University of Warsaw

Doctoral Thesis

Regularity and Singularities in Nonlinear Elliptic Systems: A Study of n-Harmonic Maps and H-Systems

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Podpis:

Data:

Regularity and Singularities in Nonlinear Elliptic Systems: A Study of n-Harmonic Maps and H-Systems

Bogdan Petraszczuk

Abstract

In this work, we study the regularity of maps $u \in W^{1,n}(\mathbb{B}^n(0,1), \mathbb{R}^m)$ which are weak solutions to nonlinear equations of critical growth:

$$-\mathrm{div}(|\nabla u|^{n-2}\nabla u) = F(u, \nabla u),$$

where $F \in L^1$. We investigate the regularity of weakly n-harmonic maps and H-systems within the space $W^{n/2,2}$ for $n \geq 2$, which is a subspace of $W^{1,n}$, showing that the regularity in the general $W^{1,n}$ space is still unresolved (see Miśkiewicz, Strzelecki, Petraszczuk [40, Theorem 1.1 and 1.2]). We also extend J. Frehse's example [1] of a one-point discontinuity for the above differential equation with analytic F to an arbitrary prescribed set of singularities (see Petraszczuk [42, Theorem 1.1]).

Abstrakt

W niniejszej pracy badamy regularność przekształceń $u \in W^{1,n}(\mathbb{B}^n(0,1), \mathbb{R}^m)$, które są słabymi rozwiązaniami nieliniowych równań różniczkowych o krytycznym wzroście:

$$-\mathrm{div}(|\nabla u|^{n-2}\nabla u) = F(u, \nabla u),$$

gdzie $F \in L^1$. Analizujemy regularność przekształceń słabo n-harmonicznych i H-układów w przestrzeni $W^{n/2,\ 2}$ dla $n \geq 2$, która jest podprzestrzenią $W^{1,n}$, pokazując, że regularność w ogólnej przestrzeni $W^{1,n}$ nadal pozostaje nierozwiązana (zob. Miśkiewicz, Strzelecki, Petraszczuk [40, Theorem 1.1 i 1.2]). Rozszerzamy również przykład J. Frehsego [1] dotyczący jednopunktowej nieciągłości dla powyższego równania różniczkowego z analityczną prawą stroną F na dowolnie określony zbiór osobliwości (zob. Petraszczuk [42, Theorem 1.1]).

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Chapter 1

Introduction

1.0.1 Object of studies

In the broadest terms, the focus of this dissertation is on the study of maps $u : \mathbb{R}^n \supset \mathbb{B}^n(0,1) \to \mathbb{R}^m$ of the Sobolev class $W^{1,n}$, which are weak solutions to nonlinear elliptic equations of critical growth. Specifically, we examine regularity and irregularity of solutions to equations of the form:

$$-\mathrm{div}(|\nabla u|^{n-2}\nabla u) = F(u, \nabla u), \tag{1.0.1}$$

where the nonlinear function $F(u, \nabla u)$ satisfies the growth condition $|F(u, \nabla u)| \leq C|\nabla u|^n$. Here are the main topics we work on:

- regularity of H-systems, i.e. maps $u : \mathbb{B}^n(0,1) \to \mathbb{R}^{n+1}$, $x := (x_1, x_2, \ldots, x_n)$ which are weak solutions of Equation (1.0.1) with the right hand side

  $$F(u, \nabla u) = H(u)J(u),$$

  where $H : \mathbb{R}^{n+1} \to \mathbb{R}$ is Lipschitz continuous and

  $$J(u) := \partial_{x_1} u \times \partial_{x_2} u \times \ldots \times \partial_{x_n} u$$

is a normal vector to the image $u(\mathbb{B}^n(0,1))$ at the point $u(x)$.

- Regularity of weakly n-harmonic maps, i.e. maps $u : \mathbb{B}^n(0,1) \to \mathcal{N}$, where $\mathcal{N} \subset \mathbb{R}^m$ is a smooth closed Riemannian manifold isometrically embedded in $\mathbb{R}^n$, which satisfy the Equation (1.0.1) for

$$F(u, \nabla u) = |\nabla u|^{n-2} A_u(\nabla u, \nabla u)$$

in a weak sense. Here

$$A_u(\cdot,\cdot) : T_u\mathcal{N} \times T_u\mathcal{N} \to (T_u\mathcal{N})^{\perp}$$

is the second fundamental form of $\mathcal{N}$.

- Irregularity of solutions of the general Equation (1.0.1) in two-dimensional case, i.e. for $n = 2$.

1.0.2 Motivation

Apart from the obvious motivation of studying the regularity theory of differential equation the initial motivation of above n-harmonic maps is rooted in physics. Each n-harmonic map is the "energy" critical point (see M.Fuchs [8, Section 1.2]), meaning it is a critical point of the E_n functional with respect to variations in the range, i.e.

$$\left.\frac{d}{dt}\right|_{t=0} E_n(\pi(u + t\phi)) = 0,$$

where

$$E_n(u) := \int_{\mathbb{B}^n(0,1)} |\nabla u|^n \, dx,$$

$\pi : B_\delta(\mathcal{N}) \to \mathcal{N}$ is the standard nearest point projection of a tubular neighbourhood of $\mathcal{N}$, and $\phi \in W_0^{1,n} \cap L^\infty(\mathbb{B}^n(0,1), \mathbb{R}^m)$. This serves as, for $p = 2$ a simplified model of the stable configuration of the distortion energy in liquid crystals. In the case of maps into the sphere (i.e. $|u(x)| = 1$) , physics has observed that such maps u (known as directors) are smooth, except at certain points or, in some cases, along lines. These singularities are known as defects in liquid crystals (for more information with further references see Brezis [28]).

The motivation for studying H-systems lies in their ability to present the same theoretical difficulties as n-harmonic maps but within a somewhat simplified framework. Specifically, the right-hand side of H-systems has a nice Jacobian structure, which is usually not present in general n-harmonic maps.

To justify the last topic mentioned in our study, we emphasize the well-known fact (as will be illustrated later) that there is no expectation for the solutions of Equation (1.0.1) to be regular, even with the right-hand side F is analytic. This naturally raises the following two questions:

- What are the maximal and minimal sets of possible singularities?
- Can we somehow control the singular set of a solution? More precisely, can we construct a solution u of Equation (1.0.1) with a singular set that is prescribed earlier?

1.0.3 State of knowledge

The following list is far away from being complete.

The case of p-harmonic maps

We say that $u \in W^{1,p}(\mathbb{B}^n(0,1), \mathcal{N})$ is weakly p-harmonic if u satisfies

$$-\text{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-2} A_u(\nabla u, \nabla u)$$

weakly in $\mathbb{B}^n(0,1)$. Equivalently, using density of smooth functions in $W^{1,p}$, for each test function $\phi \in W_0^{1,p}$ the solution u satisfies

$$\int_{\mathbb{B}^n(0,1)} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx = \int_{\mathbb{B}^n(0,1)} |\nabla u|^{p-2} A_u(\nabla u, \nabla u) \cdot \phi \, dx. \tag{1.0.2}$$

The future considerations are divided into three cases:

- If $p > n$ then, by Sobolev embedding theorem, each function $u \in W^{1,p}$ is Hölder continuous.
- If $p < n$ then one cannot expect any regularity of p-harmonic maps. The traditional example is the map $u : \mathbb{B}^n(0,1) \to \mathcal{S}^{n-1}$, $u(x) = \frac{x}{|x|}$ which is weakly p-harmonic for each $p \in [1, n)$ and is singular at $x = 0$. The real situation is much worse: even for $n = 3$ there exist everywhere discontinuous map 2-harmonic map $u : \mathbb{B}^3(0,1) \to \mathcal{S}^2$ (see T.Rivière [18]).
- The case $p = n$ (is also known as a critical case) is still unresolved and open for $n > 2$. For $n = 2$ Fréderic Heléin [10] proved that each weakly harmonic map u from planar domains, taking values in a arbitrary closed Riemannian manifold $\mathcal{N} \subset \mathbb{R}^m$, is smooth. If one assumes that the target manifold $\mathcal{N}$ is a sphere $\mathcal{S}^{m-1}$ then each n-harmonic map $u \in W^{1,n}(\mathbb{B}^n(0,1), \mathcal{S}^{m-1})$ is locally Hölder continuous (see e.g. P.Strzelecki [16], T.Toro and C.Y.Wang [19], L.Mou and P.Yang [21]). It is also crucial to note that Martino and Schikorra [39, Theorem 1.2] proved the continuity of weakly $n-$harmonic maps $u \in W^{1,n}$ under the additional assumption that $\nabla u \in L^{(n,2)}$. Although this result does not resolve the problem in its entirety, it is formally stronger than our results (c.f. Theorem 4.0.1 and Theorem 5.0.1), but was obtained after our findings.

H-systems

The conformal solution u of the H-system, i.e.

$$-\text{div}(|\nabla u|^{n-2}\nabla u) = H(u)J(u) \tag{1.0.3}$$

possesses a clear geometric property: it parameterize a surface M (in points where $Ju(x) \neq 0$) whose mean curvature in point $u(x)$ is $H(u(x))$ (see [9, page 22]). If one defines the volume of the cone over the image $u(\mathbb{B}^n(0,1))$ with its vertex at $0 \in \mathbb{R}^{n+1}$ as

$$V(u) = \frac{1}{n+1}\int_{\mathbb{B}^n(0,1)} u \cdot J(u)\, dx,$$

then the minimizer of the Dirichlet energy E_n, under a prescribed boundary condition for u on $\partial\mathbb{B}^n(0,1)$ and prescribed volume $V(u)$, satisfies equation (1.0.3) for a constant H, which is just an Euler-Lagrange multiplier.

For a constant H and conformal solution $u \in W^{1,n}$ of the $H-$system (1.0.3), it turns out that u is continuous (see Mou, Yang [20]). For the non-constant Lipschitz H and general $u \in W^{1,n}$ the problem remains to be unresolved yet. F.Bethuel [11] proved continuity of weak solutions $u \in W^{1,2}(\mathbb{B}^2(0,1), \mathbb{R}^3)$ of the H-system (1.0.3) under the assumption that the function $H : \mathbb{R}^3 \to \mathbb{R}$ is bounded and Lipschitz. T.Rivière [18] proved a more general result: weak solutions $u \in W^{1,2}(\mathbb{B}^2(0,1), \mathbb{R}^m)$ of the following antisymmetric system

$$\nabla u = \Omega \cdot \nabla u,$$

where Ω is an $m \times m$ antisymmetric matrix of two-forms in L^2, are continuous. Unlike n-harmonic maps, the assumption that u is also bounded trivializes the situation. More precisely, if $u \in W^{1,n} \cap L^\infty$ is a solution of (2.5.36) and $H \in C^{0,\alpha}$, then $u \in C^{0,\alpha}_{\text{loc}}$ (for the sketch of the proof, for $n = 4$ see A.Schikorra and P.Strzelecki [37] Proposition 3.5).

Irregularity of solutions for a general F

The critical $p = n$ and general $F \in L^1$ does not guarantee any regularity for mappings $u \in W^{1,n}(\mathbb{B}^n(0,1))$ as solutions to equation

$$-\mathrm{div}(|\nabla u|^{n-2}\nabla u) = F(u, \nabla u).$$

Indeed for $n > 2$ and $u(x) = \sin(\log^{\alpha}(1/|x|))$, where $0 < \alpha < 1 - \frac{2}{n}$, one can show (see N.Firoozye [17]) that $u \notin C^0$ but $\mathrm{div}(|\nabla u|^{n-2}\nabla u) \in L^1(\mathbb{R}^n)$. Another important example, provided by J.Frehse [1], shows that there is no regularity even in the planar case $n = 2$, analytic right-hand side F, and bounded $u \in W^{1,2}$ solutions.

1.0.4 Results of this work

H-systems and n-harmonic maps

We work here in a space $W^{n/2,\,2}$, which is the subspace of general $W^{1,n}$, showing that the general case is still unresolved. This extra assumption is stronger than $W^{1,n}$ but from the other side it does not trivialise the problem, since $W^{n/2,2}$ is not a subspace of L^∞ or C^0. We proved the following two theorems (c.f. Miśkiewicz, Strzelecki, Petraszczuk [40, Theorem 1.1 and 1.2]).

Theorem 1.0.1 (H-systems)**.** *Let $n > 2$ be fixed. Assume that $u \in W^{n/2,2}(\mathbb{B}^n(0,1),\ \mathbb{R}^{n+1})$ is the solution of H-system (1.0.3) for Lipschitz and bounded $H : \mathbb{R}^{n+1} \to \mathbb{R}$. Then, after the redefinition of u on a set of zero Lebesgue measure, u is locally Hölder continuous.*

Theorem 1.0.2 (Weakly n-harmonic maps)**.** *Let $n > 2$ be fixed. If $u \in W^{n/2,2}(\mathbb{B}^n(0,1),\ \mathcal{N})$ is a weakly n-harmonic map, then, after the redefinition of u on a set of zero Lebesgue measure, u is locally Hölder continuous.*

Note that for $n = 2$ we have $W^{1,n} = W^{n/2,\,2}$ which immediately translates to the above mentioned result of Fréderic Helén [10]. The extra assumption $u \in W^{n/2,\,2}$ is needed only for the following reason: the right-hand side $R = R_1 + R_2$, where R_1 is "nice" (i.e. is in

$\mathcal{H}^1$, see Lemma 4.0.7) and R_2 is small due to the assumption that $u \in W^{n/2,2}$ (for more details, see Note 2.6.9). We also note that, unlike in H-systems where boundedness of the solution immediately implies regularity, the fact that u is bounded in Theorem 1.0.2 does not trivialize the problem (see Chapter 5 for more). We prove regularity by showing that

$$\int_{B(a,r)} |\nabla u|^p \, dx \leq C r^{\mu}, \qquad p < n, \; \mu > n - p$$

for small radii $r > 0$, $B(a,r) \subset \mathbb{B}^n(0,1)$. Inspired by Iwaniec [12] and Iwaniec-Sbordone [15] we work below the critical exponent $p = n$, i.e. we work with the Morrey norm $\|\nabla u\|_{L^{n-\varepsilon,\varepsilon}}$ instead of $\|\nabla u\|_{L^n}$, where $\varepsilon > 0$ is fixed and small. Now, assuming that u is already restricted to a ball $B(a,r)$, we construct the test function ϕ as follows: define $F = |\nabla u|^{-\varepsilon} P^T \nabla u$, where P is a map into $\mathrm{SO}(m)$, such that $\nabla P \in L^n$ (for H-systems we set $P \equiv Id$, see more in Rivière–Uhlenbeck decomposition 2.6.11 then Equation (5.0.12) with further proof). The function ϕ is then taken as the gradient part of F. According to Iwaniec [12], the divergence-free part of F is small in the appropriate L^q-norm, where $q > n$, providing that F is close to the gradient of the solution u.

In the case of H-systems we show that the right hand side is small:

$$\left| \int H(u) J(u) \phi \, dx \right| \leq \text{small multiple of } r^{\varepsilon} \|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,r))}.$$

This is shown by splitting the right hand side onto two parts:

$$H(u) du_1 = d\alpha + \delta\beta,$$

where part with $d\alpha$ is a *good* part (it has a Jacobian structure) and part with $\delta\beta$ is a *bad* part (the Jacobian structure is lost). Luckily, by the Coifman-Rochberg-Weiss commutator estimate, $\|\delta\beta\|_{L^n} \leq \|H(u)\|_{\mathrm{BMO}} \|du_1\|_{L^n}$, which shows that the *bad* part is small compared to the $\|du\|_{L^n}$. Then to deal with $|\delta\beta| \cdot |\nabla u|^{n-1}$ on the right hand side, one needs the extra assumption that $u \in W^{n/2,2}$: the Gagliardo-Nirenberg inequality (see P.Strzelecki [29] for even n and Van Schaftingen [41] for odd n) with a BMO term gives us

$$\|\nabla u\|_{L^n}^n \leq C\|\nabla u^{n/2}\|_{L^2}^2 \|u\|_{\mathrm{BMO}}^{n-2}$$

for even n and

$$\|\nabla u\|_{L^n}^n \leq C[\nabla^k u]_{W^{1/2,\,2}}^2 \cdot \|u\|_{\mathrm{BMO}}^{n-2}$$

for odd n. Now, it turns out that both $\|\nabla u^{n/2}\|_{L^2}^2$ and $[\nabla^k u]_{W^{1/2,\,2}}^2$ are just error terms, i.e. they are small in front of the 'bad' term. This, together with the inequality,

$$\|H(u)\|_{\mathrm{BMO}} + \|u\|_{\mathrm{BMO}} \leq C\|\nabla u\|_{L^{n-\varepsilon,\varepsilon}}$$

allows us to match the estimate of the right-hand side with estimate of the left-hand side, providing, after a standard iterative argument, Hölder continuity of u.

In the case of n-harmonic maps the situation similar, however some technical details are different. To deal somehow with the Riemannian structure of the right-hand side we apply the antisymmetrization of the second fundamental form (see Fact 2.5.4). Then we apply the Rivière–Uhlenbeck decomposition to get an equivalent equation with a counterpart of Jacobian structure. The rest of the proof follows similarly to the approach used for H-systems.

Irregularity for solutions for general $F \in L^1$

As it was mentioned above, general solutions of the Equation (1.0.1) when the right-hand side $F \in L^1$ can be irregular. The celebrated Frehse example demonstrates that even bounded solutions of equation with an analytic right-hand side can fail to exhibit regularity. However, his example (among others) exhibits a singularity at only a single point. We have shown that, under the same assumptions as in J.Frehse work, the actual situation is even more problematic.

Theorem 1.0.3. *Fix a small radius $0 < r < \frac{1}{e}$ and consider the ball $B := B(0, r) \subset \mathbb{R}^2$.*

For every compact subset K within the ball B, there exists a solution $u \in W^{1,2}(B, \mathbb{R}^2) \cap L^\infty$ to the nonlinear elliptic system:

$$\Delta u = F(u, \nabla u), \tag{1.0.4}$$

where

$$F(u, \nabla u) = (F_1, F_2) = \Big(-2|\nabla u|^2 \frac{u_1 + u_2}{1 + |u|^2},\ 2|\nabla u|^2 \frac{u_1 - u_2}{1 + |u|^2}\Big).$$

This solution u is singular on K and smooth elsewhere in B.

The solution u can be constructed by a formula: we fix a countable dense subset $P = \{p_1, p_2, \ldots\} \subset K$ and denote

$$f(x) = \sum_{i=1}^{\infty} a_i \log\Big(\frac{1}{|x - p_i|}\Big),$$

where $a_i > 0$, $a_i \to 0$ is a geometric sequence. The actual solution $u = (u_1, u_2)$ is described by the following formula

$$u_1 = \sin \log f, \qquad u_2 = \cos \log f.$$

Let us describe here the road-map of the proof. We work with approximations of u, namely:

$$u_1^N = \sin \log f_N, \qquad u_2^N = \cos \log f_N,$$

where

$$f_N = \sum_{i=1}^{N} a_i \log\Big(\frac{1}{|x - p_i|}\Big).$$

Let us denote the finite singular set as $P^N = \{p_1, p_2, \ldots, p_N\}$ and its complement as $\tilde{B}^N := B \setminus P^N$. Here is the plan of the proof (for more details see section 3.0.1).

Step 1: u^N solves the Equation (1.0.4) on $B \setminus P^N$. This verification is trivial - one can compute classical derivatives of u on $\tilde{B}^N$ and show that the differential equation (1.0.4) holds (cf. Equations (3.0.6) - (3.0.8))

Step 2: Maps u^N and u are elements of Sobolev $W^{1,2}(B)$ space. This is highly technical and difficult part and the proof of it has to be done with care. Firstly we show an easier part: for every non-sharp exponent $p < 2$ every $u^N \in W^{1,p}(B)$. Moreover the sequence $\{u^N\}_{N\in\mathbb{N}}$ is bounded in $W^{1,p}(B)$. Then together with the Rellich-Kondrachov theorem, up to a subsequence, $u^N \to u$ and $\nabla u \to \nabla u$ almost everywhere on the ball B (see Lemma 3.0.2). After that we show that u indeed belongs to the Sobolev $W^{1,2}(B)$ (see Lemma 3.0.3).

Step 3: u^N and u are weak solutions to the Equation (1.0.4). Here we use the fact that $W^{1,2}$ contains unbounded functions: because of that one can remove the singularity of u^N at p_i using cut-off functions produced by suitable truncations of $\log\log|x-p_i|^{-1}$.

Step 4: u is discontinuous at points of K and continuous on $B \setminus K$. Denote

$$h(x) := \log\Big(\sum_{i=1}^{\infty} a_i \log\frac{1}{|x-p_i|}\Big).$$

We show that for each $\delta > 0$ and each $x \in K$ the image of the ball $B(x,\delta)$ under the function h contains an interval I of length at least 2π. This will guarantee that sine and cosine terms of the function h will take all values from the interval $[-1,1]$. Thus the oscillation of u will be non-zero at each point $x \in K$, showing the desired discontinuity. To achieve this we construct in the ball $B(x,\delta)$ two sets disjoint and measurable sets E, U such that:

- $|E| > 0$ and values of h on E are relatively small, i.e. there is some fixed $M > 0$ such that $h|_E < M$.
- $|U| > 0$ and values of h on U are relatively big, i.e. $h|_U > M + 2\pi$.

The first set E exists because of the integrability of h on B. The second set U could be defined as follows: inside of the region $B(x,\delta) \cap K$ we choose an arbitrary $p_j \in P$ and we take such small radius $\tilde{\delta} << \delta$ that $a_j \log\frac{1}{x-p_j} > M + 2\pi$ on the ball $B(p_j, \tilde{\delta})$. This together with the fact that $h(x) > a_j \log\frac{1}{x-p_j}$ shows the correctness of our choice of U. Once this construction is done we apply the ACL property of $W^{1,2}$ to the function h which ensures that h is continuous on almost every line parallel to the coordinate axes (the result remains true if we rotate the coordinate axes). Thus, along some of the lines connecting sets E and U the function h has to change continuously from values M to $M + 2\pi$. This shows for each $\delta > 0$ we have $u(B(x, \delta)) \supseteq [-1, 1] \times [-1, 1]$.

Chapter 2

Notation, basic tools, theorems and facts

This chapter is, in a sense, preparatory — essential definitions and notations will be introduced, and standard theorems used in this work will also be recalled. This chapter will also contain information about genesis of n-harmonic maps, their basic properties and difficulties.

We start with an outline of this chapter so that the reader can navigate it with ease:

- Hölder and Lipschitz functions — In this section, the reader will find Jensen's and Young's inequalities, along with basic definitions of Lipschitz and Hölder functions, and their fundamental properties.

- Standard Sobolev $W^{k,p}$ spaces — Here, we recapitulate all the necessary theorems and facts about standard Sobolev spaces and fractional Sobolev spaces. Additionally, we introduce the crucial Van Schaftingen inequality (refer to 2.2.10), which is essential for this work in odd dimensions n.

- BMO and Hardy $\mathcal{H}^1$-spaces — Here the BMO and $\mathcal{H}^1$ spaces are introduced. The section also provides an explanation of why these spaces are necessary and useful for this work, supported by examples.

- Morrey spaces — This section contains Dirichlet-Growth theorem and basic information about Morrey spaces.

- Riemannian manifolds, p-harmonic maps, second fundamental form, examples — Here, the reader can find all the necessary facts about the second fundamental form required in the subsequent chapters. Additionally, a historical note on n-harmonic maps and H-systems is presented, along with an analysis of different cases.

- Analytical tools — In this section, we review different facts about differential forms and Hodge decomposition. The commutator theorem and Uhlenbeck decomposition are also revisited. Additionally, we recall the Gagliardo-Nirenberg inequality with a BMO term, along with a note on how to combine it with J. Van Schaftingen's inequality for our purpose.

Notation 2.0.1.

- $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}}_{n \text{ times}}$ *stands for n-dimensional real space,*

- *For* $x, y \in \mathbb{R}^n$ *we write*
$$|x-y| = \Big(\sum_{i=1}^{n} |x_i - y_i|^2\Big)^{1/2},$$
where $x = (x_1, x_2, \ldots, x_n)$ *and* $y = (y_1, y_2, \ldots, y_n)$*.*

- *For a multi-index* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ *we denote* $\partial_\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}$*, where* $|\alpha| = \alpha_1 + \ldots + \alpha_n$*.*

- *The space of Lebesgue measurable and p-integrable functions is denoted by* L^p*.*

- *If it does not lead to misunderstanding, we write shortly:*
$$\|f\|_{L^p(\Omega)} = \|f\|_p.$$

- $\mathcal{S}^{n-1}$ *stands for the n-dimensional unit sphere, i.e.*
$$\mathcal{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}.$$

- *For any measurable set S of positive Lebesgue measure, we set*

$$\fint_S f(x)\,dx := \frac{1}{|S|}\int_S f(x)\,dx,$$

where $|S|$ is the measure of S.

- *The open ball centered at some point $x \in \mathbb{R}^n$ with radius $r > 0$ will be denoted by $B(x,r)$ or $B_r(x)$. In some cases, we write simply B_r or $B(r)$.*

2.1 Hölder and Lipschitz functions

Here we would like to gather basic useful facts on Hölder and Lipschitz functions. Those two kinds of spaces appear naturally in studying the regularity of solutions of differential equations and the reason for that is quite simple — most of the time we can not solve these equations but we can estimate the gradient of solution (if such exists). Taking into account that the gradient of a function is associated with its oscillation through inequalities like

$$|u(x+h) - u(x)| \lesssim h \cdot \|\nabla u\|,$$

(possibly with various norms of the gradient, or with values of u at specific points replaced by averages), it is clear that Lipschitz and Hölder spaces will appear. This section uses R.Fiorenza [36] as the main source.

Fix some open and connected subset $\Omega \subseteq \mathbb{R}^n$ and function $f : \Omega \to \mathbb{R}^m$. The function f is **Hölder continuous** with exponent $\alpha > 0$, i.e. $f \in C^{0,\alpha}(\Omega)$ if there exists a constant $C > 0$ such that for each $x, y \in \Omega$

$$|f(x) - f(y)| \leq C \cdot |x - y|^\alpha.$$

The function f is said to be **Lipschitz continuous** if f is Hölder continuous with exponent $\alpha = 1$. Analogously we define the $C^{k,\alpha}(\Omega)$ as the space of those functions $f : \Omega \to \mathbb{R}$ having continuous derivatives up through order k and such that the k-th partial derivatives are Hölder continuous with exponent α. More precisely:

$$C^{k,\alpha}(\Omega) := \{f : \Omega \to \mathbb{R} : \ f \in C^k(\Omega), \ \max_{|\beta|=k} \|\nabla^\beta f\|_{C^{0,\alpha}(\Omega)} < \infty\},$$

where

$$\|f\|_{C^{0,\alpha}(\Omega)} = \sup_{x,y\in\Omega, x\neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

Recall that if $f \in C^{0,\alpha}(\Omega)$, then $f \in C^{0,\beta}(\Omega)$ for each $0 < \beta \leq \alpha$ (see [36, Proposition 1.1.2]). The converse is not true (see [36, Example 1.1.4]).

Definition 2.1.1 (Local Hölder continuity)**.** Let $\Omega \subset \mathbb{R}^n$ be open and let $f : \Omega \to \mathbb{R}^m$. We say that f is **locally Hölder continuous** with exponent $\alpha > 0$ if for each compact $K \subset \Omega$ the function f is α-Hölder on K. The space of locally α-Hölder is denoted by $C^{0,\alpha}_{\text{loc}}(\Omega)$.

Sometimes in PDE's, we aim to extend the regularity of a solution (i.e. solution of some differential equation) to the boundary. This extension might be a characteristic of the functional space of the solution, but it could also be a property stemming from both the differential equation describing the solution and its associated functional space. It is a well known fact (cf. [36, Proposition 1.1.9]) that a function $f \in C^{0,\alpha}(\Omega)$ can be extended to $f \in C^{0,\alpha}(\overline{\Omega})$. This result breaks down when one assumes only the locality of Hölder continuity. Roughly speaking, *local* Hölder continuity allows for singularities at the boundary. In fact it is even worse — the assumption that function is continuous at the boundary does not guarantee that it could be extended as a Hölder continuous function to the boundary (see example below).

Example 2.1.2. *Let* $\Omega = (0, \frac{1}{2})$ *and* $f(x) = \frac{1}{\log x}$ *for* $x \in (0, \frac{1}{2})$. *Clearly, with* $f(0) = 0$ *and* $f(\frac{1}{2}) = \frac{1}{\log 2^{-1}}$ *the function* f *is continuous on* $\overline{\Omega}$ *and is locally Hölder continuous in* Ω. *Moreover in* Ω *the function* f *is locally Lipschitz continuous (on each compact set of* Ω

the derivative of f is bounded). Given any compact set $K \subset\subset \overline{\Omega}$ containing $x = 0$, one can show that there is no α such that $f \in C^{0,\alpha}(K)$.

We will conclude this section with several well known inequalities which are useful in this work.

Proposition 2.1.3 (Hölder inequality). *Let $\Omega \subset \mathbb{R}^n$ and $f, g : \Omega \to \mathbb{R}^m$ be two functions such that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, where $p, q \geq 1$ are Hölder conjugate, i.e.*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then $f \cdot g \in L^1(\Omega)$ and

$$\|f \cdot g\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \cdot \|g\|_{L^q(\Omega)}. \tag{2.1.1}$$

Proposition 2.1.4 (Young inequality). *Let $a, b > 0$ be any two numbers and $p, q \geq 1$ be Hölder conjugate. Then*

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q. \tag{2.1.2}$$

2.2 Sobolev and fractional Sobolev spaces

This is one of the most important technical sections in this work. Here the standard results on Sobolev spaces needed in the Chapters 4, 5 are gathered. The part of this section corresponding to fractional Sobolev spaces strongly relies on the so called Hitchhiker's guide [34] and Van Schaftingen [41], while the part corresponding to standard Sobolev spaces mainly depends on Evans [31] (see Chapter 5) and Brezis [30] (see Chapters 8, 9). [1]

[1] The [34] is roughly speaking an introduction to fractional Sobolev spaces with their standard applications and [41] is comparatively recent paper demonstrating one of the crucial higher-order fractional spaces estimates.

2.2.1 Standard Sobolev $W^{k,p}$ spaces

For an open set $\Omega \subset \mathbb{R}^n$ we say that a smooth function $f : \Omega \to \mathbb{R}^m$ is a **test** function if there exists a compact subset $K \subset\subset \Omega$ such that $\operatorname{supp} f \subset K$. The set of test functions on Ω is denoted by $D(\Omega)$. Functionals over $D(\Omega)$ are called **distributions**. The set of distributions on Ω is denoted classically by $D'(\Omega)$. A distribution $T \in D'(\Omega)$ is **regular** or **represented by an** $f \in L^p(\Omega)$ if for each test function $\phi \in D(\Omega)$ we have

$$T(\phi) = \int_\Omega f(x)\phi(x)\,dx,$$

we also write then $T = T_f$. For a given regular distribution T and the differential operator ∂_α we define $\partial_\alpha T \in D'(\Omega)$ by the formula:

$$\partial_\alpha T(\phi) := (-1)^{|\alpha|} T(\partial\phi), \qquad \phi \in D(\Omega).$$

Note that the derivative of a regular distribution does not have to be regular — it is enough to take any "jump" function (see [31, Chapter 5, Example 2]).

Definition 2.2.1 (Sobolev space)**.** Let $\Omega \subset \mathbb{R}^n$ be open. For $k \in \mathbb{N},\ p \geq 1$ the Sobolev $W^{k,p}(\Omega)$ space is the space of those distributions $T \in D'(\Omega)$ which are together with all their derivatives up to order k represented by some functions from $L^p(\Omega)$. More precisely

$$W^{k,p}(\Omega) := \{T \in D'(\Omega) :\ \forall_{\alpha:\ |\alpha| \leq k}\ \exists f \in L^p(\Omega)\ \partial_\alpha T = T_f\} \tag{2.2.1}$$

From now on we identify functions f from $L^p(\Omega)$ with distributions T_f. On the $W^{k,p}$ space we define norm for $1 \leq p < \infty$

$$\|u\|_{W^{k,p}(\Omega)} := \Big(\sum_{|\alpha| \leq k} \|\partial_\alpha u\|_{L^p} \Big)^{1/p} \tag{2.2.2}$$

and for $p = \infty$ the above norm is just a L^∞ norm of all derivatives, i.e.

$$\|u\|_{W^{k,\infty}(\Omega)} := \sum_{|\alpha|\le k} \text{ess sup}_\Omega |\partial_\alpha u|$$

We also note that $(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$ is a Banach space and that the space of smooth functions is dense in $W^{k,p}(\Omega)$ (see [31], Section (5.2), Theorem 2 and Section (5.3), Theorem 1 respectively). The closure of $C_0^\infty(\Omega)$ in the $W^{k,p}(\Omega)$ norm is denoted by $W_0^{k,p}(\Omega)$.

For $\Omega \subset \mathbb{R}^n$ our main focus lies in establishing a comprehensive characterization of the Sobolev $W^{1,p}(\Omega)$ space. For a fixed $1 \le p < n$ we define the **Sobolev conjugate** of p as $p^* = \frac{np}{n-p}$. Note that for those p we have $p^* > p$.

Theorem 2.2.2 (Gagliardo–Nirenberg–Sobolev inequality for $1 \le p < n$)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, bounded with smooth boundary. Assume $1 \le p < n$, and let $u \in W^{1,p}(\Omega)$. Then $u \in L^{p^*}(\Omega)$ and*

$$\|u\|_{L^{p^*}(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)}, \tag{2.2.3}$$

where the constant C depends only on n and Ω.

The proof can be found in [31] Section 5.6, Theorem 2 and Theorem 3 — for $W_0^{1,p}$ space. The estimate 2.2.3 is not explicitly stated in [31] but appears there in the proof (the same note holds for Theorem 2.2.3).

Theorem 2.2.3 (Morrey's inequality for $n < p < \infty$)**.** *Assume $n < p \le \infty$ and $\Omega \subset \mathbb{R}^n$ be open, bounded with smooth boundary. If $u \in W^{1,p}(\Omega)$, then, up to the modification of u on set of zero measure, $u \in C^{0,\gamma}(\Omega)$, for $\gamma = 1 - \frac{n}{p}$. Moreover*

$$\|u\|_{C^{0,\gamma}(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)}, \tag{2.2.4}$$

where $C = C(n, p, \Omega)$.

For the proof see [31, Section 5.6, Theorem 5].

Theorem 2.2.4 (The borderline case $p = n$)**.** *Assume $\Omega \subset \mathbb{R}^n$ is open, bounded with* *smooth boundary. If $u \in W^{1,n}(\Omega)$ then u is in every L^q for $1 \le q < \infty$. Moreover u belongs to the **BMO** space (cf. Section 2.3.2).*

For finishing our analyzing of the $W^{1,p}$ space, as the culmination, we state the Poincaré inequality.

Theorem 2.2.5 (Generalized Poincaré inequality)**.** *Assume $\Omega \subset \mathbb{R}^n$ is bounded, open, convex, symmetric with respect to some point $x \in \Omega$ and assume $u \in W^{1,p}(\Omega)$. For each $V \subset \Omega$ with a non-zero measure we have*

$$\int_\Omega |u(x) - u_V|^p \, dx \le 2^n (\operatorname{diam} \Omega)^p \frac{|\Omega|}{|V|} \int_\Omega |\nabla u(x)|^p dx, \tag{2.2.5}$$

where $u_V := \frac{1}{|V|} \int_V u(x)dx$ is the mean value of u on V.

Whenever Ω is a ball $B(x,r) \subset \mathbb{R}^n$ and $V = \Omega$, we have then a standard Poincaré inequality:

$$\|u - u_B\|_{L^p(B)} \le Cr \|\nabla u\|_{L^p(\Omega)}. \tag{2.2.6}$$

At the end we state here the Sobolev embedding theorem (c.f. [25], Theorem 4.12).

Theorem 2.2.6 (The Sobolev Embedding Theorem)**.** *Assume $B := B(x,r) \subset \mathbb{R}^n$ is the n-dimensional ball centered at the point $x \in \mathbb{R}^n$ with radius $r > 0$. Fix two integers $j \ge 0$, $m \ge 1$ and let moreover $1 \le p < \infty$. Then if $mp < n$ and either $n - mp < n$ we have*

$$W^{j+m,p}(B) \hookrightarrow W^{j,q}(B), \tag{2.2.7}$$

for $p \le q \le p^* = \frac{np}{n-mp}$.

Note 2.2.7. *The above generalization gives us in particular that $W^{l,\frac{n}{l}}(B) \hookrightarrow W^{1,n}(B)$, where $l \in \{1, \dots, n\}$. Indeed it is enough to put $j := 1$, $m := l - 1$. Taking $l = \frac{n}{2}$ we get $W^{n/2,2}(B) \hookrightarrow W^{1,n}(B)$ for every even $n > 0$.*

2.2.2 Fractional Sobolev $W^{k+s,p}$ spaces

Let us temporarily fix $s \in (0,1)$, $p \geq 1$ within the space $W^{s,p}$. The conventional definition of Sobolev space 2.2.1 does not apply to such a parameter s. This prompts a natural question: Can we extend the standard Sobolev space to a **fractional** Sobolev space? Equally important is to ask: What does this extension mean, and how could it be used in applications?

In this chapter we fix some open, bounded domain $\Omega \subset \mathbb{R}^n$. For most applications, without loss of generality one may assume that Ω is in fact a ball $B := B(x,r)$ centered in some point $x \in \mathbb{R}^n$ with radius $r > 0$. The **Gagliardo** semi-norm $[\,\cdot\,]_{W^{s,p}(B)}$ of function $u : B \to \mathbb{R}$ is defined in the following way:

$$[u]_{W^{s,p}(B)} = \Big(\int_B \int_B \frac{|(u(x) - u(y)|^p}{|x-y|^{n+sp}} dx\, dy \Big)^{1/p}. \tag{2.2.8}$$

Each constant function $u \equiv$ const has a zero Gagliardo norm — this is the reason that Gagliardo norm is in fact a semi-norm. Note that each smooth function $\phi : \overline{B} \to \mathbb{R}$ has a finite Gagliardo norm. Indeed such ϕ is in fact Lipschitz with constant $C := \|\nabla \phi\|_{L^\infty(B)}$ and thus we have

$$\begin{aligned} [\phi]^p_{W^{s,p}(B)} &= \int_B \int_B \frac{|(\phi(x) - \phi(y)|^p}{|x-y|^{n+sp}} dx\, dy \leq \int_B \int_B \frac{|x-y|^p \cdot \|\nabla \phi\|_{L^\infty(B)}}{|x-y|^{n+sp}} dx\, dy \\ &= C \int_B \int_B \frac{1}{|x-y|^{n-p(s+1)}} dx\, dy < \infty, \end{aligned} \tag{2.2.9}$$

where in last inequality we used fact that the kernel $k(x) = \frac{1}{|x|^p}$ is in $L^1(B(0,1))$ if and only if $p < n-1$ (in our case $p \geq 1$ and $s+1 > 1$, therefore $p(s+1) > 1$).

Definition 2.2.8 (Fractional Sobolev space for $s \in (0,1)$)**.** For $s \in (0,1)$, $p \geq 1$ we define the **fractional** $W^{s,p}(\Omega)$ as

$$W^{s,p}(\Omega) := \{u \in L^p(\Omega) : [u]_{W^{s,p}(\Omega)} < \infty\} \tag{2.2.10}$$

with a norm:

$$\|u\|_{W^{s,p}(\Omega)} := (\|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\Omega)})^{1/p}. \tag{2.2.11}$$

For every $0 < s < s' < 1$ we have $W^{s',p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$ and also $W^{s',p}(\Omega) \hookrightarrow W^{1,p}(\Omega)$ (for the proof, see [34], Propositions 2.1, 2.2).

To substantiate that the selection in Definition 2.2.8 truly extends standard Sobolev spaces, it is crucial to observe that as s approaches 1, we recover $W^{1,p}$, and as s tends to 0, we obtain L^p. The case $s \to 1$ is related to the following result holding for every $u \in W^{1,p}(\Omega)$:

$$\lim_{s\to 1^-}(1-s)[u]_{W^{s,p}(\Omega)} = C\|\nabla u\|^p_{L^p}; \tag{2.2.12}$$

for the proof see [23, Corollary 2 and Remark 5].

A similar result was proved by Maz'ya and Shaposhnikova [26]:

$$\lim_{s\to 0^+} s[u]_{W^{s,p}(\mathbb{R}^n)} = C\|u\|^p_{L^p(\mathbb{R}^n)}. \tag{2.2.13}$$

Also the $W^{s,p}$ space for non integer $s > 1$ is just the space of constant functions (see Brezis [24]). To summarize, if we define an equivalent norm on $W^{s,p}(\mathbb{R}^n)$:

$$|||u|||_{W^{s,p}(\mathbb{R}^n)} = (\|u\|^p_p + s(1-s)[u]_{W^{s,p}})^{1/p}, \tag{2.2.14}$$

we obtain $\lim_{s\to 1^-}|||u|||_{W^{s,p}} \simeq \|u\|_{W^{1,p}}$ and $\lim_{s\to 0^+}|||u|||_{W^{s,p}} \simeq \|u\|_{L^p}$.

It turns out that fractional Sobolev spaces have a lot of theoretical applications: they appears as the natural spaces for describing traces of functions from $W^{1,p}(\Omega)$ (see [34, Comment after Remark 2.5 with furher references]), the space $W^{s,2}(\mathbb{R}^n)$ is Hilbert and is strictly related to the fractional Laplacian operator (see [34, Section 3, Proposition 3.6]). For arbitrary $k \in \mathbb{R}_+$ we write $k = [k] + (k - [k]) := m + s$ where $[k]$ is the integer part of k. We generalize now the Definition (2.2.8) to the $W^{m+s,p}$ space.

Definition 2.2.9 (Fractional Sobolev $W^{m+s,p}$ space)**.** With notation as above the $W^{m+s,p}(\Omega)$

space is defined as follows

$$W^{m+s,\,p}(\Omega) := \{u \in W^{m,\,p}(\Omega) :\, \partial_\alpha u \in W^{s,\,p}(\Omega) \text{ for any}\, \alpha \text{ s.t. } |\alpha| = m\} \tag{2.2.15}$$

with the norm

$$\|u\|_{W^{m+s,p}(\Omega)} := \big(\|u\|_{W^{m,p}(\Omega)} + \sum_{|\alpha|=m} \|D^\alpha u\|^p_{W^{s,p}(\Omega)}\big)^{1/p} \tag{2.2.16}$$

The general $W^{m+s,p}$ space exhibits properties very similar to those of $W^{s,p}$ (see for example [34, Corollary 2.3 and Theorem 2.4]). At the end of this section we state a recent result on embeddings between fractional Sobolev spaces and standard Sobolev spaces.

Theorem 2.2.10 (J.Shaftingen)**.** *Fix arbitrary* $m \in \mathbb{N} \setminus \{0\}$*,* $s \in (0,1)$ *and* $p \geq 1$ *and the function* $u \in W^{m+s,p}(\mathbb{R}^n)$*. Then if moreover* $u \in \mathrm{BMO}(\mathbb{R}^n)$ *we have* $u \in W^{m,p_1}(\mathbb{R}^n)$*, where* $p_1 = \frac{(m+s)p}{m}$ *and*

$$\|D^m u\|^{p_1}_{L^{p_1}(\mathbb{R}^n)} \leq C\|u\|^{p_1-p}_{\mathrm{BMO}(\mathbb{R}^n)}[D^m u]^p_{W^{s,p}(\mathbb{R}^n)}, \tag{2.2.17}$$

where C *does not depend on* u*.*

For the proof see Van Schaftingen [41], Section 4, Theorem 10.

2.3 BMO and Hardy $\mathcal{H}^1$-spaces

Hardy spaces are a class of function spaces that were introduced by G.H. Hardy in the early 20th century. The study of that spaces plays a crucial role in harmonic analysis and in PDE's. It quickly becomes evident that the L^1 space is too large a functional space for seeking certain regularities in differential equations: There is a well-known result, presented by Fefferman around 1950s, indicating that controlling the Laplacian of a function in L^1 does not imply the control of all second derivatives in L^1. However, if L^1 space is replaced by $\mathcal{H}^1 \subset L^1$, then the desired estimate holds true. More precisely, for the Poisson equation on a unit ball B with right-hand side f being in L^p space for $1 < p < \infty$

$$\Delta u = f \quad \text{in } B(0,1), \quad u = 0 \text{ on } \partial B, \tag{2.3.1}$$

if one assume $u \in W^{1,2}(B(0,1))$ and $f \in L^p$ for $1 < p < \infty$ then in fact $u \in W^{2,p}$ with the estimate (c.f. Giaquinta, Martinazzi [35, Theorem 7.4])

$$\|D^2 u\|_{L^p} \lesssim \|f\|_{L^p}. \tag{2.3.2}$$

This assertion does not hold true when considering the function f in either L^1 or L^∞. In the first scenario, it suffices to examine $u(x) = \log\log\left(\frac{e}{|x|}\right)$ which belongs to $W_0^{1,2}(B^2(0,1))$ with $\Delta u \in L^1$. However, $D^2 u \notin L^1(B^2(0,1))$ (see [35, Example 7.5]). In the second case, let $u : B^2(0,1) \to \mathbb{C}$ with $u(r,\phi) = r^2 \log(r) e^{i\phi}$. As in the previous result, we have $\Delta u \in L^\infty$ but $D^2 u \notin L^\infty$ (see [35, Example 7.6]). Assuming $f \in \mathcal{H}^1$ in the first case ensures $D^2 u \in L^1$, and similarly, if $f \in \mathrm{BMO}$, then $D^2 u \in L^\infty$. For both cases, we obtain inequality (2.3.2) with appropriate norms on the left and right-hand sides.

It is noteworthy that the spaces BMO and $\mathcal{H}^1$ play a crucial role in this work, offering more insightful information compared to the spaces L^1 and L^∞. These function spaces provide additional details and nuances that contribute significantly to our analysis. We now proceed to define these spaces, enumerating the most essential theorems required for our investigation. We strongly rely here on E.M.Stein [14] Chapters, III, IV.

2.3.1 Hardy $\mathcal{H}^1$ space

Here we slightly change the space of test functions described at the beginning of Section 2.2.1. Let S be the set of all $\phi : \mathbb{R}^n \to \mathbb{R}$ that are together with all their derivatives smooth and rapidly decreasing i.e. for any multi indices α, β the seminorm given by

$$\|\phi\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta \phi(x)| \tag{2.3.3}$$

is finite. Note that $C_0^\infty \subset S$.

Definition 2.3.1 (Maximal function)**.** For a fixed $\phi \in S$ and function $f : \mathbb{R}^n \to \mathbb{R}$ we define a **maximal function** represented by ϕ

$$M_\phi(x) = \sup_{t>0} |(\phi_t * f)(x)|, \tag{2.3.4}$$

where $\phi_t(x) = t^{-n}\phi(x/t)$.

Definition 2.3.2 (Hardy $\mathcal{H}^p$ space)**.** A function $f : \mathbb{R}^n \to \mathbb{R}$ belongs to the Hardy $\mathcal{H}^p(\mathbb{R}^n)$ space for some fixed $0 < p \leq \infty$ if and only if there exists a test function $\phi \in S$ with $\int_{\mathbb{R}^n} \Phi \, dx \neq 0$ such that $M_\phi f \in L^p$. We also define norm on $\mathcal{H}^p$ as the L^p norm of $M_\phi f$, i.e.

$$\|f\|_{\mathcal{H}^p} := \|M_\phi f\|_{L^p(\mathbb{R}^n)} \tag{2.3.5}$$

The norm defined in (2.3.5) remains independent of the choice of $\phi \in S$ (see [14], Chapter III, Remark 1 in Section 1.8). In the case of $p > 1$, the $\mathcal{H}^p$ spaces coincide with the L^p spaces. This equivalence can be established by combining the results of Theorem 1 and Remark 1.2.1 in Chapter III of [14]. Unlike L^1 spaces, L^p spaces show distinct characteristics. This pattern holds true for $\mathcal{H}^p$ and $\mathcal{H}^1$ spaces as well. In the case of $\mathcal{H}^1$ we can show that it is a proper subspace of L^1, i.e. $\mathcal{H}^1 \subset L^1$, see Remarks 2.3.3 and 5.4, Chapter III in [14]. An interested reader is referred to [14], Sections 2.2 and 4.3 of Chapter III, where the classic results on the atomic decomposition and Riesz representation of Hardy spaces are demonstrated.

The Jacobian of the map $u \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ plays as a classical example of a function from the Hardy $\mathcal{H}^1$ space. An obvious L^1 estimate of the Jacobian is derived through the Hölder inequality, demonstrating

$$\|J(u)\|_{L^1} := \|\det(\nabla u)\|_{L^n} \leq n \cdot \|\nabla u\|^n_{L^n}. \tag{2.3.6}$$

Also using the Stokes theorem it is clear that $\int_{\mathbb{R}^n} J(u) \, dx = 0$. This suggests that $J(u)$, because of its special structure, might belong to a slightly better space that the general

L^1. In 1993 R.Coifman, P.Lions, Y.Meller and S.Semmes in [13] proved that $Ju \in \mathcal{H}^1$.

Theorem 2.3.3 (Coifman, Lions, Meller, Semmes). *Let $u \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ then $J(u) \in \mathcal{H}^1(\mathbb{R}^n)$ and*

$$\|J(u)\|_{\mathcal{H}^1} \leq C\|\nabla u\|_{L^n}. \tag{2.3.7}$$

Moreover whenever $V \in L^p(\mathbb{R}^n, \mathbb{R}^n)$, $W \in L^q(\mathbb{R}^n, \mathbb{R}^n)$, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, are two vector fields such that $\operatorname{div} V = 0$ and $\operatorname{curl} W = 0$ in the sense of distributions, one has $V \cdot W \in \mathcal{H}^1$ with estimate

$$\|V \cdot W\|_{\mathcal{H}^1} \leq C\|V\|_p\|W\|_q. \tag{2.3.8}$$

Note 2.3.4 (Jacobian in H-systems). *In the H-systems (see 2.5.35) we deal with the Jacobian structure of the map belonging to the $W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$. A pertinent question arises: beyond the evident inclusion in the $\mathcal{H}^1$ space, what specific improvements does Theorem 2.3.3 offer? The issue with $f \in L^1$ arises in integral expressions of the type $I = \int fg\,dx$. The trivial estimate $I \leq \|f\|_1\|g\|_\infty$ is unfortunately too weak—mainly because we often don't have a bounded function g, and this estimate overly relies on the 'absolute value' behavior of both functions. However, what if f behaves like a combination of atoms (each of them indeed has the cancellation moment), and g is 'nearly' constant but unbounded? In this case f is just an element from $\mathcal{H}^1$ and g from* BMO *and thus the Fefferman-Stein duality theorem 2.3.6 brings*

$$|I| \leq \|f\|_{\mathcal{H}^1}\|g\|_{\mathrm{BMO}}. \tag{2.3.9}$$

We also need some useful lemma about Fourier transform of Hardy function.

Fact 2.3.5. Suppose $f \in \mathcal{H}^1(\mathbb{R}^n)$, then $\frac{1}{\|x\|^n}\mathcal{F}(f) \in L^1(\mathbb{R}^n)$.

The atom estimates in a less generality could be found in a preprint of Szarek and Wolniewicz [43]. Additionally, a sketch of the proof for a more general result, known as the extension of the Paley inequality, can be found in A.Torchinski [4] (Exercise 6.12). Given the uncertainty about the specific location of the precise proof for this fact, we provide it here.

Proof. Every function from $\mathcal{H}^1$ has an atomic decomposition

$$f = \sum_{i=1}^{\infty} \lambda \cdot a_k, \tag{2.3.10}$$

where $a_k \in L^\infty$ are atoms, i.e.

- $\operatorname{supp} a_k \subset B(x_k, r_k)$,
- for almost $x \in B(x_k, r_k)$ it holds $|a(x)| \leq |B(x_k, r_k)|^{-1}$,
- $\int_{B(x_k, r_k)} a_k \, dx = 0$.

Thus it is enough to prove Fact 2.3.5 for atoms. Assume $f = a_k$. We write

$$\begin{aligned}
|\mathcal{F}(a_k)(\xi)| &= \left| \int_{B(r_k)} e^{-ix\cdot\xi} a_k(x)\, dx \right| = \left| \int_{B(r_k)} e^{-ix\cdot\xi} a_k(x) - a_k(x)\, dx \right| \\
&= \|a_k\|_{L^\infty(B(r_k))} \int_{B(r_k)} |e^{-ix\cdot\xi} - 1|\, dx \\
&\leq r_k^{-n} \int_{B(r_k)} |e^{-ix\cdot\xi} - 1|\, dx \\
&\leq r_k^{-n} \int_{B(r_k)} |x \cdot \xi|\, dx \leq r_k^{-n} |\xi| C r_k^n \\
&\leq C|\xi|,
\end{aligned} \tag{2.3.11}$$

where C is the maximum over $|x|$ on $B(r_k)$. Note that the first equality stems from Lagrange's mean value theorem (this is the reason why we subtract $a(x)$ in the above equation). Fix small $\varepsilon > 0$ and big enough R (for example $R > |B(x_r)|$) and write

$$\begin{aligned}
\|\frac{1}{|\xi|^n} \mathcal{F}(a)\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \frac{1}{|\xi|^n} |\mathcal{F}(a)(\xi)|\, d\xi \\
&= \int_{|\xi|<\varepsilon} + \int_{\varepsilon<|\xi|<R} + \int_{|\xi|>R} \frac{1}{|\xi|^n} |\mathcal{F}(a)(\xi)|\, d\xi \\
&:= I_1 + I_2 + I_3.
\end{aligned} \tag{2.3.12}$$

Note that I_2 is trivially bounded. To estimate I_1 we use inequality 2.3.11 which avoids the singularity of $1/|x|^n$ in zero, i.e.

$$I_1 \leq \int_{|\xi|<\varepsilon} \frac{1}{|\xi|^n} C|\xi| d\xi = \int_{|\xi|<\varepsilon} \frac{1}{|\xi|^{n-1}} d\xi < \infty. \tag{2.3.13}$$

The estimation of I_3 is much more easier - we use Schwartz inequality

$$\begin{aligned} I_3 &\leq \underbrace{\Big(\int_{|\xi|>R} \frac{1}{|\xi|^{2n}}\, d\xi\Big)^{1/2}}_{:=C<\infty} \Big(\int_{|\xi|>R} \mathcal{F}(a_k)^2\, d\xi\Big)^{1/2} \\ &\leq C\|\mathcal{F}(a_k)^2\|_{L^2(\mathbb{R}^n)} \overset{\text{Plancherel identity}}{=} C\|a_k\|_{L^2(\mathbb{R}^n)} \\ &= C\|a_k\|_{L^2(B(r_k))} < \infty. \end{aligned} \tag{2.3.14}$$

This shows proves Fact 2.3.5 for any atom a_k, hence it holds for any function from the $\mathcal{H}^1$. □

2.3.2 BMO spaces

Earlier, we mentioned that, much like how $\mathcal{H}^1$ can stand in for L^1, BMO spaces do the same for L^∞. BMO spaces are good for handling functions that are 'almost' constant, meaning they stay close to their average value within every open ball. Let us fix a function $u \in L^1_{\text{loc}}(\mathbb{R}^n)$. We say that the function f has a **bounded mean oscillation** if there is a constant $A > 0$ such that for each ball $B \subset \mathbb{R}^n$

$$\frac{1}{B}\int_B |u - u_B|\, dx \leq A \tag{2.3.15}$$

For such function u we write $u \in \text{BMO}(\mathbb{R}^n)$ and also Equation (2.3.15) naturally define the semi-norm on the BMO space as

$$[u]_{\text{BMO}(\mathbb{R}^n)} := \inf_A A. \tag{2.3.16}$$

It is clear that $[u]_{\text{BMO}} = 0$ if and only if u is a constant. In the same spirit we define the BMO(Ω) on every domain $\Omega \subset \mathbb{R}^n$ as the supremum of (2.3.15) over all $B \subset \Omega$. The triangle inequality applied to the expression (2.3.15) implies $L^\infty \subset$ BMO. At the same time a 'slowly' increasing function $u(x) = \log(|x|)$ is an element of BMO. Therefore, the inclusion is strict. Similarly, using the Poincare inequality in the expression (2.3.15) we get $W^{1,n}(\Omega) \subset \mathrm{BMO}(\Omega)$. Together with all observations made above (see also Note 2.3.4) we come to the main result in this Section — duality between BMO and $\mathcal{H}^1$ spaces (for the proof we refer to Stein [14], Chapter IV, Section 1.2).

Theorem 2.3.6 (Fefferman-Stein duality theorem)**.** *If $f \in \mathcal{H}^1(\mathbb{R}^n)$ and $g \in \mathrm{BMO}(\mathbb{R}^n)$, then*

$$l(f) = \int_{\mathbb{R}^n} f(x)g(x)\,dx \tag{2.3.17}$$

is an element of $(\mathcal{H}^1)^*$ and

$$|l(f)| \leq \|f\|_{\mathcal{H}^1(\mathbb{R}^n)} \|g\|_{\mathrm{BMO}(\mathbb{R}^n)}. \tag{2.3.18}$$

Moreover every $l \in (\mathcal{H}^1)^*$ is of the above type, thus $BMO = (\mathcal{H}^1)^*$.

2.4 Morrey spaces

We start this section with a useful theorem (see R.Moser [27] Lemma 2.1).

Theorem 2.4.1 (Dirichlet Growth Theorem)**.** *Let $\alpha > 0$ and $p \geq 1$. Let $B(x,r) \subset \mathbb{R}^n$ and $u \in W^{1,p}(B(x_0,R))$. If for each ball $B(r) \subset B(x,R)$ we have*

$$r^{p-n-\alpha} \int_{B_r} |\nabla u|^p \,dx \leq C \tag{2.4.1}$$

for some constant C, then for almost every $x_1, x_2 \in B(x_0,R)$,

$$|u(x_1) - u(x_2)| \leq \tilde{C}|x_1 - x_2|^{\alpha/p}. \tag{2.4.2}$$

Equation (2.4.1) could be read as follows: for some positive α and each ball $B(r) \subset$

$B(x_0, R)$

$$r^{p-n} \int_{B(r)} |\nabla u|^p \, dx \leq Cr^{\alpha}, \tag{2.4.3}$$

so if the gradient of u along with $\frac{1}{r^{p-n}}$ on the ball $B(r)$ exhibits behavior resembling the radius $r > 0$ raised to a positive power α then u is considered to be regular.

The above theorem encourages the study of the so-called Morrey spaces. Let us fix here some proper subset Ω of $\mathbb{R}^n$ (one can assume that Ω is in fact ball $B(x, r)$).

Definition 2.4.2 (Morrey spaces)**.** Let $1 \leq p < \infty$ and $\lambda \geq 0$. The **Morrey space** $L^{p,\lambda}(\Omega)$ is defined as follows

$$L^{p,\lambda}(\Omega) = \{f \in L^p(\Omega) : \sup_{x \in \Omega,\, r>0} \frac{1}{r^{\lambda}} \int_{\tilde{B}(x,r)} |f(y)|^p \, dy < \infty\}, \tag{2.4.4}$$

where $\tilde{B}(x, r) := \Omega \cap B(x, r)$.

This is the Banach space with the norm

$$\|u\|_{L^{p,\lambda}(\Omega)} := \Big(\sup_{x \in \Omega, r>0} \frac{1}{r^{\lambda}} \int_{\tilde{B}(x,r)} |f(y)|^p \Big)^{1/p}. \tag{2.4.5}$$

If $\lambda > n$ then we can write $\lambda = n + \alpha$, for some positive α, so for each ball $B(y, r) \subset \Omega$

$$\frac{1}{r^n} \int_{B(y,r)} |f(x)|^p \, dx < Cr^{\alpha}.$$

The left hand side, as $r \to 0^+$, reduces to $|f(y)|^p$ (by Lebesgue theorem), while the right-hand side is zero. Consequently, almost everywhere in Ω, we have $f(y) = 0$. Therefore, for $\lambda > n$ the space $L^{p,\lambda}(\Omega) = \{0\}$ is trivial. If $\lambda = 0$ we just get the standard L^p space.

2.5 Riemannian manifolds, p-harmonic maps, second fundamental form, examples

2.5.1 Riemannian manifolds, second fundamental form, examples

Let $\mathcal{N}$ be a Riemannian manifold of class at least C^2 isometrically embedded into $\mathbb{R}^n$. Throughout this work we additionally assume $\mathcal{N}$ is **closed** manifold, i.e. compact and without boundary. Denote a tubular neighborhood of $\mathcal{N}$ by $\mathcal{N}_\varepsilon$ and recall that the projection operator $\pi : \mathcal{N}_\varepsilon \to \mathcal{N}$ is well defined (i.e. for every $x \in \mathcal{N}_\varepsilon$ the point $y = \pi(x) \in \mathcal{N}$ is unique) and smooth. Taking now two vector fields v, w on $\mathcal{N}_\varepsilon$ such that they are tangential vector fields on $\mathcal{N}$. For any $p \in \mathcal{N}$ one can show that $\nabla^2_{v,w}\pi(p) := v(p) \cdot D^2\pi(p) \cdot w(p)$ is the projection of the vector $\nabla_v w(p)$ onto $T_p\mathcal{N}^\perp$.

Definition 2.5.1. The **second fundamental form** of a manifold $\mathcal{N}$ is denoted by A and is defined as follows

$$A_p(v, w) := -\nabla^2_{v,w}\pi(p), \tag{2.5.1}$$

where $p \in \mathcal{N}$ and $v, w \in T_p\mathcal{N}$.

For a fixed point $p \in \mathcal{N}$ the $A_p : T_p\mathcal{N} \times T_p\mathcal{N} \to T_p^\perp\mathcal{N}$ is 2-linear and symmetric transformation. The following note shows how to formally extend the second fundamental form A_p on the whole $\mathbb{R}^n$. This type of extension will be useful in the Chapter 5.

Note 2.5.2 (Extension of A)**.** *Each vector $v \in \mathbb{R}^n$ we can write as the sum $v = v_1 + v_2$, where $v_1 \in T_p\mathcal{N}$ and $v_2 \in T_p^\perp\mathcal{N}$. Taking now two vectors $v, w \in \mathbb{R}^n$ we get as above v_1, v_2, w_1, w_2 hence we define*

$$A_p(v, w) := A_p(v_1, w_1). \tag{2.5.2}$$

Also using smoothness of the projection operator π on $\mathcal{N}_\varepsilon$ we naturally extend the definition of A_p for $p \in \mathcal{N}_\varepsilon$ and then using a cut-off function $\zeta \in C_0^\infty(\mathbb{R}^n)$ such that $\zeta = 1$ on $\mathcal{N}_{\varepsilon/2}$ and zero outside of $\mathcal{N}_{\frac{3\varepsilon}{4}}$ we extend A_p to a map $p \to A_p$ with a compact support.

Note 2.5.3 (Sphere case)**.** *In the case of a round sphere* $\mathcal{N} = \mathcal{S}^{n-1}$ *embedded into* $\mathbb{R}^n$ *the second fundamental form is given by*

$$A_p(v,w) = p\langle v, w\rangle. \tag{2.5.3}$$

In the section 5 we strongly rely on the following fact.

Fact 2.5.4 (Antisymmetrization of A)**.** Suppose that $u : B(y,r) \subset \mathbb{R}^m \to \mathcal{N} \subset \mathbb{R}^n$ be some function (one can assume $u \in C^\infty$) and $\mathcal{N}$ be a Riemannian manifold. The derivative $\partial_\alpha u(x)$ belongs to the $T_{u(x)}\mathcal{N}$. Define for some fixed $x \in B(x,r)$

$$A_{ij} := A_{u(x)}(e_i, e_j) \in T^{\perp}_{u(x)}\mathcal{N}, \tag{2.5.4}$$

where e_i, e_j are standard vectors in $\mathbb{R}^m$ (see Note 2.5.2). Equation (2.5.4) together with above observation on $\partial_\alpha u$ implies

$$\langle \partial_\alpha u, A_{i,j}\rangle = \sum_{l=0}^{m} (\partial_\alpha u)^l A^l_{ij} = 0, \tag{2.5.5}$$

where the notation v^i means the i-th coordinate of vector v. Because the above expression is zero we can write as follows

$$(\partial_\alpha u)^k \sum_{l=0}^{m} (\partial_\alpha u)^l A^l_{ij} = 0 \tag{2.5.6}$$

for arbitrary fixed $k \in \{1, ..., n\}$. Therefore

$$\sum_{k=0}^{n}\sum_{l=0}^{m} (\partial_\alpha u)^k (\partial_\alpha u)^l A^l_{ij} = 0. \tag{2.5.7}$$

The above expression is equal to zero for each fixed α. Hence, by summing the left hand sides of Equality (2.5.7) additionally with respect to the α, we still get zero as the result.

More precisely

$$\sum_{\alpha=0}^{n}\sum_{k=0}^{n}\sum_{l=0}^{m}(\partial_\alpha u)^k(\partial_\alpha u)^l A_{ij}^l = 0. \tag{2.5.8}$$

Let

$$A_{u(x)}(\nabla u(x), \nabla u(x)) := \sum_{\alpha=0}^{n} A_{u(x)}(\partial_\alpha u(x), \partial_\alpha u(x)). \tag{2.5.9}$$

Let us work with i-th coordinate of $A_{u(x)}(\nabla u(x), \nabla u(x))$. Writing the vector in equation 2.5.9 in a standard basis and using Equation (2.5.8) we get

$$\begin{aligned} A_u^i(\nabla u, \nabla u) &= \sum_{\alpha=1}^{m} A^i(\partial_\alpha u, \partial_\alpha u) \\ &= \sum_{\alpha,j,l} A_{j,l}^i(\partial_\alpha u)^j(\partial_\alpha u)^l \\ &= \sum_{\alpha,j,l} A_{j,l}^i(\partial_\alpha u)^j(\partial_\alpha u)^l - \sum_{\alpha,j,l} A_{i,l}^j(\partial_\alpha u)^j(\partial_\alpha u)^l \\ &= \sum_{\alpha,j,l}(A_{j,l}^i - A_{i,l}^j)(\partial_\alpha u)^j(\partial_\alpha u)^l \\ &= \sum_{j,l}(A_{j,l}^i - A_{i,l}^j)\sum_{\alpha}(\partial_\alpha u)^j(\partial_\alpha u)^l \\ &= \sum_{j,l}(A_{j,l}^i - A_{i,l}^j)\nabla u^j \cdot \nabla u^l. \end{aligned} \tag{2.5.10}$$

Denoting

$$\Omega_{i,j} := \sum_{l}(A_{j,l}^i - A_{i,l}^j)\nabla u^l \tag{2.5.11}$$

we finally get

$$A_u^i(\nabla u, \nabla u) = \sum_{j}\Omega_{ij} \cdot \nabla u^j. \tag{2.5.12}$$

Clearly, the matrix Ω is asymmetric, i.e. $\Omega_{i,j} = -\Omega_{j,i}$.

2.5.2 Harmonic and p-harmonic maps, H-systems

We say that $u : \mathbb{R}^n \to \mathbb{R}$ is **harmonic** if for each point $x \in \mathbb{R}^n$

$$\Delta u(x) = 0. \tag{2.5.13}$$

The u is said to be **weakly harmonic** if the Laplace operator in the above equation is distributional Laplacian meaning for each test function $\phi \in D(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla \phi \, dx = 0. \tag{2.5.14}$$

Recall that harmonic maps are critical points of the Dirichlet energy $E(u) = \int |\nabla u|^2 \, dx$. If one imposes an extra constraint on the image of u, e.g. by requiring that $u : \mathbb{R}^n \to \mathcal{N} \subset \mathbb{R}^m$, where cN is a Riemannian manifold, then critical points of $E(u)$ satisfy

$$-\Delta u = A_u(\nabla u, \nabla u). \tag{2.5.15}$$

If one defines p-Dirichlet energy functional $E_p(u) = \int |\nabla u|^p \, dx$, then analogously if no assumption on the image of u is made, Equation (2.5.14) becomes

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0. \tag{2.5.16}$$

For critical points of E_p, $p \neq 2$ under the constraint $u : \mathbb{R}^n \to \mathcal{N}$ we have that Equation (2.5.15) becomes (see M.Fuchs [8] Section 1.2)

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) = |\nabla u|^{p-2} A_u(\nabla u, \nabla u). \tag{2.5.17}$$

Definition 2.5.5 (weakly p-harmonic maps)**.** Let $u : B(x,r) \subset \mathbb{R}^n \to \mathcal{N} \subset \mathbb{R}^m$ be a function of the class $W^{1,n}(B(x,r), \mathbb{R}^m)$, where $\mathcal{N}$ is a closed Riemannian manifold. The u is said to be **weakly p-harmonic** if

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) \perp T_u \mathcal{N} \tag{2.5.18}$$

in the sense of distributions, i.e. for each test function $\phi \in D(B(x,r))$

$$\int_{B(x,r)} |\nabla u|^{p-2} \nabla u \nabla \phi \, dx = \int_{B(x,r)} |\nabla u|^{p-2} A_u(\nabla u, \nabla u) \phi \, dx. \tag{2.5.19}$$

The highly nonlinear structure of the above equation presents significant challenges: even in the case of one-dimensional differential equations its nonlinear structure introduces various difficulties such as non-existence of solutions, solution blow-up in finite time, and simply difficulties in finding a direct solution. We focus on the Hölder continuity of $n-$harmonic maps and H-systems (see Chapters 4, 5).

- When $p > n$ the $u \in W^{1,p}(\mathbb{R}^n)$ is also Hölder continuous, so in this case the regularity of u follows directly from the assumptions.

- For $p < n$, the p-harmonic maps are, broadly speaking, not continuous in general context — the map $u : \mathbb{B}^n(0,1) \to \mathcal{S}^{n-1}$, $u(x) = \frac{x}{\|x\|}$ is weakly p-harmonic for every $p \in [1, n)$ and is singular at $x = 0$. An additional classical example, demonstrating that the situation is in fact even worse, is due to T. Riviére [18]: for $n = 3$ there exist everywhere discontinuous weakly harmonic map $u : \mathbb{B}^3(0,1) \to \mathcal{S}^2$. When considering weakly p-harmonic maps that also act as **minimizers** of the p-Dirichlet energy E_p, the situation becomes somewhat more favorable: **stationary** p-harmonic maps $u : \mathbb{R}^m \supset M \to \mathcal{N} \subset \mathbb{R}^n$ are Hölder continuous, outside a set of Hausdorff dimension $m - [p] - 1$ (see independent results by Hardt and Lin [5], Fuchs [8], Luckhaus [6]).

- The critical case $p = n$ is still unsolved within the widest scope of contexts. For $n = 2$ Fréderic Heléin [10] proved that each weakly harmonic map u from planar domains, taking values in a closed Riemannian manifold $\mathcal{N} \subset \mathbb{R}^m$, is smooth. For arbitrary $n > 2$ if the target is in fact sphere, i.e. the weakly n-harmonic map $u \in W^{1,n}(\mathbb{B}^n(0,1), \mathcal{S}^{m-1})$ is locally Hölder continuous (see Paweł Strzelecki [16]). More generally if $\mathcal{N}$ is compact homogeneous space with a left-invariant metric, then, all n-harmonic maps $u \in W^{1,n}(\mathbb{B}^n(0,1), \mathcal{N})$ are locally of class $C^{0,\alpha}$. Finally, we note that Martino and Schikorra [39, Theorem 1.2] proved the continuity of weakly $n-$harmonic maps $u \in W^{1,n}$ under the additional assumption that ∇u is in Lorentz $L^{(n,2)}$ space. As it was mentioned in the Introduction, this result is formally stronger that ours (c.f. Theorem 4.0.1, Theorem 5.0.1), but it was proved after ours.

Note 2.5.6 (Why the general $W^{1,n}$ case is indeed 'critical'?)**.** *Under the assumption $u \in W^{1,n}$ the right-hand side of the n-harmonic Equation (2.5.17) is simply in L^1. Indeed,*

$$|A_u(\nabla u, \nabla u)| \leq \|A_u\||\nabla u|^2 \leq C(\mathcal{N})|\nabla u|^2, \tag{2.5.20}$$

where the last inequality holds because A_p is continuous with respect to the $p \in \mathcal{N}$ and $\mathcal{N}$ is compact. In general, the solution u of the equation

$$-\mathrm{div}(|\nabla u|^{n-2}\nabla u) = F \tag{2.5.21}$$

for $F \in L^1$ could be discontinuous: we saw already discontinuous solutions for $n = 2$, $u(x) = \log\log(\frac{e}{x})$ in Section 2.3. When $n > 2$, the function $u(x) = \sin(\log^\alpha(1/|x|))$ for $0 < \alpha < 1 - \frac{2}{n}$, satisfies:

$$u \notin C^0 \text{ but } \mathrm{div}(|\nabla u|^{n-2}\nabla u) \in \mathcal{H}^1_{\mathrm{loc}}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n). \tag{2.5.22}$$

It shows that an approach based on the structure of F is needed here.

Note 2.5.7 (Case $n = 2$)**.** *The 'planar' case differs significantly from the scenario when $n > 2$, primarily due to the disappearance of the nonlinear term $|\nabla u|^{n-2}$. Consequently, the left-hand side of 2-harmonic maps simplifies to a Poisson equation. What if the right-hand side of Equation (2.5.14) is an element of Hardy $\mathcal{H}^1$ space instead of general L^1? Equivalently, what if the map $u \in W^{1,2}(B)$ has additionally $\Delta u \in \mathcal{H}^1(\mathbb{R}^n)$? As mentioned earlier, the $\mathcal{H}^1$norm of the Laplacian governs the L^1 norm of second derivatives. Indeed if $\nabla u \in \mathcal{H}^1(\mathbb{R}^n)$, then*

$$\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = R_i R_j(\nabla u), \tag{2.5.23}$$

where R_i is the Riesz transformation, i.e.

$$\mathcal{F}(R_i u)(x) = -i\frac{x_i}{|x|}(\mathcal{F}u)(x). \tag{2.5.24}$$

Using the fact that $R_i : \mathcal{H}^1 \to \mathcal{H}^1$ are continuous (in the broader context every Calderón-Zygmund operator with 'proper' kernels is continuous on $\mathcal{H}^1$) we get that every second order partial derivative is in L^1.

Remarkably, for $n = 2$, a more refined estimate holds: if the Laplacian of a planar function f is in $\mathcal{H}^1$, then f is, in fact continuous. Indeed

$$\mathcal{F}(\Delta u)(x) = |x|^2 \mathcal{F}(u), \tag{2.5.25}$$

thus using Fact 2.3.5 we get

$$\mathcal{F}(u) = \frac{1}{|x|^2}\mathcal{F}(\Delta u)(x) \in L^1. \tag{2.5.26}$$

The above equation implies that the Fourier transform of u is in L^1, hence u is continuous. The right-hand side of the harmonic map equation is simply the second fundamental form, which, as demonstrated in Fact 2.5.4, exhibits antisymmetry:

$$A_u(\nabla u, \nabla u) = \Omega \cdot \nabla u, \quad \Omega_{i,j} = -\Omega_{j,i} \tag{2.5.27}$$

Unfortunately, the matrix Ω may not be divergence free and thus it is impossible to immediately use Theorem 2.3.3. However, after an application of the Uhlenbeck–Riviére decomposition (see Theorem 2.6.11) we can without loss of generality assume $\operatorname{div}\Omega = 0$ in the sense of distributions. Hence $\Omega \cdot \nabla u \in \mathcal{H}^1$ which finishes the proof.

Note 2.5.8 (Sphere case $\mathcal{N} = \mathcal{S}^{m-1}$)**.** *Let $u : W^{1,n}(B^n(0,1), \mathcal{S}^{m-1})$ satisfy*

$$\operatorname{div}(|\nabla u|^{n-2}\nabla u) = |\nabla u|^{n-2} A_u(\nabla u, \nabla u). \tag{2.5.28}$$

The second fundamental form of the round sphere (see Note 2.5.3) is of the form

$$A_u(\nabla u, \nabla u) = u \cdot \langle \nabla u, \nabla u\rangle = u \cdot |\nabla u|^2. \tag{2.5.29}$$

Hence the above differential equation becomes

$$\operatorname{div}(|\nabla u|^{n-2}\nabla u) = |\nabla u|^n u. \tag{2.5.30}$$

The crucial observation (initially noticed by F. Hélein for $n = 2$) is that by denoting $V_i = |\nabla u|^{n-2}\nabla u_i$ one can show (c.f. P. Strzelecki [16, Equation 6] or Fact 2.5.4) that

$$\operatorname{div} V_i = \sum_{k=1}^{m} \nabla u_k \cdot (u_k V_i - u_i V_k) \tag{2.5.31}$$

and that

$$\operatorname{div}(u_k V_i - u_i V_k) = 0. \tag{2.5.32}$$

For $n = 2$ the above decomposition is even more straight:

$$|\nabla u|^2 u^i = \sum_{j=1}^{m} \underbrace{(u^i \nabla u^j - u^j \nabla u^i)}_{:=\Omega_{ij}} \cdot \nabla u^j, \tag{2.5.33}$$

where the matrix Ω is divergence free. By the extension argument [16, Corolarry 3], the right-hand side of Equation (3.0.26) turns out to be a restriction of a function $h \in \mathcal{H}^1$ to $\mathbb{B}^n(0,1)$. Together with Fact 2.3.5 we obtain the continuity of u for flat domains. For the general case where $n > 2$, although the $\mathcal{H}^1$ structure of the right-hand side is utilized, the proof is not as straightforward as in the planar case.

2.5.3 H-systems

Definition 2.5.9 (H-systems)**.** Let $H : \mathbb{R}^{n+1} \to \mathbb{R}$ be bounded and $u \in W^{1,n}(\mathbb{B}^n(0,1), \mathbb{R}^{n+1})$ be some vector function. Let us define the vector $J(u(x))$, perpendicular to the image of $u(\mathbb{B}^n)$ at the point $u(x)$:

$$J(u(x)) := \frac{\partial u(x)}{\partial x_1} \times \frac{\partial u(x)}{\partial x_2} \times \ldots \times \frac{\partial u(x)}{\partial x_n}. \tag{2.5.34}$$

The map u is said to be a weak solution to the H**-system** if

$$-\text{div}(|\nabla u|^{n-2}\nabla u) = H(u)J(u) \tag{2.5.35}$$

in the sense of distributions $D(\mathbb{B}^n, \mathbb{R}^{n+1})$. In other words, for each $\phi \in W_0^{1,n}(\mathbb{B}^n, \mathbb{R}^{n+1})$

$$\int_{B^n} |\nabla u|^{n-2}\nabla u \nabla \phi \, dx = \int_{B^n} H(u)J(u)\phi \, dx \tag{2.5.36}$$

It is clear that H-systems are closely related to n-harmonic maps but their prescribed Jacobian structure on the right-hand side streamlines the investigation. However, they lead to the same analytical difficulties. Equation (2.5.35) possesses a clear geometric property. The conformal solutions of the H-system parameterize a surface M (in points where $Ju(x) \neq 0$) whose mean curvature at the point $u(x)$, is equal to $H(u(x))$ (see [9] page 42). The existence of solutions with a **Plateau boundary condition** could be found in F.Duzaar and J.Grotowski [22]. The following list of results is far from complete.

Note 2.5.10 (Planar case for H-systems)**.** *Similarly to Note 2.5.7 for $n = 2$ things trivialise a bit. H-system becomes just a Poisson equation*

$$\Delta u = H(u)u_x \times u_y. \tag{2.5.37}$$

The Jacobian $J(u)$ for functions from $W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ is an element from the Hardy $\mathcal{H}^1(\mathbb{R}^n)$ space. This implies that if H is constant, then $H(u)J(u) \in \mathcal{H}^1(\mathbb{R}^2)$ and so $u \in C^0$. Therefore constant H, all solutions of (2.5.37) are regular. Unfortunately, for a non-constant function H, things are not as straightforward. One of the reason is that the product $H(u)J(u)$ may not lie in $\mathcal{H}^1(\mathbb{R}^n)$. For a bounded H any $W^{1,2}$-solution u of (2.5.37) is Hölder continuous (see Riviére [18]). If one assumes H to be Lipschitz continuous then every $u \in W^{1,2} \cap L^\infty$ solution belongs to $C^{1,\alpha}$ (see F.Tomi [2]).

Note 2.5.11 (Case $n > 2$)**.** *The high dimension case is much harder. For a general $u \in W^{1,n}$ and Lipschitz H, the problem is unsolved. Similarly to the planar case, each* *solution* $u \in W^{1,n}$ *of Equation (2.5.35), where the function* H *is constant, turns out to be continuous (see Mou, Jang [20]).*

It is crucial to note that the assumption that u *is additionally bounded trivializes the entire problem. More precisely, if* $u \in W^{1,n} \cap L^{\infty}$ *is a solution of 2.5.36 and* $H \in C^{0,\alpha}$*, then* $u \in C^{0,\alpha}_{loc}$ *(for the sketch of the proof see A.Schikorra and P.Strzelecki [37] Proposition 3.5). The key point is that, since* u *is bounded, one can test equation (2.5.36) with a cut-off version of* u *on some small ball* $B(a, 2r) \subset B^n(0,1)$ *getting the "decay" inequality of type*

$$\|\nabla u\|_{L^n(B(a,r))} \leq \tau \|\nabla u\|_{L^n(B(a,2r))} \tag{2.5.38}$$

for some $\tau \in (0,1)$*. The last fact implies Hölder continuity of* u *due to scaling technique.*

2.6 Analytical tools

2.6.1 Differential forms and Hodge star operator

We often employ differential forms as a convenient useful 'language' in which lengthy mathematical formulas become more visible and clear. Given the technical nature of this topic, we aim to ensure clarity in the upcoming sections by briefly presenting some necessary notation and examples.

The space of k-forms on an open set $U \subset \mathbb{R}^n$ is denoted by $\Lambda^k(U)$. For arbitrary vector $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ the projection on the i-th coordinate will be denoted by dx_i, i.e.

$$dx_i(v) = v_i. \tag{2.6.1}$$

The dx_i is a 1-form. More generally, for a set of indices $I = \{i_1, i_2, \dots, i_k\} \in \mathbb{N}^k$ such that $1 \leq i_1 < i_2 < \dots < i_k \leq n$ we define $dx_I : (\mathbb{R}^n)^k \to \mathbb{R}$ by the formula

$$dx_I(v_1, v_2, \dots, v_k) = \det((v_{m,j})_{m,j=1,\dots,k}). \tag{2.6.2}$$

For two given forms $\alpha \in \Lambda^k(U)$ and $\beta \in \Lambda^l(U)$ we denote their **wedge product** by

$\alpha \wedge \beta \in \Lambda^{k+l}(U)$. Recall that

$$dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}. \tag{2.6.3}$$

Fix an open $V \subset \mathbb{R}^m$ and a function $f : V \to U$. The **pullback** of $\omega \in \Lambda^k(U)$ via function f is denoted by $f^*(\omega)$ and is defined as the form in $\Lambda^k(V)$ such that for arbitrary vectors $v_1, \ldots, v_k \in \mathbb{R}^m$

$$f^*\omega(x;\, v_1, \ldots, v_k) := \omega(f(x), Df(x)v_1, Df(x)v_2, \ldots, Df(x)v_k). \tag{2.6.4}$$

Similarly to Equation (2.6.3) we get

$$f^*(\alpha \wedge \beta) = f^*(\alpha) \wedge f^*(\beta) \in \Lambda^{k+l}(V). \tag{2.6.5}$$

Note 2.6.1 (Jacobian in the language of forms)**.** *In the H-systems (see Equation 2.5.35) we deal with Jacobian structure on the right-hand side. To see why it is linked with differential forms let $u(x) = (u_1(x), u_2(x), u_3(x))$ and observe*

$$\begin{aligned} J(u) &= \partial_1 u \times \partial_2 u \times \partial_3 u \\ &= \left[\det \begin{pmatrix} \partial_1 u_2 & \partial_1 u_3 \\ \partial_2 u_2 & \partial_2 u_3 \end{pmatrix}, -\det \begin{pmatrix} \partial_1 u_1 & \partial_1 u_3 \\ \partial_2 u_1 & \partial_2 u_3 \end{pmatrix}, \det \begin{pmatrix} \partial_1 u_1 & \partial_1 u_2 \\ \partial_2 u_1 & \partial_2 u_2 \end{pmatrix} \right]. \end{aligned} \tag{2.6.6}$$

Every coordinate of $J(u)$ has a Jacobian structure, i.e. $J(u)^i$ is the determinant of ∇u without ∇u_i. We claim that for general $n \in \mathbb{N}$

$$J(u)^i dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n = du_1 \wedge du_2 \wedge \ldots \wedge du_{i-1} \wedge du_{i+1} \wedge \ldots \wedge du_{n+1}. \tag{2.6.7}$$

Without loss of generality fix $i = n+1$. It is enough to prove equation 2.6.7 evaluated on standard vectors $e_1, e_2, \ldots, e_n \in \mathbb{R}^{n+1}$. The left hand side is

$$J(u)^{n+1} dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n(e_1, e_2, \ldots, e_n) = J(u)^{n+1}$$

and the right-hand side

$$\begin{aligned} du_1 \wedge du_2 \wedge \ldots \wedge du_n(e_1, e_2, \ldots, e_n) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot du_1(e_{\sigma(1)}) \cdot du_2(e_{\sigma(2)}) \cdot \ldots \cdot du_n(e_{\sigma(n)}) \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot \frac{\partial u_1}{\partial x_{\sigma(1)}} \cdot \frac{\partial u_2}{\partial x_{\sigma(2)}} \cdot \ldots \cdot \frac{\partial u_n}{\partial x_{\sigma(n)}} \\ &= \det\big(\nabla u_1 \mid \nabla u_2 \mid \ \ldots \ \mid \nabla u^n\big) = J(u)^{n+1}. \end{aligned} \tag{2.6.8}$$

We say that $\mathrm{vol} = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n \in \Lambda^n(\mathbb{R}^n)$ is the **volume** form. If $\omega = f(x)dx_I$ is a differential form then

$$\int_U \omega := \int_U f(x) dx_{i_1} dx_{i_2} \ldots dx_{i_k}. \tag{2.6.9}$$

Hence

$$\int_U \mathrm{vol} = |U|. \tag{2.6.10}$$

The **Hodge star** operator of the form $\omega \in \Lambda^k(U)$ is denoted by $\star\omega \in \Lambda^{n-k}(U)$. Recall that

$$\omega \wedge \star\omega = |f|^2 \mathrm{vol}. \tag{2.6.11}$$

Definition 2.6.2. The codifferential operator $\delta : \Omega^k(U) \to \Omega^{k-1}(U)$ is a linear operator satisfying

$$\delta = (-1)^k \star^{-1} d \star,$$

where d is the exterior derivative and $\star$ is the Hodge star operator.

Notation 2.6.3. *We say that $\omega = f(x)dx_I \in L^p(U)$ if $f \in L^p(U)$. Similarly if X is some Banach space, then $\omega \in X$ if and only if $f \in X$.*

2.6.2 Hodge decomposition and stability theorem

Consider a vector field $X : U \subset \mathbb{R}^n \to \mathbb{R}^n$ defined on open set U. Let $X \in L^p$ for $1 < p < \infty$. It is known that X could be decomposed as follows

$$X = \nabla u + Y, \tag{2.6.12}$$

for some $u \in W^{1,p}(U)$ and divergence-free $Y \in L^p(U)$ (see for example Moser [27] Theorem 2.2). This decomposition is called **Hodge decomposition** of X. Moreover it can be shown that

$$\|\nabla u\|_{L^p(U)} + \|Y\|_{L^p(U)} \leq C\|X\|_{L^p(U)}, \tag{2.6.13}$$

where $C = C(n,p)$. It is worth noting that

- for $p = 1$ the Hodge decomposition is not continuous as the operator $H : L^1 \to L^1 \times L^1$,
- if X is divergence-free, then $\nabla u = 0$. Similarly if $X = \nabla w$ for some $w : \mathbb{R}^n \to \mathbb{R}$, then Y is zero.

Let $u : B^n(x,r) \subset \mathbb{R}^n \to \mathbb{R}$ be fixed function of class $W^{1,p}$ for some $1 < p < \infty$. For a fixed small $\varepsilon \in (-1,\, p-1)$ consider a vector field $G = |\nabla u|^\varepsilon \nabla u \in L^{\frac{p}{1+\varepsilon}}(B^n(x,r))$. Clearly, for $\varepsilon = 0$ our $G = \nabla u$, so as it was mentioned above, the divergence free vector field Y in the Hodge decomposition of u is zero. It turns out that if $|\varepsilon| \simeq 0$ then the divergence-free component V is small in $L^{\frac{p}{1+\varepsilon}}$ norm, which shows that G is 'close' to the gradient ∇u. Such vector fields were studied by Tadeusz Iwaniec in his research on quasiregular mappings. They play a crucial role in our work. We give a precise quantitative statement of this fact in the following theorem (for the proof see Iwaniec [12] Theorem 8.2).

Theorem 2.6.4 (Stability of Hodge decomposition of G)**.** *Let $u \in W^{1,p}(B(x,r))$ for some $p > 1$. Then*

$$G = \nabla \alpha + V,$$

where $\alpha \in W^{1,p/(1+\varepsilon)}(B(x,r))$, $V \in L^{p/(1+\varepsilon)}(B(x,r))$ with $\operatorname{div} V = 0$ in the sense of distributions. Moreover

$$\|V\|_{L^{p/(1+\varepsilon)}(B(x,r))} \leq C|\varepsilon| \|\nabla u\|^{1+\varepsilon}_{L^p(B(x,r))} \tag{2.6.14}$$

2.6.3 Commutator theorem

Definition 2.6.5. We say that K is a Calderón-Zygmund singular operator (or shortly, a Calderón-Zygmund operator) if

$$\begin{aligned}(Kf)(x) = \lim_{\varepsilon\to 0}(K_\varepsilon f)(x) &:= \lim_{\varepsilon\to 0}\Big(\frac{\Omega}{|y|^n} * f\cdot 1|_{\{\mathbb{R}^n\setminus B_\varepsilon(0)\}}\Big)(x) \\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B_\varepsilon(0)} \frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy.\end{aligned} \tag{2.6.15}$$

Here Ω is homogeneous of degree zero, such that it satisfies the cancellation condition $\int_{\mathcal{S}^{n-1}} \Omega\,dx = 0$ and $|\Omega(x)-\Omega(y)| \le C|x-y|$ on $\mathcal{S}^{n-1}$.

The typical example is $\Omega(x) = \frac{x_i}{|x|}$. Recall that K with such smooth kernel Ω is an i-th Riesz transform (see another possible definition in Equation 2.5.24). Let us define another operator known as the commutator operator: fix any $b \in BMO(\mathbb{R}^n)$ and Calderón-Zygmund singular operator K. Let

$$C(f) = [b, K](f) = bK(f) - K(b\cdot f). \tag{2.6.16}$$

R. R. Coifman, R. Rochberg and Guido Weiss (see [3] Theorem I) proved that the commutator operator 2.6.16 is bounded from L^p to itself for $1 < p < \infty$. Equivalently (c.f. [13, Equation 25]) for $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ it holds

$$(Kf)g + f(K^t g) \in \mathcal{H}^1(\mathbb{R}^n), \tag{2.6.17}$$

where K^t is the transposed on K and p, q are Hölder conjugate. For the proof see [13, Theorem III.1 and Remark III.2]. We give the precise statement of that fact in the following theorem.

Theorem 2.6.6 (Commutator theorem)**.** *Let $b : \mathbb{R}^n \to \mathbb{R}$ be a function from $BMO(\mathbb{R}^n)$ and K be a Calderón-Zygmund operator. Then the operator*

$$C(f) = [b, K](f) \tag{2.6.18}$$

is bounded from L^p to itself, for $p \in (1, \infty)$. Moreover

$$\|C\|_{L^p \to L^p} \le C(K,p)\|b\|_{BMO(\mathbb{R}^n)}, \tag{2.6.19}$$

where $C(K,p)$ is a constant depending on K and p.

2.6.4 Gagliardo–Nirenberg inequalities with a BMO term

It is a known fact that from the information about a function being in $W^{k,p}(B^n)$, one can infer the integrability of certain derivatives with a parameter even higher than p. This type of information, coupled with appropriate inequalities, holds significant importance in PDEs, as it enables working with higher integrability without imposing unnecessary assumptions on the studied functions. In this work we need a following version of Gagliardo-Nirenberg inequality involving the BMO-term proved by P.Strzelecki in [29] (see Theorem 1.2).

Theorem 2.6.7 (General version of Gagliardo-Nirenberg inequality with a BMO term)**.** *Fix some $n, k, m \in \mathbb{N}$ such that $1 \le m < k$. Let $u \in W^{k,p}(\mathbb{R}^n)$ for some $p > 1$. If moreover $u \in BMO(\mathbb{R}^n)$, then $\nabla^m u \in L^q(\mathbb{R}^n)$, where $q := \frac{k \cdot p}{m}$. It also holds*

$$\|\nabla^m u\|_{L^q(\mathbb{R}^n)} \le C\|u\|_{BMO}^{1-\frac{m}{k}} \|\nabla^k u\|_{L^p(\mathbb{R}^n)}^{m/k}. \tag{2.6.20}$$

We give now two crucial notes.

Note 2.6.8 (Combining J.Shaftingen and G-N inequalities with BMO term)**.** *Let $u \in W^{n/2,2}(\mathbb{R}^n)$ for some $n > 2$. If n is even, then this space is just a standard Sobolev space. Applying Theorem 2.6.7 with parameters $m = 1$ we get $q = n$ and*

$$\|\nabla u\|_{L^n(\mathbb{R}^n)} \le C\|u\|_{BMO(\mathbb{R}^n)}^{1-\frac{2}{n}} \|\nabla^{n/2} u\|_{L^2(\mathbb{R}^n)}^{\frac{2}{n}}, \tag{2.6.21}$$

or equivalently

$$\|\nabla u\|_{L^n(\mathbb{R}^n)}^n \le C\|u\|_{BMO(\mathbb{R}^n)}^{n-2} \|\nabla^{n/2} u\|_{L^2(\mathbb{R}^n)}^2, \tag{2.6.22}$$

Assume now that $n > 2$ is not even. Then we can write $n = 2k + 1$ for some natural k. The space $W^{n/2,2}(\mathbb{R}^n)$ becomes a fractional Sobolev $W^{k+1/2,2}(\mathbb{R}^n)$ space. Applying now Theorem 2.2.10 we firstly get that $u \in W^{k,n/k}(\mathbb{R}^n) \hookrightarrow W^{1,n}(\mathbb{R}^n)$. The second part of Van Schaftingen inequality gives us

$$\|\nabla^k u\|_{L^{n/k}(\mathbb{R}^n)}^{n/k} \leq C\|u\|_{BMO(\mathbb{R}^n)}^{n/k-2} \cdot [\nabla^k u]_{W^{1/2,2}(\mathbb{R}^n)}^2. \tag{2.6.23}$$

We want to combine inequalities (2.6.22) and (2.6.23) but technically, the k in the above inequality is not the same as $n/2$ in 2.6.22. Hence we use again Theorem 2.6.7 to the $W^{k,n/k}(\mathbb{R}^n)$ with $m = 1$ getting again $q = n$ and

$$\|\nabla u\|_{L^n(\mathbb{R}^n)} \leq C\|u\|_{BMO(\mathbb{R}^n)}^{1-\frac{1}{k}} \|\nabla^k u\|_{L^{n/k}(\mathbb{R}^n)}^{\frac{1}{k}}. \tag{2.6.24}$$

Putting in above inequality the result from 2.6.23 we get

$$\begin{aligned}
\|\nabla u\|_{L^n(\mathbb{R}^n)} &\leq \|u\|_{BMO(\mathbb{R}^n)}^{1-\frac{1}{k}} \cdot \left(\|u\|_{BMO(\mathbb{R}^n)}^{1-\frac{2k}{n}} \cdot [\nabla^k u]_{W^{1/2,2}(\mathbb{R}^n)}^{\frac{2k}{n}}\right)^{\frac{1}{k}} \\
&\leq C\|u\|_{BMO(\mathbb{R}^n)}^{1-\frac{1}{k}} \|u\|_{BMO(\mathbb{R}^n)}^{\frac{1}{k}-\frac{2}{n}} [\nabla^k u]_{W^{1/2,2}(\mathbb{R}^n)}^{\frac{2}{n}} \\
&\leq C\|u\|_{BMO(\mathbb{R}^n)}^{1-\frac{2}{n}} [\nabla^k u]_{W^{1/2,2}(\mathbb{R}^n)}^{\frac{2}{n}}.
\end{aligned} \tag{2.6.25}$$

Thus we finally get

$$\|\nabla u\|_{L^n(\mathbb{R}^n)}^n \leq C\|u\|_{BMO(\mathbb{R}^n)}^{n-2} [\nabla^k u]_{W^{1/2,2}(\mathbb{R}^n)}^2. \tag{2.6.26}$$

Note 2.6.9 (Why $W^{n/2,2}$ instead of $W^{1,n}$)**.** *In Theorems 4.0.1, 5.0.1 we consider Sobolev $W^{n/2,2}$ space instead of general $W^{1,n}$. The right-hand side (further referred as RHS) satisfies $RHS = RHS_1 + RHS_2$, where $RHS_1 \in \mathcal{H}^1$ is a 'good' part and RHS_2 is a 'bad' part. Thanks to the assumption $u \in W^{n/2,2}$ and Coifman-Rochberg-Weiss commutator theorem (see Theorem 2.6.6), the 'bad' part RHS_2 turns out to be small. Technically, we require inequalities (2.6.26) and (2.6.22) only in one specific part of the proofs. The remainder of the proof proceeds under the general assumption of $W^{1,n}$.*

2.6.5 Rivière–Uhlenbeck decomposition

The Rivière–Uhlenbeck decomposition of an antisymmetric matrix is one of the most important tools used in Chapter 5. Thus, we will provide additional details on this subject. It has already been observed that in the case of $\mathcal{N} = \mathcal{S}^{m-1}$ the proof of regularity of n-harmonic maps (see Note 2.5.8) relies on the crucial observation that the second fundamental form A of that sphere can be made divergence-free. This fact gives a batter integrability of the right-hand side of the n-harmonic equations. However this condition is too restrictive for general manifolds. The only attribute we have when working with second fundamental form is its naturally occurring antisymmetric structure. The reasonable question appears: is there any way to modify A such that it becomes divergence-free? How will that change affect the solution, or in other words, what is the cost of the potential modification? If the reader is interested only on the main result, everything in this section except of Notation 2.6.10 and Theorem 2.6.11 could be skipped. Considering that almost all operations are performed on vectors and matrices, we will significantly abbreviate the notation. This can easily lead to confusion during formal derivative calculations. To avoid such type of misunderstanding we introduce the notation firstly.

Notation 2.6.10. *The space of all square $m \times m$ matrices is denoted by $GL(m)$. The space of all antisymmetric $m \times m$ matrices we denote by $so(m)$. The special orthogonal group is denoted by* SO*, i.e.* $\mathrm{SO}(m)$ *denotes the set of orthogonal matrices with determinant* 1*. With that notation, the space of square antisymmetric matrices whose elements are one-forms on $\mathbb{R}^n$ is just $so(m) \otimes \Lambda^1 \mathbb{R}^n$. For two one-forms $\omega_1 = \sum_{i=1}^n f_i dx_i$, $\omega_2 = \sum_{i=1}^n g_i dx_i \in W^{1,p}(\mathbb{R}^n)$ we write*

$$\omega_1 \cdot \omega_2 = \sum_{i=1}^n f_i g_i dx_i \tag{2.6.27}$$

what corresponds to the scalar product for vectors. For a given operator $P \in W^{1,p}(\mathbb{R}^n, GL(m))$ we write

$$\nabla P := (dP_{i,j} \in \Lambda^1(\mathbb{R}^n))_{i,j=1}^m.$$

Hence, for $P, A \in W^{1,p}(\mathbb{R}^n, GL(m))$ by writing $\nabla P \cdot \nabla A$ we mean the standard matrix *multiplication, where the multiplication of individual elements follows the method outlined in Equation (2.6.27). Also we define the L^p norm of a matrix in the following way*

$$\|A\|_{L^p} := \sum_{i,j} \|a_{i,j}\|_{L^p}, \tag{2.6.28}$$

where the sum goes through all elements of the matrix A. Hence by writing $A \in L^p$ we mean that the norm $\|A\|_{L^p}$ is finite.

2.6.6 Main decomposition theorem

This type of decomposition has its roots in differential geometry. When working with connections $\nabla : T\mathcal{N} \times T\mathcal{N} \to T\mathcal{N}$ on an n-dimensional Riemannian manifold $\mathcal{N}$, and given an orthonormal frame $e_i(x)$ chosen locally in $T_x\mathcal{N}$ for some neighbourhood of $x \in \mathcal{N}$, one can express this connection as an $n \times n$ antisymmetric matrix $\Omega = (\omega_{i,j})$ of 1-forms. For another orthonormal frame $\tilde{e}_i(x)$ one can describe the representation of this connection in this new basis by the formula

$$\tilde{\Omega} = P^{-1}dP + P^{-1}\Omega P,$$

where P is orthogonal matrix representing the change of basis from e_i to $\tilde{e}_i$. The "good" choice of a new orthonormal frame aims to satisfy $\delta\tilde{\Omega} = 0$, known as the Coulumb condition. This leads to the appearance of the nonlinear differential equation:

$$\operatorname{div}(\tilde{\Omega}) = 0$$

and it is no longer obvious when such a frame exists.

In this part of the section we will present important theorem that is used in Chapter 5 (c.f. Miśkiewicz, Strzelecki, Petraszczuk [40, Theorem 2.13]) . That theorem is a modified version of the results presented in the paper of Paweł Goldstein and Anna

Zatorska-Goldstein [38, Theorems 1.2, 1.3].

Theorem 2.6.11 (Rivière–Uhlenbeck decomposition)**.** *Let $n > 2$ and $1 < p < \frac{n}{2}$ be fixed. Let moreover $\Omega \in L^n(B(a_0, R_0), \mathrm{so}(m) \otimes \mathbb{R}^n)$ for some ball $B := B(a_0, R_0) \subset \mathbb{R}^n$. There exists $\varepsilon_0(n, m) > 0$ such that if*

$$\|\Omega\|_{L^n(B)} < \varepsilon_0$$

then there is a map $P \in W^{1,n}(B, \mathrm{SO}(m))$ satisfying $P|_{\partial B} = Id$ in the sense of trace, for which the matrix $\tilde{\Omega} = (\tilde{w}_{ij})_{i,j}$:

$$\tilde{\Omega} := P^{-1}dP + P^{-1}\Omega P \tag{2.6.29}$$

is divergence free in the sense of distributions. More precisely, for each form $\tilde{\omega}_{i,j}$ we have

$$\delta\tilde{\omega}_{i,j} = 0 \quad \text{in the sense of distributions,} \tag{2.6.30}$$

where δ denotes the codifferential operator (see Definition 2.6.2). Moreover, we have the following estimates on dP:

$$\|dP\|_{L^n(B)} \le C(n, m)\|\Omega\|_{L^n(B)}, \tag{2.6.31}$$

$$\|dP\|_{L^{p,n-p}(B)} \le C(n, m)\|\Omega\|_{L^{2p,n-2p}(B)}. \tag{2.6.32}$$

Before we proceed to the proof, let us first examine how the Rivière–Uhlenbeck decomposition works in the specific case of two dimensions, as this will provide a clearer understanding.

Rivière–Uhlenbeck decomposition in 2-dimensions — special case

Here we would like to show that the Rivière–Uhlenbeck decomposition in the 2D case is just a Hodge decomposition. Let $\Omega : \mathbb{R}^n \to \mathrm{so}(2) \otimes \Lambda^1\mathbb{R}^n$. We want to find an orthonormal operator $P:\mathbb{R}^n \to \mathrm{SO}(2)$ such that the matrix

$$\tilde{\Omega} = P^T dP + P^T \Omega P$$

is divergence free. Note now that in that case Ω has to be of the following form:

$$\Omega = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

for some 1-form ω. Note also that every element from $\mathrm{SO}(2)$ is in fact a rotation, i.e. for some real variable function $\alpha:\mathbb{R}^n \to \mathbb{R}$

$$P = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

With this we can compute the matrix $\tilde{\Omega}$:

$$dP = \begin{pmatrix} -\sin\alpha & -\cos\alpha \\ \cos\alpha & -\sin\alpha \end{pmatrix} d\alpha,$$

so

$$P^T dP = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} d\alpha.$$

Also we compute

$$P^T \Omega P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \omega.$$

We finally get

$$\tilde{\Omega} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (d\alpha + \omega).$$

Now the condition $\delta\tilde{\Omega} = 0$ reduces to

$$\delta(d\alpha + \omega) = 0. \tag{2.6.33}$$

Thus for a given $\omega \in \Lambda^1\mathbb{R}^n$ we need to find $\alpha \in \Lambda^1\mathbb{R}^n$ such that Equation (2.6.33) holds — this is just the Hodge decomposition. More precisely, applying the Hodge decomposition to ω we get $\omega = d\beta + \delta\gamma$. Thus taking $d\alpha = -d\beta$, Equation (2.6.33) becomes now

$$\delta(d\alpha + \omega) = \delta(\delta\beta) = \delta^2\beta = 0,$$

which shows that Rivière–Uhlenbeck decomposition in the 2D exists even without the small energy assumption.

Proof of Theorem 2.6.11

The proof of the special case $p = 2$, using only elementary calculus methods, can be found in A.Schikorra [32, Theorem 2.1]. Let us write the proof of the above theorem.

Proof. Without loss of generality we assume that $B = \mathbb{B}^n(0,1)$. The reason why we can not apply directly the decomposition described in [38, Theorem 1.2, Theorem 1.3] is that the case $\Omega \in L^n$ is not covered there. Specifically, Theorem 1.2 in [38] requires $\Omega \in W^{1,n}$, and Theorem 1.3 in [38] requires $\Omega \in L^{2p,\ n-2p}$ and $d\Omega \in L^{p,\ n-2p}$. Moreover, the issue arises from the fact that inequality (2.6.31) appears in [38, Theorem 1.2] under the assumption that $\frac{n}{2} \leq p < n$, while inequality (2.6.32) appears in [38, Theorem 1.3] under the disjoint assumption $1 < p < \frac{n}{2}$. To resolve this issue, we take a smooth convolution kernel $\phi_\varepsilon \in C_0^\infty(\mathbb{B}^n(0,1))$ and set $\Omega_i := \Omega * \phi_{1/i}$. We are going to prove the following technical lemma.

Lemma 2.6.12. *Ω_i defined as above is smooth in $\mathbb{B}^n(0,1)$ and satisfies the following inequality*

$$\|\Omega_i\|_{L^n} \leq \|\Omega\|_{L^n} < \varepsilon. \tag{2.6.34}$$

Proof. Such construction ensures that $\Omega_i \in C^\infty(\mathbb{B}^n(0,1))$. To prove the next part we denote $\tau_y : L^n \to L^n$ such that $(\tau_y g)(x) := g(x-y)$ and recall that τ_y is isometry and that the operator $\mathbb{R}^n \ni y \to \tau_y g \in L^n$ is continuous. Thus we can write for each scalar functions f, g

$$(g * f)(x) = \int_{\mathbb{R}^n} (\tau_y g)(x) f(y)\, dy$$

and so

$$g * f = \int (\tau_y g) f(y)\, dy.$$

This leads to the following estimation of Ω_i:

$$\begin{aligned}
\|\Omega_i\|_{L^n(\mathbb{B}^n(0,1))} &= \|\Omega * \phi_{1/i}\|_{L^n(\mathbb{B}^n(0,1))} \le \int_{\mathbb{R}^n} \|(\tau_y\Omega)\phi_{1/i}(y)\|_{L^n(\mathbb{B}^n(0,1))}\, dy \\
&= \int_{\mathbb{R}^n} \|(\tau_y\Omega)\|_{L^n(\mathbb{B}^n(0,1))} \phi_{1/i}(y)\, dy \\
&= \int_{\mathbb{R}^n} \|\Omega\|_{L^n(\mathbb{B}^n(0,1))} \phi_{1/i}(y)\, dy \\
&= \|\Omega\|_{L^n(\mathbb{B}^n(0,1))} \int_{\mathbb{R}^n} \phi_{1/i}(y)\, dy \\
&= \|\Omega\|_{L^n(\mathbb{B}^n(0,1))},
\end{aligned} \tag{2.6.35}$$

which ends the proof. □

Using the above Lemma and that Ω has regularity even more than required, we can apply [38, Thorem 1.2] to the operator Ω_i. Because the referenced theorem is written in the language of differential forms, and we are translating it into the language of derivatives, we will provide a comment for clarification.

Note 2.6.13. *The [38, Theorem 1.2] implies the existence of orthonormal operators* $P_i \in W^{2,p}(\mathbb{B}^n(0,1), \mathrm{SO}(m))$ *and antisymmetric operators of* $(n-2)$*-forms*

$\zeta_i \in W^{2,p} \cap W_0^{1,p}(\mathbb{B}^n(0,1), \mathrm{so}(m) \otimes \Lambda^{n-2}\mathbb{R}^n)$ *for* $\frac{n}{2} \le p < n$, *such that*

$$\begin{cases} P_i^{-1} dP_i + P_i^{-1}\Omega_i P_i = *d\zeta_i & \text{on } \mathbb{B}^n(0,1), \\ d * \zeta_i = 0 & \text{on } \mathbb{B}^n(0,1), \\ \zeta_i = 0 & \text{on } \partial\mathbb{B}^n(0,1), \\ P_i = \mathrm{Id} & \text{on } \partial\mathbb{B}^n(0,1). \end{cases} \tag{2.6.36}$$

Moreover

$$\|d\zeta_i\|_{L^n(\mathbb{B}^n(0,1))} + \|dP_i\|_{L^n(\mathbb{B}^n(0,1))} \le C(n,m)\|\Omega_i\|_{L^n(\mathbb{B}^n(0,1))}. \tag{2.6.37}$$

Note now that the higher order estimates described in the [38, Theorem 1.2] will not hold for initial Ω *(see further in the proof). We do the same note on operators* P_i *and* ζ_i *— in the limit case we only control the* $W^{1,p}$ *of it (see further computations).*

One can note that $*d\zeta$ *could be identified with the matrix of divergence free one-forms. Indeed*

$$\delta(*d\zeta_i) = (-1)^{n-1} * d * \circ * d\zeta_i = (-1)^k * d^2\zeta_i = 0,$$

where k *is some natural number depending on the sign* n *and we recall* $(*)^2 = id$, *and* $d^2 = 0$. *From now on we denote the divergence free matrix* $*d\zeta_i$ *by* $\tilde{\Omega}_i$.

The precise choice of parameter p *will be presented in the Lemma 2.6.15.*

One of the main points of that proof is the following lemma.

Lemma 2.6.14. *There exists a subsequence* P_{i_k}, Ω_{i_k} *and orthonormal operator* $P \in W^{1,n}(\mathbb{B}^n(0,1), SO(m))$ *such that*

$$\tilde{\Omega}_{i_k} := P_{i_k}^{-1} dP_{i_k} + P_{i_k}^{-1}\Omega_{i_k} P_{i_k} \rightharpoonup P^{-1}dP + P^{-1}\Omega P := \tilde{\Omega} \tag{2.6.38}$$

weakly in $L^1(\mathbb{B}^n(0,1))$. *Moreover the matrix operator* $\tilde{\Omega}$ *is divergence free in the sense of distributions and* $\|dP\|_{L^n} \le \|\Omega\|_{L^n}$.

Proof.

P_i are uniformly bounded in $W^{1,n}$

The key observation is that P_i are uniformly bounded in $W^{1,n}(\mathbb{B}^n(0,1))$. Indeed, since P_i has values in $SO(m)$, each column is an element of $\mathcal{S}^{m-1}$, thus $\|P_i\|_{L^n(\mathbb{B}^n(0,1))} < C(n)$. The L^n norm of the derivative of P_i is uniformly bounded because of inequalities (2.6.37) and (2.6.34), i.e.

$$\|dP_i\|_{L^n(\mathbb{B}^n(0,1))} \leq \|\Omega_i\|_{L^n(\mathbb{B}^n(0,1))} \leq \|\Omega\|_{L^n(\mathbb{B}^n(0,1))}.$$

In summary we get

$$\|P_i\|_{W^{1,n}(\mathbb{B}^n(0,1))} < \infty.$$

Construction of P

The $W^{1,n}$ space is reflexive so from each bounded set we can choose a weakly convergent (to some element of $W^{1,n}$) subsequence $P_{i_k} \rightharpoonup P$. That means $\nabla P_{i_k} \rightharpoonup \nabla P$ and $P_{i_k} \rightharpoonup P$ weakly in L^n. BY Rellich–Kondrachov theorem we get $W^{1,n} \hookrightarrow\hookrightarrow L^q$ for any $q \geq 1$ and thus $P_{i_k} \to P$ strongly in any L^q. The strong convergence of P_{i_k} in L^q implies the pointwise convergence a.e. in $\mathbb{B}^n$ (upon passing to a subsequence). Without loss of generality we denote $P_{i_k} := P_i$. This shows in particular that the limit point P is also a map valued in $SO(m)$.

Showing that $\tilde{\Omega}_i \rightharpoonup \tilde{\Omega}$ weakly in L^1

Let us show two things: $P_i^{-1} dP_i \rightharpoonup P^{-1}dP$ and $P_i^{-1}\Omega_i P_i \rightharpoonup P^{-1}\Omega P$ weakly in L^1. Let $\phi \in L^\infty(\mathbb{B}^n(0,1))$, we write

$$
\begin{aligned}
I :&= \int_{B^n(0,1)} P_i^{-1} \cdot dP_i \cdot \phi \, dx - \int_{B^n(0,1)} P^{-1} \cdot dP \cdot \phi \, dx \\
&= \int_{B^n(0,1)} P_i^{-1} \cdot dP_i \cdot \phi \, dx - \int_{B^n(0,1)} P^{-1} \cdot dP_i \cdot \phi \, dx + \int_{B^n(0,1)} P^{-1} \cdot dP_i \cdot \phi \, dx \\
&\quad - \int_{B^n(0,1)} P^{-1} \cdot dP \cdot \phi \, dx \\
&= \int_{B^n(0,1)} dP_i (P_i^{-1} - P^{-1}) \phi \, dx + \int_{B^n(0,1)} P^{-1} (dP_i - dP) \phi \, dx.
\end{aligned}
\tag{2.6.39}
$$

Thus we can estimate

$$
\begin{aligned}
|I| &\leq \left| \int_{B^n(0,1)} dP_i (P_i^{-1} - P^{-1}) \phi \, dx \right| + \left| \int_{B^n(0,1)} P^{-1} (dP_i - dP) \phi \, dx \right| \\
&\leq \|dP_i\|_{L^n(\mathbb{B}^n(0,1))} \|\phi\|_{L^\infty(\mathbb{B}^n(0,1))} \|P_i^T - P^T\|_{L^{n'}(\mathbb{B}^n(0,1))} \\
&\quad + \|P^T\|_{L^\infty(\mathbb{B}^n(0,1))} \cdot \left| \int_{B^n(0,1)} (dP_i - dP) \phi \, dx \right| \\
&:= I^* + II^*,
\end{aligned}
\tag{2.6.40}
$$

where the last inequality follows from the Hölder inequality and the fact that $P^{-1} = P^T$. Each of the terms I^*, II^* converges to zero for $i \to \infty$. The first convergence follows from the fact that $\|dP_i\|_n \leq C$ and that P_i converges strongly to P in any L^q. The second term converges to zero because $dP_i \rightharpoonup dP$ weakly in L^n (the bounded ϕ is also in any L^q for bounded domains, thus in particular $\phi \in (L^n)^*$). This concludes our demonstration of weak convergence of the $P_i^T dP_i$ to $P^T dP$. In the same way we show the convergence of $P_i^{-1} \Omega_i P_i \rightharpoonup P^{-1} \Omega P$ weakly in L^1 :

$$
\begin{aligned}
\|P_i^{-1} \Omega_i P_i - P^{-1} \Omega P\|_{L^1} &= \|P_i^{-1} \Omega_i P_i - P_i^{-1} \Omega P + P_i^{-1} \Omega P - P^{-1} \Omega P\|_{L^1} \\
&\leq \|P_i^{-1} (\Omega_i P_i - \Omega P)\|_{L^1} + \|(P_i^T - P^T) \Omega P\|_{L^1} \\
&\leq \|P_i\|_{L^\infty} \|\Omega_i P_i - \Omega P\|_{L^1} + \|P_i^T - P^T\|_{L^{n'}} \|\Omega P\|_{L^n} \\
&:= (*) + (**).
\end{aligned}
\tag{2.6.41}
$$

The term $(**)$ converges to zero because $\Omega \in L^n$ and $P_i \to P$ strongly in any L^q in particular in L^n. We estimate the first term as follows:

$$\begin{aligned}\|\Omega_i P_i - \Omega P\|_{L^1} &\leq \|\Omega_i(P_i - P)\|_{L^1} + \|(\Omega_i - \Omega)P\|_{L^1} \\ &\leq \|\Omega_i\|_{L^n}\|P_i - P\|_{L^n} + \|\Omega_i - \Omega\|\|P\|_\infty \\ &\to 0,\end{aligned} \tag{2.6.42}$$

where the last convergence follows from the fact that $\Omega_i \to \Omega$ strongly in L^1. Hence finally we get as follows

$$\Omega_i \rightharpoonup \Omega$$

weakly in $L^1(\mathbb{B}^n(0,1))$. Note that for the weak limit we have

$$\|\nabla P\|_{L^n} \leq \liminf_{i\to\infty} \|\nabla P_i\|_{L^n} \leq \|\Omega\|_{L^n}.$$

The last thing which is needed to be shown is that $\tilde{\Omega}$ is divergence free. This is a simple consequence of the weak convergence of $\tilde{\Omega}_i$. Indeed, let $\phi \in C_0^\infty$, then for each i we have

$$\int \tilde{\Omega}_i \cdot \nabla\phi \, dx = 0.$$

Thus

$$\lim_{i\to\infty} \int \tilde{\Omega}_i \cdot \nabla\phi \, dx = 0.$$

On the other hand, using the definition of the weak convergence we get the following equalities

$$\lim_{i\to\infty} \int \tilde{\Omega}_i \cdot \nabla\phi \, dx = \int \tilde{\Omega} \cdot \nabla\phi \, dx = 0.$$

This ends the proof of the lemma. □

At the end of the proof we have to justify the Morrey type inequality (2.6.32) for our Ω, $\tilde{\Omega}$ and P from the previous lemma.

Lemma 2.6.15. *Let $1 < q < \frac{n}{2}$. The orthonormal operator P, the antisymmetric operator*

Ω *and the divergence free operator* $\tilde{\Omega}$ *defined in Lemma 2.6.14 satisfy the following Morrey type inequality*

$$\|\nabla P\|_{L^{q,\,n-q}} + \|\tilde{\Omega}\|_{L^{q,\,n-p}} \leq C(n,m)\|\Omega\|_{L^{2q,n-2q}}$$

Proof. The reason we cannot use [38, Theorem 1.3] for the operator Ω_i is that there is no clear way to justify that the decomposition will be the same as the one obtained in the previous lemma. To avoid this kind of difficulty we need to operate with the same sequences P_i and $\tilde{\Omega}_i$. Therefore, we apply [38, Lemma 5.4] to the Ω_i. Following [38], we denote

$$V_\varepsilon^\alpha := \{f \in L^{2q,\,n-2q+\alpha} :\ df \in L^{q,\,n-2q},\ \|f\|_{L^{2q,n-2q}} < \varepsilon\},$$

where $\alpha > 0$ is a small fixed parameter. Let us now prove that all assumptions of that lemma are satisfied.

Step 1: every Ω_i is in V_ε^α

For every function f, using the Hölder inequality we get

$$\|f\|_{L^{2q,\,n-2q+\alpha}(\mathbb{B}^n(0,1))} \leq \|f\|_{L^{\frac{2nq}{2q-\alpha}}(\mathbb{B}^n(0,1))}. \tag{2.6.43}$$

Hence, since $\Omega_i \in C^\infty$ and $d\Omega_i$ we get

$$\|\Omega_i\|_{L^{2q,\,n-2q+\alpha}(\mathbb{B}^n(0,1))} \leq \|\Omega_i\|_{L^{\frac{2nq}{2q-\alpha}}(\mathbb{B}^n(0,1))} < \infty,$$

$$\|d\Omega_i\|_{L^{2q,\,n-2q+\alpha}(\mathbb{B}^n(0,1))} \leq \|d\Omega_i\|_{L^{\frac{2nq}{2q-\alpha}}(\mathbb{B}^n(0,1))} < \infty.$$

Taking now $\alpha = 0$ we get

$$\|\Omega_i\|_{L^{2q,\,n-2q}(\mathbb{B}^n(0,1))} \leq \|\Omega_i\|_{L^n(\mathbb{B}^n(0,1))} < \varepsilon.$$

Step 2: P_i are in $L_2^{q,n-2q+\alpha}$ and $\tilde{\Omega}_i \in L_1^{q,n-2q+\alpha}$

We begin with demonstrating that $P_i \in L_2^{q,n-2q+\alpha}$, thus we need to show that $\nabla P_i \in L^{q,n-q+\alpha}$ and that $\nabla^2 P_i \in L^{q,n-2q+\alpha}$. From the first part we know that $P_i \in W^{2,p}$ for arbitrary chosen $\frac{n}{2} \leq p < n$. Consequently, $\nabla P_i \in L^{p^*}$, where $p^* := \frac{np}{n-p}$. As p approaches n, $p^* \to \infty$. On the other hand, the $L^{\frac{nq}{q-\alpha}}$-norm of ∇P_i controls its $L^{q,n-q+\alpha}$-norm (see inequality (2.6.43)). Therefore, it is sufficient to select p such that $p^* \geq \frac{nq}{q-\alpha}$. For small α, the fraction $\frac{nq}{q-\alpha} \approx n$, making such a choice of p possible. With the second derivative the problem is even easier: we fix some ball $B(a,r) \subset \mathbb{B}^n(0,1)$ and write

$$\frac{1}{r^{n-2q+\alpha}} \int_{B(a,r)} |\nabla^2 P_i|^q \, dx \leq C(n) \int_{B(0,1)} |\nabla^2 P_i|^{\frac{qn}{2q-\alpha}} \, dx \tag{2.6.44}$$

noting also that for small fixed value of α we have $\frac{qn}{2q-\alpha} \approx \frac{n}{2}$. The above expression is also finite for the same reason presented earlier — we ensure the parameter p is close to n, guaranteeing $p > \frac{n}{2}$.

We handle the $L_1^{q,n-2q+\alpha}$ norm of the $\tilde{\Omega}_i$ in a similar manner as above.

Step 3: the $L^{2q,n-2q}$-smallness assumption on $\tilde{\Omega}_i$ and P_i

The $L^{2q,n-2q}$-smallness assumption [38, Inequality 5.15] on $\tilde{\Omega}_i$ and P_i is satisfied — this is the direct consequence of inequality (2.6.43) and the fact that one can take $\varepsilon := \kappa$ at the start of the proof.

Therefore, once the assumptions of the referenced Lemma are satisfied, one obtains, for each $1 < p < \frac{n}{2}$

$$\|\nabla P_i\|_{L^{p,n-p}} + \|\tilde{\Omega}_i\|_{L^{p,n-p}} \leq C(n,m)\|\Omega_i\|_{L^{2p,n-2p}}, \tag{2.6.45}$$

in particular

$$\|\nabla P_i\|_{L^{p,n-p}} \leq C(n,m)\|\Omega_i\|_{L^{2p,n-2p}}. \tag{2.6.46}$$

Now we need to show that the same inequality holds in the limit case, i.e.

$$\|\nabla P\|_{L^{p,n-p}} \leq C(n,m)\|\Omega\|_{L^{2p,n-2p}}.$$

For this, we are going to prove that

$$\|\nabla P\|_{L^{p,n-p}} \leq \liminf \|\nabla P_i\|_{L^{p,n-p}} \leq \liminf \|\Omega_i\|_{L^{2p,n-2p}} \leq \|\Omega\|_{L^{2p,n-2p}}. \tag{2.6.47}$$

The first inequality could be proved as follows: pick arbitrary ball $B(a,r) \subset \mathbb{B}^n(0,1)$ and note that $\nabla P_i \rightharpoonup \nabla P$ weakly in $L^p(\mathbb{B}^n(0,1))$ for each $p \leq n$. Thus in particular we get $\nabla P_i \rightharpoonup \nabla P$ weakly in $L^p(B(a,r))$. The last show that $\int_{B(a,r)} |\nabla P|^p\,dx \leq \liminf \int_{B(a,r)} |\nabla P_i|^p\,dx$ so trivially we get the following inequality

$$\frac{1}{r^{n-p}} \int_{B(a,r)} |\nabla P|^p\,dx \leq \liminf \frac{1}{r^{n-p}} \int_{B(a,r)} |\nabla P_i|^p\,dx.$$

As the consequence we get

$$\|\nabla P\|_{L^{p,n-p}} \leq \liminf \|\nabla P_i\|_{L^{p,n-p}}.$$

The second inequality in (2.6.47) follows from inequality (2.6.46). The last inequality is derived from Lemma 2.6.12. Here is the reasoning:

$$\|\Omega_i\|_{L^p(B(a,r))} \leq \|\Omega\|_{L^p(B(a,r))}.$$

By employing the methods used to prove the first inequality of (2.6.47), we finally get

$$\|\Omega_i\|_{L^{2p,n-2p}(\mathbb{B}^n(0,1))} \leq \|\Omega\|_{L^{2p,n-2p}(\mathbb{B}^n(0,1))}.$$

The proof of the lemma is complete now. □

Combining results of Lemmas 2.6.14, 2.6.15, we finish the proof of Theorem 2.6.11. □

Chapter 3

Prescribing singularities for a nonlinear elliptic system in the plane

This chapter, based on author's paper [42], stands somewhat independently from the following two chapters and serves as a valuable introduction to them: it shows how crucial the structure of nonlinearity of the right-hand side is for the geometry of singularities. On one hand, we have the celebrated Hélein [10] result, which shows that weakly harmonic maps $u : \mathbb{R}^2 \supseteq \Omega \to \mathcal{N} \subseteq \mathbb{R}^m$, where $\mathcal{N}$ is a closed Riemannian manifold, are smooth. On the other hand, in 1973, J.Frehse [1], provided an illustrative example of a two-dimensional solution to the equation

$$\Delta u = F(u, \nabla u), \tag{3.0.1}$$

where F is an analytic function with quadratic growth in ∇u, which is bounded and discontinuous in singe point. Here, we aim to extend the Frehse one-point singularity example, showing that solutions of the general equation (3.0.1) could have highly intricate singularity structure. More precisely, for the ball $B := \mathbb{B}^2(0, e^{-1})$ and $u \in W^{1,2} \cap L^\infty(B, \mathbb{R}^2)$ with the right hand side F defined as follows

$$F(u, \nabla u) = (F_1, F_2) = \left(-2|\nabla u|^2 \frac{u_1 + u_2}{1 + |u|^2},\, 2|\nabla u|^2 \frac{u_1 - u_2}{1 + |u|^2} \right). \tag{3.0.2}$$

and u defined by the equations:

$$u_1(x) = \sin(\log\log(|x|^{-1})), \quad u_2(x) = \cos(\log\log(|x|^{-1})), \tag{3.0.3}$$

it can be demonstrated that $\Delta u = F(u, \nabla u)$ holds weakly in B. Using the same right hand side F as in (3.0.2) and modifying u in (3.0.3), we demonstrate a method to prescribe singularities on any predetermined compact subset of the domain (see next Theorem).

Theorem 3.0.1. *Fix a small radius $0 < r < \frac{1}{e}$ and consider the ball $B := B(0, r) \subset \mathbb{R}^2$. For every compact subset K within the ball B, there exists a solution $u \in W^{1,2}(B, \mathbb{R}^2) \cap L^\infty$ to a nonlinear elliptic system:*

$$\Delta u = F(u, \nabla u), \tag{3.0.4}$$

where F defined in (3.0.2). This solution u is singular on K and smooth elsewhere.

In higher-dimensional case, the substitute of Equation (3.0.1), i.e. the p-Laplace elliptic differential equation of type

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = F(u, \nabla u),$$

where $|F(u, \nabla u)| < C|\nabla u|^p$ and $u \in W^{1,p}$, as it was mentioned in Section 2.5.2 also can exhibit solutions with singularities. Below we will prove the main theorem relying on the author's accepted paper [42].

Proof. Let us fix a countable dense subset $P := \{p_1, p_2, \ldots\} \subset K$. We provide an exact formula for a solution u, which is singular in $\overline{P} = K$ and smooth in $\tilde{B} := B \setminus \overline{P}$.

3.0.1 The structure of the proof

In the spirit of [1], we denote

$$f(x) = \log(1/|x|).$$

Recall that f is harmonic outside of the origin and

$$\Delta f = -2\pi\delta_0$$

in the sense of distributions $D'(B)$. Let u be defined as in (3.0.3). The most natural approach to extending the singularity of such u to all points of a set P could seem to define

$$\tilde{u}(x) = \sum_{i=1}^{\infty} a_i u(x - p_i),$$

where $a_i > 0$ is a sequence decreasing sufficiently fast (e.g., geometrically) to zero. It is then straightforward to demonstrate that $\tilde{u} \in W^{1,2}(B)$. Unfortunately, due to the nonlinear structure of the right-hand side (3.0.2), verifying $\Delta\tilde{u} = \tilde{F}(\tilde{u}, \nabla\tilde{u})$, where $\tilde{F}$ exhibits quadratic growth in the gradient, becomes a highly intricate task. To solve this issue, we actually change u in a different way, by modifying the function f first. We set

$$f(x) = \sum_{i=1}^{\infty} a_i \log\Big(\frac{1}{|x - p_i|}\Big).$$

With this new f, we consider

$$u_1 = \sin\log f, \qquad u_2 = \cos\log f, \tag{3.0.5}$$

and their approximations

$$u_1^N = \sin\log f_N, \qquad u_2^N = \cos\log f_N,$$

where

$$f_N = \sum_{i=1}^{N} a_i \log\Big(\frac{1}{|x - p_i|}\Big).$$

Before proceeding with further computations, we present a road-map for the whole argument which consists of four main steps.

Everywhere below, we write $P^N := \{p_1, \dots, p_n\}$ and $\tilde{B}^N := B \setminus P^N$.

Our first step is to show that each u^N is a classical solution to Equation (3.0.4) within $\tilde{B}^N$. This verification is straightforward: one can directly compute the derivatives on $\tilde{B}^N$ and check that the equation holds.

Next, in the second step, we prove that all the maps u^N and their limit u belong to the Sobolev space $W^{1,2}(B,\mathbb{R}^2)$, as stated in Lemma 3.0.3. This is one of the most technically challenging parts of the proof. To achieve this, we verify that u is well defined and that, up to the choice of a subsequence, $u^N \to u$ and $\nabla u^N \to \nabla u$ almost everywhere (see Lemma 3.0.2). This pointwise convergence is, in turn, proved by demonstrating that all u^N belong to $W^{1,p}(B)$ for every $p < 2$, cf. the computations following Equation (3.0.13), and the sequence ∇u^N is bounded in $L^p(B)$, cf. Equation (3.0.12). Then we apply the Rellich-Kondrachov theorem to deduce the existence of such a subsequence u^N.

We then show that the map u indeed does belong to $W^{1,2}(B)$ (see Lemma 3.0.3). Here is the mechanism behind the computations: we use the fact, see (3.0.6),

$$\nabla u \in L^2(B) \iff |\nabla f|^2/f^2 \in L^1(B)$$

and that, cf. (3.0.16)–(3.0.17),

$$|\nabla f|^2 \lesssim \Big(\sum_{i=1}^{\infty} \frac{a_i}{|x-p_i|}\Big)^2.$$

The key idea is to first consider the case of a single singularity $P = \{p_i\}$, where

$$|\nabla f|^2/f^2 = \frac{a_i^2|x-p_i|^{-2}}{(a_i \log|x-p_i|^{-1})^2} = \frac{1}{|x-p_i|^2 \log^2|x-p_i|}.$$

For any fixed $\varepsilon > 0$, we compute:

$$\begin{aligned}\int_{B(0,1)} \frac{1}{|x-p_i|^2\log^2|x-p_i|}dx &= \Big(\int_{B(p_i,\varepsilon)} + \int_{B\setminus B(p_i,\varepsilon)}\Big)\frac{1}{|x-p_i|^2\log^2|x-p_i|}dx \\ &\simeq \Big(\int_0^{\varepsilon} + \int_{\varepsilon}^1\Big)\frac{1}{r\log^2 r}dr < \infty.\end{aligned}$$

The last inequality holds due to the Cauchy condensation test, as the related sum

$$\sum_{n=1}^{\infty} \frac{1}{n \log^s n} < \infty$$

converges for all $s > 1$. Once this special case is handled, we apply standard techniques to estimate $|\nabla u|^2$ for $P = \{p_1,\ p_2,\ \ldots\}$.

The third step is to show that both u^N and u are weak solutions to (3.0.4) on the the entire domain B (see Lemma 3.0.4). The argument here uses the fact that $W^{1,2}$ contains unbounded functions; because of that one can remove the singularity of u^N at p_i using cut-off functions produced by suitable truncations of $\log \log |x - p_i|^{-1}$.

Finally, in Lemma 3.0.5 we show that u is discontinuous at points of K and continuous on $\tilde{B}$. The continuity at points $x \in B \setminus K$ follows from the uniform convergence of the defining series in a neighbourhood of each of these x. However, proving discontinuity at each $x \in K$ is more subtle. It is not enough to show discontinuity of the function inside the sine or cosine term, as the composition with periodic functions could, theoretically at least, mask such a discontinuity. To address this, let

$$h(x) := \log \Big(\sum_{i=1}^{\infty} a_i \log \frac{1}{|x - p_i|} \Big).$$

The core of the argument is to show that for every $x \in K$ and for arbitrarily small $\delta > 0$, the image $h(B(x, \delta))$ contains an interval I of length at least 2π. This is achieved through an application of the ACL property of functions in $W^{1,2}(B)$, cf. Ziemer [7, Theorem 2.4.4], as follows. Within each $B(x, \delta)$, we find two disjoint measurable sets $E, U \subset B(x, \delta)$ such that:

- $|E| > 0$ and $h|_E < M$ for some constant $M > 0$,
- $h|_U > M + 2\pi$, with $U = B(p_j, \tilde{\delta})$, where $p_j \in P$, $|x - p_j| < \tilde{\delta}$ for a small $\tilde{\delta}$, $0 < \tilde{\delta} \ll \delta$.

The existence of E follows just from the integrability of h. By [7, Theorem 2.4.4], h is of class $W^{1,2}$ – and therefore Hölder continuous! – on almost every line parallel to the coordinate axes (also if we rotate the coordinate system). Therefore, along some of the lines connecting E to U the function h must continuously change from M to $M' > M + 2\pi$. This implies that for each $\delta > 0$ the image $u(B(x,\delta))$ contains the full range $[-1,1]\times[-1,1]$. For more details, see the proof of Lemma 3.0.5.

3.0.2 The details of the proof.

We now proceed with the detailed computations and estimates.

Step 1: u^N solves the equation on $B \setminus P^N$. Computing derivatives of u^N in $\tilde{B}^N = B \setminus P^N$, we get

$$\partial_\alpha u_1^N = f_N^{-1}(\partial_\alpha f_N)\cos\log f_N,$$

$$\partial_\alpha u_2^N = -f_N^{-1}(\partial_\alpha f_N)\sin\log f_N.$$

In the same way we compute the second derivatives

$$\partial_\alpha^2 u_1^N = f_N^{-1}(\partial_\alpha^2 f_N)\cos\log f_N - f_N^{-2}(\partial_\alpha f_N)^2(\cos\log f_N + \sin\log f_N),$$

$$\partial_\alpha^2 u_2^N = -f_N^{-1}(\partial_\alpha^2 f_N)\sin\log f_N + f_N^{-2}(\partial_\alpha f_N)^2(\sin\log f_N - \cos\log f_N).$$

Now we can compute $|\nabla u^N|^2$ and Δu_1^N, Δu_2^N as follows:

$$\begin{aligned} |\nabla u^N|^2 &= |\nabla u_1^N|^2 + |\nabla u_2^N|^2 \\ &= f_N^{-2}|\nabla f_N|^2\cos^2\log f_N + f_N^{-2}|\nabla f_N|^2\sin^2\log f_N \\ &= f_N^{-2}|\nabla f_N|^2, \end{aligned} \tag{3.0.6}$$

$$\begin{aligned} \Delta u_1^N &= f_N^{-1}\Delta f_N\cos\log f_N - f_N^{-2}|\nabla f_N|^2(\cos\log f_N + \sin\log f_N) \\ &= -2|\nabla u^N|^2(u_1^N + u_2^N)\frac{1}{1+|u^N|^2}, \end{aligned} \tag{3.0.7}$$

$$\begin{aligned} \Delta u_2^N &= f_N^{-1}\Delta f_N\sin\log f_N + f_N^{-2}|\nabla f_N|^2(\sin\log f_N - \cos\log f_N) \\ &= 2|\nabla u^N|^2(u_1^N - u_2^N)\frac{1}{1+|u^N|^2}. \end{aligned} \tag{3.0.8}$$

The last equation holds on $\tilde{B}^N$ since $|u^N| = 1$ and $\Delta f_N = 0$.

Step 2: Sobolev space estimates for u^N and u. It is necessary to verify that each u^N and $u = \lim u^N$ indeed belong to $W^{1,2} \cap L^\infty$ on the entire ball B, and that both u^N and u serve as weak solutions to Equation (3.0.4) within B. Before proceeding with that proof, we will first establish the following technical lemma, as previously mentioned.

Lemma 3.0.2. *Let $N \in \mathbb{N}$ and $1 \le p < 2$ be fixed. Let moreover $a_n \searrow 0$ be a geometric sequence of positive numbers. Then, the function u^N belongs to the $W^{1,p}(B)$. Moreover, we have the following pointwise convergences*

$$u^N(x) \to u(x), \tag{3.0.9}$$

$$\nabla u^N(x) \to \nabla u(x) \tag{3.0.10}$$

for almost every $x \in B$.

Proof. In the first part of the proof, we will show that $u^N \in W^{1,p}(B)$. It is sufficient to demonstrate that $u_1^N \in W^{1,p}(B)$, as the proof for u_2^N follows in the same manner. The function u_1^N is bounded on B, which implies $u_1^N \in L^p(B)$. On $\tilde{B}^N$, we have

$$\nabla u_1^N = \cos\left(\log \sum_{i=1}^N a_i \log \frac{1}{|x-p_i|}\right) \cdot \frac{1}{\sum_{i=1}^N a_i \log |x-p_i|^{-1}} \cdot \sum_{i=1}^N a_i \frac{x-p_i}{|x-p_i|^2}. \tag{3.0.11}$$

For each $x \in B$, we have $|x - p_i| < r < \frac{1}{e}$. Thus, $\rho := \inf_i \left(\inf_{x\in B} \log \frac{1}{|x-p_i|}\right) > 0$. Using this, we get

$$\begin{aligned} \|\nabla u_1^N\|_{L^p(B)} &\le \frac{1}{\rho \sum_{i=1}^N a_i} \sum_{i=1}^N a_i \left\| \frac{x-p_i}{|x-p_i|^2} \right\|_{L^p(B)} \le C \sum_{i=1}^N a_i \left\| \frac{1}{|x-p_i|} \right\|_{L^p(B)} \\ &\le C \sum_{i=1}^\infty a_i \le C < \infty, \end{aligned} \tag{3.0.12}$$

for some constant C. This shows that the sequence ∇u_1^N is bounded in $L^p(B)$. Next, let us fix an arbitrary test function $\phi \in C_0^\infty(B)$. To complete the proof that u_1^N belongs to the Sobolev $W^{1,p}$ space, we need to show that

$$\int_B u_1^N \cdot \nabla\phi \, dx = -\int_B \nabla u_1^N \cdot \phi \, dx. \tag{3.0.13}$$

To do this, we write

$$\int_{B\setminus \cup B_\varepsilon(p_i)} u_1^N \cdot \nabla\phi \, dx = -\int_{B\setminus \cup B_\varepsilon(p_i)} \nabla u_1^N \cdot \phi \, dx + \int_{\partial \cup B_\varepsilon(p_i)} u_1^N \cdot \phi \cdot \nu \, dx \tag{3.0.14}$$

where ν denotes the unit normal vector to $\partial(\cup B_\varepsilon(p_i))$. Now we estimate:

$$\left| \int_{\partial \cup B_\varepsilon(p_i)} u_1^N \cdot \phi \cdot \nu \, dx \right| \leq \|u_1^N\|_\infty \|\phi\|_\infty \|\nu\|_\infty \cdot 2\pi N\varepsilon \to 0.$$

Thus, by letting $\varepsilon \to 0$ in Equation (3.0.14), we obtain that condition (3.0.13) is satisfied.

Next, we prove the pointwise convergences stated in (3.0.9), (3.0.10). The first converges is straightforward: since the sequence f_N is bounded in $W^{1,p}(B)$, by the Rellich-Kondrachov theorem, we can write (after possibly passing to a subsequence) that $f_N \to f$ in $L^q(B)$, where $q = \frac{2p}{2-p} > 1$. In particular $f_N \to f$ almost everywhere. This shows that

$$u_1^N = \sin(\log f_N) \to \sin(\log f) = u_1 \tag{3.0.15}$$

almost everywhere. To prove the pointwise convergence of the gradients (3.0.10), we first choose a subsequence of f^N such that the pointwise convergence (3.0.15) holds, and denote this subsequence again by f^N. This ensures the following convergence for almost every $x \in B$:

$$\cos(\log f^N(x)) \cdot \frac{1}{f^N(x)} \to \cos(\log f(x)) \cdot \frac{1}{f(x)}.$$

Thus, to show the pointwise convergence of ∇u^N described in (3.0.11), it is sufficient to show that (up to a subsequence)

$$g^N(x) := \sum_{i=1}^N a_i \frac{x - p_i}{|x - p_i|^2}$$

converges to

$$g(x) := \sum_{i=1}^{\infty} a_i \frac{x - p_i}{|x - p_i|^2}$$

almost everywhere. It is clear that g^N is a Cauchy sequence in $L^1(B)$: for fixed $N, M \in \mathbb{N}$ such that $N < M$, using the fact that the sequence of functions $h_i(x) = \frac{x-p_i}{|x-p_i|^2}$ is bounded in $L^p(B)$, we obtain the following inequality

$$\|g^M - g^N\|_{L^1(B)} = \sum_{i=N+1}^{M} a_i \left\| \frac{x - p_i}{|x - p_i|^2} \right\|_{L^1(B)} \leq C \sum_{N}^{M} a_i.$$

This shows that the above difference can be made arbitrary small. Since $L^1(B)$ is complete, g^N converges (possibly up to a subsequence) to a limit in L^1. Clearly, the limit is exactly the function g. The L^1-convergence implies the pointwise convergence of g^N to g. Consequently, for that subsequence, we have $\nabla u^N \to \nabla u$ almost everywhere in B. □

Lemma 3.0.3. *Let $u : B \to \mathbb{R}^2$ be the function defined in Equation (3.0.5). Assuming that $a_n \searrow 0$ is a geometric sequence of positive numbers, it follows that $u \in W^{1,2} \cap L^\infty(B)$.*

Proof. The fact that u is bounded and that $u \in L^2(B)$ is obvious. Unlike the straightforward proof for $\nabla u \in L^p$ when $p < 2$ (as discussed in Lemma 3.0.2), demonstrating that $\nabla u \in L^2(B)$ requires a more delicate argument. Using computations in Equation (3.0.6) we need to prove that $|\nabla f|^2/f^2 \in L^1(B)$. Using an elementary computation we get

$$\partial_\alpha f = \sum_{i=1}^{\infty} a_i \frac{(x - p_i)_\alpha}{|x - p_i|^2}. \tag{3.0.16}$$

Hence

$$|\nabla f|^2 \lesssim \Big(\sum_{i=1}^{\infty} \frac{a_i}{|x - p_i|} \Big)^2. \tag{3.0.17}$$

We fix some small positive $\beta > 0$ and use Cauchy-Schwartz inequality to get

$$\Big(\sum_{i=1}^{\infty} \frac{a_i}{|x - p_i|} \Big)^2 = \Big(\sum_{i=1}^{\infty} a_i^\beta \frac{a_i^{1-\beta}}{|x - p_i|} \Big)^2 \leq \Big(\sum_{i=1}^{\infty} a_i^{2\beta} \Big) \cdot \sum_{i=1}^{\infty} \frac{a_i^{2-2\beta}}{|x - p_i|^2}.$$

We know that

$$C = \sum_{i=1}^{\infty} a_i^{\beta} < \infty. \tag{3.0.18}$$

Thus we get

$$\begin{aligned} |\nabla f|^2/f^2 &\leq \frac{C \sum_{i=1}^{\infty} a_i^{2-2\beta} |x-p_i|^{-2}}{\left(\sum_{j=0}^{\infty} a_j \log |x-p_j|^{-1}\right)^2} \\ &= \sum_{i=1}^{\infty} \frac{a_i^{2-2\beta}}{|x-p_i|^2} \cdot \frac{1}{\underbrace{\left(\sum_{j=0}^{\infty} a_j \log |x-p_j|^{-1}\right)^2}_{:=M}} \qquad = (*). \end{aligned}$$

Since the ball B is small enough, we can assume that each term $a_j \log \frac{1}{|x-p_j|}$ is strictly positive. This observation allows us to estimate the denominator M as follows

$$M \geq a_i^2 \log^2 \frac{1}{|x-p_i|} + 2a_i a_1 \log \frac{1}{|x-p_i|} \log \frac{1}{|x-p_1|}.$$

Assuming $a_1 = 1$, $p_1 = 0$ and denoting $\lambda := \inf_{x \in B}(\log \frac{1}{|x|}) > 0$ we get

$$M \geq a_i^2 \log^2 \frac{1}{|x-p_i|} + 2\lambda a_i \log \frac{1}{|x-p_i|}.$$

We fix $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and recall that for each numbers $x, y > 0$ the next inequality holds

$$x \cdot y \leq \frac{x^p}{p} + \frac{y^p}{q} \leq x^p + y^q.$$

Thus, by applying the aforementioned inequalities, we obtain

$$\begin{aligned} (*) &\leq \sum_{i=1}^{\infty} \frac{a_i^{2-2\beta}}{|x-p_i|^2} \cdot \frac{1}{a_i^2 \log^2 |x-p_i|^{-1} + 2\lambda a_i \log |x-p_i|^{-1}} \\ &\leq \sum_{i=1}^{\infty} \frac{a_i^{2-2\beta}}{|x-p_i|^2} \cdot \frac{1}{(a_i^2 \log^2 |x-p_i|^{-1})^{1/q} \cdot (2\lambda a_i \log |x-p_i|^{-1})^{1/p}} \\ &\leq \tilde{C} \sum_{i=1}^{\infty} \frac{a_i^{1-2\beta-1/q}}{|x-p_i|^2 \log^{1+1/q} |x-p_i|^{-1}} \end{aligned} \tag{3.0.19}$$

Writing

$$b_i := \int_B \frac{1}{|x-p_i|^2 \log^{1+1/q} |x-p_i|^{-1}}\, dx$$

and integrating the above inequality over the ball B, we obtain

$$\tilde{C} \sum_{i=1}^{\infty} a_i^{1-2\beta-1/q} \int_B \frac{1}{|x-p_i|^2 \log^{1+1/q} |x-p_i|^{-1}}\, dx = \sum_{i=1}^{\infty} a_i^{1-2\beta-1/q} b_i. \tag{3.0.20}$$

Note that the whole sequence b_i is bounded. Indeed each of the b_i is finite because $1+\frac{1}{q} > 1$. Let us fix arbitrary index i and radius $\tilde{r} << r$, let moreover $B_i := B(p_i, \tilde{r})$. Using the change of variable we write

$$\begin{aligned} b_i &= \left(\int_{B_i} + \int_{B \setminus B_i} \right) \frac{1}{|x-p_i|^2 \log^{1+1/q} |x-p_i|^{-1}}\, dx \\ &= \int_{B(0,\tilde{r})} \frac{1}{|x|^2 \log^{1+1/q} |x|^{-1}}\, dx + \int_{B \setminus B_i} \frac{1}{|x-p_i|^2 \log^{1+1/q} |x-p_i|^{-1}}\, dx \\ &:= \mathrm{I} + \mathrm{II}. \end{aligned}$$

The first term I is bounded by constant $C = C(\tilde{r})$. The second term II is also bounded because each $x \in B \setminus B_i$ satisfy

$$0 < \varepsilon < |x - p_i| < r$$

for some fixed constant $\varepsilon > 0$ and so

$$\mathrm{II} < \int_B \frac{1}{\varepsilon^2 \log^{1+1/q} \varepsilon^{-1}}\, dx \le C(\varepsilon, r).$$

Hence by combining those two inequalities it follows that $b_i \le C(\varepsilon, r, \tilde{r})$. These constant is independent of index i and so the sequence b_i is bounded. In that case the expression in (3.0.20) is bounded if

$$\sum_{i=1}^{\infty} a_i^{1-2\beta-1/q} < \infty. \tag{3.0.21}$$

To ensure that it is possible to find such parameters β, q, p that (3.0.21) and (3.0.18) are satisfied, one might choose $\beta = \frac{1}{4}$ and $q = 6$, leading to $1 - 2\beta - 1/q = 1/3$. Consequently, a_i must fulfill the conditions

$$\sum_{i=1}^{\infty} a_i^{1/2} < \infty, \quad \sum_{i=1}^{\infty} a_i^{1/3} < \infty. \tag{3.0.22}$$

Observe that if $a_i = q^i$ for a certain $q \in (0,1)$, then both $a_i^{1/2} = (q^{1/2})^i$ and $a_i^{1/3} = (q^{1/3})^i$ form geometric sequence with $q^{1/2} \in (0,1)$ and $q^{1/3} \in (0,1)$, respectively. Consequently, this ensures that the conditions in (3.0.22) are satisfied.

Similarly to Lemma 3.0.2, we need to show that for each test function $\phi \in C_0^\infty(B)$, the following equation is satisfied:

$$\int_B u \cdot \nabla\phi \, dx = -\int_B \nabla u \cdot \phi \, dx. \tag{3.0.23}$$

Using the fact that u^N is bounded and that the sequence ∇u^N is bounded in L^p for every $p < 2$, we can write:

$$\begin{aligned} \int_B u(x) \cdot \nabla\phi(x) \, dx &= \int_B \lim_N u^N(x) \cdot \nabla\phi(x) \, dx = \lim_N \int_B u^N(x) \cdot \nabla\phi(x) \, dx \\ &= \lim_N \int_B \nabla u^N(x) \cdot \phi(x) \, dx = \int_B \lim_N \nabla u^N(x) \cdot \phi(x) \, dx \\ &= \int_B \nabla u(x) \cdot \phi(x) \, dx. \end{aligned} \tag{3.0.24}$$

Hence $u \in W^{1,2}$ on the whole ball B. □

Step 3: removal of singularities. Now we prove that u^N and u are weak solutions to Equation (3.0.4) on the whole ball B.

Lemma 3.0.4. *Assume u defined in Equation (3.0.5) satisfies Lemma 3.0.3. Then, for each $\phi \in C_0^\infty(B)$ the next equation holds*

$$\int_B \nabla u \cdot \nabla\phi \, dx = \int_B F(u, \nabla u)\phi \, dx, \tag{3.0.25}$$

where F is defined in (3.0.2).

Proof. Firstly we show that u^N satisfies (3.0.25) with

$$F^N := F(u^N, \nabla u^N).$$

We take advantage of the fact that $W^{1,2}$ includes unbounded functions. Recall that the function

$$\zeta_i(x) := \begin{cases} \log\log \frac{1}{|x-p_i|} & |x-p_i| < \frac{1}{e}, \\ 0 & |x-p_i| \geq \frac{1}{e} \end{cases}$$

belongs to $W^{1,2}(B)$ for $i \in \{1, \dots, N\}$. We construct the sequence $\zeta_i^k \in W^{1,2}(B)$ for $k \in \mathbb{N}$ in the following way:

$$\zeta_i^k(x) := \max\{\min\{\zeta_i(x) - k,\, 1\},\, 0\}.$$

Note that $\zeta_i^k(x) \neq 0$ on the ball $B(p_i, r_k)$ for $r_k := e^{-e^k}$ and $\zeta_i^k(p_i) = 1$. Also we note that

$$\nabla\zeta_i^k = \begin{cases} \nabla\zeta_i & \text{if } \ \zeta_i \in (k,\, k+1), \\ 0 & \text{elsewhere.} \end{cases}$$

For each $x \in B$ we have a pointwise convergence $\zeta_i^k(x) \to 0$ and

$$\begin{aligned} \|\nabla\zeta_i^k\|_{L^2(B)}^2 &= \int_B |\nabla\zeta_i^k|^2\, dx \\ &= \int_{B(p_i,r_k)\setminus B(p_i,r_{k+1})} |\nabla\zeta_i^k|^2\, dx \xrightarrow{k\to\infty} 0. \end{aligned}$$

In the same way we show that

$$\|\zeta_i^k\|_{L^2(B)}^2 = \int_B |\zeta_i^k|^2\, dx = \int_{B(p_i,r_k)} |\zeta_i^k|^2\, dx \xrightarrow{k\to\infty} 0.$$

Hence $\zeta^k \xrightarrow{k\to\infty} 0$ in $W^{1,2}(B)$. Let

$$\zeta^k := \zeta_1^k + \zeta_2^k + \ldots + \zeta_N^k.$$

It is worth noticing that for $k > 0$ large enough, all ζ_i^k, $i = 1, \ldots, N$ have disjoint supports. Now we decompose each $\phi \in C_0^\infty(B)$ as follows

$$\phi = \zeta^k \phi + \underbrace{\phi(1-\zeta^k).}_{\text{supported in } \tilde{B}^N}$$

We write $u^N = u_1^N = \sin\log\left[\sum_{i=1}^N a_i \log\frac{1}{|x-p_i|}\right]$ and $F^N = F_1^N$ because proving that the weak Equation (3.0.25) holds for the entire vector $u^N = (u_1^N, u_2^N)$ and F^N is equivalent to proving it for the first coordinate.

$$\begin{aligned}\int_B \nabla u^N \nabla\phi\, dx &= \int_B \nabla u^N \nabla(\zeta^k\phi + \phi(1-\zeta^k))\, dx \\ &= \int_B \nabla u^N \nabla(\zeta^k\phi)\, dx + \int_B \nabla u^N \nabla(\phi(1-\zeta^k))\, dx \\ &= \mathrm{I}_k + \mathrm{II}_k.\end{aligned}$$

The support of $\phi(1-\zeta^k)$ is contained in $\tilde{B}^N$. Hence

$$\mathrm{II}_k = \int_B F^N \phi(1-\zeta^k)\, dx \xrightarrow{k\to\infty} \int_B F^N \phi\, dx.$$

By computing the derivative in the first term and using Hölder inequality we get

$$\begin{aligned}\mathrm{I}_k &= \int_B \nabla u^N \phi \nabla\zeta^k\, dx + \int_B \nabla u^N \zeta^k \nabla\phi\, dx \\ &\leq \|\phi\|_\infty \|\nabla u^N\|_{L^2} \|\nabla\zeta^k\|_{L^2} + \|\nabla\phi\|_\infty \|\nabla u^N\|_{L^2} \|\zeta^k\|_2 \xrightarrow{k\to\infty} 0.\end{aligned}$$

This shows that, for each finite N, Equation (3.0.25) holds in a weak sense, i.e.

$$\int_B \nabla u^N \nabla\phi\, dx = \int_B F^N \phi\, dx. \tag{3.0.26}$$

To show that the same holds for initial $u = \lim_{N\to\infty} u^N$ we use a dominated convergence theorem. For almost every $x \in B$ we have $u^N(x) \to u(x)$ and $\nabla u^N(x) \to \nabla u(x)$ (see Lemma 3.0.2). Firstly we show that there exists a function $g \in L^1(B)$ such that for almost every $x \in B$ and every $N \in \mathbb{N}$

$$|F^N(x)| \leq g(x).$$

Indeed, using the fact that $\sin x, \cos x \leq 1$ it follows

$$|F^N(x)| = 2|\nabla u^N|^2 \frac{u_1^N + u_2^N}{1 + |u^N|^2} \leq 2|\nabla u^N|^2.$$

So, by estimating $|\nabla u^N|^2$, we can estimate F^N. We proceed in the same way as in the proof of Lemma 3.0.3. Equation (3.0.19) gives us

$$\begin{aligned} |\nabla u^N|^2 &\leq C \sum_{i=1}^{N} \frac{a_i^{1-2\beta-1/q}}{|x-p_i|^2 \log^{1+1/q} |x-p_i|^{-1}} \\ &\leq C \sum_{i=1}^{\infty} \frac{a_i^{1-2\beta-1/q}}{|x-p_i|^2 \log^{1+1/q} |x-p_i|^{-1}} := \Phi(x) \in L^1(B), \end{aligned} \tag{3.0.27}$$

where last inequality holds because every $|x-p_i|^2 \log^{1+1/q} \frac{1}{|x-p_i|} > 0$ on B.

To estimate the left hand side of (3.0.26) we firstly recall that it is enough to show that

$$\sup_N \left| \int_B \nabla u^N \nabla \phi \, dx \right| < \infty.$$

We write

$$\begin{aligned} \sup_N \left| \int_B \nabla u^N \nabla \phi \, dx \right| &\leq \sup_N \|\nabla u^N\|_{L^2(B)} \|\nabla \phi\|_{L^2(B)} \\ &\leq C(\phi) \|\Phi\|_{L^1(B)}, \end{aligned} \tag{3.0.28}$$

where $C(\phi)$ is a constant depending on ϕ. This completes the proof of Lemma 3.0.4. $\square$

Step 4: the pointwise behavior of u on K and on $B \setminus K$. To complete the whole proof, we need the following.

Lemma 3.0.5. *Let $u : B \to \mathbb{R}^2$ be defined by 3.0.5. Then, u is discontinuous at each $x \in K$ and continuous on $B \setminus K$.*

Proof. Denote

$$h^N := \log \sum_{i=1}^{N} a_i \log\Big(\frac{1}{|x-p_i|}\Big),$$

and its limit

$$h = \log \sum_{i=1}^{\infty} a_i \log\Big(\frac{1}{|x-p_i|}\Big),$$

recall also the notation

$$f^N = \sum_{i=1}^{N} a_i \log\Big(\frac{1}{|x-p_i|}\Big).$$

We are going to split the proof of the lemma onto two parts: continuity of u on $B \setminus K$ and discontinuity of u in every point of K.

Continuity of u on $B \setminus K$.

Here the situation is easier because of the fact that each point from $B \setminus K$ has an open neighborhood disjoint with K. More precisely we proceed as follows. Each of the terms $\log\big(\frac{1}{|x-p_i|}\big)$ is continuous on $B \setminus \{p_i\}$, and thus its final sum f^N is continuous on $B \setminus \{p_1, p_2, \ldots, p_N\}$. In particular each of f^N is continuous on $B \setminus K$. Let $x \in B \setminus K$ be fixed and because of the fact that $B \setminus K$ is an open set we get the existence of ball $B(x,R) \subset B \setminus K$. For arbitrary $y \in B(x,R)$ we write

$$|f^N(y) - f(y)| = \left| \sum_{i=N+1}^{\infty} a_i \log\Big(\frac{1}{|y-p_i|}\Big) \right| \le \sum_{i=N+1}^{\infty} a_i \left| \log\Big(\frac{1}{|y-p_i|}\Big) \right| = (\star) \tag{3.0.29}$$

Note that

$$|y-p_i| \ge \inf_{p_i \in P} |y-p_i| = \inf_{z \in K} |y-z| \ge \inf_{z \in K, y \in B(x,R)} |y-z| = \operatorname{dist}(K, B(x,R)) := \delta > 0.$$

Hence for each $y \in B(x,R)$ and each $p_i \in P$ we get

$$\left| \log\Big(\frac{1}{|y-p_i|}\Big) \right| \le \left| \log\Big(\frac{1}{\delta}\Big) \right| = C.$$

As a result we derive

$$(\star) \leq C\|\{a_i\}_{i=1}^{\infty}\|_{l^1} = C < \infty,$$

where constant C is independent on $y \in B(x,R)$. This shows that the sequence f^N converges uniformly to f on the ball $B(x,R)$ and thus f is continuous on the whole $B(x,R)$. Thus for each $x \in B\setminus K$ the function f is the limit of continuous functions f^N, and therefor is continuous on all of $B \setminus K$. As a conclusion

$$u(x) = (\cos h(x),\ \sin h(x))$$

is also continuous on $B \setminus K$ as the composition of continuous functions.

Discontinuity of u in every point $x \in K$. Showing the discontinuity of u via the discontinuity of h requires additional care, since trigonometric functions such as sin and cos can mask discontinuities due to their periodicity. In fact this is the main problem here. Let us fix now $x \in K$. The goal is now to show the following: there exists $\delta' > 0$ such that for every $\delta < \delta'$ the image of the ball $B(x,\delta)$ under h satisfies $h(B(x,\delta)) \supset I$, where $I \subset \mathbb{R}$ is an interval with length $|I| > 2\pi$. This ensures that u takes every value of $[-1,1] \times [-1,1]$ on the ball $B(x,\delta)$, and hence u is discontinuous at x. We fix some small $\delta > 0$ and the ball $B(x,\delta)$ such that $B(x,\delta) \subset B$. Here is the argument of the proof:

We will construct two measurable sets $E, U \subset B(x,\delta)$ such that $h|_E$ takes only small values, while $h|_U$ takes sufficiently large values. Then, using the ACL (absolutely continuous on lines) characterization of Sobolev functions, we will conclude that there exists a line segment l connecting a point in E to a point in U, along which h is continuous. This will imply that h varies continuously along l, transitioning from small values on E to large values on U. We now proceed with a careful and precise argument. Recall that from the proof in Lemma 3.0.3 we have that $h \in W^{1,2}(B)$. To construct the first measurable set we proceed as follows: since $h \in L^1(B(x,\delta))$, there exists a measurable set $E \subset B(x,\delta)$ with $|E| > 0$, and a constant $M > 0$, such that $h|_E \leq M$ almost everywhere. To construct the second measurable set we rely on density of P in K. Since P is dense in K there exists $p_j \in P \cap B(x, \delta)$. Using the fact that each term $a_i \log \frac{1}{|x-p_i|}$ is strictly positive (see argument in Lemma 3.0.3), we obtain the estimate:

$$h(x) = \log \sum_{i=1}^{\infty} a_i \log\left(\frac{1}{|x-p_i|}\right) > \log\left(a_j \log \frac{1}{|x-p_j|}\right).$$

Now fix $M' = M + 2\pi$. Since the right-hand side becomes arbitrary large as $x \to p_j$, we can choose a sufficiently small radius $\tilde{\delta} > 0$ such that $h|_{B(p_j,\tilde{\delta})} > M'$ almost everywhere. We then define the second measurable set as $U := B(p_j, \tilde{\delta})$.

From the ACL characterization (see first Ziemer [7, Theorem 2.1.4 and Remark 2.1.5]) we know that $h \in W^{1,2}(l)$ for almost all line segments $l \in B(x, \delta)$ parallel to the coordinate axis.

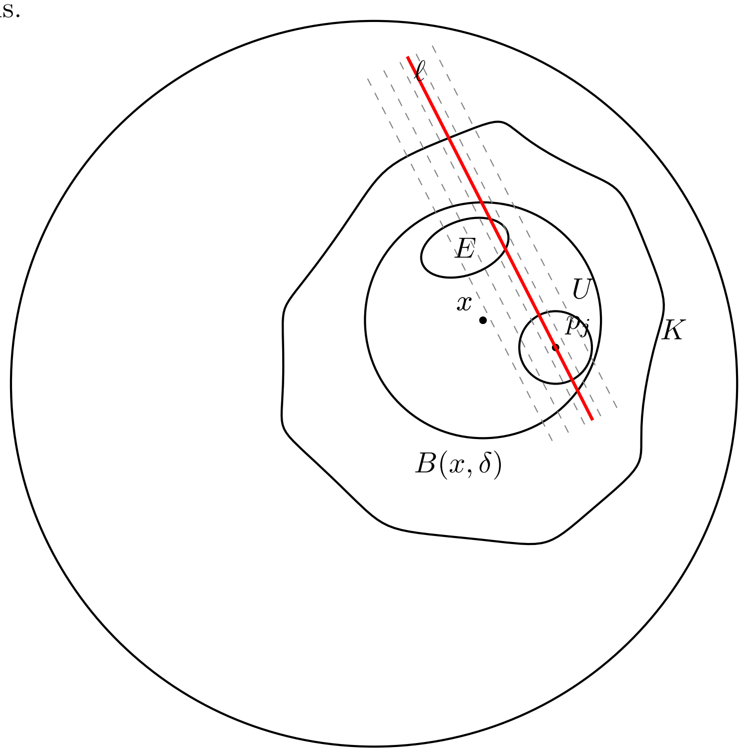

Figure 1: Illustration related to Lemma 3.0.5. The sets $E, U \subset B(x, \delta)$ represent regions where the function h attains relatively small and large values, respectively. The point $p_j \in P$ lies at the center of the ball $U = B(p_j, \tilde{\delta})$. The bold line ℓ is one along which h is continuous, while the dashed parallel lines indicate a slicing family of B aligned with ℓ.

This result remains valid under arbitrary rotations of the family of lines, since the Sobolev $W^{1,p}$ space is invariant under rotations. Let us now construct such the rotation α such that the resulting family of lines $L = \{l_j\}_j \in J$ indexed by a set J of positive measure, satisfies the following: each line l_j starts in E, ends in U, and $h|_{l_j}$ is continuous for every $j \in J$. The construction is easy: We cover the whole ball $\overline{B}$ by the finite number of balls B_i with radius $\tilde{\delta}$ (it is possible due to the compactness of $\overline{B}$), denote $F = \{B_i\}_{i=1}^m$. There exists a ball $B_k \in F$ such that $B_k \cap E \neq \emptyset$ and $|B_k \cap E| > 0$; otherwise, this would contradict the fact that E has a positive measure. Now it is enough to take the line going through centres of sets U and B_k. This line defines the angle α for which the above construction holds.

Now, take an arbitrary such line $l \in L$. By the ACL characterization, we know that $h \in W^{1,2}(l \cap B)$. Since l is one dimensional, the Sobolev embedding theorem gives the embedding $W^{1,2}(l) \hookrightarrow C^{1/2}$ (see Ziemer [7, Theorem 2.4.4]). Thus, h is continuous along the line l, and hence $h|_{l \cap B}$ varies continuously from M to $M' > M + 2\pi$. As a result, for each small $\delta > 0$ we have $u(B(x, \delta)) = [-1, 1] \times [-1, 1]$. This shows that u is discontinuous at each point $x \in K$. □

□

Chapter 4

Regularity of H-systems

Theorem 4.0.1. *Let $n > 2$ be fixed. Let $u \in W^{n/2,\,2}(\mathbb{B}^n(0,1),\ \mathbb{R}^{n+1})$ be a solution of H-system equation (2.5.36) for Lipschitz and bounded $H : \mathbb{R}^{n+1} \to \mathbb{R}$. More precisely for arbitrary test function $\phi \in D(\mathbb{B}^n(0,1), \mathbb{R}^{n+1})$ the following equation holds*

$$\int_{\mathbb{B}^n(0,1)} |\nabla u|^{n-2} \nabla u \nabla \phi \, dx = \int_{\mathbb{B}^n(0,1)} H(u) J(u) \phi \, dx. \tag{4.0.1}$$

Then u is locally Hölder continuous.

The space $W^{n/2,2}$ is a subspace of $W^{1,n}$ (see Note 2.6.9) and, importantly, is not a subspace of L^∞, so the problem does not trivialise like in Note 2.5.11 (i.e. one cannot test Equation (4.0.1) by its solution).

To show the Hölder continuity of a solution u, we will use the Dirichlet Growth Theorem described in Theorem 2.4.1. Therefore, our analysis reduces to studying the local behaviour of the gradient of the solution. Specifically, we aim to establish the inequality of the form

$$\int_{B(a,r)} |\nabla u|^p \, dx \lesssim r^\alpha, \quad p < n,\ \alpha > n - p.$$

We work under the critical integrability exponent $p = n$: we fix $\varepsilon > 0$ and we show that the $\|\nabla u\|_{L^{n-\varepsilon,\varepsilon}(B(a,r))}$ decays like r^s for some $s > 0$.

Note 4.0.2. *To avoid any miscommunication that may arise from a choice of parameter*

ε, we would like to make the following note: the precise value of ε is specified in inequality (4.0.54) at the end of the proof, and this value ***does not*** *depend on the solution u.*

For that, inspired by Iwaniec [12], assuming that u has been already cut on the ball $B(a,r)$, we define the vector field $G = |\nabla u|^{\varepsilon}\nabla u$. Due to the stability theorem 2.6.4 one show that $G = \nabla\phi + V$, where V is small in L^q-norm for $q > n$. We choose ϕ as the test function to Equation (4.0.1). Applying the Morrey embedding theorem one shows that the left hand side satisfies

$$\int_{\mathbb{B}^n(0,1)} |\nabla u|^{n-2}\nabla u\nabla\phi\,dx \geq \int_{B(a,r)} |\nabla u|^{n-\varepsilon}\,dx + \text{ small other terms.}$$

Then the right hand side satisfies

$$\left|\int H(u)J(u)\phi\,dx\right| \leq \text{small multiple of } r^{\varepsilon}\|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}.$$

To demonstrate this, we decompose the right-hand side by Hodge decomposition into two parts: the Jacobian part (the 'good' part), which has one coordinate in L^q for $q > n$, and the 'bad' part, which does not have a Jacobian structure. The Jacobian structure of the 'good' part allows us to use $\mathcal{H}^1-$ BMO duality theorem to obtain more refined estimates. We estimate the 'bad' part using the commutator theorem. The assumption $u \in W^{n/2,2}$ is needed to estimate the critical norm $\|\nabla u\|^n_{L^n}$, which appears on the right-hand side.

Proof. Fix some point $a \in \mathbb{B}^n(0,1)$ and small radius $0 < r < \frac{1}{8}\text{dist}(a, \partial\mathbb{B}^n(0,1))$. Let $\zeta \in C_0^{\infty}(B(a,3r))$ be a cut-off function, i.e. $\zeta \equiv 1$ on $B(a,2r)$, $\zeta \equiv 0$ on $\mathbb{R}^n \setminus B(a,3r)$ and $|\nabla\zeta| \leq \frac{C}{r}$ for some constant $C > 0$. Define now a cut off version of u as follows

$$\tilde{u} = \zeta(u - [u]_{B(a,3r)}), \tag{4.0.2}$$

where $[u]_{B(a,3r)} := \fint_{B(a,3r)} u\,dx$. Before delving into the proof, it is necessary to establish the following useful lemma.

Lemma 4.0.3. *Under the notation as above, we have*

$$\|\nabla\tilde{u}\|_{L^p(\mathbb{R}^n)} \leq C(p,n)\|\nabla u\|_{L^p(B(a,3r))} \tag{4.0.3}$$

for some constant $C(p,n)$.

Proof.

$$\begin{aligned}
\|\nabla\tilde{u}\|^p_{L^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\nabla\tilde{u}|^p\,dx = \int_{B(a,3r)} |\nabla(\zeta(u-[u]_A))|^p\,dx \\
&= \int_{B(a,3r)} |(u-[u]_A)\nabla\zeta + \zeta\nabla u|^p\,dx \\
&\leq \left(\frac{C}{r}\right)^p \int_{B(a,3r)} |u-[u]_A|^p + \int_{B(a,3r)} |\nabla u|^p\,dx = (*).
\end{aligned} \tag{4.0.4}$$

Applying the Poincaré inequality to the first term we get the following inequality

$$\begin{aligned}
(*) &\leq \left(\frac{C}{r}\right)^p r^p \int_{B(a,3r)} |\nabla u|^p\,dx + \int_{B(a,3r)} |\nabla u|^p\,dx \\
&\leq C(p,n)\int_{B(a,3r)} |\nabla u|^p\,dx.
\end{aligned} \tag{4.0.5}$$

□

In order to find a proper test function ϕ for Equation (4.0.1) we define a vector field $G = |\nabla\tilde{u}|^{-\varepsilon}\nabla\tilde{u}$ for some fixed $\varepsilon > 0$ (the precise value of parameter ε will be given at the end of the proof). Applying Hodge decomposition to the vector field G and using Theorem 2.6.4 we get

$$G = \nabla\phi + V, \tag{4.0.6}$$

for some $\phi \in W^{1,q}(\mathbb{R}^n)$ and divergence free $V \in L^q(\mathbb{R}^n)$, where $1 \leq q \leq \frac{n}{1-\varepsilon}$. We use the cut-off version of ϕ on the $B(a,2r)$ to test Equation (4.0.1) (see the argument below).

Properties of the test function ϕ

To justify our choice, we first observe that $q > n$, implying that ϕ is Hölder continuous on $\mathbb{R}^n$ with the exponent $1 - \frac{n}{q}$ (see Theorem 2.2.3). Secondly, by employing inequality

(2.6.13) and acknowledging the continuity of the Hodge decomposition, we deduce that

$$\|\nabla\phi\|_{L^q(\mathbb{R}^n)} + \|V\|_{L^q} \leq C(n,q)\|G\|_{L^q(\mathbb{R}^n)} \leq C(n)\|G\|_{L^q(\mathbb{R}^n)}$$

for some general constant $C(n)$ that depends only on n. Given that the balls are for now centered at the point a, we will simply denote $B(r) := B(a,r)$ for convenience. With that notation, using Lemma 4.0.3, we get

$$\begin{aligned}\|G\|^q_{L^q(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (|\nabla\tilde{u}|^{-\varepsilon}|\nabla\tilde{u}|)^q\,dx = \int_{B(3r)} |\nabla\tilde{u}|^{(1-\varepsilon)q}\,dx \\ &\leq C(\varepsilon,n)\int_{B(3r)} |\nabla u|^{(1-\varepsilon)q}\,dx \leq C(n)\int_{B(3r)} |\nabla u|^{(1-\varepsilon)q}\,dx,\end{aligned} \tag{4.0.7}$$

where the last inequality follows from following computations: by computing the precise value of $C(\varepsilon,n)$ one gets $C(\varepsilon,n) = 1 + n\cdot 2^{n/(1-\varepsilon)q}$. The function $f(\varepsilon,n) = 1 + n\cdot 2^{n/(1-\varepsilon)q}$ is continuous on the compact set $[0,\frac{1}{4}]\times[n-\varepsilon,\frac{n}{1-\varepsilon}]$, thus it attains the supremum $C(n)$.

Thus the function ϕ satisfies

$$\|\nabla\phi\|_{L^q(\mathbb{R}^n)} \leq C(n)\Big(\int_{B(3r)} |\nabla u|^{(1-\varepsilon)q}\,dx\Big)^{1/q}. \tag{4.0.8}$$

Hölder continuity of ϕ on $\mathbb{R}^n$ leads to the boundedness of ϕ within the ball $B(3r)$. Before we proceed to estimate the supremum norm of the function ϕ on the ball $B(3r)$ we make a technical note.

Note 4.0.4. *Because the function ϕ is defined as the gradient part of the vector G in (4.0.6) we observe that ϕ is defined up to a constant. We use that property by considering ϕ of zero mean value on the ball $B(3r)$, i.e. $[\phi]_{B(3r)} = 0$. Indeed if this is not the case, we define then*

$$\tilde{\phi}(x) = \phi(x) - [\phi]_{B(3r)}.$$

Thus from here on we assume that ϕ has a zero mean value on the ball $B(3r)$.

Lemma 4.0.5.

$$\|\phi\|_{L^\infty(B(3r))} \le C(n,\varepsilon) r^\varepsilon \|\nabla u\|^{1-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(3r))}, \tag{4.0.9}$$

where $C(n,\varepsilon)$ *could potentially approach infinity as* ε *tends toward zero, i.e.* $\lim_{\varepsilon\to 0} C(n,\varepsilon) = \infty$ *for every unbounded solution* u.

Proof. □

Using the Morrey embedding Theorem 2.2.3 we get

$$\|\phi\|_{C^{0,\varepsilon}(B(3r))} \le C(n,\varepsilon)\|\nabla\phi\|_{L^q(B(3r))}.$$

The zero mean value of ϕ together with continuity of ϕ implies that the function ϕ has a zero value, for instance, $\phi(y) = 0$ for some $y \in B(3r)$.

$$\begin{aligned} \|\phi\|_{L^\infty(B(3r))} &= \sup_{x\in B(3r)} |\phi(x)| \le \sup_{x\in B(3r)} |\phi(x)-\phi(y)| + |\phi(y)| \\ &\le \sup_{x\in B(3r)} \|\phi\|_{C^{0,\varepsilon}(B(3r))}|x-y|^{1-\frac{n}{q}} \\ &\le C(n,\varepsilon)(3r)^{1-\frac{n}{q}}\|\nabla\phi\|_{L^q(B(3r))} \\ &\le C(n,\varepsilon)(3r)^{1-\frac{n}{q}}\Big(\int_{B(3r)} |\nabla u|^{(1-\varepsilon)q}\,dx\Big)^{1/q}. \end{aligned} \tag{4.0.10}$$

If one sets $q = \frac{n}{1-\varepsilon}$, the above inequality will contain the L^n-norm of the gradient of u, rendering it ineffective. One of the reason is that the critical exponent does not allow to efficiently manipulate Hölder inequalities. To avoid that problem we put $q = \frac{n-\varepsilon}{1-\varepsilon} > n$, getting

$$
\begin{aligned}
\|\phi\|_{L^\infty(B(3r))} &\leq C(n,\varepsilon)(3r)^{1-\frac{n(1-\varepsilon)}{n-\varepsilon}}\Big(\int_{B(3r)}|\nabla u|^{n-\varepsilon}\,dx\Big)^{(1-\varepsilon)/(n-\varepsilon)}\\
&= C(n,\varepsilon)(r^\varepsilon)^{(n-1)/(n-\varepsilon)}\Big(\int_{B(3r)}|\nabla u|^{n-\varepsilon}\,dx\Big)^{(1-\varepsilon)/(n-\varepsilon)}\\
&= C(n,\varepsilon)\cdot r^\varepsilon\cdot(r^\varepsilon)^{-(1-\varepsilon)/(n-\varepsilon)}\cdot\Big(\int_{B(3r)}|\nabla u|^{n-\varepsilon}\,dx\Big)^{(1-\varepsilon)/(n-\varepsilon)} \qquad (4.0.11)\\
&= C(n,\varepsilon)r^\varepsilon\Big(\frac{1}{r^\varepsilon}\int_{B(3r)}|\nabla u|^{n-\varepsilon}\,dx\Big)^{(1-\varepsilon)/(n-\varepsilon)}\\
&\leq C(n,\varepsilon)r^\varepsilon\|\nabla u\|^{1-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(3r))}.
\end{aligned}
$$

The above constant $C(n,\varepsilon)$ also depends on ε, as setting $\varepsilon = 0$ yields $\phi = u \in W^{1,n}$, where ϕ may become unbounded. Moreover in such scenario, the constant $C(n,\varepsilon)$ could potentially approach infinity as ε tends toward zero, as $W^{1,n} \not\subset L^\infty$.

Estimation of the left hand side

We prove the following lemma.

Lemma 4.0.6. *Let ϕ be a function described in Equation (4.0.6) and ζ_1 be a cut-off function on the ball $B(a,2r)$, i.e. $\zeta_1 \in C^\infty(B(a,2r))$, $\zeta_1 \equiv 1$ on the ball $B(a,r)$, $\zeta_1 = 0$ on $\mathbb{R}^n \setminus B(a,2r)$, and $|\nabla\zeta_1| \leq \frac{C}{r}$ for some constant C. Then*

$$
\int_{\mathbb{B}^n(0,1)}|\nabla u|^{n-2}\nabla u\nabla(\zeta_1\phi)\,dx \geq \int_{B(a,r)}|\nabla u|^{n-\varepsilon}\,dx - \frac{1}{5}\int_{B(a,3r)}|\nabla u|^{n-\varepsilon}\,dx - C(n)\int_A |\nabla u|^{n-\varepsilon}\,dx, \tag{4.0.12}
$$

where $A := B(a,2r)\setminus B(a,r)$.

Proof. The proof is straightforward - it is enough to compute the gradient of $\zeta_1\phi$ and to estimate each of the terms using Hölder and Poincaré inequalities. The reason we introduce the new cut-off function ζ_1 has a technical nature - all computations of the left hand side will be localized to the ball $B(a,2r)$, which is crucial for getting the final decay inequality (see Equation (4.0.55)). More precisely, in the calculations below we use the identity $\nabla\phi = G - V$. While the vector G is supported in $B(a,3r)$ the same is not true for V, whose support generally extends beyond $B(a, 3r)$. By introducing the new cut-off function ζ_1, it is now possible to substitute ∇u on $\nabla \tilde{u}$ on the support of ζ_1. Here is the way we handle computations:

$$\begin{aligned}\int_{\mathbb{B}^n(0,1)} |\nabla u|^{n-2}\nabla u \nabla(\zeta_1\phi)\, dx &= \int_{B(2r)} |\nabla u|^{n-2}\phi\nabla u\cdot\nabla\zeta_1\, dx + \int_{B(2r)} |\nabla u|^{n-2}\zeta_1\nabla u \underbrace{\nabla\phi}_{:=G-V}\, dx \\ &= \int_{B(2r)} |\nabla u|^{n-2}\phi\nabla u\cdot\nabla\zeta_1\, dx + \int_{B(2r)} |\nabla u|^{n-2}\zeta_1\nabla u\cdot G\, dx \\ &\quad - \int_{B(2r)} |\nabla u|^{n-2}\zeta_1\nabla u\cdot V\, dx := \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned} \tag{4.0.13}$$

We are going to estimate each of the above terms separately. For the term II we just write

$$\begin{aligned}\mathrm{II} &= \int_{B(2r)} |\nabla u|^{n-2}\zeta_1\nabla u\cdot|\nabla u|^{-\varepsilon}\nabla u\, dx \\ &= \int_{B(2r)} |\nabla u|^{n-\varepsilon}\zeta_1\, dx \geq \int_{B(r)} |\nabla u|^{n-\varepsilon}\zeta_1\, dx \\ &\geq \int_{B(r)} |\nabla u|^{n-\varepsilon}\, dx.\end{aligned} \tag{4.0.14}$$

We are going to estimate the term III by applying the T.Iwaniec stability theorem (see Theorem 2.6.4). Indeed by inequality (2.6.14) we get

$$\|V\|_{L^{q/(1+\varepsilon)}(B(2r))} \leq C|\varepsilon|\|\nabla u\|^{1+\varepsilon}_{L^q(B(2r))}, \tag{4.0.15}$$

consequently recalling the fact that $\zeta_1 \leq 1$ and using Hölder inequality with parameters

$\frac{n-\varepsilon}{n-1}$, $\frac{n-\varepsilon}{1-\varepsilon}$ we get

$$
\begin{aligned}
\mathrm{III} &\leq \int_{B(2r)} |\nabla u|^{n-2}|\zeta_1||\nabla u||V|\,dx \leq \int_{B(2r)} |\nabla u|^{n-1}|V|\,dx \\
&\leq \Big(\int_{B(2r)} |\nabla u|^{n-\varepsilon}\,dx\Big)^{\frac{n-1}{n-\varepsilon}} \Big(\int_{B(2r)} |V|^{\frac{n-\varepsilon}{1-\varepsilon}}\,dx\Big)^{\frac{1-\varepsilon}{n-\varepsilon}} \\
&\leq \Big(\int_{B(2r)} |\nabla u|^{n-\varepsilon}\,dx\Big)^{\frac{n-1}{n-\varepsilon}} C(n)|\varepsilon| \Big(\int_{B(2r)} |\nabla u|^{n-\varepsilon}\,dx\Big)^{\frac{1-\varepsilon}{n-\varepsilon}} \\
&= C(n)|\varepsilon| \int_{B(2r)} |\nabla u|^{n-\varepsilon}\,dx \leq \frac{1}{10}\int_{B(2r)} |\nabla u|^{n-\varepsilon}\,dx
\end{aligned}
\tag{4.0.16}
$$

for $\varepsilon > 0$ sufficiently small (depending only on n). For the last term we observe that the support of $\nabla\zeta_1$ is $A := B(2r) \setminus B(r)$ and there it holds $|\nabla\zeta_1| \lesssim \frac{1}{r}$. Applying additionally the Poicaré and Hölder inequalities to the term I and Note (4.0.4) we get the following inequalities:

$$
\begin{aligned}
\mathrm{I} &\leq \frac{C}{r}\int_A |\nabla u|^{n-1}|\phi|\,dx \leq \frac{C}{r}\Big(\int_A |\nabla u|^{n-\varepsilon}\,dx\Big)^{\frac{n-1}{n-\varepsilon}} \Big(\int_A |\phi|^{\frac{n-\varepsilon}{1-\varepsilon}}\,dx\Big)^{\frac{1-\varepsilon}{n-\varepsilon}} \\
&\leq \frac{C}{r}\Big(\int_A |\nabla u|^{n-\varepsilon}\,dx\Big)^{\frac{n-1}{n-\varepsilon}} \Big(\int_{B(3r)} |\phi - [\phi]_{B(3r)}|^{\frac{n-\varepsilon}{1-\varepsilon}}\,dx\Big)^{\frac{1-\varepsilon}{n-\varepsilon}} \\
&\leq C\Big(\int_A |\nabla u|^{n-\varepsilon}\,dx\Big)^{\frac{n-1}{n-\varepsilon}} \Big(\int_{B(3r)} |\nabla\phi|^{\frac{n-\varepsilon}{1-\varepsilon}}\,dx\Big)^{\frac{1-\varepsilon}{n-\varepsilon}} \\
&\overset{\text{Inequality (4.0.8)}}{\leq} C\Big(\int_A |\nabla u|^{n-\varepsilon}\,dx\Big)^{\frac{n-1}{n-\varepsilon}} \Big(\int_{B(3r)} |\nabla u|^{n-\varepsilon}\Big)^{\frac{1-\varepsilon}{n-\varepsilon}} \\
&\leq C(n)\int_A |\nabla u|^{n-\varepsilon}\,dx + \frac{1}{10}\int_{B(3r)} |\nabla u|^{n-\varepsilon}\,dx,
\end{aligned}
\tag{4.0.17}
$$

where last inequality is the Young inequality (see Evans [31], Appendix B.2., inequality d.). Combining aforementioned estimates we finally get

$$
\begin{aligned}
&\int_{\mathbb{B}^n(0,1)} |\nabla u|^{n-2}\nabla u\nabla(\zeta_1\phi)\,dx \geq \mathrm{II} - |\mathrm{III}| - |\mathrm{I}| \\
&\geq \int_{B(r)} |\nabla u|^{n-\varepsilon}\,dx - \frac{1}{10}\int_{B(2r)} |\nabla u|^{n-\varepsilon}\,dx - C(n)\int_A |\nabla u|^{n-\varepsilon}\,dx \\
&- \frac{1}{10}\int_{B(3r)} |\nabla u|^{n-\varepsilon}\,dx \\
&\geq \int_{B(r)} |\nabla u|^{n-\varepsilon}\,dx - \frac{1}{5}\int_{B(3r)} |\nabla u|^{n-\varepsilon}\,dx - C(n)\int_A |\nabla u|^{n-\varepsilon}\,dx.
\end{aligned}
\tag{4.0.18}
$$

□

Estimation of the right hand side

Unlike the approach taken to estimate the left-hand side, where the entire system was considered, in this instance, we will focus on a specific coordinate of the right-hand side vector. This is because the proof for other coordinates follows similarly. The right hand side (or shortly R) of Equation (4.0.1) is the vector field in $\mathbb{R}^{n+1}$ such that for each $i = 1, \ldots, n+1$ it holds (cf. Note 2.6.1)

$$(Ju)^i dx_1 \wedge \ldots \wedge dx_n = du_1 \wedge \ldots \wedge du_{i-1} \wedge du_{i+1} \wedge \ldots \wedge du_{n+1}.$$

We fix now $i = n+1$ and write

$$\int_{\mathbb{B}^n(0,1)} H(u)(Ju)_{n+1} \cdot \zeta_1 \cdot \phi_{n+1}\, dx = \int_{\mathbb{B}^n(0,1)} H(u)\zeta_1 \cdot \phi_{n+1}\, du_1 \wedge \ldots \wedge du_n. \tag{4.0.19}$$

Denote $b := H(u)\zeta_1$ and recall that $d\tilde{u} = du$ on the ball $B(a, 2r)$, we simply get

$$\begin{aligned} R &= \int_{B(a,2r)} \phi_{n+1} b\, d\tilde{u}_1 \wedge \ldots \wedge d\tilde{u}_n = \int_{B(a,2r)} \phi_{n+1} \left(b \cdot d(\tilde{u}_1)\right) \wedge \ldots \wedge d\tilde{u}_n \\ &= \int_{B(a,2r)} \phi_{n+1}\, d\alpha \wedge \ldots \wedge d\tilde{u}_n + \int_{B(a,2r)} \phi_{n+1}\, \delta\beta \wedge \ldots \wedge d\tilde{u}_n \\ &:= R_1 + R_2, \end{aligned} \tag{4.0.20}$$

where $b \cdot d\tilde{u}_1 = d\alpha + \delta\beta$ is the Hodge decomposition. We are going to prove the following lemma.

Lemma 4.0.7.

$$|R_1| \leq C \|\nabla u\|_{L^n(B(a,3r))} \int_{B(a,3r)} |\nabla u|^{n-\varepsilon}\, dx \tag{4.0.21}$$

$$|R_2| \leq C(n,\varepsilon) r^{\varepsilon} \|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))} \tilde{\Psi}(a, 3r), \tag{4.0.22}$$

where $\tilde{\Psi} : \mathbb{B}^n(0,1) \times (0,1) \to \mathbb{R}$ is an absolutely continuous function (see inequality (4.0.51)).

Let us highlight the primary challenge in estimating R_1 and R_2. The term R_1 does not deviate significantly from the general right-hand side R, as it also encompasses the Jacobian structure, which facilitates the application of the Fefferman–Stein duality theorem—effectively addressing the issue. Conversely, the term R_2, unlike R_1, lacks the Jacobian structure, thereby precluding a straightforward application of the $\mathcal{H}^1$-BMO duality theorem. Fortunately, in our scenario, it is possible to represent co-differential part $\delta\beta$ as a commutator operator, represented by an element from the BMO space. This allows for the application of the commutator theorem to effectively estimate the L^n norm of it. We write the precise proof below.

Proof. We start with an examination of the simpler term, R_1. To aid the reader's comprehension, we include a brief technical note on our approach to analyzing this term. One might initially consider directly applying the Fefferman-Stein (F-S) duality theorem to the unmodified form of R_1, resulting in the inequality

$$R_1 \leq \|\phi\|_{\mathrm{BMO}} \|d\alpha \wedge \ldots \wedge d\tilde{u}_n\|_{H^1}. \tag{4.0.23}$$

However, this strategy is fundamentally flawed as it does not provide a sufficiently good method to further estimate the $\mathcal{H}^1$ norm of the Jacobian introduced. The core issue arises because $d\alpha$ along with all coordinates of du reside in L^n, which restricts the use of the Hölder inequality with parameters $p_1, p_2, \ldots, p_n = \frac{1}{n}$. To solve that issue, we perform integration by parts on R_1 replacing ϕ on $\tilde{u}_n$ (no boundary values appear) and then we use the fact that $\nabla\phi \in L^q$ for $q > n$ which permits operations beyond the critical exponent.

Here is the detailed method by which we proceed:

$$
\begin{aligned}
R_1 &= -\int_{\mathbb{B}^n(0,1)} \tilde{u}^n \, d\alpha \wedge d\tilde{u}^2 \wedge \ldots \wedge d\tilde{u}^{n-1} \wedge d\phi^{n+1} \\
&\leq \|\tilde{u}^n\|_{\mathrm{BMO}(\mathbb{R}^n)} \|d\alpha \wedge d\tilde{u}^2 \wedge \ldots \wedge d\tilde{u}^{n-1} \wedge d\phi^{n+1}\|_{H^1(\mathbb{R}^n)} \\
&\leq \|\tilde{u}^n\|_{\mathrm{BMO}(\mathbb{R}^n)} \|\nabla\phi\|_{L^{\frac{n-\varepsilon}{1-\varepsilon}}(\mathbb{R}^n)} \|\nabla\tilde{u}\|^{n-2}_{L^{n-\varepsilon}(\mathbb{R}^n)} \|d\alpha\|_{L^{n-\varepsilon}(\mathbb{R}^n)} \\
&\leq C(n)\|\tilde{u}^n\|_{BMO(\mathbb{R}^n)} \Big(\int_{B(3r)} |\nabla u|^{n-\varepsilon}\, dx\Big)^{\frac{1-\varepsilon}{n-\varepsilon}} \|\nabla\tilde{u}\|^{n-2}_{L^{n-\varepsilon}(\mathbb{R}^n)} \|d\alpha\|_{L^{n-\varepsilon}(\mathbb{R}^n)}
\end{aligned}
\tag{4.0.24}
$$

where the last inequality follows from Equation (4.0.8). We now note that

$$
\begin{aligned}
\|d\alpha\|_{L^{n-\varepsilon}(\mathbb{R}^n)} \leq \|b \cdot d\tilde{u}_1\|_{L^{n-\varepsilon}(B(3r))} &\leq \|b\|_{L^\infty(B(3r))} \|\nabla\tilde{u}\|_{L^{n-\varepsilon}(B(3r))} \\
&\leq C(n,H)\|\nabla\tilde{u}\|_{L^{n-\varepsilon}(B(3r))} \\
&\leq C(n,H)\|\nabla u\|_{L^{n-\varepsilon}(B(3r))}.
\end{aligned}
\tag{4.0.25}
$$

The last inequality follows from the Lemma 4.0.3. Hence, by utilizing the Sobolev embedding theorem BMO $\hookrightarrow W^{1,n}$, we arrive at the final estimation of R_1:

$$
\begin{aligned}
R_1 &\leq C(n,H)\|\tilde{u}\|_{\mathrm{BMO}(\mathbb{R}^n)} \int_{B(3r)} |\nabla u|^{n-\varepsilon}\, dx \\
&\leq C\|\nabla\tilde{u}\|_{L^n(\mathbb{R}^n)} \int_{B(3r)} |\nabla u|^{n-\varepsilon}\, dx \\
&\leq C\|\nabla u\|_{L^n(B(3r))} \int_{B(3r)} |\nabla u|^{n-\varepsilon}\, dx.
\end{aligned}
\tag{4.0.26}
$$

Let us now take care of R_2 which is a more complicated term. Estimating this term cannot be approached in the same manner as R_1 due to the absence of the Jacobian structure. Consequently, integrating by parts or applying the Hölder inequality to achieve similar outcomes is not feasible. To express the co-differential part $\delta\beta$ as a commutator operator, we introduce the projection onto the gradient part operator. Specifically, for every vector field X in L^p we define

$$T(X) = T(d\alpha + \delta\beta) = d\alpha.$$

The operator T is a Calderon-Zygmund operator — it can be shown that $\alpha(y) = \nabla\Gamma * X(y)$, where $\Gamma = \frac{C(n)}{|x|^{n-2}}$ is the fundamental solution of Laplace's equation. Hence (cf. Definition 2.6.5)

$$d\alpha = \nabla^2\Gamma * X(y) = \Big(\frac{C(n)}{|x|^n}\Big) * X(y).$$

With this notation we express the co-differential part $\delta\beta$ as follows

$$\begin{aligned} \delta\beta = d\alpha - bd\tilde{u}_1 &= T(bd\tilde{u}_1) - bT(d\tilde{u}_1) \\ &= [b,T](d\tilde{u}_1). \end{aligned} \tag{4.0.27}$$

We estimate the L^n norm of $\delta\beta$ using Coifman-Rochberg-Weiss commutator theorem 2.6.6 getting

$$\begin{aligned} \|\delta\beta\|_{L^n(\mathbb{R}^n)} = \|[b,T](d\tilde{u}_1)\|_{L^n(\mathbb{R}^n)} &\le \|[b,T]\|\|d\tilde{u}_1\|_{L^n(\mathbb{R}^n)} \\ &\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|d\tilde{u}_1\|_{L^n(\mathbb{R}^n)}. \end{aligned} \tag{4.0.28}$$

Now we can write

$$\begin{aligned} R_2 &\le C\|\phi\|_{L^\infty}\|\|\nabla\tilde{u}\|_{L^n}^{n-1}\|\delta\beta\|_{L^n} \\ &\le C\|\phi\|_{L^\infty}\|\nabla\tilde{u}\|_{L^n}^{n}\|b\|_{\mathrm{BMO}}. \end{aligned} \tag{4.0.29}$$

Note that when using C-R-W commutator theorem instead of trivially estimating $\delta\beta$ by the L^n norm of the $\nabla\tilde{u}$, there appears an extra term $\|b\|_{\mathrm{BMO}}$. The supremum norm of ϕ will be estimated by Lemma 4.0.5, so the last thing we need to show is that $\|b\|_{\mathrm{BMO}}$ is small enough. To do so we write

$$\begin{aligned} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} &= \|\zeta_1 H(u)\|_{\mathrm{BMO}(B(a,2r)} \\ &\le \|\zeta_1\|_{\mathrm{BMO}(B(a,2r))}\|H(u)\|_{L^\infty(B(a,2r))} + \|\zeta_1\|_{L^\infty(B(a,2r)))}\|H(u)\|_{\mathrm{BMO}(B(a,2r)} \\ &= (*). \end{aligned} \tag{4.0.30}$$

Functions H and ζ_1 are bounded also the BMO norm of ζ_1 is bounded by a constant:

$$\|\zeta_1\|_{\mathrm{BMO}(B(a,2r))} \leq \|\nabla\zeta_1\|_{L^n(B(a,2r))} \leq \frac{C}{r^n}|B(a,2r)|^{\frac{1}{n}} \leq C. \tag{4.0.31}$$

The BMO norm of $H(u)$ will be estimated as follows: by leveraging the assumption that H is Lipschitz continuous, we obtain

$$\begin{aligned}
\|H(u)\|_{\mathrm{BMO}(B(a,2r))} &:= \sup_{B\subset B(a,2r)} \fint_B \left| H(u(x)) - \fint_B H(u(y))\,dy \right| dx \\
&= \sup_{B\subset B(a,2r)} \fint_B \left| \fint_B H(u(x)) - H(u(y))\,dy \right| dx \\
&\leq \sup_{B\subset B(a,2r)} \fint_B \fint_B \Big| H(u(x)) - H(u(y)) \Big|\,dy\,dx \\
&\leq \mathrm{Lip}(H) \sup_{B\subset B(a,2r)} \fint_B \fint_B \Big| u(x) - u(y) \Big|\,dy\,dx \\
&= C \sup_{B\subset B(a,2r)} \fint_B \fint_B \Big| u(x) - [u]_B + [u]_B - u(y) \Big|\,dy\,dx \\
&\leq C \sup_{B\subset B(a,2r)} \left(\fint_B \fint_B \Big| u(x) - [u]_B \Big|\,dy\,dx + \fint_B \fint_B \Big| u(y) - [u]_B \Big|\,dy\,dx \right) \\
&\leq C\|u\|_{\mathrm{BMO}(B(a,2r))}.
\end{aligned}$$

The last thing needed to be shown is that the BMO norm of u on ball $B(a,2r)$ is estimated by the Morrey $L^{n-\varepsilon,\varepsilon}$ norm of the gradient of u. Indeed, applying Hölder inequality and then the Gagliardo-Nirenberg-Sobolev inequality (Theorem 2.2.3) with parameters $p^* = n$

and $p = n$ we get the following inequality

$$\begin{aligned}
\|u\|_{\mathrm{BMO}(B(a,2r))} &= \sup_{B\subset B(a,2r)} ⨍_B \Big|u(x) - [u]_B\Big|\,dx \\
&\le \sup_{B\subset B(a,2r)} \Big(⨍_B \Big|u(x) - [u]_B\Big|^n\,dx\Big)^{1/n} \\
&\le C \sup_{B\subset B(a,2r)} |B|^{-\frac{1}{n}}\Big(\int_B |\nabla u|^{n/2}\,dx\Big)^{2/n} \\
&\le C \sup_{B\subset B(a,2r)} |B|^{-\frac{1}{n}}\Big[\Big(\int_B |\nabla u|^{n-\varepsilon}\,dx\Big)^{\frac{n}{2(n-\varepsilon)}}|B|^{\frac{n-2\varepsilon}{2(n-\varepsilon)}}\Big]^{2/n} \\
&= C \sup_{B\subset B(a,2r)} |B|^{-\frac{1}{n}}\Big(\int_B |\nabla u|^{n-\varepsilon}\,dx\Big)^{\frac{1}{(n-\varepsilon)}}|B|^{\frac{n-2\varepsilon}{n(n-\varepsilon)}} \\
&= C \sup_{B\subset B(a,2r)} |B|^{-\frac{\varepsilon}{n(n-\varepsilon)}}\Big(\int_B |\nabla u|^{n-\varepsilon}\,dx\Big)^{\frac{1}{(n-\varepsilon)}} \\
&\le C \sup_{B\subset B(a,2r)} r^{-\frac{\varepsilon}{n-\varepsilon}}\Big(\int_B |\nabla u|^{n-\varepsilon}\,dx\Big)^{\frac{1}{n-\varepsilon}} \\
&= C \sup_{B\subset B(a,2r)} \Big(r^{-\varepsilon}\int_B |\nabla u|^{n-\varepsilon}\,dx\Big)^{\frac{1}{n-\varepsilon}} \le C\|\nabla u\|_{L^{n-\varepsilon,\,\varepsilon}(B(a,2r))}.
\end{aligned} \tag{4.0.32}$$

Thus, turning back to the initial inequality (4.0.30) we get

$$\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \le C\|\nabla u\|_{L^{n-\varepsilon,\,\varepsilon}(B(a,2r))} \tag{4.0.33}$$

and so by combining inequalities (4.0.29), (4.0.33), (4.0.9) we have

$$|R_2| \le C(n,\varepsilon)r^{\varepsilon}\|\nabla u\|^{2-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}\|\nabla\tilde{u}\|^n_{L^n(B(a,2r))}. \tag{4.0.34}$$

For now, let us divide our analysis into two parts, focusing separately on odd and even dimensions n.

- **The number n is even**: using here inequality (2.6.22) in Note 2.6.8 the estimate of R_2 becomes

$$|R_2| \le C(n,\varepsilon)r^{\varepsilon}\|\nabla u\|^{2-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}\|\tilde{u}\|^{n-2}_{\mathrm{BMO}(B(a,3r))}\|\nabla^{n/2}\tilde{u}\|^2_{L^2(B(a,3r))}. \tag{4.0.35}$$

Initially we note that

$$\|\tilde{u}\|_{\mathrm{BMO}(B(a,3r))} \le C(n)\|\nabla u\|_{L^{n-\varepsilon,\,\varepsilon}(B(a,3r))}. \tag{4.0.36}$$

Indeed, by substituting $\tilde{u}$ into inequality (4.0.32) and approaching the estimations in the same manner as demonstrated in that inequality, using also Lemma 4.0.3, we obtain

$$\begin{aligned}\|\tilde{u}\|_{\mathrm{BMO}(B(a,3r))} &\le \sup_{B\subset B(a,3r)} \Big(\fint_B \Big|\tilde{u}(x) - [\tilde{u}]_B\Big|^n \, dx\Big)^{1/n} \\ &\le C \sup_{B\subset B(a,3r)} |B|^{-\frac{1}{n}} \Big(\int_B |\nabla \tilde{u}|^{n/2}\, dx\Big)^{2/n} \\ &\le C \sup_{B\subset B(a,3r)} |B|^{-\frac{1}{n}} \Big(\int_B |\nabla u|^{n/2}\, dx\Big)^{2/n} \\ &\le C\|\nabla u\|_{L^{n-\varepsilon,\,\varepsilon}(B(a,3r))}.\end{aligned} \tag{4.0.37}$$

We also need to ensure that the last term of inequality (4.0.35) is an error term, i.e. to show that the function $r \to \|\nabla^{n/2}\tilde{u}\|^2_{L^2(B(a,3r))}$ is absolutely continuous. Applying the Leibniz rule to the higher order partial derivative we can write

$$\begin{aligned}\|\nabla^{n/2}\tilde{u}\|^2_{L^2(B(a,3r))} &= \Big\| \sum_{\{\alpha,\beta:|\alpha|+|\beta|=n/2\}} \nabla^{\alpha}\zeta\nabla^{\beta}u\Big\|^2_{L^2(B(a,3r))} \\ &\le \sum_{\{\alpha,\beta:|\alpha|+|\beta|=n/2\}} \|\nabla^{\alpha}\zeta\nabla^{\beta}u\|^2_{L^2(B(a,3r))} \\ &= (*).\end{aligned} \tag{4.0.38}$$

Slightly abusing notation and recalling that all norms on $\mathbb{R}^n$ are equivalent, we can rewrite $(*)$ as follows:

$$\begin{aligned}
(*) &= \sum_{\alpha=0}^{n/2} \|\nabla^{n/2-\alpha}\zeta \cdot \nabla^{\alpha} u\|^2_{L^2(B(a,3r))} \\
&\leq \sum_{\alpha=0}^{n/2} \frac{C}{r^{n-2\alpha}} \|\nabla^{\alpha} u\|^2_{L^2(B(a,3r))} \\
&= \frac{C}{r^n} \|u - [u]_{B(a,3r)}\|^2_{L^2(B(a,3r))} + \sum_{\alpha=1}^{n/2} \frac{C}{r^{n-2\alpha}} \int_{B(a,3r)} |\nabla^{\alpha} u|^2 \, dx \\
&\leq \frac{C}{r^{n-2}} \|\nabla u\|^2_{L^2(B(3r))} + \sum_{|\alpha|=1}^{n/2} \frac{C}{r^{n-2\alpha}} \Big(\int_{B(3r))} |\nabla^{\alpha} u|^{n/\alpha} \, dx \Big)^{\frac{2\alpha}{n}} (|B(3r))|)^{\frac{n-2\alpha}{n}} \\
&\leq \|\nabla u\|^2_{L^n(B(3r))} + \sum_{|\alpha|=1}^{n/2} C(n) \|\nabla^{\alpha} u\|^2_{L^{n/\alpha}(B(3r))} \\
&\leq \sum_{|\alpha|=1}^{n/2} C(n) \|\nabla^{\alpha} u\|^2_{L^{n/\alpha}(B(3r))} := \Psi(a, 3r).
\end{aligned} \tag{4.0.39}$$

Thus finally we get

$$|R_2| \leq C(n, \varepsilon) r^{\varepsilon} \|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))} \Psi(a, 3r). \tag{4.0.40}$$

- **The number n is odd**: let in this case $n = 2k+1$ for some $k > 1$. Using inequality (2.6.26) we have

$$\|\nabla \tilde{u}\|^n_{L^n(\mathbb{R}^n)} \leq C \|\tilde{u}\|^{n-2}_{\mathrm{BMO}(\mathbb{R}^n)} \cdot [\nabla^k \tilde{u}]^2_{W^{1/2,\,2}(\mathbb{R}^n)}.$$

We denote for $s \in (0,1)$ the space $H^s := W^{s,p}$. Like in the previous case we have to show that the term $[\nabla^k \tilde{u}]_{H^{1/2}(\mathbb{R}^n)}$ is an error term, i.e. we want to show that the function

$$\Phi(a, r) := [\nabla^k \tilde{u}]_{H^{1/2}(\mathbb{R}^n)}$$

satisfies $\lim_{r \to 0} \Phi(a, r) = 0$. This case is more complicated than the previous one because of the non-trivial norm structure of H^s. We proceed similarly to [33, Proposition 3.5], with slight modifications. Here is the precise way we take a facile approach to this problem:

$$[\nabla^k \tilde{u}]_{H^s(\mathbb{R}^n)} = [\nabla^k(\zeta(u-C))]_{H^s(\mathbb{R}^n)} \leq \sum_{\alpha+\beta=k} [\nabla^\alpha \zeta \cdot \nabla^\beta u]_{H^s(\mathbb{R}^n)}. \tag{4.0.41}$$

We denote

$$w(x) := \nabla^\alpha \zeta(x) \cdot \nabla^\beta u(x).$$

The support of w contains in the ball $B(a, 3r)$ and moreover

$$|w(x)| \leq \frac{C}{r^\alpha} |\nabla^\beta u(x)|.$$

Note 4.0.8. *We are working on the ball $B(a, 4r)$ instead of the ball $B(a, 3r)$ mainly because the integral 4.0.43 explodes for x near by boundary. Hence, by this extension we omit this problem simply because the function w is zero outside of the ball $B(a, 3r)$.*

Using the definition of the H^s norm we write

$$\begin{aligned}
[w]_{H^{1/2}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|w(x)-w(y)|^2}{|x-y|^{n+1}}\,dxdy \\
&= \int_{\mathbb{R}^n}\Big(\int_{B(a,4r)} + \int_{\mathbb{R}^n\setminus B(a,4r)} \frac{|w(x)-w(y)|^2}{|x-y|^{n+1}}\,dxdy\Big) \\
&= \int_{\mathbb{R}^n}\int_{B(a,4r)} \frac{|w(x)-w(y)|^2}{|x-y|^{n+1}}\,dxdy + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus B(a,4r)} \frac{|w(x)-w(y)|^2}{|x-y|^{n+1}}\,dxdy \\
&= \Big(\int_{B(a,4r)} + \int_{\mathbb{R}^n\setminus B(a,4r)}\Big)\int_{B(a,4r)} \frac{|w(x)-w(y)|^2}{|x-y|^{n+1}}\,dxdy \\
&\quad + \Big(\int_{B(a,4r)} + \int_{\mathbb{R}^n\setminus B(a,4r)}\Big)\int_{\mathbb{R}^n\setminus B(a,4r)} \frac{|w(x)-w(y)|^2}{|x-y|^{n+1}}\,dxdy \\
&= \int_{B(a,4r)}\int_{B(a,4r)} \frac{|w(x)-w(y)|^2}{|x-y|^{n+1}}\,dxdy + 2\int_{B(a,4r)}\int_{\mathbb{R}^n\setminus B(a,4r)} \frac{|w(x)-w(y)|^2}{|x-y|^{n+1}}\,dxdy \\
&\quad + \int_{\mathbb{R}^n\setminus B(a,4r)}\int_{\mathbb{R}^n\setminus B(a,4r)} \frac{|w(x)-w(y)|^2}{|x-y|^{n+1}}\,dxdy \\
&= \int_{B(a,4r)}\int_{B(a,4r)} \frac{|w(x)-w(y)|^2}{|x-y|^{n+1}}\,dxdy + 2\int_{B(a,4r)}\int_{\mathbb{R}^n\setminus B(a,4r)} \frac{|w(x)|^2}{|x-y|^{n+1}}\,dxdy \\
&= \int_{B(a,4r)}\int_{B(a,4r)} \frac{|w(x)-w(y)|^2}{|x-y|^{n+1}}\,dxdy + 2\int_{B(a,4r)}|w(x)|^2\int_{\mathbb{R}^n\setminus B(a,4r)} \frac{1}{|x-y|^{n+1}}\,dydx \\
&:= \mathrm{I} + 2\mathrm{II}.
\end{aligned} \tag{4.0.42}$$

Let us firstly take care of the second term II. One can show (see for example [33, Proposition 3.3]) that

$$\int_{\mathbb{R}^n\setminus B(a,4r)} \frac{1}{|x-y|^{n+1}}\,dy \le Cr^{-1}. \tag{4.0.43}$$

Thus we write

$$\begin{aligned}
|\mathrm{II}| &\le Cr^{-1}\int_{B(a,4r)}|w(x)|^2\,dx \le Cr^{-1}\int_{B(a,4r)}\frac{1}{r^{2\alpha}}|\nabla^\beta u(x)|^2\,dx \\
&\le \frac{C}{r^{2\alpha+1}}\int_{B(a,4r)}|\nabla^\beta u(x)|^2\,dx = \frac{C}{r^{2(k-\beta)+1}}\int_{B(a,4r)}|\nabla^\beta u(x)|^2\,dx \\
&= \frac{C}{r^{n-2\beta}}\int_{B(a,4r)}|\nabla^\beta u(x)|^2\,dx = (*).
\end{aligned} \tag{4.0.44}$$

Note that this kind of expressions we got in the even dimension case. Applying Hölder inequality to the above integral to eliminate $\frac{1}{r^{n-2\beta}}$ we obtain

$$(*) \leq C\Big(\int_{B(a,4r)} |\nabla^\beta u(x)|^{\frac{\beta}{n}}\, dx \Big)^{2\beta/n}. \tag{4.0.45}$$

Now we have to ensure that the above integral is finite. This can be done as follows: the 'last' derivative of u belongs to the $H^{1/2}$, i.e. $\nabla^k u \in H^{1/2}$, thus the Sobolev embedding theorem for the fractional H^s space (s.f. [34, Theorem 6.5]) implies $\nabla^k u \in L^{p^*}$, where $p^* = \frac{np}{n-sp}$. In our case $s = \frac{1}{2}$, $p = 2$ and so $p^* = \frac{2n}{n-1} > 2$ — this is the improvement we were searching. Hence $\nabla^k u \in L^{\frac{2n}{n-1}}$ which implies inductively that $\nabla^{k-1} u \in W^{1,\frac{2n}{n-1}}$. Again using the Sobolev embedding theorem 2.2.3 we get $\nabla^{k-1} u \in L^{p^*}$, where this time $p^* = \frac{np}{n-p}$. Hence

$$p^* = \frac{n \cdot \frac{2n}{(n-1)}}{n - \frac{2n}{(n-1)}} = \frac{2n}{n-3}.$$

This gives us again that $\nabla^{k-2} u \in W^{1,\frac{2n}{n-3}}$. By the induction we show that

$$\nabla^{k-j} u \in L^{\frac{2n}{n-(2j+1)}}.$$

Therefore we write $\beta = k - j$, for some $j \in \mathbb{N}$, so

$$\nabla^\beta u \in L^{\frac{2n}{n-1-2j}} = L^{\frac{2n}{n-2(k-\beta)-1}} = L^{\frac{n}{\beta}},$$

where the last equality follows from the assumption $n = 2k + 1$.

To estimate the first term I we write

$$\begin{aligned}
|w(x) - w(y)| &= |\nabla^\alpha \zeta(x) \cdot \nabla^\beta u(x) - \nabla^\alpha \zeta(y) \cdot \nabla^\beta u(y)| \\
&= |\nabla^\alpha \zeta(x) \cdot \nabla^\beta u(x) - \nabla^\alpha \zeta(x) \cdot \nabla^\beta u(y) + \nabla^\alpha \zeta(x) \cdot \nabla^\beta u(y) - \nabla^\alpha \zeta(y) \cdot \nabla^\beta u(y)| \\
&\leq |\nabla^\alpha \zeta(x)| \cdot |\nabla^\beta u(x) - \nabla^\beta u(y)| + |\nabla^\beta u(y)| \cdot |\nabla^\alpha \zeta(x) - \nabla^\alpha \zeta(y)| \\
&\leq \frac{C}{r^\alpha} |\nabla^\beta u(x) - \nabla^\beta u(y)| + |\nabla^\beta u(y)| \cdot |x-y| \cdot \|\nabla^{\alpha+1} \zeta\|_{L^\infty} \\
&\leq \frac{C}{r^\alpha} |\nabla^\beta u(x) - \nabla^\beta u(y)| + \frac{C}{r^{\alpha+1}} |x-y| |\nabla^\beta u(y)| = \mathrm{II}_1 + \mathrm{II}_2. \qquad (4.0.46)
\end{aligned}$$

Note that for some constant $C > 0$ we have $|w(x) - w(y)|^2 \le C(\mathrm{II}_1^2 + \mathrm{II}_2^2)$. With this we estimate II as follows

$$\begin{aligned} \mathrm{II} \le \frac{C}{r^{2\alpha}} \int_{B(a,4r)} \int_{B(a,4r)} \frac{|\nabla^\beta u(x) - \nabla^\beta u(y)|^2}{|x-y|^{n+1}}\, dxdy \\ + \frac{C}{r^{2\alpha+2}} \int_{B(a,4r)} |\nabla^\beta u(x)|^2 \int_{B(a,4r)} \frac{1}{|x-y|^{n-1}}\, dydx = (*) \end{aligned} \tag{4.0.47}$$

The second term II_2^2 is easy, indeed

$$\int_{B(a,4r)} \frac{1}{|x-y|^{n-1}}\, dydx \le Cr,$$

thus

$$\mathrm{II}_2^2 \le \frac{C}{r^{n-2\beta}} \int_{B(a,4r)} |\nabla^\beta u(x)|^2\, dx \le C\Big(\int_{B(a,4r)} |\nabla^\beta u(x)|^{\frac{n}{\beta}}\, dx \Big)^{\frac{2\beta}{n}} < \infty.$$

To estimate the first part II_1^2 we distinguish here two cases $\alpha = 0$ and $\alpha \ne 0$. When $\alpha = 0$ we have $\beta = k$ and so

$$\mathrm{II}_1^2 = \int_{B(a,4r)} \int_{B(a,4r)} \frac{|\nabla^k u(x) - \nabla^k u(y)|^2}{|x-y|^{n+1}}\, dxdy < \infty.$$

Therefore, due to the absolute continuity of the integral, the above expression is absolutely continuous with respect to r. If $\alpha \ne 0$, then the singular term $\frac{1}{r^{2\alpha}}$ appears. In order to estimate it properly we need the following estimate on convex domains (for the proof see [33, Proposition 3.1]).

Fact 4.0.9. Let D be a convex domain and let $s > 1$. Assume $u \in H^s(D)$, then the following estimate holds true

$$\int_D \int_D \frac{|u(x) - u(y)|^2}{|x-y|^{n+1}}\, dxdy \le Cr \int_D |\nabla u(x)|^2\, dx.$$

Applying this fact we get

$$\begin{aligned} \mathrm{II}_1^2 &\leq \frac{C}{r^{2\alpha}} r \int_{B(a,4r)} |\nabla^{\beta+1} u(x)|^2 \, dx = \frac{C}{r^{2\alpha-1}} \int_{B(a,4r)} |\nabla^{\beta+1} u(x)|^2 \, dx \\ &= \frac{C}{r^{n-2(\beta+1)}} \int_{B(a,4r)} |\nabla^{\beta+1} u(x)|^2 \, dx < \infty, \end{aligned} \tag{4.0.48}$$

where the last inequality follows in the same way as inequality (4.0.45). Clearly, $\Phi(a,r) = [\nabla^k \tilde{u}]_{H^{1/2}(\mathbb{R}^n)}$ is absolutely continuous due to the absolutely continuity of the above estimations. Hence

$$|R_2| \leq C(n,\varepsilon) r^{\varepsilon} \|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))} \Psi(a,3r), \tag{4.0.49}$$

where $\Psi(a,r)$ is the upper bound of $\Phi(a,3r)$ derived from the previous computations.

This ends the proof of the Lemma 4.0.7. □

Thus we have the following / final inequality on the right hand side

$$|R| = |R_1| + |R_2| \leq C(n,\varepsilon) r^{\varepsilon} \|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))} \tilde{\Psi}(a,3r), \tag{4.0.50}$$

where

$$\tilde{\Psi}(a,3r) := \|\nabla u\|^n_{L^n(B(a,3r))} + \Psi(a,3r). \tag{4.0.51}$$

Last step — choice of ε and decay inequality

Combining the lower bound estimate of the left hand side (see inequality (4.0.12)) and upper bound estimate of the right hand side (see inequality (4.0.50)) we obtain

$$\begin{aligned} \int_{B(a,r)} |\nabla u|^{n-\varepsilon} \, dx - \frac{1}{5} \int_{B(a,3r)} |\nabla u|^{n-\varepsilon} \, dx - C(n) \int_A |\nabla u|^{n-\varepsilon} \, dx \\ \leq C(n,\varepsilon) r^{\varepsilon} \|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))} \tilde{\Psi}(a,3r), \end{aligned} \tag{4.0.52}$$

hence

$$\int_{B(a,r)} |\nabla u|^{n-\varepsilon}\,dx \le C(n)\int_A |\nabla u|^{n-\varepsilon}\,dx + \frac{1}{5}\int_{B(a,3r)} |\nabla u|^{n-\varepsilon}\,dx + C(n,\varepsilon)r^{\varepsilon}\|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}\tilde{\Psi}(a,3r). \quad (4.0.53)$$

Now we recall that $A := B(a,2r)\setminus B(a,r)$, so the integral over A could be rewritten as $\int_{B(a,2r)} - \int_{B(a,r)}$. Taking the integral over the ball $B(a,r)$ on the left hand side and using the monotonicity of the integral, i.e. $\int_{B(a,2r)} \le \int_{B(a,3r)}$ we get the following inequality

$$(1 + C(n))\int_{B(a,r)} |\nabla u|^{n-\varepsilon}\,dx \le \Big(\frac{1}{5} + C(n)\Big)\int_{B(a,3r)} |\nabla u|^{n-\varepsilon}\,dx + C(n,\varepsilon)r^{\varepsilon}\|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}\tilde{\Psi}(a,3r).$$

Dividing both sides on $1 + C(n)$ and noting that $\lambda := \frac{1/5+C(n)}{1+C(n)} < 1$ we get

$$\int_{B(a,r)} |\nabla u|^{n-\varepsilon}\,dx \le \lambda\int_{B(a,3r)} |\nabla u|^{n-\varepsilon}\,dx + C(n,\varepsilon)r^{\varepsilon}\|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}\tilde{\Psi}(a,3r).$$

Now we divide by r^{ε}, getting

$$\frac{1}{r^{\varepsilon}}\int_{B(a,r)} |\nabla u|^{n-\varepsilon}\,dx \le \frac{\lambda 3^{\varepsilon}}{(3r)^{\varepsilon}}\int_{B(a,3r)} |\nabla u|^{n-\varepsilon}\,dx + C(n,\varepsilon)\|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}\tilde{\Psi}(a,3r).$$

We choose $\varepsilon > 0$ such that

$$\lambda_1 := 3^{\varepsilon}\lambda < 1. \quad (4.0.54)$$

The ε chosen in that way depends only on n. Hence the above inequality is now of the form

$$\frac{1}{r^{\varepsilon}}\int_{B(a,r)} |\nabla u|^{n-\varepsilon}\,dx \le \lambda_1\|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))} + C(n,\varepsilon)\tilde{\Psi}(a,3r)\|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}.$$

The above gives us

$$\frac{1}{r^{\varepsilon}}\int_{B(a,r)} |\nabla u|^{n-\varepsilon}\, dx \le (\lambda_1 + C(n,\varepsilon)\tilde{\Psi}(a,3r))\|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))},$$

hence using the absolute continuity of the function $\tilde{\Psi}(a,3r)$ we choose $r>0$ so small that $\lambda_2 := \lambda_1 + C(n,\varepsilon)\tilde{\Psi}(a,3\tilde{r}) < 1$ holds for each ball $B(a,3\tilde{r})$, where $0<\tilde{r}<r$. With this notation we write

$$\frac{1}{r^{\varepsilon}}\int_{B(a,r)} |\nabla u|^{n-\varepsilon}\, dx \le \lambda_2\|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}.$$

Using the fact that if $B(z,\rho)\subset B(a,r)$ then $B(z,3\rho)\subset B(a,3r)$, we can in fact take the supremum of the left hand side. In this way we obtain decay inequality of the following form

$$\|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,r))} \le \lambda_2\|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}. \tag{4.0.55}$$

Now, we have to make one more standard technical step to finish the proof. Applying the above decay inequality k-times we get

$$\|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,r))} \le \lambda_2^k\|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3^k r))}.$$

Let k be the largest "correct" iteration, i.e. $R_0 := 3^k r < \frac{1}{8}\mathrm{dist}(a,\partial\mathbb{B}^n(0,1))$ but $3^{k+1}r > \frac{1}{8}\mathrm{dist}(a,\partial\mathbb{B}^n(0,1))$. We write

$$\lambda_2^k = (3^k)^{\log_3\lambda_2} = (3^k r^k)^{\log_3\lambda_2} r^{-\log_3\lambda_2}.$$

Denoting now $\mu := -\log_3\lambda_2$ we get

$$\lambda_2^k = R^{-\mu}r^{\mu} = Cr^{\mu},$$

where $C>0$ is constant. The above decay inequality gives us now that (noting also that $\|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(0,1))} < \infty$)

$$\|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,r))} \le Cr^{\mu}.$$

Turning back to the definition of the Morrey's norm we obtain that for each sufficiently small ball $B(a,r)$

$$\int_{B(a,r)} |\nabla u|^{n-\varepsilon}\, dx \leq C r^{\varepsilon+\mu}.$$

Therefore by the Dirichlet Growth theorem 2.4.1 the map u is locally Hölder continuous.

□

Chapter 5

Regularity of n-harmonic maps

Theorem 5.0.1. *Let $\mathcal{N}$ be a Riemannian manifold isometrically embedded into $\mathbb{R}^m$, $u \in W^{n/2,\,2}(\mathbb{B}^n(0,1),\,\mathcal{N})$ be a weakly n-harmonic map for some fixed natural $n > 2$, i.e. u is a weak solution of the differential equation*

$$-\text{div}(|\nabla u|^{n-2}\nabla u) = |\nabla u|^{n-2} A_u(\nabla u, \nabla u), \tag{5.0.1}$$

where $A_u : T_u\mathcal{N} \times T_u\mathcal{N} \to T_u^{\perp}\mathcal{N}$ is the second fundamental form of $\mathcal{N}$. More precisely for each test function $\phi \in W_0^{1,n}(\mathbb{B}^n(0,1),\,\mathbb{R}^m)$ the following equation is satisfied

$$\int_{\mathbb{B}^n(0,1)} |\nabla u|^{n-2}\nabla u \nabla\phi\, dx = \int_{\mathbb{B}^n(0,1)} |\nabla u|^{n-2} A_u(\nabla u, \nabla u)\phi\, dx. \tag{5.0.2}$$

Then u is locally Hölder continuous.

The strategy of that proof is the same as in the proof of H-systems: we aim to show that a solution u of Equation (5.0.1) satisfies locally the following inequality

$$\int_{B(a,r)} |\nabla u|^p\, dx \leq Cr^{\mu}.$$

The main difference here lies in the structure of the right hand side, where the Jacobian structure is lost. Now because u is bounded, one could try to use the solution u as the test function to equation 5.0.2. However, the naive L^1 estimate of the right hand side together with $\|u\|_\infty \le C < \infty$ is useless:

$$\|\nabla u\|_{L^n}^n = \int |\nabla u|^{n-2} \nabla u \cdot \nabla u \, dx = \int |\nabla u|^{n-2} A_u(\nabla u, \nabla u) u \, dx \le C \|\nabla u\|_{L^n}^n.$$

In order to find a better estimate of the right hand side, one has to exploit its structure. Using the antisymmetrization trick of the bilinear form A (see Fact 2.5.4), we write $A(\nabla u, \nabla u) = \Omega \nabla u$, where $\Omega_{i,j} = \sum_l (A^i_{j,l} - A^j_{il}) \nabla u^l$ is an 'almost' gradient field. More precisely, the L^n-norm of the divergence-free 'error' part of the Hodge decomposition of Ω is small by the Coifman–Rochberg–Weiss commutator. The antisymmetric structure of the right-hand side provides hope for further improvements. Applying the Rivière–Uhlenbeck decomposition to the Ω (see Theorem 2.6.11) we can replace the matrix Ω with a divergence free matrix $\tilde{\Omega}$ given by the formula $\tilde{\Omega} = P^T \nabla P + P^T \Omega P$, where P is some $\mathrm{SO}(\mathbb{R}^m)$ valued function. This leads to the modified equation of the form

$$\operatorname{div}(|\nabla u|^{n-2} P^T \nabla u) = |\nabla u|^{n-2} \tilde{\Omega} \cdot P^T \nabla u$$

see Lemma 5.0.3 below in the proof. The L^n-norm of new matrix $\tilde{u}$ is also small, i.e. one can show that $\|\tilde{\Omega}\|_{L^n} \le \|u\|_{\mathrm{BMO}} \|\nabla u\|_{L^n}$ (see Lemma 5.0.6). This significantly improves the L^1 estimate of the right hand side — testing by the solution u now produces much more refined estimate than its trivial $\|\nabla u\|_{L^n}^n$. However, the following problem arises: testing the left-hand side of the new equation with the solution u does not yield $\|\nabla u\|_{L^n}^n$. To achieve this, one has to test with a map w such that $P^T \nabla u = \nabla w + V$. Unfortunately, this introduces a new problem: ∇u is not necessarily bounded, and consequently, w also can be unbounded. The solution, with a slight modification, of this problem is the same as for the H-systems: we fix some small parameter $\varepsilon > 0$ and, inspired by Iwaniec-Sbordone [15], work under the critical exponent by testing the equation with the gradient part of $G = |\nabla u|^{-\varepsilon} P^T \nabla u$. With this choice, we can properly estimate both the left-hand side and the right-hand side, so that an application of Dirichlet Growth Theorem is finally possible.

Proof. **Antisymmetrization of the right hand side, extensions**

We start with analyzing the right hand side of Equation (5.0.1). Following Note (2.5.2) we extend the second fundamental form A_p to the bilinear form on whole $\mathbb{R}^m \times \mathbb{R}^m$ for each fixed $p \in \mathcal{N}$. Also we extend A_p with respect to the p variable to a function $C_0^\infty(\mathbb{R}^m)$. Using the antisymmetric structure of the operator A_u (c.f. Fact 2.5.4) we can write

$$A_u(\nabla u, \nabla u) = \Omega \cdot \nabla u,$$

where $\Omega \colon \mathbb{R}^n \to \mathrm{so}(m) \otimes \Lambda^1\mathbb{R}^n$ is an antisymmetric matrix of one-forms. More precisely, by denoting $A_u(e_i, e_j) := A_{i,j}$, elements of Ω are defined as follows

$$\Omega_{i,j} := \sum_{l=1}^{n} (A^i_{j,l} - A^j_{i,l}) \nabla u^l.$$

In that way Equation (5.0.1) reduces to the following equation:

$$-\mathrm{div}(|\nabla u|^{n-2} \nabla u) = |\nabla u|^{n-2} \Omega \cdot \nabla u. \tag{5.0.3}$$

Cutting a solution u, further estimates on Ω

Fix a point $a \in \mathbb{B}^n$ and let $r < \frac{1}{8}\mathrm{dist}(a, \partial\mathbb{B}^n(0,1))$. Let $\zeta \in C_0^\infty(\mathbb{R}^n)$ be a cutoff function such that $\zeta \equiv 1$ on $B(a, 2r)$, $\zeta \equiv 0$ outside of the $B(a, 3r)$ and $|\nabla^i \zeta(x)| \leq \frac{C}{r^i}$ for each $x \in B(a, 3r)$. Similarly to the proof of regularity for H-systems we define a the cutoff version of u on the ball $B(a, 3r)$ in the following way

$$\tilde{u}(x) = \zeta(x)(u(x) - [u]_{B(a,3r)}) + [u]_{B(a,3r)}. \tag{5.0.4}$$

However there is a small difference between this cutoff function and the one described in Equation (4.0.2): this function $\tilde{u}$ is equal to the original solution u on the ball $B(a, 2r)$, not just their derivatives. With that we redefine $\Omega_{i,j}$ by putting instead of u the function

$\tilde{u}$, i.e.

$$\Omega_{i,j} := \sum_{l=1}^{n} (A^i_{j,l}(\tilde{u}) - A^j_{i,l}(\tilde{u}))\nabla \tilde{u}^l, \tag{5.0.5}$$

where $A_{j,l}(\tilde{u}) := A_{\tilde{u}}(e_j, e_l)$. Thanks to the extensions of A, all $\Omega_{i,j}$ are well defined. Observe that Equation (5.0.3) still holds on the ball $B(a, 2r)$. The advantage of such redefining is that now $\Omega = 0$ outside of the ball $B(a, 3r)$, hence we localize the investigation locally to that ball. We are going to prove here the following very useful lemma.

Lemma 5.0.2 (Estimation on Ω)**.** *With the notation as above we have*

$$\|\Omega\|_{L^n(\mathbb{R}^n)} \leq C(n, \mathcal{N})\|\nabla \tilde{u}\|_{L^n(\mathbb{R}^n)} \leq C(n, \mathcal{N})\|\nabla u\|_{L^n(B(a,3r))}, \tag{5.0.6}$$

and for fixed $p > 0$

$$\|\Omega\|_{L^{n-p,p}(\mathbb{R}^n)} \leq C(n, \mathcal{N})\|\nabla \tilde{u}\|_{L^{n-p,p}(\mathbb{R}^n)} \leq C(n, \mathcal{N})\|\nabla u\|_{L^{n-p,p}(B(a,3r))}, \tag{5.0.7}$$

where $C(n, \mathcal{N})$ is the constant depending on n and manifold $\mathcal{N}$.

Proof. The first equation follows from the Lemma 4.0.3 and the fact that the second fundamental form A_p is smooth and compactly supported. More precisely, for fixed indices $i, j > 0$ we write

$$\begin{aligned}
\|\Omega_{i,j}\|_{L^n(\mathbb{R}^n)} &\leq \sum_{l=1}^{n} \|(A^i_{j,l}(\tilde{u}) - A^j_{i,l}(\tilde{u}))\nabla \tilde{u}^l\|_{L^n(\mathbb{R}^n)} \\
&\leq \sum_{l=1}^{n} \|(A^i_{j,l}(\tilde{u}) - A^j_{i,l}(\tilde{u}))\|_{L^\infty(\mathbb{R}^n)}\|\nabla \tilde{u}^l\|_{L^n(\mathbb{R}^n)} \\
&\leq C(n)\|\nabla u\|_{L^n(B(a,3r)} \sum_{l=1}^{n} \big(\|A_{\tilde{u}}(e_j, e_l)\|_{L^\infty(\mathbb{R}^n)} + \|A_{\tilde{u}}(e_i, e_l)\|_{L^\infty(\mathbb{R}^n)}\big) \\
&\leq C(n, \mathcal{N})\|\nabla u\|_{L^n(\mathbb{R}^n)}.
\end{aligned}$$

Hence the whole norm $\|\Omega\|_{L^n(\mathbb{R}^n)} := \sum_{i,j} \|\Omega_{i,j}\|_{L^n(\mathbb{R}^n)}$ is bounded by $C(n, \mathcal{N})\|\nabla u\|_{L^n(\mathbb{R}^n)}$. The proof of inequality (5.0.7) is a bit more complicated. As we saw above we have the pointwise inequality $|\Omega(x)| \leq C(n, \mathcal{N})|\nabla u(x)|$, thus the first part of the inequality is trivial.

Let us prove now that

$$\|\nabla\tilde{u}\|_{L^{n-p,p}(\mathbb{R}^n)} \leq C(n)\|\nabla u\|_{L^{n-p,p}(B(a,3r))}.$$

We write

$$\begin{aligned}\|\nabla\tilde{u}\|_{L^{n-p,p}(\mathbb{R}^n)} &= \|\nabla(\zeta(u-[u]_{B(a,3r)}))\|_{L^{n-p,p}(\mathbb{R}^n)}\\ &\leq \|(u-[u]_{B(a,3r)})\nabla\zeta\|_{L^{n-p,p}(\mathbb{R}^n)} + \|\zeta\nabla u\|_{L^{n-p,p}(\mathbb{R}^n)}\\ &\leq \frac{C}{r}\|u-[u]_{B(a,3r)}\|_{L^{n-p,p}(B(a,3r))} + \|\nabla u\|_{L^{n-p,p}(B(a,3r))},\end{aligned} \tag{5.0.8}$$

where last inequality follows from the fact that $|\nabla\phi| \leq \frac{C}{r}$ and from the fact that for each function $f:\mathbb{R}^n \to \mathbb{R}^m$ such that $f \equiv 0$ outside of $B(a,r) \subset \mathbb{R}^n$ it holds

$$\|f\|_{L^{p,s}(\mathbb{R}^n)} = \|f\|_{L^{p,s}(B(a,r))}.$$

To estimate the Morrey norm of $u-[u]_{B(a,3r)}$ we perform the same steps as in inequality (4.0.32), i.e. we firstly apply the Hölder inequality and then the Gagliardo-Nirenberg-Sobolev inequality (c.f. Theorem 2.2.3) getting the following inequality

$$\begin{aligned}\|u-[u]_{B(a,3r)}\|_{L^{n-p,p}(B(a,3r))} &:= \sup_{\tilde{r}>0,\ y\in B(a,3r)} \Big(\frac{1}{\tilde{r}^p}\int_{B(y,\tilde{r})} |u-[u]_{B(a,3r)}|^{n-p}dx\Big)^{\frac{1}{n-p}}\\ &\leq C(n)\sup_{\tilde{r}>0, y\in B(a,3r)}\Big(\int_{B(y,\tilde{r})}|u-[u]_{B(a,3r)}|^n dx\Big)^{1/n}\\ &\leq C(n)\Big(\int_{B(a,3r)}|u-[u]_{B(a,3r)}|^n dx\Big)^{1/n}\\ &\leq C(n)r\Big(\frac{1}{r^{n-p}}\int_{B(a,3r)}|\nabla u|^p\,dx\Big)^{\frac{1}{n-p}}\\ &\leq C(n)r\sup_{\tilde{r}>0,\ y\in B(a,3r)}\Big(\frac{1}{\tilde{r}^p}\int_{B(y,\tilde{r})}|u-[u]_{B(a,3r)}|^{n-p}dx\Big)^{\frac{1}{n-p}}\\ &= C(n)r\|\nabla u\|_{L^{n-p,p}(B(a,3r))}.\end{aligned} \tag{5.0.9}$$

Hence Equation (5.0.8) reduces to the following

$$\|\nabla\tilde{u}\|_{L^{n-p,\,p}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^{n-p,\,p}(B(a,3r))} \tag{5.0.10}$$

which ends the proof of the lemma. □

Rivière–Uhlenbeck decomposition of antisymmetric matrix Ω

We improve the structure of the right hand side of Equation (5.0.3) by using Theorem 2.6.11 to the operator Ω. Here is the precise way we do that.

Small energy assumption: Firstly, we assume that the condition of small energy,

$$\|\Omega\|_{L^n(\mathbb{R}^n)} < \varepsilon_0 \tag{5.0.11}$$

is met. This assumption holds true since the L^n-norm of the Ω is entirely controlled by the L^n-norm of the ∇u within the corresponding ball (see Lemma 5.0.2, Equation 5.0.6). Hence, if this condition (5.0.11) does not hold on the ball $B(a, 3r)$ we simply take smaller radius $0 < \tilde{r} << r$ and a corresponding cut-off function ϕ such that $\|\Omega\|_{L^n(\mathbb{R}^n)} < \varepsilon_0$. The selection of $\tilde{r}$ and ϕ remains independent of the point $a \in \mathbb{B}^n(0,1)$, due to the absolute continuity of integration.

Choice of p: Fix some small parameter $\varepsilon > 0$ and let $p = \frac{n-\varepsilon}{2}$. We apply the Rivière–Uhlenbeck decomposition 2.6.11 to the matrix-valued operator Ω on the ball $B(a, 3r)$ getting $P \in W^{1,n}(B(a, 3r), \mathrm{SO}(m))$, $P = \mathrm{Id}$ on $\partial B(a, 3r)$ such that the matrix

$$\tilde{\Omega} = P^{-1}dP + P^{-1}\Omega P \tag{5.0.12}$$

is divergence free. We set $P \equiv \mathrm{Id}$ outside of the ball $B(a, 3r)$, resulting $\tilde{\Omega} = \Omega = 0$ outside of $B(a, 3r)$. Moreover we have the following estimates

$$\|dP\|_{L^n(B(a,3r))} \leq C(n,m)\|\Omega\|_{L^n(B(a,3r))} \tag{5.0.13}$$

$$\|dP\|_{L^{p,n-p}(B(a,3r))} \leq C(n,m)\|\Omega\|_{L^{2p,n-2p}(B(a,3r))}. \tag{5.0.14}$$

Combining Lemma 5.0.2 with the zero extension of dP outside of the ball $B(a,3r)$, we get

$$\|dP\|_{L^n(\mathbb{R}^n)} \leq C(n,m)\|\Omega\|_{L^n(\mathbb{R}^n)} \leq C(n,m,\mathcal{N})\|\nabla u\|_{L^n(B(a,3r))}, \tag{5.0.15}$$

and

$$\|dP\|_{L^{p,n-p}(\mathbb{R}^n)} \leq C(n,m)\|\Omega\|_{L^{2p,n-2p}(\mathbb{R}^n)} \leq C(n,m,\mathcal{N})\|\nabla u\|_{L^{\varepsilon,n-\varepsilon}(B(a,3r))}. \tag{5.0.16}$$

Connection of the new $\tilde{\Omega}$ with Equation (5.0.3): the following lemma shows how to involve the new matrix $\tilde{\Omega}$ to Equation (5.0.3).

Lemma 5.0.3. *The new map $P^{-1}\nabla u$ satisfies:*

$$-\mathrm{div}(|P^{-1}\nabla u|^{n-2}P^{-1}\nabla u) = |P^{-1}\nabla u|^{n-2}\tilde{\Omega}\cdot P^{-1}\nabla u \tag{5.0.17}$$

weakly in $\mathbb{B}^n(a,3r)$. Equivalently, by recalling that P is orthogonal a.e. in $B(a,3r)$ and that $P^{-1} = P^T$ we write

$$-\mathrm{div}(|\nabla u|^{n-2}P^T\cdot\nabla u) = |\nabla u|^{n-2}\tilde{\Omega}\cdot P^T\nabla u. \tag{5.0.18}$$

Proof. The proof relies on elementary computations. Firstly we are going to prove that

$$-\mathrm{div}(|\nabla u|^{n-2}P^T\cdot\nabla u) = |\nabla u|^{n-2}(P^T\Omega - \nabla P^T)\cdot\nabla u, \tag{5.0.19}$$

weakly in $B(a,3r)$. To be rigorous we actually computes for $\phi \in C_0^\infty(B(a,3r))$

$$\begin{aligned}\int_{B(a,3r)} |\nabla u|^{n-2} P^T \cdot \nabla u \cdot \nabla\phi \, dx &= \int_{B(a,3r)} |\nabla u|^{n-2} \nabla(P^T\phi)\cdot \nabla u \, dx \\ &\quad - \int_{B(a,3r)} |\nabla u|^{n-2}\nabla P^T \cdot \nabla u \cdot \phi \, dx \\ &= \int_{B(a,3r)} |\nabla u|^{n-2} P^T \phi \cdot \Omega \cdot \nabla u - \int_{B(a,3r)} |\nabla u|^{n-2}\nabla P^T \cdot \nabla u \cdot \phi \, dx \\ &= \int_{B(a,3r)} |\nabla u|^{n-2}(P^T\Omega - \nabla P^T)\nabla u \cdot \phi \, dx,\end{aligned} \tag{5.0.20}$$

where the first equality follows from the differentiation by parts

$$\nabla(P^T\phi) = \phi\nabla P^T + P^T\nabla\phi.$$

To finish the proof of this lemma, we need to express the gradient of the P^T. By differentiation the identity $PP^T = Id$ we obtain

$$\nabla P \cdot P^T + P\nabla P^T = 0,$$

which implies

$$\nabla P^T = -P^T \cdot \nabla P \cdot P^T$$

Hence Equation (5.0.19) is now of the following form

$$\begin{aligned}-\mathrm{div}(|\nabla u|^{n-2} P^T \cdot \nabla u) &= |\nabla u|^{n-2}(P^T\Omega - \nabla P^T)\cdot\nabla u \\ &= |\nabla u|^{n-2}(P^T\Omega + P^T\cdot\nabla P\cdot P^T)\nabla u \\ &= |\nabla u|^{n-2}(P^T\Omega P + P^T\cdot\nabla P)P^T\nabla u \\ &= |\nabla u|^{n-2}\tilde{\Omega}\cdot P^T\nabla u.\end{aligned} \tag{5.0.21}$$

□

Note 5.0.4. *We now focus on studying Equation (5.0.17). The regularity of solutions to this new equation implies the regularity of the initial solution u due to the norm equality*

$|P^T \nabla u| = |\nabla u|$.

Now that the structure of the equation is established, it is time to select an appropriate test function.

Selecting the test function ϕ

We proceed similarly to the case of H-systems: define the vector field

$$G^k = |\nabla \tilde{u}|^{-\varepsilon} (P^T \nabla \tilde{u})^k,$$

where $k = 1, \dots n$ and $\varepsilon > 0$ will be chosen, independently of u (see inequality (4.0.54)), at the end of the proof. Without loss of generality we skip writing G^k and instead we write just G. One can check that $G \in L^r$ for $1 \leq r \leq \frac{n}{1-\varepsilon}$. In particular $G \in L^q$ for $q := \frac{n-\varepsilon}{1-\varepsilon} > n$. Applying the Hodge decomposition to the vector field G we get

$$G = \nabla \phi + V,$$

where $\phi \in W^{1,q}_{\text{loc}}(\mathbb{R}^n)$, and $V \in L^q(\mathbb{R}^n, \mathbb{R}^n)$ is divergence free in the sense of distributions. Moreover by inequality (2.6.13) and Lemma 4.0.3 we show that

$$\|\nabla \phi\|_{L^q(\mathbb{R}^n)} + \|V\|_{L^q(\mathbb{R}^n)} \leq C \|G\|_{L^q(\mathbb{R}^n)} \leq C \Big(\int_{B(a,3r)} |\nabla u|^{n-\varepsilon} \, dx \Big)^{(1-\varepsilon)/(n-\varepsilon)}. \tag{5.0.22}$$

Using Lemma 4.0.9 we get that

$$\|\phi\|_{L^\infty(B(a,3r))} \leq C(\varepsilon) r^\varepsilon \Big(r^{-\varepsilon} \int_{B(a,3r)} |\nabla u|^{n-\varepsilon} \, dx \Big)^{(1-\varepsilon)/(n-\varepsilon)}, \tag{5.0.23}$$

where the constant $C(\varepsilon) \to \infty$ as $\varepsilon \to 0^+$. As the final step of this section we introduce another cut-off function $\zeta_1 \in C_0^\infty(B(a, 2r))$ such that $\zeta_1 \equiv 1$ on the ball $B(a, r)$ and $\zeta_1 \equiv 0$ outside of the ball $B(a, 2r)$. Similarly to the case of H-systems, we will test Equation (5.0.17) by the function $\zeta_1 \phi$.

Estimation of the left hand side

Lemma 5.0.5. *The left hand side of Equation (5.0.18) is bounded from the below by $\int_{B(a,r)} |\nabla u|^{n-\varepsilon}\, dx$, up to some less important terms. More precisely we have*

$$\int_{\mathbb{B}^n(0,1)} |\nabla u|^{n-2}\nabla u \cdot P^T \cdot \nabla u \cdot \nabla(\zeta_1 \phi)\, dx \geq \int_{B(a,r)} |\nabla u|^{n-\varepsilon}\, dx - C(n) \int_A |\nabla u|^{n-\varepsilon}\, dx - \frac{1}{5} \int_{B(a,3r)} |\nabla u|^{n-\varepsilon}\, dx. \tag{5.0.24}$$

The main problem here is to estimate the divergence free part V, showing that this map is actually small enough. This is mainly due to the fact that there is no trivial way to represent V as the commutator operator as we did in the H-systems. Also the estimation 5.0.22 is too crude for this case (i.e. that it will not combine well with the estimation of the right hand side). To resolve the issue, we will use a more general theorem than the commutators theorem — the Iwaniec-Sbordone stability estimate (c.f. [15, Theorem 4]).

Proof. Differentiating by parts and then using that $\nabla \phi = G - V$, we get

$$\begin{aligned} &\int_{\mathbb{B}^n(0,1)} |\nabla u|^{n-2} P^T \cdot \nabla u \cdot \nabla(\zeta_1 \phi)\, dx = \int_{B(a,2r)} |\nabla u|^{n-2} P^T \nabla u \cdot \phi \cdot \nabla \zeta_1\, dx \\ &+ \int_{B(a,2r)} |\nabla u|^{n-2} \cdot P^T \nabla u \cdot \zeta_1 \cdot G\, dx - \int_{B(a,2r)} |\nabla u|^{n-2} P^T \nabla u \cdot \zeta_1 \cdot V\, dx \\ &:= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned} \tag{5.0.25}$$

Now we are going to estimate each term independently. We start with the second term. Note firstly that on the ball $B(a, 2r)$ the map u and its cut version $\tilde{u}$ coincide, i.e. $u \equiv \tilde{u}$.

We write

$$\begin{aligned}\mathrm{II} = \int_{B(a,2r)} |\nabla u|^{n-2} P^T \nabla u \cdot \zeta_1 |\nabla u|^{-\varepsilon} P^T \nabla u \, dx &= \int_{B(a,2r)} |\nabla u|^{n-2-\varepsilon} \cdot \zeta_1 \cdot |P^T \nabla u|^2 \, dx \\ &= \int_{B(a,2r)} |\nabla u|^{n-\varepsilon} \zeta_1 \, dx \\ &\geq \int_{B(a,r)} |\nabla u|^{n-\varepsilon} \, dx.\end{aligned} \tag{5.0.26}$$

Let us now estimate the I term. By applying the Hölder and Poincaré inequalities we get

$$\begin{aligned}|\mathrm{I}| &\leq \frac{C}{r} \int_A |\nabla u|^{n-1} |\phi| \, dx \\ &\leq \frac{C}{r} \Big(\int_A |\nabla u|^{n-\varepsilon} \, dx \Big)^{\frac{n-1}{n-\varepsilon}} \Big(\int_A |\phi|^{(n-\varepsilon)/(1-\varepsilon)} \, dx \Big)^{\frac{1-\varepsilon}{n-\varepsilon}} \\ &\leq \frac{C}{r} \Big(\int_A |\nabla u|^{n-\varepsilon} \, dx \Big)^{\frac{n-1}{n-\varepsilon}} \Big(\int_{B(a,2r)} |\phi|^{(n-\varepsilon)/(1-\varepsilon)} \, dx \Big)^{\frac{1-\varepsilon}{n-\varepsilon}} \\ &\leq \Big(\int_A |\nabla u|^{n-\varepsilon} \, dx \Big)^{\frac{n-1}{n-\varepsilon}} \Big(\int_{B(a,2r)} |\nabla \phi|^{(n-\varepsilon)/(1-\varepsilon)} \, dx \Big)^{\frac{1-\varepsilon}{n-\varepsilon}}.\end{aligned} \tag{5.0.27}$$

Similarly to the H-systems we use the Young inequality to obtain

$$|\mathrm{I}| \leq C(n) \int_A |\nabla u|^{n-\varepsilon} \, dx + \frac{1}{10} \int_{B(a,3r)} |\nabla u|^{n-\varepsilon} \, dx.$$

The last step is to estimate the term III. We start with Hölder inequality applied to III, getting the following inequality

$$|\mathrm{III}| \leq \int_{B(a,2r)} |\nabla u|^{n-1} |V| \, dx \leq \|\nabla u\|^{n-1}_{L^{n-\varepsilon}(B(a,2r))} \|V\|_{L^{(n-\varepsilon)/(1-\varepsilon)}(B(a,2r))}. \tag{5.0.28}$$

Now the whole difficulty lies in the estimating of the $L^{(n-\varepsilon)/(1-\varepsilon)}$-norm of V. To do this, we define two operators: T is the projection on a divergence free part and $S^\varepsilon(f) := \|f\|^\varepsilon_{n-\varepsilon} |f|^{-\varepsilon} f$. The main point is the Iwaniec-Sbordone stability theorem (see [15, Theorem 4]), which implies

$$\|TS^\varepsilon(f) - S^\varepsilon(Tf)\|_{L^{\frac{n-\varepsilon}{1-\varepsilon}}} \leq C(n) |\varepsilon| \|f\|_{L^{n-\varepsilon}}. \tag{5.0.29}$$

If we put $f = P^T\nabla\tilde{u}$, the term V appears then in the $T(S^\varepsilon f)$. Indeed,

$$T(S^\varepsilon f) = T(\|f\|_{n-\varepsilon}^\varepsilon |f|^{-\varepsilon} f) = \|\nabla\tilde{u}\|_{n-\varepsilon}^\varepsilon T(|\nabla\tilde{u}|^{-\varepsilon} P^T\nabla\tilde{u}) = \|\nabla\tilde{u}\|_{n-\varepsilon}^\varepsilon \cdot V. \tag{5.0.30}$$

We additionally denote

$$P^T\nabla\tilde{u} = \nabla w + W,$$

where W is divergence free. Now estimate V in the following way:

$$\begin{aligned}
\|V\|_{(n-\varepsilon)/(1-\varepsilon)} &= \|\nabla\tilde{u}\|_{n-\varepsilon}^{-\varepsilon}(\|\nabla\tilde{u}\|_{n-\varepsilon}^{\varepsilon}\|V\|_{(n-\varepsilon)/(1-\varepsilon)}) = \|\nabla\tilde{u}\|_{n-\varepsilon}^{-\varepsilon}\|T(S^\varepsilon f)\|_{(n-\varepsilon)/(1-\varepsilon)} \\
&= \|\nabla\tilde{u}\|_{n-\varepsilon}^{-\varepsilon}\|T(S^\varepsilon f) - S^\varepsilon(Tf) + S^\varepsilon(Tf)\|_{(n-\varepsilon)/(1-\varepsilon)} \\
&\le \|\nabla\tilde{u}\|_{n-\varepsilon}^{-\varepsilon}(\|T(S^\varepsilon f) - S^\varepsilon(Tf)\|_{(n-\varepsilon)/(1-\varepsilon)} + \|S^\varepsilon(Tf)\|_{(n-\varepsilon)/(1-\varepsilon)}) \\
&\le \|\nabla\tilde{u}\|_{n-\varepsilon}^{-\varepsilon}(C(n)|\varepsilon|\|f\|_{n-\varepsilon} + \|S^\varepsilon(T(P^T\nabla\tilde{u}))\|_{(n-\varepsilon)/(1-\varepsilon)}) \\
&\le \|\nabla\tilde{u}\|_{n-\varepsilon}^{-\varepsilon}(C(n)|\varepsilon|\|\nabla u\|_{L^{n-\varepsilon}(B(a,3r))} + \|S^\varepsilon(W)\|_{(n-\varepsilon)/(1-\varepsilon)} \\
&= \|\nabla\tilde{u}\|_{n-\varepsilon}^{-\varepsilon}(C(n)|\varepsilon|\|\nabla u\|_{L^{n-\varepsilon}(B(a,3r))} + \|W\|_{n-\varepsilon}^{\varepsilon}\||W|^{-\varepsilon}W\|_{(n-\varepsilon)/(1-\varepsilon)}) \\
&= \|\nabla\tilde{u}\|_{n-\varepsilon}^{-\varepsilon}(C(n)|\varepsilon|\|\nabla u\|_{L^{n-\varepsilon}(B(a,3r))} + \|W\|_{n-\varepsilon}\|W\|_{n-\varepsilon}^{1-\varepsilon}) \\
&= \|\nabla\tilde{u}\|_{n-\varepsilon}^{-\varepsilon}(C(n)|\varepsilon|\|\nabla u\|_{L^{n-\varepsilon}(B(a,3r))} + \|W\|_{n-\varepsilon}).
\end{aligned} \tag{5.0.31}$$

We estimate the $L^{n-\varepsilon}$-norm of W using the commutator theorem: let $\tilde{T}$ be the projection operator on the gradient part, then

$$\begin{aligned}
\|W\|_{n-\varepsilon} &= \|P^T\nabla\tilde{u} - \nabla w\|_{n-\varepsilon} = \|[P^T, \tilde{T}](\nabla\tilde{u})\|_{n-\varepsilon} \\
&\le \|P^T\|_{\mathrm{BMO}}\|\nabla\tilde{u}\|_{n-\varepsilon} \\
&\le C(n, m, \mathcal{N})\|\nabla P\|_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}\|\nabla u\|_{L^{n-\varepsilon}(B(a,3r))} \\
&\lesssim \|\nabla u\|_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}\|\nabla u\|_{L^{n-\varepsilon}(B(a,3r))},
\end{aligned} \tag{5.0.32}$$

where last two inequalities follow from inequality (4.0.32) and inequality (5.0.16). This gives us finally the estimation of V:

$$\begin{aligned}\|V\|_{(n-\varepsilon)/(1-\varepsilon)} \leq \|\nabla u\|^{-\varepsilon}_{L^{n-\varepsilon}(B(a,3r))}\big(C(n)|\varepsilon|\|\nabla u\|_{L^{n-\varepsilon}(B(a,3r))} \\ + C\|\nabla u\|_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}\|\nabla u\|_{L^{n-\varepsilon}(B(a,3r))}\big) \\ \leq C\|\nabla u\|^{1-\varepsilon}_{L^{n-\varepsilon}(B(a,3r))}\big(|\varepsilon| + \|\nabla u\|_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}\big)\end{aligned} \tag{5.0.33}$$

With this estimate on V, inequality (5.0.28) takes of the following form:

$$\begin{aligned}|\mathrm{III}| &\leq C\|\nabla u\|^{n-1}_{L^{n-\varepsilon}(B(a,2r))}\|\nabla u\|^{1-\varepsilon}_{L^{n-\varepsilon}(B(a,3r))}\big(|\varepsilon| + \|\nabla u\|_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}\big) \\ &\leq C\|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon}(B(a,3r))}\big(|\varepsilon| + \|\nabla u\|_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}\big) \\ &= C\big(|\varepsilon| + \|\nabla u\|_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}\big) \cdot \int_{B(a,3r)} |\nabla u|^{n-\varepsilon}\, dx\end{aligned} \tag{5.0.34}$$

Now we choose $\varepsilon > 0$ and the radius $r > 0$ so small that $C\big(|\varepsilon| + \|\nabla u\|_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}\big) < \frac{1}{10}$. Hence, we finally get the desired estimation of III

$$|\mathrm{III}| \leq \frac{1}{10}\int_{B(a,3r)} |\nabla u|^{n-\varepsilon}\, dx. \tag{5.0.35}$$

Combining all the previous estimates on the I, II and III we estimate the left hand side of Equation (5.0.18) from below in the following way:

$$\begin{aligned}\int_{\mathbb{B}^n(0,1)} |\nabla u|^{n-2} P^T \cdot \nabla u \cdot \nabla(\zeta_1 \phi)\, dx \geq \mathrm{II} - |\mathrm{I}| - |\mathrm{III}| \\ = \int_{B(a,r)} |\nabla u|^{n-\varepsilon}\, dx - C(n)\int_A |\nabla u|^{n-\varepsilon}\, dx - \frac{1}{5}\int_{B(a,3r)} |\nabla u|^{n-\varepsilon}\, dx.\end{aligned} \tag{5.0.36}$$

□

Estimation of the right hand side

Similarly to the H-systems, our goal is to properly estimate the right hand side from above. As will become clear later, the main issue lies in estimating the L^n-norm of $\tilde{\Omega}$. The trivial approach

$$\|\tilde{\Omega}\|_n \lesssim \|\Omega\|_n \lesssim \|\nabla u\|_n$$

leads to the

$$\||\nabla u|^{n-2}\tilde{\Omega} \cdot P^T \nabla \tilde{u}\|_{L^n(\mathbb{B}^n(0,1))} \lesssim \|\nabla u\|^n_{L^n(B(a,3r))}.$$

However, the above estimate is useless — one can not obtain a 'decay' inequality (see inequality (5.0.47)) from that. Fortunately, the commutator theorem provides a significantly better estimate for $\tilde{\Omega}$, showing it to be much smaller than the previous naive estimate. We start with a straightforward approach:

$$\begin{aligned}
\int_{\mathbb{B}^n(0,1)} |\nabla u|^{n-2}\tilde{\Omega} \cdot P^T \nabla u \cdot \zeta_1 \phi \, dx &\leq \int_{\mathbb{B}^n(0,1)} |\nabla u|^{n-2} |\tilde{\Omega}| \cdot |P^T| \cdot |\nabla u| \cdot |\zeta_1| \cdot |\phi| \, dx \\
&\leq \int_{B(a,2r)} |\nabla u|^{n-1} |\tilde{\Omega}| \cdot |\phi| \, dx \\
&\leq \|\phi\|_{L^\infty(B(a,3r))} \int_{B(a,2r)} |\nabla u|^{n-1} |\tilde{\Omega}| \, dx \\
&\leq \|\phi\|_{L^\infty(B(a,3r))} \|\nabla u\|^{n-1}_{L^n(B(a,2r))} \|\tilde{\Omega}\|_{L^n(B(a,2r))},
\end{aligned} \tag{5.0.37}$$

where the last inequality follows from the Hölder inequality with parameters n, $\frac{n}{n-1}$. We estimate the L^∞-norm of the ϕ by using inequality (5.0.23)

$$\begin{aligned}
\|\phi\|_{L^\infty(B(a,3r))} &\leq C(\varepsilon) r^\varepsilon \Big(\frac{1}{r^\varepsilon} \int_{B(a,3r)} |\nabla u|^{n-\varepsilon} \, dx \Big)^{(1-\varepsilon)/(n-\varepsilon)} \\
&\leq C(\varepsilon) r^\varepsilon \Big(\sup_{B(x,\tilde{r}) \subset B(a,3r)} \frac{1}{\tilde{r}^\varepsilon} \int_{B(x,\tilde{r})} |\nabla u|^{n-\varepsilon} \, dx \Big)^{(1-\varepsilon)/(n-\varepsilon)} \\
&= C(\varepsilon) r^\varepsilon \|\nabla u\|^{1-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}.
\end{aligned} \tag{5.0.38}$$

Now it is time to estimate L^n-norm of $\tilde{\Omega}$.

Lemma 5.0.6. *The L^n-norm of the $\tilde{\Omega}$ satisfies the following inequality*

$$\|\tilde{\Omega}\|_{L^n(\mathbb{R}^n)} \leq C \|\nabla u\|_{L^{n-\varepsilon,\varepsilon}(B(a,3r))} \|\nabla \tilde{u}\|_{L^n(\mathbb{R}^n)}. \tag{5.0.39}$$

Proof. The key trick here lies in the commutator theorem. As usually let T be a projection on the divergence-free part operator. Now because the matrix $\tilde{\Omega}$ is divergence-free we have

$$T(\tilde{\Omega}) = \tilde{\Omega}.$$

On the other hand, $\tilde{\Omega} = P^T dP + P^T \Omega P$, hence we obtain

$$\tilde{\Omega} = T(\tilde{\Omega}) = T(P^T dP + P^T \Omega P) = T(P^T dP) + T(P^T \Omega P).$$

Taking the L^n norm of $\tilde{\Omega}$ and using the triangle inequality,

$$\|\tilde{\Omega}\|_{L^n(\mathbb{R}^n)} \leq \|T(P^T dP)\|_{L^n(\mathbb{R}^n)} + \|T(P^T \Omega P)\|_{L^n(\mathbb{R}^n)}.$$

We are going to estimate each of the norms independently. This is a perfect moment to make a technical note.

Note 5.0.7. *Assume $F = b\nabla g$, where $b \in BMO$ and $g \in W^{1,p}$ for some $p > 1$. Using the Hodge decomposition we obtain $F = \nabla f + \tilde{F}$. In order to estimate the L^p norm of $\tilde{F}$ we write*

$$\tilde{F} = b\nabla g - \nabla f = bT(\nabla g) - T(b\nabla g) = [b, T](\nabla g),$$

where, unlike the previous definition, T temporarily denotes the projection operator on the ***gradient*** *part. Thus the commutator theorem gives us the proper estimate on $\tilde{F}$:*

$$\|\tilde{F}\|_p \leq C\|b\|_{BMO}\|\nabla g\|_g.$$

We start with the first term. Note firstly that $P \equiv \mathrm{Id}$ outside of the ball $B(a, 3r)$ implies $dP \equiv 0$ outside of $B(a, 3r)$. Using now the above note we estimate $T(P^T dP)$ as follows

$$\|T(P^T dP)\|_{L^n(\mathbb{R}^n)} \leq C\|P\|_{\mathrm{BMO}(B(a,3r))}\|dP\|_{L^n(B(a,3r))}.$$

Combining again the estimate $\|P\|_{\mathrm{BMO}} \lesssim \|\nabla P\|_{L^{n-\varepsilon,\varepsilon}}$ (see first 4.0.32) and then the inequality $\|P\|_{L^{n-\varepsilon,\varepsilon}} \lesssim \|\nabla u\|_{L^{n-\varepsilon,\varepsilon}}$ (see inequality 5.0.16), and also $\|\nabla P\|_{L^n} \lesssim \|\nabla \tilde{u}\|_{L^n}$ (see inequality (5.0.15)) we get

$$\|T(P^T dP)\|_{L^n(\mathbb{R}^n)} \leq \|\nabla u\|_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}\|\nabla \tilde{u}\|_{L^n(\mathbb{R}^n)}. \tag{5.0.40}$$

Thus the estimate of the first term has the form we need.

Let us now take care of the second term. In fact it behaves similarly to the previous term. To see this we note that $(P^T\Omega P)_{ij} = \sum_{s,t} P^T_{i,s}\Omega_{s,t}P_{t,j}$. Recall that $\Omega_{i,j} = \sum_{l=1}^n (A^i_{j,l}(\tilde{u}) - A^j_{i,l}(\tilde{u}))\nabla \tilde{u}^l$, so now we can write

$$(P^T\Omega P)_{ij} = \sum_{s,t,l} P^T_{i,s}P_{t,j}(A^s_{t,l}(\tilde{u}) - A^t_{s,l}(\tilde{u}))\nabla \tilde{u}^l. \tag{5.0.41}$$

Abusing the notation, we note that $b := P^T_{i,s}P_{t,j}(A^s_{t,l}(\tilde{u}) - A^t_{s,l}(\tilde{u})) \in L^\infty$, and hence it is an element of BMO space. Once again, we can apply the C.R.W. commutator theorem as in Note 5.0.7, to obtain

$$\|T(P^T\Omega P)\|_{L^n} \lesssim \sum_{s,t,l} \|P^T_{i,s}P_{t,j}(A^s_{t,l}(\tilde{u}) - A^t_{s,l}(\tilde{u}))\|_{\mathrm{BMO}}\|\nabla \tilde{u}^l\|_{L^n}. \tag{5.0.42}$$

We deal with above BMO norm as usually — firstly we estimate it by the $L^{n-\varepsilon,\varepsilon}$, secondly, by applying the integration by parts, we estimate each of the $P_{i,j}$ and $A^h_{k,r}$ separately. Here is the precise way we do that

$$\begin{aligned}
&\|P^T_{i,s}P_{t,j}(A^s_{t,l}(\tilde{u}) - A^t_{s,l}(\tilde{u}))\|_{\mathrm{BMO}} \lesssim \|\nabla(P^T_{i,s}P_{t,j}(A^s_{t,l}(\tilde{u}) - A^t_{s,l}(\tilde{u})))\|_{L^{n-\varepsilon,\varepsilon}} \\
&\leq \|P_{t,j}(A^s_{tl}(\tilde{u}) - A^t_{s,l}(\tilde{u}))\|_{L^\infty}\|\nabla P^T_{i,s}\|_{L^{n-\varepsilon,\varepsilon}} + \|P^T_{i,s}(A^s_{t,l}(\tilde{u}) - A^t_{s,l}(\tilde{u}))\|_{L^\infty}\|\nabla P_{t,j}\|_{L^{n-\varepsilon,\varepsilon}} \\
&\quad + \|P_{t,j}P^T_{i,s}\|_{L^\infty}\|\nabla(A^s_{t,l}(\tilde{u}) - A^t_{s,l}(\tilde{u}))\|_{L^{n-\varepsilon,\varepsilon}} \\
&\lesssim \|\nabla P^T_{i,s}\|_{L^{n-\varepsilon,\varepsilon}} + \|\nabla P_{t,j}\|_{L^{n-\varepsilon,\varepsilon}} + \|\nabla(A^s_{t,l}(\tilde{u}) - A^t_{s,l}(\tilde{u}))\|_{L^{n-\varepsilon,\varepsilon}} \\
&\lesssim \|\nabla u\|_{L^{n-\varepsilon}(B(a,3r))} + \|\nabla A(\tilde{u})\|_{L^{n-\varepsilon,\varepsilon}}.
\end{aligned} \tag{5.0.43}$$

Using the chain rule, we can write $\nabla(A(\nabla\tilde{u})) = \nabla A(\nabla\tilde{u}) \cdot \nabla\tilde{u}$. As it was said at the beginning of the whole proof, A is a smooth function, so its gradient is bounded by some constant depending only on the manifold $\mathcal{N}$. Hence

$$|\nabla(A(\nabla\tilde{u}))| \leq C|\nabla\tilde{u}|.$$

This also implies that

$$\|\nabla(A(\nabla\tilde{u}))\|_{L^{n-\varepsilon,\varepsilon}} \leq C\|\nabla\tilde{u}\|_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}.$$

We showed finally that

$$\|P^T_{i,s}P_{t,j}(A^s_{t,l}(\tilde{u}) - A^t_{s,l}(\tilde{u}))\|_{\mathrm{BMO}} \lesssim \|\nabla u\|_{L^{n-\varepsilon,\varepsilon}(B(a,3r))},$$

hence the whole L^n norm of $T(P^T\Omega P)$ is bounded by $\|\nabla u\|_{L^{n-\varepsilon,\varepsilon}(B(a,3r))} \cdot \|\nabla\tilde{u}\|_{L^n(\mathbb{R}^n)}$. $\square$

Using estimate given in the above Lemma and inequality (5.0.38), inequality (5.0.37) becomes

$$\left|\int_{\mathbb{B}^n(0,1)} |\nabla u|^{n-2}\tilde{\Omega} \cdot P^T\nabla u \cdot \zeta_1\phi\, dx\right| \leq C(\varepsilon)r^{\varepsilon}\|\nabla u\|^{2-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}\|\nabla u\|^n_{L^n(B(a,3r))}. \quad (5.0.44)$$

Estimating the L^n-norm of ∇u in the same way as in the H-systems we finally get

$$\left|\int_{\mathbb{B}^n(0,1)} |\nabla u|^{n-2}\tilde{\Omega} \cdot P^T\nabla u \cdot \zeta_1\phi\, dx\right| \leq C(n,\varepsilon)\|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))}\tilde{\Phi}(a,3r), \quad (5.0.45)$$

where $\Phi(a,r)$ is the absolutely continuous function precisely described in section 4.

Last step — choice of ε and decay inequality

This section holds the same strategy and computations as in the correspond section in H-systems. Here are only the main steps. Combining inequality (5.0.45) and inequality

(5.0.24) we get

$$\int_{B(a,r)} |\nabla u|^{n-\varepsilon}\, dx - C(n) \int_A |\nabla u|^{n-\varepsilon}\, dx - \frac{1}{5} \int_{B(a,3r)} |\nabla u|^{n-\varepsilon}\, dx$$
$$\leq C(n,\varepsilon) \|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))} \tilde{\Phi}(a,3r). \quad (5.0.46)$$

By a hole filling trick we obtain a decay inequality of the following form

$$\|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,r))} \leq \lambda \|\nabla u\|^{n-\varepsilon}_{L^{n-\varepsilon,\varepsilon}(B(a,3r))} \quad (5.0.47)$$

for some $\lambda < 1$, $\varepsilon = \varepsilon(n) > 0$ small and all $r < r_0(\varepsilon)$. By iterating the above inequality we get

$$\int_{B(a,r)} |\nabla u|^{n-\varepsilon}\, dx \leq C r^{\mu+\varepsilon},$$

where $\mu = -\log_3 \lambda$. Hence, applying the Dirichlet Growth Theorem 2.4.1 one obtain the local Hölder continuity of u. □

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