(\mathbf{u_0}), \mathbf{u}(\overline{\mathbf{u}}_\mathbf{0}) \in B_R^\tau$. We have

$$\begin{aligned} \|\mathbf{u}(\mathbf{u_0}) - \mathbf{u}(\overline{\mathbf{u}}_\mathbf{0})\|_\mathbf{X} \leqslant \|S(t)(w_0 - \overline{w}_0)\|_{X_\tau} + \|v_0 - \overline{v}_0\|_{Y_\tau} + \|S(t)(m_0 - \overline{m}_0)\|_{Z_\tau} + \\ + \|\mathbf{F}(\mathbf{u}(\mathbf{u_0})) - \mathbf{F}(\mathbf{u}(\overline{\mathbf{u}}_\mathbf{0}))\|_\mathbf{X}. \end{aligned} \tag{4.28}$$

Taking into account (4.7) and the contractivity of $\mathbf{F}$ we get

$$\|\mathbf{u}(\mathbf{u_0}) - \mathbf{u}(\overline{\mathbf{u}}_\mathbf{0})\|_\mathbf{X} \leqslant C\|\mathbf{u_0} - \overline{\mathbf{u}}_\mathbf{0}\|_\mathbf{Y} + K\|\mathbf{u}(\mathbf{u_0}) - \mathbf{u}(\overline{\mathbf{u}}_\mathbf{0})\|_\mathbf{X}, \tag{4.29}$$

with $K < 1$. Thus, the proof is finished. ∎

From Theorem 4.6 we easily get the local existence Theorem for the original system (4.1). We have just to observe that

$$w(x,t) = u(x,t)z(x,t).$$

Therefore $w_0 \in H^1(\Omega)$, if $u_0 \in X_2^{1/2} = H^1(\Omega)$ and $v_0 \in W^{1,\infty}(\Omega)$. Then we apply Theorem 4.6 obtaining the regularity on (w, v, m). Finally the regularity of u can be recover having in mind that, for $t > 0$

$$u(x,t) = w(x,t)z(x,t),$$

with $w(\cdot,t), z(\cdot,t) \in W^{1,\infty}(\Omega)$.

Corollary 4.7. *Let $\gamma \in \left(\frac{1}{2} + \frac{3}{2p}, 1\right)$, $p \in (3,6)$. Suppose that the initial data satisfies*

$$\mathbf{x_0} := (u_0, v_0, m_0) \in H^1(\Omega) \times W^{1,\infty}(\Omega) \times X_p^\gamma,$$

then there exists $\tau(\|\mathbf{x_0}\|_\mathbf{Y})$ such that the problem (4.1) has a unique solution

$$\begin{aligned} u &\in C\left([0,\tau]; H^1(\Omega)\right) \cap C^1\left((0,\tau); W^{1,\infty}(\Omega)\right), \\ v &\in C\left([0,\tau]; W^{1,\infty}(\Omega)\right) \cap C^1\left((0,\tau); W^{1,\infty}(\Omega)\right), \\ m &\in C\left([0,\tau]; X_p^\gamma\right) \cap C^1\left((0,\tau); X_p^\gamma\right) \cap C\left((0,\tau); W^{2,p}(\Omega)\right). \end{aligned} \tag{4.30}$$

Moreover, the solution depends continuously on the initial data.

Remark 4.8. *We would like to point out that in our proof of local existence, by contrast with [75, Lemma 2.1], we are not using the explicit knowledge of v. In particular, our method still valid for v-equations of the form $v_t = f(m,v)$, with f satisfying suitable regularity assumptions.*

In rest of the section we will show the non-negativity of the solutions to (4.1) for non-negative initial conditions.

Theorem 4.9. *Let $T < T_{max}$, with T_{max} the maximal existence time of the solutions to (4.1) in the sense given in Corollary 4.7. If $u_0(x), v_0(x), m_0(x) \geqslant 0$ then*

$$u(x,t), v(x,t), m(x,t) \geqslant 0, \qquad \forall (x,t) \in Q_T.$$

Proof. We take, as a test function u_-, in the weak formulation of the first equation of (4.1), then

$$
\begin{aligned}
\frac{d}{2dt}\int_\Omega u_-^2 &= -\int_\Omega |\nabla u_-|^2 + \int_\Omega u_-\nabla v\cdot\nabla u_- + \mu u_-^2 - \mu u_-^3 \\
&\leqslant -\frac{1}{2}\int_\Omega |\nabla u_-|^2 + (\mu + \frac{1}{2}\|\nabla v\|^2_{\infty,Q_T})\int_\Omega u_-^2 - \mu\int_\Omega u_-^3 .
\end{aligned}
\tag{4.31}
$$

In order to estimate the last term in the right-hand side of (4.31) we use Gagliardo-Nirenberg and Young inequalities. Thus,

$$
\|u_-\|_3^3 \leqslant \epsilon\|\nabla u_-\|_2^2 + \epsilon\|u_-\|_2^2 + C(\epsilon)\|u_-\|_2^6. \tag{4.32}
$$

Finally, taking into account that $\|u_-\|^4_{2,Q_T} \leqslant C$, we can put the estimate (4.32) in (4.31) and conclude with Gronwall Lemma that $u_- \equiv 0$ in Q_T. Next, we know that

$$
v(x,t) = v_0(x)e^{-\int_0^t m(x,s)ds}, \tag{4.33}
$$

therefore $v(x,t) \geqslant 0$ in Q_T. Finally, since $ug(v) \geqslant 0$, the non-negativity of m follows from the standard maximum principle for parabolic equations. ■

From now on, in the rest of the chapter we will assume that $(u_0(x), v_0(x), m_0(x)) \geqslant 0$.

4.5. Global Existence

In order to show that $T_{max} = +\infty$, where T_{max} denotes the maximal interval of existence, we have just to show that

$$
\|(u,v,m)\|_{X_T\times Y_T\times Z_T} \leqslant C(T), \tag{4.34}
$$

with $C(T) < \infty$, for all $T > 0$. Basically the method is the following. We apply the Corollary 4.7, this gives us a solution till a time $t_1 > 0$, then we can apply again the Corollary 4.7 with initial data

$$
(u_0, v_0, m_0) = (u(\cdot,t_1), v(\cdot,t_1), m(\cdot,t_1)).
$$

Therefore, recursively we have an increasing sequence of times t_k, $k \in \mathbb{N}$ and thanks to (4.34) $t_k \to +\infty$. This method provides us with the existence of solution on $[0,T]$. It should be stressed that, a priori, we do not have uniqueness of solution on $[0,T]$ for any given $T > 0$ because the Corollary 4.7 just assure uniqueness in B_R^t for t small. However, this difficulty can be solved with the following argument. Let $\mathbf{u_1}, \mathbf{u_2}$ two solutions of (4.1). We define the set A by

$$
A := \{t \in [0,T] : \ \mathbf{u_1}(\cdot,t) \neq \mathbf{u_2}(\cdot,t) \text{ in } \mathbf{Y}\}.
$$

Assume $A \neq \varnothing$, then there exists $t^* = \inf A$ and $t^* > 0$, thanks to Corollary 4.7. Hence, $t^* - \epsilon \notin A$, for all $\epsilon > 0$. Now, applying Corollary 4.7 at time $t^* - \epsilon$ we obtain that $\mathbf{u_1}(\cdot,t) = \mathbf{u_2}(\cdot,t)$ for all $t \in [0, t^* + k]$ with $k > 0$, contradicting the definition of t^*. Therefore $A = \varnothing$, concluding the uniqueness result.

Lemma 4.10. *For every $t \in (0, T_{max})$, we have*

$$
\|u(t)\|_1 \leqslant \max\{|\Omega|, \|u_0\|_1\}\ , \quad \|v(t)\|_\infty \leqslant \|v_0\|_\infty\ , \quad \|w(t)\|_1 \leqslant C\ , \quad \|m(t)\|_\theta \leqslant C\ ,
$$

for all $\theta < 3$.

Proof. Integrating $(4.1)_1$ in space we get

$$\frac{d}{dt}\int_\Omega u = \mu \int_\Omega u(1-u-v). \tag{4.35}$$

Having in mind that $v \geqslant 0$, $u \geqslant 0$ and the inequality

$$\|u\|_1^2 \leqslant \|u\|_2^2|\Omega|,$$

we have from (4.35) that

$$\frac{d}{dt}\|u(t)\|_1 \leqslant \mu\|u(t)\|_1 - \frac{\mu}{|\Omega|}\|u(t)\|_1^2. \tag{4.36}$$

Finally, solving the differential inequality (4.36) we have the boundedness of u. The boundedness of v cames from the fact that $v_t \leqslant 0$. Taking into account that $w = uz$, we obtain

$$\|w(t)\|_1 \leqslant \|u(t)\|_1\|z(t)\|_\infty.$$

Since $\|z(t)\|_\infty \leqslant 1$ we get the boundedness of w. The last bound it is just the parabolic regularity, see for instance, [40, Lemma 4.1]. ∎

Lemma 4.11. *Let $\tau > 0$ as in Corollary 4.7. Then for every $t \in (\tau/2, T_{max})$ and every $p \in (1,\infty)$ we have*

$$\|u(t)\|_p \leqslant C(p). \tag{4.37}$$

Proof. From $(4.15)_1$, we know that

$$w_t = z\nabla\cdot(z^{-1}\nabla w) + \mu w(1 - wz^{-1} - v) + w\chi(v)mv. \tag{4.38}$$

On multiplying (4.38) by $pw^{p-1}z^{-1}$ we get

$$w_t p w^{p-1} z^{-1} = \nabla\cdot(z^{-1}\nabla w)pw^{p-1} + \mu p w^p z^{-1}(1 - z^{-1}w - v) + pw^p\chi(v)mvz^{-1}. \tag{4.39}$$

After integrating (4.39) in space, we deduce

$$\begin{aligned}\frac{d}{dt}\int_\Omega z^{-1}w^p = &-\frac{4(p-1)}{p}\int_\Omega z^{-1}|\nabla w^{p/2}|^2 + \mu p\int_\Omega w^p z^{-1}(1 - z^{-1}w - v) + \\ &+ (p-1)\int_\Omega z^{-1}\chi(v)w^p mv.\end{aligned} \tag{4.40}$$

Having in mind that $z^{-1} \geqslant 1$ and the estimate $\|v(t)\|_\infty \leqslant C$ then,

$$\frac{d}{dt}\int_\Omega z^{-1}w^p \leqslant -\frac{4(p-1)}{p}\int_\Omega |\nabla w^{p/2}|^2 + \mu p\int_\Omega z^{-1}w^p + C(p-1)\int_\Omega w^p m. \tag{4.41}$$

In what follows we estimate the last integral in the right-hand side with the use of Hölder and Gagliardo-Nirenberg inequalities

$$\begin{aligned}\mu p\int_\Omega w^p m &\leqslant \mu p\|w^{p/2}\|_{2j}^2\|m\|_{j'} \\ &\leqslant C\|w^{p/2}\|_{1,2}^{2\theta}\|w^{p/2}\|_{2/p}^{2(1-\theta)}\|m\|_{j'} \\ &\leqslant \epsilon\|w^{p/2}\|_{1,2}^2 + C\left(\|w^{p/2}\|_{2/p}^{2(1-\theta)}\right)^{q'}\end{aligned} \tag{4.42}$$

where $\theta \in (0,1)$ and q', j' stands for the dual exponents of q and j respectively. Observe that in order to fulfill the requirements of the Gagliardo-Nirenberg inequality we have to pick $j < 3$. On the other hand, from Lemma 4.10 we should take $j > 3/2$, thus $j \in (3/2, 3)$ and

$$\mu p \int_\Omega w^p m \leqslant \epsilon \|w^{p/2}\|_{1,2}^2 + C. \tag{4.43}$$

Adding $k\|w^{p/2}\|_2^2$ on both sides of (4.41) and thanks to the estimate (4.43) we get

$$\frac{d}{dt}\int_\Omega z^{-1}w^p + k\|w^{p/2}\|_2^2 \leqslant \left(\epsilon - \frac{4(p-1)}{p}\right)\|\nabla w^{p/2}\|_2^2 + (\mu pC + k + \epsilon)\|w^{p/2}\|_2^2 + C. \tag{4.44}$$

In order to have uniform estimates in time we apply again Gagliardo-Nirenberg inequality together with the young inequality to the last term in (4.44), this gives us

$$\frac{d}{dt}\int_\Omega z^{-1}w^p + \frac{k}{2}\|w^{p/2}\|_2^2 \leqslant \left(2\epsilon - \frac{4(p-1)}{p}\right)\|\nabla w^{p/2}\|_2^2 + C. \tag{4.45}$$

Finally using the fact that $1 \leqslant z^{-1} \leqslant C$ we obtain

$$\frac{d}{dt}\int_\Omega z^{-1}w^p + C_1\int_\Omega z^{-1}w^p \leqslant C_2 \tag{4.46}$$

with C_1, C_2 positive constants. Hence

$$\int_\Omega z^{-1}(t)w^p(t) \leqslant C, \quad \forall t \in (\tau/2, T_{max}). \tag{4.47}$$

Taking into account that $z^{-1} \geqslant 1$, we conclude the proof. ■

Remark 4.12. *In order to cover the case $p = \infty$ in Lemma 4.11 we can use the Moser's technique, see [2]. Since, such a technique is standard and we have done something similar with a different method in the previous chapter we skip the proof.*

Lemma 4.13. *Let p, γ as in Corollary 4.7. Then, for every $t \in (\tau/2, T_{max})$ we have*

$$\|m(t)\|_{X_p^\gamma} \leqslant C. \tag{4.48}$$

Proof. For $t \in (\tau/2, T_{max})$ we know that

$$m(t) = S(t)m(\tau/2) + \int_{\tau/2}^t S(t-s)ug(v)\, ds. \tag{4.49}$$

Therefore,

$$\begin{aligned} \|m(t)\|_{X_p^\gamma} &\leqslant \|A_p^\beta S(t)m(\tau/2)\|_p + \int_{\tau/2}^t \|A_p^\beta S(t-s)ug(v)\|_p\, ds \\ &\leqslant C\left(\frac{\tau}{2}\right)^{-\beta}\|m(\tau/2)\|_p + \int_{\tau/2}^t (t-s)^{-\beta}e^{-\delta(t-s)}\|ug(v)\|_p\, ds. \end{aligned} \tag{4.50}$$

Finally, taking into account Lemma 4.11 we can easily conclude the Lemma. ■

Lemma 4.14. *For every $t \in (0, T_{max})$ we have*

$$\|v(t)\|_{1,\infty} \leqslant Ct. \tag{4.51}$$

Proof. Calculating explicitly the solution to $(4.1)_2$ we get

$$v(x,t) = v_0(x)e^{-\int_0^t m}. \tag{4.52}$$

Thus,

$$\nabla v = e^{-\int_0^t m}\left(\nabla v_0 - v_0\int_0^t \nabla m\right). \tag{4.53}$$

Next, taking into account Lemma 4.13 and the local existence Theorem, we can obtain the desired result bounding (4.52) and (4.53) in $L^\infty(\Omega\times(0,T_{\max}))$. ■

In order to proof the last Lemma of the section we will use the following generalization of Gronwall Lemma. This Lemma can be found, for example in [74]. We state it for the reader's convenience.

Lemma 4.15. *For all $t\geqslant 0$, let three functions λ,ϕ,u be given such that λ is integrable and nonnegative, ϕ is absolutely continuous and u is continuous. If $u(t)\leqslant\phi(t)+\int_0^t\lambda(s)u(s)ds$, then*

$$u(t)\leqslant\int_0^t\phi'(s)exp\left(\int_s^t\lambda(r)dr\right)ds+\phi(0)exp\left(\int_0^t\lambda(r)dr\right).$$

Lemma 4.16. *For $t\in(\tau/2,T_{max})$ we have*

$$\|u(t)\|_{1,2}\leqslant C(t). \tag{4.54}$$

Proof. First we observe that since $u=wz$ and $\|z(t)\|_{1,\infty}\leqslant C$ for all $t\in(0,T_{max})$ then it is enough to prove that $\|w(t)\|_{1,2}\leqslant C$ for all $t\in(\tau/2,T_{max})$. Next, we know that for $t\geqslant\tau/2$ we have

$$w(x,t)=S(t)w(\tau/2)+\int_{\tau/2}^t S(t-s)\theta(w,v,m)\,ds\,, \tag{4.55}$$

with θ given in (4.17). Then, thanks to the information provided by the preceding lemmas, we obtain

$$\begin{aligned}\|w(t)\|_{1,2}&\leqslant C+\int_{\tau/2}^t(t-s)^{-1/2}e^{-\delta(t-s)}\|\theta(w,v,m)\|_2\,ds\\&\leqslant C_1+C_2\int_{\tau/2}^t(t-s)^{-1/2}e^{-\delta(t-s)}s\|\nabla w\|_2\,ds\\&\leqslant C_1+C_2\int_{\tau/2}^t(t-s)^{-1/2}e^{-\delta(t-s)}s\|w\|_{1,2}\,ds.\end{aligned} \tag{4.56}$$

Finally the result follows from the generalized Gronwall Lemma.

Remark 4.17. *A particular consequence of the previous lemmas are the uniform bounds in time*

$$\|u\|_{L^\infty(\Omega\times(0,\infty))}\leqslant C\,,\quad\|v\|_{L^\infty(\Omega\times(0,\infty))}\leqslant C\,,\quad\|m\|_{L^\infty(\Omega\times(0,\infty))}\leqslant C\,.$$

Therefore, it is not possible to have blow-up in L^∞-norm even at infinity.

Remark 4.18. *Under the assumption*

$$(T)\quad m(x,t)>\delta>0 \text{ for all } (x,t)\in\Omega\times[\tau,+\infty).$$

all the bounds of the previous Lemmas are independent of time. In fact, as we will see in the next section, under the assumption (T) we will prove even more.

4.6. Asymptotic behaviour

In order to prove the convergence to the steady states, we will need an additional estimate on the solutions to (4.1). Basically, the key of this convergence is the strong decay of $\|\nabla v(t)\|_2^2$ in time. In the proof of this decay we will use a tricky calculus that unfortunately oblige us to pick $v_0(x) > 0$ for all $x \in \Omega$. Moreover we assume and additional hypothesis. Later on, we will provide sufficient conditions on the initial data u_0 in order to fulfill this additional hypothesis.

Lemma 4.19. *Let $\tau > 0$, $v_0 > 0$. Assume that*

$$m(x,t) \geqslant \delta > 0, \quad \forall t \geqslant \tau \tag{4.57}$$

then for all $t \geqslant \tau$ we have

$$\int_\Omega |\nabla v(t)|_2^2 \leqslant Ce^{-kt}, \tag{4.58}$$

for all $0 < k < \delta$.

Proof. On one hand, from $(4.1)_2$ we know

$$\frac{d}{dt}\int_\Omega |\nabla v^{1/2}|^2 = -\int_\Omega m|\nabla v^{1/2}|^2 - \frac{1}{2}\int_\Omega \nabla m \cdot \nabla v. \tag{4.59}$$

On the other hand from $(4.1)_3$ we have

$$\int_\Omega m_t v = -\int_\Omega \nabla m \cdot \nabla v - \int_\Omega mv + \int_\Omega ug(v)v. \tag{4.60}$$

Therefore,

$$\frac{1}{2}\int_\Omega \nabla m \cdot \nabla v = \frac{d}{2dt}\int_\Omega mv - \frac{1}{2}\int_\Omega v_t m + \frac{1}{2}\int_\Omega mv - \frac{1}{2}\int_\Omega ug(v)v. \tag{4.61}$$

Using (4.61) in (4.59) we get

$$\frac{d}{dt}\int_\Omega |\nabla v^{1/2}|^2 + \int_\Omega m|\nabla v^{1/2}|^2 = \frac{d}{2dt}\int_\Omega mv + \frac{1}{2}\int_\Omega m^2 v + \frac{1}{2}\int_\Omega mv - \frac{1}{2}\int_\Omega ug(v)v. \tag{4.62}$$

Now, thanks to (4.57), (4.62) is estimated as follows

$$\begin{aligned} \frac{d}{dt}\left(e^{kt}\int_\Omega |\nabla v^{1/2}|^2\right) &\leqslant e^{kt}\frac{d}{dt}\int_\Omega mv + \frac{e^{kt}}{2}\int_\Omega m^2 v + \frac{e^{kt}}{2}\int_\Omega mv - \frac{e^{kt}}{2}\int_\Omega ug(v)v \\ &\leqslant \frac{d}{dt}\left(e^{kt}\int_\Omega mv\right) + \frac{e^{kt}}{2}\int_\Omega m^2 v + \frac{e^{kt}}{2}\int_\Omega mv. \end{aligned} \tag{4.63}$$

After integrating between τ and t (4.63) we obtain

$$\begin{aligned} e^{kt}\int_\Omega |\nabla v^{1/2}(t)|^2 \leqslant e^{k\tau}\int_\Omega |\nabla v^{1/2}(\tau)|^2 + \frac{e^{kt}}{2}\int_\Omega mv - \frac{e^{k\tau}}{2}\int_\Omega m(\tau)v(\tau) + \\ + \frac{1}{2}\int_\tau^t \left(\int_\Omega m^2 v\right) e^{ks} ds + \frac{1}{2}\int_\tau^t \left(\int_\Omega mv\right) e^{ks} ds. \end{aligned} \tag{4.64}$$

Since $v(x,t) = v(x,\tau)e^{-\int_\tau^t m}$ and $m(x,t) \geqslant \delta$ then

$$v(x,t) \leqslant Ce^{-\delta t}. \tag{4.65}$$

Putting the estimate (4.65) in (4.64) and keeping in mind the uniform estimates of m in $L^\infty(\Omega \times (0,+\infty))$ we get

$$e^{kt} \int_\Omega |\nabla v^{1/2}(t)|^2 \leqslant C. \tag{4.66}$$

Finally, from (4.66) and taking into account that $1/v \geqslant 1/M$, with $M = \sup_\Omega v_0(x)$ we conclude the Lemma. ∎

Lemma 4.20. *Let $g(v) = 1$.*

- *If $\mu = 0$ and $u_0 > \epsilon > 0$ then (4.57) is satisfied.*
- *If $\mu > 0$, $u_0 > \epsilon > 0$ and $v_0 < 1$ then (4.57) is satisfied.*

Proof. We know that

$$w_t = z\nabla \cdot (z^{-1}\nabla w) + \mu w(1 - wz^{-1} - v) + w\chi(v)mv. \tag{4.67}$$

Let us take $\delta > 0$, a constant to be fixed later on. On multiplying (4.67) by $(w-\delta)_-$ and integrating in the space variable we get

$$\begin{aligned} \frac{d}{2dt}\int_\Omega z^{-1}(w-\delta)_-^2 = &- \int_\Omega z^{-1}|\nabla(w-\delta)_-|^2 + \mu \int_\Omega wz^{-1}(w-\delta)_-(1-wz^{-1}-v) + \\ &+ \int_\Omega w\chi(v)mvz^{-1}(w-\delta)_- - \frac{1}{2}\int_\Omega z^{-1}\chi(v)mv(w-\delta)_-^2. \end{aligned} \tag{4.68}$$

Assume $\mu = 0$ then the right-hand side of (4.68) is nonpositive, thus

$$z(t)^{-1}(w(t)-\delta)_-^2 \leqslant z_0^{-1}(w_0-\delta)_-^2. \tag{4.69}$$

Since $u_0 > \epsilon > 0$ then $w_0 > \epsilon' > 0$. Thus, taking $\delta = \epsilon'$ we get $w(t) \geqslant \epsilon'$ for all $t \geqslant 0$. So $u(t) \geqslant \epsilon'$ for all $t \geqslant 0$.
Assume $\mu > 0$. In order to show that $u(t) \geqslant \epsilon'$ for all $t \geqslant 0$ we have just to prove that

$$\mu wz^{-1}(w-\delta)_-(1-wz^{-1}-v) \leqslant 0. \tag{4.70}$$

Observe that we can claim that $wz^{-1} < \epsilon$ just taking δ sufficiently small. Otherwise, the left-hand side of (4.70) would be zero. Therefore if $v_0 < 1$ then $v_0 + \epsilon < 1$ and $v + \epsilon < 1$ because v is non-increasing in time. Thus, we have proved (4.70). Next, thanks to the property $u(t) \geqslant \epsilon' > 0$ for all $t \geqslant 0$ we can conclude, from the maximum principle that $m \geqslant \widetilde{m}$ with $\widetilde{m}$ the solution to the parabolic problem

$$\begin{cases} \widetilde{m}_t - \Delta\widetilde{m} + \widetilde{m} = \epsilon' & \text{in } \Omega \times (0,\infty), \\ \frac{\partial \widetilde{m}}{\partial n} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ \widetilde{m}(x,0) = m_0(x) & \text{in } \Omega. \end{cases} \tag{4.71}$$

Now, (4.57) holds taking into account that $\widetilde{m} > \delta$ for all $t \geqslant \tau > 0$ because of the strong maximum principle. ∎

Remark 4.21. *Let us point out that in Lemma 4.20 we have proved additionally that under conditions $u_0 > 0$ and $0 < v_0 < 1$ if $\mu \neq 0$ then $u(t) \geqslant \epsilon'$ independently of the value of g.*

Now, we state our first large time behaviour result that refers to the case $g(v) = 1$

Theorem 4.22. *Let $g(v) = 1$, $\tau > 0$, $t \geqslant \tau$, any given initial data $(u_0, v_0, m_0) \geqslant 0$, $v_0 > 0$, $u_0 > 0$ in the class* $\mathbf{Y}$ *and $v_0 < 1$ if $\mu > 0$. Then the solution to (4.1) (u, v, m) satisfies that,*

- *if $\mu = 0$,*

$$\|u(t) - \overline{u}\|_2^2 \leqslant Ce^{-\theta t}, \quad \|v(t)\|_\infty \leqslant Ce^{-\delta t}, \quad \|m(t) - \overline{u}\|_2^2 \leqslant Ce^{-\alpha t} \tag{4.72}$$

  with $\theta, \delta, \alpha > 0$ and $\overline{u} = \frac{1}{|\Omega|}\int_\Omega u_0$,

- *if $\mu > 0$,*

$$\|u(t) - 1\|_2^2 \leqslant Ce^{-\theta t}, \quad \|v(t)\|_\infty \leqslant Ce^{-\delta t}, \quad \|m(t) - 1\|_2^2 \leqslant Ce^{-\rho t}, \tag{4.73}$$

  with $\rho > 0$.

Proof. First, Lemma 4.20 gives us (4.57) and from there, using the explicit formula of v we get $v(x,t) \leqslant v_0(x)e^{-\delta t}$. In what follows we distinguish between two cases.
Case 1.- Assume $\mu = 0$. We know that

$$(u - \overline{u})_t = \nabla \cdot (\nabla u - u\chi(v)\nabla v). \tag{4.74}$$

On multiplying (4.74) by $u - \overline{u}$ and integrating in space we get

$$\begin{aligned}\frac{d}{2dt}\int_\Omega (u-\overline{u})^2 &= -\int_\Omega |\nabla u|^2 + \int_\Omega u\chi(v)\nabla v \cdot \nabla u \\ &\leqslant (\epsilon - 1)\int_\Omega |\nabla u|^2 + C\int_\Omega |\nabla v|^2.\end{aligned} \tag{4.75}$$

Next, the Poincare-Wintinger inequality gives us

$$\frac{2(1-\epsilon)}{C_{poin}}\int_\Omega (u-\overline{u})^2 \leqslant 2(1-\epsilon)\int_\Omega |\nabla u|^2, \tag{4.76}$$

where C_{poin} is a constant coming from the Poincare inequality. Putting the estimate (4.76) in (4.75) we obtain

$$\frac{d}{dt}\int_\Omega (u-\overline{u})^2 + \theta\int_\Omega (u-\overline{u})^2 \leqslant C\int_\Omega |\nabla v|^2 \tag{4.77}$$

with $\theta < \min\{k, \frac{2(1-\epsilon)}{C_{poin}}\}$. Therefore, owing to Lemma 4.19, we obtain

$$\frac{d}{dt}\left(e^{\theta t}\int_\Omega (u-\overline{u})^2\right) \leqslant e^{(\theta - k)t}. \tag{4.78}$$

Since, $\theta < k$ then

$$\|u(t) - \overline{u}\|_2^2 \leqslant Ce^{-\theta t}. \tag{4.79}$$

From $(4.1)_3$ we have

$$(m - \overline{u})_t = \Delta m - (m - \overline{u}) + u - \overline{u}. \tag{4.80}$$

After multiplying (4.80) by $m-\overline{u}$ and applying Hölder inequality we get

$$\frac{d}{dt}\int_\Omega (m-\overline{u})^2+\alpha\int_\Omega (m-\overline{u})^2 \leqslant C\int_\Omega (u-\overline{u})^2, \tag{4.81}$$

with $\alpha<\min\{1,\theta\}$. Now, this case can be easily concluded if we take into account (4.79).
Case 2.- $\mu>0$. On multiplying $(4.1)_1$ by $u-1$ and integrating in the space variable we obtain

$$\begin{aligned}\frac{d}{2dt}\int_\Omega (u-1)^2 &= -\int_\Omega |\nabla u|^2+\int_\Omega u\chi(v)\nabla v\cdot\nabla u-\mu\int_\Omega u(u-1)^2-\mu\int_\Omega u^2v+\mu\int_\Omega uv\\ &\leqslant (\epsilon-1)\int_\Omega |\nabla u|^2+C\int_\Omega |\nabla v|^2-\mu\delta\int_\Omega (u-1)^2+C\int_\Omega e^{-\delta t}.\end{aligned} \tag{4.82}$$

Owing to Lemma 4.20 we have

$$\frac{d}{dt}\left(e^{\theta t}\int_\Omega (u-1)^2\right)\leqslant Ce^{(\theta-k)t}+Ce^{(\theta-\delta)t}. \tag{4.83}$$

For any $\theta<\min\{\mu\delta,k\}$. Next, (4.83) leads us to

$$\|u-1\|_2^2\leqslant Ce^{-\theta t}. \tag{4.84}$$

For the estimates on m we have just to repeat the calculations that we have done for the case $\mu=0$. ∎

The next Theorem shows that the convergence to the steady-states is, in fact, in stronger norms.

Theorem 4.23. *Let $g(v)=1$, $p\in(2,+\infty)$, $\beta<1$ and $\mu\geqslant 0$ then, under conditions of Theorem 4.22, we have*

$$\|m-u_\mu\|_{X_p^\beta}\leqslant Ce^{-\theta t},\quad \|v\|_{1,\infty}\leqslant Ce^{-\delta t},\quad \|u-u_\mu\|_{1,\infty}\leqslant Ce^{-\theta t},\quad t\geqslant\tau>0. \tag{4.85}$$

for any $\theta,\delta>0$ and

$$z_\mu:=\begin{cases}\overline{z} & \text{if } \mu=0,\\ 1 & \text{if } \mu>0.\end{cases} \tag{4.86}$$

Proof. We will provide the proof only for $\mu>0$. The case $\mu=0$ follows in a similar way. From the uniform bound in L^∞ for m we get

$$\|m(t)-1\|_p^p=\int_\Omega |m(t)-1|^p\leqslant C\int_\Omega |m(t)-1|^2\leqslant Ce^{-\alpha t} \tag{4.87}$$

Observe that arguing in the same manner for u we obtain $\|u(t)-1\|_p^p\leqslant e^{-\gamma t}$. Since 1 is the solution to

$$\begin{cases} z_t-\Delta z+z=1 & \text{in } \Omega\times(0,+\infty),\\ \frac{\partial z}{\partial n}=0 & \text{on } \partial\Omega\times(0,+\infty),\\ z(x,0)=1 & \text{in } \Omega.\end{cases}$$

then $1=S(t)1+\int_\tau^t S(t-s)1$. Therefore,

$$\begin{aligned}\|m(t)-1\|_{X_p^\beta} &\leqslant C\tau^{-\beta}e^{-\delta t}\|m(\tau)-1\|_p+\int_\tau^t \|S(t-s)(w-1)\|_{X_p^\beta}\\ &\leqslant C\tau^{-\beta}e^{-\delta t}+\int_\tau^t (t-s)^{-\beta}e^{-\delta t}\|u-1\|_p\\ &\leqslant C\tau^{-\beta}e^{-\delta t}+Ce^{-\min\{\delta,\theta\}t}\int_\tau^t (t-s)^{-\beta}\\ &\leqslant C\tau^{-\beta}e^{-\delta t}+Ce^{-\min\{\delta,\theta\}t}(t-\tau)^{1-\beta}.\end{aligned} \tag{4.88}$$

Since, for $p > 3$ and $\beta \in \left(\frac{1}{2} + \frac{3}{2p}, 1\right)$, $X_p^\beta \hookrightarrow C^1(\overline{\Omega})$ then

$$\|\nabla m\|_\infty \leqslant Ce^{-kt}, \tag{4.89}$$

with $k < \min\{\delta, \theta\}$. For v, we know that $v \leqslant v_0 e^{-\delta t}$. On the other hand,

$$\nabla v = e^{-\int_0^t m}\left(\nabla v_0 - v_0 \int_0^t \nabla m\right). \tag{4.90}$$

Therefore, taking into account (4.89), we have

$$\|\nabla v\|_\infty \leqslant e^{-\delta t}\left(\|\nabla v_0\|_\infty + \frac{C\|v_0\|_\infty}{k}(1 - e^{-kt})\right). \tag{4.91}$$

We claim that there exist $\theta' > 0$ such that

$$\|w - 1\|_2^2 \leqslant Ce^{-\theta' t}. \tag{4.92}$$

Observe that it is enough to prove that $\|w - u\|_2^2 \leqslant Ce^{-\theta'' t}$ for any $\theta'' > 0$.

$$\begin{aligned}
\int_\Omega (w-u)^2 &= \int_\Omega w^2(1 - z^{-1})^2 \\
&\leqslant C\int_\Omega (1 - z^{-1})^2 \\
&= C\int_\Omega (e^{\int_0^v \chi} - e^0)^2 \\
&\leqslant \int_\Omega C\left|\int_0^v \chi(s)ds\right|^2 \\
&\leqslant \int_\Omega C\left(\max_{s\in[0,\max_\Omega v_0(x)]} \chi(s)\right)^2 v^2
\end{aligned} \tag{4.93}$$

and thanks to (4.73), the estimate(4.92) holds. Thus,

$$\|w - 1\|_p \leqslant Ce^{-\rho t}. \tag{4.94}$$

Moreover, we know that

$$w(t) - 1 = S(t)(w(\tau) - 1) + \int_\tau^t S(t-s)(\theta(w, v, m) - 1)ds, \tag{4.95}$$

where θ was defined in (4.17). Now, we pick $p > 3$ and $\beta < 1$ such that $X_p^\beta \hookrightarrow W^{1,\infty}(\Omega)$, then
(4.96)

$$\begin{aligned}
\|w - 1\|_{X_p^\gamma} &\leqslant C\tau^{-\beta}e^{-\delta t}\|w(\tau) - 1\|_p + C\int_\tau^t (t-s)^{-\beta}e^{-\delta(t-s)}\|\theta(w, v, m) - 1\|_p ds \\
&\leqslant C\tau^{-\beta}e^{-\delta t} + C\int_\tau^t (t-s)^{-\beta}e^{-\delta(t-s)}(\|w - 1\|_p + \|\nabla v\|_\infty + \|1 - u\|_p + 2\|v\|_\infty)ds
\end{aligned}$$

and from (4.96) it is not difficult to deduce that

$$\|u - 1\|_{1,\infty} \leqslant Ce^{-\theta''' t}.$$

■

Remark 4.24. *We point out that if the m-equation i.e.* $(4.9)_3$ *is of the form* $m_t = \Delta m - \beta m + \alpha u$ *with* $\alpha > 0$, $\beta > 0$ *then, Theorem 4.23 differs slightly. In particular, we have*

$$\|m - (\beta/\alpha)u_\mu\|_{X_p^\beta} \leqslant Ce^{-\theta t}, \quad \|v\|_{1,\infty} \leqslant Ce^{-\delta t}, \quad \|u - u_\mu\|_{1,\infty} \leqslant Ce^{-\theta t}, \quad t \geqslant \tau > 0. \tag{4.97}$$

In the rest of the section we deal with the case $g(v) = v$. By contrast with the case $g(v) = 1$ we will just prove convergence to the steady-states without any rate. In order to do that we will use the Lemma 3.31.

Theorem 4.25. *Let* $g(v) = v$, $\tau > 0$, $t \geqslant \tau$ *and any given initial data* $u_0, v_0, m_0 \geqslant 0$, $v_0 > 0$, $u_0 > 0$ *in the class* $\mathbf{Y}$. *Then the solution to (4.1) satisfies,*

- *if* $\mu = 0$,

$$\lim_{t\to+\infty} \|u(t) - \overline{u}\|_2^2 = 0, \quad \lim_{t\to+\infty} \|v(t)\|_2 = 0, \quad \lim_{t\to+\infty} \|m(t)\|_2^2 = 0 \tag{4.98}$$

- *if* $\mu > 0$,

$$\lim_{t\to+\infty} \|u(t) - 1\|_2^2 = 0, \quad \lim_{t\to+\infty} \|v(t)\|_2 = 0, \quad \lim_{t\to+\infty} \|m(t)\|_2^2 = 0, \tag{4.99}$$

under the additional condition $v_0 < 1$.

Proof. On multiplying $(4.1)_3$ by m and thanks to the uniform bound on u in $L^\infty(\Omega \times (\tau, +\infty))$ we obtain

$$\frac{d}{2dt}\int_\Omega m^2 + \int_\Omega m^2 + \int_\Omega |\nabla m|^2 \leqslant C\int_\Omega vm. \tag{4.100}$$

Next, taking into account that $mv = -v_t$, we can integrate (4.100) in $(\tau, +\infty)$ and obtain

$$\int_\tau^{+\infty}\int_\Omega m^2 \leqslant C\,, \qquad \int_\tau^{+\infty}\int_\Omega |\nabla m|^2 \leqslant C. \tag{4.101}$$

From (4.101) easily follows

$$\int_\tau^{+\infty}\left|\frac{d}{dt}\int_\Omega m^2\right| \leqslant C \tag{4.102}$$

Therefore, thanks to (4.101) and (4.102), we can apply Lemma 3.31 with $y(t) = \int_\Omega m^2$. For the convergence of v we take (4.62)

$$\frac{d}{dt}\int_\Omega |\nabla v^{1/2}|^2 + \int_\Omega uv^2 + \int_\Omega m|\nabla v^{1/2}|^2 = \frac{d}{2dt}\int_\Omega mv + \frac{1}{2}\int_\Omega m^2 v + \int_\Omega mv. \tag{4.103}$$

Thanks to the uniform bound of m in $L^\infty(\Omega \times (\tau, +\infty))$ we can argue exactly as in (4.100) and we deduce that

$$\int_\tau^{+\infty}\int_\Omega uv^2 \leqslant C. \tag{4.104}$$

Now, having in mind Remark 4.21, we get from (4.104)

$$\int_\tau^{+\infty}\int_\Omega v^2 \leqslant C. \tag{4.105}$$

The bound (4.105) together with

$$\int_\tau^{+\infty}\left|\frac{d}{2dt}\int_\Omega v^2\right| \leqslant C\int_\tau^{+\infty}\int_\Omega -v_t \leqslant C \tag{4.106}$$

and Lemma 4.21 conclude that $\lim_{t\to+\infty}\|v(t)\|_2^2 = 0$. Now, we distinguish between two cases.
Case 1.- $\mu = 0$. We know that w satisfies

$$(w-\overline{u})_t = z\nabla\cdot(z^{-1}\nabla w) + \mu w(1 - wz^{-1} - v) + w\chi(v)mv. \tag{4.107}$$

On multiplying (4.107) by $w - \overline{u}$ and integrating in space we get

$$\begin{aligned}\frac{d}{2dt}\int_\Omega z^{-1}(w-\overline{u})^2 &= -\int_\Omega z^{-1}|\nabla w|^2 + \int_\Omega wz^{-1}\chi(v)mv(w-\overline{u}) - \\ &\quad - \frac{1}{2}\int_\Omega z^{-1}\chi(v)mv(w-\overline{u})^2 \\ &\leqslant -\int_\Omega z^{-1}|\nabla w|^2 + C.\int_\Omega mv\end{aligned} \tag{4.108}$$

Taking into account that $mv = -v_t$ then, after integrating in time (4.108), we have

$$\int_\tau^{+\infty}\int_\Omega z^{-1}|\nabla w|^2 \leqslant C. \tag{4.109}$$

Having in mind (4.109) it is not difficult to obtain

$$\int_\tau^{\infty}\left|\frac{d}{dt}\int_\Omega z^{-1}(w-\overline{u})^2\right| \leqslant C. \tag{4.110}$$

Since $1 \leqslant z \leqslant M$, then in order to apply Lemma 3.31 it is enough to prove

$$\int_0^{+\infty}\int_\Omega (w-\overline{u})^2. \tag{4.111}$$

In fact, having in mind that

$$\int_\tau^{+\infty}\int_\Omega (w-\overline{u}+\overline{u}-\overline{w})^2 \leqslant \int_0^{+\infty}\int_\Omega (w-\overline{w})^2 + \int_0^{+\infty}\int_\Omega (\overline{u}-\overline{w})^2 \tag{4.112}$$

we have just to show that $\int_\tau^{+\infty}\int_\Omega(\overline{u}-\overline{w})^2 \leqslant C$. Therefore

$$\begin{aligned}\int_\tau^{+\infty}\int_\Omega(\overline{u}-\overline{w})^2 &= \frac{1}{|\Omega|}\int_\tau^{+\infty}\left(\int_\Omega w(1-z^{-1})\right)^2 \\ &\leqslant \frac{C}{|\Omega|}\int_\tau^{+\infty}\int_\Omega (1-z^{-1})^2 \\ &= \frac{C}{|\Omega|}\int_\tau^{+\infty}\int_\Omega (e^{\int_0^v \chi} - e^0)^2 \\ &\leqslant \frac{C}{|\Omega|}\int_\tau^{+\infty}\int_\Omega C\left|\int_0^v \chi(s)ds\right|^2 \\ &\leqslant \frac{C}{|\Omega|}\int_\tau^{+\infty}\int_\Omega Cv^2\end{aligned} \tag{4.113}$$

From (4.113) and taking into account (4.105) we conclude

$$\lim_{t\to+\infty}\|w-\overline{u}\|_2^2 = 0. \tag{4.114}$$

Finally, since $w \to u$ strong in L^2 then $u \to \overline{u}$ strong in L^2.
Case 2.- $\mu > 0$. On multiplying (4.67) by $w-1$ we obtain
(4.115)
$$\begin{aligned}\frac{d}{2dt}\int_\Omega z^{-1}(w-1)^2 = &-\int_\Omega z^{-1}|\nabla w|^2 + \mu\int_\Omega wz^{-1}(1-wz^{-1}-v)(w-1)+\\ &+\int_\Omega wz^{-1}\chi(v)mv(w-1) - \frac{1}{2}\int_\Omega z^{-1}\chi(v)mv(w-1)^2\\ &\leqslant -\int_\Omega z^{-1}|\nabla w|^2 - \mu\int_\Omega wz^{-1}(w-1)^2 + \mu\int_\Omega w^2z^{-1}(1-z^{-1})(w-1)-\\ &-\mu\int_\Omega wz^{-1}v(w-1) + C\int_\Omega -v_t.\end{aligned}$$

Having in mind that $w \geqslant \theta > 0$ (see Remark 4.21) and applying Young's inequality in (4.108) we get

$$\begin{aligned}\frac{d}{2dt}\int_\Omega z^{-1}(w-1)^2 + (\mu\theta - 2\epsilon)\int_\Omega z^{-1}(w-1)^2 + \int_\Omega z^{-1}|\nabla w|^2 \leqslant C\int_\Omega (1-z^{-1})^2+\\ + C\int_\Omega v^2 + C\int_\Omega -v_t.\end{aligned} \tag{4.116}$$

After integrating (4.116) in time and owing to (4.113), (4.105) we infer

$$\int_\tau^{+\infty}\int_\Omega z^{-1}(w-1)^2 + \int_\tau^{+\infty}\int_\Omega z^{-1}|\nabla w|^2 \leqslant C \tag{4.117}$$

From (4.117) and (4.115) we can deduce

$$\int_\tau^{\infty}\left|\frac{d}{dt}\int_\Omega z^{-1}(w-\overline{u})^2\right| \leqslant C \tag{4.118}$$

and conclude, thanks to Lemma 3.31, that $w \to 1$ strong in L^2. Finally, the convergence of w together with the strong convergence in L^2 of w to u finishes the proof of the Theorem.

∎

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