University of Warsaw
Faculty of Mathematics, Informatics and Mechanics

Aneta Wróblewska-Kamińska

An application of Orlicz spaces in partial differential equations

PhD dissertation

Supervisor

dr hab. Piotr Gwiazda, prof. UW

Institute of Applied Mathematics and Mechanics
University of Warsaw

July 2012

Author's declaration:
aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

July 16, 2012 ....
date *Aneta Wróblewska-Kamińska*

Supervisor's declaration:
the dissertation is ready to be reviewed

July 16, 2012 ....
date *dr hab. Piotr Gwiazda, prof. UW*

Abstract

Our purpose is to investigate mathematical properties of some systems of nonlinear partial differential equations where the nonlinear term is monotone and its behaviour - coercivity/growth conditions are given with the help of some general convex function defining Orlicz spaces.

Our first result is the existence of weak solutions to unsteady flows of non-Newtonian incompressible nonhomogeneous (with non-constant density) fluids with nonstandard growth conditions of the stress tensor. We are motivated by the problem of anisotropic behaviour of fluids which are also characterised by rapid shear thickness. Since we are interested in flows with the rheology more general than power-law-type, we describe the growth conditions with the help of an x–dependent convex function and formulate our problem in generalized Orlicz (Musielak-Orlicz) spaces.

As a second result we give a proof of the existence of weak solutions to the problem of the motion of one or several nonhomogenous rigid bodies immersed in a homogenous non-Newtonian fluid. The nonlinear viscous term in the equation is described with the help of a general convex function defining isotropic Orlicz spaces. The main ingredient of the proof is convergence of the nonlinear term achieved with the help of the pressure localisation method.

The third result concerns the existence of weak solutions to the generalized Stokes system with the nonlinear term having growth conditions prescribed by an anisotropic $\mathcal{N}-$function. Our main interest is directed to relaxing the assumptions on the $\mathcal{N}-$function and in particular to capture the shear thinning fluids with rheology close to linear. Additionally, for the purpose of the existence proof, a version of the Sobolev–Korn inequality in Orlicz spaces is proved.

Last but not least, we study also a general class of nonlinear elliptic problems, where the given right-hand side belongs only to the L^1 space. Moreover the vector field is monotone with respect to the second variable and satisfies a non-standard growth condition described by an x-dependent convex function that generalizes both $L^{p(x)}$ and classical Orlicz settings. Using truncation techniques and a generalized Minty method in the functional setting of non reflexive spaces we prove existence of renormalized solutions for general L^1-data. Under an additional strict monotonicity assumption uniqueness of the renormalized solution is established. Sufficient conditions are specified which guarantee that the renormalized solution is already a weak solution to the problem.

Streszczenie

Naszym celem jest zbadanie matematycznych własności pewnych układów nieliniowych równań różniczkowych cząstkowych, dla których człon nielinowy jest monotoniczny a jego warunki wzrostu i koercytywności zadane są za pomocą pewnej ogólnej funkcji wypukłej, definiującej przestrzenie Orlicza.
Naszym pierwszym rezultatem jest istnienie słabych rozwiązań dla niestacjonarnego przepływu nieściśliwej, niejednorodnej (gęstość nie jest stała) cieczy nienewtonowskiej z niestandardowymi warunkami wzrostu dla tensora naprężeń. Motywacją do badań jest problem anizotropowego zachowania płynów charakteryzujących się wzrostem lepkości wraz ze wzrostem wartości naprężenia. Jesteśmy zainteresowani reologią ogólniejszą niż typu potęgowego, dlatego zadajemy warunki wzrostu za pomocą wypukłej funkcji zależnej od x i formułujemy problem w uogólnionych przestrzeniach Orlicza (Musielaka-Orlicza).
Jako kolejny rezultat przedstawiamy dowód istnienia słabych rozwiązań dla problemu ruchu jednego lub kilku niejednorodnych ciał sztywnych zanurzonych w jednorodnej nieściśliwej cieczy nienewtonowskiej. Nieliniowy człon lepkościowy w równaniu jest opisany przy wykorzystaniu ogólnej funkcji wypukłej definiującej izotropowe przestrzenie Orlicza. Główna część dowodu polega na wykazaniu zbieżności członu nielinowego, co osiągamy za pomocą metody lokalnego ciśnienia.
Trzecia część badań dotyczy istnienia słabych rozwiązań dla uogólnionego systemu Stokesa z nielinowym członem o warunkach wzrostu opisanych przez anizotropową $\mathcal{N}$–funkcję. Nasza uwaga skierowana jest na osłabienie założeń na $\mathcal{N}$–funkcję, ponieważ chcielibyśmy uwzględnić w naszych badaniach płyny nienewtonowskie, których lepkość maleje pod wpływem ścinania i których reologia zbliżona jest do liniowej. Ponadto, w celu przeprowadzenia dowodu, wyprowadzamy nierówność typu Korna-Sobolewa dla przestrzeni Orlicza.
W ostatniej części pracy studiujemy ogólną klasę nieliniowych problemów eliptycznych, gdzie dana prawa strona należy jedynie do przestrzeni L^1. Co więcej, pole wektorowe jest monotoniczne względem drugiej zmiennej i spełnia niestandardowe warunki wzrostu zadane przez, zależną od x, funkcję wypukłą. Tak postawiony problem uogólnia zarówno rozważania dla zagadnienia sformułowanego w przestrzeni $L^{p(x)}$ jak i w klasycznych przestrzeniach Orlicza. Wykorzystując metodę obcięć oraz "trik Minty'iego" uogólniony dla przestrzeni nierefleksywnych udowadniamy istnienie rozwiązań zrenormalizowanych z danymi w L^1. Przy dodatkowym założeniu ścisłej monotoniczności wykazujemy również jednoznaczność rozwiązań. Podajemy także warunki gwarantujące, że rozwiązanie zrenormalizoane jest słabym rozwiązaniem problemu.

Acknowledgment

I would like to emphasise my gratitude to two people without whom this work would not come into being – to my parents. Thank you for your support, your love you give us and which is between both of you. I am also thankful to Przemek that I have met him on my way and that he is here, next to me.
I appreciate Piotr Gwiazda – my supervisor – his scientific and personal support, his open mind and patience, that he always believes in my abilities and takes care of all the aspects of my scientific development.
I would also like to thank Šarka Nečasova and Eduard Feireisl for their help in preparing Chapter V, scientific support and fruitful discussions during my stay at the Academy of Sciences of the Czech Republic in Prague. For the cooperation on Chapter VII I would like to thank Petra Wittbold and Aleksandra Zimmermann. I am grateful also to Agnieszka Świerczewska-Gwiazda for cooperation on Chapter VI and her scientific support during my studies.
Finally, I would like to thank all the friends I met on my scientific path. I will not mention them by name as the list would be long.
I was supported by Foundation for Polish Science as a Ph.D student in the International Ph.D. Projects Programme operated within the Innovative Economy Operational Programme 2007-2013 (Ph.D. Programme: Mathematical Methods in Natural Sciences) and by grant no 6113/B/H03/2011/40 of National Science Centre.

Podziękowania

Chciałabym podziękować w sposób szczególny osobom, bez których ta praca nigdy by nie powstała - moim rodzicom. Za ich nieustające wsparcie, miłość, którą nam dają i ofiarowują sobie nawzajem.
Dziękuję Przemkowi za to, że spotkałam go na swojej drodze i za to że jest tu obok mnie.
Dziękuję swojemu promotorowi - Piotrowi Gwieździe za nieustanną wiarę w moje możliwości, wsparcie naukowe i osobiste, za zawsze otwarty umysł i dużą cierpliwość oraz opiekę nad moim naukowym rozwojem.
Wyrazy wdzięczności kieruję również do Šarki Nečasovej i Eduarda Feireisla za pomoc w tworzeniu rozdziału V, wsparcie naukowe i owocne dyskusje podczas mojego pobytu w Czeskiej Akademii Nauk w Pradze. Za współpracę nad ostatnim rozdziałem pracy dziękuję Petrze Wittbold i Aleksandrze Zimmermann. Za współpracę nad VI rozdziałem oraz czujne oko nad całym doktoratem chciałabym podziękować Agnieszce Świerczewskiej-Gwieździe.
Dziękuję również wszystkim przyjaciołom, których spotkałam na swojej naukowej ścieżce. Nie wymienię ich tutaj z imienia, ponieważ lista byłaby zbyt długa.
Na koniec dziękuję Fundacji Nauki Polskiej za wsparcie, które otrzymałam jako stypendystka w ramach międzynarodowych projektów doktoranckich realizowanego ze środków funduszy europejskich Programu Operacyjnego Innowacyjna Gospodarka 2007-2013 oraz Narodowemu Centrum Nauki za wsparcie na realizację projektu badawczego w postaci grantu nr 6113/B/H03/2011/40.

Contents

CHAPTER I

Introduction

Our main goal is to contribute to the mathematical theory of fluid mechanics and abstract theory of renormalized solutions to elliptic equations. In particular we are interested in existence of different types of solutions to nonlinear partial differential equations. The studies will be undertaken for the case of rather general growth conditions for the highest order term. This formulation requires a general framework for the function space setting. The problems will be considered in Orlicz and Musielak-Orlicz spaces. The level of generality of our considerations will have a crucial significance on the applied methods. Hence we will investigate isotropic and anisotropic cases as well as space homogeneous and nonhomogeneous cases of growth conditions. This is a natural generalization of the numerous recent studies appearing on Lebesgue, generalized Lebesgue and Sobolev spaces, which may be considered as a particular case of our approach. Together with the advance in methods for partial differential equations we will develop the theory of function spaces. The framework of Sobolev-Orlicz spaces is well developed only in the case of classical Orlicz spaces, namely defined by an $\mathcal{N}$–function (a continuous, convex, superlinear, nonnegative function, which will be defined in Chapter III) dependent on the absolute value of the vector and independent of the space variable x, usually considered under some additional condition on the growth of an $\mathcal{N}$–function (we mean here the so-called Δ_2–conditions on M or on the Fenchel-Young conjugate M^*, which we define precisely later). This is the analytical basis which can be used also in other fields applying the Orlicz space functional setting like variational inequalities, homogenization of elliptic and parabolic equations and many others. One can distinguish various cases of $\mathcal{N}$–functions:

- isotropic $\mathcal{N}$–function, i.e. $M : \mathbb{R}_+ \to \mathbb{R}_+$
- anisotropic $\mathcal{N}$–function, namely dependent on the whole vector $M : \mathbb{R}^n \to \mathbb{R}_+$,
- inhomogeneous in space, namely x dependent $\mathcal{N}$–function $M : \Omega \times \mathbb{R}^n \to \mathbb{R}_+$,
- rapidly or slowly growing $\mathcal{N}$–functions (lack of the Δ_2–assumption on M or on the conjugate M^*).

Extending the analytical tool in these directions is not only beneficial to the topics considered in this thesis but can also contribute to other problems mentioned above, where the phenomena of anisotropy and/or space nonhomogeneity may be of an interest. It is important to underline that in the case when an $\mathcal{N}$–function is isotropic, homogeneous, both M and M^* satisfy the Δ_2–condition, then most of the properties, even such fine properties like continuity of singular Riesz operators or Marcinkiewicz interpolation theorem follow analogously to the case of L^p spaces, cf. [**29**].

We consider a large class of problems capturing flows of non-Newtonian fluids with non-standard rheology. We want to include the phenomena of viscosity changing under various stimuli like shear rate, magnetic or electric field. This forces us to use space nonhomogeneous anisotropic Orlicz spaces. Our investigations are directed to existence and properties of solutions.

Substantial part of our considerations is motivated by a significant shear thickening phenomenon. Therefore we want to investigate the processes where the growth of the viscous stress tensor is faster than polynomial. Hence $\mathcal{N}$–function defining a space does not satisfy the Δ_2–condition.

Within the thesis we consider the existence of weak solutions to four problems. At the beginning our attention is directed to incompressible fluids with non-constant density. We include the case of different growth of the stress tensor in various directions of the shear stress and possible dependence on some outer field.

The second problem concerns the motion of rigid bodies in shear thickening fluid. The bodies have a nonhomogeneous structure and are immersed in a homogenous incompressible fluid. Omitting in this case the assumption of Δ_2–conditions has physical motivations. The requirement for avoiding collisions is a high enough integrability of the shear stress (at least in L^4). Hence it is natural to consider an $\mathcal{N}$–function of high growth e.g. exponential.

The presence of convective term in both of the mentioned problems allowed us to consider only shear thickening fluids. If we assume that the flow is slow, then it is reasonable to neglect the convective term. Therefore we are able to investigate the flow of shear thinning fluid described by a generalized Stokes system. The growth of the viscous stress tensor can be close to linear and is prescribed by an anisotropic $\mathcal{N}$–function whose complementary does not satisfy the Δ_2–condition.

Last but not least we concentrate on a general class of elliptic equations with right hand side integrable only in L^1 space. We extend the theory of renormalized solutions to the setting of Orlicz spaces given by a nonhomogeneous anisotropic $\mathcal{N}$–function with non polynomial upper bound.

In order to give the reader better insight into the results we give here short overview of the considered problems.

The main part of the thesis deals with a problem of the flow of a non-Newtonian fluid with non-standard rheology. Therefore we consider materials whose properties can be described not only by the dependence on constant viscosity. In our research we take under consideration the fact that it can change significantly under various stimuli like shear rate, magnetic or electric field. Our investigation concerns existence and properties of solutions to systems of equations coming from fluid mechanics. We concentrate on the case of an incompressible fluid for which equations can take the following form

$$
\begin{aligned}
\partial_t\varrho + \operatorname{div}_x(\varrho\boldsymbol{u}) &= 0 \quad \text{in} \quad (0,T)\times\Omega,\\
\partial_t(\varrho\boldsymbol{u}) + \operatorname{div}_x(\varrho\boldsymbol{u}\otimes\boldsymbol{u}) - \operatorname{div}_x\mathbf{S}(\mathbf{D}\boldsymbol{u}) + \nabla_x p &= \varrho\boldsymbol{f} \quad \text{in} \quad (0,T)\times\Omega,\\
\operatorname{div}_x\boldsymbol{u} &= 0 \quad \text{in} \quad (0,T)\times\Omega,
\end{aligned} \tag{I.0.1}
$$

where $\boldsymbol{u}$ denotes the velocity field of a fluid, ϱ - its density; p is a pressure; Ω is a bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary; $T < \infty$; $\boldsymbol{f}$ is a given outer force; $\mathbf{D}\boldsymbol{u} = \frac{1}{2}(\nabla_x\boldsymbol{u} + \nabla_x^T\boldsymbol{u})$ is the symmetric part of the velocity field. The first equation is the continuity equation, the second – momentum equation and the last one stands for incompressibility condition. We assume no-slip boundary condition (zero Dirichlet boundary condition).

In order to close the system we have to state the constitutive relation, rheology, which describes the relation between $\mathbf{S}$ and $\mathbf{D}\boldsymbol{u}$. In our considerations we do not want to assume that $\mathbf{S}$ has only polynomial-structure, i.e. $\mathbf{S} \approx (\kappa + |\mathbf{D}\boldsymbol{u}|)^{p-2}\mathbf{D}\boldsymbol{u}$ or $\mathbf{S} \approx (\kappa + |\mathbf{D}\boldsymbol{u}|^2)^{(p-2)/2}\mathbf{D}\boldsymbol{u}$ (where $\kappa > 0$). Standard growth conditions of the stress tensor, namely polynomial growth, see e.g. **[58, 92]**

$$
\begin{aligned}
|\mathbf{S}(\mathbf{D}\boldsymbol{u})| &\leqslant c(1 + |\mathbf{D}\boldsymbol{u}|^2)^{(p-2)/2}|\mathbf{D}\boldsymbol{u}|\\
\mathbf{S}(\mathbf{D}\boldsymbol{u}) : \mathbf{D}\boldsymbol{u} &\geqslant c(1 + |\mathbf{D}\boldsymbol{u}|^2)^{(p-2)/2}|\mathbf{D}\boldsymbol{u}|^2
\end{aligned} \tag{I.0.2}
$$

can not suffice to describe nonstandard behaviour of the fluid. Motivated by the significant shear thickening phenomenon we want to investigate the processes where the growth is faster than polynomial and possibly different in various directions of the shear stress. Also the case of growth close to linear can be covered in this way. A viscosity of the fluid is not assumed to be constant and can depend on density and full symmetric part of the velocity gradient. Therefore we formulate the growth conditions of the stress tensor using a general convex function M called an $\mathcal{N}-$function (the definitions of an $\mathcal{N}-$function M and its complementary function M^* appear in Section III.1) similarly as in **[72, 74, 75, 76, 78, 79, 131, 133, 134, 135]**. Now we are able to describe the effect of rapidly shear thickening and shear thinning fluids. Therefore we formulate growth/coercivity conditions in the following way:

$$
\mathbf{S}(x,\mathbf{D}\boldsymbol{u}) : \mathbf{D}\boldsymbol{u} \geqslant c\left\{M(x,\mathbf{D}\boldsymbol{u}) + M^*(x,\mathbf{S}(x,\mathbf{D}\boldsymbol{u}))\right\} \tag{I.0.3}
$$

where M is an $\mathcal{N}$–function, and therefore a quite general convex function.

In classical case, i.e. with polynomial growth conditions, the proper space setting is standard Lebesgue and Sobolev spaces. In our considerations condition (I.0.3) forces us to use Orlicz, Orlicz-Sobolev spaces, defined by the $\mathcal{N}$–function. We want to emphasise that we do not want to assume that M satisfies the so-called Δ_2–condition. Therefore we lose a wide range of facilitating properties of function spaces that one normally works with. Namely, if M does not satisfy the Δ_2–condition then our spaces are not reflexive, separable, smooth functions are not dense with respect to the norm. The lack of such assumption is a reason of many delicate and deep handicaps. Therefore we need to obtain the result using more sophisticated methods than in the classical case.

In Chapter IV we investigate the evolutionary equation for the flow of an incompressible non-Newtonian fluid which can take the form of the system (I.0.1). The first step of the proof of existence of a weak solution is the Galerikn approximation for the considered problem and existence of an approximate solution. The main difficulty then is to show the proper convergences in nonlinear terms. The result is achieved by a monotonicity method adapted to non-reflexive spaces [**131, 75**] and the compensated compactness method. We want here to extend the existence theory for flows of non-Newtonian incompressible fluids to a more general class than polynomial growth conditions by formulating the problem in Orlicz setting as in [**72, 75, 131**]. Moreover, we want to complete the theory the reader can find therein, by including the continuity equation $(\mathrm{IV}.1.1)_1$ to the considered system and dependence of $\mathbf{S}$ on density of the fluid (density is not assumed to be constant). Additionally we are able to obtain better time regularity of solution than in [**58, 59, 72, 75**], namely in a Nikolskii space. The existence of a weak solution accordant to Definition IV.1.1 is stated in Theorem IV.1.2. Chapter IV is based on [**133**] by Wróblewska-Kamińska and partially on methods and results from [**131, 75**].

Using the result mentioned above, in Chapter V we consider the problem of motion of one or several nonhomogeneous rigid bodies immersed in a homogeneous non-Newtonian fluid occupying a bounded domain. Therefore the fluid flow in the system is of (I.0.1)-type which is completed with the equations describing the motion of rigid bodies. We use here the fact, proved by Starovoitov, that two rigid objects do not collide if they are immersed in a fluid of viscosity significantly increasing with increasing shear rate. The method we use in order to solve the problem is, in the first step, to replace the rigid object by a fluid of high viscosity becoming singular in the limit. This idea was developed by Hoffman [**80**] and San Marin at al. [**113**]. Since we consider an incompressible fluid, the existence and estimates for the pressure function are not crucial from the point of existence of weak solutions. This is due to the fact that in a weak formulation the pressure function disappears. In this case we have to localise the problem only in the fluid part of the system. Therefore we need to deliver the decomposition and local estimates also for the pressure function. To this end we use the Riesz transform which in general is not continuous from Orlicz space to itself (it is the case if the $\mathcal{N}$–function and its complementary satisfy the Δ_2–condition). Therefore the space where the part of our pressure function is regular is larger than the space containing the nonlinear viscous term. Moreover we are not able to use theorems of Marcinkiewicz type and interpolation theory in the same form as in Lebesgue or Sobolev spaces. For this reason the passage in terms associated with the regular part of the pressure function is much more demanding than in [**56**]. The result concerning existence of a weak solution to the above problem is formulated in Theorem V.3.1. Chapter V is based on the result achieved in [**134, 135**] by Wróblewska-Kamińska.

In the above two problems the presence of a convective term $\mathrm{div}(\boldsymbol{u}\otimes\boldsymbol{u})$ enforces at least polynomial growth of tensor $\mathbf{S}$ with respect to $\mathbf{D}\boldsymbol{u}$. With these assumptions we are able to investigate only the case of shear thickening fluids. This motivates us to consider the generalized Stokes system:

$$\begin{aligned} \partial_t \boldsymbol{u} - \mathrm{div}_x \mathbf{S}(\mathbf{D}\boldsymbol{u}) + \nabla_x p = \boldsymbol{f} &\quad \text{in } (0,T)\times\Omega, \\ \mathrm{div}_x \boldsymbol{u} = 0 &\quad \text{in } (0,T)\times\Omega. \end{aligned} \tag{I.0.4}$$

In particular the considerations of the above problem, which the reader can find in Chapter VI, allow us to investigate the case of shear thinning fluids, whose viscosity decreases when the shear rate increases. Let us notice that if we assume that the flow is slow, the density is constant and so the system stated in (I.0.1) can be reduced to (I.0.4). The problem is considered in anisotropic Orlicz spaces. In the proof we need to provide the type of the Korn-Sobolev inequality for anisotropic Orlicz spaces when the Δ_2–condition is not satisfied. We show also that the closure of smooth functions with compact support with respect to two topologies is equal: the convergence of symmetric gradients in modular and in weak star topology in Orlicz space. Then we are able to give the formula for integration by parts. The existence of a weak solution to the problem (I.0.4) is stated in Theorem VI.1.1. The result of Chapter VI the reader can also find in [**76**] by Gwiazda, Świerczewska-Gwiazda and Wróblewska-Kamińska.

The last part of our research, namely Chapter VII is addressed to the theory of renormalized solutions to elliptic problems associated with the differential inclusion

$$\beta(\cdot,u) - \mathrm{div}\ (\boldsymbol{a}(\cdot,\nabla u) + \boldsymbol{F}(u)) \ni f,$$

where $f \in L^1(\Omega)$. The vector field $\boldsymbol{a}(\cdot,\cdot)$ is monotone in the second variable and satisfies a non-standard growth condition described by an x-dependent convex function, i.e.

$$\boldsymbol{a}(x,\boldsymbol{\xi})\cdot\boldsymbol{\xi} \geqslant c_a\{M^*(x,\boldsymbol{a}(x,\boldsymbol{\xi})) + M(x,\boldsymbol{\xi})\} - a_0(x) \tag{I.0.5}$$

for a.a. $x \in \Omega$ and all $\boldsymbol{\xi} \in \mathbb{R}^d$, where a_0 is some nonnegative integrable function. The above condition generalizes both $L^{p(x)}$ and classical Orlicz settings.

The concept of renormailzed solutions allows us to solve the problem of well-posedness under very general assumptions which do not provide existence of weak solutions. This notion was introduced by P.-L. Lions and DiPerna in [**44**] for the study of the Boltzmann equation. The concept was also applied to fluid mechanics models by P.-L. Lions, cf. [**91**] and plays a crucial role in existence and regularity theory of systems capturing density dependent flows.

The studies will be undertaken for the case of rather general growth conditions of the highest order nonlinear term. The results obtained in the frame of this thesis generalize the existing theory for equations with only L^1 integrable right-hand side. Up to our knowledge, growth and coercivity conditions for nonlinear term are more general than already known results. Namely we capture a wider class of operators by stating the problem in nonhomogeneous anisotropic Orlicz spaces. This is a natural generalization of numerous recent studies appearing on $L^{p(x)}$ spaces, which may be considered as a particular case of our framework. Applying the methods of renormalized solutions is crucial due to L^1 terms appearing in the equations. Our main result of this part, existence of a renormalized solution to (I.0.5) for any L^1-data f, the results on uniqueness of renormalized solutions (see Definition VII.2.3) and on existence of weak solutions (see Definition VII.2.1), are formulated in Theorem VII.3.1, Theorem VII.3.2 and in Proposition VII.3.3 respectively. Chapter VII is based on the joint work of Gwiazda, Wittbold, Wróblewska-Kamińska and Zimmermann [**79**].

For a detailed description of the above problems, the state of the art and motivation we refer the reader to Chapters IV, V, VI, VII respectively.

In order to present some of well known results concerning application of Orlicz space setting we recall some existing analytical results concerning the abstract parabolic problems in non-separable Orlicz spaces with zero Dirichlet boundary condition. Donaldson in [**46**] assumed that the nonlinear operator is an elliptic second-order, monotone operator in divergence form. The growth and coercivity conditions were more general than the standard growth conditions in L^p, namely the $\mathcal{N}$–function formulation was stated. Under the assumptions on the $\mathcal{N}-$function M: $\xi^2 << M(|\xi|)$ (i.e., ξ^2 grows essentially less rapidly than $M(|\xi|)$) and M^* satisfies the Δ_2-condition, existence result to parabolic equation was established. These restrictions on the growth of M were abandoned in [**50**].

The review paper [**97**] by Mustonen summarises the monotone-like mappings techniques in Orlicz and Orlicz–Sobolev spaces. The authors need essential modifications of such notions as: monotonicity, pseudomonotonicity, operators of type (M), (S_+), et al. The reason is that Orlicz–Sobolev spaces are not reflexive in general. Moreover, the nonlinear differential operators in divergence form with standard growth conditions are neither bounded nor everywhere defined.

One of the main problems in our considerations is that the $\triangle_2$–condition can not be satisfied and we lose many facilitating properties. An interesting obstacle here is the lack of the classical integration by parts formula, cf. [**65**, Section 4.1]. To extend it for the case of generalized Orlicz spaces we would essentially need that $C^\infty-$functions are dense in $L_M(Q)$ and $L_M(Q) = L_M(0,T;L_M(\Omega))$. The first one only holds if M satisfies the Δ_2-condition. The second one is not the case in Orlicz and generalized Orlicz spaces. We recall the proposition from [**46**] (although it is stated for Orlicz spaces with $M = M(|\boldsymbol{\xi}|)$).

Proposition I.0.1. *Let I be the time interval, $\Omega \subset \mathbb{R}^d$, $M = M(|\boldsymbol{\xi}|)$ an $\mathcal{N}$–function, $L_M(I\times\Omega), L_M(I;L_M(\Omega))$ the Orlicz spaces on $I\times\Omega$ and the vector valued Orlicz space on I respectively. Then*

$$L_M(I\times\Omega) = L_M(I;L_M(\Omega)),$$

if and only if there exist constants k_0, k_1 such that

$$k_0 M^{-1}(s)M^{-1}(r) \leqslant M^{-1}(sr) \leqslant k_1 M^{-1}(s)M^{-1}(r) \tag{I.0.6}$$

for every $s \geqslant 1/|I|$ and $r \geqslant 1/|\Omega|$.

One can conclude that (I.0.6) means that M must be equivalent to some power p, $1 < p < \infty$. Hence, if (I.0.6) should hold, very strong assumptions must be satisfied by M. Surely they would provide $L_M(Q)$ to be separable and reflexive.

CHAPTER II

A few words about notation

Within the whole thesis we will use the following notation: Ω stands for bounded domain in $\mathbb{R}^d$, $(0,T)$ is a time interval and $Q := (0,T) \times \Omega$.

The following notation for function spaces is introduced

$$
\begin{aligned}
\mathcal{D}(\Omega) &:= \{\varphi \in C^\infty(\Omega) \,|\, \varphi \text{ has compact support contained in } \Omega\} \\
\mathcal{V}(\Omega) &:= \{\varphi \in \mathcal{D}(\Omega) \,|\, \operatorname{div}\varphi = 0\}.
\end{aligned} \tag{II.0.7}
$$

Moreover, by $L^p, W^{1,p}$ we mean the standard Lebesgue and Sobolev spaces respectively and

$$
\begin{aligned}
L^2_{\operatorname{div}}(\Omega) &:= \text{ the closure of } \mathcal{V} \text{ w.r.t. the } \|\cdot\|_{L^2}\text{-norm} \\
W^{1,p}_{0,\operatorname{div}}(\Omega) &:= \text{ the closure of } \mathcal{V} \text{ w.r.t. the } \|\nabla(\cdot)\|_{L^p}\text{-norm.}
\end{aligned} \tag{II.0.8}
$$

Let $W^{-1,p'} = (W^{1,p}_0)^*$, $W^{-1,p'}_{\operatorname{div}} = (W^{1,p}_{0,\operatorname{div}})^*$. By p' we mean the conjugate exponent to p, namely $\frac{1}{p} + \frac{1}{p'} = 1$.

We will use $C_{\text{weak}}([0,T]; L^2(\Omega))$ in order to denote the space of functions $\boldsymbol{u} \in L^\infty(0,T; L^2(\Omega))$ which satisfy $(\boldsymbol{u}(t), \varphi) \in C([0,T])$ for all $\varphi \in L^2(\Omega)$.

If X is a Banach space of scalar functions, then X^d or $X^{d\times d}$ denotes the space of vector- or tensor-valued functions where each component belongs to X. The symbols $L^p(0,T;X)$ and $C([0,T];X)$ mean the standard Bochner spaces.

Finally, we recall that the Nikolskii space $N^{\alpha,p}(0,T;X)$ corresponding to the Banach space X and the exponents $\alpha \in (0,1)$ and $p \in [1,\infty]$ is given by

$$
N^{\alpha,p}(0,T;X) := \{f \in L^p(0,T;X) \ : \ \sup_{0<h<T} h^{-\alpha} \|\tau_h f - f\|_{L^p(0,T-h;X)} < \infty\},
$$

where $\tau_h f(t) = f(t+h)$ for a.a. $t \in [0, T-h]$.

By (a,b) we mean $\int_\Omega a(x)\cdot b(x)dx$ and $\langle a, b\rangle$ denotes the duality pairing.

By "$\cdot$" we denote the scalar product of two vectors, i.e.

$$
\boldsymbol{\xi} \cdot \boldsymbol{\eta} = \sum_{i=1}^d \xi_i \eta_i
$$

for $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$ and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_d) \in \mathbb{R}^d$ and ":" stands for the scalar product of two tensors, i.e.

$$
\boldsymbol{\xi} : \boldsymbol{\eta} = \sum_{i,j=1}^d \xi_{i,j} \eta_{i,j}
$$

for $\boldsymbol{\xi} = [\xi_{i,j}]_{i=1,\ldots,d,\, j=1,\ldots,d} \in \mathbb{R}^{d\times d}$ and $\boldsymbol{\eta} = [\eta_{i,j}]_{i=1,\ldots,d,\, j=1,\ldots,d} \in \mathbb{R}^{d\times d}$.

CHAPTER III

Orlicz spaces

III.1. Notation

In the following chapter we introduce the notation and present some properties of Orlicz spaces. Since within the whole thesis we use various generalizations of Orlicz spaces: isotropic and anisotropic Orlicz spaces, Musielak-Orlicz spaces, we start with basic definition of an $\mathcal{N}$–function and then generalize it.

Definition III.1.1. A function $M : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *an isotropic* $\mathcal{N}$*−function* if it is a continuous, real-valued, non-negative, convex function, which has superlinear growth near zero and infinity, i.e., $\lim_{\tau\to 0} \frac{M(\tau)}{\tau} = 0$ and $\lim_{\tau\to\infty} \frac{M(\tau)}{\tau} = \infty$, and $M(\tau) = 0$ if and only if $\tau = 0$.

Definition III.1.2. The *complementary function* M^* to a function M is defined by

$$M^*(\varsigma) = \sup_{\tau\in\mathbb{R}_+} (\tau\varsigma - M(\tau))$$

for $\varsigma \in \mathbb{R}_+$.

Definition III.1.3. A function $M : \mathbb{R}^n \to \mathbb{R}_+$ is said to be *an anisotropic* $\mathcal{N}$*−function* if it is a continuous, real-valued, non-negative, convex function, which has superlinear growth near zero and infinity, i.e., $\lim_{|\boldsymbol{\xi}|\to 0} \frac{M(\boldsymbol{\xi})}{|\boldsymbol{\xi}|} = 0$ and $\lim_{|\boldsymbol{\xi}|\to\infty} \frac{M(\boldsymbol{\xi})}{|\boldsymbol{\xi}|} = \infty$, $M(-\boldsymbol{\xi}) = M(\boldsymbol{\xi})$ and $M(\boldsymbol{\xi}) = 0$ if and only if $\boldsymbol{\xi} = 0$.

Definition III.1.4. The *complementary function* M^* to an anisotropic $\mathcal{N}$–function function M is defined by

$$M^*(\boldsymbol{\xi}) = \sup_{\boldsymbol{\eta}\in\mathbb{R}^n} (\boldsymbol{\eta} \cdot \boldsymbol{\xi} - M(\boldsymbol{\xi}))$$

for $\boldsymbol{\xi} \in \mathbb{R}^n$.

Definition III.1.5. Let Ω be a bounded domain in $\mathbb{R}^d$. A function $M : \Omega\times\mathbb{R}^n \to \mathbb{R}_+$ is said to be *a generalized* $\mathcal{N}$*−function* if it satisfies the following conditions

(1) M is a Carathéodory function such that $M(x, \boldsymbol{\xi}) = M(x, -\boldsymbol{\xi})$ a.e. in Ω and $M(x, \boldsymbol{\xi}) = 0$ if and only if $\boldsymbol{\xi} = 0$,
(2) $M(x, \boldsymbol{\xi})$ is a convex function w.r.t. $\boldsymbol{\xi}$,
(3)

$$\lim_{|\boldsymbol{\xi}|\to 0} \frac{M(x, \boldsymbol{\xi})}{|\boldsymbol{\xi}|} = 0 \quad \text{for every } x \in \Omega, \tag{III.1.1}$$

(4)

$$\lim_{|\boldsymbol{\xi}|\to\infty} \frac{M(x,\boldsymbol{\xi})}{|\boldsymbol{\xi}|} = \infty \quad \text{for every } x \in \Omega. \tag{III.1.2}$$

Definition III.1.6. The *complementary function* M^* to a generalized $\mathcal{N}$–function M is defined by

$$M^*(x,\boldsymbol{\xi}) = \sup_{\boldsymbol{\xi}\in\mathbb{R}^n} (\boldsymbol{\xi}\cdot\boldsymbol{\eta} - M(x,\boldsymbol{\xi})) \tag{III.1.3}$$

for $\boldsymbol{\eta} \in \mathbb{R}^n$, $x \in \Omega$.

Remark III.1.7. Within the thesis we use two forms of a generalized $\mathcal{N}$–function, depending on the considered problem, i.e. $M(x,\boldsymbol{\xi}) : \Omega \times \mathbb{R}^{d\times d}_{\text{sym}} \to \mathbb{R}$ and $M(x,\boldsymbol{\xi}) : \Omega \times \mathbb{R}^d \to \mathbb{R}$.

The complementary function M^* is also an $\mathcal{N}$–function (see [**117**]).

Let I be a time interval, $\Omega \subset \mathbb{R}^d$ be a bounded set and $Q = I\times\Omega$. The *generalized Orlicz class* $\mathcal{L}_M(Q;\mathbb{R}^n)$ is the set of all measurable functions $\boldsymbol{\xi} : Q \to \mathbb{R}^n$ such that

$$\int_Q M(x,\boldsymbol{\xi}(t,x))\,\mathrm{d}x\mathrm{d}t < \infty.$$

Note that $\mathcal{L}_M(Q;\mathbb{R}^n)$ is a convex set and it need not be a linear space.

Let us denote m^* such that

$$m^*(r) = \operatorname*{ess\,inf}_{x\in\Omega} \inf_{\boldsymbol{\xi}\in\mathbb{R}^n,|\boldsymbol{\xi}|=r} M^*(x,\boldsymbol{\xi}).$$

and let us assume that m^* is an $\mathcal{N}$– function and there exists an $\mathcal{N}$–function $m = m(|\boldsymbol{\xi}|)$ complementary to m^*. Then we have $m^*(|\boldsymbol{\xi}|) \leqslant M^*(x,\boldsymbol{\xi})$ and $M(x,\boldsymbol{\xi}) \leqslant m(|\boldsymbol{\xi}|)$. Therefore M maps bounded sets into bounded sets, which shows that

$$L^\infty(Q;\mathbb{R}^n) \subseteq \mathcal{L}_M(Q;\mathbb{R}^n). \tag{III.1.4}$$

In order to provide existence of such functions m^* and m, which are $\mathcal{N}$–functions, it is enough to assume that M and M^* satisfy (III.1.1-III.1.2) uniformly w.r.t. $x \in \Omega$, namely M satisfies

$$\lim_{|\boldsymbol{\xi}|\to 0} \sup_{x\in\Omega} \frac{M(x,\boldsymbol{\xi})}{|\boldsymbol{\xi}|} = 0, \qquad \lim_{|\boldsymbol{\xi}|\to\infty} \inf_{x\in\Omega} \frac{M(x,\boldsymbol{\xi})}{|\boldsymbol{\xi}|} = \infty$$

and the same assumption concerns M^*.

The *generalized Orlicz space* (or Musielak-Orlicz space) $L_M(Q;\mathbb{R}^n)$ is defined as the set of all measurable functions $\boldsymbol{\xi} : Q \to \mathbb{R}^n$ which satisfy

$$\int_Q M(x,\lambda\boldsymbol{\xi}(t,x))\,\mathrm{d}x\mathrm{d}t \to 0 \quad \text{as } \lambda \to 0.$$

Remark III.1.8. If we consider an isotropic or an anisotropic $\mathcal{N}$–function, then in analogous way we can define respectively *isotropic and anisotropic Orlicz spaces*. Obviously corresponding definitions and properties which are stated below can be rewritten for less general case of isotropic and anisotropic $\mathcal{N}$–functions.

Definition III.1.9. Let $\boldsymbol{\xi} \in L_M(Q;\mathbb{R}^n)$. Then the Luxemburg norm is defined by

$$\|\boldsymbol{\xi}\|_M = \inf\left\{\lambda > 0 \mid \int_Q M\left(x, \frac{\boldsymbol{\xi}(t,x)}{\lambda}\right) \mathrm{d}x\mathrm{d}t \leqslant 1\right\}. \tag{III.1.5}$$

Definition III.1.10. Let $\boldsymbol{\xi} \in L_M(Q;\mathbb{R}^n)$. Then the Orlicz norm is defined by

$$\|\boldsymbol{\xi}\|_M^o = \sup\left\{\int_Q \boldsymbol{\xi}\cdot\boldsymbol{\eta}\,\mathrm{d}x\mathrm{d}t \mid \boldsymbol{\eta} \in \mathcal{L}_{M^*},\ \int_Q M(x,\boldsymbol{\eta}(t,x))\,\mathrm{d}x\mathrm{d}t \leqslant 1\right\}. \tag{III.1.6}$$

Orlicz and Luxemburg norm are equivalent. The proof in a less general case, namely for $M(x,\boldsymbol{\xi}) := M(x,|\boldsymbol{\xi}|)$, can be found in [**96**].

In general, $L_M(Q;\mathbb{R}^n)$ is neither separable nor reflexive. Finally, because of the superlinear growth of M (see (III.1.2)), there holds

$$L_M(Q;\mathbb{R}^n) \subseteq L^1(Q;\mathbb{R}^n). \tag{III.1.7}$$

Let us denote by $E_M(Q;\mathbb{R}^n)$ the closure of all bounded measurable functions defined on Q with respect to the Luxemburg norm $\|\cdot\|_{M,Q}$. It turns out that $E_M(Q;\mathbb{R}^n)$ is the largest linear space contained in the Orlicz class $\mathcal{L}_M(Q;\mathbb{R}^n)$ such that

$$E_M(Q;\mathbb{R}^n) \subseteq \mathcal{L}_M(Q;\mathbb{R}^n) \subseteq L_M(Q;\mathbb{R}^n),$$

where the inclusion is in general strict.

The space $E_M(Q;\mathbb{R}^n)$ is separable and $C_0^\infty(Q;\mathbb{R}^d)$ is dense in $E_M(Q;\mathbb{R}^n)$.

Theorem III.1.11. *The generalized Orlicz space is a Banach space with respect to the Orlicz norm (III.1.6) or the equivalent Luxemburg norm (III.1.5).*

Proof. We will prove the completeness w.r.t Orlicz norm. Let $\{\boldsymbol{\xi}_j\}_{j=1}^\infty$ be a Cauchy sequence in $L_M(Q;\mathbb{R}^n)$ such that for all $\varepsilon > 0$ there exists $J_\varepsilon > 0$ such that

$$\sup\left\{\int_Q \boldsymbol{\eta}\cdot(\boldsymbol{\xi}_i - \boldsymbol{\xi}_j)\,\mathrm{d}x\mathrm{d}t \mid \boldsymbol{\eta} \in \mathcal{L}_{M^*}(Q), \int_Q M^*(x,\boldsymbol{\eta})\,\mathrm{d}x\mathrm{d}t \leqslant 1\right\} < \varepsilon \tag{III.1.8}$$

holds for all $i,j > J_\varepsilon$. Let $\lambda > 0$ be such that

$$\int_Q M^*(x,\boldsymbol{\eta})\,\mathrm{d}x\mathrm{d}t \leqslant 1 \text{ for all } \boldsymbol{\eta} \in L^\infty(Q;\mathbb{R}^n), \|\boldsymbol{\eta}\|_\infty \leqslant \lambda.$$

By plugging

$$\boldsymbol{\eta} = \begin{cases} \lambda\frac{\boldsymbol{\xi}_i - \boldsymbol{\xi}_j}{|\boldsymbol{\xi}_i - \boldsymbol{\xi}_j|} & \text{if } \boldsymbol{\xi}_i \neq \boldsymbol{\xi}_j \\ 0 & \text{otherwise} \end{cases}$$

into (III.1.8) we obtain

$$\int_\Omega |\boldsymbol{\xi}_i - \boldsymbol{\xi}_j|\,\mathrm{d}x\mathrm{d}t \leqslant \frac{\varepsilon}{\lambda} \text{ for all } i,j \geqslant J_\varepsilon.$$

Therefore $\{\boldsymbol{\xi}_j\}_{j=1}^\infty$ is a Cauchy sequence in $L^1(Q;\mathbb{R}^n)$. Hence, by the Fatou lemma

$$\int_Q |(\boldsymbol{\xi} - \boldsymbol{\xi}_j)\cdot\boldsymbol{\eta}|\,\mathrm{d}x\mathrm{d}t = \int_Q \lim_{i\to\infty} |(\boldsymbol{\xi}_i - \boldsymbol{\xi}_j)\cdot\boldsymbol{\eta}|\,\mathrm{d}x\mathrm{d}t \leqslant \liminf_{i\to\infty}\int_Q |(\boldsymbol{\xi}_i - \boldsymbol{\xi}_j)\cdot\boldsymbol{\eta}|\,\mathrm{d}x\mathrm{d}t < \varepsilon.$$

Thus $\boldsymbol{\xi} \in L_M(Q;\mathbb{R}^n)$ and $\|\boldsymbol{\xi} - \boldsymbol{\xi}_j\|_M \to 0$ with $j \to \infty$. This completes the proof. □

Proposition III.1.12 (Fenchel-Young inequality). *Let M be an $\mathcal{N}-$function and M^* the complementary to M. Then the following inequality is satisfied*

$$|\boldsymbol{\xi}\cdot\boldsymbol{\eta}| \leqslant M(x,\boldsymbol{\xi}) + M^*(x,\boldsymbol{\eta}) \tag{III.1.9}$$

for all $\boldsymbol{\xi},\boldsymbol{\eta}\in\mathbb{R}^n$ and a.a. $x\in\Omega$.

Lemma III.1.13 (Generalized Hölder inequality). *Let M be an $\mathcal{N}$–function and M^* its complementary. Then*

$$\left|\int_\Omega \boldsymbol{\xi}\cdot\boldsymbol{\eta}\,\mathrm{d}x\right| \leqslant 2\|\boldsymbol{\xi}\|_M\|\boldsymbol{\eta}\|_{M^*}, \tag{III.1.10}$$

where $\boldsymbol{\xi}\in L_M(Q;\mathbb{R}^n)$ and $\boldsymbol{\eta}\in L_{M^*}(Q;\mathbb{R}^n)$.

Proof. From Proposition III.1.12 by putting $\boldsymbol{\xi} = \frac{\boldsymbol{\xi}(t,x)}{\|\boldsymbol{\xi}\|_M}$, $\boldsymbol{\eta} = \frac{\boldsymbol{\eta}(t,x)}{\|\boldsymbol{\eta}\|_{M^*}}$ we obtain

$$\int_Q \left|\frac{\boldsymbol{\xi}(t,x)}{\|\boldsymbol{\xi}\|_M}\frac{\boldsymbol{\eta}(t,x)}{\|\boldsymbol{\eta}\|_{M^*}}\right| \mathrm{d}x\mathrm{d}t \leqslant \int_Q M\left(x,\frac{\boldsymbol{\xi}(t,x)}{\|\boldsymbol{\xi}\|_M}\right)\mathrm{d}x\mathrm{d}t + \int_Q M^*\left(x,\frac{\boldsymbol{\eta}(t,x)}{\|\boldsymbol{\eta}\|_{M^*}}\right)\mathrm{d}x\mathrm{d}t \leqslant 2.$$

We finish the proof of (III.1.10) by multiplying the above inequality by $\|\boldsymbol{\xi}\|_M\|\boldsymbol{\eta}\|_{M^*}$. □

Theorem III.1.14. *The space $L_{M^*}(Q;\mathbb{R}^n)$ is a dual space of $E_M(Q;\mathbb{R}^n)$, namely $(E_M(Q;\mathbb{R}^n))^* = L_{M^*}(Q;\mathbb{R}^n)$.*

Before we prove Theorem III.1.14, we will state the following

Lemma III.1.15. *Let $\boldsymbol{\eta}\in L_{M^*}(Q;\mathbb{R}^n)$. The linear functional $F_{\boldsymbol{\eta}}$ defined by*

$$F_{\boldsymbol{\eta}}(\boldsymbol{\xi}) = \int_Q \boldsymbol{\xi}\cdot\boldsymbol{\eta}\,\mathrm{d}x\mathrm{d}t \tag{III.1.11}$$

belongs to the space $(E_M(Q;\mathbb{R}^n))^*$ and its norm in that space fulfills

$$\|F_{\boldsymbol{\eta}}\| \leqslant 2\|\boldsymbol{\eta}\|_{M^*}. \tag{III.1.12}$$

Proof. It follows from Hölder inequality (III.1.10) that

$$|F_{\boldsymbol{\eta}}(\boldsymbol{\xi})| \leqslant 2\|\boldsymbol{\xi}\|_M\|\boldsymbol{\eta}\|_{M^*}$$

holds for all $\boldsymbol{\xi}\in L_M(Q;\mathbb{R}^n)$ confirming the inequality (III.1.12). □

Proof. (of the Theorem III.1.14) Lemma III.1.15 has already shown that any element $\boldsymbol{\eta}\in L_{M^*}(Q;\mathbb{R}^n)$ defines a bounded linear functional $F_{\boldsymbol{\eta}}$ on $E_M(Q;\mathbb{R}^n)$ which is given by (III.1.11). It remains to show that every bounded linear functional on $E_M(Q;\mathbb{R}^n)$ is of the form $F_{\boldsymbol{\eta}}$ for any $\boldsymbol{\eta}\in L_{M^*}(Q;\mathbb{R}^n)$.

Let $F\in(E_M(Q;\mathbb{R}^n))^*$. We define a measure λ on the measurable subsets S of Q

$$\lambda(S) = F(\boldsymbol{\tau}\mathbb{I}_S)$$

where $\mathbb{I}_S$ denotes the characteristic function of S, $\boldsymbol{\tau}\in\mathbb{R}^n$, $|\boldsymbol{\tau}| = 1$. Let

$$A(r) = \sup_{x\in\Omega,|\boldsymbol{\xi}|=r} M(x,\boldsymbol{\xi})$$

be an auxiliary function and $r \in [0, \infty)$. This function is required to generalise the approach presented in [**1**]. Since

$$\int_Q M\left(x, A^{-1}\left(\frac{1}{|S|}\right)\mathbb{I}_S\boldsymbol{\tau}\right) \mathrm{d}x\mathrm{d}t \leqslant \int_S \sup_{(t,x)\in S} M\left(x, A^{-1}\left(\frac{1}{|S|}\right)\boldsymbol{\tau}\right) \mathrm{d}x\mathrm{d}t \leqslant \int_S \frac{1}{|S|} \leqslant 1, \tag{III.1.13}$$

we have

$$|\lambda(S)| = |F(\boldsymbol{\tau}\mathbb{I}_S)| \leqslant \|F\|\|\boldsymbol{\tau}\mathbb{I}_S\|_M \leqslant \frac{c\|F\|}{A^{-1}(1/|S|)}. \tag{III.1.14}$$

Since the right-hand side of (III.1.14) converges to zero when $|S|$ converges to zero, the measure λ is absolutely continuous w.r.t. Lebesgue measure. By Radon-Nikodym and Riesz theorems, cf. [**128**], λ can be expressed in the form

$$\lambda(S) = \int_S \boldsymbol{\eta}(t,x)\,\mathrm{d}x\mathrm{d}t$$

for some $\boldsymbol{\eta}$ integrable on Q. Therefore

$$F(\boldsymbol{\xi}) = \int_\Omega \boldsymbol{\xi}\cdot\boldsymbol{\eta}\,\mathrm{d}x\mathrm{d}t$$

holds for measurable bounded functions $\boldsymbol{\xi}$.

If $\boldsymbol{\xi} \in E_M(Q;\mathbb{R}^n)$ we can find a sequence of measurable functions $\boldsymbol{\xi}_i$ which converges a.e. to $\boldsymbol{\xi}$ and satisfies $|\boldsymbol{\xi}_i| \leqslant |\boldsymbol{\xi}|$ on Q. Since $|\boldsymbol{\xi}_i\cdot\boldsymbol{\eta}|$ converges a.e to $|\boldsymbol{\xi}\cdot\boldsymbol{\eta}|$, Fatou's lemma yields

$$\left|\int_Q \boldsymbol{\xi}\cdot\boldsymbol{\eta}\,\mathrm{d}x\mathrm{d}t\right| \leqslant \int_Q |\boldsymbol{\xi}\cdot\boldsymbol{\eta}|\,\mathrm{d}x\mathrm{d}t \leqslant \liminf_{i\to\infty}\int_Q |\boldsymbol{\xi}_i\cdot\boldsymbol{\eta}|\,\mathrm{d}x\mathrm{d}t \leqslant \liminf_{i\to\infty} 2\|\boldsymbol{\xi}_i\|_M\|\boldsymbol{\eta}\|_{M^*} \leqslant 2\|\boldsymbol{\xi}\|_M\|\boldsymbol{\eta}\|_{M^*}.$$

Hence the linear functional

$$F_{\boldsymbol{\eta}}(\boldsymbol{\xi}) = \int_Q \boldsymbol{\xi}\cdot\boldsymbol{\eta}\,dx$$

is bounded on $E_M(Q)$ when $\boldsymbol{\eta} \in L_{M^*}(Q)$. Since $F_{\boldsymbol{\eta}}$ and F achieve the same values on the measurable, simple functions (a set which is dense in $E_M(Q)$) they agree on $E_M(Q)$ and the proof is completed. □

The functional

$$\varrho(\boldsymbol{\xi}) = \int_Q M(x, \boldsymbol{\xi}(x))\,\mathrm{d}x\mathrm{d}t$$

is a modular in the space of measurable functions $\boldsymbol{\xi} : Q \to \mathbb{R}^n$ in the sense of [**87**, p. 208].

A sequence $\{\boldsymbol{z}^j\}_{j=1}^\infty$ converges *modularly* to $\boldsymbol{z}$ in $L_M(Q;\mathbb{R}^n)$ if there exists $\lambda > 0$ such that

$$\int_Q M\left(x, \frac{\boldsymbol{z}^j - \boldsymbol{z}}{\lambda}\right) \mathrm{d}x\mathrm{d}t \to 0 \quad \text{as } j \to \infty.$$

We will write $\boldsymbol{z}^j \xrightarrow{M} \boldsymbol{z}$ for the modular convergence in $L_M(Q;\mathbb{R}^n)$.

Definition III.1.16. We say that an $\mathcal{N}-$function M satisfies the Δ_2*–condition* if for some nonnegative, integrable on Ω function g_M and a constant holds $C_M > 0$

$$M(x, 2\boldsymbol{\xi}) \leqslant C_M M(x,\boldsymbol{\xi}) + g_M(x) \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^n \text{ and a.a. } x \in \Omega. \tag{III.1.15}$$

Proposition III.1.17. *If an $\mathcal{N}$–function M does not satisfy the Δ_2–condition, then*
- *The space $L_M(Q;\mathbb{R}^n)$ is not separable.*
- *The space $L_M(Q;\mathbb{R}^n)$ is not reflexive.*
- *The space of smooth functions C^∞ is not dense in the space $L_M(Q;\mathbb{R}^n)$.*

The proof can be found in [**1**]) for the case of isotropic $\mathcal{N}$–functions.

Proposition III.1.18. *In particular, if (III.1.15) holds, then*

$$E_M(Q;\mathbb{R}^n) = L_M(Q;\mathbb{R}^n)$$

(see [**1, 87, 118**]*).*

The Δ_2–condition is rather restrictive. Nevertheless, for a measurable function $p : \Omega \to (1,\infty)$ the $L^{p(x)}$ spaces (generalized Lebesgue spaces) are included in the generalized Orlicz spaces framework with $M(x,\xi) = |\xi|^{p(x)}$ and with the classical assumption $1 < \operatorname{ess\,inf}_{x\in\Omega} p(x) \leqslant p(x) \leqslant \operatorname{ess\,sup}_{x\in\Omega} p(x) < \infty$ both $|\cdot|^{p(x)}$ and $|\cdot|^{p'(x)}$, where $p'(x) = p(x)/(p(x)-1)$ a.e. in Ω, satisfy the Δ_2–condition.

More information for the case of x-dependent generalized $\mathcal{N}-$function can be found in [**117, 118, 132**] and for less general $\mathcal{N}-$functions in [**87, 96**].

III.2. Properties and useful facts

Let us recall some general properties of Orlicz spaces, see e.g. [**96**] and technical facts which can be found also in [**72, 75, 133**].

We recall an analogue to the Vitali's lemma, however for the modular convergence instead of the strong convergence in L^p.

Lemma III.2.1. *Let $\boldsymbol{z}^j : Q \to \mathbb{R}^n$ be a measurable sequence. Then $\boldsymbol{z}^j \xrightarrow{M} \mathbf{z}$ in $L_M(Q;\mathbb{R}^n)$ modularly if and only if $\boldsymbol{z}^j \to \boldsymbol{z}$ in measure and there exists some $\lambda > 0$ such that the sequence $\{M(\cdot,\lambda\boldsymbol{z}^j)\}$ is uniformly integrable, i.e.,*

$$\lim_{R\to\infty}\left(\sup_{j\in\mathbb{N}}\int_{\{(t,x):|M(x,\lambda\boldsymbol{z}^j)|\geqslant R\}} M(x,\lambda\boldsymbol{z}^j)\,\mathrm{d}x\mathrm{d}t\right) = 0.$$

PROOF. Note that $\boldsymbol{z}^j \to \boldsymbol{z}$ in measure if and only if $M\left(\cdot,\frac{\boldsymbol{z}^j-\boldsymbol{z}}{\lambda}\right) \to 0$ in measure for all $\lambda > 0$. Moreover the convergence $\boldsymbol{z}^j \to \boldsymbol{z}$ in measure implies that for all measurable sets $A \subset Q$ it holds

$$\liminf_{j\to\infty}\int_A M(x,\boldsymbol{z}^j)\,\mathrm{d}x\mathrm{d}t \geqslant \int_A M(x,\boldsymbol{z})\,\mathrm{d}x\mathrm{d}t.$$

Note also that the convexity of M implies

$$\int_A M\left(x, \frac{\boldsymbol{z}^j - \boldsymbol{z}}{\lambda}\right) \mathrm{d}x\mathrm{d}t \leqslant \int_A M\left(x, \frac{\boldsymbol{z}^j}{2\lambda}\right) \mathrm{d}x\mathrm{d}t + \int_A M\left(x, \frac{\boldsymbol{z}}{2\lambda}\right) \mathrm{d}x\mathrm{d}t.$$

Hence by the classical Vitali's lemma for $f^j(x) = M\left(x, \frac{\boldsymbol{z}^j - \boldsymbol{z}}{\lambda}\right)$ we obtain that $f^j \to 0$ strongly in $L^1(Q)$. □

Lemma III.2.2. *Let M be an $\mathcal{N}-$function such that*

$$\lim_{|\boldsymbol{\xi}| \to \infty} \inf_{x \in \Omega} \frac{M(x, \boldsymbol{\xi})}{|\boldsymbol{\xi}|} = \infty$$

and for all $j \in \mathbb{N}$ let $\int_Q M(x, \boldsymbol{z}^j) \,\mathrm{d}x\mathrm{d}t \leqslant c$. Then the sequence $\{\boldsymbol{z}^j\}_{j=1}^{\infty}$ is uniformly integrable.

Proof. Let us define $\delta(R) = \min_{|\boldsymbol{\xi}|=R} \inf_{x\in\Omega} \frac{M(x,\boldsymbol{\xi})}{|\boldsymbol{\xi}|}$. Then for all $j \in \mathbb{N}$ it holds

$$\int_{\{(t,x):|\boldsymbol{z}^j(t,x)| \geqslant R\}} M(x, \boldsymbol{z}^j(t,x)) \,\mathrm{d}x\mathrm{d}t \geqslant \delta(R) \int_{\{(t,x):|\boldsymbol{z}^j(t,x)| \geqslant R\}} |\boldsymbol{z}^j(t,x)| \,\mathrm{d}x\mathrm{d}t.$$

Since the left-hand side is bounded, then we obtain

$$\sup_{j \in \mathbb{N}} \int_{\{(t,x):|\boldsymbol{z}^j(t,x)| \geqslant R\}} |\boldsymbol{z}^j(t,x)| dxdt \leqslant \frac{c}{\delta(R)}.$$

Using condition (III.1.2) we obtain uniform integrability. □

Proposition III.2.3. *Let M be an $\mathcal{N}-$function and M^* its complementary function. Suppose that the sequences $\boldsymbol{\psi}^j : Q \to \mathbb{R}^n$ and $\boldsymbol{\phi}^j : Q \to \mathbb{R}^n$ are uniformly bounded in $L_M(Q;\mathbb{R}^n)$ and $L_{M^*}(Q;\mathbb{R}^n)$ respectively. Moreover $\boldsymbol{\psi}^j \xrightarrow{M} \boldsymbol{\psi}$ modularly in $L_M(Q;\mathbb{R}^n)$ and $\boldsymbol{\phi}^j \xrightarrow{M^*} \boldsymbol{\phi}$ modularly in $L_{M^*}(Q;\mathbb{R}^n)$. Then $\boldsymbol{\psi}^j \cdot \boldsymbol{\phi}^j \to \boldsymbol{\psi} \cdot \boldsymbol{\phi}$ strongly in $L^1(Q)$.*

Proof. Due to Lemma III.2.1 the modular convergence of $\{\boldsymbol{\psi}^j\}$ and $\{\boldsymbol{\phi}^j\}$ implies the convergence in measure of these sequences and consequently also the convergence in measure of the product. Hence it is sufficient to show the uniform integrability of $\{\boldsymbol{\psi}^j \cdot \boldsymbol{\phi}^j\}$. Notice that it is equivalent with the uniform integrability of the term $\left\{\frac{\boldsymbol{\psi}^j}{\lambda_1} \cdot \frac{\boldsymbol{\phi}^j}{\lambda_2}\right\}$ for any $\lambda_1, \lambda_2 > 0$. The assumptions of the proposition give that there exist some $\lambda_1, \lambda_2 > 0$ such that the sequences

$$\left\{M\left(x, \frac{\boldsymbol{\psi}^j}{\lambda_1}\right)\right\} \quad \text{and} \quad \left\{M^*\left(x, \frac{\boldsymbol{\phi}^j}{\lambda_2}\right)\right\}$$

are uniformly integrable. Hence let us use the same constants and estimate with the help of the Fenchel-Young inequality

$$\left|\frac{\boldsymbol{\psi}^j}{\lambda_1} \cdot \frac{\boldsymbol{\phi}^j}{\lambda_2}\right| \leqslant M\left(x, \frac{\boldsymbol{\psi}^j}{\lambda_1}\right) + M^*\left(x, \frac{\boldsymbol{\phi}^j}{\lambda_2}\right).$$

Obviously the uniform integrability of the right-hand side provides the uniform integrability of the left-hand side and this yields the assertion. □

Proposition III.2.4. *Let ϱ^j be a standard mollifier, i.e., $\varrho \in C^\infty(\mathbb{R})$, ϱ has a compact support and $\int_{\mathbb{R}} \varrho(\tau)\, d\tau = 1, \varrho(t) = \varrho(-t)$. We define $\varrho^j(t) = j\varrho(jt)$. Moreover let $*$ denote a convolution in the variable t. Then for any function $\boldsymbol{\psi} : Q \to \mathbb{R}^d$ such that $\boldsymbol{\psi} \in L^1(Q;\mathbb{R}^n)$ it holds*

$$(\varrho^j * \boldsymbol{\psi})(t,x) \to \boldsymbol{\psi}(t,x) \quad \textit{in measure.}$$

Proof. For a.a. $x \in \Omega$ the function $\boldsymbol{\psi}(\cdot,x) \in L^1(0,T)$ and $\varrho^j * \boldsymbol{\psi}(\cdot,x) \to \boldsymbol{\psi}(\cdot,x)$ in $L^1(0,T)$ and hence $\varrho^j * \boldsymbol{\psi} \to \boldsymbol{\psi}$ in measure on the set $(0,T)\times\Omega$. □

Proposition III.2.5. *Let ϱ^j be defined as in Proposition III.2.4, let M be an $\mathcal{N}$−function and $\boldsymbol{\psi} : Q \to \mathbb{R}^n$ be such that $\boldsymbol{\psi} \in \mathcal{L}_M(Q;\mathbb{R}^n)$. Then the sequence $\{M(x, \varrho^j * \boldsymbol{\psi})\}$ is uniformly integrable.*

Proof. We start with an abstract fact concerning uniform integrability. Namely, the following two conditions are equivalent for any measurable sequence $\{z^j\}$

(a) $\forall \varepsilon > 0 \quad \exists \delta > 0 : \quad \sup\limits_{j\in\mathbb{N}} \sup\limits_{|A|\leqslant\delta} \int_A |z^j(x)|\, \mathrm{d}x\mathrm{d}t \leqslant \varepsilon,$

(b) $\forall \varepsilon > 0 \quad \exists \delta > 0 : \quad \sup\limits_{j\in\mathbb{N}} \int_Q \left| |z^j(x)| - \frac{1}{\sqrt{\delta}} \right|_+ \mathrm{d}x\mathrm{d}t \leqslant \varepsilon,$

where we use the notation

$$|\xi|_+ = \max\{0, \xi\}.$$

The implication $(a) \Rightarrow (b)$ is obvious. To show that also $(b) \Rightarrow (a)$ holds let us estimate

$$\begin{aligned}\sup_{j\in\mathbb{N}} \sup_{|A|\leqslant\delta} \int_A |z^j|\, \mathrm{d}x\mathrm{d}t &\leqslant \sup_{|A|\leqslant\delta} |A| \cdot \frac{1}{\sqrt{\delta}} + \sup_{j\in\mathbb{N}} \int_Q \left| |z^j| - \frac{1}{\sqrt{\delta}} \right|_+ \mathrm{d}x\mathrm{d}t \\ &\leqslant \sqrt{\delta} + \sup_{j\in\mathbb{N}} \int_Q \left| |z^j| - \frac{1}{\sqrt{\delta}} \right|_+ \mathrm{d}x\mathrm{d}t.\end{aligned}$$

Notice that since M is a convex function, then the following inequality holds for all $\delta > 0$

$$\int_Q \left| M(x, \boldsymbol{\psi}) - \frac{1}{\sqrt{\delta}} \right|_+ \mathrm{d}x\mathrm{d}t \geqslant \int_Q \left| M(x, \varrho^j * \boldsymbol{\psi}) - \frac{1}{\sqrt{\delta}} \right|_+ \mathrm{d}x\mathrm{d}t. \tag{III.2.1}$$

Finally, since $\boldsymbol{\psi} \in \mathcal{L}_M(Q;\mathbb{R}^n)$, then also $\int_Q |M(x,\boldsymbol{\psi}) - \frac{1}{\sqrt{\delta}}|_+ \,\mathrm{d}x\mathrm{d}t$ is finite and hence taking supremum over $j \in \mathbb{N}$ in (III.2.1) we prove the assertion. □

Remark III.2.6. The same proofs for Propositions III.2.4 and III.2.5 work if instead of a standard mollifier ϱ^j we will take

$$\tilde{\sigma}_h^+ = \frac{1}{h} \mathbb{1}(\tau)_{[0,h]} \quad \text{or} \quad \tilde{\sigma}_h^- = \frac{1}{h} \mathbb{1}(\tau)_{[-h,0]}$$

with $h > 0$.

Lemma III.2.7. *Let Ω be a bounded domain, $(0,T)$ be time interval, $Q = (0,T)\times\Omega$ and M be an isotropic $\mathcal{N}$−function satisfying Definition III.1.1 s.t. $M((|\cdot|)^{1/p})$ is* *convex. If $f(t,x) \in L_M(Q)$, i.e. $\|f\|_{M,Q} < \infty$, then $f \in L_M(0,T;L^p(\Omega))$, i.e.*

$$\|f\|_{L_M(0,T;L^p(\Omega))} := \inf\left\{\lambda > 0 : \quad \int_0^T M\left(\frac{\|f(t,\cdot)\|_{L^p(\Omega)}}{\lambda}\right)\mathrm{d}t \leqslant 1\right\} < \infty$$

PROOF. If $f \in L_M(Q)$, then there exists $0 < \lambda < \infty$ such that

$$\int_0^T \int_\Omega M\left(\frac{|\Omega|^{\frac{1}{p}}|f(t,x)|}{\lambda}\right)\mathrm{d}x\mathrm{d}t \leqslant 1.$$

Employing the Jensen inequality, using the non-negativity, the convexity of M and $M((|\cdot|)^{1/p})$, and that $M(0) = 0$ we infer the following

$$\int_0^T M\left(\frac{1}{\lambda}\left(\int_\Omega |f(t,x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\right)\mathrm{d}t = \int_0^T M\left(\left(\frac{|\Omega|}{\lambda^p|\Omega|}\int_\Omega |f(t,x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\right)\mathrm{d}t$$
$$\leqslant \frac{1}{|\Omega|}\int_0^T\int_\Omega M\left(\left(\frac{|\Omega|}{\lambda^p}|f(t,x)|^p\right)^{\frac{1}{p}}\right)\mathrm{d}x\mathrm{d}t = \frac{1}{|\Omega|}\int_0^T\int_\Omega M\left(\left(\frac{|\Omega|}{\lambda^p}|f(t,x)|^p\right)^{\frac{1}{p}}\right)\mathrm{d}x\mathrm{d}t$$
$$= \frac{1}{|\Omega|}\int_0^T\int_\Omega M\left(\frac{|\Omega|^{\frac{1}{p}}|f(t,x)|}{\lambda}\right)\mathrm{d}x\mathrm{d}t < 1.$$

Since $M(|\cdot|^{1/p})$ is convex and $f \in L_M((0,T)\times\Omega)$, we notice that $f \in L^p((0,T)\times\Omega)$, hence $f \in L^p(0,T;L^p(\Omega))$. Consequently $t \mapsto f(t,x)$ is measurable which provides Bochner measurability of the function f. Therefore we obtain the statement and $f \in L_M(0,T;L^p(\Omega))$. □

Now we want to introduce the Riesz transform in an Orlicz space, which will be used later as a tool in the local pressure method in Chapter V.

Let $\beta,\gamma \in (0,\infty)$ and $\tau \in [0,\infty)$. Let us denote by $L_{\tau\log^\beta}(\Omega)$ the Orlicz space associated with the $\mathcal{N}-$function $M(\tau) = \tau(\log(\tau+1))^\beta$ and by $L_{e(\gamma)}(\Omega)$ the Orlicz space associated with the $\mathcal{N}-$function which asymptotically, i.e. for sufficiently large τ, behaves like $\widetilde{M}(\tau) = \exp(\tau^\gamma)$. Note that $L_{\tau\log^\beta}(\Omega) = E_{\tau\log^\beta}(\Omega)$ and

$$\big(E_{e(\gamma)}(\Omega)\big)^* = L_{\tau\log^{1/\gamma}}(\Omega) \quad\text{and}\quad \big(L_{\tau\log^\beta}(\Omega)\big)^* = L_{e(1/\beta)}(\Omega),$$

hold, see [**87**].

Let $\mathcal{R}_{i,j}$ stand for a "double" Riesz transform of an integrable function g on $\mathbb{R}^3$, which can be given by a Fourier transform $\mathcal{F}$ as

$$\mathcal{R}_{i,j}[g] = \mathcal{F}^{-1}\left(\frac{\xi_i\xi_j}{|\xi|^2}\right)\mathcal{F}[g] = \nabla_{x_i}\nabla_{x_j}\Delta^{-1}g\,, \quad i,\, j = 1,2,3, \tag{III.2.2}$$

where

$$\Delta^{-1}g(x) = \mathcal{F}^{-1}\left(\frac{-1}{|\xi|^2}\right)\mathcal{F}[g] = \int_{\mathbb{R}^3}\frac{g(y)}{|x-y|}\mathrm{d}y.$$

Lemma III.2.8. *Let Ω be a bounded domain, let $b : \mathbb{R}^3 \to \mathbb{R}$ be a multiplier, α be a multi-index such that $|\alpha| \leqslant 2$ and*

$$|\xi|^{|\alpha|}|D^\alpha b(\xi)| \leqslant C < \infty\,.$$

Then for any $\beta > 0$ there exists a constant $c(\beta)$ such that for all $g \in L_{\tau \log^{\beta+1}}(\Omega)$

$$\|(\mathcal{F}^{-1}b\mathcal{F})[g]\|_{\tau \log^\beta} \leqslant c(\beta)\|g\|_{\tau \log^{\beta+1}} \tag{III.2.3}$$

where g is extended to be 0 on $\mathbb{R}^3\backslash\Omega$.

We recall here the proof given by Erban in [**52**].

Proof. The standard Mikhlin multiplier theorem (see e.g. [**19**, Chapter 6]) provides that $\mathcal{F}^{-1}b\mathcal{F}$ is bounded as a mapping

$$\mathcal{F}^{-1}b\mathcal{F} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \quad \text{and} \quad \mathcal{F}^{-1}b\mathcal{F} : L^1(\mathbb{R}^3) \to L^{1,\infty}(\mathbb{R}^3),$$

where $L^{1,\infty}$ stands for a Lorenz space[1]. Employing the result from [**71**, Theorem B.2] (see also [**32**]) we conclude that there exists a constant $c(\beta)$ such that (III.2.3) is satisfied. □

Corollary III.2.9. *Let Ω be a bounded domain. Then for any $\beta > 0$ and $g \in L_{\tau \log^{\beta+1}}(\Omega)$*

$$\|\mathcal{R}_{i,j}[g]|_\Omega\|_{\tau \log^\beta} \leqslant c(\beta)\|g\|_{\tau \log^{\beta+1}}. \tag{III.2.4}$$

Remark III.2.10. Let M be an arbitrary isotropic $\mathcal{N}-$function. If $f \in L_M(\mathbb{R}^3)$, then $\|f|_B\|_{L_M(B)} \leqslant \|f\|_{L_M(\mathbb{R}^3)}$. Indeed,

$$\begin{aligned}
&\|f|_B\|_{L_M(B)}\\
&= \inf\left\{\lambda > 0 \,:\, \int_B M\left(\frac{f\mathbb{1}_B}{\lambda}\right)\,\mathrm{d}x \leqslant 1\right\} = \inf\left\{\lambda > 0 \,:\, \int_{\mathbb{R}^3} M\left(\frac{f}{\lambda}\right)\mathbb{1}_B\,\mathrm{d}x \leqslant 1\right\}\\
&\leqslant \inf\left\{\lambda > 0 \,:\, \int_{\mathbb{R}^3} M\left(\frac{f}{\lambda}\right)\,\mathrm{d}x \leqslant 1\right\} = \|f\|_{L_M(\mathbb{R}^3)},
\end{aligned}$$

where the inequality is provided by non-negativity of M.

Proposition III.2.11. *Let M be an isotropic $\mathcal{N}$–function and let M satisfy the Δ_2–condition. Then $M(\tau) \leqslant C|\tau|^\alpha$, $\tau \geqslant \tau_0$, for some $C > 0$ and $\alpha > 0$, and its complementary function M^* satisfies $M^*(\zeta) \geqslant D|\zeta|^\beta$, $\zeta \geqslant \zeta_0 > 0$ for some $D > 0$ and $\beta > 1$.*

The above proposition can be found with the proof in [**105**, Chapter II] as Corollary 5.

Proposition III.2.12. *Let M^* be a generalized $\mathcal{N}$-function and let M^* satisfy Δ_2-condition* (III.1.15) *with the function $g_M \in L^\infty(\Omega)$. Then there exist $\nu > 0$ and $c > 0$ such that*

$$M(x, \boldsymbol{\xi}) \geqslant c|\boldsymbol{\xi}|^{1+\nu}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^d$ such that $|\boldsymbol{\xi}| \geqslant |\boldsymbol{\xi}_0|$ for some $\boldsymbol{\xi}_0$ with $|\boldsymbol{\xi}_0| > 0$.

[1] i.e. $g \in L^{1,\infty}$ iff $\sup_\sigma \sigma m(\sigma, g) < \infty$, where $m(\sigma, g) = |\{x : |g(x)| > \sigma\}|$

Proof. Let

$$m^*(r) = \operatorname*{ess\,sup}_{x\in\Omega} \sup_{\boldsymbol{\xi}\in\mathbb{R}^d,\,|\boldsymbol{\xi}|=r} M^*(x,\boldsymbol{\xi})$$

Obviously m^* is an $\mathcal{N}-$function and satisfies Δ_2–condition for sufficiently large r. Using Proposition III.2.11 we infer that there exists a complementary $\mathcal{N}-$function $m = m(|\boldsymbol{\xi}|)$ to m^* and constants $\nu > 0$ and $c > 0$ such that $m(|\boldsymbol{\xi}(x)|) \geqslant c|\boldsymbol{\xi}|^{1+\nu}$ for $\boldsymbol{\xi} \in \mathbb{R}^d$ s.t. $|\boldsymbol{\xi}| \geqslant |\boldsymbol{\xi}_0|$. According to the definition of m^*, $M^*(x,\boldsymbol{\xi}(x)) \leqslant m^*(|\boldsymbol{\xi}(x)|)$ for a.a. $x \in \Omega$. Thus $m(|\boldsymbol{\xi}|) \leqslant M(x,\boldsymbol{\xi})$ and for all measurable functions $\boldsymbol{\xi} : \Omega \to \mathbb{R}^d$, we obtain

$$M(x,\boldsymbol{\xi}) \geqslant c|\boldsymbol{\xi}|^{1+\nu}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^d$ such that $|\boldsymbol{\xi}| \geqslant |\boldsymbol{\xi}_0|$. □

Remark III.2.13. Let us remark that at most polynomial growth i.e if $M(x,\boldsymbol{\xi}) \leqslant c_1|\boldsymbol{\xi}|^q$ for some c_1 and $q \in (1,\infty)$, does not imply, that M satisfies Δ_2–condition. For the counterexample see [**105**].

Theorem III.2.14. *Let Ω be a bounded domain with a Lipschitz boundary. Let M be an isotropic $\mathcal{N}$–function satisfying Δ_2-condition and such that M^γ is quasiconvex for some $\gamma \in (0,1)$. Then, for any $f \in L_M(\Omega)$ such that*

$$\int_\Omega f \,\mathrm{d}x = 0,$$

the problem of finding a vector field $\boldsymbol{v} : \Omega \to \mathbb{R}^d$ such that

$$\begin{aligned} \operatorname{div}\boldsymbol{v} &= f \quad \text{in } \Omega \\ \boldsymbol{v} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

has at least one solution $\boldsymbol{v} \in L_M(\Omega;\mathbb{R}^d)$ and $\nabla\boldsymbol{v} \in L_M(\Omega;\mathbb{R}^{d\times d})$. Moreover, for some positive constant c

$$\int_\Omega M(|\nabla\boldsymbol{v}|)\,\mathrm{d}x \leqslant c\int_\Omega M(|f|)\,\mathrm{d}x.$$

For the proof see e.g. [**127, 42**].

CHAPTER IV

Existence result for unsteady flows of nonhomogeneous non-Newtonian fluids

IV.1. Introduction and formulation of the problem

We wish to investigate and understand mathematical properties of the motion of incompressible, nonhomogeneous non-Newtonian fluid, which can be described by the system of equations:

$$
\begin{aligned}
\partial_t \varrho + \mathrm{div}_x(\varrho \boldsymbol{u}) &= 0 \quad \text{in} \quad Q,\\
\partial_t(\varrho \boldsymbol{u}) + \mathrm{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \mathrm{div}_x \mathbf{S}(t, x, \varrho, \mathbf{D}\boldsymbol{u}) + \nabla_x p &= \varrho \boldsymbol{f} \quad \text{in} \quad Q,\\
\mathrm{div}_x \boldsymbol{u} &= 0 \quad \text{in} \quad Q,\\
\boldsymbol{u}(0,x) &= \boldsymbol{u}_0 \quad \text{in} \quad \Omega,\\
\varrho(0,x) &= \varrho_0 \quad \text{in} \quad \Omega,\\
\boldsymbol{u}(t,x) &= 0 \quad \text{on} \quad (0,T) \times \partial\Omega,
\end{aligned} \tag{IV.1.1}
$$

where $\varrho : Q \to \mathbb{R}$ is the mass density, $\boldsymbol{u} : Q \to \mathbb{R}^3$ denotes the velocity field, $p : Q \to \mathbb{R}$ the pressure, $\mathbf{S}$ the stress tensor, $\boldsymbol{f} : Q \to \mathbb{R}^3$ given outer sources. The set $\Omega \subset \mathbb{R}^3$ is a bounded domain with a regular boundary $\partial\Omega$ (of class, say $C^{2+\nu}$, $\nu > 0$, to avoid unnecessary technicalities connected with smoothness). We denote by $Q = (0,T) \times \Omega$ the time-space cylinder with some given $T \in (0, +\infty)$. The tensor $\mathbf{D}\boldsymbol{u} = \frac{1}{2}(\nabla_x \boldsymbol{u} + \nabla_x^T \boldsymbol{u})$ is a symmetric part of the velocity gradient.

It is supposed that the initial density is bounded, i.e.,

$$\varrho(0,\cdot) = \varrho_0 \in L^\infty(\Omega) \tag{IV.1.2}$$

and

$$0 < \varrho_* \leqslant \varrho_0(x) \leqslant \varrho^* < +\infty \quad \text{for a.a. } x \in \Omega. \tag{IV.1.3}$$

There have been many studies concerning the mathematical analysis of time-dependent flows of nonhomogeneous, incompressible fluids depending on or independent of density.

Our interest is directed to the phenomena of viscosity increase under various stimuli: shear rate, magnetic or electric field. Particularly we want to focus on shear thickening (STF) and magnetorheological (MR) fluids. Both types of fluid have the ability of transferring rapidly from liquid to solid-like state and this phenomenon is completely reversible, and the time scale for the transmission is of the order of a millisecond. The magnetorheological fluids [**136**] found their application in modern suspension system, clutches or crash-protection systems in cars and shock absorbers providing seismic protection.

In particular we are interested in fluids having viscosity which increases dramatically with increasing shear rate or applied stress, i.e. we want to consider shear thickening fluids, which can behaves like a solid when it encounters mechanical stress or shear. STF moves like a liquid until an object strikes or agitates it forcefully. Then, it hardens in a few milliseconds. This is the opposite of a shear-thinning fluid, like paint, which becomes thinner when it is agitated or shaken. The fluid is a colloid, consists of solid particles dispersed in a liquid (e.g. silica particles suspended in polyethylene glycol). The particles repel each other slightly, so they float easily throughout the liquid without clumping together or settling to the bottom. But the energy of a sudden impact overwhelms the repulsive forces between the particles – they stick together, forming masses called hydroclusters. When the energy from the impact dissipates, the particles begin to repel one another again. The hydroclusters fall apart, and the apparently solid substance reverts to a liquid.

Possible application for fluids with changeable viscosity appears in military armour. The so-called STF-fabric produced by simple impregnation process of e.g. Kevlar makes it applicable to any high-performance fabric. The resulting material is thin and flexible, and provides protection against the risk of needle, knife or bullet contact that face police officers and medical personnel [**49, 81, 90**].

As follows from (IV.1.1) we assume that the traceless part S of the Cauchy stress tensor depends on the density and due to the principle of objectivity the extra stress tensor depends on the velocity gradient only through the symmetric part $\mathsf{D}\boldsymbol{u}$. On one hand we want to be able to consider constitutive relations which are invariant w.r.t. translations and rotations perpendicular to one chosen direction and on the other hand allow that in this specific direction properties of the material can be different than with respect to others.

One of the example is a magnetorheological fluid, which consists of the magnetic particles suspended within the carrier oil distributed randomly in suspension under normal circumstances. When a magnetic field is applied, the microscopic particles align themselves along the lines of magnetic flux. In the fluid contained between two poles, the resulting chains of particles restrict the movement of the fluid, perpendicular to the direction of flux, effectively increasing its viscosity. Consequently mechanical properties of the fluid are anisotropic.

On the other hand we can consider the constitutive relation for fluids with dependence on outer field, in particular, we mean electrorheological fluids. In this case, from representation theorem it follows that the most general form for the stress tensor S (cf. [**111**]) is given by

$$\mathsf{S} = \alpha_1 \boldsymbol{E}\otimes\boldsymbol{E} + \alpha_2\mathsf{D} + \alpha_3\mathsf{D}^2 + \alpha_4(\mathsf{D}\boldsymbol{E}\otimes\boldsymbol{E} + \boldsymbol{E}\otimes\mathsf{D}\boldsymbol{E}) + \alpha_5(\mathsf{D}^2\boldsymbol{E}\otimes\boldsymbol{E} + \boldsymbol{E}\otimes\mathsf{D}^2\boldsymbol{E})$$

where α_i, $i = 1,\dots,5$ may be functions of invariants

$$|\boldsymbol{E}|^2,\ \operatorname{tr}\mathsf{D}^2,\ \operatorname{tr}\mathsf{D}^3,\ \operatorname{tr}(\mathsf{D}\boldsymbol{E}\otimes\boldsymbol{E}),\ \operatorname{tr}(\mathsf{D}^2\boldsymbol{E}\otimes\boldsymbol{E}).$$

Then it is easy to show that for $i = 1, 3, 5$, $\alpha_i = 0$ the stress tensor in the form

$$\mathsf{S} = |\mathrm{tr}\,\mathsf{D}^2|^3\mathsf{D} + |\mathrm{tr}\,(\mathsf{D}^2\boldsymbol{E}\otimes\boldsymbol{E})|^6(\mathsf{D}\boldsymbol{E}\otimes\boldsymbol{E} + \boldsymbol{E}\otimes\mathsf{D}\boldsymbol{E}) \tag{IV.1.4}$$

is thermodynamically admissible (i.e. $\mathsf{S} : \mathsf{D} \geqslant 0$), satisfies a principle of material frame-indifference and is monotone. Moreover, without loss of generality for $\boldsymbol{E} = (1, 0, 0)$ it can be calculated that the standard growth conditions: $|\mathsf{S}(\mathsf{D}, \boldsymbol{E})| \leqslant c(1 + |\mathsf{D}|)^{p-1}$, $\mathsf{S}(\mathsf{D}, \boldsymbol{E}) : \mathsf{D} \geqslant c|\mathsf{D}|^p$ is not satisfied, because the tensor S possesses growth of different powers in various directions of D. From mechanical point of view though the minimal assumptions are satisfied. For this reason we can not exclude constitutive relation of anisotropic behaviour like (IV.1.4).

In our considerations we do not want to assume that S has only p-structure, i.e. $\mathsf{S} \approx \mu(\varrho)(\kappa + |\mathsf{D}\boldsymbol{u}|)^{p-2}\mathsf{D}\boldsymbol{u}$ or $\mathsf{S} \approx \mu(\varrho)(\kappa + |\mathsf{D}\boldsymbol{u}|^2)^{(p-2)/2}\mathsf{D}\boldsymbol{u}$ (where $\kappa > 0$ and μ is a nonnegative bounded function). Standard growth conditions of the stress tensor, namely polynomial growth, see e.g. **[58, 92]**

$$\begin{aligned} |\mathsf{S}(x,\boldsymbol{\xi})| &\leqslant c(1 + |\boldsymbol{\xi}|^2)^{(p-2)/2}|\boldsymbol{\xi}| \\ \mathsf{S}(x,\boldsymbol{\xi}) : \boldsymbol{\xi} &\geqslant c(1 + |\boldsymbol{\xi}|^2)^{(p-2)/2}|\boldsymbol{\xi}|^2 \end{aligned} \tag{IV.1.5}$$

can not suffice to describe our model. Motivated by this significant shear thickening phenomenon we want to investigate the processes where growth is faster than polynomial and possibly different in various directions of the shear rate. We do not assume that a viscosity the fluid is constant. Moreover, we take under considerations the case of the viscosity depending on density and full symmetric part of the gradient. Therefore we formulate the growth conditions of the stress tensor with the help of general convex function M called a generalized $\mathcal{N}-$function similarly like in **[72, 74, 75, 76, 78, 79, 131, 133, 134, 135]**. Now we are able to describe the effect of rapidly shear thickening fluids.

We assume also that the stress tensor $\mathsf{S} : (0, T) \times \Omega \times \mathbb{R}_+ \times \mathbb{R}^{3\times3}_{\mathrm{sym}} \to \mathbb{R}^{3\times3}_{\mathrm{sym}}$ satisfies ($\mathbb{R}^{3\times3}_{\mathrm{sym}}$ stands for the space of 3×3 symmetric matrices):

S1: $\mathsf{S}(t, x, \varrho, \mathsf{K})$ is a Carathéodory function (i.e., measurable function of t, x for all $\varrho > 0$ and $\mathsf{K} \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$ and continuous function of ϱ and K for a.a. $x \in \Omega$) and $\mathsf{S}(t, x, \varrho, \mathbf{0}) = \mathbf{0}$.

S2: There exist a positive constant c_c, $\mathcal{N}-$functions M and M^* (which denotes the complementary function to M) such that for all $\mathsf{K} \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$, $\varrho > 0$ and a.a. t, $x \in Q$ it holds

$$\mathsf{S}(t, x, \varrho, \mathsf{K}) : \mathsf{K} \geqslant c_c\{M(x, \mathsf{K}) + M^*(x, \mathsf{S}(t, x, \varrho, \mathsf{K}))\}. \tag{IV.1.6}$$

S3: S is monotone, i.e. for all $\mathsf{K}_1, \mathsf{K}_2 \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$, $\varrho > 0$ and a.a. $x \in \Omega$

$$[\mathsf{S}(t, x, \varrho, \mathsf{K}_1) - \mathsf{S}(t, x, \varrho, \mathsf{K}_2)] : [\mathsf{K}_1 - \mathsf{K}_2] \geqslant 0.$$

We can observe that the case of stress tensors having convex potentials (additionally vanishing at $\mathbf{0}$ and symmetric w.r.t. the origin) significantly simplifies verifying condition **S2**. For finding $\mathcal{N}-$functions M and M^* we take an advantage of the following relation

$$M(\boldsymbol{\xi}) + M^*(\nabla M(\boldsymbol{\xi})) = \boldsymbol{\xi} : \nabla M(\boldsymbol{\xi}) \tag{IV.1.7}$$

holding for all $\boldsymbol{\xi} \in \mathbb{R}^{3\times 3}_{\rm sym}$, cf. [**109**]. This corresponds to the case when the Fenchel-Young inequality for $\mathcal{N}$–functions becomes an equality. Once we have a given function S, for simplicity consider it in the form $\mathsf{S}(\mathbf{D}\boldsymbol{u}) = 2\mu(|\mathbf{D}\boldsymbol{u}|^2)\mathbf{D}\boldsymbol{u}$, then choosing $M(x,\boldsymbol{\xi}) = M(\boldsymbol{\xi}) = \int_0^{|\boldsymbol{\xi}|^2} \mu(\boldsymbol{\alpha})\ d\boldsymbol{\alpha}$ provides that (IV.1.6) is satisfied with a constant $c = 1$. For such chosen M we only need to verify whether the $\mathcal{N}-$function–conditions, i.e, behaviour in/near zero and near infinity, are satisfied. The monotonicity of S follows from the convexity of the potential.

Our assumptions can capture shear dependent viscosity function which includes power-law and Carreau-type models which are quite popular among rheologists, in chemical engineering, and colloidal mechanics (see [**94**] for more references). Nevertheless we want to investigate also more general constitutive relations like non-polynomial growth $\mathsf{S} \approx |\mathbf{D}\boldsymbol{u}|^p \ln(1 + |\mathbf{D}\boldsymbol{u}|)$ or of anisotropic behaviour e.g. $\mathsf{S}_{i,j} \approx |\cdot|^{p_{ij}}[\mathbf{D}\boldsymbol{u}]_{i,j}$, i, $j = 1, 2, 3$.

The appropriate spaces to capture such formulated problem are generalized Orlicz spaces, often called Orlicz-Musielak spaces. We also allow the stress tensor to depend on x, this provides the possibility to consider electro- and magnetorheological fluids and significant influence of magnetic and magnetic field on the increase of viscosity. Thus we use the generalized Orlicz spaces, often called Orlicz-Musielak spaces (see [**96**] for more details). For definitions and preliminaries of $\mathcal{N}-$functions and Orlicz spaces see Section III.1. Contrary to [**96**] we consider the $\mathcal{N}-$function M not dependent only on $|\boldsymbol{\xi}|$, but on whole tensor $\boldsymbol{\xi}$. It results from the fact that the viscosity may differ in different directions of symmetric part of velocity gradient $\mathbf{D}\boldsymbol{u}$. Hence we want to consider the growth condition for the stress tensor dependent on the whole tensor $\mathbf{D}\boldsymbol{u}$, not only on $|\mathbf{D}\boldsymbol{u}|$. The spaces with an $\mathcal{N}$–function dependent on vector-valued argument were investigated in [**117, 118, 126**].

An example of a generalized Orlicz space is a generalized Lebesgue space, in this case $M(x,\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p(x)}$. These kind of spaces were applied in [**111**] to the description of flow of electrorheological fluid. The standard assumption in this work was $1 < p_0 \leqslant p(x) \leqslant p_\infty < \infty$, where $p \in C^1(\Omega)$ is a function of an electric field E, i.e. $p = p(|E|^2)$, and $p_0 > \frac{3d}{d+2}$ in case of steady flow, where $d \geqslant 2$ is the space dimension. The $\triangle_2$–condition is then satisfied and consequently the space is reflexive and separable. One of the main problems in our model is that the $\triangle_2$–condition is not satisfied and we lose the above properties. Later in [**41**] the above result was improved by Lipschitz truncations methods for $L^{p(x)}$ setting for S, where $\frac{2d}{d+2} < p(\cdot) < \infty$ was log-Hölder continuous and S was strongly monotone.

The mathematical analysis of time dependent flow of homogeneous non-Newtonian fluids with standard polynomial growth conditions was initiated by Ladyzhenskaya [**88, 89**] where the global existence of weak solutions for $p \geqslant 1 + (2d)/(d+2)$ was proved for Dirichlet boundary conditions. Later the steady flow was considered by Frehse at al. in [**60**], where the existence of weak solutions was established for the constant exponent $p > \frac{2d}{d+2}$, $d \geqslant 2$ by Lipschitz truncation methods.

Wolf in [**130**] proved existence of weak solutions to unsteady motion of an incompressible fluid with shear rate dependent viscosity for $p > 2(d+1)/(d+2)$

without assumptions on the shape and size of Ω employing an L^∞–test function and local pressure method. Finally, the existence of global weak solutions with Dirichlet boundary conditions for $p > (2d)/(d+2)$ was achieved in [**43**] by Lipschitz truncation and local pressure methods.

Most of the available results concerning nonhomogeneous incompressible fluids deal with the polynomial dependence between $\mathbf{S}$ and $|\mathbf{D}\boldsymbol{u}|$. The analysis of nonhomogeneous Newtonian ($p = 2$ in (IV.1.5)) fluids was investigated by Antontsev, Kazhikhov and Monakhov [**10**] in the seventies. P.L. Lions in [**91**] presented the concept of renormalized solutions and obtained new convergence and continuity properties of the density.

The first result for unsteady flow of nonhomogenous non-Newtonian fluids goes back to Fernández–Cara [**57**], where existence of Dirichlet weak solutions was obtained for $p \geqslant 12/5$ if $d = 3$, later existence of space-periodic weak solutions for $p \geqslant 2$ with some regularity properties of weak solutions whenever $p \geqslant 20/9$ (if $d = 3$) was obtained by Guilién-González in [**70**]. Frehse and Ružička showed in [**59**] existence of a weak solution for generalized Newtonian fluid of power-law type for $p > 11/5$. Authors needed also existence of the potential of $\mathbf{S}$. Recent results concerning fluids where the growth condition is as in (IV.1.5) for $p \geqslant 11/5$ belong to Frehse, Málek and Ružička [**58**]. The novelty of this paper is the consideration of the full thermodynamic model for a nonhomogeneous incompressible fluid. Particularly in [**58, 59**] the reader can find the concept of integration by parts formula, which we adapted to our case. Also more details concerning references can be found therein.

First results concerning non-Newtonian fluid with the assumption that $\mathbf{S}$ is strictly monotone and satisfies conditions **S1.-S2.** were established by Gwiazda et al. [**72**] for the case of unsteady flow. The stronger assumption on $\mathbf{S}$ was crucial for the applied tools (Young measures). This restriction was abandoned in [**131**] by Wróblewsk-Kamińska for the case of steady flow and in [**75**] by Gwiazda et al. for unsteady flow. The authors used generalization of Minty trick for non-reflexive spaces. The above existence results were established for $p \geqslant 11/5$ in [**75**], but without including in the system the dependence on density.

Summarising, we want to extend the existence theory for flows of non-Newtonian incompressible fluids to a more general class than polynomial growth conditions [**58, 59**] by formulating the problem in nonhomogeneous in space anisotropic Orlicz setting as in [**72, 75, 131**]. Moreover, we want to complete the results the reader can find therein by including continuity equation $(\mathrm{IV.1.1})_1$ to the considered system and dependence of $\mathbf{S}$ on density of the fluid, namely we do not assume that density is constant. Additionally we are able to obtain better regularity of solution in time than in [**58, 59, 72, 75, 131**], namely in the Nikolskii space.

In order to state the main result of the chapter we start with the following definition of a weak solution:

Definition IV.1.1. Let ϱ_0 satisfies assumptions (IV.1.2), (IV.1.3), $\boldsymbol{u}_0 \in L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3)$ and $\boldsymbol{f} \in L^{p'}(0,T;L^{p'}(\Omega;\mathbb{R}^3))$. Let $\mathbf{S}$ satisfy conditions **S1.-S3.** with an $\mathcal{N}$–function

M such that for some $\underline{c} > 0$, $\widetilde{C} \geqslant 0$ and $p \geqslant \frac{11}{5}$ M satisfies

$$M(x, \boldsymbol{\xi}) \geqslant \underline{c}|\boldsymbol{\xi}|^p - \widetilde{C}$$

for a.a. $x \in \Omega$ and all $\boldsymbol{\xi} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$.

We call the pair $(\varrho, \boldsymbol{u})$ a *weak solution* to (IV.1.1) if

$$\begin{aligned}
&0 < \varrho_* \leqslant \varrho(t,x) \leqslant \varrho^* \quad \text{for a.a. } (t,x) \in Q,\\
&\varrho \in C([0,T]; L^q(\Omega)) \quad \text{for arbitrary } q \in [1,\infty),\\
&\partial_t \varrho \in L^{5p/3}(0,T; (W^{1,5p/(5p-3)})^*)\\
&\boldsymbol{u} \in L^\infty(0,T; L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3)) \cap L^p(0,T; W^{1,p}_{0,\mathrm{div}}(\Omega;\mathbb{R}^3)) \cap N^{1/2,2}(0,T; L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3)),\\
&\mathbf{D}\boldsymbol{u} \in L_M(Q; \mathbb{R}^{3\times 3}_{\mathrm{sym}}) \quad \text{and} \quad (\varrho \boldsymbol{u}, \boldsymbol{\psi}) \in C([0,T]) \text{ for all } \boldsymbol{\psi} \in L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3)
\end{aligned}$$

and

$$\int_0^T \langle \partial_t \varrho, z\rangle - (\varrho \boldsymbol{u}, \nabla_x z)\,\mathrm{d}t = 0 \tag{IV.1.8}$$

for all $z \in L^r(0,T; W^{1,r}(\Omega))$ with $r = 5p/(5p-3)$, i.e.

$$\int_{s_1}^{s_2}\int_\Omega \varrho \partial_t z + (\varrho \boldsymbol{u}) \cdot \nabla_x z \,\mathrm{d}x\mathrm{d}t = \int_\Omega \varrho z(s_2) - \varrho z(s_1)\,\mathrm{d}x$$

for all z smooth and s_1, $s_2 \in [0,T]$, $s_1 < s_2$ and

$$\begin{aligned}
&-\int_0^T\int_\Omega \varrho \boldsymbol{u}\cdot \partial_t \boldsymbol{\varphi} - \varrho \boldsymbol{u}\otimes \boldsymbol{u} : \nabla_x \boldsymbol{\varphi} + \mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}) : \mathbf{D}\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t\\
&= \int_0^T\int_\Omega \varrho \boldsymbol{f}\cdot \boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t + \int_\Omega \varrho_0 \boldsymbol{u}_0 \cdot \boldsymbol{\varphi}(0)\,\mathrm{d}x \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}((-\infty,T);\mathcal{V}),
\end{aligned}$$

and initial conditions are achieved in the following way

$$\lim_{t\to 0^+} \|\varrho(t) - \varrho_0\|_{L^q(\Omega)} + \|\boldsymbol{u}(t) - \boldsymbol{u}_0\|^2_{L^2(\Omega)} = 0 \quad \text{for arbitrary } q \in [1,\infty). \tag{IV.1.9}$$

Theorem IV.1.2. *Let M be an $\mathcal{N}-$function satisfying for some $\underline{c} > 0$, $\widetilde{C} \geqslant 0$ and*

$$p \geqslant \frac{11}{5} \tag{IV.1.10}$$

the condition

$$M(x,\boldsymbol{\xi}) \geqslant \underline{c}|\boldsymbol{\xi}|^p - \widetilde{C} \tag{IV.1.11}$$

for a.a. $x \in \Omega$ and all $\boldsymbol{\xi} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$. Let us assume that the conjugate function

$$M^* \text{ satisfies the } \Delta_2-\text{condition and } \lim_{|\boldsymbol{\xi}|\to\infty} \inf_{x\in\Omega} \frac{M^*(x,\boldsymbol{\xi})}{|\boldsymbol{\xi}|} = \infty. \tag{IV.1.12}$$

Moreover, let $\mathbf{S}$ satisfy conditions **S1.-S3.** *and $\boldsymbol{u}_0 \in L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3)$, $\varrho_0 \in L^\infty(\Omega)$ with $0 < \varrho_* \leqslant \varrho_0(x) \leqslant \varrho^* < +\infty$ for a.a. $x \in \Omega$ and $\boldsymbol{f} \in L^{p'}(0,T; L^{p'}(\Omega;\mathbb{R}^3))$. Then there exists a weak solution to (IV.1.1).*

In the following chapter we consider the flow in the domain of space dimension $d = 3$, just for the brevity. The existence result can be easily extended to the case of arbitrary $d \geqslant 2$ and $p \geqslant \frac{3d+2}{d+2}$. The chapter is based on [**133**] by Wróblewska-Kamińska and partially on [**131**] by Wróblewska-Kamińska and [**75**] by Gwiazda, Świerczewska-Gwiazda, Wróblewska-Kamińska, see also [**78**].

In Section IV.2 our main result of existence of weak solutions to the system (IV.1.1) is proved.

IV.2. Proof of Theorem IV.1.2 - Existence of weak solutions

IV.2.1. Uniform estimates. Let $\{\boldsymbol{\omega}^n\}_{n=1}^{\infty}$ be a basis of $W^{1,p}_{0,\mathrm{div}}(\Omega;\mathbb{R}^3)$ constructed with the help of eigenfunctions of the problem

$$(\!(\boldsymbol{\omega}_i, \boldsymbol{\varphi})\!)_s = \lambda_i(\boldsymbol{\omega}_i, \boldsymbol{\varphi}) \quad \text{for all } \boldsymbol{\varphi} \in W^{s,2}_{0,\mathrm{div}},$$

where

$$W^{s,2}_{0,\mathrm{div}} = \text{the closure of } \mathcal{V} \text{ w.r.t. the } W^{s,2}(\Omega)\text{-norm}$$

and $(\!(\cdot,\cdot)\!)_s$ denotes the scalar product in $W^{s,2}_{0,\mathrm{div}}$. We assume that $s > 3$ and then the Sobolev embedding theorem provides

$$W^{s-1,2}(\Omega) \hookrightarrow C(\Omega). \tag{IV.2.1}$$

Moreover the basis is orthonormal in $L^2(\Omega;\mathbb{R}^3)$ (see [**94**, Appendix]).

We denote

$$L^{2,n}_{\mathrm{div}} := \mathrm{span}\{\boldsymbol{\omega}^1, ..., \boldsymbol{\omega}^n\}$$

and define orthonormal projection $P^n : L^2_{\mathrm{div}} \to L^{2,n}_{\mathrm{div}}$ by $P^n\boldsymbol{u} := \sum_{i=1}^n (\boldsymbol{u}, \boldsymbol{\omega}^i)\boldsymbol{\omega}^i$ for every $n \in \mathbb{N}$. Let us seek for an approximate solution $\boldsymbol{u}^n$ of the system (IV.1.1) in the following form of finite sums

$$\boldsymbol{u}^n(t,x) := \sum_{j=1}^n \alpha_j^n(t)\boldsymbol{\omega}^j(x) \tag{IV.2.2}$$

for $n = 1, 2, \ldots$ with the unknown coefficients $\alpha_j^n \in C([0,T])$, $j = 1, 2, \ldots, n$, while ϱ^n is the solution of the continuous problem

$$\begin{aligned} \partial_t \varrho^n + \mathrm{div}_x(\varrho^n \boldsymbol{u}^n) &= 0, \\ \varrho^n(0) &= \varrho_0^n \end{aligned} \tag{IV.2.3}$$

with $\varrho_0^n \in C^1(\Omega)$ and $\boldsymbol{u}^n$ solves the Galerkin system

$$\begin{aligned} (\varrho^n \partial_t \boldsymbol{u}^n, \boldsymbol{\omega}^j) + (\varrho^n(\nabla_x \boldsymbol{u}^n)\boldsymbol{u}^n, \boldsymbol{\omega}^j) + (\mathbf{S}(t,x,\varrho^n,\mathbf{D}\boldsymbol{u}^n), \mathbf{D}\boldsymbol{\omega}^j) &= (\varrho^n \boldsymbol{f}^n, \boldsymbol{\omega}^j) \\ \boldsymbol{u}^n(0) &= P^n(\boldsymbol{u}_0) \end{aligned} \tag{IV.2.4}$$

for all $1 \leqslant j \leqslant n$ and a.a. $t \in [0,T]$. We assume additionally that

$$\begin{aligned} &\boldsymbol{u}_0^n \to \boldsymbol{u}_0 \quad \text{strongly in } L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3), \\ &\varrho_0^n \to \varrho_0 \quad \text{strongly in } L^\infty(\Omega), \\ &\varrho_0^n \in C^1(\Omega) \text{ and } \varrho_* \leqslant \varrho_0^n \leqslant \varrho^* \end{aligned} \tag{IV.2.5}$$

and

$$\boldsymbol{f}^n \to \boldsymbol{f} \quad \text{strongly in } L^{p'}(0,T;L^{p'}(\Omega;\mathbb{R}^3)). \tag{IV.2.6}$$

Let us note that since our approximate solution $\boldsymbol{u}^n$ satisfies (IV.2.3), (IV.2.4)$_1$ for $1 \leqslant j \leqslant n$ is equivalent to

$$\begin{aligned}\left\langle \partial_t(\varrho^n \boldsymbol{u}^n), \boldsymbol{\omega}^j \right\rangle - (\varrho^n \boldsymbol{u}^n \otimes \boldsymbol{u}^n, \nabla_x \boldsymbol{\omega}^j) + (\mathbf{S}(t,x,\varrho^n,\mathbf{D}\boldsymbol{u}^n), \mathbf{D}\boldsymbol{\omega}^j)& \\ = (\varrho^n \boldsymbol{f}^n, \boldsymbol{\omega}^j)&\end{aligned} \tag{IV.2.7}$$

and consequently after integrating over the time interval $(0,T)$ we have

$$\begin{aligned}\int_0^T \left\langle \partial_t(\varrho^n \boldsymbol{u}^n), \boldsymbol{\omega}^j \right\rangle - (\varrho^n \boldsymbol{u}^n \otimes \boldsymbol{u}^n, \nabla_x \boldsymbol{\omega}^j) + (\mathbf{S}(t,x,\varrho^n,\mathbf{D}\boldsymbol{u}^n), \mathbf{D}\boldsymbol{\omega}^j)\,\mathrm{d}t& \\ = \int_0^T (\varrho^n \boldsymbol{f}^n, \boldsymbol{\omega}^j)\,\mathrm{d}t&\end{aligned} \tag{IV.2.8}$$

for all $1 \leqslant j \leqslant n$ and (IV.2.3) satisfies also

$$\int_0^T \langle \partial_t \varrho^n, z\rangle - (\varrho^n \boldsymbol{u}^n, \nabla_x z)\ \mathrm{d}t = 0 \tag{IV.2.9}$$

for all $z \in L^q(0,T;W^{1,q}(\Omega))$ with $q \in [1,\infty)$.

Before we prove existence of the approximate solution we want to show that some uniform w.r.t. n a priori estimates are valid and to present some of their consequences which we will use later.

In the first step we concentrate on equations (IV.2.3). Since (IV.2.1) holds, we will use standard techniques for the transport equation and apply the method of characteristics. We notice that (IV.2.3) is an equation of the first order w.r.t. $\varrho^n(t,x)$. We solve the Cauchy problem

$$\begin{aligned}\frac{\mathrm{d}y^n(t,x)}{\mathrm{d}t} &= \boldsymbol{u}^n(t,y^n(t,x)) \\ y^n(0,x) &= x,\end{aligned} \tag{IV.2.10}$$

with the help of Carathéodory's theory. The system (IV.2.10) defines the so-called characteristics associated with (IV.2.3). Note that for every $t \in [0,T]$ the map $x \mapsto y^n(t,x)$ is a diffeomorphism of $\bar\Omega$ onto $\bar\Omega$. Using this fact and $\operatorname{div}_x \boldsymbol{u}^n = 0$ we can see that the solution of (IV.2.3) is given by

$$\varrho^n(t,y^n(t,x)) = \varrho_0^n(x). \tag{IV.2.11}$$

Since (IV.2.11) is satisfied and according to assumptions on ϱ_0^n we obtain that

$$0 < \varrho_* \leqslant \varrho^n(t,x) \leqslant \varrho^* < +\infty \quad \text{for all } (t,x) \in Q. \tag{IV.2.12}$$

For later consideration let us note that the Alaoglu-Banach theorem provides existence of a subsequence such that

$$\begin{aligned}\varrho^n &\rightharpoonup \varrho \quad \text{weakly in } L^q(Q) \text{ for any } q \in [1,\infty), \\ \varrho^n &\overset{*}{\rightharpoonup} \varrho \quad \text{weakly–}(*) \text{ in } L^\infty(Q).\end{aligned} \tag{IV.2.13}$$

If we multiply (IV.2.4) by α_j^n, sum up over j and use (IV.2.3), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho^n|\boldsymbol{u}^n|^2\,\mathrm{d}x + (\mathbf{S}(t,x,\varrho^n,\mathbf{D}\boldsymbol{u}^n),\mathbf{D}\boldsymbol{u}^n) = (\varrho^n\boldsymbol{f}^n,\boldsymbol{u}^n) \tag{IV.2.14}$$

Using the Hölder, the Korn-Poincaré and the Young inequalities, the assumption (IV.1.11) and (IV.2.12) we are able to estimate the right-hand side of (IV.2.14) in the following way

$$|(\varrho^n\boldsymbol{f}^n,\boldsymbol{u}^n)| \leqslant C_1(\Omega,c_c,\underline{c},\varrho^*,p)\|\boldsymbol{f}^n\|^{p'}_{L^{p'}(\Omega)} + \frac{c_c}{2}\int_\Omega M(x,\mathbf{D}\boldsymbol{u}^n)\,\mathrm{d}x. \tag{IV.2.15}$$

Integrating (IV.2.14) over the time interval $(0,s_0)$, using estimates (IV.2.15) and (IV.2.12), the coercivity conditions (IV.1.6) on $\mathbf{S}$, continuity of P^n uniformly w.r.t. n and strong convergence $\boldsymbol{f}^n\to\boldsymbol{f}$ in $L^{p'}(0,T;L^{p'}(\Omega;\mathbb{R}^3))$ we obtain

(IV.2.16)
$$\int_\Omega \frac{1}{2}\varrho^n(s_0)|\boldsymbol{u}^n(s_0)|^2\,\mathrm{d}x + \int_0^{s_0}\int_\Omega \frac{c_c}{2}M(x,\mathbf{D}\boldsymbol{u}^n) + c_cM^*(x,\mathbf{S}(t,x,\varrho^n,\mathbf{D}\boldsymbol{u}^n))\,\mathrm{d}x\mathrm{d}t$$
$$\leqslant C_2(\Omega,c_c,\underline{c},\varrho^*,p,\|\boldsymbol{f}\|_{L^{p'}(0,T;L^{p'}(\Omega))}) + \frac{1}{2}\varrho^*\|\boldsymbol{u}_0\|^2_{L^2(\Omega)},$$

where C_2 is a nonnegative constant independent of n and dependent on the given data. Noticing that $E_{M^*}(Q;\mathbb{R}^{3\times3}_{\mathrm{sym}})$ is separable, $(E_{M^*})^* = L_M$ and using the Alaoglu-Banach theorem we obtain for suitable subsequence, as a direct consequence of (IV.2.16), that

$$\mathbf{D}\boldsymbol{u}^n \overset{*}{\rightharpoonup} \mathbf{D}\boldsymbol{u} \quad \text{weakly–(*) in } L_M(Q;\mathbb{R}^{3\times3}_{\mathrm{sym}}). \tag{IV.2.17}$$

Moreover, the condition (IV.1.11) provides that $\{\mathbf{D}\boldsymbol{u}^n\}_{n=1}^\infty$ is uniformly bounded in the space $L^p(Q;\mathbb{R}^{3\times3})$ for $p\geqslant\frac{11}{5}$

$$\int_0^T \|\mathbf{D}\boldsymbol{u}^n\|^p_{L^p(\Omega)}\,\mathrm{d}t \leqslant C \tag{IV.2.18}$$

and hence there exists a subsequence such that

$$\mathbf{D}\boldsymbol{u}^n \rightharpoonup \mathbf{D}\boldsymbol{u} \quad \text{weakly in} \quad L^p(Q;\mathbb{R}^{3\times3}). \tag{IV.2.19}$$

According to the Korn inequality we also obtain

$$\int_0^T \|\nabla_x\boldsymbol{u}^n\|^p_{L^p(\Omega)}\,\mathrm{d}t \leqslant C \tag{IV.2.20}$$

and

$$\boldsymbol{u}^n \rightharpoonup \boldsymbol{u} \quad \text{weakly in} \quad L^p(0,T,W^{1,p}_{0,\mathrm{div}}(\Omega;\mathbb{R}^3)). \tag{IV.2.21}$$

Using (IV.2.16) we deduce that

$$\|\mathbf{S}(t,x,\varrho^n,\mathbf{D}\boldsymbol{u}^n)\|_{L^1(Q)} \leqslant C. \tag{IV.2.22}$$

Moreover, we get that the sequence $\{\mathbf{S}(t,x,\varrho^n,\mathbf{D}\boldsymbol{u}^n)\}_{n=1}^\infty$ is uniformly bounded in Orlicz class $\mathcal{L}_{M^*}(Q;\mathbb{R}^{3\times3})$. Consequently for a subsequence we infer that

$$\mathbf{S}(\cdot,\varrho^n,\mathbf{D}\boldsymbol{u}^n) \overset{*}{\rightharpoonup} \overline{\mathbf{S}} \quad \text{weakly–(*) in } L_{M^*}(Q;\mathbb{R}^{3\times3}_{\mathrm{sym}}). \tag{IV.2.23}$$

Applying Lemma III.2.2 an using assumption $(\text{IV}.1.12)_2$ we conclude the uniform integrability of the sequence. Consequently there exists a tensor $\overline{\mathbf{S}} \in L^1(Q;\mathbb{R}^{3\times 3})$ and a subsequence $\{\mathbf{S}(\cdot,\varrho^n,\mathbf{D}\boldsymbol{u}^n)\}_{n=1}^{\infty}$ such that

$$\mathbf{S}(\cdot,\varrho^n,\mathbf{D}\boldsymbol{u}^n) \rightharpoonup \overline{\mathbf{S}} \quad \text{weakly in} \quad L^1(Q;\mathbb{R}^{3\times 3}). \tag{IV.2.24}$$

Furthermore (IV.2.16) and (IV.2.12) provide

$$\begin{aligned} &\sup_{t\in[0,T]} \|\boldsymbol{u}^n(t)\|^2_{L^2(\Omega)} \leqslant C,\\ &\sup_{t\in[0,T]} \|\varrho^n(t)|\boldsymbol{u}^n(t)|^2\|_{L^1(\Omega)} \leqslant C, \end{aligned} \tag{IV.2.25}$$

where C is a positive constant dependent on the size of data, but independent of n. It follows immediately that for some subsequence

$$\boldsymbol{u}^n \overset{*}{\rightharpoonup} \boldsymbol{u} \quad \text{weakly–(*) in } L^\infty(0,T;L^2_{\text{div}}(\Omega;\mathbb{R}^3)). \tag{IV.2.26}$$

In particular, from $(\text{IV}.2.25)_1$ follows that there exists constant C_B s.t.

$$\|\boldsymbol{u}^n\|_{L^q(0,T;L^2_{\text{div}}(\Omega))} \leqslant C_B \quad \text{for } q \geqslant 1. \tag{IV.2.27}$$

Since the sequence $\{\boldsymbol{u}^n\}_{n=1}^{\infty}$ is uniformly bounded in $L^p(0,T;W^{1,p}_{0,\text{div}}(\Omega;\mathbb{R}^3))$ the Gagliardo-Nirenberg-Sobolev inequality provides uniform boundedness in the space $L^p(0,T;L^{3p/(3-p)})$. Standard interpolation (see e.g. [**108**, Proposition 1.41]) of $L^\infty(0,T;L^2)$ and $L^p(0,T;L^{3p/(3-p)})$ (this particular argument deals with the case $p<3$, the case $p \geqslant 3$ can be treated easier e.g. with the Poincaré or the Morrey inequality) gives us

$$\int_0^T \|\boldsymbol{u}^n\|^r_{L^r(\Omega)} dt \leqslant C_B \quad \text{for } 1 \leqslant r \leqslant 5p/3 \tag{IV.2.28}$$

for some constant C_B, therefore from (IV.2.12) and (IV.2.28) we infer also

$$\int_0^T \|\varrho^n\boldsymbol{u}^n\|^{5p/3}_{L^{5p/3}(\Omega)} dt \leqslant C. \tag{IV.2.29}$$

Consequently we can take a subsequence satisfying

$$\boldsymbol{u}^n \rightharpoonup \boldsymbol{u} \quad \text{weakly in } L^{5p/3}(0,T;L^{5p/3}(\Omega;\mathbb{R}^3)) \tag{IV.2.30}$$

and there exist subsequence $\{\varrho^n\boldsymbol{u}^n\}_{n=1}^{\infty}$ and $\overline{\varrho\boldsymbol{u}} \in L^{5p/3}(0,T;L^{5p/3}(\Omega;\mathbb{R}^3))$ such that

$$\varrho^n\boldsymbol{u}^n \rightharpoonup \overline{\varrho\boldsymbol{u}} \quad \text{weakly in} \quad L^{5p/3}(0,T;L^{5p/3}(\Omega;\mathbb{R}^3)). \tag{IV.2.31}$$

Using (IV.2.12), (IV.2.20) and (IV.2.28) and applying the Hölder inequality, we obtain

$$\int_0^T |(\varrho^n\boldsymbol{u}^n\otimes\boldsymbol{u}^n,\nabla_x\boldsymbol{u}^n)|\,\mathrm{d}t \leqslant C \quad \Longleftrightarrow \quad p \geqslant \frac{11}{5}$$

(here is the restriction for the exponent p stated in (IV.1.10)).

Using (IV.2.29) it follows from (IV.2.9) that

$$\int_0^T \|\partial_t\varrho^n\|^{5p/3}_{(W^{1,5p/(5p-3)})^*}\,\mathrm{d}t \leqslant C. \tag{IV.2.32}$$

Hence the Alaoglu-Banach theorem provides existence of a subsequence such that

$$\partial_t \varrho^n \rightharpoonup \partial_t \varrho \quad \text{weakly in } L^{5p/3}(0,T;(W^{1,5p/(5p-3)})^*). \tag{IV.2.33}$$

IV.2.2. Existence of approximate solution. On the basis of estimates proved in Subsection IV.2.1 we will show the existence of solutions of (IV.2.4) and (IV.2.3) using Schauder's fixed point theorem for the operator

$$\Lambda : B \subset Y \to B : \tilde{\boldsymbol{u}}^n \to \boldsymbol{u}^n$$

where $Y := L^q(0,T;L^q(\Omega;\mathbb{R}^3)) \cap L^q(0,T;L^{2,n}_{\mathrm{div}}(\Omega;\mathbb{R}^3))$, $q = 2p'$ is equipped with the norm of the space $L^q(0,T;L^q(\Omega;\mathbb{R}^3))$ and B is the closed ball which will be defined later. For given $\tilde{\boldsymbol{u}}^n \in B$ the element $\Lambda\tilde{\boldsymbol{u}}^n = \boldsymbol{u}^n$ is a solution of the problem

$$\begin{aligned} \partial_t \tilde{\varrho}^n + \mathrm{div}_x(\tilde{\varrho}^n \tilde{\boldsymbol{u}}^n) &= 0, \\ \tilde{\varrho}^n(0) &= \varrho_0^n, \end{aligned} \tag{IV.2.34}$$

$$\begin{aligned} (\tilde{\varrho}^n \partial_t \boldsymbol{u}^n, \boldsymbol{\omega}^j) + (\tilde{\varrho}^n[\nabla_x \boldsymbol{u}^n]\tilde{\boldsymbol{u}}^n, \boldsymbol{\omega}^j) + (\mathbf{S}(t,x,\tilde{\varrho}^n,\mathbf{D}\tilde{\boldsymbol{u}}^n), \mathbf{D}\boldsymbol{\omega}^j) &= (\tilde{\varrho}^n \boldsymbol{f}^n, \boldsymbol{\omega}^j), \\ \boldsymbol{u}^n(0) &= P^n(\boldsymbol{u}_0). \end{aligned} \tag{IV.2.35}$$

It means that in the first step we find solution $\tilde{\varrho}^n$ of the linear problem (IV.2.34) and next we look for the vector $\boldsymbol{u}^n$, solution of the linearization (IV.2.35) of the system (IV.2.4).

The equation (IV.2.35) can be rewritten as a system of ordinary differential equations (the reader can find the details in [**10, 92, 93**]). We obtain local in time solvability according to Peano's existence theorem for ordinary differential equations. The global solvability is provided by the a'priori estimates (IV.2.16) where $\boldsymbol{u}^n$ is replaced by $\tilde{\boldsymbol{u}}^n$ in suitable places.

Let us take $\tilde{\boldsymbol{u}}^n \in B := \overline{B_{C_B}(0)}$, where $B_{C_B}(0)$ is a ball and C_B is a constant from (IV.2.28). Inequalities $2p' \leqslant 5p/3$ for $p \geqslant 11/5$ assure that $Y \supset B$. Previous estimates (IV.2.27) and (IV.2.28) provide that Λ maps B into B. Using (IV.2.25)$_1$ and (IV.2.20) we deduce that $\boldsymbol{u}^n \in L^\infty(0,T;L^{2,n}_{\mathrm{div}}(\Omega;\mathbb{R}^3)) \cap L^p(0,T;W^{1,p}_{0,\mathrm{div}}(\Omega;\mathbb{R}^3))$. The continuity of the operator Λ results from the theorem on continuous dependence of the solutions of the Cauchy problem (IV.2.35) on the coefficients and right-hand side. Now the main difficulty is to show compactness of the operator Λ. Similarly as in [**10, 59**] our plan is to prove that

$$\int_0^{T-\delta} \|\boldsymbol{u}^n(s+\delta) - \boldsymbol{u}^n(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s \to 0 \quad \text{as } \delta \to 0 \tag{IV.2.36}$$

is satisfied. According to [**115**, Theorem 5] and parabolic embedding theorem $\Lambda(B)$ is a compact subset of Y. Applying Schauder's fixed point theorem we deduce that there exists a fixed point $\tilde{\boldsymbol{u}}^n$ and the corresponding density $\tilde{\varrho}^n$ which solve the system (IV.2.3), (IV.2.4).

To show (IV.2.36) we will follow [**10**, Chap.3. Lemma 1.2] with some modifications concerning a change from L^2-structure for L^p-structure and additional one concerning the nonlinear term controlled by nonstandard condition (IV.1.6).

Let us fix δ and s, $0 < \delta < T$, $0 \leqslant s \leqslant T-\delta$. Next we test (IV.2.35) at time t by $\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)$ and integrate the equation over time interval $(s, s+\delta)$ w.r.t. time t. Using the integration by parts formula w.r.t. time, the equality $\partial_t \tilde{\varrho}^n = -\mathrm{div}_x(\tilde{\varrho}^n \tilde{\boldsymbol{u}}^n)$ and obvious identity

$$\tilde{\varrho}^n(s+\delta)\boldsymbol{u}^n(s+\delta)-\tilde{\varrho}(s)\boldsymbol{u}^n(s) = \tilde{\varrho}^n(s+\delta)[\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)]+[\tilde{\varrho}^n(s+\delta)-\tilde{\varrho}^n(s)]\boldsymbol{u}^n(s)$$

we get

(IV.2.37)

$$\begin{aligned}
&\int_\Omega \tilde{\varrho}^n(s+\delta)|\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)|^2 + [\tilde{\varrho}^n(s+\delta)-\tilde{\varrho}^n(s)]\boldsymbol{u}^n(s)\cdot[\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)]\,\mathrm{d}x \\
&+\int_s^{s+\delta}\int_\Omega \mathrm{div}_x(\tilde{\varrho}^n(t)\tilde{\boldsymbol{u}}^n(t))\boldsymbol{u}^n(t)\cdot[\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)]\,\mathrm{d}x\mathrm{d}t \\
&+\int_s^{s+\delta}\int_\Omega \tilde{\varrho}^n(t)[\nabla_x\boldsymbol{u}^n(t)]\tilde{\boldsymbol{u}}^n(t)\cdot[\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)]\,\mathrm{d}x\mathrm{d}t \\
&+\int_s^{s+\delta}\int_\Omega \mathbf{S}(t,x,\tilde{\varrho}^n(t),\mathbf{D}\tilde{\boldsymbol{u}}^n(t)) : \mathbf{D}[\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)]\,\mathrm{d}x\mathrm{d}t \\
&=\int_s^{s+\delta}\int_\Omega \tilde{\varrho}^n(t)\boldsymbol{f}^n(t)\cdot[\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)]\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Now, let us test (IV.2.34) at time t by $\boldsymbol{u}^n(s)\cdot(\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s))$ and integrate the equation over time interval $(s, s+\delta)$ w.r.t. t to obtain

$$\begin{aligned}
&\int_\Omega [\tilde{\varrho}^n(s+\delta)-\tilde{\varrho}^n(s)]\boldsymbol{u}^n(s)\cdot[\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)]\,\mathrm{d}x \\
&= -\int_s^{s+\delta}\int_\Omega \mathrm{div}_x(\tilde{\varrho}^n(t)\tilde{\boldsymbol{u}}^n(t))\boldsymbol{u}^n(s)\cdot[\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)]\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Substituting the above relation into (IV.2.37) and using some obvious manipulations, i.e.

$$\begin{aligned}
&(\mathrm{div}_x(\tilde{\varrho}^n\tilde{\boldsymbol{u}}^n)\boldsymbol{u}^n(s), [\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)]) = \\
&-(\tilde{\varrho}^n(t)[\nabla_x\boldsymbol{u}^n(s)]\tilde{\boldsymbol{u}}^n(t), [\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)]) \\
&-(\tilde{\varrho}^n(t)\boldsymbol{u}^n(s)\otimes\tilde{\boldsymbol{u}}^n(t), \nabla_x[\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)])
\end{aligned} \tag{IV.2.38}$$

and (IV.2.12) we get

$$
\begin{aligned}
&\|\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)\|^2_{L^2(\Omega)}\,\mathrm{d}x \leqslant \\
&\frac{1}{\varrho_*}\{|-\int_s^{s+\delta}\int_\Omega \tilde{\varrho}^n(t)\boldsymbol{u}^n(s)\otimes\tilde{\boldsymbol{u}}^n(t)\cdot\nabla_x[\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)]\,\mathrm{d}x\mathrm{d}t \\
&+\int_s^{s+\delta}\int_\Omega \tilde{\varrho}^n(t)\boldsymbol{u}^n(t)\otimes\tilde{\boldsymbol{u}}^n(t)\cdot\nabla_x[\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)]\,\mathrm{d}x\mathrm{d}t \\
&-\int_s^{s+\delta}\int_\Omega \tilde{\varrho}^n(t)[\nabla_x\boldsymbol{u}^n(s)]\tilde{\boldsymbol{u}}^n(t)\cdot[\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)]\,\mathrm{d}x\mathrm{d}t \\
&-\int_s^{s+\delta}\int_\Omega \mathbf{S}(t,x,\tilde{\varrho}^n(t),\mathbf{D}\tilde{\boldsymbol{u}}^n(t)):\mathbf{D}[\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)]\,\mathrm{d}x\mathrm{d}t \\
&+\int_s^{s+\delta}\int_\Omega \tilde{\varrho}^n(t)\boldsymbol{f}^n(t)\cdot[\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)]\,\mathrm{d}x\mathrm{d}t|\}.
\end{aligned}
\tag{IV.2.39}
$$

Next we integrate over $(0,T-\delta)$ w.r.t. time s and we intend to show that for any of the ten addends $I_k(s)$, $k=1,2,\dots,10$ on the right-hand side of (IV.2.39), the following inequalities are valid

$$
\int_0^{T-\delta} I_k(s)\,\mathrm{d}s \leqslant \kappa_k\theta(\delta) \quad \text{for } k=1,2,\dots,10,
\tag{IV.2.40}
$$

where $\theta(\delta)\to 0$ as $\delta\to 0$ and constant κ_k is independent of δ. To estimate the first six integrals let us employ (IV.2.12), the Hölder inequality, the assumption that $q=2p'$ and the fact that Λ maps B into B. Employing additionally the Young and Jensen inequality and following obvious relation $\int_0^{T-\delta}\frac{1}{\delta}\int_s^{s+\delta}a(t)\,\mathrm{d}t\mathrm{d}s \leqslant \int_0^T a(s)\,\mathrm{d}s$ for $a(t)\geqslant 0$ for one of representative terms we obtain

$$
\begin{aligned}
&|\int_0^{T-\delta}\int_s^{s+\delta}\int_\Omega \tilde{\varrho}^n(t)\boldsymbol{u}^n(s)\otimes\tilde{\boldsymbol{u}}^n(t)\cdot\nabla_x\boldsymbol{u}^n(s+\delta)\,\mathrm{d}x\mathrm{d}t\mathrm{d}s| \\
&\leqslant \varrho^*\int_0^{T-\delta}\int_s^{s+\delta}\|\boldsymbol{u}^n(s)\|_{L^q(\Omega)}\|\tilde{\boldsymbol{u}}^n(t)\|_{L^q(\Omega)}\|\nabla_x\boldsymbol{u}^n(s+\delta)\|_{L^p(\Omega)}\,\mathrm{d}t\mathrm{d}s \\
&\leqslant \delta\varrho^*\int_0^{T-\delta}\left\{\frac{1}{q}\|\boldsymbol{u}^n(s)\|^q_{L^q(\Omega)}+\frac{1}{q}\left|\frac{1}{\delta}\int_s^{s+\delta}\|\tilde{\boldsymbol{u}}^n(t)\|_{L^q(\Omega)}\,\mathrm{d}t\right|^q+\frac{1}{p}\|\nabla_x\boldsymbol{u}^n(s+\delta)\|^p_{L^p(\Omega)}\right\}\mathrm{d}s \\
&\leqslant \delta\varrho^*\int_0^{T-\delta}\left\{\frac{1}{q}\|\boldsymbol{u}^n(s)\|^q_{L^q(\Omega)}+\frac{1}{q}\frac{1}{\delta}\int_s^{s+\delta}\|\tilde{\boldsymbol{u}}^n(t)\|^q_{L^q(\Omega)}\,\mathrm{d}t+\frac{1}{p}\|\nabla_x\boldsymbol{u}^n(s+\delta)\|^p_{L^p(\Omega)}\right\}\mathrm{d}s \\
&\leqslant \delta\varrho^*\left(\frac{1}{q}\|\boldsymbol{u}^n(s)\|^q_{L^q(0,T;L^q(\Omega))}+\frac{1}{q}\|\tilde{\boldsymbol{u}}^n(s)\|^q_{L^q(0,T;L^q(\Omega))}+\frac{1}{p}\|\nabla_x\boldsymbol{u}^n\|^p_{L^p(0,T:L^p(\Omega))}\right) \\
&\leqslant \kappa_1\delta.
\end{aligned}
$$

Next we deal with nonlinear viscous term. Using the Fubini theorem, the Fenchel-Young inequality (Proposition III.1.12) and the Jensen inequality we get the following estimates

$$
\begin{aligned}
&|\int_0^{T-\delta}\int_s^{s+\delta}\int_\Omega \mathbf{S}(t,x,\tilde{\varrho}^n(t),\mathbf{D}\tilde{\boldsymbol{u}}^n(t)) : \mathbf{D}\boldsymbol{u}^n(s+\delta)\,\mathrm{d}x\mathrm{d}t\mathrm{d}s| \\
&= \delta\int_0^{T-\delta}\int_\Omega \left\{\left|\frac{1}{\delta}\int_s^{s+\delta} \mathbf{S}(t,x,\tilde{\varrho}^n(t),\mathbf{D}\tilde{\boldsymbol{u}}^n(t))dt\ \cdot \mathbf{D}\boldsymbol{u}^n(s+\delta)\right|\right\}\mathrm{d}x\mathrm{d}s \\
&\leqslant \delta\int_0^{T-\delta}\int_\Omega \left\{M^*\left(x,\frac{1}{\delta}\int_s^{s+\delta}\mathbf{S}(t,x,\tilde{\varrho}^n(t),\mathbf{D}\tilde{\boldsymbol{u}}^n(t))dt\right)+M\left(x,\mathbf{D}\boldsymbol{u}^n(s+\delta)\right)\right\}\mathrm{d}x\mathrm{d}s \\
&\leqslant \delta\int_\Omega\int_0^{T-\delta}\left\{\frac{1}{\delta}\int_s^{s+\delta} M^*\left(x,\mathbf{S}(t,x,\tilde{\varrho}^n(t),\mathbf{D}\tilde{\boldsymbol{u}}^n(t))\right)dt+M\left(x,\mathbf{D}\boldsymbol{u}^n(s+\delta)\right)\right\}\mathrm{d}s\mathrm{d}x \\
&\leqslant \int_\Omega\left\{\int_0^T M^*\left(x,\mathbf{S}(t,x,\tilde{\varrho}^n(s),\mathbf{D}\tilde{\boldsymbol{u}}^n(s))\right)ds+\int_0^{T-\delta} M\left(x,\mathbf{D}\boldsymbol{u}^n(s+\delta)\right)\mathrm{d}s\right\}\mathrm{d}x \\
&\leqslant \kappa_2\delta,
\end{aligned}
$$

where κ_2 is uniform w.r.t. n.

Using assumptions on $\boldsymbol{f}^n$ and (IV.2.20) we deduce

$$
\begin{aligned}
&|\int_0^{T-\delta}\int_s^{s+\delta}\int_\Omega \tilde{\varrho}^n(t)\boldsymbol{f}^n(t)\cdot\boldsymbol{u}^n(s+\delta)\,\mathrm{d}x\mathrm{d}t\mathrm{d}s| \\
&\leqslant \delta\varrho^*\int_0^{T-\delta}\left\{\frac{1}{p'}\left|\frac{1}{\delta}\int_s^{s+\delta}\|\boldsymbol{f}(t)\|_{L^{p'}(\Omega)}\,\mathrm{d}t\right|^{p'}+\|\boldsymbol{u}^n(s+\delta)\|_{L^p(\Omega)}\right\}\mathrm{d}s \\
&\leqslant \delta\varrho^*\left(\frac{1}{p'}\|\boldsymbol{f}^n(s+\delta)\|^{p'}_{L^{p'}(0,T;L^{p'}(\Omega))}+\frac{1}{p}\|\boldsymbol{u}^n\|^p_{L^p(0,T;L^p(\Omega))}\right)\leqslant\kappa_3\delta
\end{aligned}
$$

We proceed with the second source term in a similar way. Summarising all of the above estimates for integrals on the right-hand side of (IV.2.39) we prove (IV.2.36) and existence of approximate solution $\boldsymbol{u}^n$.

Remark IV.2.1. Since we already know that $\{\boldsymbol{u}^n\}_{n=1}^\infty$ is uniformly bounded in $L^2(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3))$ and above considerations show that for any approximate solution of (IV.2.3), (IV.2.4) we obtain

$$
\frac{1}{\delta^{1/2}}\left(\int_0^{T-\delta}\|\boldsymbol{u}^n(s+\delta)-\boldsymbol{u}^n(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s\right)^{1/2}\leqslant\kappa
$$

where κ is independent of n and δ. Therefore, as a byproduct, we obtain that $\{\boldsymbol{u}^n\}_{n=1}^\infty$ is uniformly bounded in Nikolskii space $N^{1/2,2}(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3))$.

IV.2.3. Strong convergence of ϱ^n and $\boldsymbol{u}^n$. Since at this moment we have existence of approximate solution to (IV.2.3 - IV.2.4) and the previous considerations show (IV.2.36) uniformly w.r.t. n, we get by [**115**, Theorem 3] that

$$
\boldsymbol{u}^n\to\boldsymbol{u}\quad\text{strongly in } L^2(Q;\mathbb{R}^3). \tag{IV.2.41}
$$

Using (IV.2.12), (IV.2.13) and (IV.2.33) the Aubin-Lions lemma provides that

(IV.2.42) $$\varrho^n \to \varrho \quad \text{strongly in}\, C([0,T]; W^{-1,5p/3}(\Omega)).$$

If we employ the same methods like Lions et al. in [**45**], [**91**, Chapter 2], we are able to deduce that

(IV.2.43) $$\varrho^n \to \varrho \quad \text{strongly in}\, C([0,T]; L^q(\Omega))\, \text{for all}\, q \in [1,\infty)\, \text{and a.e. in}\, Q,$$

and also

(IV.2.44) $$\lim_{t\to 0^+} \|\varrho(t) - \varrho_0\|_{L^q(\Omega)} = 0 \quad \text{for all}\, q \in [1,\infty),$$

which is the first part of the initial condition (IV.1.9). To give the reader a view of main steps we list some of them.

Using the fact that $\mathrm{div}_x \boldsymbol{u}^n = 0$ we see that the so-called strong and weak form of the transport equation coincide, i.e. equation (IV.2.3) is equivalent to $\partial_t \varrho^n - \boldsymbol{u}_n \nabla_x \varrho^n = 0$ in a weak sense. Consequently with the concept of renormalized solutions to the equation (IV.2.9), it is possible to strengthen (IV.2.42). First, we need the time-space version of the Friedrichs commutator lemma (see [**54**, Corollary 10.3],[**45**]). Since $\varrho \in L^q(0,T;L^q(\Omega))$ for $q \in [1,\infty)$ and $\boldsymbol{u} \in L^p(0,T;W^{1,p}_{\mathrm{div}}(\Omega;\mathbb{R}^3))$, then

$$\mathrm{div}_x\left(\sigma_\epsilon * (\varrho^n \boldsymbol{u}^n)\right) - \mathrm{div}_x\left((\sigma_\epsilon * \varrho^n)\boldsymbol{u}^n\right) \to 0 \quad \text{in } L^r(Q)$$

for r such that $\frac{1}{q} + \frac{1}{p} = \frac{1}{r} \in (0,1]$, where σ_ϵ is the standard mollifying operator acting on the space variable.

Additionally since $\varrho^n \geqslant \varrho_*$ and continuity equation (IV.2.3) is satisfied, then ϱ^n satisfies renormalized continuity equation, namely

(IV.2.45) $$\partial_t b(\varrho^n) + \mathrm{div}_x\left(b(\varrho^n)\boldsymbol{u}^n\right) = 0$$

in a weak sense for $b \in C^1([0,\infty)) \cap W^{1,\infty}(0,\infty)$ which vanishes near zero (see [**45**], [**54**, Appendix]). Next we are able to prove that

$$\varrho^n \in C([0,T]; L^q(\Omega)) \quad \text{for } q \in [1,\infty).$$

With the above information at hand following [**45**] or [**91**] we can prove (IV.2.43).

The task now is to show that

(IV.2.46) $$\varrho^n \boldsymbol{u}^n \rightharpoonup \varrho \boldsymbol{u} \quad \text{weakly in}\, L^q(0,T;L^q(\Omega;\mathbb{R}^3))\, \text{for all}\, q \in [1, 5p/3].$$

Indeed, (IV.2.43) provides that ϱ^n converges strongly to ϱ in $L^{\frac{5p}{3+\gamma}}(0,T;L^{\frac{5p}{3+\gamma}}(\Omega;\mathbb{R}^3)$, where $\gamma \in [0,\infty)$. This together with (IV.2.30) implies

$$\begin{aligned}\lim_{n\to\infty} \int_0^T \int_\Omega \varrho^n \boldsymbol{u}^n \cdot \boldsymbol{\varphi}\, \mathrm{d}x\mathrm{d}t = \lim_{n\to\infty} \int_0^T (\varrho^n, \boldsymbol{u}^n \cdot \boldsymbol{\varphi})\, \mathrm{d}t = \int_0^T (\varrho, \boldsymbol{u} \cdot \boldsymbol{\varphi})\, \mathrm{d}t \\ = \int_0^T \int_\Omega \varrho \boldsymbol{u} \cdot \boldsymbol{\varphi}\, \mathrm{d}x\mathrm{d}t\end{aligned}$$

for every $\boldsymbol{\varphi} \in (L^{\frac{5p}{6+\varepsilon}}(0,T;L^{\frac{5p}{6+\varepsilon}}(\Omega;\mathbb{R}^3)))^*$, where $\varepsilon(\gamma) \in [0, \frac{5p}{3})$. Therefore (IV.2.31) infers that (IV.2.46) holds. Finally from (IV.2.33) and (IV.2.46) we conclude that ϱ and $\boldsymbol{u}$ satisfy (IV.1.8).

Additionally previous considerations imply, by using the test function of the form $\mathbb{1}_{(t_1,t_2)}h$, $h \in W^{1,5p/(5p-3)}$ in (IV.1.8), partial integration w.r.t. time and the density of $W^{1,5p/(5p-3)}$ in L^1, that $\varrho \in C([0,T]; L^\infty_{\text{weak}})$, i.e. for all $h \in L^1$ and all $0 \leqslant t_0 \leqslant T$ we have

$$\lim_{t \to t_0} (\varrho(t), h) = (\varrho(t_0), h). \tag{IV.2.47}$$

Using (IV.2.41) and (IV.2.28) we infer by interpolation inequalities that

$$\boldsymbol{u}^n \to \boldsymbol{u} \quad \text{strongly in } L^r(Q; \mathbb{R}^3) \text{ for all } r \in [1, 5p/3) \text{ and a.e. in } Q. \tag{IV.2.48}$$

Summarising (IV.2.48), (IV.2.12) and (IV.2.30), (IV.2.43),

$$\varrho^n \boldsymbol{u}^n \otimes \boldsymbol{u}^n \rightharpoonup \varrho \boldsymbol{u} \otimes \boldsymbol{u} \quad \text{weakly in } L^{r'}(0,T; W^{-1,r'}) \text{ for } r \text{ sufficiently large,}$$

i.e. $\frac{1}{q} + \frac{6}{5p} + \frac{1}{r} < 1$, with arbitrary $q \in [1, \infty)$. Density argument and (IV.2.21) provides

$$\varrho^n \boldsymbol{u}^n \otimes \boldsymbol{u}^n \rightharpoonup \varrho \boldsymbol{u} \otimes \boldsymbol{u} \quad \text{weakly in } L^{p'}(0,T; W^{-1,p'}_{\text{div}}) \text{ for } p \geqslant 11/5. \tag{IV.2.49}$$

In particular we obtain

$$\lim_{n\to\infty} \int_0^T \int_\Omega \varrho^n \boldsymbol{u}^n \otimes \boldsymbol{u}^n : \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \varrho \boldsymbol{u} \otimes \boldsymbol{u} : \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t \quad \text{for } \boldsymbol{\varphi} \in \mathcal{D}((-\infty, T); \mathcal{V}). \tag{IV.2.50}$$

IV.2.4. Integration by parts. For any function z (for which integrals below have sense) and for $h > 0$ we denote

$$(\tilde{\sigma}_h^+ * z)(t,x) := \frac{1}{h} \int_0^h z(t+\tau, x) \, \mathrm{d}\tau,$$
$$(\tilde{\sigma}_h^- * z)(t,x) := \frac{1}{h} \int_{-h}^0 z(t+\tau, x) \, \mathrm{d}\tau,$$

where $*$ means convolution w.r.t. time variable. Let us define also

$$D^{+h} z := \frac{z(t+h, x) - z(t,x)}{h},$$
$$D^{-h} z := \frac{z(t,x) - z(t-h, x)}{h}.$$

Then it is easy to observe that

$$\partial_t (\tilde{\sigma}_h^+ * z) = D^{+h} z \qquad \text{and} \qquad \partial_t (\tilde{\sigma}_h^- * z) = D^{-h} z. \tag{IV.2.51}$$

Let us take $h > 0$ and $0 < s_0 < s < T$ such that $h \leqslant \min\{s_0, T - s\}$. We multiply each equation in the system (IV.2.7) by

$$\tilde{\sigma}_h^+ * ((\tilde{\sigma}_h^- * \alpha_j^i(t)) \, \mathbb{1}_{(s_0,s)}),$$

next we sum up over $j = 1, \ldots, i$, where $i \leqslant n$ and integrate this sum over time interval $(0,T)$. Noticing that $\tilde{\sigma}_h^+ * ((\tilde{\sigma}_h^- * \boldsymbol{u}^i) \, \mathbb{1}_{(s_0,s)}) = \sum_{j=1}^i \tilde{\sigma}_h^+ * ((\tilde{\sigma}_h^- * \alpha_j^i(t)) \, \mathbb{1}_{(s_0,s)}) \boldsymbol{\omega}^j(x)$ let

$$\boldsymbol{u}^{h,i} \overset{\text{def}}{=} \tilde{\sigma}_h^+ * ((\tilde{\sigma}_h^- * \boldsymbol{u}^i) \, \mathbb{1}_{(s_0,s)})$$

with $h \leqslant \min\{s_0, T-s\}$. Since

$$\int_0^T \left\langle \partial_t(\varrho^n \boldsymbol{u}^n), \boldsymbol{u}^{h,i} \right\rangle \mathrm{d}t = \int_0^T \left\langle \partial_t(\tilde{\sigma}_h^- * (\varrho^n \boldsymbol{u}^n)), ((\tilde{\sigma}_h^- * \boldsymbol{u}^i)\mathbb{1}_{(s_0,s)}) \right\rangle \mathrm{d}t,$$

and $i \leqslant n$ we get in the limit as $n \to \infty$

(IV.2.52)

$$\int_{s_0}^s \left\langle (\partial_t(\tilde{\sigma}_h^- * \varrho \boldsymbol{u})), (\tilde{\sigma}_h^- * \boldsymbol{u}^i) \right\rangle \mathrm{d}t = \int_0^T \int_\Omega (\varrho \boldsymbol{u} \otimes \boldsymbol{u}) : \nabla_x \boldsymbol{u}^{h,i} \, \mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega \bar{\mathbf{S}} : \mathbf{D}\boldsymbol{u}^{h,i} \, \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \varrho \boldsymbol{f} \cdot \boldsymbol{u}^{h,i} \, \mathrm{d}x\mathrm{d}t.$$

Indeed, let us notice that for fixed h and i it is provided that $\boldsymbol{u}^{h,i}$, $\mathbf{D}\boldsymbol{u}^{h,i} \in L^\infty$. Then the convergence process in the first term on the left-hand side of (IV.2.52) is provided by the fact that $\tilde{\sigma}_h^- * \boldsymbol{u}^i$ is locally Lipschitz w.r.t. time variable and (IV.2.46) holds. In terms on the left-hand side we use respectively (IV.2.49), (IV.2.23) (obviously $L^\infty \subset E_M$) and (IV.2.6) with (IV.2.43).

Our aim now is to use a test function in (IV.2.52)

$$\boldsymbol{u}^h \stackrel{\text{def}}{=} \tilde{\sigma}_h^+ * ((\tilde{\sigma}_h^- * \boldsymbol{u}) \ \mathbb{1}_{(s_0,s)})$$

with $0 < h < \min\{s_0, T-s\}$. For this purpose define the truncation operator $\bar{\mathcal{T}}_m : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}$ such that

$$\bar{\mathcal{T}}_m(\mathbf{K}) = \begin{cases} \mathbf{K} & |\mathbf{K}| \leqslant m, \\ m\frac{\mathbf{K}}{|\mathbf{K}|} & |\mathbf{K}| > m. \end{cases}$$

Observe the following identity

$$\begin{aligned} \int_{s_0}^s \left\langle (\partial_t(\tilde{\sigma}_h^- * (\varrho \boldsymbol{u})), (\tilde{\sigma}_h^- * \boldsymbol{u}^i) \right\rangle \mathrm{d}t = {} & \int_0^T \int_\Omega (\varrho \boldsymbol{u} \otimes \boldsymbol{u}) : \nabla_x \boldsymbol{u}^{h,i} \, \mathrm{d}x\mathrm{d}t \\ & + \int_0^T \int_\Omega (\bar{\mathcal{T}}_m(\bar{\mathbf{S}}) - \bar{\mathbf{S}}) : \mathbf{D}\boldsymbol{u}^{h,i} \, \mathrm{d}x\mathrm{d}t \\ & - \int_0^T \int_\Omega \bar{\mathcal{T}}_m(\bar{\mathbf{S}}) : \mathbf{D}\boldsymbol{u}^{h,i} \, \mathrm{d}x\mathrm{d}t \\ & + \int_0^T \int_\Omega \varrho \boldsymbol{f} \cdot \boldsymbol{u}^{h,i} \, \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{IV.2.53}$$

Let us concentrate now on the right-hand side of (IV.2.53) and investigate the first and the last term.

The sequence $\{\boldsymbol{u}^{h,i}\}_{i=1}^\infty$ is weakly convergent to $\boldsymbol{u}^h$ in $L^p(0,T; W^{1,p}_{0,\mathrm{div}}(\Omega;\mathbb{R}^3))$ when $i \to \infty$. Note that if $p \geqslant \frac{11}{5}$, then since ϱ is bounded we infer that $\int_0^T \int_\Omega (\varrho \boldsymbol{u} \otimes \boldsymbol{u}) \cdot \nabla_x \boldsymbol{u}^{h,i} \, \mathrm{d}x\mathrm{d}t \to \int_0^T \int_\Omega (\varrho \boldsymbol{u} \otimes \boldsymbol{u}) \cdot \nabla_x \boldsymbol{u}^h \, \mathrm{d}x\mathrm{d}t$ as $i \to \infty$.

Since $\boldsymbol{f} \in L^{p'}(0,T; L^{p'}(\Omega;\mathbb{R}^3))$ we treat in the same way the source term to obtain that $\int_0^T \int_\Omega \varrho \boldsymbol{f} : \boldsymbol{u}^{h,i} \, \mathrm{d}x\mathrm{d}t \to \int_0^T \int_\Omega \varrho \boldsymbol{f} : \boldsymbol{u}^h \, \mathrm{d}x\mathrm{d}t$ as $i \to \infty$.

Now we show convergence process in the second term on the right-hand side of (IV.2.53). We fix $k \in \mathbb{N}$ and using the Fenchel-Young inequality, the convexity of M and that M^* satisfies the Δ_2-condition (see (IV.1.12)) with some nonnegative integrable function g_{M^*} (see (III.1.15)) we estimate the integral

$$
\begin{aligned}
\int_0^T \int_\Omega |(\bar{\mathcal{T}}_m(\bar{\mathbf{S}}) - \bar{\mathbf{S}}) : \mathbf{D}\boldsymbol{u}^{h,i}| \,\mathrm{d}x\mathrm{d}t &\leqslant \int_0^T \int_\Omega M^*(x, 2^k(\bar{\mathcal{T}}_m(\bar{\mathbf{S}}) - \bar{\mathbf{S}})) \,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega M(x, \frac{1}{2^k}\mathbf{D}\boldsymbol{u}^{h,i}) \,\mathrm{d}x\mathrm{d}t \\
&\leqslant C^k_{M^*} \int_0^T \int_\Omega M^*(x, \bar{\mathcal{T}}_m(\bar{\mathbf{S}}) - \bar{\mathbf{S}}) \,\mathrm{d}x\mathrm{d}t \\
&\quad + k \int_0^T \int_\Omega g_{M^*}(x) \mathbb{1}_{\{|\bar{\mathbf{S}}(t,x)| > m\}} \,\mathrm{d}x\mathrm{d}t \\
&\quad + \frac{1}{2^k} \int_0^T \int_\Omega M(x, \mathbf{D}\boldsymbol{u}^{h,i}) \,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{IV.2.54}
$$

Inequality (IV.2.16) and Proposition III.2.5 provide that for each $0 < h \leqslant \min\{s_0, T - s\}$ it holds

$$
\sup_h \sup_{i \in \mathbb{N}} \int_0^T \int_\Omega M(x, \mathbf{D}\boldsymbol{u}^{h,i}) \,\mathrm{d}x\mathrm{d}t < C,
$$

where C is a nonnegative constant independent of i and h. Consequently we infer that

$$
\lim_{k \to \infty} \frac{1}{2^k} \sup_h \sup_{i \in \mathbb{N}} \int_0^T \int_\Omega M(x, \mathbf{D}\boldsymbol{u}^{h,i}) \,\mathrm{d}x\mathrm{d}t = 0.
$$

Due to the convexity and symmetry of M^* and that $M^*(x, 0) = 0$ a.e. it holds that

$$
M^*(x, \bar{\mathcal{T}}_m(\bar{\mathbf{S}}) - \bar{\mathbf{S}}) \leqslant M^*(x, \bar{\mathbf{S}}).
$$

Since M^* satisfies the Δ_2-condition and $\bar{\mathbf{S}}$ is an element of $\mathcal{L}_{M^*}(Q; \mathbb{R}^{3\times3}_{\mathrm{sym}})$, the above inequality yields by the Lebesgue convergence theorem that $\int_Q M^*(x, \bar{\mathcal{T}}_m(\bar{\mathbf{S}}) - \bar{\mathbf{S}}) \,\mathrm{d}x\mathrm{d}t$ converges to zero as $m \to \infty$. Hence

$$
\lim_{k \to \infty} \lim_{m \to \infty} \int_{s_0}^s \int_\Omega C^k_{M^*} M^*(x, \bar{\mathcal{T}}_m(\bar{\mathbf{S}}) - \bar{\mathbf{S}}) + k g_{M^*}(x) \mathbb{1}_{\{|\bar{\mathbf{S}}(t,x)| > m\}} \,\mathrm{d}x\mathrm{d}t = 0.
$$

Then we can pass to the limits in the second and the third term on the right-hand side of (IV.2.53) (together with (IV.2.54)) consecutively with $i \to \infty$, $m \to \infty$ and $k \to \infty$.

Now we will concentrate on the left hand-side term of (IV.2.52). Let us notice that as $\varrho\boldsymbol{u} \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))$, $\tilde{\sigma}_h^- * \varrho\boldsymbol{u}$ is a Lipschitz function w.r.t. the time variable, hence $\partial_t(\tilde{\sigma}_h^- * \varrho\boldsymbol{u}) \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))$. By (IV.2.26) and letting $i \to \infty$

we obtain

$$\begin{aligned} L_h &:= \int_{s_0}^{s}\int_\Omega (\partial_t(\tilde{\sigma}_h^- * (\varrho\boldsymbol{u}))\cdot(\tilde{\sigma}_h^- * \boldsymbol{u})\,\mathrm{d}x\mathrm{d}t \\ &= \int_{s_0}^{s}\int_\Omega \Big((\varrho\boldsymbol{u}\otimes\boldsymbol{u}) : \nabla_x\boldsymbol{u}^h - \boldsymbol{\mathsf{S}} : \mathbf{D}\boldsymbol{u}^h + \varrho\boldsymbol{f}\cdot\boldsymbol{u}^h\Big)\,\mathrm{d}x\mathrm{d}t =: R_h. \end{aligned} \tag{IV.2.55}$$

In order to pass with $h \to 0^+$ we conclude form (IV.2.51) that

$$L_h = \int_{s_0}^{s}\int_\Omega (D^{-h}(\varrho\boldsymbol{u}))\cdot(\tilde{\sigma}_h^- * \boldsymbol{u})\,\mathrm{d}x\mathrm{d}t.$$

Moreover notice that

$$\begin{aligned} L_h &= \int_{s_0}^{s}\int_\Omega (\varrho D^{-h}\boldsymbol{u})\cdot(\tilde{\sigma}_h^- * \boldsymbol{u}) + ((D^{-h}\varrho)\boldsymbol{u}(t-h))\cdot(\tilde{\sigma}_h^- * \boldsymbol{u})\,\mathrm{d}x\mathrm{d}t \\ &= \int_{s_0}^{s}\int_\Omega \varrho\cdot\frac{1}{2}\partial_t|\tilde{\sigma}_h^- * \boldsymbol{u}|^2 + \big(\tilde{\sigma}_h^- * (\varrho\boldsymbol{u})\big)\cdot\big(\nabla_x\left(\boldsymbol{u}(t-h)\cdot(\tilde{\sigma}_h^- * \boldsymbol{u})\right)\big)\,\mathrm{d}x\mathrm{d}t, \end{aligned}$$

where we used (IV.2.51) and relation $D^{-h}\varrho = -\mathrm{div}_x(\tilde{\sigma}_h^- * (\varrho\boldsymbol{u}))$, which is provided by the fact that the couple $(\varrho,\ \boldsymbol{u})$ solves the continuity equation $\partial_t\varrho + \mathrm{div}_x(\varrho\boldsymbol{u}) = 0$ in a weak sense. Inserting $z = \frac{1}{2}|\tilde{\sigma}_h^- * \boldsymbol{u}|^2$ into the weak formulation of the continuity equation, which means that for all $s_0,\ s \in [0,T]$, $s_0 < s$

$$\int_{s_0}^{s}\int_\Omega (\varrho(\tau)\cdot\partial_t z(\tau) + \varrho(\tau)\boldsymbol{u}(\tau)\cdot\nabla_x z(\tau))\,\mathrm{d}x\mathrm{d}\tau = \int_\Omega \varrho(s)\cdot z(s) - \varrho(s_0)\cdot z(s_0)\,\mathrm{d}x$$

(for all $z \in L^r(0,T;W^{1,r})$ with $r = 5p/(5p-3)$ and $\partial_t z \in L^{1+\delta}(0,T;L^{1+\delta})$) we obtain

$$\begin{aligned} L_h = &\int_\Omega \varrho(s)\cdot(\frac{1}{2}|\tilde{\sigma}_h^- * \boldsymbol{u}(s)|^2)\,\mathrm{d}x - \int_\Omega \varrho(s_0)\cdot(\frac{1}{2}|\tilde{\sigma}_h^- * \boldsymbol{u}(s_0)|^2)\,\mathrm{d}x \\ &- \int_{s_0}^{s}\int_\Omega (\varrho\boldsymbol{u})\cdot(\frac{1}{2}\nabla_x|\tilde{\sigma}_h^- * \boldsymbol{u}|^2)\,\mathrm{d}x\mathrm{d}t \\ &+ \int_{s_0}^{s}\int_\Omega \big(\tilde{\sigma}_h^- * (\varrho\boldsymbol{u})\big)\cdot\big(\nabla_x\left[\boldsymbol{u}(t-h)\cdot(\tilde{\sigma}_h^- * \boldsymbol{u})\right]\big)\,\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Let us notice that $\tilde{\sigma}_h^- * \boldsymbol{u}$ converges strongly (locally in time) to $\boldsymbol{u}$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^3))$ and in $L^{5p/3}(0,T;L^{5p/3}(\Omega;\mathbb{R}^3))$ and $\nabla_x\tilde{\sigma}_h^- * \boldsymbol{u}$ converges strongly (locally in time) to $\nabla_x\boldsymbol{u}$ in $L^p(0,T;L^p(\Omega;\mathbb{R}^{3\times 3}))$ as $h \to 0^+$. The same arguments are valid for translation $\tau_{-h}\boldsymbol{u} = \boldsymbol{u}(t-h)$. Then by the Hölder inequality letting $h \to 0^+$ in the above we obtain for almost all s_0 and s in $(0,T)$

$$\begin{aligned}\lim_{h\to 0^+} L_h &= \int_{s_0}^{s}\int_\Omega (\varrho\boldsymbol{u})\cdot(\frac{1}{2}\nabla_x|\boldsymbol{u}|^2)\,\mathrm{d}x\mathrm{d}t\\ &\quad+\frac{1}{2}\int_\Omega \varrho(s,x)|\boldsymbol{u}(s,x)|^2\,\mathrm{d}x-\frac{1}{2}\int_\Omega \varrho(s_0,x)|\boldsymbol{u}(s_0,x)|^2\,\mathrm{d}x\\ &=\int_{s_0}^{s}\int_\Omega \varrho\boldsymbol{u}\otimes\boldsymbol{u}:\nabla_x\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t\\ &\quad+\frac{1}{2}\int_\Omega \varrho(s,x)|\boldsymbol{u}(s,x)|^2\,\mathrm{d}x-\frac{1}{2}\int_\Omega \varrho(s_0,x)|\boldsymbol{u}(s_0,x)|^2\,\mathrm{d}x.\end{aligned} \tag{IV.2.56}$$

Next we consider the right-hand side of (IV.2.55) and pass with $h\to 0^+$. First we investigate the convergence of the term $\int_{s_0}^{s}\int_\Omega(\varrho\boldsymbol{u}\otimes\boldsymbol{u}:\nabla_x\boldsymbol{u}^h)\,\mathrm{d}x\mathrm{d}t$. Since condition (IV.1.11) provides that $\mathbf{D}\boldsymbol{u}\in L^p(0,T;L^p(\Omega;\mathbb{R}^{3\times3}))$ and due to the Korn inequality $\nabla_x\boldsymbol{u}\in L^p(0,T;L^p(\Omega;\mathbb{R}^{3\times3}))$, we have that also the sequence $\nabla_x\boldsymbol{u}^h = \nabla_x\left(\tilde\sigma_h^+ * ((\tilde\sigma_h^- * \boldsymbol{u})\mathbb{1}_{(s_0,s)})\right)$ is uniformly bounded in $L^p(0,T;L^p(\Omega;\mathbb{R}^{3\times3}))$. Hence we obtain, for subsequence if needed,

$$\lim_{h\to0^+}\int_{s_0}^{s}\int_\Omega(\varrho\boldsymbol{u}\otimes\boldsymbol{u}:\nabla_x\boldsymbol{u}^h)\,\mathrm{d}x\mathrm{d}t=\int_{s_0}^{s}\int_\Omega(\varrho\boldsymbol{u}\otimes\boldsymbol{u}:\nabla_x\boldsymbol{u})\,\mathrm{d}x\mathrm{d}t.$$

Since $\boldsymbol{f}\in L^{p'}(0,T;L^{p'}(\Omega;\mathbb{R}^3))$ and ϱ satisfies (IV.2.12) in the same way we conclude

$$\lim_{h\to0^+}\int_{s_0}^{s}\int_\Omega(\varrho\boldsymbol{f})\cdot\boldsymbol{u}^h\,\mathrm{d}x\mathrm{d}t=\int_{s_0}^{s}\int_\Omega\varrho\boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t. \tag{IV.2.57}$$

Let us concentrate now on the term

$$\int_0^T\int_\Omega\overline{\mathbf{S}}:(\tilde\sigma_h^+ * ((\tilde\sigma_h^- * \mathbf{D}\boldsymbol{u})\mathbb{1}_{(s_0,s)}))\,\mathrm{d}x\mathrm{d}t=\int_{s_0}^{s}\int_\Omega(\tilde\sigma_h^- * \overline{\mathbf{S}}):(\tilde\sigma_h^- * \mathbf{D}\boldsymbol{u})\,\mathrm{d}x\mathrm{d}t.$$

Sequences $\{\tilde\sigma_h^- * \overline{\mathbf{S}}\}_h$ and $\{\tilde\sigma_h^- * \mathbf{D}\boldsymbol{u}\}_h$ converge in measure on Q due to Proposition III.2.4. Moreover, since M and M^* are convex nonnegative functions, then the weak lower semicontinuity and estimate (IV.2.16) provide that the integrals

$$\int_0^T\int_\Omega M(x,\mathbf{D}\boldsymbol{u})\,\mathrm{d}x\mathrm{d}t\quad\text{and}\quad\int_0^T\int_\Omega M^*(x,\overline{\mathbf{S}})\,\mathrm{d}x\mathrm{d}t \tag{IV.2.58}$$

are finite. Hence Proposition III.2.5 implies that the sequences $\{\tilde\sigma_h^- * \overline{\mathbf{S}}\}_h$ and $\{\tilde\sigma_h^- * \mathbf{D}\boldsymbol{u}\}_h$ are uniformly integrable and hence according to Lemma III.2.1 we have

$$\begin{aligned}\tilde\sigma_h^- * \mathbf{D}\boldsymbol{u}&\xrightarrow{M}\mathbf{D}\boldsymbol{u}\quad\text{modularly in } L_M(Q;\mathbb{R}^{3\times3}_{\mathrm{sym}}),\\ \tilde\sigma_h^- * \overline{\mathbf{S}}&\xrightarrow{M^*}\overline{\mathbf{S}}\quad\text{modularly in } L_{M^*}(Q;\mathbb{R}^{3\times3}_{\mathrm{sym}}).\end{aligned}$$

Applying Proposition III.2.3 allows to conclude

$$\lim_{h\to0^+}\int_{s_0}^{s}\int_\Omega(\tilde\sigma_h^- * \overline{\mathbf{S}}):(\tilde\sigma_h^- * \mathbf{D}\boldsymbol{u})\,\mathrm{d}x\mathrm{d}t=\int_{s_0}^{s}\int_\Omega\overline{\mathbf{S}}:\mathbf{D}\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t. \tag{IV.2.59}$$

Summarising arguments (IV.2.56), (IV.2.59) and (IV.2.57) we are able to pass to the limit in (IV.2.52) and we obtain

(IV.2.60)
$$\frac{1}{2}\int_\Omega \varrho(s,x)|\boldsymbol{u}(s,x)|^2\,\mathrm{d}x + \int_{s_0}^{s}\int_\Omega \overline{\mathbf{S}} : \mathbf{D}\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t = \int_{s_0}^{s}\int_\Omega \varrho\boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_\Omega \varrho(s_0,x)|\boldsymbol{u}(s_0,x)|^2\,\mathrm{d}x.$$

IV.2.5. Continuity w.r.t. time in the weak topology and the initial condition. Using the already proved properties of the density and the velocity field, namely $\varrho \in C([0,T], L^q(\Omega))$ for $q \in [1,\infty)$ and $\boldsymbol{u} \in C(0,T; L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3))$, it we are lead to a conclusion that $(\varrho(\cdot)\boldsymbol{u}(\cdot), \tilde{\boldsymbol{\varphi}})$ is continuous at $s_1 \in (0,T)$ for all $\tilde{\boldsymbol{\varphi}} \in W^{s,2}_{0,\mathrm{div}}$, in other words, $\varrho\boldsymbol{u} \in C(0,T; (W^{s,2}_{0,\mathrm{div}})^*_{\mathrm{weak}})$ or

$$\lim_{s_2\to s_1} (\varrho(s_2)\boldsymbol{u}(s_2) - \varrho(s_1)\boldsymbol{u}(s_1), \tilde{\boldsymbol{\varphi}}) = 0.$$

Since $\boldsymbol{u} \in L^\infty(0,T; L^2_{\mathrm{div}}(\Omega;\mathbb{R}^3))$, $\varrho \in C([0,T]; L^q(\Omega))$ for $q \in [1,\infty)$ and $W^{s,2}_{0,\mathrm{div}}$ is dense in L^2_{div}, we observe that $\varrho\boldsymbol{u} \in C([0,T]; L^2_{\mathrm{div,weak}}(\Omega;\mathbb{R}^3))$. As a consequence we have

$$\lim_{s_1\to 0}(\varrho(s_1)\boldsymbol{u}(s_1) - \varrho_0\boldsymbol{u}_0, \tilde{\boldsymbol{\varphi}}) = 0 \quad \text{for all } \tilde{\boldsymbol{\varphi}} \in L^2_{\mathrm{div}}. \tag{IV.2.61}$$

Integrating (IV.2.14) over time interval $(0,s_1)$, using that $(\mathbf{S}(t,x,\varrho^n,\mathbf{D}\boldsymbol{u}^n), \mathbf{D}\boldsymbol{u}^n)$ is nonnegative (because of monotonicity and that $\mathbf{S}(\cdot,\cdot,\cdot,\mathbf{0}) = \mathbf{0}$) and taking the limit as $n\to\infty$ we obtain

$$(\varrho(s_1), |\boldsymbol{u}(s_1)|^2) - (\varrho(0), |\boldsymbol{u}(0)|^2) \leqslant 2\int_0^{s_1} (\varrho\boldsymbol{f}, \boldsymbol{u})\,\mathrm{d}t. \tag{IV.2.62}$$

If we employ obvious identity

$$\|\sqrt{\varrho(s_1)}(\boldsymbol{u}(s_1) - \boldsymbol{u}_0)\|^2_{L^2(\Omega)} = (\varrho(s_1), |\boldsymbol{u}(s_1)|^2) - 2(\varrho(s_1)\boldsymbol{u}(s_1), \boldsymbol{u}_0) + (\varrho(s_1), |\boldsymbol{u}_0|^2),$$

then the second part of property (IV.1.9) is an easy consequence of (IV.2.62) and

(IV.2.63)
$$\begin{aligned}
&\|\sqrt{\varrho(s_1)}(\boldsymbol{u}(s_1) - \boldsymbol{u}_0)\|^2_{L^2(\Omega)} = (\varrho(s_1), |\boldsymbol{u}(s_1)|^2) - 2(\varrho(s_1)\boldsymbol{u}(s_1), \boldsymbol{u}_0) + (\varrho(s_1), |\boldsymbol{u}_0|^2)\\
&= (\varrho(s_1), |\boldsymbol{u}(s_1)|^2) - (\varrho_0, |\boldsymbol{u}_0|^2) - 2(\varrho(s_1)\boldsymbol{u}(s_1) - \varrho_0\boldsymbol{u}_0, \boldsymbol{u}_0) + (\varrho(s_1) - \varrho_0, |\boldsymbol{u}_0|^2)\\
&\leqslant 2\int_0^{s_1} (\varrho\boldsymbol{f}, \boldsymbol{u})\,\mathrm{d}t - 2(\varrho(s_1)\boldsymbol{u}(s_1) - \varrho_0\boldsymbol{u}_0, \boldsymbol{u}_0) + (\varrho(s_1) - \varrho_0, |\boldsymbol{u}_0|^2).
\end{aligned}$$

Letting $s_1 \to 0^+$ in (IV.2.63) using (IV.2.61), (IV.2.47) and $(\varrho\boldsymbol{f}, \boldsymbol{u}) \in L^1(0,T; L^1(\Omega))$ we can conclude that

$$\lim_{s_1\to 0} \|\sqrt{\varrho(s_1)}(\boldsymbol{u}(s_1) - \boldsymbol{u}_0)\|^2_{L^2(\Omega)} = 0. \tag{IV.2.64}$$

Hence this implies together with (IV.2.12) the second part of (IV.1.9). Above arguments and (IV.2.63), (IV.2.64) provide also the fact which we will use later:

$$\lim_{s_1\to 0}(\varrho(s_1), |\boldsymbol{u}(s_1)|^2) = (\varrho_0, |\boldsymbol{u}_0|^2). \tag{IV.2.65}$$

IV.2.6. Monotonicity method. Using the property (IV.2.65) and letting $s_0 \to 0$ in (IV.2.60) we obtain

$$\frac{1}{2}\int_\Omega \varrho(s,x)|\boldsymbol{u}(s,x)|^2\,\mathrm{d}x + \int_0^s\int_\Omega \overline{\mathbf{S}} : \mathbf{D}\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t = \int_0^s\int_\Omega \varrho \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_\Omega \varrho_0(x)|\boldsymbol{u}_0(x)|^2\,\mathrm{d}x.$$

Additionally integrating (IV.2.14) over the interval $(0,s)$ allows to conclude by (IV.2.5), (IV.2.6), (IV.2.43), (IV.2.46), (IV.2.48) that

$$\frac{1}{2}\int_\Omega \varrho(s,x)|\boldsymbol{u}(s,x)|^2\,\mathrm{d}x + \lim_{n\to\infty}\int_0^s\int_\Omega \mathbf{S}(t,x,\varrho^n,\mathbf{D}\boldsymbol{u}^n) : \mathbf{D}\boldsymbol{u}^n\,\mathrm{d}x\mathrm{d}t = \int_0^s\int_\Omega \varrho \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_\Omega \varrho_0(x)|\boldsymbol{u}_0(x)|^2\,\mathrm{d}x.$$

Consequently we obtain

$$\limsup_{n\to\infty}\int_0^s\int_\Omega \mathbf{S}(t,x,\varrho^n,\mathbf{D}\boldsymbol{u}^n) : \mathbf{D}\boldsymbol{u}^n\,\mathrm{d}x\mathrm{d}t \leqslant \int_0^s\int_\Omega \overline{\mathbf{S}} : \mathbf{D}\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t. \tag{IV.2.66}$$

By Q^s we will mean the set $(0,s)\times\Omega$. Since $\mathbf{S}$ is monotone, then we have

$$\int_{Q^s}(\mathbf{S}(t,x,\varrho^n,\mathbf{w}) - \mathbf{S}(t,x,\varrho^n,\mathbf{D}\boldsymbol{u}^n)) : (\mathbf{w} - \mathbf{D}\boldsymbol{u}^n)\,\mathrm{d}x\mathrm{d}t \geqslant 0 \tag{IV.2.67}$$

for all $\mathbf{w} \in L^\infty(Q;\mathbb{R}^{3\times 3})$. Observe that also $\mathbf{S}(t,x,\varrho^n,\mathbf{w}) \in L^\infty(Q;\mathbb{R}^{3\times 3})$. We prove this by contradiction, i.e. let us suppose that $\mathbf{S}(t,x,\varrho^n,\mathbf{w})$ is unbounded. Then, since M is nonnegative, by (IV.1.6), it holds

$$|\mathbf{w}| \geqslant \frac{M^*(x,\mathbf{S}(t,x,\varrho^n,\mathbf{w}))}{|\mathbf{S}(t,x,\varrho^n,\mathbf{w})|}.$$

The right-hand side tends to infinity as $|\mathbf{S}(t,x,\varrho^n,\mathbf{w})| \to \infty$ by $(\text{IV.1.12})_2$, which contradicts that $\mathbf{w} \in L^\infty(Q;\mathbb{R}^{3\times 3})$. Now employing continuity of $\mathbf{S}$ w.r.t. the third variable and (IV.2.12) we obtain uniform boundedness of $\{\mathbf{S}(t,x,\varrho^n,\mathbf{w})\}_{n=1}^\infty$ w.r.t n. Together with boundedness of Q^s this gives uniform integrability of a sequence $\{M^*(\mathbf{S}(t,x,\varrho^n,\mathbf{w}))\}_{n=1}^\infty$. Lemma III.2.1 and (IV.2.43) provide modular convergence of the sequence. Since M^* satisfies the Δ_2–condition, then the modular and strong convergence in L_{M^*} coincide (see [**87**]) and hence $\mathbf{S}(t,x,\varrho^n,\mathbf{w}) \to \mathbf{S}(t,x,\varrho,\mathbf{w})$ strongly in L_{M^*}. Therefore by (IV.2.17) we deduce

$$\lim_{n\to\infty}\int_{Q^s}\mathbf{S}(t,x,\varrho^n,\mathbf{w}) : \mathbf{D}\boldsymbol{u}^n\,\mathrm{d}x\mathrm{d}t = \int_{Q^s}\mathbf{S}(t,x,\varrho,\mathbf{w}) : \mathbf{D}\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t. \tag{IV.2.68}$$

Before passing to the limit with $n \to \infty$, we rewrite (IV.2.67)

$$\int_{Q^s} \mathbf{S}(t,x,\varrho^n,\mathbf{D}\boldsymbol{u}^n) : \mathbf{D}\boldsymbol{u}^n \,\mathrm{d}x\mathrm{d}t$$
$$\geqslant \int_{Q^s} \mathbf{S}(t,x,\varrho^n,\mathbf{D}\boldsymbol{u}^n) : \mathbf{w}\,\mathrm{d}x\mathrm{d}t + \int_{Q^s} \mathbf{S}(t,x,\varrho^n,\mathbf{w}) : (\mathbf{D}\boldsymbol{u}^n - \mathbf{w})\,\mathrm{d}x\mathrm{d}t.$$

hence (IV.2.19), (IV.2.24), (IV.2.66), (IV.2.68) give

$$\int_{Q^s} \overline{\mathbf{S}} : \mathbf{D}\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t \geqslant \int_{Q^s} \overline{\mathbf{S}} : \mathbf{w}\,\mathrm{d}x\mathrm{d}t + \int_{Q^s} \mathbf{S}(t,x,\varrho,\mathbf{w}) : (\mathbf{D}\boldsymbol{u} - \mathbf{w})\,\mathrm{d}x\mathrm{d}t$$

and consequently

$$\int_{Q^s} (\mathbf{S}(t,x,\varrho,\mathbf{w}) - \overline{\mathbf{S}}) : (\mathbf{w} - \mathbf{D}\boldsymbol{u})\,\mathrm{d}x\mathrm{d}t \geqslant 0. \tag{IV.2.69}$$

Let $k > 0$ and denote by

$$Q_k = \{(t,x) \in Q^s : |\mathbf{D}\boldsymbol{u}(t,x)| \leqslant k \ \text{ a.e. in } Q^s\}$$

and let $0 < j < i$ be arbitrary and $h > 0$

$$\mathbf{w} = (\mathbf{D}\boldsymbol{u})\mathbb{1}_{Q_i} + h\mathbf{v}\mathbb{1}_{Q_j},$$

where $\mathbf{v} \in L^\infty(Q;\mathbb{R}^{3\times 3})$ is arbitrary. By (IV.2.69), we have

$$-\int_{Q^s\setminus Q_i} (\mathbf{S}(t,x,\varrho,\mathbf{0}) - \overline{\mathbf{S}}) : \mathbf{D}\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t + h\int_{Q_j} (\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u} + h\mathbf{v}) - \overline{\mathbf{S}}) : \mathbf{v}\,\mathrm{d}x\mathrm{d}t \geqslant 0.$$

Note that $\mathbf{S}(t,x,\varrho,\mathbf{0}) = \mathbf{0}$. Obviously

$$\int_{Q^s\setminus Q_i} \overline{\mathbf{S}} : \mathbf{D}\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t = \int_Q (\overline{\mathbf{S}} : \mathbf{D}\boldsymbol{u})\mathbb{1}_{Q^s\setminus Q_i}\,\mathrm{d}x\mathrm{d}t.$$

By Proposition III.1.12 and (IV.2.58) we obtain

$$\int_Q \overline{\mathbf{S}} : \mathbf{D}\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t < \infty.$$

Then as $i \to \infty$ we get

$$(\overline{\mathbf{S}} : \mathbf{D}\boldsymbol{u})\mathbb{1}_{Q^s\setminus Q_i} \to 0 \quad \text{ a.e. in } Q.$$

Hence by the Lebesgue dominated convergence theorem

$$\lim_{i\to\infty} \int_{Q^s\setminus Q_i} \overline{\mathbf{S}} : \mathbf{D}\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t = 0.$$

Letting $i \to \infty$ in (IV.2.6) and dividing by h, we get

$$\int_{Q_j} (\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u} + h\mathbf{v}) - \overline{\mathbf{S}}) : \mathbf{v}\,\mathrm{d}x\mathrm{d}t \geqslant 0.$$

Since $\mathbf{D}\boldsymbol{u} + h\mathbf{v} \to \mathbf{D}\boldsymbol{u}$ a.e. in Q_j when $h \to 0^+$ and $\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u} + h\mathbf{v})$ is uniformly bounded in $L^\infty(Q_j;\mathbb{R}^{3\times 3})$, $|Q_j| < \infty$, by the Vitali lemma we conclude

$$\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u} + h\mathbf{v}) \to \mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}) \text{ in } L^1(Q_j;\mathbb{R}^{3\times 3})$$

and

$$\int_{Q_j} (\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}+h\mathbf{v}) - \overline{\mathbf{S}}) : \mathbf{v} dxdt \to \int_{Q_j} (\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}) - \overline{\mathbf{S}}) : \mathbf{v} dxdt$$

when $h \to 0^+$. Consequently,

$$\int_{Q_j} (\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}) - \overline{\mathbf{S}}) : \mathbf{v} dxdt \geqslant 0$$

for all $\mathbf{v} \in L^\infty(Q;\mathbb{R}^{3\times 3})$. The choice

$$\mathbf{v} = \begin{cases} -\frac{\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}) - \overline{\mathbf{S}}}{|\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}) - \overline{\mathbf{S}}|} & \text{for} \quad \mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}) \neq \overline{\mathbf{S}}, \\ 0 & \text{for} \quad \mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}) = \overline{\mathbf{S}}, \end{cases}$$

yields

$$\int_{Q_j} |\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}) - \overline{\mathbf{S}}| dxdt \leqslant 0.$$

Hence

$$\mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u}) = \overline{\mathbf{S}} \quad \text{a.e. in } Q_j. \tag{IV.2.70}$$

Since j was arbitrary, (IV.2.70) holds a.e. in Q^s. Since it holds for almost all s such that $0 < s < T$, we conclude that $\overline{\mathbf{S}} = \mathbf{S}(t,x,\varrho,\mathbf{D}\boldsymbol{u})$ a.e. in Q.

CHAPTER V

Existence result for the motion of several rigid bodies in an incompressible non-Newtonian fluid

V.1. Introduction

We want to investigate the mathematical properties of motion of one or several non-homogenous rigid bodies immersed in a homogeneous incompressible viscous fluid which occupies a bounded domain $\Omega \subset \mathbb{R}^3$. In particular we are interested in fluids having viscosity which increases dramatically with increasing shear rate or applied stress, i.e. we want to consider shear thickening fluids and as in Chapter IV we formulate the growth conditions of the stress tensor using quite general convex function M called an isotropic $\mathcal{N}-$function. Fore more references and more detailed description of our motivation we refer the reader to the Chapter IV.

We assume that the viscous stress tensor $\mathbf{S}$ depends on the symmetric part of the gradient of the velocity field $\boldsymbol{u}$ in the following way: $\mathbf{S} : \mathbb{R}^{3\times 3}_{\text{sym}} \to \mathbb{R}^{3\times 3}_{\text{sym}}$ satisfies ($\mathbb{R}^{3\times 3}_{\text{sym}}$ stands for the space of 3×3 symmetric matrices):

$$\mathbf{S}(\mathbf{0}) = \mathbf{0}, \quad \mathbf{S} = \mathbf{S}(\mathbf{D}\boldsymbol{u}) \quad \text{is continuous,} \tag{V.1.1}$$

$$\Big(\mathbf{S}(\boldsymbol{\xi}) - \mathbf{S}(\boldsymbol{\eta})\Big) : \Big(\boldsymbol{\xi} - \boldsymbol{\eta}\Big) \geqslant 0 \text{ for all } \boldsymbol{\xi} \neq \boldsymbol{\eta},\ \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^{3\times 3}_{\text{sym}} \tag{V.1.2}$$

and there exist a positive constant c, an isotropic $\mathcal{N}$–functions M (Definition III.1.1) and M^* (M^* denotes the complementary function to M) such that for all $\boldsymbol{\xi} \in \mathbb{R}^{3\times 3}_{\text{sym}}$ it holds

$$\mathbf{S}(\boldsymbol{\xi}) : \boldsymbol{\xi} \geqslant c_c\{M(|\boldsymbol{\xi}|) + M^*(|\mathbf{S}(\boldsymbol{\xi})|)\}. \tag{V.1.3}$$

Additionally we assume that the $\mathcal{N}$–function M satisfies an additional growth condition

$$c_1|\cdot|^p \leqslant M(\cdot) \leqslant c_2 \exp^{\frac{1}{\beta+1}}(|\cdot|) \quad \text{for } p \geqslant 4,\ \beta > 0 \tag{V.1.4}$$

where $c_1,\ c_2$ are some positive constants, the complementary function

$$M^* \text{ satisfies the } \Delta_2 - \text{condition} \tag{V.1.5}$$

and

$$M(|\cdot|^{\frac{1}{4}}) \quad \text{is convex.} \tag{V.1.6}$$

The appropriate spaces to capture such formulated problem are isotropic Orlicz spaces.

The motion of the body during and before the contacts with boundary of the domain Ω was studied by Starovoitov. In particular, the author gives sufficient conditions which imply the impossibility of the collision with rigid object, see [**120**, Theorem 3.2.], i.e.:

c1: the domain $\Omega \subset \mathbb{R}^3$ as well as the rigid bodies in its interior have boundaries of class $C^{1,1}$;

c2: the $p - th$ power of the velocity gradient is integrable, with $p \geqslant 4$.

Therefore any contact of rigid body with the boundary of domain or with other one or several rigid bodies does not occur. We just need to assume that it was not present in initial time and consider certain class of non-Newtonian fluids, where the contact can be eliminated by the phenomenon of shear-thickening.

We want to investigate the motion of several rigid bodies in a non-Newtonian incompressible fluid. To construct the solution we use penalization method developed by Hoffmann and Starovoitov [**80**], and San Martin et al. [**113**], which is based on the idea of approximating rigid objects of the system by the fluid of very high viscosity becoming singular in limiting consideration. To avoid some technical difficulties we assume that fluid density is constant in the approximate "fluid" part (this assumption is avoided in [**113**], where 2-D case of a Newtonian fluid is investigated).

There are two main difficulties we have to face in the proof of the existence result, more precisely in the proof of the sequential stability of the approximate solutions:
1. strong compactness of the approximate velocities on the time-space cylinder in L^2 space;
2. passing to the limit in the nonlinear term – i.e. in viscous stress tensor by means of the monotonicity method.

To solve the first problem, similarly as San Martin et al. in [**113**], we use the Aubin-Lions argument applied to a suitable projection of the velocity field onto the "space of rigid velocities". It is worth to notice that no-collision result by Starovoitov [**120**] significantly simplifies our analysis. Namely we are ensured that bodies do not penetrate each other and the boundary. We will notice that positive distance between the bodies and boundary is always kept and any sharp cones do not appear in the fluid part.

As in the previous chapter, the principal difficulty here is caused by the fact that we consider the problem in Orlicz-space setting and we do not assume that the $\triangle_2$–condition is satisfied as we want to investigate the case of shear thickening fluids of rheology more general then of power-law type. For this reason the spaces we work with lose many facilitating properties, which have been mentioned in Chapters I,III.

The latter problem, inherent to the theory of non-Newtonian fluids, is that we have to identify the nonlinear term on "fluid" part of time-space cylinder. Therefore the problem is more delicate as the monotonicity argument must be localised to the "fluid" part of the system. We take the idea of Wolf [**130**], localise the pressure and represent it as a sum of a regular and a harmonic part. Following Feireisl et al. [**56**] we construct the pressure function with the help of Riesz transform which gives a result more suitable for non-standard growth conditions and such an approach can be easily adapted to more general constitutive relations for $\mathbf{S}$. The main difference from any previous works in this direction is, due to nonstandard growth conditions, that we are in Orlicz-space setting. Besides the difficulties mentioned above, the

Riesz transform in general can not be well defined on Orlicz space to itself. If M and M^* do not satisfy the Δ_2-condition it can happen that it is continuous from one Orlicz space to another one, with a modular of essentially slower growth. Therefore the pressure localisation method appears to us to be more difficult.

We want to emphasise that we achieve the existence result for the problem of motion of rigid bodies in non-Newtonian fluids with non-polynomial growth conditions. This allows us to consider a situation of non-power-law fluids, where constitutive relation can be more general than (IV.1.5) considered in [**56**].

Our main result, formulated below in Theorem V.3.1, concerns the existence of weak solutions of the associated evolutionary system, where, in accordance with [**120**], collisions of two or more rigid objects do not appear in a finite time unless they were present initially, which considerably simplified analysis of the problem. The chapter is based on [**134, 135**] by Wróblewska-Kamińska.

The chapter is organised as follows: some preliminary considerations, weak formulation and basic notation of the investigated problem are summarised in Section V.2. The main result is formulated in Section V.3 as Theorem V.3.1. The remaining part of the chapter contains the proof of the existence result. In Section V.4 the approximate problem is introduced by replacing the bodies by the fluid of high viscosity. Section V.5 contains the artificial viscosity limit. In Section V.6 previous arguments are recalled to provide the limit for the regularized velocity field.

V.2. Preliminaries, weak formulation

We state the following problem: let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with a sufficiently smooth boundary $\partial\Omega$, occupied by an incompressible fluid containing rigid bodies. Each rigid body in the considered system is identified with the connected subset of Euclidean space $\mathbb{R}^3$. The initial position of the rigid bodies is given through a family of domains

$$S_i \subset \mathbb{R}^3, \ i = 1, \dots, n,$$

which are diffeomorphic to the unit ball in $\mathbb{R}^3$. To avoid additional difficulties the boundaries of all rigid bodies are supposed to be sufficiently regular, namely there exists $\delta_0 > 0$ such that for any $x \in \partial S_i$ there are two closed balls B^{int}, B^{ext} of radius δ_0 such that

$$x \in B^{\text{int}} \cap B^{\text{ext}}, \quad B^{\text{int}} \subset \overline{S}_i, \quad B^{\text{ext}} \subset \mathbb{R}^3 \setminus S_i. \tag{V.2.1}$$

The same assumption concerns the considered physical space $\Omega \subset \mathbb{R}^3$, occupied by the fluid and containing all rigid bodies. In particular, Ω is supposed to be a bounded domain such that for any $x \in \partial\Omega$ there are two closed balls B^{int}, B^{ext} of radius δ_0 such that

$$x \in B^{\text{int}} \cap B^{\text{ext}}, \quad B^{\text{int}} \subset \overline{\Omega}, \quad B^{\text{ext}} \subset \mathbb{R}^3 \setminus \Omega. \tag{V.2.2}$$

The motion of the rigid body S_i is represented by the associated mapping $\boldsymbol{\eta}_i$

$$\boldsymbol{\eta}_i = \boldsymbol{\eta}_i(t,x), \ t \in [0,T), \ x \in \mathbb{R}^3, \ \boldsymbol{\eta}_i(t,\cdot) : \mathbb{R}^3 \to \mathbb{R}^3 \text{ is an isometry for all } t \in [0,T)$$
$$\text{and } \boldsymbol{\eta}_i(0,x) = x \text{ for all } x \in \mathbb{R}^3, \quad i = 1, \dots, n.$$

Therefore, the position of the body S_i at a time $t \in [0,T)$ is given by the following formula

$$S_i(t) = \boldsymbol{\eta}_i(t, S_i), \;\; i = 1, \ldots, n. \tag{V.2.3}$$

In the above terms we introduce domains Q^f and Q^s respectively as a fluid and a rigid part of the time-space cylinder in the following way:

$$Q^S := \bigcup_{i=1,\ldots,n} \big\{(t,x) \mid t \in [0,T], \; x \in \overline{S_i}(t)\big\} \quad Q^f := Q \backslash Q^S.$$

In the present work the concept of weak solutions is based on the Eulerian reference system and on a class of test functions which depend on the position of the rigid bodies. This idea was introduced by Judakov [**85**] (see also Desjardins and Esteban [**39, 40**], Galdi [**66, 67**], Hoffmann and Starovoitov [**80**], San Martin et al. [**113**] , Serre [**114**]). Let us denote the velocity field of the system by $\boldsymbol{u} : Q \to \mathbb{R}^3$ and introduce decomposition for a fluid and a rigid velocity as follows

$$\boldsymbol{u}^f = \boldsymbol{u} \text{ on } Q^f \quad \text{ and } \quad \boldsymbol{u}^S = \boldsymbol{u} \text{ on } Q^S.$$

In our considerations we assume no-slip boundary conditions for the velocity on all surfaces and the velocity of the fluid on the boundary of each rigid body S_i $(i = 1, \ldots, n)$ is supposed to coincide with the velocity of rigid object. Namely

$$\boldsymbol{u}^f(t,x) = 0 \text{ on } \partial\Omega \quad \text{ and } \quad \boldsymbol{u}^f(t,x) = \boldsymbol{u}_i^S(t,x) \text{ on } \partial S_i(t)$$

for all $t \in [0,T]$ and $i = 1, \ldots, n$. To be more precise, if we consider the mass density $\varrho = \varrho(t,x)$ and the velocity field $\boldsymbol{u} = \boldsymbol{u}(t,x)$ at a time $t \in (0,T)$ and the spatial position $x \in \Omega$, then those functions satisfy the following integral identities

$$\int_0^T \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \boldsymbol{u} \cdot \nabla_x \varphi\Big) \, \mathrm{d}x\mathrm{d}t = -\int_\Omega \varrho_0 \varphi \, \mathrm{d}x \tag{V.2.4}$$

for any test function $\varphi \in C_c^1([0,T) \times \Omega)$, and

$$\begin{aligned} &\int_0^T \int_\Omega \Big(\varrho \boldsymbol{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \boldsymbol{u} \otimes \boldsymbol{u} : \mathbf{D}\boldsymbol{\varphi} - \mathbf{S} : \mathbf{D}\boldsymbol{\varphi}\Big) \, \mathrm{d}x\mathrm{d}t \\ &= -\int_0^T \int_\Omega \varrho \nabla_x F \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t - \int_\Omega \varrho_0 \boldsymbol{u}_0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x \end{aligned} \tag{V.2.5}$$

for any test function $\boldsymbol{\varphi} \in C_c^1([0,T) \times \Omega; \mathbb{R}^3)$,

$$\boldsymbol{\varphi}(t, \cdot) \in [\mathcal{R}M](t), \tag{V.2.6}$$

which is associated with the position of rigid bodies, i.e.

$$\begin{aligned} [\mathcal{R}M](t) = \big\{ &\boldsymbol{\phi} \in C_c^1(\Omega; \mathbb{R}^3) \mid \mathrm{div}_x \boldsymbol{\phi} = 0 \text{ in } \Omega, \\ &\mathbf{D}\boldsymbol{\phi} \text{ has compact support on } \Omega \backslash \cup_{i=1}^n \overline{S}_i(t)\big\}. \end{aligned} \tag{V.2.7}$$

The symbol $\mathbf{S}$ denotes the viscous stress tensor determined through (V.1.1 - V.1.6), $\nabla_x F$ is a given potential driving force and ϱ_0, $\boldsymbol{u}_0$ stand for the initial distribution of the density and the velocity, respectively.

The tensor $\mathbf{D}\boldsymbol{u}$ is called also a deformation rate tensor as $\boldsymbol{u}$ stands for velocity field. The kernel of this tensor consists of rigid vector field. Assume that S is a connected domain in $\mathbb{R}^3$ and $\boldsymbol{u} : S \to \mathbb{R}^3$ is a velocity field. Then $\mathbf{D}\boldsymbol{u} = \mathbf{0}$ in S if and only if there exists a vector $\boldsymbol{a} \in \mathbb{R}^3$ and an antisymmetric tensor $\mathbf{A} \in \mathbb{R}^3 \times \mathbb{R}^3$ such that $\boldsymbol{u}(x) = \boldsymbol{a} + \mathbf{A}x$ for $x \in S$. The proof of this fact can be found for instance in [**124**]. Velocity of the above form corresponds to rigid motion. Thus, it is possible to specify rigid bodies by the condition that the deformation rate tensor vanishes in the domains corresponding to the bodies.

In order to close the system we have to specify the relation between the velocity $\boldsymbol{u}$ and the motion of solids given by isometries $\boldsymbol{\eta}_i$. This can be formulated as follows. As the mappings $\boldsymbol{\eta}_i(t, \cdot)$ are isometries on $\mathbb{R}^3$, they can be written in the form

$$\boldsymbol{\eta}_i(t, x) = x_i(t) + \mathbf{O}_i(t)x,$$

where $\mathbf{O}_i(t) \in SO(3)$ (i.e. it is a matrix satisfying $\mathbf{O}_i^T\mathbf{O}_i = \mathbf{Id}$). The position $x_i(t)$ denotes the position of the center of mass of S_i at a time t and

$$x_i(t) = \frac{1}{m_i}\int_{\overline{S}_i(t)} \varrho_{S_i}(t, x)x \,\mathrm{d}x,$$

where

$$m_i = \int_{\overline{S}_i(t)} \varrho_{S_i}(t, x) \,\mathrm{d}x$$

is the total mass of ith rigid body of a mass density ϱ_{S_i}. We say that the velocity field $\boldsymbol{u}$ is compatible with the family of motions $\{\boldsymbol{\eta}_1, \dots, \boldsymbol{\eta}_n\}$ if

$$\text{(V.2.8)}\quad \boldsymbol{u}(t, x) = \boldsymbol{u}^{S_i}(t, x) = \boldsymbol{U}_i(t) + \mathbf{Q}_i(t)(x - x_i(t)) \text{ for a.a. } x \in \overline{S}_i(t),\ i = 1, \dots, n$$

for a.a. $t \in [0, T)$, where $\boldsymbol{u}^{S_i}$ is solid velocity, $\boldsymbol{U}_i(t)$ denotes the translation velocity and $\mathbf{Q}$ - the angular velocity of the body s.t.

$$\text{(V.2.9)}\qquad \frac{\mathrm{d}}{\mathrm{d}t}x_i(t) = \boldsymbol{U}_i(t),\ \Big(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{O}_i(t)\Big)\mathbf{O}_i^T(t) = \mathbf{Q}_i(t) \text{ a.a. on } (0, T).$$

V.3. Main result

Let us formulate now the main existence result of this chapter.

Theorem V.3.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ and let the following assumptions be satisfied:*

- *The initial position of the rigid bodies is given through a family of open sets*

  $$S_i \subset \Omega \subset \mathbb{R}^3,\ S_i \text{ diffeomorphic to the unit ball for } i = 1, \dots, n,$$

  where both ∂S_i, $i = 1, \dots, n$, and $\partial\Omega$ belong to the regularity class specified by (V.2.1), (V.2.2).
- $\mathrm{dist}[\overline{S}_i, \overline{S}_j] > 0$ *for* $i \neq j$, $\mathrm{dist}[\overline{S}_i, \mathbb{R}^3\backslash\Omega] > 0$ *for any* $i, j = 1, \dots, n$.
- *The viscous stress tensor $\mathbf{S}$ satisfies hypotheses (V.1.1 - V.1.3).*
- *The isotropic $\mathcal{N}$–function M satisfies conditions (V.1.4 - V.1.6) with $p \geqslant 4$ and the complementary function M^* to M satisfies the Δ_2–condition.*
- *The given forces $F \in W^{1,\infty}(\Omega)$.*

- *The initial distribution of the density is given by*

$$\varrho_0 = \begin{cases} \varrho_f = \text{const} > 0 \ \textit{in}\ \Omega \backslash \bigcup_{i=1}^n \overline{S}_i, \\ \\ \varrho_{S_i} \ \textit{on}\ S_i,\ \textit{where}\ \varrho_{S_i} \in L^\infty(\Omega),\ \operatorname{ess\,inf}_{S_i} \varrho_{S_i} > 0,\ i = 1, \dots, n, \end{cases}$$

while the initial velocity field $\boldsymbol{u}_0$ *satisfies*

$$\boldsymbol{u}_0 \in L^2(\Omega; \mathbb{R}^3),\ \operatorname{div}_x \boldsymbol{u}_0 = 0 \ \textit{in}\ \mathcal{D}'(\Omega),\ \mathbf{D}\boldsymbol{u}_0 = 0 \ \textit{in}\ \mathcal{D}'(S_i; \mathbb{R}^{3\times 3}) \ \textit{for}\ i = 1, \dots, n.$$

Then there exist a density function ϱ,

$$\varrho \in C([0,T]; L^1(\Omega)),\ 0 < \operatorname*{ess\,inf}_{\Omega} \varrho(t,\cdot) \leqslant \operatorname*{ess\,sup}_{\Omega} \varrho(t,\cdot) < \infty \ \textit{for all}\ t \in [0,T],$$

a family of isometries $\{\boldsymbol{\eta}_i(t,\cdot)\}_{i=1}^n$, $\boldsymbol{\eta}_i(0,\cdot) = \mathbf{Id}$, *and a velocity field* $\boldsymbol{u}$,

$$\boldsymbol{u} \in L^\infty(0,T; L^2_{\operatorname{div}}(\Omega;\mathbb{R}^3)) \cap L^p(0,T; W_0^{1,p}(\Omega;\mathbb{R}^3)), \quad \mathbf{D}\boldsymbol{u} \in L_M(Q;\mathbb{R}^{3\times 3}),$$

compatible with $\{\boldsymbol{\eta}_i\}_{i=1}^n$ *in the sense specified in (V.2.8), (V.2.9), such that* ϱ, $\boldsymbol{u}$ *satisfy the integral identity (V.2.4) for any test function* $\varphi \in C_c^1([0,T) \times \overline{\Omega})$, *and the integral identity (V.2.5) for any* $\boldsymbol{\varphi}$ *satisfying (V.2.6), (V.2.7).*

The aim of this chapter is to prove Theorem V.3.1.

V.4. Approximate problem

The first step of the proof is to approximate the rigid objects by a fluid of a very high viscosity. For this reason we introduce a penalization problem and the construction of weak solutions is based on a two-level approximation scheme that consists of solving the system of equations:

$$\partial_t \varrho + \operatorname{div}_x(\varrho[\boldsymbol{u}]_\delta) = 0, \tag{V.4.1}$$

$$\partial_t(\varrho\boldsymbol{u}) + \operatorname{div}_x(\varrho\boldsymbol{u} \otimes [\boldsymbol{u}]_\delta) + \nabla_x p = \operatorname{div}_x([\mu_\varepsilon]_\delta \mathbf{S}) + \varrho\nabla_x F - \chi_\varepsilon \boldsymbol{u}, \tag{V.4.2}$$

$$\partial_t \mu_\varepsilon + \operatorname{div}_x(\mu_\varepsilon[\boldsymbol{u}]_\delta) = 0, \tag{V.4.3}$$

$$\operatorname{div}_x \boldsymbol{u} = 0, \tag{V.4.4}$$

where p is a scalar function denoting the pressure. Moreover we regularise the vector field in (V.4.1) and (V.4.3) with a standard regularizing kernel. Namely for δ such that $\delta < \delta_0$ (δ_0 is as in Section V.2) the symbol

$$[\boldsymbol{u}]_\delta = \omega_\delta * \boldsymbol{u}$$

stands for a spatial convolution with

$$\omega_\delta(x) = \frac{1}{\delta^3}\omega\left(\frac{|x|}{\delta}\right), \tag{V.4.5}$$

where $\omega \in C^\infty(\mathbb{R}^3)$, $\operatorname{supp}\omega \subset \mathrm{B}(0,1)$,

$$\omega(x) > 0 \text{ for } x \in B(0,1), \quad \omega(x) = \omega(-x), \quad \int_{B(0,1)} \omega(x)\,\mathrm{d}x = 1.$$

As Ω is bounded, we can assume that $\Omega \subset [-L, L]^3$ for a certain $L > 0$ and consider system (V.4.1 - V.4.4) on the spatial torus

$$\mathcal{T} = [(-L, L)|_{\{-L,L\}}]^3.$$

Then all quantities are assumed to be spatially periodic with period $2L$, in particular we extend the initial velocity field u_0 by 0 outside of Ω and density by ϱ_f – constant density of the fluid. We also extend the outer force in such a way that $F \in W^{1,\infty}(\mathcal{T})$.

The system (V.4.1 - V.4.4) is supplemented with the initial conditions

$$0 < \varrho(0, \cdot) = \varrho_{0,\delta} = \varrho_f + \sum_{i=1}^{n} \varrho_{S_i,\delta}, \tag{V.4.6}$$

where

$$\varrho_{0,\delta} \to \varrho_0 \text{ strongly in } L^1(\mathcal{T}) \quad \text{as } \delta \to 0$$

and

$$\varrho_{S_i} \in \mathcal{D}(S_i),\ \varrho_{S_i,\delta}(x) = 0 \text{ whenever } \operatorname{dist}[x, \partial S_i] < \delta < \delta_0,\ \text{for } i = 1, \dots, n. \tag{V.4.7}$$

Similarly, we prescribe $\varepsilon-$dependent artificial "viscosity" $\mu : (0, T) \times \mathcal{T} \to \mathbb{R}$ with initial data given by

$$\mu(0, \cdot) = \mu_{0,\varepsilon} = 1 + \frac{1}{\varepsilon} \sum_{i=1}^{n} \mu_{S_i}, \tag{V.4.8}$$

where

$$\begin{aligned} &\mu_{S_i} \in \mathcal{D}(S_i),\ \mu_{S_i}(x) = 0 \text{ whenever } \operatorname{dist}[x, \partial S_i] < \delta, \\ &\mu_{S_i}(x) > 0 \text{ for } x \in S_i,\ \operatorname{dist}[x, \partial S_i] > \delta \quad \text{for } i = 1, \dots, n. \end{aligned} \tag{V.4.9}$$

The "viscosity" μ can be identified as the penalization introduced by Hoffmann and Starovoitov [**80**] and San Martin et al. [**113**], where the rigid bodies are replaced by the fluid of high viscosity becoming singular for $\varepsilon \to 0$.

Furthermore, we penalize also the region out of the set Ω and we take

$$\chi_\varepsilon = \frac{1}{\varepsilon}\chi,\ \chi \in \mathcal{D}(\mathcal{T}),\ \chi > 0 \text{ on } \mathcal{T}\backslash\Omega,\ \chi = 0 \text{ in } \overline{\Omega}. \tag{V.4.10}$$

The parameters ε and δ are small positive numbers. In the above formulation an additional parameter $\delta_0 > \delta > 0$ has been introduced to keep the density constant in the approximate fluid region in order to construct the local pressure.

For fixed $\varepsilon > 0$ and $\delta > 0$ we report the following existence result that can be proved by means of the monotonicity argument for nonreflexive spaces (for the existence result without regularization of the velocity field see Chapter IV or [**133**] and for partial results in the Sobolev space setting see Frehse et al. [**58, 59**] and in the Orlicz space setting Gwiazda et al. [**72, 75**] and Wróblewska-Kamińska [**131**]):

Proposition V.4.1. *Suppose that* $p \geqslant 4$. *Let the initial distribution of* ϱ, μ *be given through (V.4.6 - V.4.9), with fixed* $\varepsilon > 0$, $\delta_0 > \delta > 0$. *Moreover, assume that*

$$\boldsymbol{u}(0, \cdot) = \boldsymbol{u}_0,\ x \in \mathcal{T},\ \boldsymbol{u}_0 \in L^2(\mathcal{T}; \mathbb{R}^3),\ \operatorname{div}_x \boldsymbol{u}_0 = 0 \text{ in } \mathcal{D}'(\mathcal{T}; \mathbb{R}^3), \tag{V.4.11}$$

and $\chi_\varepsilon, F \in C^\infty(\mathcal{T})$*, where* χ_ε *is determined by (V.4.10).*

Then the problem (V.4.1 - V.4.4), supplemented with the initial data (V.4.6 - V.4.9), possesses a (weak) solution ϱ*,* μ*,* $\boldsymbol{u}$ *satisfying*

$$\varrho, \mu \in C([0,T]; L^1(\mathcal{T})),$$

$$0 < \operatorname{ess\,inf}_\Omega \varrho(t,\cdot) \leqslant \operatorname{ess\,sup}_\Omega \varrho(t,\cdot) < \infty \text{ for all } t \in [0,T],$$

$$0 < \operatorname{ess\,inf}_\Omega \mu(t,\cdot) \leqslant \operatorname{ess\,sup}_\Omega \mu(t,\cdot) < \infty \text{ for all } t \in [0,T],$$

$$\boldsymbol{u} \in L^\infty(0,T; L^2(\mathcal{T};\mathbb{R}^3)) \cap L^p(0,T; W^{1,p}(\mathcal{T};\mathbb{R}^3)), \quad \mathbf{D}\boldsymbol{u} \in L_M(Q;\mathbb{R}^{3\times 3}).$$

In addition, the solution satisfies the energy inequality

$$\text{(V.4.12)} \quad \begin{aligned} &\int_{\mathcal{T}} \frac{1}{2}\varrho|\boldsymbol{u}|^2(z)\,\mathrm{d}x + \int_s^z\int_{\mathcal{T}} [\mu_\varepsilon]_\delta \mathbf{S} : \nabla_x \boldsymbol{u}\,\mathrm{d}x\mathrm{d}t + \int_s^z\int_{\mathcal{T}} \chi_\varepsilon |\boldsymbol{u}|^2\,\mathrm{d}x\mathrm{d}t \\ &\qquad\leqslant \int_{\mathcal{T}} \frac{1}{2}\varrho|\boldsymbol{u}|^2(s)\,\mathrm{d}x + \int_s^z\int_{\mathcal{T}} \varrho\nabla_x F\cdot \boldsymbol{u}\,\mathrm{d}x\mathrm{d}t \end{aligned}$$

for a.a. $0 \leqslant s < z \leqslant T$ *including* $s = 0$*.*

The weak formulation of the equation (V.4.2) is represented by the integral identity

$$\text{(V.4.13)} \quad \int_0^T\int_{\mathcal{T}} \varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho(\boldsymbol{u}\otimes[\boldsymbol{u}]_\delta) : \nabla_x\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t$$

$$= \int_0^T\int_{\mathcal{T}} [\mu_\varepsilon]_\delta\mathbf{S} : \mathbf{D}\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_{\mathcal{T}} \varrho\nabla_x F\cdot\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_{\mathcal{T}} \chi_\varepsilon\boldsymbol{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t$$

$$- \int_{\mathcal{T}} \varrho_{0,\delta}\boldsymbol{u}_0\cdot\boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}x$$

which is satisfied for any test function $\boldsymbol{\varphi} \in \mathcal{D}([0,T)\times\mathcal{T};\mathbb{R}^3)$, $\operatorname{div}_x\boldsymbol{\varphi} = 0$.

Using the continuity equation (V.4.1), assumption (V.1.3), the Young and the Sobolev inequality and condition (V.1.4) we easily deduce also that the following inequality is satisfied

(V.4.14)

$$\begin{aligned} &\int_{\mathcal{T}} \frac{1}{2}\varrho|\boldsymbol{u}|^2(z)\,\mathrm{d}x + \int_0^z\int_{\mathcal{T}} c_c[\mu_\varepsilon]_\delta M^*(|\mathbf{S}|)\,\mathrm{d}x\mathrm{d}t + \int_0^z\int_{\mathcal{T}} c_c\left([\mu_\varepsilon]_\delta - \frac{1}{2}\right) M(|\mathbf{D}\boldsymbol{u}|)\,\mathrm{d}x\mathrm{d}t \\ &\qquad + \int_0^z\int_{\mathcal{T}} \chi_\varepsilon|\boldsymbol{u}|^2\,\mathrm{d}x\mathrm{d}t \leqslant C(F,\boldsymbol{u}_0,\varrho_0) \end{aligned}$$

for a.a. $0 \leqslant z \leqslant T$.

Let us notice that due to the method of characteristics applied to (V.4.3), the artificial "viscosity" $\mu_\varepsilon \geqslant 1$ on $[0,T]\times\mathcal{T}$. Hence the l.h.s. of (V.4.14) is nonnegative.

V.5. Artificial viscosity limit

V.5.1. Notation. For a family $\{S_i\}_{i=1}^n$ of precompact subsets of Ω, we denote

$$\mathrm{d}[\{S_i\}_{i=1}^n] = \inf\left\{ \inf_{i,j=1,\dots,n,\ i\neq j} \mathrm{dist}[S_i, S_j], \inf_{i=1,\dots,n} \mathrm{dist}[S_i, \partial\Omega] \right\}. \tag{V.5.1}$$

We define a signed distance to the boundary of a subset S of Ω by

$$\mathbf{db}_S(x) = \mathrm{dist}[x, \mathbb{R}^3 \backslash S] - \mathrm{dist}[x, S].$$

We say that a sequence of sets S_n converges to S in the sense of boundaries and denote it by

$$S_n \xrightarrow{b} S,$$

if

$$\mathbf{db}_{S_n}(x) \to \mathbf{db}_S(x) \text{ uniformly for } x \text{ belonging to compact subsets of } \mathbb{R}^3. \tag{V.5.2}$$

In similar way as San Martin et al. [**113**] and Feireisl et al. [**56**], we introduce $[S]_\delta$ called the $\delta-$kernel and $]S[_\delta$ - the $\delta-$neighbourhood of the set S, i.e.:

$$[S]_\delta = \mathbf{db}_S^{-1}((\delta,\infty)), \quad]S[_\delta = \mathbf{db}_S^{-1}((-\delta,\infty)). \tag{V.5.3}$$

Moreover, we define for $k \geqslant 0$

$$W^{k,2}_{0,\mathrm{div}} = \mathrm{closure}_{W^{k,p}(\Omega;\mathbb{R}^3)} \left\{ \boldsymbol{v} \in \mathcal{D}(\Omega;\mathbb{R}^3) \mid \mathrm{div}_x \boldsymbol{v} = 0 \right\},$$

and

$$\mathcal{K}^{k,p}(S) = \left\{ \boldsymbol{v} \in W^{k,p}_{0,\mathrm{div}} \mid \mathbf{D}\boldsymbol{v} = 0 \text{ in } \mathcal{D}'(S;\mathbb{R}^3) \right\}, \text{ where } S \text{ is an open subset of } \Omega.$$

For $p = 2$ and the Hilbert space $W^{k,2}_{0,\mathrm{div}}$ the symbol

$$\mathcal{P}^k(S) \text{ denotes the orthogonal projection of } W^{k,2}_{0,\mathrm{div}} \text{ onto the closed subspace } \mathcal{K}^{k,2}(S). \tag{V.5.4}$$

V.5.2. Uniform estimates and the continuity equation. Let us denote by $\{\varrho_\varepsilon, \mu_\varepsilon, \boldsymbol{u}_\varepsilon\}_{\varepsilon>0}$ the family of approximate solutions associated with the problem (V.4.1 - V.4.9). For the brevity of the notation we omit the dependence of this sequence on δ. The existence of such a family of solutions is assured by Proposition V.4.1. In the first step we fix $\delta > 0$ and identify the limit for $\varepsilon \to 0$. The limit for δ will be shortly shown in Section V.6.

At first we show briefly how the continuity equation (V.4.1) behaves as $\varepsilon \to 0$. As we noticed already, the method of characteristics applied to (V.4.3) gives us that $\mu_\varepsilon \geqslant 1$. Hence following the estimates (V.4.14) we infer that

$$\int_{(0,T)\times\mathcal{T}} M\left(\mathbf{D}\boldsymbol{u}_\varepsilon\right) \mathrm{d}x\mathrm{d}t \leqslant c \tag{V.5.5}$$

and together with the assumption (V.1.4) this gives

$$\int_{(0,T)\times\mathcal{T}} |\mathbf{D}\boldsymbol{u}_\varepsilon|^p \,\mathrm{d}x\mathrm{d}t \leqslant c. \tag{V.5.6}$$

Let us notice that the estimate (V.4.14) provides that

$$\int_0^z \int_{\mathcal{T}\setminus\Omega} |\boldsymbol{u}| \,\mathrm{d}x \leqslant c.$$

Without loss of generality we can assume that $|\mathcal{T}\setminus\Omega| > 0$, therefore employing the general version of the Korn inequality (see [**54**, Theorem 10.16]) we obtain

$$\|\boldsymbol{u}_\varepsilon\|_{L^p(0,T;W^{1,p}(\mathcal{T}))} \leqslant c$$

By the Alaoglu-Banach theorem we obtain that for a subsequence

$$\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u} \text{ weakly in } L^p(0,T;W^{1,p}(\mathcal{T};\mathbb{R}^3)) \tag{V.5.7}$$

and additionally $\mathrm{div}_x\boldsymbol{u} = 0$ a.e. on $(0,T)\times\mathcal{T}$. Next, the regularized sequence $\{[\boldsymbol{u}_\varepsilon]_\delta\}_{\varepsilon>0}$ satisfies

(V.5.8)
$[\boldsymbol{u}_\varepsilon]_\delta \overset{*}{\rightharpoonup} [\boldsymbol{u}]_\delta$ weakly-(*) in $L^p(0,T;W^{1,\infty}(\mathcal{T};\mathbb{R}^3))$ and $\mathrm{div}_x[\boldsymbol{u}]_\delta = 0$ a.e. in $(0,T)\times\mathcal{T}$.

Furthermore, employing (V.4.10) together with (V.4.12), we infer

$$\boldsymbol{u} = 0 \text{ a.e. in the set } (0,T)\times(\mathcal{T}\setminus\Omega) \text{ as } \varepsilon\to 0. \tag{V.5.9}$$

Since Ω is regular (see (V.2.2)), we get in the sense of traces

$$\boldsymbol{u}|_{\partial\Omega} = 0$$

and therefore

$$\boldsymbol{u} \in L^p(0,T;W^{1,p}_{0,\mathrm{div}}(\Omega;\mathbb{R}^3))$$

(we mean here $\boldsymbol{u}|_{(0,T)\times\Omega}$).

Let us recall now the stability result for solutions to the transport equation obtained in [**55**, Proposition 5.1]:

Proposition V.5.1. *Let $\boldsymbol{v}_n = \boldsymbol{v}_n(t,x)$ be a sequence of vector fields such that*

$$\{\boldsymbol{v}_n\}_{n=1}^\infty \text{ is bounded in } L^2(0,T;W^{1,\infty}(\mathbb{R}^3;\mathbb{R}^3)).$$

Let $\boldsymbol{\eta}_n(t,\cdot):\mathbb{R}^3\to\mathbb{R}^3$ be the solution operator corresponding to the family of characteristic curves generated by $\boldsymbol{v}_n$, i.e.

$$\frac{\partial}{\partial t}\boldsymbol{\eta}_n(t,x) = \boldsymbol{v}_n(t,\boldsymbol{\eta}_n(t,x)),\ \boldsymbol{\eta}_n(0,x) = x \text{ for every } x\in\mathbb{R}^3.$$

Then passing to subsequences, as the case may be, we have

$$\boldsymbol{v}_n \overset{*}{\rightharpoonup} \boldsymbol{v} \text{ weakly-(*) in } L^2(0,T;W^{1,\infty}(\mathbb{R}^3;\mathbb{R}^3))$$

and

$$\boldsymbol{\eta}_n(t,\cdot)\to\boldsymbol{\eta}(t,\cdot) \text{ in } C_{\mathrm{loc}}(\mathbb{R}^3) \text{ uniformly for } t\in[0,T]$$

as $n\to\infty$, where $\boldsymbol{\eta}$ is the unique solution of

$$\frac{\partial}{\partial t}\boldsymbol{\eta}(t,x) = \boldsymbol{v}(t,\boldsymbol{\eta}(t,x)),\ \boldsymbol{\eta}(0,x) = x,\ x\in\mathbb{R}^3.$$

In addition, let $S_n \subset \mathbb{R}^3$ be the sequence of sets s.t. $S_n \xrightarrow{b} S$ and let us define $\boldsymbol{\eta}_n(t, S_n) \equiv S_n(t)$. Then

$$S_n(t) \xrightarrow{b} S(t) \tag{V.5.10}$$

with $S(t) \equiv \boldsymbol{\eta}(t, S)$, meaning $\mathbf{db}_{S_n(t)} \to \mathbf{db}_{S(t)}$ in $C_{\mathrm{loc}}(\mathbb{R}^3)$ uniformly with respect to $t \in [0, T]$.

Now let us notice that since $\{\varrho_\varepsilon\}_{\varepsilon>0}$ solve the transport equation (V.4.1) with regular transport coefficients ($[\boldsymbol{u}_\varepsilon]_\delta \in L^\infty(0, T; W^{1,\infty}(\mathcal{T}))$, we can use Proposition V.5.1 and (V.5.8) to conclude that

$$\varrho_\varepsilon \to \varrho \text{ in } C([0, T] \times \mathcal{T}). \tag{V.5.11}$$

Moreover due to the method of characteristics for all $t \in [0, T]$

$$\inf_{x \in \mathcal{T}} \varrho_{0,\delta} \leqslant \inf_{x \in \mathcal{T}} \varrho_\varepsilon(t, x) \leqslant \sup_{x \in \mathcal{T}} \varrho_\varepsilon(t, x) \leqslant \sup_{x \in \mathcal{T}} \varrho_{0,\delta}, \tag{V.5.12}$$

and

$$\inf_{x \in \mathcal{T}} \varrho_{0,\delta} \leqslant \inf_{x \in \mathcal{T}} \varrho(t, x) \leqslant \sup_{x \in \mathcal{T}} \varrho(t, x) \leqslant \sup_{x \in \mathcal{T}} \varrho_{0,\delta}. \tag{V.5.13}$$

Employing once more inequality (V.4.12) we obtain

$$\boldsymbol{u}_\varepsilon \overset{*}{\rightharpoonup} \boldsymbol{u} \text{ weakly-(*) in } L^\infty(0, T; L^2(\mathcal{T}; \mathbb{R}^3)) \text{ as } \varepsilon \to 0. \tag{V.5.14}$$

Using a strong-weak argument together with (V.5.8), (V.5.11) we obtain, that the limit density ϱ satisfies the equation of continuity in a weak sense

$$\partial_t \varrho + \mathrm{div}_x(\varrho [\boldsymbol{u}]_\delta) = 0 \text{ in } (0, T) \times \mathcal{T} \tag{V.5.15}$$

provided ϱ has been extended by ϱ_f outside of Ω. Once more, according to Proposition V.5.1 and assumption (V.4.7) we notice that the density is constant in the approximation of the fluid region, i.e.

$$\varrho = \varrho_f \text{ on the set } \Big((0, T) \times \Omega\Big) \setminus \bigcup_{t \in [0,T]} \bigcup_{i=1}^{n} \boldsymbol{\eta}(t, [S_i]_\delta), \tag{V.5.16}$$

where $[S_i]_\delta$ is the δ-kernel (see (V.5.3)) and $\boldsymbol{\eta}$ is a solution of

$$\partial_t \boldsymbol{\eta}(t, x) = [\boldsymbol{u}]_\delta(t, \boldsymbol{\eta}(t, x)), \;\; \boldsymbol{\eta}(0, x) = x. \tag{V.5.17}$$

V.5.3. Position of the rigid bodies. Next we identify the position of rigid bodies.

Let us remark as in [**113**] that if $\boldsymbol{u}$ is a rigid velocity field in the set S, then $[\boldsymbol{u}]_\delta = \boldsymbol{u}$ for all x in S for which $\mathbf{db}_S(x) > \delta$.

The replacement of $\boldsymbol{u}_\varepsilon$ by $[\boldsymbol{u}_\varepsilon]_\delta$ in (V.4.3) allows to obtain better results on characteristics of transport equations. Moreover, we are able to obtain a rigid motion as $\varepsilon \to 0$, without passing to the limit w.r.t. δ due to the above remark.

Here we follow [**56**] and just for convenience of the reader we recall briefly some of the steps.

Step 1: First let us recall that $]\cdot[_\delta$, $[\cdot]_\omega$ denote respectively the $\delta-$neighbourhood and the $\omega-$kernel defined in (V.5.3). We notice that the kernels $[S_i]_\omega$ and their images $\boldsymbol{\eta}(t,[S_i]_\omega)$ are non-empty connected open sets since $0<\delta<\omega<\delta_0/2$ (δ_0 has been introduced in (V.2.1)).

Directly from the hypothesis (V.4.8) and (V.4.9) we infer

(V.5.18) $\mu_\varepsilon(0,x)\to\infty$ as $\varepsilon\to 0$ uniformly for $x\in[S_i]_{\omega'},\ i=1,\dots,n,\ \omega>\omega'>\delta.$

Since $\boldsymbol{\eta}_\varepsilon$ is determined as the unique solution (due to regularity of $[\boldsymbol{u}_\varepsilon]_\delta$) of the problem

(V.5.19) $$\partial_t\boldsymbol{\eta}_\varepsilon(t,x)=[\boldsymbol{u}_\varepsilon]_\delta(t,\boldsymbol{\eta}_\varepsilon(t,x)),\ \boldsymbol{\eta}_\varepsilon(0,x)=x,$$

convergence (V.5.18) provides that

(V.5.20) $$\mu_\varepsilon(t,x)\to\infty \text{ uniformly for } t\in[0,T],\ x\in\boldsymbol{\eta}_\varepsilon(t,[S_i]_{\omega'}),\ i=1,\dots,n.$$

According to (V.5.10) in Proposition V.5.1

(V.5.21)
$$\overline{\boldsymbol{\eta}(t,[S_i]_\omega)}\subset\boldsymbol{\eta}_\varepsilon(t,[S_i]_{\omega'}) \text{ for sufficiently small } \varepsilon>0 \text{ and for } \delta_0/2>\omega>\omega'>\delta.$$

Hence from (V.5.20) we deduce

(V.5.22)
$$\mu_\varepsilon(t,x)\to\infty \text{ as } \varepsilon\to 0 \text{ uniformly for } t\in[0,T],\ x\in\boldsymbol{\eta}(t,[S_i]_\omega),\ \text{for } i=1,\dots,n.$$

Therefore we infer that

$$[\mu_\varepsilon]_\delta\to\infty$$

uniformly on compact subsets of

$$\{t\in[0,T],\ x\in]\boldsymbol{\eta}(t,[S_i]_\omega)[_\delta\},\ i=1,\dots,n.$$

Consequently, we deduce from the estimate (V.4.14) that

(V.5.23) $$\mathbf{D}\boldsymbol{u}_\varepsilon\to\mathbf{D}\boldsymbol{u}=0 \text{ a.a. on the set } \bigcup_{t\in[0,T]}\bigcup_{i=1}^{n}]\boldsymbol{\eta}(t,[S_i]_\omega)[_\delta \text{ for any } \omega>\delta,$$

where $\boldsymbol{\eta}$ is determined by (V.5.17).

Step 2: Using now (V.5.23) we deduce that the limit velocity $\boldsymbol{u}$ coincides with a rigid velocity field $\boldsymbol{u}^{S_i}$ on the $\delta-$neighbourhood of each of the sets $\boldsymbol{\eta}(t,[S_i]_\omega)$, where $\omega>\delta$, $i=1,\dots,n$. Since the rigid velocity fields coincide with their regularizations, namely $[\boldsymbol{u}^{S_i}]_\delta=\boldsymbol{u}^{S_i}$, we conclude that

(V.5.24) $\boldsymbol{u}(t,x)=\boldsymbol{u}^{S_i}(t,x)=[\boldsymbol{u}]_\delta(t,x)$ for $t\in[0,T],\ x\in\boldsymbol{\eta}(t,[S_i]_\delta),\ i=1,\dots,n.$

Accordingly, by (V.5.17), (V.5.24) we infer the existence of a family of isometries $\boldsymbol{\eta}_i(t,\cdot)$, $t\in[0,T]$, $i=1,\dots,n$, $\boldsymbol{\eta}(0,\cdot)=\boldsymbol{Id}$, such that

(V.5.25) $$\boldsymbol{\eta}_i(t,[S_i]_\delta)=\boldsymbol{\eta}(t,[S_i]_\delta) \text{ for all } t\in[0,T],\ i=1,\dots,n.$$

Moreover by (V.5.23) the mappings $\{\boldsymbol{\eta}_i\}_{i=1}^n$ are compatible with the velocity field $\boldsymbol{u}$ and with the rigid bodies $\{S_i\}_{i=1}^n$ in the sense stated in (V.2.8), (V.2.9). In particular, hypothesis (V.5.16), (V.5.24) and the assumption $\varrho_f=\text{const}$ provide that (V.5.15) reduces to

(V.5.26) $$\partial_t\varrho+\operatorname{div}_x(\varrho\boldsymbol{u})=0 \text{ in } (0,T)\times\mathcal{T}.$$

Step 3: Now we concentrate on the momentum equation. Since $\boldsymbol{\eta}_i$ for $i = 1, \dots, n$ are isometries, (V.5.25) implies

$$]\boldsymbol{\eta}_i(t, [S_i]_\delta)[_\delta = \boldsymbol{\eta}_i(t, S_i), \ i = 1, \dots, n.$$

Hence $[\mu_\varepsilon]_\delta$ converges uniformly locally to 1 in the complementary of $\bigcup_{i=1}^n \overline{S_i(t)}$ for any $t \in [0, T]$. According to estimates (V.4.14) and properties of regularization we notice that

$$\boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta \rightharpoonup \overline{\boldsymbol{u} \otimes [\boldsymbol{u}]_\delta} \text{ weakly in } L^2(0, T; L^2(\Omega; \mathbb{R}^3)). \tag{V.5.27}$$

Together with (V.5.11) and by a weak-strong argument we obtain

$$\varrho_\varepsilon \boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta \rightharpoonup \varrho(\overline{\boldsymbol{u} \otimes [\boldsymbol{u}]_\delta}) \text{ weakly in } L^2(0, T; L^2(\Omega; \mathbb{R}^3)). \tag{V.5.28}$$

Employing again the estimate (V.4.14) and recalling that $\mu_\varepsilon \geqslant 1$ in $[0, T) \times \mathcal{T}$ we get

$$\mathbf{S}_\varepsilon \overset{*}{\rightharpoonup} \overline{\mathbf{S}} \text{ weakly–(*) in } L_{M^*}([0, T] \times \mathcal{T}; \mathbb{R}^{3\times 3}), \tag{V.5.29}$$

due to properties of an $\mathcal{N}$−function M^* (convexity and superlinear growth) the Dunford-Pettis lemma provides

$$\mathbf{S}_\varepsilon \rightharpoonup \overline{\mathbf{S}} \text{ weakly in } L^1([0, T] \times \mathcal{T}; \mathbb{R}^{3\times 3}) \tag{V.5.30}$$

Moreover by (V.5.11), (V.5.13) and (V.5.14) we infer

$$\varrho_\varepsilon \boldsymbol{u}_\varepsilon \rightharpoonup \varrho \boldsymbol{u} \text{ weakly-(*) in } L^\infty(0, T; L^2(\mathcal{T}; \mathbb{R}^3)). \tag{V.5.31}$$

Finally letting $\varepsilon \to 0$ in the momentum equation (V.4.13) we deduce that

(V.5.32)

$$\int_0^T \int_\Omega \varrho \boldsymbol{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho(\overline{\boldsymbol{u} \otimes [\boldsymbol{u}]_\delta}) : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_\Omega \overline{\mathbf{S}} : \mathbf{D}\boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega \varrho \nabla_x F \cdot \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t - \int_\Omega \varrho_{0,\delta} \boldsymbol{u}_0 \cdot \boldsymbol{\varphi}(0, \cdot) \, \mathrm{d}x$$

for any test function $\boldsymbol{\varphi} \in C_c^1([0, T) \times \Omega)$, $\boldsymbol{\varphi}(t, \cdot) \in [\mathcal{R}M](t)$, where $[\mathcal{R}M](t)$ is defined by (V.2.7) with

$$S_i(t) = \boldsymbol{\eta}_i(t, S_i), \ i = 1, \dots, n.$$

V.5.4. Convergence of the velocities. Our next goal is to identify the weak limit in (V.5.27), namely we want to show that

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \text{ in } L^2(0, T; L^2(\Omega; \mathbb{R}^3)). \tag{V.5.33}$$

Let us notice that due to (V.5.7) and the Sobolev embedding theorem we obtain the desired convergence in space but there is still a possibility for oscillations of the velocity fields $\{\boldsymbol{u}_\varepsilon\}_{\varepsilon>0}$ in time.

As it was already pointed out, according to the result obtained by Starovoitov [**120**, Theorem 3.1], the collisions of two rigid objects do not appear. It is provided by the fact we consider the fluid which is incompressible and the velocity gradients are assumed to be bounded in the Lebesgue space L^p, with $p \geqslant 4$. Originally in [**120**] this statement was proven only for one body in a bounded domain, but it is easy to observe that this result can be extended to the case of several bodies (what is also mentioned therein). Hence we infer

$$\text{(V.5.34)} \qquad \mathrm{d}[\bigcup_{i=1}^{n} S_i(t)] = d(t) > 0 \text{ uniformly for } t \in [0,T],$$

(where d is defined by (V.5.1) in Section V.5.1). Setting $S_i^\varepsilon(t) = \boldsymbol{\eta}_\varepsilon(t, S_i)$ (see (V.5.19)) and according to Proposition V.5.1 we have

$$\text{(V.5.35)} \qquad \mathrm{d}[\bigcup_{i=1}^{n} S_i^\varepsilon(t)] = d_\varepsilon \to d \text{ in } C[0,T].$$

Since the contacts of rigid bodies or bodies with boundary do not occur, to prove compactness of the sequence $\{\boldsymbol{u}_\varepsilon\}_{\varepsilon>0}$, we can use the same method as in [**113, 56**], namely by employing projection of momentum on a space of rigid velocities.

Since

$$S_i^\varepsilon(t) \xrightarrow{b} S_i(t) \text{ uniformly with respect to } t \in [0,T],\ i = 1,\ldots,n,$$

we obtain, for any fixed $\sigma > 0$, and all $\varepsilon < \varepsilon_0(\sigma)$ small enough

$$\text{(V.5.36)} \qquad S_i(t) \subset]S_i^\varepsilon(t)[_\sigma, \quad S_i^\varepsilon(t) \subset]S_i(t)[_\sigma, \text{ for all } t \in [0,T],\ i = 1,\ldots,n.$$

Let us now recall the following result of Feireisl et al. [**56**].

Lemma V.5.2. *Given a family of smooth open sets $\{S_i\}_{i=1}^n \subset \Omega$, $0 < k < 1/2$, there exists a function $h : (0,\sigma_0) \to \mathbb{R}^+$ s.t. $h(\sigma) \to 0$ when $\sigma \to 0$ and for arbitrary $\boldsymbol{v} \in W^{1,p}_{0,\mathrm{div}}(\Omega;\mathbb{R}^3)$:*

(V.5.37)

$$\left\| \boldsymbol{v} - \mathcal{P}^k\Big(\bigcup_{i=1}^{n}]S_i[_\sigma\Big)\boldsymbol{v} \right\|_{W^{1,k}(\Omega;\mathbb{R}^3)} \leqslant c\Big(\| \, \mathbf{D}(\boldsymbol{v}) \, \|_{L^2(\bigcup_{i=1}^n S_i;\mathbb{R}^{3\times 3})} + h(\sigma)\|\boldsymbol{v}\|_{W^{1,p}(\Omega;\mathbb{R}^3)}\Big)$$

with a constant $0 < c < \infty$. Moreover, h and c are independent of the position of S_i inside Ω as long as $\mathrm{d}[\bigcup_{i=1}^n S_i] > 2\sigma_0$.

The projection $\mathcal{P}^k$ is defined by (V.5.4).

Next using the local-in-time Aubin-Lions argument we show the following

Lemma V.5.3. *For all $\sigma > 0$ sufficiently small, and $0 < k < 1/2$, we have*

$$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega \varrho_\varepsilon \boldsymbol{u}_\varepsilon \cdot \mathcal{P}^k\Big(\bigcup_{i=1}^{n}]S_i(t)[_\sigma\Big)[\boldsymbol{u}_\varepsilon]\,\mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \varrho \boldsymbol{u} \cdot \mathcal{P}^k\Big(\bigcup_{i=1}^{n}]S_i(t)[_\sigma\Big)[\boldsymbol{u}]\,\mathrm{d}x\mathrm{d}t.$$

The idea of the proof follows [**56, 113**].

Proof. Let us fix $\sigma > 0$. According to (V.5.36) there exists $\varepsilon_0(\sigma)$ such that for all $\varepsilon < \varepsilon_0$ it holds

$$\bigcup_{i=1}^{n} S_i(t) \subset \bigcup_{i=1}^{n}]S_i^\varepsilon(t)[_{\sigma/2}, \ \bigcup_{i=1}^{n} S_i^\varepsilon(t) \subset \cup_{i=1}^{n}]S(t)[_{\sigma/2} \text{ for all } t \in [0,T].$$

If we apply the Proposition V.5.1 to the sequence $\{[\boldsymbol{u}_\varepsilon]_\delta\}_{\varepsilon>0}$ with $\boldsymbol{u}_\varepsilon \in L^\infty(0,T;L^2)$ we notice that $\boldsymbol{\eta}_\varepsilon$ is Lipschitz continuous in time. Hence we infer that there exists $\tau > 0$ (dependent on σ) and a subdivision of the time interval $0 < \tau < 2\tau < ... < J\tau = T$ such that for arbitrary $t \in I_j := [j\tau, (j+1)\tau]$ we have

$$\bigcup_{i=1}^n S_i(t) \subset \bigcup_{i=1}^n]S_i(j\tau)[_{\sigma/2}, \qquad \bigcup_{i=1}^n S_i(j\tau) \subset \bigcup_{i=1}^n]S_i(t)[_{\sigma/2}. \tag{V.5.38}$$

To be more precise, if we take Lip as a Lipschitz constant of the function $t \to \boldsymbol{\eta}(t,x)$, then there exists $\tau < \sigma/(2Lip)$ which satisfies (V.5.38).

Our goal now is to infer from the momentum equation (V.4.13) that

(V.5.39)
$$\mathcal{P}^0\Big(\cup_{i=1}^n]S_i(t)[_\sigma\Big)[\varrho_\varepsilon \boldsymbol{u}_\varepsilon] \text{ is precompact in } L^2(I_j;\Big[\mathcal{K}^{k,2}\Big(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\Big) \cap W^{s,2}_{0,\mathrm{div}}\Big]^*)$$

for any $k < 1$ and $s > \frac{5}{2}$ (then $W^{s-1,2} \subset L^\infty$).

First, let us fix one of the intervals I_m, $j = 1, \dots, J$ and in the momentum equation let us take as a test function $\boldsymbol{\xi}$, which is equal to zero if $t \notin I_m$ and such that

$$\boldsymbol{\xi} \in \mathcal{K}^{1,2}\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right) \cap W^{s,2}_{0,\mathrm{div}} \text{ for all } t \in I_j.$$

Using estimates (V.4.14) we deduce from the momentum equation (V.4.13) that

$$\left|\int_{I_m}\int_\Omega \varrho_\varepsilon \boldsymbol{u}_\varepsilon \partial_t \boldsymbol{\xi}\, \mathrm{d}x\mathrm{d}t\right| \leqslant C\|\boldsymbol{\xi}\|_{L^\infty(I_j;W^{1,2}\cap W^{s,2}_{0,\mathrm{div}})} \quad \text{for all } \varepsilon > \varepsilon_0.$$

According to the above relation we infer that

$$\left\{\partial_t \mathcal{P}^0\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\varrho_\varepsilon \boldsymbol{u}_\varepsilon]\right\}_{\varepsilon>0}$$
$$\text{is bounded in } L^1(I_j;\left[\mathcal{K}^{k,2}\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right) \cap W^{s,2}_{0,\mathrm{div}}\right]^*).$$

Moreover, since $\varrho_\varepsilon \boldsymbol{u}_\varepsilon$ is bounded also in $L^2(I_m \times \mathcal{T})$, then the sequence

$$\{\mathcal{P}^0\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\varrho_\varepsilon \boldsymbol{u}_\varepsilon]\}_\varepsilon \text{ is bounded in } L^2(I_j;\mathcal{K}^{0,2}\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right)).$$

Since the inclusion

$$\mathcal{K}^{0,2}\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right) \subset \left[\mathcal{K}^{k,2}\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right)\right]^* \text{ is compact for } 0 < k < 1,$$

the Aubin-Lions argument provides that the sequence

$$\{\mathcal{P}^0\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\varrho_\varepsilon \boldsymbol{u}_\varepsilon]\}_{\varepsilon>0} \text{ is precompact in } L^2(I_j;\left[\mathcal{K}^{k,2}\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right)\right]^*).$$

Furthermore by (V.5.31) we have that

$$\mathcal{P}^0\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\varrho_\varepsilon \boldsymbol{u}_\varepsilon] \to \mathcal{P}^0\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\varrho \boldsymbol{u}] \tag{V.5.40}$$

strongly in $L^2(I_j;\left[\mathcal{K}^{k,2}\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right)\right]^*)$ for $0 < k < 1$.

The relation (V.5.38) provides

$$\mathcal{P}^0\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right)\mathcal{P}^k\left(\cup_{i=1}^n]S_i(t)[_\sigma\right) = \mathcal{P}^k\left(\cup_{i=1}^n]S_i(t)[_\sigma\right) \tag{V.5.41}$$

for all $t \in I_j$ and $0 < k < 1$. Since $\mathcal{P}^0\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right)$ is self-adjoint in $L^2(\Omega)$ and by

$$\int_{I_j}\int_\Omega \varrho_\varepsilon \boldsymbol{u}_\varepsilon \cdot \mathcal{P}^k\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\boldsymbol{u}_\varepsilon]\,\mathrm{d}x\mathrm{d}t$$
$$= \int_{I_j}\int_\Omega \mathcal{P}^0\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right)[\varrho_\varepsilon \boldsymbol{u}_\varepsilon] \cdot \mathcal{P}^k((\cup_{i=1}^n]S_i(t)[_\sigma)[\boldsymbol{u}_\varepsilon]\,\mathrm{d}x\mathrm{d}t$$
$$= \int_{I_j}\left(\mathcal{P}^0\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right)[\varrho_\varepsilon \boldsymbol{u}_\varepsilon], \mathcal{P}^k\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\boldsymbol{u}_\varepsilon]\right)_{L^2(\Omega)}\mathrm{d}t$$

Then by (V.5.40) and as $\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u}$ in $L^2(0,T;L^2(\mathcal{T}))$ we get

$$\lim_{\varepsilon\to 0}\int_{I_j}\left(\varrho_\varepsilon \boldsymbol{u}_\varepsilon, \mathcal{P}^k\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\boldsymbol{u}_\varepsilon]\right)_{L^2(\Omega)}\mathrm{d}t$$
$$= \lim_{\varepsilon\to 0}\int_{I_j}\left(\mathcal{P}^0\left(\cup_{i=1}^n]S_i(j\tau)[_{\sigma/2}\right)[\varrho_\varepsilon \boldsymbol{u}_\varepsilon], \mathcal{P}^k\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\boldsymbol{u}_\varepsilon]\right)_{L^2(\Omega)}\mathrm{d}t$$
$$= \int_{I_j}\left(\varrho \boldsymbol{u}, \mathcal{P}^k\left(\cup_{i=1}^n]S_i(t)[_\sigma\right)[\boldsymbol{u}]\right)_{L^2(\Omega)}\mathrm{d}t$$

Summing up the relation as above from $j = 1$ to $j = J$ we obtain the desired conclusion of Lemma V.5.3 □

Combining Lemma V.5.2 and Lemma V.5.3 we deduce

$$\lim_{\varepsilon\to 0}\int_0^T\int_\Omega \varrho_\varepsilon|\boldsymbol{u}_\varepsilon|^2\,\mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega \varrho|\boldsymbol{u}|^2\,\mathrm{d}x\mathrm{d}t \tag{V.5.42}$$

which can be shown exactly step by step as in [**56**, Section 5.2] or [**98**, Section 6.1]. Therefore we achieve the conclusion (V.5.33).

Indeed, for a fixed $k \in (0,1/2)$ and for sufficiently small $\varepsilon > 0$, $\sigma > 0$ we set

$$\int_0^T\int_\Omega\left(\varrho_\varepsilon|\boldsymbol{u}_\varepsilon|^2 - \varrho|\boldsymbol{u}|^2\right)\mathrm{d}x\mathrm{d}t = I_1^\varepsilon(\sigma) + I_2(\sigma) - I_3^\varepsilon(\sigma),$$

where

$$I_1^\varepsilon(\sigma) = \int_0^T\int_\Omega\left(\varrho_\varepsilon \boldsymbol{u}_\varepsilon \cdot \mathcal{P}^k\Big(\cup_{i=1}^n]S(t)[_\sigma\Big)[\boldsymbol{u}_\varepsilon] - \varrho \boldsymbol{u} \cdot \mathcal{P}^k\Big(\cup_{i=1}^n]S(t)[_\sigma\Big)[\boldsymbol{u}]\right)\mathrm{d}x\mathrm{d}t$$

$$I_2(\sigma) = \int_0^T\int_\Omega \varrho \boldsymbol{u} \cdot \left(\mathcal{P}^k\Big(\cup_{i=1}^n]S_i(t)[_\sigma\Big)[\boldsymbol{u}] - \boldsymbol{u}\right)\mathrm{d}x\mathrm{d}t,$$

and

$$I_3^\varepsilon(\sigma) = \int_0^T\int_\Omega \varrho_\varepsilon \boldsymbol{u}_\varepsilon \cdot \left(\mathcal{P}^k\Big(\cup_{i=1}^n]S_i(t)[_\sigma\Big)[\boldsymbol{u}_\varepsilon] - \boldsymbol{u}_\varepsilon\right)\mathrm{d}x\mathrm{d}t.$$

Next let us notice that Lemma V.5.3 provides

$$\lim_{\varepsilon\to 0} I_1^\varepsilon(\sigma) = 0 \quad \text{for all } \sigma \text{ sufficiently small.}$$

As $\boldsymbol{u} \in L^2(0,T;\mathcal{K}^{k,p}(\cup_{i=1}^n S_i(t)))$ by Lemma V.5.2 we infer

$$\int_0^T \int_\Omega |(\mathcal{P}^k\Big(\cup_{i=1}^n S_i(t)\Big)[\boldsymbol{u}] - \boldsymbol{u})|^2 \,\mathrm{d}x\mathrm{d}t \leqslant h(\sigma)^2 \int_0^T \int_\Omega \|\boldsymbol{u}\|^2_{W^{1,p}(\Omega;\mathbb{R}^3)} \,\mathrm{d}x\mathrm{d}t,$$

provided that there are no contacts of two bodies or of the rigid body and the boundary of the set Ω. Therefore we obtain

$$I_2(\sigma) \to 0 \quad \text{as } \sigma \to 0$$

Recalling that $\{\varrho_\varepsilon \boldsymbol{u}_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$, we have

$$I_3^\varepsilon(\sigma) \leqslant c\int_0^T \|\mathcal{P}^k\Big(\cup_{i=1}^n]S_i(t)[_\sigma\Big)[\boldsymbol{u}_\varepsilon] - \boldsymbol{u}_\varepsilon)\|^2_{W^{k,2}(\Omega;\mathbb{R}^3)} \,\mathrm{d}t.$$

Since $]S_i(t)[_\sigma \subset]S_i^\varepsilon(t)[_{2\sigma}$ we obtain also

$$I_3^\varepsilon(\sigma) \leqslant c\int_0^T \|(\mathcal{P}^k\Big(\cup_{i=1}^n]S_i^\varepsilon(t)[_{2\sigma}\Big)[\boldsymbol{u}_\varepsilon] - \boldsymbol{u}_\varepsilon)\|^2_{W^{k,2}(\Omega;\mathbb{R}^3)} \,\mathrm{d}t$$

for $\varepsilon > 0$ sufficiently small. Applying again Lemma V.5.2, with $\boldsymbol{u}_\varepsilon(t,\cdot)$ for arbitrary $t \in [0,T]$ and $\varepsilon > 0$ sufficiently small we have that

$$\int_0^T \|(\mathcal{P}^k\Big(\cup_{i=1}^n]S_i^\varepsilon(t)[_{2\sigma}\Big)[\boldsymbol{u}_\varepsilon]-\boldsymbol{u}_\varepsilon)\|^2_{W^{k,2}(\Omega;\mathbb{R}^3)} \,\mathrm{d}t \leqslant c\Big(\sum_{i=1}^n \int_0^T \int_{S^i} |\mathbf{D}\boldsymbol{u}_\varepsilon|^2 \,\mathrm{d}x\mathrm{d}t\Big) + cTh(2\sigma).$$

The first term on the right hand side converges to zero as $\mathbf{D}\boldsymbol{u}_\varepsilon \to 0$ on $\bigcup_{i=1}^n S_i(t)$ and a.a. $t \in [0,T]$, and $\{\mathbf{D}\boldsymbol{u}_\varepsilon\}_{\varepsilon>0}$ is uniformly integrable in $L^2(Q)$ since (V.5.6) holds. Finally if we pass to the limit with $\varepsilon \to 0$ we obtain that

$$\limsup_{\varepsilon\to\infty} I^\varepsilon(\sigma) \leqslant C(T)h(2\sigma) + I_2(\sigma)$$

whenever σ is small enough. Letting $\sigma \to 0$ we achieve the relation (V.5.42).

V.5.5. Convergence in the nonlinear viscous term. Our main goal now is to prove convergence in the nonlinear viscous term in the "fluid" part of the time-space cylinder $(0,T)\times\Omega$. As $\mu = 1$ on the fluid part and boundaries $S_n \to S$, we can choose for a sufficiently small epsilon small cylinders contained in the fluid part of the time-space cylinder Q_f. Thus in order to obtain this result we consider the equation (V.4.13) on the set $I \times B$ such that $I \subset (0,T)$ is a time interval and a spatial ball $|B| \leqslant 1$ and $B \subset \Omega\setminus \cup_{i=1}^n S_i(t)$ for $t \in I$. By (V.5.16) we can assume that $\varrho = \varrho_f$ in $I \times B$. In particular, we have

$$\int_0^T \int_\Omega \varrho_f \boldsymbol{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} + (\varrho_f \boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta - \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon)) : \nabla_x \boldsymbol{\varphi} \,\mathrm{d}x\mathrm{d}t = 0 \tag{V.5.43}$$

for any $\boldsymbol{\varphi} \in \mathcal{D}(I\times B;\mathbb{R}^3)$, $\mathrm{div}_x \boldsymbol{\varphi} = 0$.

We cannot test the above equation with a function with non-zero support on Q^s, as neither the penalizing term $\mu_\varepsilon \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon)$ nor $\mu_\varepsilon \mathbf{D}\boldsymbol{u}_\varepsilon$ can be controlled. At this stage of our investigations, the problem must be localised in the fluid part separately from the rigid bodies. Therefore we introduce a "local" pressure

$$p = p_{\mathrm{reg}} + \partial_t p_{\mathrm{harm}}, \tag{V.5.44}$$

where p_{reg} enjoys the same regularity properties as the sum of the convective and the viscous terms in case of power-law fluids (see [**56**]), while p_{harm} is a harmonic function. If the tensor $\mathbf{S}$ satisfies only conditions (V.1.1)-(V.1.3) and an isotropic $\mathcal{N}-$function M does not satisfy the Δ_2-condition, then the regularity of p_{reg} can be lower than the regularity of the viscous term, what in fact makes the problem different from any previous considerations in this field.

The concept of local pressure was developed by Wolf [**130**, Theorem 2.6]. However our construction is based on Riesz transform as in [**56**] and it is more suitable for application to problems with non-standard growth conditions.

We start with formulation of the following lemma:

Lemma V.5.4. *Let $B \subset \mathbb{R}^3$ be a bounded domain with a regular C^3 boundary and $I = (t_0, t_1)$ be a time interval. Let m^* and m' be $\mathcal{N}-$functions given by $m^*(\tau) = \tau \log^{\beta+1}(\tau+1)$ for some $\beta > 0$ and $m'(\tau) = \tau \log^{\beta}(\tau+1)$ for $\tau \in \mathbb{R}_+$. Moreover let M^* be an $\mathcal{N}-$function such that $c_1 m^*(\tau) \leqslant M^*(\tau) < c_2|\tau|^2$ for some positive constants c_1, c_2. Assume that $\boldsymbol{U} \in L^\infty(I; L^2(B;\mathbb{R}^3))$, $\mathrm{div}_x \boldsymbol{U} = 0$, and $\mathbf{T} \in L_{M^*}(I \times B; \mathbb{R}^{3\times 3})$ satisfy the integral identity*

$$\int_I \int_B \Big(\boldsymbol{U} \cdot \partial_t \boldsymbol{\varphi} + \mathbf{T} : \nabla_x \boldsymbol{\varphi} \Big) \, \mathrm{d}x \mathrm{d}t = 0 \tag{V.5.45}$$

for all $\boldsymbol{\varphi} \in \mathcal{D}(I \times B; \mathbb{R}^3)$, $\mathrm{div}_x \boldsymbol{\varphi} = 0$.

Then there exist two functions

$$p_{\mathrm{reg}} \in L^1(I; L_{m'}(B)),$$

$$p_{\mathrm{harm}}(t,\cdot) \in \mathcal{D}'(B), \ \Delta_x p_{\mathrm{harm}} = 0 \ \text{in } \mathcal{D}'(I \times B), \ \int_B p_{\mathrm{harm}}(t,\cdot)\, \mathrm{d}x = 0$$

satisfying

$$\int_I \int_B \Big(\boldsymbol{U} \cdot \partial_t \boldsymbol{\varphi} + \mathbf{T} : \nabla_x \boldsymbol{\varphi} \Big) \, \mathrm{d}x \mathrm{d}t = \int_I \int_B \Big(p_{\mathrm{harm}} \partial_t \mathrm{div}_x \boldsymbol{\varphi} + p_{\mathrm{reg}} \mathrm{div}_x \boldsymbol{\varphi} \Big) \, \mathrm{d}x \mathrm{d}t \tag{V.5.46}$$

for any $\boldsymbol{\varphi} \in \mathcal{D}(I \times B; \mathbb{R}^3)$. Additionally,

$$\|p_{\mathrm{reg}}\|_{L_1(I; L_{m'}(B))} \leqslant c(m') \|\mathbf{T}\|_{L_{M^*}(I \times B; \mathbb{R}^{3\times 3})} \tag{V.5.47}$$

and

$$p(t,\cdot)|_{B'} \in C^\infty(B'), \ \text{where } B' \subset\subset B, \tag{V.5.48}$$

$$\|p_{\mathrm{harm}}\|_{L^\infty(I; L^1(B))} \leqslant c(m', I, B) \Big(\|\mathbf{T}\|_{L_{M^*}(I \times B; \mathbb{R}^3)} + \|\boldsymbol{U}\|_{L^\infty(I; L^2(B;\mathbb{R}^3))} \Big). \tag{V.5.49}$$

PROOF. To begin with, the "regular" component of the pressure p_{reg} is identified as

$$p_{\mathrm{reg}}(t,\cdot) = \mathcal{R} : \mathbf{T} = \sum_{i,j=1}^{3} \mathcal{R}_{i,j}[T_{i,j}](t,\cdot) \text{ in } \mathbb{R}^3 \text{ for a.a. } t \in I,$$

where $\mathcal{R}$ denotes the "double" Riesz transform (see (III.2.2)) and $\mathbf{T} = [T_{i,j}]_{i=1,2,3;j=1,2,3}$ has been extended to be zero outside of B. Using (III.2.4) we obtain that the mappings

$$\mathcal{R}_{i,j}|_B : L_{m^*}(B) \to L_{m'}(B) \quad \text{are bounded for } i,j = 1,2,3.$$

As a consequence we get (V.5.47) in the following way

(V.5.50)
$$\|p_{\mathrm{reg}}\|_{L^1(I;L_{m'}(B))} = \|\mathcal{R} : \mathbf{T}\|_{L^1(I;L_{m'}(B))} \leqslant c_1(m')\|\mathbf{T}\|_{L^1(I;L_{M^*}(B))} \leqslant c_2(m')\|\mathbf{T}\|_{L_{M^*}(I\times B)},$$

where we use the fact that $L_{M^*}(I \times B;\mathbb{R}^{3\times 3}) \subseteq L^1(I;L_{M^*}(B;\mathbb{R}^{3\times 3}))$ (see the proof of [**46**, Corollary 1.1.0]).

Moreover,

$$\int_B p_{\mathrm{reg}}\Delta\psi \,\mathrm{d}x = \int_B \mathbf{T} : \nabla^2\psi \;\mathrm{d}x \text{ for any } \psi \in \mathcal{D}(B). \tag{V.5.51}$$

On the other hand, (V.5.45) provides that we can redefine $\boldsymbol{U}$ w.r.t. time on the set of zero measure such that the mappings

$$t \mapsto \int_B \boldsymbol{U}\cdot\boldsymbol{\psi}\,\mathrm{d}x \in C([t_0,t_1]) \text{ for any } \quad \boldsymbol{\psi} \in \mathcal{D}(B;\mathbb{R}^3),\ \mathrm{div}_x\boldsymbol{\psi} = 0.$$

Particularly, we infer that the Helmholtz projection $\boldsymbol{H}[\boldsymbol{U}]$ belongs to the space $C_{\mathrm{weak}}([t_0,t_1];L^2(B;\mathbb{R}^3))$. Therefore after taking in (V.5.45) $\boldsymbol{\phi}(t,x) = \eta(t)\boldsymbol{\psi}(x)$ such that $\eta \in \mathcal{D}([t_0,t_1))$, $\boldsymbol{\psi} \in \mathcal{D}(B;\mathbb{R}^3)$, $\mathrm{div}_x\boldsymbol{\psi} = 0$ it follows that

$$\int_I \Big[\int_B (\boldsymbol{U}(t,\cdot) - \boldsymbol{U}(t_0,\cdot))\cdot\boldsymbol{\psi}\,\mathrm{d}x\Big]\partial_t\eta\,\mathrm{d}t - \int_I \Big[\int_B \Big(\int_{t_0}^t \mathbf{T}(s,\cdot)\,\mathrm{d}s\Big) : \nabla_x\boldsymbol{\psi}\,\mathrm{d}x\Big]\partial_t\eta\,\mathrm{d}t = 0.$$

Employing Lemma 2.2.1 from [**119**], there exists a pressure $p = p(t,\cdot)$ such that

(V.5.52)
$$\int_B (\boldsymbol{U}(t,\cdot) - \boldsymbol{U}(t_0,\cdot))\cdot\boldsymbol{\psi}\,\mathrm{d}x - \int_B \Big(\int_{t_0}^t \mathbf{T}(s,\cdot)\;\mathrm{d}s\Big) : \nabla_x\boldsymbol{\psi}\,\mathrm{d}x = \int_B p(t,\cdot)\mathrm{div}_x\boldsymbol{\psi}\,\mathrm{d}x$$

for all $t \in I$ and all $\boldsymbol{\psi} \in \mathcal{D}(B;\mathbb{R}^3)$. Note that the term on the right-hand side is measurable and integrable w.r.t. time variable, since the left-hand side is measurable and integrable. Moreover for a.a. $t \in I$

$$\int_B p(t,\cdot)\,\mathrm{d}x = 0 \quad \text{and} \quad p(t,\cdot) \in \mathcal{D}'(B). \tag{V.5.53}$$

Testing (V.5.52) by $\partial_t\zeta$, $\zeta \in \mathcal{D}(I)$ and integrating over the time interval I and setting $\boldsymbol{\varphi}(t,x) = \zeta(t)\boldsymbol{\psi}(x)$ we conclude that

$$\int_I\int_B \Big(\boldsymbol{U}\cdot\partial_t\boldsymbol{\varphi} + \mathbf{T} : \nabla_x\boldsymbol{\varphi}\Big)\,\mathrm{d}x\mathrm{d}t = \int_I\int_B p\partial_t\mathrm{div}_x\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t \tag{V.5.54}$$

for any $\boldsymbol{\varphi} \in \mathcal{D}(I \times B;\mathbb{R}^3)$.

Let us define the harmonic pressure as

$$p_{\mathrm{harm}}(t,\cdot) = p(t,\cdot) + \Big(\int_{t_0}^t \Big[p_{\mathrm{reg}}(\tau,\cdot) - \frac{1}{|B|}\int_B p_{\mathrm{reg}}(\tau,\cdot)\,\mathrm{d}x\Big]\;\mathrm{d}\tau\Big). \tag{V.5.55}$$

Now we intend to show that $p_{\rm harm}(t,\cdot)$ is a harmonic function for any t. To this end, we take $\boldsymbol{\psi} = \nabla_x \gamma$, $\gamma \in \mathcal{D}(B)$ in (V.5.52) and compare the resulting expression with (V.5.51), (V.5.55) and use that $\mathrm{div}_x \boldsymbol{U} = 0$. If we insert (V.5.55) in (V.5.54), we infer (V.5.46).

Finally Weyl's lemma (see e.g. [**121**]) ensures that the function $p_{\rm harm}$ is regular locally in B, i.e. $p_{\rm harm} \in C^\infty(B')$, where $B' \subset\subset B$. Hence we obtain (V.5.48).

Moreover according to (V.5.55), (V.5.52) we show that (V.5.49) holds. Indeed, let us recall first the following result concerning the Bogovski operator in the space of bounded mean oscillations BMO [1] : Let $v : B \to \mathbb{R}^3$, $f : B \to \mathbb{R}$, $f \in L^\infty(B)$ and $\int_B f = 0$. Then there exists at least one solution satisfying $\mathrm{div}_x v = f$ in the sense of distributions. Furthermore

$$\|v\|_{BMO} + \|\nabla_x v\|_{BMO} \leqslant c\|f\|_\infty$$

and $N \cdot v|_{\partial B} = 0$ in the sense of generalized traces for some constant $C > 0$ (more details can be found in [**35**] and see also in [**38, 127**], it can be shown also via Calderón-Zygmund operators and results of Peetre). Moreover let us notice that $BMO(B) \subseteq L_{\widetilde{m}}(B)$ with $\widetilde{m}(\tau) = \exp(\tau) - 1$ for $\tau \in (0,\infty)$ (see [**14**, Chapter 5.7]), $L_{\widetilde{m}}(B) \subseteq L_M$ and $L_{\widetilde{m}}(B) \subseteq L_{m'^*}$, where m'^* is a complementary function to the $\mathcal{N}$−function m'. Then we use in (V.5.52) a test function $\boldsymbol{\psi}$ such that

$$\mathrm{div}_x \boldsymbol{\psi} = \left(\operatorname{sgn} p - \frac{1}{|B|} \int_B \operatorname{sgn} p \right) \in L^\infty(B).$$

Considering (V.5.52) with the above results, the Hölder inequality, generalized Hölder inequality (III.1.10) and noticing that $L_{M^*} \subseteq L_{m^*} \subseteq L_{m'}$ we obtain that

$$\operatorname*{ess\,sup}_{t\in I} \|p(t,\cdot)\|_{L^1(B)} \leqslant c(B,M) \left\{ \|\boldsymbol{U}\|_{L^\infty(I;L^2(B))} + \|\mathbf{T}\|_{L_M(I\times B)} \right\}. \tag{V.5.56}$$

Therefore (V.5.55) and (V.5.47) provide (V.5.49). □

Remark V.5.5. The assumption for the lower bound for an $\mathcal{N}$−function M^*, i.e. $m^*(\tau) = \tau \log^{\beta+1}(\tau+1) \leqslant M^*(\tau)$ for $\tau \in \mathbb{R}_+$, $\beta > 0$, implies that we have to assume also that $M(\tau) \leqslant c \exp(\tau^{\frac{1}{\beta+1}}) - c$ for some nonnegative constant c (see (V.1.4)).

Now we apply Lemma V.5.4 with the $\mathcal{N}$−function M^*, with $\boldsymbol{U} := \varrho_f \boldsymbol{u}_\varepsilon$ and $\mathbf{T} := \varrho_f \boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta - \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon)$. Accordingly, for any $\varepsilon > 0$, there exist two scalar functions $p^\varepsilon_{\rm reg}$, $p^\varepsilon_{\rm harm}$ such that

$$p^\varepsilon_{\rm reg} \in L^1(I; L_{m'}(B)),\ p^\varepsilon_{\rm harm} \in L^\infty(I; L^1(B)) \ \text{ are uniformly bounded} \tag{V.5.57}$$

and $p^\varepsilon_{\rm harm}$ is a harmonic function w.r.t. x, i.e.

$$\Delta p^\varepsilon_{\rm harm} = 0, \qquad \int_B p^\varepsilon_{\rm harm}(t,\cdot) = 0, \quad \forall\, t \in I.$$

[1] $BMO(\Omega)$ is a space of locally integrable functions such that $\sup_B \frac{1}{|B|} \int_B |f(x) - \frac{1}{|B|}\int_B f(y)\mathrm{d}\,\mathrm{y}|\,\mathrm{dx} < \infty$, where supremum is taken over all balls in Ω

Moreover the following is satisfied

(V.5.58)

$$\int_0^T \int_\Omega \Big[(\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\text{harm}}) \cdot \partial_t \boldsymbol{\varphi} + (\varrho_f \boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta - \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) + p^\varepsilon_{\text{reg}} \mathbf{I}) : \nabla_x \boldsymbol{\varphi} \Big] \, \mathrm{d}x\mathrm{d}t = 0$$

for any test function $\boldsymbol{\varphi} \in \mathcal{D}(I \times B; \mathbb{R}^3)$.

The standard estimates provide that $p^\varepsilon_{\text{harm}}$ is uniformly bounded in $L^\infty(I; W^{2,2}_{loc}(B))$, moreover we already know that $\boldsymbol{u}_\varepsilon \in L^p(I, W^{1,p}(B))$. The equation (V.5.58) provides that

$$\|\partial_t (\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\text{harm}})\|_{L^1(I;(W^{s,2}_0(B))^*)} < c,$$

where $s > 5/3$. Then the Lions-Aubin argument gives us that

$$\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\text{harm}} \to \varrho_f \boldsymbol{u} + \nabla_x p_{\text{harm}} \text{ in } L^2(I; L^2(B'; \mathbb{R}^3)), \tag{V.5.59}$$

for arbitrary $B' \subset\subset B$ as $\varepsilon \to 0$.

By (V.5.33), the velocity field $\{\boldsymbol{u}_\varepsilon\}_{\varepsilon>0}$ is precompact in $L^2(0,T; L^2(\Omega; \mathbb{R}^3))$, hence we infer that

$$\nabla_x p^\varepsilon_{\text{harm}} \to \nabla_x p_{\text{harm}} \text{ in } L^2(I; L^2(B'; \mathbb{R}^3)). \tag{V.5.60}$$

As the argument is valid for any B', our goal now is to let $\varepsilon \to 0$ in (V.5.58). First we recall that the sequence $\{\mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon)|_{i\times B}\}_{\varepsilon>0}$ satisfies

$$\mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) \overset{*}{\rightharpoonup} \overline{\mathbf{S}} \text{ weakly–(*) in } L_{M^*}(I \times B; \mathbb{R}^{3\times3}_{\text{sym}}), \tag{V.5.61}$$

$$\text{or} \quad \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) \rightharpoonup \overline{\mathbf{S}} \text{ weakly in } L^1(I \times B; \mathbb{R}^{3\times3}_{\text{sym}}).$$

Let us recall that $\boldsymbol{u}_\varepsilon|_{I\times B}$ and $\mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon)|_{I\times B}$ are uniformly bounded in $L^p(I; W^{1,p}(B; \mathbb{R}^3))$ and $L_{M^*}(I \times B; \mathbb{R}^{3\times3})$ respectively. Hence classical embedding theorem provides that $\mathbf{T}^\varepsilon = (\varrho_f \boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta - \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon))|_{I\times B}$ is uniformly bounded in $L_{M^*}(I \times B; \mathbb{R}^{3\times3})$. Therefore there exists some $\mathbf{T} \in L_{M^*}(I \times B)$ such that

$$\mathbf{T}^\varepsilon \overset{*}{\rightharpoonup} \mathbf{T} \quad \text{weakly–(*) in } L_{M^*}(I \times B; \mathbb{R}^{3\times3}).$$

Moreover, since $\mathcal{R}_{i,j}$ is a linear operator, using the properties of difference quotients, we show that for any function $\phi \in W^{1,r}(B)$ possessing compact support contained in an open set B, there holds

$$\|\mathcal{R}_{i,j}[\phi]|_B\|_{W^{1,r}(B)} \leqslant c\|\phi\|_{W^{1,r}(B)} \quad \text{for any } r \in (0,\infty),$$

where on the left-hand side ϕ is prolonged by zero, preserving the norm. Hence the functions $\mathcal{R}_{i,i}\partial_{x_i}\varphi_i$, $i = 1,2,3$ are sufficiently regular in order to obtain

$$\begin{aligned} &\int_I \int_B p^\varepsilon_{\text{reg}} \mathbf{I} : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t = \int_I \int_B (\mathcal{R} : \mathbf{T}^\varepsilon) \mathbf{I} : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t \\ &= \int_I \int_B \sum_{i=1}^3 T^\varepsilon_{i,i} \mathcal{R}_{i,i}[\partial_{x_i}\varphi_i] \, \mathrm{d}x\mathrm{d}t \to \int_I \int_B \sum_{i=1}^3 T_{i,i} \mathcal{R}_{i,i}[\partial_{x_i}\varphi_i] \, \mathrm{d}x\mathrm{d}t \quad \text{as } \varepsilon \to 0 \end{aligned} \tag{V.5.62}$$

Finally by (V.5.33), (V.5.61) and (V.5.59), (V.5.62) passing with $\varepsilon \to 0$ in (V.5.58) we get
(V.5.63)

$$\int_I \int_B \Big[(\varrho_f \boldsymbol{u} + \nabla_x p_{\text{harm}}) \cdot \partial_t \boldsymbol{\varphi} + (\varrho_f \boldsymbol{u} \otimes [\boldsymbol{u}]_\delta - \overline{\mathbf{S}}) : \nabla_x \boldsymbol{\varphi} \Big] + \sum_{i=1}^{3} T_{i,i} \mathcal{R}_{i,i} \partial_{x_i} \varphi_i \, \mathrm{d}x\mathrm{d}t = 0$$

for any test function $\boldsymbol{\varphi} \in \mathcal{D}(I \times B; \mathbb{R}^3)$.

Our aim is to use (V.5.58) and (V.5.63) with strong convergence (V.5.60) to characterise nonlinear viscous term using monotonicity methods for nonreflexive spaces as in Chapter IV and in [**75, 131, 133**].

To this end we take for any $s_0, s_1 \in I$ and sufficiently small h

$$\boldsymbol{\varphi} = \sigma_h * (\mathbb{1}_{(s_0,s_1)}(\sigma_h * r(\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\text{harm}}))) \quad \text{with any } r \in \mathcal{D}(B)$$

as a test function in (V.5.58). Here $*$ stands for convolution in the time variable with regularising kernel σ_h (i.e. $\sigma \in C^\infty(\mathbb{R})$, $\text{supp}\sigma \in B_1(0)$, $\sigma(-t) = \sigma(t)$, $\int_{\mathbb{R}} \sigma(t)\,\mathrm{d}t = 1$, $\sigma_h(t) = \frac{1}{h}\sigma(\frac{1}{h})$). Since $\boldsymbol{u}_\varepsilon|_B \in L^2(I; W^{1,2}(B))$ and $p^\varepsilon_{\text{harm}} \in L^\infty(I; W^{1,2}_{\text{loc}}(B))$ we infer that

$$\sigma_h * (\mathbb{1}_{(s_0,s_1)}(\sigma_h * r(\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\text{harm}}))) \in C(I; L^2(B; \mathbb{R}^3)) \quad \text{for any } r \in \mathcal{D}(B).$$

Then we obtain that

(V.5.64)

$$\int_{s_0}^{s_1} \int_B \sigma_h * \big(\mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) - \varrho_f \boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta - p^\varepsilon_{\text{reg}} \mathbf{I}\big) : \sigma_h * (\nabla_x (r(\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\text{harm}}))) \, \mathrm{d}x\mathrm{d}t$$
$$= \frac{1}{2} \int_B r|\sigma_h * (\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\text{harm}})|^2 \, \mathrm{d}x \, \Big|_{t=s_0}^{t=s_1} \quad \text{for any } s_0,\ s_1 \in I.$$

Let us pass to the limit with $h \to 0$ and $\varepsilon \to 0$ and start with the right-hand side of (V.5.64). The relation (V.5.59) provides that

$$\lim_{\varepsilon\to 0} \lim_{h \to 0} \int_B r|\sigma_h * (\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\text{harm}})|^2 \, \mathrm{d}x \, \Big|_{t=s_0}^{t=s_1} = \lim_{\varepsilon \to 0} \int_B r|(\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\text{harm}})|^2 \, \mathrm{d}x \, \Big|_{t=s_0}^{t=s_1}$$
$$= \frac{1}{2} \int_B r|\varrho_f \boldsymbol{u} + \nabla_x p_{\text{harm}}|^2 \, \mathrm{d}x \, \Big|_{t=s_0}^{t=s_1}.$$

for any Lebesgue point in $[0, T]$. As p_{harm} is a harmonic function on B, standard elliptic estimates provide that

$$r(\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\text{harm}}) \in L^p(I; W^{1,p}(B; \mathbb{R}^3)) \cap L^\infty(I; L^2(B; \mathbb{R}^3)),$$

while

$$\varrho_f \boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta \in L^{p'}(I; L^{p'}(B; \mathbb{R}^3)).$$

Employing (V.5.59) we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{h\to 0}\int_{s_0}^{s_1}\int_B &(\sigma_h * (\varrho_f \boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta)) : (\sigma_h * (\nabla_x (r(\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\text{harm}})))) \,\mathrm{d}x\mathrm{d}t\\ &= \lim_{\varepsilon\to 0}\int_{s_0}^{s_1}\int_B (\varrho_f \boldsymbol{u}_\varepsilon \otimes [\boldsymbol{u}_\varepsilon]_\delta) : (\nabla_x (r(\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\text{harm}}))) \,\mathrm{d}x\mathrm{d}t\\ &= \int_{s_0}^{s_1}\int_B (\varrho_f \boldsymbol{u} \otimes [\boldsymbol{u}]_\delta) : (\nabla_x (r(\varrho_f \boldsymbol{u} + \nabla_x p_{\text{harm}}))) \,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Now we concentrate on the third term on the left-hand side of (V.5.64). Since $p^\varepsilon_{\text{harm}}$ is harmonic and $\operatorname{div}_x \boldsymbol{u}_\varepsilon = 0$, we obtain

$$\begin{aligned}\int_{s_0}^{s_1}\int_B &\sigma_h * \left(p^\varepsilon_{\text{reg}}\mathbf{I}\right) : \sigma_h * \nabla_x (r(\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\text{harm}})) \,\mathrm{d}x\mathrm{d}t\\ &= \int_{s_0}^{s_1}\int_B \left(\sigma_h * p^\varepsilon_{\text{reg}}\right)(\sigma_h * \operatorname{div}_x(r(\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\text{harm}}))) \,\mathrm{d}x\mathrm{d}t\\ &= \int_{s_0}^{s_1}\int_B \sigma_h * p^\varepsilon_{\text{reg}} \sigma_h * (\nabla_x r \cdot (\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\text{harm}})) \,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{V.5.65}$$

Let us consider the first term on the right-hand side of (V.5.65). Employing the estimate (V.5.5) and assumption (V.1.6) by Lemma III.2.7 we infer

$$\|\mathbf{D}\boldsymbol{u}_\varepsilon\|_{L_M(0,T;L^4(\mathcal{T}))} < \infty.$$

The generalized version of Korn inequality [**54**, Theorem 10.16] gives us

$$\|\boldsymbol{u}_\varepsilon\|_{L_M(0,T;W^{1,4}(\mathcal{T}))} < \infty.$$

Since $4 > \dim(B) = 3$,

$$\|\boldsymbol{u}_\varepsilon\|_{L_M(0,T;C(\mathcal{T}))} < \infty.$$

Hence $\boldsymbol{u}_\varepsilon|_{I\times B} \in L_M(I;W^{1,4}(B;\mathbb{R}^3)) \subset L_M(I;C(B;\mathbb{R}^3)) \subset L_M(I\times B;\mathbb{R}^3)$. Using the definition of $p^\varepsilon_{\text{reg}}$ and the property $(\mathcal{R}_{i,j})^* = (\mathcal{R}_{j,i})$ we get

$$\begin{aligned}\int_{s_0}^{s_1}\int_B &(\sigma_h * p^\varepsilon_{\text{reg}})(\sigma_h * (\nabla_x r \cdot (\varrho_f \boldsymbol{u}_\varepsilon))) \,\mathrm{d}x\mathrm{d}t\\ &= \int_{s_0}^{s_1}\int_B \{\sigma_h * \sum_{i,j=1}^3 \mathcal{R}_{i,j}[T_{i,j}](t,x)\}\{\sigma_h * (\nabla_x r \cdot (\varrho_f \boldsymbol{u}_\varepsilon))\} \,\mathrm{d}x\mathrm{d}t\\ &= \int_{s_0}^{s_1}\int_B \sum_{i,j=1}^3 \{\sigma_h * T^\varepsilon_{i,j}(t,x)\}\{\sigma_h * \mathcal{R}_{j,i}[\nabla_x r \cdot (\varrho_f \boldsymbol{u}_\varepsilon)]\} \,\mathrm{d}x\mathrm{d}t\end{aligned}$$

Since $\nabla_x r \cdot (\varrho_f \boldsymbol{u}_\varepsilon) \in L_M(I;W^{1,4}(B))$ and $r \in \mathcal{D}(B)$, in particular $\operatorname{supp} \nabla_x r \subset\subset B$, using the properties of difference quotients, see e.g [**48**] and if we extend $\nabla_x r \cdot (\varrho_f \boldsymbol{u}_\varepsilon)$ by zero on the whole space $\mathbb{R}^3$ preserving the norm, then we deduce that

$$\|\mathcal{R}_{j,i}[\nabla_x r \cdot (\varrho_f \boldsymbol{u}_\varepsilon)]|_B\|_{C(B)} \leqslant c_1 \|\mathcal{R}_{j,i}[\nabla_x r \cdot (\varrho_f \boldsymbol{u}_\varepsilon)]|_B\|_{W^{1,4}(B)} \leqslant c_2 \|\nabla_x r \cdot (\varrho_f \boldsymbol{u}_\varepsilon)\|_{W^{1,4}(B)}$$

for all $t \in I$. Consequently we have

$$\mathcal{R}_{i,j}[\nabla_x r \cdot (\varrho_f \boldsymbol{u}_\varepsilon)]|_B \in L_M(I;C(B)) \quad \text{for } i,j = 1,2,3.$$

Let us denote

$$\mathbf{b}^\varepsilon = [b^\varepsilon_{i,j}]_{i=1,2,3,\,j=1,2,3} := \left[T^\varepsilon_{i,j}\right]_{i=1,2,3,\,j=1,2,3} \tag{V.5.66}$$

and notice that $\{\mathbf{b}^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L_{M^*}(I\times B;\mathbb{R}^{3\times3})$. Hence

$$\mathbf{b}^\varepsilon \rightharpoonup \mathbf{b} \text{ weakly in } L^1(I\times B;\mathbb{R}^{3\times3}). \tag{V.5.67}$$

Moreover let us denote

$$\mathbf{w}^\varepsilon = \left[w^\varepsilon_{j,i}\right]_{i=1,2,3,\,j=1,2,3} := [\mathcal{R}_{j,i}[\nabla_x r\cdot(\varrho_f \boldsymbol{u}_\varepsilon)]]_{i=1,2,3,\,j=1,2,3} \tag{V.5.68}$$

which is uniformly bounded in $L_M(I;W^{1,4}(B;\mathbb{R}^{3\times3}))$.

Now let us converge with $h\to 0$. Since for any $\varepsilon>0$, $\mathbf{b}^\varepsilon\in L_{M^*}(I\times B;\mathbb{R}^{3\times3})$ and $\mathbf{w}^\varepsilon\in L_M(I;C(B;\mathbb{R}^{3\times3}))\subset L_M(I\times B;\mathbb{R}^{3\times3})$, then there exist $\lambda_b,\ \lambda_w\in(0,\infty)$ such that $\mathbf{b}^\varepsilon/\lambda_b\in\mathcal{L}_{M^*}(I\times B;\mathbb{R}^{3\times3})$ and $\mathbf{w}^\varepsilon/\lambda_w\in\mathcal{L}_M(I\times B;\mathbb{R}^{3\times3})$. Due to Proposition III.2.4 we obtain that

$$\sigma_h * \mathbf{b}^\varepsilon \to \mathbf{b}^\varepsilon \quad \text{in measure as } h\to 0^+,$$
$$\sigma_h * \mathbf{w}^\varepsilon \to \mathbf{w}^\varepsilon \quad \text{in measure as } h\to 0^+.$$

and $\{M^*(\sigma_h*\mathbf{b}^\varepsilon/\lambda_b)\}_{h>0}$, $\{M(\sigma_h*\mathbf{w}^\varepsilon/\lambda_w)\}_{h>0}$ are uniformly integrable by Proposition III.2.5. Therefore by Lemma III.2.1 we obtain

$$\sigma_h * \mathbf{b}^\varepsilon \xrightarrow{M^*} \mathbf{b}^\varepsilon \quad \text{modularly in } L_{M^*}(I\times B;\mathbb{R}^{3\times3}) \text{ as } h\to 0,$$
$$\sigma_h * \mathbf{w}^\varepsilon \xrightarrow{M} \mathbf{w}^\varepsilon \quad \text{modularly in } L_M(I\times B;\mathbb{R}^{3\times3}) \text{ as } h\to 0.$$

Consequently by Proposition III.2.3 we get

$$\lim_{h\to0}\int_{s_0}^{s_1}\int_B(\sigma_h*\mathbf{b}^\varepsilon):(\sigma_h*\mathbf{w}^\varepsilon)\,\mathrm{d}x\mathrm{d}t = \int_{s_0}^{s_1}\int_B \mathbf{b}^\varepsilon:\mathbf{w}^\varepsilon\,\mathrm{d}x\mathrm{d}t \tag{V.5.69}$$

Using the following interpolation

$$\|\mathbf{w}^\varepsilon\|_{W^{\alpha,r}(B)} \leqslant c\|\mathbf{w}^\varepsilon\|^{1-\lambda}_{W^{1,4}(B)}\|\mathbf{w}^\varepsilon\|^{\lambda}_{L^2(B)} \tag{V.5.70}$$

with $\alpha = 1-\lambda$ and $\frac{1}{r}=\frac{\lambda}{2}+\frac{1-\lambda}{4}$ (see [**125**] Section 2.3.1, 2.4.1, 4.3.1, 4.3.2), we can find such λ ($\lambda\in(0,\frac{1}{7})$ for space dimension 3) that $W^{\alpha,r}(B)$ is continuously embedded in $L^\infty(B)$ (see [**1**]). Therefore for any fixed $K>0$

$$\begin{aligned}\int_I\int_B M(K(\mathbf{w}^\varepsilon-\mathbf{w}))\,\mathrm{d}x\mathrm{d}t &\leqslant |B|\int_I M(K\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|_{L^\infty(B)})\,\mathrm{d}t\\ &\leqslant |B|\int_I M\left(cK\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|^{1-\lambda}_{W^{1,4}(B)}\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|^{\lambda}_{L^2(B)}\right)\,\mathrm{d}t = I_1.\end{aligned} \tag{V.5.71}$$

As $\mathbf{w}^\varepsilon\to\mathbf{w}$ strongly in $L^2(I\times B)$ (as $\varrho_f=\text{const}$ in $I\times B$)

$$\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|_{L^2(B)}\to 0 \quad \text{in measure on } I.$$

As (V.5.68) holds, $\mathbf{w}^\varepsilon-\mathbf{w}\in L^1(I;W^{1,4}(B))$ and consequently

$$\left|\{t\in I\ :\ \|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|_{W^{1,4}(B)}>\alpha\}\right|\leqslant\frac{c}{\alpha}$$

for some c independent of ε. Then

$$\|\mathbf{w}^\varepsilon(t) - \mathbf{w}(t)\|^{1-\lambda}_{W^{1,4}(B)}\|\mathbf{w}^\varepsilon(t) - \mathbf{w}(t)\|^{\lambda}_{L^2(B)} \to 0 \quad \text{in measure on } I.$$

Continuity of M gives that

$$\text{(V.5.72)}\quad M\left\{cK\|\mathbf{w}^\varepsilon(t) - \mathbf{w}(t)\|^{1-\lambda}_{W^{1,4}(B)}\|\mathbf{w}^\varepsilon(t) - \mathbf{w}(t)\|^{\lambda}_{L^2(B)}\right\} \to 0 \quad \text{in measure on } I.$$

Next we show uniform integrability of

$$\left\{M\left(cK\|\mathbf{w}^\varepsilon(t) - \mathbf{w}(t)\|^{1-\lambda}_{W^{1,4}(B)}\|\mathbf{w}^\varepsilon(t) - \mathbf{w}(t)\|^{\lambda}_{L^2(B)}\right)\right\}_{\varepsilon>0}$$

in $L^1(I)$. Let us denote $R = cK\|\mathbf{w}^\varepsilon - \mathbf{w}\|_{L^\infty(I;L^2(B))}$ and let us notice that for any subset $E \subset I$ an $\lambda \in (0,1)$

$$\begin{aligned}
&\int_E M\left(R\|\mathbf{w}^\varepsilon(t) - \mathbf{w}(t)\|^{1-\lambda}_{W^{1,4}(B)}\right) \mathrm{d}t \\
&\quad = \int_{\{t\in E\,:\,\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|_{W^{1,4}(B)}\leqslant R^{2/\lambda}\}} M\left(R\|\mathbf{w}^\varepsilon(t) - \mathbf{w}(t)\|^{1-\lambda}_{W^{1,4}(B)}\right) \mathrm{d}t \\
&\quad + \int_{\{t\in E\,:\,\|\mathbf{w}^\varepsilon(t)-\mathbf{w}(t)\|_{W^{1,4}(B)}> R^{2/\lambda}\}} M\left(R\|\mathbf{w}^\varepsilon(t) - \mathbf{w}(t)\|^{1-\lambda}_{W^{1,4}(B)}\right) \mathrm{d}t \\
&\quad \leqslant |E| M\left(R^{(1+\frac{2}{\lambda}(1-\lambda))}\right) + \int_E M\left(\|\mathbf{w}^\varepsilon(t) - \mathbf{w}(t)\|^{1-\frac{\lambda}{2}}_{W^{1,4}(B)}\right) \mathrm{d}t.
\end{aligned}$$

Let us notice that the first term on the right-hand side depends linearly on the measure of the set E. It remains to show that $\left\{M\left(\|\mathbf{w}^\varepsilon(t) - \mathbf{w}(t)\|^{1-\frac{\lambda}{2}}_{W^{1,4}(B)}\right)\right\}_{\varepsilon>0}$ is uniformly integrable in $L^1(I)$. Indeed, as M is an $\mathcal{N}-$function (in particular is convex) and for $\lambda \in (0,1)$ the following assertion holds by de l'Hôpital's rule

$$\frac{M(\tau)}{M(\tau^{1-\frac{\lambda}{2}})} \to \infty \quad \text{as } \tau \to \infty.$$

Consequently $M\left(\|\mathbf{w}^\varepsilon(t) - \mathbf{w}(t)\|^{1-\frac{\lambda}{2}}_{W^{1,4}(B)}\right)$ is uniformly integrable in $L^1(I)$. Summarising we obtain that

(V.5.73)

$$M\left(cK\|\mathbf{w}^\varepsilon(t) - \mathbf{w}(t)\|^{1-\lambda}_{W^{1,4}(B)}\|\mathbf{w}^\varepsilon(t) - \mathbf{w}(t)\|^{\lambda}_{L^2(B)}\right) \text{ is uniformly integrable in } L^1(I).$$

By (V.5.72) and (V.5.73) the Vitali lemma provides that the right-hand side of (V.5.71) converges to 0. Consequently

$$\text{(V.5.74)}\qquad K\mathbf{w}^\varepsilon \xrightarrow{M} K\mathbf{w} \quad \text{modularly in } L_M(I \times B).$$

According to Lemma III.2.1 $\{M(K\mathbf{w}^\varepsilon)\}_{\varepsilon>0}$ is uniformly integrable in $L^1(I \times B)$ and passing to subsequence if necessary

$$\text{(V.5.75)}\qquad \mathbf{w}^\varepsilon \to \mathbf{w} \quad \text{a.e. in } I \times B.$$

Our next step is to show the uniform integrability of $\{\mathbf{b}^\varepsilon : \mathbf{w}^\varepsilon\}_{\varepsilon>0}$ in $L^1(I \times B)$. By the Fenchel-Young inequality and convexity of M^* for $K > 1$ it follows that

$$\begin{aligned} \left|\int_I \int_B \mathbf{b}^\varepsilon : \mathbf{w}^\varepsilon \,\mathrm{d}x\mathrm{d}t\right| &= \int_I \int_B \frac{1}{K}\mathbf{b}^\varepsilon : K\mathbf{w}^\varepsilon \,\mathrm{d}x\mathrm{d}t \\ &\leqslant \int_I \int_B \frac{1}{K} M^*(\mathbf{b}^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + \int_I \int_B M(K\mathbf{w}^\varepsilon)\,\mathrm{d}x\mathrm{d}t \end{aligned}$$

As K is arbitrary and $\{M(K\mathbf{w}^\varepsilon)\}_{\varepsilon>0}$ is uniformly integrable in $L^1(I \times B)$ we obtain the assertion that $\{\mathbf{b}^\varepsilon : \mathbf{w}^\varepsilon\}_{\varepsilon>0}$ is uniformly integrable in $L^1(I \times B)$. Moreover as (V.5.67) and (V.5.75) hold we infer that

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_{s_0}^{s_1} \int_B \sum_{i,j=1}^{3} \left(T_{i,j}^\varepsilon(t,x)\right) \left(\mathcal{R}_{j,i}\left[\nabla_x r \cdot (\boldsymbol{u}_\varepsilon)\right]\right) \mathrm{d}x\mathrm{d}t \\ &\qquad = \int_{s_0}^{s_1} \int_B \sum_{i,j=1}^{3} \left(T_{i,j}(t,x)\right) \left(\mathcal{R}_{j,i}\left[\nabla_x r \cdot (\boldsymbol{u})\right]\right) \mathrm{d}x\mathrm{d}t. \end{aligned}$$

Then the second term on the right hand side of (V.5.65) can be treated in a similar way, since $p^\varepsilon_{\mathrm{harm}}$ is a harmonic function and $r \in B$. Finally we infer

$$\begin{aligned} &\lim_{\varepsilon\to 0} \lim_{h\to\infty} \int_{s_0}^{s_1} \int_B \sigma_h * \left(p^\varepsilon_{\mathrm{reg}}\mathbf{I}\right) : \sigma_h * \nabla_x \left(r(\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\mathrm{harm}})\right) \mathrm{d}x\mathrm{d}t \\ &\qquad = \int_{s_0}^{s_1} \int_B \sum_{i,j=1}^{3} T_{i,j}(t,x) \mathcal{R}_{i,j}[\nabla_x r \cdot (\varrho_f \boldsymbol{u} + \nabla_x p_{\mathrm{harm}})]\,\mathrm{d}x\mathrm{d}t. \end{aligned}$$

It remains to show how the viscous term behaves in the limit $h \to 0$ and $\varepsilon \to 0$, i.e.

(V.5.76)
$$\begin{aligned} &\int_{s_0}^{s_1} \int_B \sigma_h * \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) : \sigma_h * \nabla_x \left(r \varrho_f \boldsymbol{u}_\varepsilon\right) \mathrm{d}x\mathrm{d}t \\ &\qquad = \int_{s_0}^{s_1} \int_B \sigma_h * \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) : \sigma_h * \left(\nabla_x r \varrho_f \boldsymbol{u}_\varepsilon + r(\varrho_f \nabla_x \boldsymbol{u}_\varepsilon)\right) \mathrm{d}x\mathrm{d}t \end{aligned}$$

As $\mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon)|_{I\times B} \in L_{M^*}(I \times B; \mathbb{R}^{3\times 3})$ and $\boldsymbol{u}_\varepsilon|_{I\times B} \in L_M(I; W^{1,4}(B; \mathbb{R}^3)) \subseteq L_M(I \times B; \mathbb{R}^3)$, we proceed in a similar way as in (V.5.69) passing to the limi with $h \to 0$. Then we proceed exactly as with $\mathbf{b}^\varepsilon$ and $\mathbf{w}^\varepsilon$ in order to converge with $\varepsilon \to 0$. Therefore we obtain

(V.5.77)
$$\begin{aligned} &\lim_{\varepsilon\to 0} \lim_{h\to 0} \int_{s_0}^{s_1} \int_B (\sigma_h * \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon)) : \sigma_h * \left(\nabla_x r \varrho_f \boldsymbol{u}_\varepsilon + r(\varrho_f \nabla_x \boldsymbol{u}_\varepsilon)\right) \mathrm{d}x\mathrm{d}t \\ &\qquad = \lim_{\varepsilon\to 0} \int_{s_0}^{s_1} \int_B \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) : \left(\nabla_x r \varrho_f \boldsymbol{u}_\varepsilon + r(\varrho_f \nabla_x \boldsymbol{u}_\varepsilon)\right) \mathrm{d}x\mathrm{d}t \\ &\qquad = \int_{s_0}^{s_1} \int_B \overline{\mathbf{S}} : \nabla r \varrho_f \boldsymbol{u}\,\mathrm{d}x\mathrm{d}t + \lim_{\varepsilon\to 0} \int_{s_0}^{s_1} \int_B \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) r(\varrho_f \nabla_x \boldsymbol{u}_\varepsilon)\,\mathrm{d}x\mathrm{d}t \end{aligned}$$

Summarising (V.5.77) and previous consideration, passing to the limit first with $h \to 0$ and next with $\varepsilon \to 0$ in (V.5.64) we have

(V.5.78)
$$\lim_{\varepsilon\to 0}\int_{s_0}^{s_1}\int_B \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) : (r(\varrho_f \nabla_x \boldsymbol{u}_\varepsilon))\,\mathrm{d}x\mathrm{d}t = \frac{1}{2}\int_B r^2 |\varrho_f \boldsymbol{u} + \nabla_x p_{\text{harm}}|^2\,\mathrm{d}x\,\Big|_{t=s_0}^{t=s_1}$$
$$- \int_{s_0}^{s_1}\int_B \overline{\mathbf{S}} : \nabla r \varrho_f \boldsymbol{u}\,\mathrm{d}x\mathrm{d}t + \int_{s_0}^{s_1}\int_B (\varrho_f \boldsymbol{u}\otimes[\boldsymbol{u}]_\delta) : (\nabla_x (r(\varrho_f \boldsymbol{u} + \nabla_x p_{\text{harm}})))\,\mathrm{d}x\mathrm{d}t$$
$$+ \int_{s_0}^{s_1}\int_B \sum_{i,j=1}^{3} T_{i,j}(t,x)\mathcal{R}_{j,i}\left[\nabla_x r\cdot \boldsymbol{u}\right]\mathrm{d}x\mathrm{d}t$$

Using
$$\sigma_h * \sigma_h * r(x)(\varrho_f \boldsymbol{u}_\varepsilon + \nabla_x p^\varepsilon_{\text{harm}})$$
as a test function in the limit equation (V.5.63) and after passing with $\varepsilon \to 0$ and $h \to 0$ we are allow to conclude that

$$\int_{s_0}^{s_1}\int_B \left(\overline{\mathbf{S}} - \varrho_f \boldsymbol{u}\otimes[\boldsymbol{u}]_\delta\right) : \nabla_x (r(\varrho_f \boldsymbol{u} + \nabla_x p_{\text{harm}}))$$
$$\text{(V.5.79)}\qquad - \sum_{i,j=1}^{3} T_{i,j}\mathcal{R}_{j,i}\left[\nabla_x r\cdot(\varrho_f \boldsymbol{u} + \nabla_x p_{\text{harm}})\right]\mathrm{d}x\mathrm{d}t$$
$$= \frac{1}{2}\int_B r|(\varrho_f \boldsymbol{u} + \nabla_x p_{\text{harm}})|^2\,\mathrm{d}x\,\Big|_{t=s_0}^{t=s_1} \quad \text{for any } s_0,\ s_1 \in I.$$

Finally we conclude from (V.5.78) and (V.5.79) that
$$\limsup_{\varepsilon\to 0}\int_{s_0}^{s_1}\int_B r\mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) : \nabla_x \boldsymbol{u}_\varepsilon\,\mathrm{d}x\mathrm{d}t \leqslant \int_{s_0}^{s_1}\int_B r\overline{\mathbf{S}} : \nabla_x \boldsymbol{u}\,\mathrm{d}x\mathrm{d}t \quad \text{for a.a. } s_0,\ s_1 \in I$$
and by the monotonicity argument for nonreflexive spaces used in Chapter IV or in [**75, 131, 133**] we obtain

$$\text{(V.5.80)}\qquad \mathbf{S}(\mathbf{D}\boldsymbol{u}_\varepsilon) \to \mathbf{S}(\mathbf{D}\boldsymbol{u}) \text{ a.e. in } I\times B.$$

V.5.6. Conclusion. Considerations given in two preceding sections provide, that (V.5.32) reduces to

$$\text{(V.5.81)}\qquad \int_0^T\int_\Omega \varrho\boldsymbol{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho(\boldsymbol{u}\otimes[\boldsymbol{u}]_\delta) : \nabla_x\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t$$
$$= \int_0^T\int_\Omega \mathbf{S}(\mathbf{D}\boldsymbol{u}) : \mathbf{D}\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_{\mathcal{T}} \varrho\nabla_x F\cdot\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t - \int_{\mathcal{T}} \varrho_{0,\delta}\boldsymbol{u}_0\cdot\boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}x$$
for any test function $\boldsymbol{\varphi} \in C^1([0,T)\times\Omega)$, $\boldsymbol{\varphi}(t,\cdot) \in [\mathcal{R}M](t)$, with
$$[\mathcal{R}M](t) = \{\boldsymbol{\phi} \in C^1_c(\Omega;\mathbb{R}^3) \mid \operatorname{div}_x\boldsymbol{\phi} = 0 \text{ in } \Omega,$$
$$\mathbf{D}\boldsymbol{\phi} \text{ has compact support on } \Omega\backslash \cup_{i=1}^n \overline{S}_i(t)\},$$
where
$$S_i(t) = \eta_i(t, S_i),\ i = 1,\dots,n.$$

Furthermore, the limit solution satisfies the energy inequality

(V.5.82)
$$\int_\Omega \frac{1}{2}\varrho|\boldsymbol{u}|^2(\tau)\,\mathrm{d}x + \int_s^\tau \int_\Omega \mathbf{S} : \mathbf{D}\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t \leqslant \int_{\mathcal{T}} \frac{1}{2}\varrho|\boldsymbol{u}|^2(s)\,\mathrm{d}x + \int_s^\tau \int_\Omega \varrho\nabla_x F \cdot \boldsymbol{u}\,\mathrm{d}x\mathrm{d}t$$

for any τ and a.a. $s \in (0,T)$ including $s = 0$.

V.6. The limit passage $\delta \to 0$

In the last section we pass to the limit with $\delta \to 0$ in the system of equations (V.5.26), (V.5.81) and in the corresponding family of isometries $\{\boldsymbol{\eta}_i\}_{i=1}^n$ describing the motion of rigid bodies. Hence we denote the associated sequences of solutions by $\{\varrho_\delta, \boldsymbol{u}_\delta, \{\eta_i^\delta\}_{i=1}^n\}_{\delta>0}$.

Observe now that the initial data $\varrho_{S_i,\delta}$ in (V.4.6) can be taken in such a way that
$$\|\varrho_{S_i,\delta}\|_{L^\infty(\Omega)} \leqslant c,\ \varrho_f + \varrho_{S_i,\delta} \to \varrho_{S_i} \text{ as } \delta \to 0 \text{ in } L^1(\Omega),\ i = 1,\ldots,n,$$
where $\{\varrho_{S_i}\}_{i=1}^n$ are the initial distributions of the mass on the rigid bodies in Theorem V.3.1. Then the theory for transport equation developed by DiPerna and Lions [**45**] provides that
$$\varrho_\delta \to \varrho \quad \text{strongly in } C([0,T];L^1(\Omega)) \text{ as } \delta \to 0.$$
According to energy inequality (V.5.82), we obtain that for a subsequence if necessary
$$\boldsymbol{u}_\delta \to \boldsymbol{u} \text{ weakly in } L^p(0,T;W^{1,p}(\Omega;\mathbb{R}^3))$$
where $\boldsymbol{u}_\delta$ as well as the limit velocity $\boldsymbol{u}$ are divergence-free. Hence the continuity equation (V.5.26) reduces to a transport equation
$$\partial_t\varrho + \boldsymbol{u}\cdot\nabla_x\varrho = 0.$$

Following step by step the arguments given in previous sections we complete the rest of the convergence process. The compactness of the velocity and convergence in convective term can be done by combining arguments form previous sections and Chapter IV, see also [**39, 56**]. The convergence in nonlinear viscous term is completed by the same arguments as in Section V.5.5.

CHAPTER VI

Generalized Stokes system

VI.1. Introduction

Our interest is directed to the generalized Stokes system

$$\partial_t \boldsymbol{u} - \mathrm{div}_x \mathbf{S}(t,x,\mathbf{D}\boldsymbol{u}) + \nabla_x p = \boldsymbol{f} \quad \text{in } (0,T)\times\Omega, \tag{VI.1.1}$$

$$\mathrm{div}_x \boldsymbol{u} = 0 \quad \text{in } (0,T)\times\Omega, \tag{VI.1.2}$$

$$\boldsymbol{u}(0,x) = \boldsymbol{u}_0 \quad \text{in } \Omega, \tag{VI.1.3}$$

$$\boldsymbol{u}(t,x) = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{VI.1.4}$$

where $\Omega \subset \mathbb{R}^d$ is an open, bounded set with a sufficiently smooth boundary $\partial\Omega$, $(0,T)$ is the time interval with $T < \infty$, $Q = (0,T)\times\Omega$, $\boldsymbol{u} : Q \to \mathbb{R}^d$ is the velocity of a fluid, $p : Q \to \mathbb{R}$ the pressure and $\mathbf{S} + \mathbf{I}p$ is the Cauchy stress tensor. We assume that $\mathbf{S}$ satisfies the following conditions

- (S1) $\mathbf{S}$ is a Carathéodory function (i.e., measurable w.r.t. t and x and continuous w.r.t. the last variable).
- (S2) There exists an anisotropic $\mathcal{N}$–function $M : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}_+$ (Definition III.1.3) and a constant $c > 0$ such that for all $\boldsymbol{\xi} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$

$$\mathbf{S}(t,x,\boldsymbol{\xi}) : \boldsymbol{\xi} \geqslant c(M(\boldsymbol{\xi}) + M^*(\mathbf{S}(t,x,\boldsymbol{\xi}))) \tag{VI.1.5}$$

  where M is an anisotropic $\mathcal{N}-$ function
- (S3) For all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and for a.a. $t,\ x \in Q$

$$(\mathbf{S}(t,x,\boldsymbol{\xi}) - \mathbf{S}(t,x,\boldsymbol{\eta})) : (\boldsymbol{\xi} - \boldsymbol{\eta}) \geqslant 0.$$

By conditions $(S1)-(S3)$ we can capture a wide class of models. Our particular interest is directed here to the rheology close to linear in at least one direction. We do not assume that the $\mathcal{N}-$ function satisfies the Δ_2-condition in case of star-shaped domains. For other domains we need to assume some conditions on the upper growth of M, however this does not contradict with a goal of describing the rheology close to linear. There is a wide range of fluid dynamics models obeying these conditions, we mention here two constitutive relations: Prandtl-Eyring model, cf. **[53]**, where the stress tensor $\mathbf{S}$ is given by

$$\mathbf{S} = \eta_0 \frac{\mathrm{ar}\sinh(\lambda|\mathbf{D}\boldsymbol{u}|)}{\lambda|\mathbf{D}\boldsymbol{u}|}\mathbf{D}\boldsymbol{u}$$

and modified Powell-Eyring model cf. **[103]**

$$\mathbf{S} = \eta_\infty \mathbf{D}\boldsymbol{u} + (\eta_0 - \eta_\infty)\frac{\ln(1 + \lambda|\mathbf{D}\boldsymbol{u}|)}{(\lambda|\mathbf{D}\boldsymbol{u}|)^m}\mathbf{D}\boldsymbol{u}$$

where η_∞, η_0, λ, m are material constants. Our attention in the present chapter is particularly directed to the case $\eta_\infty = 0$ and $m = 1$.

Both models are broadly used in geophysics, engineering and medical applications, e.g. for modelling of glacier ice, cf. [**83**], blood flow, cf. [**106, 107**] and many others, cf. [**31, 101, 116, 137**].

Our considerations concern the simplified system of equations of conservation of mass and momentum. Indeed, the convective term $\mathrm{div}_x(\boldsymbol{u} \otimes \boldsymbol{u})$ is not present in the equations. The motivation for considering such a simplified model is twofold. If the flow is assumed to be slow, then the inertial term $\mathrm{div}_x(\boldsymbol{u} \otimes \boldsymbol{u})$ can be assumed to be very small and therefore neglected, hence the whole system reduces to a generalized Stokes system (VI.1.1)-(VI.1.2). Another situation is the case of simple flows, e.g. Poisseuille type flow, between two fixed parallel plates, which is driven by a constant pressure gradient (see [**82**]). With regards to blood flows the importance of considering simple flows arises since the geometry of vessels can be simplified to a flow in a pipe. The analysis of both models in steady case (also without convective term) through variational approach was undertaken by Fuchs and Seregin in [**63, 64**].

The equations (VI.1.1)-(VI.1.2) with additional convective term $\mathrm{div}_x(\boldsymbol{u} \otimes \boldsymbol{u})$ in (VI.1.1) have been extensively studied in Chaters IV V and e.g. in [**72, 75, 131, 133, 135**]. The appearance of the convective term enforced the restriction for the growth of an $\mathcal{N}$-function, namely $M(\cdot) \geqslant c|\cdot|^q$ for some exponent $q \geqslant \frac{3d+2}{d+2}$. Such a formulation allowed to capture shear thickening fluids, even very rapidly thickening (e.g. exponential growth). In the present chapter we are able to skip the assumption on the lower growth of M (and consequently the bound for M^*), which opens a possibility to include flows of different behaviour, in particular shear thinning fluids.

The present chapter consists of a new analytical approach to the existence problem. In the previous studies the main reason to assume that M^* satisfies the Δ_2-condition was providing that the solution is bounded in an appropriate Sobolev space $W^{1,q}(\Omega)$ which is compactly embedded in $L^2(\Omega)$. However, as a byproduct, we gained that $L_{M^*}(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}}) = E_{M^*}(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ is a separable space. The naturally arising question is whether the existence of solutions can still be proved after omitting the convective term and relaxing the assumptions on M and M^*. The preliminary studies in this direction were done for an abstract parabolic equation, cf. [**73**]. Also the convergence of a full discretization of quasilinear parabolic equation can be found in [**51**] by Emmrich and Wróblewska-Kamińska. In the present chapter we give a non-trivial extension of these considerations for the system of equations.

We study the problem in two different cases. In the first case the domain is star-shaped and the $\mathcal{N}$-function is anisotropic with absolutely no restriction on the growth. In the second case arbitrary domains with a sufficiently smooth boundary are considered. We define two functions $\underline{m}$, $\overline{m} : \mathbb{R}_+ \to \mathbb{R}_+$ as follows

$$\underline{m}(r) := \min_{\boldsymbol{\xi}\in\mathbb{R}^{d\times d}_{\mathrm{sym}}, |\boldsymbol{\xi}|=r} M(\boldsymbol{\xi}),$$
$$\overline{m}(r) := \max_{\boldsymbol{\xi}\in\mathbb{R}^{d\times d}_{\mathrm{sym}}, |\boldsymbol{\xi}|=r} M(\boldsymbol{\xi}).$$

The existence result is formulated under the control of the spread between $\underline{m}$ and $\overline{m}$.

We define the space of functions with symmetric gradient in $L_M(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$, namely

$$BD_M(\Omega) := \{\boldsymbol{u} \in L^1(\Omega;\mathbb{R}^d) \mid \mathbf{D}\boldsymbol{u} \in L_M(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})\}.$$

The space $BD_M(\Omega)$ is a Banach space with a norm

$$\|\boldsymbol{u}\|_{BD_M(\Omega)} := \|\boldsymbol{u}\|_{L^1(\Omega)} + \|\mathbf{D}\boldsymbol{u}\|_M$$

and it is a subspace of the space of bounded deformations $BD(\Omega)$, i.e.

$$BD(\Omega) := \{\boldsymbol{u} \in L^1(\Omega;\mathbb{R}^d) \mid [\mathbf{D}\boldsymbol{u}]_{i,j} \in \mathcal{M}(\Omega), \text{ for } i,j = 1,\dots,n\},$$

where $\mathcal{M}(\Omega)$ denotes the space of bounded measures on Ω and $[\mathbf{D}\boldsymbol{u}]_{i,j} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$. According to [**123**, Theorem 1.1.] there exists a unique continuous operator γ_0 from $BD(\Omega)$ onto $L^1(\partial\Omega;\mathbb{R}^d)$ such that the generalized Green formula

(VI.1.6)
$$2\int_\Omega \phi[\mathbf{D}\boldsymbol{u}]_{i,j}\,\mathrm{d}x = -\int_\Omega \left(u_j\frac{\partial\phi}{\partial x_i} + u_i\frac{\partial\phi}{\partial x_j}\right)\mathrm{d}x + \int_{\partial\Omega}\phi\left(\gamma_0(u_i)\nu_j + \gamma_0(u_j)\nu_i\right)\mathrm{d}\mathcal{H}^{d-1}$$

holds for every $\phi \in C^1(\overline{\Omega})$, where $\boldsymbol{\nu} = (\nu_1,\dots,\nu_d)$ is the unit outward normal vector on $\partial\Omega$ and $\gamma_0(u_i)$ is the i-th component of $\gamma_0(\boldsymbol{u})$ and $\mathcal{H}^{d-1}$ is the $(d-1)-$Hausdorff measure. Such a γ_0 is a generalization of the trace operator in Sobolev spaces to the case of BD space. If additionally $\boldsymbol{u} \in C(\overline{\Omega};\mathbb{R}^d)$, then $\gamma_0(\boldsymbol{u}) = \boldsymbol{u}|_{\partial\Omega}$. In case of $\boldsymbol{u} \in W^{1,1}_0(\Omega;\mathbb{R}^d)$ this coincides with the classical trace operator in Sobolev spaces.

Understanding the trace in this generalized sense we define the subspace and the subset of $BD_M(\Omega)$ as follows

$$BD_{M,0}(\Omega) := \{\boldsymbol{u} \in BD_M(\Omega) \mid \gamma_0(\boldsymbol{u}) = 0\},$$

$$\mathcal{BD}_{M,0}(\Omega) := \{\boldsymbol{u} \in BD_M(\Omega) \mid \mathbf{D}\boldsymbol{u} \in \mathcal{L}_M(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}) \text{ and } \gamma_0(\boldsymbol{u}) = 0\}.$$

Let us define also

$$BD_M(Q) := \{\boldsymbol{u} \in L^1(Q;\mathbb{R}^d) \mid \mathbf{D}\boldsymbol{u} \in L_M(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})\}$$

and the corresponding subspace

$$BD_{M,0}(Q) := \{\boldsymbol{u} \in BD_M(Q) \mid \gamma_0(\boldsymbol{u}) = 0\}$$

where γ_0 has the following meaning

(VI.1.7)
$$\begin{aligned}2\int_Q \phi[\mathbf{D}\boldsymbol{u}]_{i,j}\,\mathrm{d}x\mathrm{d}t &= -\int_Q \left(u_j\frac{\partial\phi}{\partial x_i} + u_i\frac{\partial\phi}{\partial x_j}\right)\mathrm{d}x\mathrm{d}t\\ &\quad + \int_{(0,T)\times\partial\Omega}\phi\left(\gamma_0(u_i)\nu_j + \gamma_0(u_j)\nu_i\right)\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t\end{aligned}$$

for all $\phi \in C^1(\bar{Q})$. If $\boldsymbol{u} \in BD_M(Q)$, then for a.a. $t \in (0,T)$ we have $\boldsymbol{u}(t,\cdot) \in BD_M(\Omega)$. For such vector fields it is equivalent that $\boldsymbol{u} \in BD_{M,0}(Q)$ and that $\boldsymbol{u}(t,\cdot) \in BD_{M,0}(\Omega)$ for a.a. $t \in (0,T)$. By [**123**, Proposition 1.1.] there exists an extension operator from $BD(\Omega)$ to $BD(\mathbb{R}^d)$ and consequently we are able to extend the functions from $BD_{M,0}(Q)$ by zero to the function in $BD_M([0,T]\times\mathbb{R}^d)$.

In what follows, the closure of $\mathcal{D}(\Omega;\mathbb{R}^d)$ with respect to two topologies will be considered, i.e.

(1) modular topology of $L_M(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})$, which we denote by Y_0^M, namely

(VI.1.8)
$$Y_0^M = \{\boldsymbol{u}\in L^\infty(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d)),\, \mathbf{D}\boldsymbol{u}\in L_M(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}}) \mid \exists\, \{\boldsymbol{u}^j\}_{j=1}^\infty \subset \mathcal{D}((-\infty,T);\mathcal{V}) : \boldsymbol{u}^j \xrightharpoonup{*} \boldsymbol{u} \text{ in } L^\infty(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d)) \text{ and } \mathbf{D}\boldsymbol{u}^j \xrightarrow{M} \mathbf{D}\boldsymbol{u} \text{ modularly in } L_M(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})\}$$

(2) weak–(*) topology of $L_M(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})$, which we denote by Z_0^M, namely

(VI.1.9)
$$Z_0^M = \{\boldsymbol{u}\in L^\infty(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d)),\, \mathbf{D}\boldsymbol{u}\in L_M(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}}) \mid \exists\, \{\boldsymbol{u}^j\}_{j=1}^\infty \subset \mathcal{D}((-\infty,T);\mathcal{V}) : \boldsymbol{u}^j \xrightharpoonup{*} \boldsymbol{u} \text{ in } L^\infty(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d)) \text{ and } \mathbf{D}\boldsymbol{u}^j \xrightharpoonup{*} \mathbf{D}\boldsymbol{u} \text{ weakly star in } L_M(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})\}.$$

The main result of this chapter concerns the existence of weak solutions to the initial boundary value problem (VI.1.1)–(VI.1.4).

Theorem VI.1.1. *Let condition D1. or D2. be satisfied*

(D1) Ω is a bounded star-shaped domain,
(D2) Ω is a bounded non-star-shaped domain and

$$\overline{m}(r) \leqslant c_m((\underline{m}(r))^{\frac{d}{d-1}} + |r|^2 + 1) \tag{VI.1.10}$$

for all $r\in\mathbb{R}_+$, and $\underline{m}$ satisfies Δ_2-condition.

Let M be an $\mathcal{N}-$function and $\mathbf{S}$ satisfy conditions (S1)-(S3). Then, for given $\boldsymbol{u}_0 \in L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d)$ and $\boldsymbol{f}\in E_{\underline{m}^*}(Q;\mathbb{R}^d)$ there exists $\boldsymbol{u}\in Z_0^M$ such that

$$\int_Q -\boldsymbol{u}\cdot\partial_t\boldsymbol{\varphi} + \mathbf{S}(t,x,\mathbf{D}\boldsymbol{u}) : \mathbf{D}\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t = \int_Q \boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t - \int_\Omega \boldsymbol{u}_0\boldsymbol{\varphi}(0)\,\mathrm{d}x$$

for all $\boldsymbol{\varphi}\in\mathcal{D}(-\infty,T;\mathcal{V})$.

This chapter is organized as follows: Section VI.2 is devoted to the Sobolev-Korn-type inequality in Orlicz spaces. In Section VI.3 we concentrate on showing that the spaces Y_0^M and Z_0^M defined above coincide and how this fact is used in the integration by parts formula. The last section contains the proof of Theorem VI.1.1, which essentially bases on the facts proved in previous sections.

VI.2. Variant of the Sobolev-Korn inequality

Numerous classical results (e.g. Poincaré, Sobolev, Korn inequalities) have been generalized from Lebesgue and Sobolev spaces to Orlicz spaces. Among others we find results of Cianchi on the Sobolev inequality, see [**33, 34**]. Other interesting results concern the embeddings of a very particular type of Orlicz-Sobolev space, namely $BLD(\Omega) := \{\boldsymbol{u}\in L^1(\Omega;\mathbb{R}^d) \mid |\mathbf{D}\boldsymbol{u}|\in L_m(\Omega)\}$ where $L_m(\Omega)$ is defined by the function $m(\xi) = \xi\ln(\xi+1), \xi\in\mathbb{R}_+$, cf. Bildhauer and Fuchs [**62**].

The Korn inequality is a standard tool used in problems arising from fluid mechanics to provide an estimate of the gradient by symmetric gradient in appropriate norms. The generalization of the Korn inequality, namely

$$\int_\Omega m(|\nabla_x \boldsymbol{u}|)\,\mathrm{d}x \leqslant c\int_\Omega m(|\mathbf{D}\boldsymbol{u}|)\,\mathrm{d}x$$

is valid for the case of m and m^* satisfying the Δ_2-condition, see e.g. [**61**]. Since this is not the case of our considerations we will concentrate on generalizing the result of Strauss, cf. [**122**], namely

$$\|\boldsymbol{u}\|_{L^{\frac{d}{d-1}}(\Omega)} \leqslant \|\mathbf{D}\boldsymbol{u}\|_{L^1(\Omega)}$$

to the case of integrability of appropriate $\mathcal{N}-$functions.[1] Indeed the following fact holds:

Lemma VI.2.1. *Let m be an $\mathcal{N}-$function and Ω be a bounded domain, $\bar{\Omega} \subset [-\frac14, \frac14]^d$, and $\boldsymbol{u} \in \mathcal{BD}_{M,0}(\Omega)$. Then*

$$\|m(|\boldsymbol{u}|)\|_{L^{\frac{d}{d-1}}(\Omega)} \leqslant C_d\|m(|\mathbf{D}\boldsymbol{u}|)\|_{L^1(\Omega)}. \tag{VI.2.1}$$

The proof is presented in two parts. First, we show the validity of (VI.2.1) for $\boldsymbol{u} \in X$, where

$$X(\Omega) := \{\boldsymbol{\varphi} \in C^1_c(\Omega;\mathbb{R}^d);\ \int_\Omega m(|\mathbf{D}\boldsymbol{\varphi}|)\,\mathrm{d}x < \infty\}$$

and then the result is extended for $\boldsymbol{u} \in \mathcal{BD}_{M,0}(\Omega)$.

Proof. *Step 1.*
Assume that $\boldsymbol{u} \in X(\Omega)$ and $\operatorname{supp}\boldsymbol{u} \subset [-\frac14, \frac14]^d$. Let us denote $\delta_d = (1,1,...,1)$. Then by the mean value theorem in the integral form (see e.g. [**4**]) it follows

$$u_i(x) = \int_{-\frac12}^0 \sum_{j=1}^d \partial_j u_i(x+s\delta_d)\,\mathrm{d}s = -\int_0^{\frac12}\sum_{j=1}^d \partial_j u_i(x+s\delta_d)\,\mathrm{d}s$$

and

$$\sum_{i=1}^d u_i(x) = \int_{-\frac12}^0 \sum_{i,j=1}^d \partial_j u_i(x+s\delta_d)\,\mathrm{d}s = -\int_0^{\frac12}\sum_{i,j=1}^d \partial_j u_i(x+s\delta_d)\,\mathrm{d}s.$$

Hence

$$\begin{aligned} 2\sum_{i=1}^d u_i(x) &= \int_{-\frac12}^0 \sum_{i,j=1}^d \left(\partial_j u_i(x+s\delta_d) + \partial_i u_j(x+s\delta_d)\right)\,\mathrm{d}s \\ &= -\int_0^{\frac12}\sum_{i,j=1}^d \left(\partial_j u_i(x+s\delta_d) + \partial_i u_j(x+s\delta_d)\right)\,\mathrm{d}s \end{aligned}$$

[1]In the current section the $\mathcal{N}-$function has the same properties as before with only one difference - it is defined on $\mathbb{R}_+$. To help the reader distinguish this case, we will denote it in this chapter with a small letter m, contrary to M defined on $\mathbb{R}^{d\times d}_{\mathrm{sym}}$.

and consequently we obtain

$$4|\sum_{i=1}^{d} u_i(x)| \leqslant \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{i,j=1}^{d} |\partial_j u_i(x+s\delta_d) + \partial_i u_j(x+s\delta_d)| \,\mathrm{d}s.$$

Applying $\mathcal{N}$-function $m : \mathbb{R}_+ \to \mathbb{R}_+$ to the above inequality, using convexity of m and the Jensen inequality (here we use the fact that the support of $\boldsymbol{u}$ is in $[-\frac{1}{4}, \frac{1}{4}]^d$) we observe that

$$\begin{aligned}
&\left(m \left(|\sum_{i=1}^{d} u_i(x)| \right) \right)^{\frac{1}{d-1}} \\
&\leqslant \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} m \left(\frac{1}{4} \sum_{i,j=1}^{d} |\partial_j u_i(x+s\delta_d) + \partial_i u_j(x+s\delta_d)| \right) \mathrm{d}s \right)^{\frac{1}{d-1}}.
\end{aligned}$$

Let $e_k = (0,...,0,1,0,...,0)$ be a unit vector along the x_k-axis and $f_k = \delta_d - e_k = (1,...,1,0,1,...,1)$ for $k \in \{1,...,d-1\}$. Obviously

$$\begin{aligned}
\sum_{i=1}^{d} u_i(x) &= \int_{-\frac{1}{2}}^{0} \sum_{i,j=1, i\neq k, j\neq k}^{d} \partial_j u_i(x+sf_k)\,\mathrm{d}s + \int_{-\frac{1}{2}}^{0} \partial_k u_k(x+se_k)\,\mathrm{d}s \\
&= -\int_{0}^{\frac{1}{2}} \sum_{i,j=1, i\neq k, j\neq k}^{d} \partial_j u_i(x+sf_k)\,\mathrm{d}s - \int_{0}^{\frac{1}{2}} \partial_k u_k(x+se_k)\,\mathrm{d}s.
\end{aligned}$$

Consequently

(VI.2.2)

$$\begin{aligned}
&\left(m(|\sum_{i=1}^{d} u_i(x)|) \right)^{\frac{1}{d-1}} \leqslant \\
&\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} m \left(\frac{1}{4} \sum_{i,j=1, i\neq k, j\neq k}^{d} |\partial_j u_i(x+sf_k) + \partial_i u_j(x+sf_k)| + \frac{1}{2} |\partial_k u_k(x+se_k)| \right) \mathrm{d}s \right]^{\frac{1}{d-1}} \\
&\leqslant \left(\frac{1}{2} \right)^{\frac{1}{d-1}} \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} m \left(\frac{1}{2} \sum_{i,j=1, i\neq k, j\neq k}^{d} |\partial_j u_i(x+sf_k) + \partial_i u_j(x+sf_k)| \right) \right. \\
&\left. + m\left(|\partial_k u_k(x+se_k)| \right) ds \right]^{\frac{1}{d-1}} \\
&\leqslant \left(\frac{1}{2} \right)^{\frac{1}{d-1}} C \left[\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} m \left(\frac{1}{2} \sum_{i,j=1, i\neq k, j\neq k}^{d} |\partial_j u_i(x+sf_k) + \partial_i u_j(x+sf_k)| \right) \right)^{\frac{1}{d-1}} \right. \\
&\left. + \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} m\left(|\partial_k u_k(x+se_k)| \right) \mathrm{d}s \right)^{\frac{1}{d-1}} \right].
\end{aligned}$$

Next, we multiply expression $\left(m(|\sum_{i=1}^d u_i(x)|)\right)^{\frac{1}{d-1}}$ by itself d times and conclude that

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\left(m\left(|\sum_{i=1}^d u_i(x)|\right)\right)^{\frac{d}{d-1}} \mathrm{d}x_1...\mathrm{d}x_d \\
&\leqslant C\int_{\mathbb{R}^d}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} m\left(\frac{1}{4}\sum_{i,j=1}^d |\partial_j u_i(x+s\delta_d)+\partial_i u_j(x+s\delta_d)|\right) \mathrm{d}s\right)^{\frac{1}{d-1}} \\
&\prod_{k=1}^{d-1}\Big[\left(\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} m\left(\frac{1}{2}\sum_{i,j=1,i\neq k,j\neq k}^d |\partial_j u_i(x+sf_k)+\partial_i u_j(x+sf_k)|\right)\right)\right)^{\frac{1}{d-1}} \\
&+\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} m\left(|\partial_k u_k(x+se_k)|\right) \mathrm{d}s\right)^{\frac{1}{d-1}}\Big] \mathrm{d}x_1...\mathrm{d}x_d \\
&= C\sum_{\sigma}\int_{\mathbb{R}^d}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} m\left(\frac{1}{4}\sum_{i,j=1}^d |\partial_j u_i(x+s\delta_d)+\partial_i u_j(x+s\delta_d)|\right) \mathrm{d}s\right)^{\frac{1}{d-1}} \\
&\prod_{k=1,k\in\sigma}^{d-1}\left(\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} m\left(\frac{1}{2}\sum_{i,j=1,i\neq k,j\neq k}^d |\partial_j u_i(x+sf_k)+\partial_i u_j(x+sf_k)|\right)\right)\right)^{\frac{1}{d-1}} \\
&\prod_{k=1,k\notin\sigma}^{d-1}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} m\left(|\partial_k u_k(x+se_k)|\right) \mathrm{d}s\right)^{\frac{1}{d-1}} \mathrm{d}x_1...\mathrm{d}x_d
\end{aligned}
\tag{VI.2.3}
$$

where σ runs over possible subsets of $\{1,2,...,d-1\}$. Since $\operatorname{supp}\boldsymbol{u}\subset[-\frac{1}{4},\frac{1}{4}]^d$, then by the Fubini theorem it is easy to notice that

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\left(m\left(|\sum_{i=1}^d u_i|\right)\right)^{\frac{d}{d-1}} \mathrm{d}x_1...\mathrm{d}x_d \\
&\leqslant C\sum_{\sigma}\left(\int_{\mathbb{R}^d} m\left(\frac{1}{4}\sum_{i,j=1}^d |\partial_j u_i(x)+\partial_i u_j(x)|\right) \mathrm{d}x\right)^{\frac{1}{d-1}} \\
&\prod_{k=1,k\in\sigma}^{d-1}\left(\int_{\mathbb{R}^d} m\left(\frac{1}{2}\sum_{i,j=1,i\neq k,j\neq k}^d |\partial_j u_i(x)+\partial_i u_j(x)|\right) \mathrm{d}x\right)^{\frac{1}{d-1}} \\
&\prod_{k=1,k\notin\sigma}^{d-1}\left(\int_{\mathbb{R}^d} m\left(|\partial_k u_k(x)|\right) \mathrm{d}x\right)^{\frac{1}{d-1}}.
\end{aligned}
\tag{VI.2.4}
$$

In a similar way, by integration over lines $(1,-1,1,....,-1)$ etc., instead of these we can obtain the same bound for any $\|m(\sum_{i=1}^d v_i(x)u_i)\|^{d/(d-1)}_{L^{d/(d-1)}(\mathbb{R}^d)}$ where $v_i\in\{\pm1,0\}$.

Now, let v_i vary by setting $v_i(x) = \operatorname{sgn} u_i(x)$, and then

$$\int_{\mathbb{R}^d} \left(m\left(\sum_{i=1}^d |u_i(x)| \right) \right)^{\frac{d}{d-1}} \mathrm{d}x_1 ... \mathrm{d}x_d \leqslant \int_{\mathbb{R}^d} \left(m\left(\sum_{i=1}^d |v_i(x)u_i(x)| \right) \right)^{\frac{d}{d-1}} \mathrm{d}x_1 ... \mathrm{d}x_d$$

has the same bound (up to a constant 2^d). Indeed, let $\Upsilon = \{\gamma = (\gamma_1, \gamma_2, \gamma_3) : \gamma_i \in \{-1, 0, 1\}, i = 1, 2, 3\}$, $A_\gamma = \{x \in \mathbb{R}^n : \operatorname{sgn} u_i(x) = v_i(x) = \gamma_i, i = 1, 2, 3\}$. Estimates (VI.2.2), (VI.2.4) are also valid if we integrate over any measurable subset of $\mathbb{R}^d$ instead of the whole $\mathbb{R}^d$. One easily observes that $\{A_\gamma\}_\gamma$ is a division of $\mathbb{R}^d$ on measurable subsets. Obviously

$$|\sum_{i=1}^d v_i(x)u_i(x)| = \sum_{i=1}^d v_i(x)u_i(x) \geqslant 0.$$

Thus

$$\begin{aligned} \int_{\mathbb{R}^d} |\sum_{i=1}^d v_i(x)u_i(x)| \,\mathrm{d}x &= \sum_{\gamma\in\Upsilon} \int_{A_\gamma} |\sum_{i=1}^d v_i(x)u_i(x)| \,\mathrm{d}x \\ &= \sum_{\gamma\in\Upsilon} \int_{A_\gamma} \sum_{i=1}^d |v_i(x)u_i(x)| \,\mathrm{d}x = I_1 \end{aligned}$$

where $v_i(x)$ is constant on any subset of division $\{A_\gamma\}_\gamma$. Hence all expressions in summation over γ are positive and independent of $v_i(x)$. Therefore we obtain

$$I_1 = \sum_{\gamma\in\Upsilon} \int_{A_\gamma} \sum_{i=1}^d |u_i(x)| \,\mathrm{d}x = \int_{\mathbb{R}^d} \sum_{i=1}^d |u_i(x)| \,\mathrm{d}x$$

and on the other hand

$$I_1 \leqslant 2^d \int_{\mathbb{R}^d} m(|\sum_{i=1}^d v_i(x)u_i(x)|) \,\mathrm{d}x.$$

Finally we deduce that

$$\int_{\mathbb{R}^d} m(\sum_{i=1}^d |u_i(x)|) \,\mathrm{d}x \leqslant 2^d \int_{\mathbb{R}^d} m(|\sum_{i=1}^d v_i u_i(x)|) \,\mathrm{d}x.$$

In the end, since the geometric mean of nonnegative numbers is no greater than the arithmetic mean, we estimate the right hand side of (VI.2.4)

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\left(m(|\sum_{i=1}^d u_i|)\right)^{\frac{d}{d-1}} \mathrm{d}x_1 ... \mathrm{d}x_d \\
&\leqslant C\sum_{\sigma}\frac{1}{d}\Big[\int_{\mathbb{R}^d} m\left(\frac{1}{4}\sum_{i,j=1}^d |\partial_j u_i(x)+\partial_i u_j(x)|\right)\mathrm{d}x \\
&+\sum_{k=1,k\in\sigma}^{d-1}\int_{\mathbb{R}^d} m\left(\frac{1}{2}\sum_{i,j=1,i\neq k,j\neq k}^{d}|\partial_j u_i(x)+\partial_i u_j(x)|\right)\mathrm{d}x \\
&+\sum_{k=1,k\notin\sigma}^{d-1}\int_{\mathbb{R}^d} m\left(\frac{1}{2}|\partial_k u_k(x)|\right)\mathrm{d}x\Big]^{\frac{d}{d-1}} = I_2.
\end{aligned}
\tag{VI.2.5}
$$

Since m is convex and $m(0)=0$, then

$$
\begin{aligned}
I_2 &\leqslant C\sum_{\sigma}\frac{1}{d}\Big[\frac{1}{2}\int_{\mathbb{R}^d} m\left(\frac{1}{2}\sum_{i,j=1}^d |\partial_j u_i(x)+\partial_i u_j(x)|\right)\mathrm{d}x \\
&+\sum_{k=1,k\in\sigma}^{d}\int_{\mathbb{R}^d} m\left(\frac{1}{2}\sum_{i,j=1,i\neq k,j\neq k}^{d-1}|\partial_j u_i(x)+\partial_i u_j(x)|\right)\mathrm{d}x \\
&+\sum_{k=1,k\notin\sigma}^{d-1}\int_{\mathbb{R}^d} m\left(\frac{1}{2}|\partial_k u_k(x)|\right)\mathrm{d}x\Big]^{\frac{d}{d-1}} \\
&\leqslant \Big[K(d)\int_{\mathbb{R}^d} m\left(\frac{1}{2}\sum_{i,j=1}^d |\partial_j u_i(x)+\partial_i u_j(x)|\right)\mathrm{d}x\Big]^{\frac{d}{d-1}}.
\end{aligned}
\tag{VI.2.6}
$$

Step 2.
Let $[-\frac{1}{4},\frac{1}{4}]\supset\tilde{\Omega}\supset\bar{\Omega}$ and $\mathcal{D}(\tilde{\Omega};\mathbb{R}^d)$ be the set of smooth functions in $\mathbb{R}^d$ with support in $\tilde{\Omega}$. Step 1 provides that $\boldsymbol{u}\in\mathcal{D}(\tilde{\Omega};\mathbb{R}^d)$ with $\operatorname{supp}\boldsymbol{u}\in[-\frac{1}{4},\frac{1}{4}]^d$ satisfies

$$
\|m(|\boldsymbol{u}|)\|_{L^{\frac{d}{d-1}}(\mathbb{R}^n)} \leqslant C_d\|m(|\mathbf{D}\boldsymbol{u}|)\|_{L^1(\mathbb{R}^d)}.
\tag{VI.2.7}
$$

To deduce the validity of (VI.2.7) for all $\boldsymbol{u}\in\mathcal{BD}_{M,0}(\Omega)$, we extend $\boldsymbol{u}$ by zero outside of the set Ω. Obviously $\boldsymbol{u}\in\mathcal{BD}_{M,0}(\tilde{\Omega})$. Now $\boldsymbol{u}$ can be regularized as follows

$$
\boldsymbol{u}^{\varepsilon}(x) := \varrho_{\varepsilon} * \boldsymbol{u}(x)
$$

where $\varepsilon<\frac{1}{2}\operatorname{dist}(\partial\tilde{\Omega},\Omega)$ and ϱ_ε is a standard mollifier, the convolution being done w.r.t. x. Since $\boldsymbol{u}^\varepsilon(x)$ is smooth with compact support in $\tilde{\Omega}$, inequality (VI.2.7) is provided for $\boldsymbol{u}^\varepsilon$. Passing to the limit with $\varepsilon\to 0$ yields that $\boldsymbol{u}^\varepsilon\to\boldsymbol{u}$, $\mathbf{D}\boldsymbol{u}^\varepsilon\to\mathbf{D}\boldsymbol{u}$ a.e. in $\mathbb{R}^d$ and the continuity of an $\mathcal{N}$–function m provides that $m(|\boldsymbol{u}^\varepsilon|)\to m(|\boldsymbol{u}|)$, $m(|\mathbf{D}\boldsymbol{u}^\varepsilon|)\to m(|\mathbf{D}\boldsymbol{u}|)$ a.e. in $\mathbb{R}^d$.

To conclude the strong convergence in $L^1(\Omega)$ of the sequence $\{m(|\boldsymbol{u}^\varepsilon|)\}_{\varepsilon>0}$ we start with an abstract fact concerning uniform integrability. Observe that the following two conditions are equivalent for any measurable sequence $\{z^j\}_{j=1}^\infty$

$$\forall\,\varepsilon>0\quad\exists\,\delta>0:\quad \sup_{j\in\mathbb{N}}\ \sup_{A\subset\tilde\Omega,|A|\leqslant\delta}\int_A |z^j(x)|\,\mathrm{d}x\leqslant\varepsilon, \tag{VI.2.8}$$

$$\forall\,\varepsilon>0\quad\exists\,\delta>0:\quad \sup_{j\in\mathbb{N}}\int_{\tilde\Omega}\left||z^j(x)|-\frac{1}{\sqrt{\delta}}\right|_+\mathrm{d}x\leqslant\varepsilon, \tag{VI.2.9}$$

where

$$|\xi|_+=\max\{0,\xi\}.$$

The implication (VI.2.8)⇒(VI.2.9) is obvious. To show that also (VI.2.9)⇒(VI.2.8) holds let us estimate

$$\begin{aligned}\sup_{j\in\mathbb{N}}\sup_{|A|\leqslant\delta}\int_A|z^j|\,\mathrm{d}x&\leqslant\sup_{|A|\leqslant\delta}|A|\cdot\frac{1}{\sqrt{\delta}}+\sup_{j\in\mathbb{N}}\int_{\tilde\Omega}\left||z^j|-\frac{1}{\sqrt{\delta}}\right|_+\mathrm{d}x\\&\leqslant\sqrt{\delta}+\sup_{j\in\mathbb{N}}\int_{\tilde\Omega}\left||z^j|-\frac{1}{\sqrt{\delta}}\right|_+\mathrm{d}x.\end{aligned}$$

Since m is a convex function, the following inequality holds for all $\delta>0$

$$\int_{\tilde\Omega}\left|m(|\boldsymbol{u}|)-\frac{1}{\sqrt{\delta}}\right|_+dx\geqslant\int_{\tilde\Omega}\left|m(|\varrho^\varepsilon*\boldsymbol{u}|)-\frac{1}{\sqrt{\delta}}\right|_+dx. \tag{VI.2.10}$$

Finally, since $\int_{\tilde\Omega}m(|\boldsymbol{u}|)dx<\infty$, then also $\int_{\tilde\Omega}|m(|\boldsymbol{u}|)-\frac{1}{\sqrt{\delta}}|_+dx$ is finite and hence taking supremum over $\varepsilon>0$ in (VI.2.10) we prove uniform integrability of $\{m(|\boldsymbol{u}^\varepsilon|)\}_{\varepsilon>0}$. The same considerations are valid for $\{m(|\mathbf{D}\boldsymbol{u}^\varepsilon|)\}_{\varepsilon>0}$. Finally, by the Vitali lemma we conclude that

$$\begin{aligned}m(|\boldsymbol{u}^\varepsilon|)&\to m(|\boldsymbol{u}|)\quad\text{strongly in }L^1(\mathbb{R}^d),\\ m(|\mathbf{D}\boldsymbol{u}^\varepsilon|)&\to m(|\mathbf{D}\boldsymbol{u}|)\quad\text{strongly in }L^1(\mathbb{R}^d).\end{aligned}$$

Consequently, the limit $\boldsymbol{u}$ satisfies inequality (VI.2.7).

□

Remark VI.2.2. If Ω is bounded, we can rescale the space variables. Then we have

$$\|m(|\boldsymbol{u}|)\|_{L^{\frac{d}{d-1}}(\Omega)}\leqslant C_d\|m(|C_r\mathbf{D}\boldsymbol{u}|)\|_{L^1(\Omega)}.$$

where C_r is a constant dependent on the jacobian of rescaling.

VI.3. Domains and closures

In the present section we concentrate on the issue of closures of smooth functions w.r.t. various topologies. In the introduction we defined the spaces Y_0^M and Z_0^M. Our interest is directed to the equivalence between these two spaces. The simplest proof is provided in the case of star-shaped domains. For extending the result for arbitrary domains with regular boundary, the set Ω is considered as a sum of star-shaped domains. In this case the Sobolev-Korn inequality (VI.2.1) provides an essential estimate. Another requirement appearing for non-star-shaped domains is the constrain on the spread between $\underline{m}$ and $\overline{m}$ and also on the growth of $\underline{m}$, i.e. the condition (D2) in Theorem VI.1.1.

In the present section integration by parts is also considered as the main issue, where the equivalence between the spaces Y_0^M and Z_0^M is crucial.

Lemma VI.3.1 (star-shaped domains)**.** *Let $M : \mathbb{R}^{d\times d}_{\rm sym} \to \mathbb{R}_+$ be an $\mathcal{N}$-function, Ω be a bounded star-shaped domain and Y_0^M, Z_0^M be the function spaces defined by (VI.1.8) and (VI.1.9). Then $Y_0^M = Z_0^M$.*

Moreover, if $\boldsymbol{u} \in Y_0^M$, $\boldsymbol{\chi} \in \mathcal{L}_{M^*}(Q;\mathbb{R}^{d\times d}_{\rm sym})$, $\boldsymbol{f} \in \mathcal{L}_{\underline{m}^*}(Q;\mathbb{R}^d)$ and

$$\partial_t \boldsymbol{u} - \operatorname{div}_x \boldsymbol{\chi} = \boldsymbol{f} \quad \text{in } \mathcal{D}'(Q), \tag{VI.3.1}$$

then

$$\frac{1}{2}\|\boldsymbol{u}(s)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\boldsymbol{u}(s_0)\|^2_{L^2(\Omega)} + \int_{s_0}^{s}\int_\Omega \boldsymbol{\chi} : \mathbf{D}\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t = \int_{s_0}^{s}\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t$$

for a.a. $s_0,\ s:\ 0 < s_0 < s < T$.

Proof. Since the modular topology is stronger than weak-star, obviously we have $Y_0^M \subset Z_0^M$. Therefore we focus on proving the opposite inclusion, namely

$$Z_0^M \subset Y_0^M. \tag{VI.3.2}$$

To reach this goal we want to extend $\boldsymbol{u}$ by zero outside of Ω to the whole $\mathbb{R}^d$ and then mollify it. To extend $\boldsymbol{u}$ we observe that $Z_0^M \subset BD_{M,0}(Q)$. By definition it is obvious that each $\boldsymbol{u} \in Z_0^M$ is an element of $BD(Q)$, hence let us concentrate on showing that it vanishes on the boundary. Recall that for $\boldsymbol{u}$ formula (VI.1.7) is satisfied. Take a sequence $\{\boldsymbol{u}^k\}_{k=1}^\infty$ of compactly supported smooth functions with the properties prescribed in the definition of the space Z_0^M. After inserting this sequence into (VI.1.7) we obtain

$$2\int_Q \phi[\mathbf{D}\boldsymbol{u}^k]_{i,j}\,\mathrm{d}x\mathrm{d}t = -\int_Q \left(u_j^k\frac{\partial\phi}{\partial x_i} + u_i^k\frac{\partial\phi}{\partial x_j}\right)\mathrm{d}x\mathrm{d}t \tag{VI.3.3}$$

Now we can easily pass to the weak-star limit in (VI.3.3) because of the linearity of all terms. As a consequence we conclude that the boundary term vanishes.

Next we introduce $\boldsymbol{u}^\lambda$, where the index λ for any function $\boldsymbol{v}$ is understood as follows

$$\boldsymbol{v}^\lambda(t,x) := \boldsymbol{v}(t, \lambda(x-x_0)+x_0) \tag{VI.3.4}$$

where x_0 is a vantage point of Ω and $\lambda \in (0,1)$. Let $\varepsilon_\lambda = \frac{1}{2}\mathrm{dist}\,(\partial\Omega, \lambda\Omega)$ where $\lambda\Omega := \{y = \lambda(x-x_0)+x_0 \mid x \in \Omega\}$. Define then

$$\boldsymbol{u}^{\delta,\lambda,\varepsilon}(t,x) := \sigma_\delta * ((\varrho_\varepsilon * \boldsymbol{u}^\lambda(t,x))\ \mathbb{1}_{(s_0,s)}) \tag{VI.3.5}$$

where $\varrho_\varepsilon(x) = \frac{1}{\varepsilon^d}\varrho(\frac{x}{\varepsilon})$ is a standard regularizing kernel on $\mathbb{R}^d$ (i.e. $\varrho \in C^\infty(\mathbb{R}^d)$, ϱ has a compact support in $B(0,1)$ and $\int_{\mathbb{R}^d}\varrho(x)\,\mathrm{d}x = 1$, $\varrho(x) = \varrho(-x)$) and the convolution is done w.r.t. space variable x, $\varepsilon < \frac{\varepsilon_\lambda}{2}$ and $\sigma_\delta(t) = \frac{1}{\delta}\sigma(\frac{t}{\delta})$ is a regularizing kernel on $\mathbb{R}$ (i.e. $\sigma \in C^\infty(\mathbb{R})$, σ has a compact support and $\int_{\mathbb{R}}\sigma(\tau)d\tau = 1, \sigma(t) = \sigma(-t)$) and convolution is done w.r.t. time variable t with $\delta < \min\{s_0, T-s\}$. The approximated function $\boldsymbol{u}^{\delta,\lambda,\varepsilon}$ also has zero trace.

First we pass to the limit with $\varepsilon \to 0$ and hence

$$\mathbf{D}\boldsymbol{u}^{\delta,\lambda,\varepsilon} \xrightarrow{\varepsilon\to 0} \mathbf{D}\boldsymbol{u}^{\delta,\lambda} \text{ in } L^1(Q;\mathbb{R}^{d\times d}).$$

For a.a. $t \in [0,T]$ the function $\mathbf{D}\boldsymbol{u}^{\delta,\lambda,\varepsilon}(t,\cdot) \in L^1(\Omega;\mathbb{R}^{d\times d})$ and

$$\varrho_\varepsilon * \mathbf{D}\boldsymbol{u}^{\delta,\lambda}(t,\cdot) \xrightarrow{\varepsilon\to 0} \mathbf{D}\boldsymbol{u}^{\delta,\lambda}(t,\cdot) \text{ in } L^1(\Omega;\mathbb{R}^{d\times d})$$

and hence

$$\varrho_\varepsilon * \mathbf{D}\boldsymbol{u}^{\delta,\lambda} \xrightarrow{\varepsilon\to 0} \mathbf{D}\boldsymbol{u}^{\delta,\lambda} \text{ in measure on the set } [0,T]\times\Omega.$$

To show the uniform integrability of $\{M(\mathbf{D}\boldsymbol{u}^{\delta,\lambda,\varepsilon})\}_{\varepsilon>0}$ we use the analogous argumentation as in the proof of Lemma VI.2.1, i.e. the equivalence of the following two conditions for any measurable sequence $\{z^j\}$

(a) $\forall\, \epsilon > 0 \quad \exists\, \theta > 0 : \quad \sup\limits_{j\in\mathbb{N}} \sup\limits_{A\subset Q,\ |A|\leqslant\theta} \int_A |z^j(t,x)|\, \mathrm{d}x\mathrm{d}t \leqslant \epsilon,$

(b) $\forall\, \epsilon > 0 \quad \exists\, \theta > 0 : \quad \sup\limits_{j\in\mathbb{N}} \int_Q \left| |z^j(t,x)| - \frac{1}{\sqrt{\delta}} \right|_+ \mathrm{d}x\mathrm{d}t \leqslant \epsilon.$

Notice that since M is a convex function, then the following inequality holds for all $\theta > 0$

$$\int_Q \left| M(\mathbf{D}\boldsymbol{u}^{\delta,\lambda}) - \frac{1}{\sqrt{\theta}} \right|_+ \mathrm{d}x\mathrm{d}t \geqslant \int_Q \left| M(\varrho_\varepsilon * \mathbf{D}\boldsymbol{u}^{\delta,\lambda}) - \frac{1}{\sqrt{\theta}} \right|_+ \mathrm{d}x\mathrm{d}t. \tag{VI.3.6}$$

Finally, since $\beta\mathbf{D}\boldsymbol{u}^{\delta,\lambda} \in \mathcal{L}_M(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ for some $\beta > 0$, then also $\int_Q |M(\beta\mathbf{D}\boldsymbol{u}^{\delta,\lambda}) - \frac{1}{\sqrt{\theta}}|_+ \,\mathrm{d}x\mathrm{d}t$ is finite and hence taking supremum over $\varepsilon \in (0, \frac{\varepsilon_\lambda}{2})$ in (VI.3.6) we prove that the sequence $\{M(\beta\mathbf{D}\boldsymbol{u}^{\delta,\lambda,\varepsilon})\}_{\varepsilon>0}$ is uniformly integrable.

Finally, Lemma III.2.1 provides that

$$\mathbf{D}\boldsymbol{u}^{\delta,\lambda,\varepsilon} \xrightarrow{\varepsilon\to 0} \mathbf{D}\boldsymbol{u}^{\delta,\lambda} \text{ modularly in } L_M(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}}).$$

Next, we pass to the limit with $\lambda \to 1$ and obtain that

$$\mathbf{D}\boldsymbol{u}^{\delta,\lambda} \xrightarrow{\lambda\to 1} \mathbf{D}\boldsymbol{u}^{\delta} \text{ in } L^1(Q;\mathbb{R}^{d\times d})$$

and

$$\mathbf{D}\boldsymbol{u}^{\delta,\lambda} \xrightarrow{\lambda\to 1} \mathbf{D}\boldsymbol{u}^{\delta} \text{ modularly in } L_M(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}}).$$

To converge with $\delta \to 0^+$ we employ similar arguments as for convergence with $\varepsilon \to 0^+$. Finally we observe that $Y_0^M = Z_0^M$.

The forthcoming part of the proof is devoted to the integration by parts formula. Let us define now

$$\boldsymbol{u}^{\delta,\lambda,\varepsilon}(t,x) := \sigma_\delta * ((\sigma_\delta * \varrho_\varepsilon * \boldsymbol{u}^\lambda(t,x))\, \mathbb{1}_{(s_0,s)}) \tag{VI.3.7}$$

where $\varepsilon < \frac{\varepsilon_\lambda}{2}$ and $\sigma < \frac{1}{2}\min\{s_0, T-s\}$. We test each equation in (VI.3.1) by $\boldsymbol{u}^{\delta,\lambda,\varepsilon}$ (which is a sufficiently regular test function)

(VI.3.8)
$$\int_{s_0}^{s}\int_\Omega (\boldsymbol{u} * \sigma_\delta)\cdot \partial_t(\boldsymbol{u}^{\lambda,\varepsilon} * \sigma_\delta)\,\mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega \boldsymbol{\chi} : \mathbf{D}\boldsymbol{u}^{\delta,\lambda,\varepsilon}\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}^{\delta,\lambda,\varepsilon}\,\mathrm{d}x\mathrm{d}t.$$

The left-hand side of (VI.3.8) is equivalent to $\int_{s_0}^{s}\int_\Omega (\boldsymbol{u} * \sigma_\delta)\cdot(\boldsymbol{u}^{\lambda,\varepsilon} * \partial_t\sigma_\delta)\,\mathrm{d}x\mathrm{d}t$, hence to pass to the limit with $\varepsilon \to 0$ and $\lambda \to 1$ we use the fact that $\boldsymbol{u}^{\lambda,\varepsilon} \overset{*}{\rightharpoonup} \boldsymbol{u}$ in $L^\infty(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d))$. To handle the right-hand side of (VI.3.8) we use the results shown in the first part of the proof. For proving the convergence of the term $\int_0^T\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}^{\delta,\lambda,\varepsilon}\,\mathrm{d}x\mathrm{d}t$ we apply Lemma VI.2.1 to $\underline{m}$ and observe that

$$\left(\int_\Omega (\underline{m}(|\boldsymbol{u}^{\delta,\lambda,\varepsilon}(t,x)|))^{\frac{d}{d-1}}\,\mathrm{d}x\right)^{\frac{d-1}{d}} \leqslant C_d\int_\Omega \underline{m}(|\mathbf{D}\boldsymbol{u}^{\delta,\lambda,\varepsilon}(t,x)|)\,\mathrm{d}x)$$

for a.a. $t \in [0,T]$. Consequently the Hölder inequality implies that

$$\int_0^T\int_\Omega (\underline{m}(|\boldsymbol{u}^{\delta,\lambda,\varepsilon}(t,x)|))\,\mathrm{d}x\mathrm{d}t \leqslant C_{\Omega,d}\int_0^T\int_\Omega (\underline{m}(|\mathbf{D}\boldsymbol{u}^{\delta,\lambda,\varepsilon}(t,x)|)\,\mathrm{d}x\mathrm{d}t.$$

Using definition of $\underline{m}$ we obtain

$$\int_0^T\int_\Omega (\underline{m}(|\boldsymbol{u}^{\delta,\lambda,\varepsilon}(t,x)|))\,\mathrm{d}x\mathrm{d}t \leqslant C_{\Omega,d}\int_0^T\int_\Omega M(\mathbf{D}\boldsymbol{u}^{\delta,\lambda,\varepsilon}(t,x))\,\mathrm{d}x\mathrm{d}t. \tag{VI.3.9}$$

Hence (VI.3.9) provides that modular convergences

$$\mathbf{D}\boldsymbol{u}^{\delta,\lambda,\varepsilon} \xrightarrow{\varepsilon\to 0} \mathbf{D}\boldsymbol{u}^{\delta,\lambda}, \quad \mathbf{D}\boldsymbol{u}^{\delta,\lambda} \xrightarrow{\lambda\to 1} \mathbf{D}\boldsymbol{u}^{\delta} \text{ in } L_M(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})$$

imply that

$$\boldsymbol{u}^{\delta,\lambda,\varepsilon} \xrightarrow{\varepsilon\to 0} \boldsymbol{u}^{\delta,\lambda}, \quad \boldsymbol{u}^{\delta,\lambda} \xrightarrow{\lambda\to 1} \boldsymbol{u}^{\delta} \text{ modularly in } L_{\underline{m}}(Q;\mathbb{R}^d).$$

Using Proposition III.2.3 for $\mathcal{N}$-functions $\underline{m}^*$ and $\underline{m}$ we obtain

$$\lim_{\varepsilon\to 0,\lambda\to 1}\int_Q \boldsymbol{f}\cdot\boldsymbol{u}^{\delta,\lambda,\varepsilon}\,\mathrm{d}x\mathrm{d}t = \int_Q \boldsymbol{f}\cdot\boldsymbol{u}^{\delta}\,\mathrm{d}x\mathrm{d}t.$$

In a similar way Proposition III.2.3 for $\mathcal{N}-$functions M and M^* provides the convergence

$$\lim_{\varepsilon\to 0,\lambda\to 1}\int_Q \boldsymbol{\chi} : \mathbf{D}\boldsymbol{u}^{\delta,\lambda,\varepsilon}\,\mathrm{d}x\mathrm{d}t = \int_Q \boldsymbol{\chi} : \mathbf{D}\boldsymbol{u}^{\delta}\,\mathrm{d}x\mathrm{d}t.$$

Note that for all $0 < s_0 < s < T$ it follows

$$\begin{aligned}\int_{s_0}^{s}\int_\Omega (\sigma_\delta * \boldsymbol{u})\cdot\partial_t(\sigma_\delta * \boldsymbol{u})\,\mathrm{d}x\mathrm{d}t &= \int_{s_0}^{s}\frac{1}{2}\frac{d}{dt}\|\sigma_\delta * \boldsymbol{u}\|^2_{L^2(\Omega)}\,\mathrm{d}t\\ &= \frac{1}{2}\|\sigma_\delta * \boldsymbol{u}(s)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\sigma_\delta * \boldsymbol{u}(s_0)\|^2_{L^2(\Omega)}.\end{aligned}$$

We pass to the limit with $\delta \to 0$ and obtain for almost all s_0, s (namely for all Lebesgue points of the function $\boldsymbol{u}(t)$) the following identity

$$\lim_{\delta\to 0}\int_{s_0}^{s}\int_\Omega (\boldsymbol{u} * \sigma_\delta)\cdot \partial_t(\boldsymbol{u} * \sigma_\delta)\,\mathrm{d}x\mathrm{d}t = \frac{1}{2}\|\boldsymbol{u}(s)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\boldsymbol{u}(s_0)\|^2_{L^2(\Omega)} \tag{VI.3.10}$$

holds. Observe now the term

$$\int_0^T\int_\Omega \boldsymbol{\chi} : (\sigma_\delta * ((\sigma_\delta * \mathbf{D}\boldsymbol{u})\ \mathbb{1}_{(s_0,s)}))\,\mathrm{d}x\mathrm{d}t = \int_{s_0}^{s}\int_\Omega (\sigma_\delta * \boldsymbol{\chi}) : (\sigma_\delta * \mathbf{D}\boldsymbol{u})\,\mathrm{d}x\mathrm{d}t.$$

Both of the sequences $\{\sigma_\delta * \boldsymbol{\chi}\}_\delta$ and $\{\sigma_\delta * \mathbf{D}\boldsymbol{u}\}_\delta$ converge in measure on Q. Moreover, the assumptions $\boldsymbol{u} \in Y_0^M$ and $\boldsymbol{\chi} \in \mathcal{L}_{M^*}(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ provide that the integrals

$$\int_0^T\int_\Omega M(\mathbf{D}\boldsymbol{u})\,\mathrm{d}x\mathrm{d}t \quad \text{and} \quad \int_0^T\int_\Omega M^*(\boldsymbol{\chi})\,\mathrm{d}x\mathrm{d}t$$

are finite. Hence using the same method as before we conclude that the sequences $\{M^*(\sigma_\delta * \boldsymbol{\chi})\}_\delta$ and $\{M(\sigma_\delta * \mathbf{D}\boldsymbol{u})\}_\delta$ are uniformly integrable and by Lemma III.2.1 we have

$$\begin{aligned} \sigma_\delta * \mathbf{D}\boldsymbol{u} &\xrightarrow{M} \mathbf{D}\boldsymbol{u} \quad \text{modularly in } L_M(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}}),\\ \sigma_\delta * \boldsymbol{\chi} &\xrightarrow{M^*} \boldsymbol{\chi} \quad \text{modularly in } L_{M^*}(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}}). \end{aligned}$$

Applying Proposition III.2.3 allows to conclude

$$\lim_{\delta\to 0}\int_{s_0}^{s}\int_\Omega (\sigma_\delta * \boldsymbol{\chi}) : (\sigma_\delta * \mathbf{D}\boldsymbol{u})\,\mathrm{d}x\mathrm{d}t = \int_{s_0}^{s}\int_\Omega \boldsymbol{\chi} : \mathbf{D}\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t. \tag{VI.3.11}$$

In the same manner we treat the source term, just instead of the $\mathcal{N}-$function M we consider $\underline{m}$. Hence we have

$$\int_0^T\int_\Omega \boldsymbol{f}\cdot(\sigma_\delta * ((\sigma_\delta * \boldsymbol{u})\ \mathbb{1}_{(s_0,s)}))dxdt = \int_{s_0}^{s}\int_\Omega (\sigma_\delta * \boldsymbol{f})\cdot(\sigma_\delta * \boldsymbol{u})dxdt.$$

Then we observe that

$$\begin{aligned} \sigma^\delta * \boldsymbol{u} &\xrightarrow{\underline{m}} \boldsymbol{u} \quad \text{modularly in } L_{\underline{m}}(Q;\mathbb{R}^d),\\ \sigma^\delta * \boldsymbol{f} &\xrightarrow{\underline{m}^*} \boldsymbol{f} \quad \text{modularly in } L_{\underline{m}^*}(Q;\mathbb{R}^d). \end{aligned}$$

and we conclude that

$$\lim_{\delta\to 0}\int_{s_0}^{s}\int_\Omega (\sigma^\delta * \boldsymbol{f})\cdot(\sigma^\delta * \boldsymbol{u})\,\mathrm{d}x\mathrm{d}t = \int_{s_0}^{s}\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t. \tag{VI.3.12}$$

Combining (VI.3.10), (VI.3.11) and (VI.3.12) we obtain after passing to the limit with ε, λ and δ in (VI.3.8) that

$$\frac{1}{2}\|\boldsymbol{u}(s)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\boldsymbol{u}(s_0)\|^2_{L^2(\Omega)} + \int_{s_0}^{s}\int_\Omega \boldsymbol{\chi} : \mathbf{D}\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t = \int_{s_0}^{s}\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t \tag{VI.3.13}$$

for almost all $0 < s_0 < s < T$. □

Lemma VI.3.2 (Non-star-shaped domains with the control of anisotropy)**.** *Let M be an $\mathcal{N}-$function such that $\overline{m}(r) \leqslant c_m((\underline{m}(r))^{\frac{d}{d-1}} + |r|^2 + 1)$ for $r \in \mathbb{R}_+$ and let $\underline{m}$ satisfy the Δ_2-condition. Let Ω be a bounded domain with a sufficiently smooth boundary, Y_0^M, Z_0^M be the function spaces defined by (VI.1.8) and (VI.1.9). Then $Y_0^M = Z_0^M$.*

Moreover, let $\boldsymbol{u} \in Y_0^M$, $\boldsymbol{\chi} \in \mathcal{L}_{M^*}(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})$, $\boldsymbol{f} \in \mathcal{L}_{\underline{m}^*}(Q;\mathbb{R}^d)$ and

$$\partial_t \boldsymbol{u} - \mathrm{div}_x \boldsymbol{\chi} = \boldsymbol{f} \quad \text{in } \mathcal{D}'(Q). \tag{VI.3.14}$$

Then

$$\frac{1}{2}\|\boldsymbol{u}(s)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\boldsymbol{u}(s_0)\|^2_{L^2(\Omega)} + \int_{s_0}^s \int_\Omega \boldsymbol{\chi} : \mathbf{D}\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t = \int_{s_0}^s \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t$$

holds for a.a. s_0, s: $0 \leqslant s_0 < s \leqslant T$.

Proof. Already for Lipschitz domains there exists a finite family of star-shaped domains $\{\Omega_i\}_{i\in J}$ such that

$$\Omega = \bigcup_{i\in J} \Omega_i$$

see e.g. [**99**]. We introduce the partition of unity θ_i with $0 \leqslant \theta_i \leqslant 1$, $\theta_i \in \mathcal{D}(\Omega_i)$, $\operatorname{supp}\theta_i = \Omega_i, \sum_{i\in J}\theta_i(x) = 1$ for $x \in \Omega$. Applying Lemma VI.2.1. to $\underline{m}$, we obtain

$$\int_\Omega (\underline{m}(|\boldsymbol{u}^{\delta,\lambda,\varepsilon}(t,x)|))^{\frac{d}{d-1}}\,\mathrm{d}x \leqslant C_d \left(\int_\Omega \underline{m}(|\mathbf{D}\boldsymbol{u}^{\delta,\lambda,\varepsilon}(t,x)|)\,\mathrm{d}x\right)^{\frac{d}{d-1}}$$

for a.a. $t \in [0,T]$ (here $\boldsymbol{u}^{\delta,\lambda,\varepsilon}$ is defined as in (VI.3.5)). Consequently

$$\int_0^T \int_\Omega (\underline{m}(|\boldsymbol{u}^{\delta,\lambda,\varepsilon}(t,x)|))^{\frac{d}{d-1}}\,\mathrm{d}x\mathrm{d}t \leqslant C_d \int_0^T \left(\int_\Omega (\underline{m}(|\mathbf{D}\boldsymbol{u}^{\delta,\lambda,\varepsilon}(t,x)|)\,\mathrm{d}x\right)^{\frac{d}{d-1}}\,\mathrm{d}t.$$

Using definition of $\underline{m}$ and the assumption that $T < \infty$ we obtain

$$\begin{aligned}\int_0^T \int_\Omega (\underline{m}(|\boldsymbol{u}^{\delta,\lambda,\varepsilon}(t,x)|))^{\frac{d}{d-1}}\,\mathrm{d}x\mathrm{d}t &\leqslant C_d \int_0^T \left(\int_\Omega M(\mathbf{D}\boldsymbol{u}^{\delta,\lambda,\varepsilon}(t,x))\,\mathrm{d}x\right)^{\frac{d}{d-1}}\,\mathrm{d}t \\ &\leqslant C_{T,d} \sup_{t\in[0,T]} \left(\int_\Omega M(\mathbf{D}\boldsymbol{u}^{\delta,\lambda,\varepsilon}(t,x))\,\mathrm{d}x\right)^{\frac{d}{d-1}}.\end{aligned} \tag{VI.3.15}$$

To show boundedness of the right-hand side of (VI.3.15) for fixed δ we use the Jensen inequality, the Fubini theorem and nonnegativity of M in the following way

$$\begin{aligned}\int_\Omega M(\mathbf{D}\boldsymbol{u}^{\delta,\lambda,\varepsilon}(t,x))\,\mathrm{d}x &\leqslant \int_\Omega\int_{B_\delta} M(\mathbf{D}\boldsymbol{u}^{\lambda,\varepsilon}(t-\tau,x))\sigma_\delta(\tau)\,\mathrm{d}\tau\mathrm{d}x \\ &= \int_{B_\delta}\int_\Omega M(\mathbf{D}\boldsymbol{u}^{\lambda,\varepsilon}(t-\tau,x))\sigma_\delta(\tau)\,\mathrm{d}x\mathrm{d}\tau \\ &\leqslant \|\sigma_\delta\|_{L^\infty(B_\delta)}\|M(\mathbf{D}\boldsymbol{u}^{\lambda,\varepsilon})\|_{L^1(B_\delta\times\Omega)} \\ &\leqslant \|\sigma_\delta\|_{L^\infty(B_\delta)}\|M(\mathbf{D}\boldsymbol{u}^{\lambda,\varepsilon})\|_{L^1(Q)}.\end{aligned} \tag{VI.3.16}$$

Since $\overline{m}(r) \leqslant c_m((\underline{m}(r))^{\frac{d}{d-1}} + |r|^2 + 1)$ and $\nabla_x\theta \in L^\infty(\Omega;\mathbb{R}^d)$ we obtain

$$(\mathbf{D}(\boldsymbol{u}^{\delta,\lambda})\theta_i^\lambda)^\varepsilon + \frac{1}{2}(\boldsymbol{u}^\delta \otimes \nabla_x\theta_i)^{\lambda,\varepsilon} + \frac{1}{2}(\nabla_x\theta_i \otimes \boldsymbol{u}^\delta)^{\lambda,\varepsilon} = \mathbf{D}(\boldsymbol{u}^\delta\theta_i)^{\lambda,\varepsilon} \in L_M((0,T)\times\Omega_i;\mathbb{R}^{d\times d}_{\text{sym}}),$$

where $\Omega_i = \operatorname{supp}\theta_i$.

We observe now the function $\boldsymbol{u}^{\delta,\lambda,\varepsilon}(t,x) = \sum_{i\in J}\varrho_\varepsilon * \left\{\left(\sigma_\delta * \boldsymbol{u}\ \mathbb{1}_{(s_0,s)}\right)\theta_i\right\}^\lambda$, where $\{\cdot\}^\lambda$ is defined by (VI.3.4). Since $\boldsymbol{u}^{\delta,\lambda,\varepsilon}$ is in general not divergence-free, we introduce for a.a. $t \in (0,T)$ the function $\boldsymbol{\varphi}^{\lambda,\varepsilon}(t,\cdot) \in L_{\underline{m}^{\frac{d}{d-1}}}(\Omega;\mathbb{R}^d)$ which for a.a. $t \in (0,T)$ is a solution to the problem

$$\begin{aligned} \operatorname{div}_x\boldsymbol{\varphi}^{\lambda,\varepsilon} &= \sum_{i\in J}\varrho_\varepsilon * \left\{\left(\sigma_\delta * \boldsymbol{u}\ \mathbb{1}_{(s_0,s)}\right)\cdot\nabla_x\theta_i\right\}^\lambda \quad \text{in } \Omega \\ \boldsymbol{\varphi}^{\lambda,\varepsilon} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Existence of such $\boldsymbol{\varphi}^{\lambda,\varepsilon}$ is provided by Theorem III.2.14 applied to the $\mathcal{N}$-function $\underline{m}^{\frac{d}{d-1}}$ which satisfies the Δ_2-condition. The quasiconvexity condition is obviously satisfied with $\gamma = \frac{d-1}{d}$. Then we follow the case of star-shaped domains to complete the proof, but instead of the sequence defined by (VI.3.5), we consider

$$\boldsymbol{\psi}^{\delta,\lambda,\varepsilon}(t,x) := \sum_{i\in J}\varrho_\varepsilon * \left\{\left(\sigma_\delta * \boldsymbol{u}\ \mathbb{1}_{(s_0,s)}\right)\theta_i\right\}^\lambda - \boldsymbol{\varphi}^{\lambda,\varepsilon}(x)$$

It remains to show that $\boldsymbol{\varphi}^{\lambda,\varepsilon}$ vanishes in the limit as $\lambda \to 1$ and $\varepsilon \to 0$. Indeed, Theorem III.2.14 implies the estimate

$$\begin{aligned} \int_\Omega \underline{m}^{\frac{d}{d-1}}(|\mathbf{D}\boldsymbol{\varphi}^{\lambda,\varepsilon}|)\,\mathrm{d}x &\leqslant \int_\Omega \underline{m}^{\frac{d}{d-1}}(|\nabla_x\boldsymbol{\varphi}^{\lambda,\varepsilon}|)\,\mathrm{d}x \\ &\leqslant c\int_\Omega \underline{m}^{\frac{d}{d-1}}(|\sum_{i\in J}\varrho_\varepsilon * \left\{\left(\sigma_\delta * \boldsymbol{u}\ \mathbb{1}_{(s_0,s)}\right)\cdot\nabla_x\theta_i\right\}^\lambda|) \end{aligned} \tag{VI.3.17}$$

for a.a. $t \in (0,T)$. Let us integrate (VI.3.17) over the time interval $(0,T)$. Since for every $i \in J$ the sequence

$$\varrho_\varepsilon * \left\{\left(\sigma_\delta * \boldsymbol{u}\ \mathbb{1}_{(s_0,s)}\right)\cdot\nabla_x\theta_i\right\}^\lambda \overset{\underline{m}^{\frac{d}{d-1}}}{\longrightarrow} \left(\sigma_\delta * \boldsymbol{u}\ \mathbb{1}_{(s_0,s)}\right)\cdot\nabla_x\theta_i \text{ modularly in } L_{\underline{m}^{\frac{d}{d-1}}}(Q)$$

as $\varepsilon \to 0$ and $\lambda \to 1$ and $\sum_{i\in J}\left(\sigma_\delta * \boldsymbol{u}\ \mathbb{1}_{(s_0,s)}\right)\cdot\nabla_x\theta_i = 0$, we immediately conclude that

$$\sum_{i\in J}\varrho_\varepsilon * \left\{\left(\sigma_\delta * \boldsymbol{u}\ \mathbb{1}_{(s_0,s)}\right)\cdot\nabla_x\theta_i\right\}^\lambda \overset{\underline{m}^{\frac{d}{d-1}}}{\longrightarrow} 0 \text{ modularly in } L_{\underline{m}^{\frac{d}{d-1}}}(Q)$$

as $\varepsilon \to 0$ and $\lambda \to 1$. Consequently

$$\mathbf{D}\boldsymbol{\varphi}^{\lambda,\varepsilon} \overset{\underline{m}^{\frac{d}{d-1}}}{\longrightarrow} 0 \text{ modularly in } L_{\underline{m}^{\frac{d}{d-1}}}(Q;\mathbb{R}^{d\times d}). \tag{VI.3.18}$$

Employing the same argumentation, instead of the function defined by (VI.3.7), we test (VI.3.14) with

$$\boldsymbol{\zeta}^{\delta,\lambda,\varepsilon}(t,x) := \sum_{i\in J}\varrho_\varepsilon * \left\{\sigma_\delta * \left(\sigma_\delta * \boldsymbol{u}\ \mathbb{1}_{(s_0,s)}\right)\theta_i\right\}^\lambda - \sigma_\delta * \left(\sigma_\delta * \boldsymbol{\varphi}^{\lambda,\varepsilon}(t,x)\ \mathbb{1}_{(s_0,s)}\right). \tag{VI.3.19}$$

Passing to the limit with $\lambda \to 1$ and $\varepsilon \to 0$ in (VI.3.8) it again remains to show that the second term on the right-hand side of (VI.3.19) converges to zero, i.e., the following three limits vanish

$$\text{(VI.3.20)}\quad \begin{aligned} &\lim_{\varepsilon\to 0,\lambda\to 1}\int_{s_0}^{s}\int_\Omega (\boldsymbol{u} * \sigma_\delta)\cdot \partial_t\left(\sigma_\delta * \boldsymbol{\varphi}^{\lambda,\varepsilon}(t,x)\ \mathbb{1}_{(s_0,s)}\right)\,\mathrm{d}x\mathrm{d}t = 0,\\ &\lim_{\varepsilon\to 0,\lambda\to 1}\int_0^T\int_\Omega \boldsymbol{\chi} : \sigma_\delta * \left(\sigma_\delta * \mathbf{D}\boldsymbol{\varphi}^{\lambda,\varepsilon}(t,x)\ \mathbb{1}_{(s_0,s)}\right)\,\mathrm{d}x\mathrm{d}t = 0,\\ &\lim_{\varepsilon\to 0,\lambda\to 1}\int_0^T\int_\Omega \boldsymbol{f}\cdot \sigma_\delta * \left(\sigma_\delta * \boldsymbol{\varphi}^{\lambda,\varepsilon}(t,x)\ \mathbb{1}_{(s_0,s)}\right)\,\mathrm{d}x\mathrm{d}t = 0. \end{aligned}$$

To show $(\text{VI.3.20})_1$ we apply Theorem III.2.14 with the $\mathcal{N}$–function $m = |\cdot|^2$ and the Poincaré inequality, which allow to conclude that

$$\text{(VI.3.21)}\quad \|\boldsymbol{\varphi}^{\lambda,\varepsilon}\|_{L^2(\Omega)} \leqslant c_1\|\nabla_x\boldsymbol{\varphi}^{\lambda,\varepsilon}\|_{L^2(\Omega)} \leqslant c_2\|\sum_{i\in J}\varrho_\varepsilon * \left\{\left(\sigma_\delta * \boldsymbol{u}\ \mathbb{1}_{(s_0,s)}\right)\cdot\nabla_x\theta_i\right\}^\lambda\|_{L^2(\Omega)}$$

for a.a. $t \in (0,T)$. Since the term on the left-hand side of $(\text{VI.3.20})_1$ is equivalent to $\int_{s_0}^s\int_\Omega(\boldsymbol{u} * \sigma_\delta)\cdot\left(\boldsymbol{\varphi}^{\lambda,\varepsilon} * \partial_t\sigma_\delta\right)\,\mathrm{d}x\mathrm{d}t$, we pass to the limit using the fact that

$$\sum_{i\in J}\varrho_\varepsilon * \left\{\left(\sigma_\delta * \boldsymbol{u}\ \mathbb{1}_{(s_0,s)}\right)\cdot\nabla_x\theta_i\right\}^\lambda \overset{*}{\rightharpoonup} 0 \text{ weakly–}(*) \text{ in } L^\infty(0,T;L^2(\Omega)),$$

thus

$$\boldsymbol{\varphi}^{\lambda,\varepsilon} \overset{*}{\rightharpoonup} 0 \text{ weakly–}(*) \text{ in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))$$

as $\varepsilon \to 0$ and $\lambda \to 1$.

Since $\mathbf{D}\boldsymbol{\varphi}^{\lambda,\varepsilon}$ converges modularly to zero in $L_{\underline{m}^{\frac{n}{n-1}}}(Q;\mathbb{R}^{d\times d}_{\text{sym}})$ (see (VI.3.18)), $\overline{m}(r) \leqslant c_m((\underline{m}(r))^{\frac{d}{d-1}} + |r|^2 + 1)$ and (VI.3.21) holds, then $M(\alpha\mathbf{D}\boldsymbol{\varphi}^{\lambda,\varepsilon})$ is uniformly integrable with some $\alpha > 0$. Moreover, by Lemma III.2.1 the modular convergence in $L_{\underline{m}^{\frac{d}{d-1}}}(Q;\mathbb{R}^{d\times d}_{\text{sym}})$ to zero implies the convergence in measure to zero. Hence using again Lemma III.2.1 with a function M we conclude that $\mathbf{D}\boldsymbol{\varphi}^{\lambda,\varepsilon} \to 0$ modularly in $L_M(Q;\mathbb{R}^{d\times d}_{\text{sym}})$ as $\varepsilon \to 0$ and $\lambda \to 1$. Therefore $(\text{VI.3.20})_2$ is satisfied.

Finally, the convergence passage in $(\text{VI.3.20})_3$ is a consequence of (VI.3.18). It implies that $\nabla_x\boldsymbol{\varphi}^{\lambda,\varepsilon} \to 0$ modularly in $L_{\underline{m}}(Q;\mathbb{R}^{d\times d})$ and since $\boldsymbol{\varphi} = 0$ on $\partial\Omega$ we obtain $\boldsymbol{\varphi}^{\lambda,\varepsilon} \to 0$ modularly in $L_{\underline{m}}(Q;\mathbb{R}^d)$ as $\varepsilon \to 0$ and $\lambda \to 1$.

Now we follow the case of star-shaped domains to complete the proof. □

VI.4. Existence result

The first part of the proof is standard. However we recall it for completeness of the chapter.

We construct Galerkin approximations to (VI.1.1) - (VI.1.4) using basis $\{\boldsymbol{\omega}_i\}_{i=1}^\infty$ consisting of eigenvectors of the Stokes operator. We define $\boldsymbol{u}^k = \sum_{i=1}^k \alpha_i^k(t)\boldsymbol{\omega}_i$, where

$\alpha_i^k(t)$ solve the system

$$\text{(VI.4.1)} \qquad \begin{aligned} \int_\Omega \frac{d}{dt}\boldsymbol{u}^k \cdot \boldsymbol{\omega}_i + \int_\Omega \mathbf{S}(t,x,\mathbf{D}\boldsymbol{u}^k) : \mathbf{D}\boldsymbol{\omega}_i \,\mathrm{d}x &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\omega}_i \,\mathrm{d}x, \\ \boldsymbol{u}^k(0) &= P^k \boldsymbol{u}_0. \end{aligned}$$

where $i = 1,\dots,k$ and by P^k we denote the orthogonal projection of $L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d)$ on $\mathrm{conv}\{\boldsymbol{\omega}_1,\dots,\boldsymbol{\omega}_k\}$. Multiplying each equation of (VI.4.1) by $\alpha_i^k(t)$, summing over $i = 1,\dots,k$ we obtain

$$\text{(VI.4.2)} \qquad \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}^k\|^2_{L^2(\Omega)} + \int_\Omega \mathbf{S}(t,x,\mathbf{D}\boldsymbol{u}^k) : \mathbf{D}\boldsymbol{u}^k \,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}^k\,\mathrm{d}x.$$

The Fenchel-Young inequality, the Hölder inequality, Lemma VI.2.1 and convexity of the $\mathcal{N}-$function provide that

$$\text{(VI.4.3)} \qquad \begin{aligned} \left|\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}^k\,\mathrm{d}x\right| &\leqslant \int_\Omega \left|\frac{2\tilde{c}}{c}\boldsymbol{f}\cdot\frac{c}{2\tilde{c}}\boldsymbol{u}^k\right|\,\mathrm{d}x \\ &\leqslant \int_\Omega \underline{m}^*\left(\frac{2\tilde{c}}{c}|\boldsymbol{f}|\right)\mathrm{d}x + \int_\Omega \underline{m}\left(\frac{c}{2\tilde{c}}|\boldsymbol{u}^k|\right)\mathrm{d}x \\ &\leqslant \int_\Omega \underline{m}^*\left(\frac{2\tilde{c}}{c}|\boldsymbol{f}|\right)\mathrm{d}x + |\Omega|^{\frac{1}{d}}\left(\int_\Omega \underline{m}\left(\frac{c}{2\tilde{c}}|\boldsymbol{u}^k|\right)dx\right)^{\frac{d-1}{d}} \\ &\leqslant \int_\Omega \underline{m}^*\left(\frac{2\tilde{c}}{c}|\boldsymbol{f}|\right)\mathrm{d}x + |\Omega|^{\frac{1}{d}} C_n \int_\Omega \underline{m}\left(\frac{c}{2\tilde{c}}|\mathbf{D}\boldsymbol{u}^k|\right)\mathrm{d}x \\ &\leqslant \int_\Omega \underline{m}^*\left(\frac{2\tilde{c}}{c}|\boldsymbol{f}|\right)\mathrm{d}x + \frac{c}{2}\int_\Omega M\left(\mathbf{D}\boldsymbol{u}^k\right)\mathrm{d}x. \end{aligned}$$

In the above considerations we choose a constant such that $\max(|\Omega|^{\frac{1}{d}}C_d, \frac{c}{2}) < \tilde{c} < \infty$, where C_d is coming from Lemma VI.2.1. The last inequality follows from the fact that M is a convex function, $M(0) = 0$ and $0 < c \leqslant 1$, which is an obvious consequence of combining (VI.1.5) with the Fenchel-Young inequality. Integrating (VI.4.2) over the time interval $(0,t)$ with $t \leqslant T$, using estimate (VI.4.3) and the coercivity condition (S2) on $\mathbf{S}$ we obtain

$$\text{(VI.4.4)} \qquad \begin{aligned} \frac{1}{2}\|\boldsymbol{u}^k(t)\|^2_{L^2(\Omega)} + \frac{c}{2}\int_0^t\int_\Omega M(\mathbf{D}\boldsymbol{u}^k)\,\mathrm{d}x\mathrm{d}t + c\int_0^t\int_\Omega M^*(\mathbf{S}(t,x,\mathbf{D}\boldsymbol{u}^k))\,\mathrm{d}x\mathrm{d}t \\ \leqslant \int_0^t\int_\Omega \underline{m}^*(\frac{2\tilde{c}}{c}|\boldsymbol{f}|)\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\|\boldsymbol{u}_0\|^2_{L^2(\Omega)}, \end{aligned}$$

for all $t \in (0,T]$. Hence there exists a subsequence such that

$$\mathbf{D}\boldsymbol{u}^k \overset{*}{\rightharpoonup} \mathbf{D}\boldsymbol{u} \quad \text{weakly–(*) in} \quad L_M(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}})$$

and

$$\mathbf{S}(\cdot,\cdot,\mathbf{D}\boldsymbol{u}^k) \overset{*}{\rightharpoonup} \boldsymbol{\chi} \quad \text{weakly–(*) in} \quad L_{M^*}(Q;\mathbb{R}^{d\times d}_{\mathrm{sym}}).$$

Moreover from (VI.4.4) we conclude the uniform boundedness of the sequence $\{\boldsymbol{u}^k\}_{k=1}^{\infty}$ in the space $L^\infty(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d))$ and as an immediate conclusion, we have at least for a subsequence

$$\boldsymbol{u}^k \overset{*}{\rightharpoonup} \boldsymbol{u} \quad \text{weakly–(*) in } L^\infty(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d)).$$

After passing to the limit we obtain the following limit identity

$$\text{(VI.4.5)} \qquad -\int_Q \boldsymbol{u}\cdot\partial_t\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t + \int_Q \boldsymbol{\chi} : \mathbf{D}\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t = \int_Q \boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t - \int_\Omega \boldsymbol{u}_0\cdot\boldsymbol{\varphi}(0,x)\,\mathrm{d}x$$

for all $\boldsymbol{\varphi}\in\mathcal{D}((-\infty,T);\mathcal{V})$.

In the remaining steps we will concentrate on characterizing the limit $\boldsymbol{\chi}$. Since the weak-star and modular limits coincide, Lemma VI.3.1 for star-shaped domains or Lemma VI.3.2 for non-star-shaped domains and the equality (VI.4.5) provide

$$\text{(VI.4.6)} \qquad \frac{1}{2}\|\boldsymbol{u}(s)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\boldsymbol{u}(s_0)\|^2_{L^2(\Omega)} + \int_{s_0}^{s}\int_\Omega \boldsymbol{\chi} : \mathbf{D}\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t = \int_{s_0}^{s}\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t$$

for a.a. $0 < s_0 < s < T$. To pass to the limit with $s_0\to 0$ we need to establish the weak continuity of $\boldsymbol{u}$ in $L^2(\Omega;\mathbb{R}^d)$ w.r.t. time. For this purpose we consider again the sequence $\{\frac{\mathrm{d}\boldsymbol{u}^k}{\mathrm{d}t}\}$ and provide uniform estimates. Let $\boldsymbol{\varphi}\in L^\infty(0,T;W^{r,2}_{0,\mathrm{div}})$, $\|\boldsymbol{\varphi}\|_{L^\infty(0,T;W^{r,2}_{0,\mathrm{div}})}\leqslant 1$, where

$$W^{r,2}_{0,\mathrm{div}} = \text{closure of } \mathcal{V} \text{ w.r.t. the } W^{r,2}(\Omega)\text{−norm}$$

where $r > \frac{d}{2}+1$ and observe that

$$\left\langle \frac{d\boldsymbol{u}^k}{dt},\boldsymbol{\varphi}\right\rangle = \left\langle \frac{d\boldsymbol{u}^k}{dt},P^k\boldsymbol{\varphi}\right\rangle = -\int_\Omega \mathbf{S}(t,x,\mathbf{D}\boldsymbol{u}^k) : \mathbf{D}(P^k\boldsymbol{\varphi})\,\mathrm{d}x + \int_\Omega \boldsymbol{f}\cdot(P^k\boldsymbol{\varphi})\,\mathrm{d}x.$$

Since $\|P^k\boldsymbol{\varphi}\|_{W^{r,2}_{0,\mathrm{div}}}\leqslant\|\boldsymbol{\varphi}\|_{W^{r,2}_{0,\mathrm{div}}}$ and $W^{r-1,2}(\Omega)\subset L^\infty(\Omega)$ we estimate as follows

(VI.4.7)

$$\begin{aligned}\Big|\int_0^T\int_\Omega \mathbf{S}(t,x,\mathbf{D}\boldsymbol{u}^k) : \mathbf{D}(P^k\boldsymbol{\varphi})\,\mathrm{d}x\mathrm{d}t\Big| &\leqslant \int_0^T \|\mathbf{S}(t,\cdot,\mathbf{D}\boldsymbol{u}^k)\|_{L^1(\Omega)}\|\mathbf{D}(P^k\boldsymbol{\varphi})\|_{L^\infty(\Omega)}\,\mathrm{d}t\\ &\leqslant c\int_0^T \|\mathbf{S}(t,\cdot,\mathbf{D}\boldsymbol{u}^k)\|_{L^1(\Omega)}\|P^k\boldsymbol{\varphi}\|_{W^{r,2}_{0,\mathrm{div}}}\,\mathrm{d}t \leqslant c\int_0^T \|\mathbf{S}(t,\cdot,\mathbf{D}\boldsymbol{u}^k)\|_{L^1(\Omega)}\|\boldsymbol{\varphi}\|_{W^{r,2}_{0,\mathrm{div}}}\,\mathrm{d}t\\ &\leqslant c\|\mathbf{S}(\cdot,\cdot,\mathbf{D}\boldsymbol{u}^k)\|_{L^1(Q)}\|\boldsymbol{\varphi}\|_{L^\infty(0,T;W^{r,2}_{0,\mathrm{div}})}\end{aligned}$$

and

$$\text{(VI.4.8)} \qquad \begin{aligned}\Big|\int_0^T\int_\Omega \boldsymbol{f}\cdot P^k\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t\Big| &\leqslant \int_0^T \|\boldsymbol{f}\|_{L^1(\Omega)}\|P^k\boldsymbol{\varphi}\|_{L^\infty(\Omega)}\,\mathrm{d}t\\ &\leqslant c\int_0^T \|\boldsymbol{f}\|_{L^1(\Omega)}\|P^k\boldsymbol{\varphi}\|_{W^{r,2}_{0,\mathrm{div}}}\,\mathrm{d}t \leqslant c\int_0^T \|\boldsymbol{f}\|_{L^1(\Omega)}\|\boldsymbol{\varphi}\|_{W^{r,2}_{0,\mathrm{div}}}\,\mathrm{d}t\\ &\leqslant c\|\boldsymbol{f}\|_{L^1(Q)}\|\boldsymbol{\varphi}\|_{L^\infty(0,T;W^{r,2}_{0,\mathrm{div}})}.\end{aligned}$$

The assumptions on $\boldsymbol{f}$ and uniform estimates for $\mathbf{S}(\cdot,\cdot,\mathbf{D}\boldsymbol{u}^k)$ in a proper Orlicz class provide integrability of the above functions. Hence we conclude that $\frac{\mathrm{d}\boldsymbol{u}^k}{\mathrm{d}t}$ is bounded in $L^1(0,T;V_r^*)$. By (VI.4.4) and assumptions on $\boldsymbol{f}$ there exists a constant $C>0$ such that

$$\sup_{k\in\mathbb{N}}\int_Q \left[M(\mathbf{S}(t,x,\mathbf{D}\boldsymbol{u}^k)+\underline{m}^*(|\boldsymbol{f}|)\right]\,\mathrm{d}x\mathrm{d}t \leqslant C.$$

Consequently using the Jensen inequality we obtain

$$\sup_{k\in\mathbb{N}}|\Omega|\int_0^T \left[\underline{m}(\|\mathbf{S}(t,\cdot,\mathbf{D}\boldsymbol{u}^k\|_{L^1(\Omega)})+\underline{m}^*(\|\boldsymbol{f}\|_{L^1(\Omega)})\right]\,\mathrm{d}t < C$$

and hence we conclude by Lemma III.2.2 that there exists a monotone, continuous function $L:\mathbb{R}_+\to\mathbb{R}_+$, with $L(0)=0$ which is independent of k and

$$\int_{s_1}^{s_2}\left(\|\mathbf{S}(t,\cdot,\mathbf{D}\boldsymbol{u}^k)\|_{L^1(\Omega)}+\|\boldsymbol{f}\|_{L^1(\Omega)}\right)\,\mathrm{d}t \leqslant L(|s_1-s_2|)$$

for any $s_1,s_2\in[0,T]$. Consequently, estimates (VI.4.7)-(VI.4.8) provide that

$$\left|\int_{s_1}^{s_2}\left\langle \frac{\mathrm{d}\boldsymbol{u}^k}{\mathrm{d}t},\boldsymbol{\varphi}\right\rangle\,\mathrm{d}t\right| \leqslant L(|s_1-s_2|)$$

for all $\boldsymbol{\varphi}$ with $\operatorname{supp}\boldsymbol{\varphi}\subset(s_1,s_2)\subset[0,T]$ and $\|\boldsymbol{\varphi}\|_{L^\infty(0,T;W^{r,2}_{0,\mathrm{div}})}\leqslant 1$. The following estimates

$$\begin{aligned}
&\|\boldsymbol{u}^k(s_1)-\boldsymbol{u}^k(s_2)\|_{(W^{r,2}_{0,\mathrm{div}})^*}\\
&\quad=\sup_{\substack{\boldsymbol{\psi}\in W^{r,2}_{0,\mathrm{div}}\\ \|\boldsymbol{\psi}\|_{W^{r,2}_{0,\mathrm{div}}}\leqslant 1}}\left|\left\langle \boldsymbol{u}^k(s_1)-\boldsymbol{u}^k(s_2),\boldsymbol{\psi}\right\rangle\right| = \sup_{\|\boldsymbol{\psi}\|_{W^{r,2}_{0,\mathrm{div}}}\leqslant 1}\left|\left\langle\int_{s_1}^{s_2}\frac{\mathrm{d}\boldsymbol{u}^k(t)}{\mathrm{d}t},\boldsymbol{\psi}\right\rangle\right|\\
&\quad\leqslant \sup\left\{\int_0^T\left|\left\langle\frac{\mathrm{d}\boldsymbol{u}^k(\tau)}{\mathrm{d}\tau},\boldsymbol{\varphi}\right\rangle\right|\,\mathrm{d}t \ :\ \|\boldsymbol{\varphi}\|_{L^\infty(0,T;W^{r,2}_{0,\mathrm{div}})}\leqslant 1,\ \operatorname{supp}\boldsymbol{\varphi}\subset(s_1,s_2)\right\}
\end{aligned}$$

imply that

$$\sup_{k\in\mathbb{N}}\|\boldsymbol{u}^k(s_1)-\boldsymbol{u}^k(s_2)\|_{(W^{r,2}_{0,\mathrm{div}})^*}\leqslant L(|s_1-s_2|). \tag{VI.4.9}$$

Since $\boldsymbol{u}\in L^\infty(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d))$, we can choose a sequence $\{s_0^i\}_i$, $s_0^i\to 0^+$ as $i\to\infty$. Thus $\{\boldsymbol{u}(s_0^i)\}_i$ is weakly convergent in $L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d)$. The estimate (VI.4.9) provides that the family of functions $\boldsymbol{u}^k:[0,T]\to(W^{r,2}_{0,\mathrm{div}})^*$ is equicontinuous. Using the uniform bound in $L^\infty(0,T;L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d))$ and the compact embedding $L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d)\subset\subset(W^{r,2}_{0,\mathrm{div}})^*$ we conclude by means of the Arzelà-Ascoli theorem that the sequence $\{\boldsymbol{u}^k\}_{k=1}^\infty$ is relatively compact in $C([0,T];(W^{r,2}_{0,\mathrm{div}})^*)$ and $\boldsymbol{u}\in C([0,T];(W^{r,2}_{0,\mathrm{div}})^*)$. Consequently we obtain that

$$\boldsymbol{u}(s_0^i)\stackrel{i\to\infty}{\longrightarrow}\boldsymbol{u}(0)\quad\text{in } (W^{r,2}_{0,\mathrm{div}})^*. \tag{VI.4.10}$$

The limit coincides with the weak limit of $\{\boldsymbol{u}(s_0^i)\}_{i=1}^{\infty}$ in $L^2_{\mathrm{div}}(\Omega;\mathbb{R}^d)$ and hence we conclude

$$\liminf_{i\to\infty}\|\boldsymbol{u}(s_0)\|_{L^2(\Omega)} \geqslant \|\boldsymbol{u}_0\|_{L^2(\Omega)}. \tag{VI.4.11}$$

Let s be any Lebesgue point of $\boldsymbol{u}$. Integrating (VI.4.2) over the time interval $(0,s)$ gives

(VI.4.12)

$$\begin{aligned}
\limsup_{k\to\infty}\int_0^s\int_\Omega \mathbf{S}(t,x,\mathbf{D}\boldsymbol{u}^k) : \mathbf{D}\boldsymbol{u}^k \,\mathrm{d}x\mathrm{d}t
&= \int_0^s\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\|\boldsymbol{u}_0\|^2_{L^2(\Omega)} - \liminf_{k\to\infty}\frac{1}{2}\|\boldsymbol{u}^k(s)\|^2_{L^2(\Omega)}\\
&\leqslant \int_0^s\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\|\boldsymbol{u}_0\|^2_{L^2(\Omega)} - \frac{1}{2}\|\boldsymbol{u}(s)\|^2_{L^2(\Omega)}\\
&\overset{(VI.4.11)}{\leqslant} \liminf_{i\to\infty}\left(\int_{s_0^i}^s\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\|\boldsymbol{u}(s_0^i)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\boldsymbol{u}(s)\|^2_{L^2(\Omega)}\right)\\
&\overset{(VI.4.6)}{=} \lim_{i\to\infty}\int_{s_0^i}^s\int_\Omega \boldsymbol{\chi} : \mathbf{D}\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t = \int_0^s\int_\Omega \boldsymbol{\chi} : \mathbf{D}\boldsymbol{u}\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

The monotonicity of $\mathbf{S}$ yields

$$\int_0^s\int_\Omega (\mathbf{S}(t,x,\bar{\mathbf{v}}) - \mathbf{S}(t,x,\mathbf{D}\boldsymbol{u}^k)) : (\bar{\mathbf{v}} - \mathbf{D}\boldsymbol{u}^k)\,\mathrm{d}x\mathrm{d}t \geqslant 0 \tag{VI.4.13}$$

for all $\bar{\mathbf{v}} \in L^\infty(Q;\mathbb{R}^{d\times d})$. Using (VI.4.12) and (VI.4.13) we follow the same steps as in Chapter IV or in [**75, 131**] to show $\boldsymbol{\chi} = \mathbf{S}(t,x,\mathbf{D}\boldsymbol{u})$ a.e. in Q.

□

CHAPTER VII

Renormalized solutions of nonlinear elliptic problems

VII.1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d \geqslant 1)$ with a sufficiently smooth boundary $\partial\Omega$. Our aim is to show existence and uniqueness of renormalized solutions to the following nonlinear elliptic inclusion

$$(E,f) \qquad \begin{aligned} \beta(x,u) - \operatorname{div}\left(\boldsymbol{a}(x,\nabla_x u) + \boldsymbol{F}(u)\right) \ni f \text{ in } \Omega,\\ u = 0 \text{ on } \partial\Omega, \end{aligned}$$

with a right-hand side $f \in L^1(\Omega)$. The function $\boldsymbol{F} : \mathbb{R} \to \mathbb{R}^d$ is assumed to be locally Lipschitz and $\boldsymbol{a} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies the following assumptions:

(A1): $\boldsymbol{a}(\cdot,\cdot)$ is a Carathéodory function.

(A2): there exist a generalized $\mathcal{N}$-function $M : \Omega \times \mathbb{R}^d \to \mathbb{R}_+$ (see Definition III.1.5 below), a constant $c_a \in (0,1]$ and a nonnegative function $a_0 \in L^1(\Omega)$ such that

$$\boldsymbol{a}(x,\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geqslant c_a\{M^*(x,\boldsymbol{a}(x,\boldsymbol{\xi})) + M(x,\boldsymbol{\xi})\} - a_0(x) \tag{VII.1.1}$$

for a.a. $x \in \Omega$ and for every $\boldsymbol{\xi} \in \mathbb{R}^d$, where M^* is the conjugate function to M (see relation (III.1.3)).

(A3): $\boldsymbol{a}(\cdot,\cdot)$ is monotone, i.e.,

$$(\boldsymbol{a}(x,\boldsymbol{\xi}) - \boldsymbol{a}(x,\boldsymbol{\eta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) \geqslant 0 \tag{VII.1.2}$$

for a.a. $x \in \Omega$ and for every $\boldsymbol{\xi},\ \boldsymbol{\eta} \in \mathbb{R}^d$.

Moreover, we assume that the complementary function

$$M^* \text{ satisfies the } \Delta_2-\text{condition} \quad \text{and} \quad \lim_{|\boldsymbol{\xi}|\to\infty} \inf_{x\in\Omega} \frac{M^*(x,\boldsymbol{\xi})}{|\boldsymbol{\xi}|} = \infty \tag{VII.1.3}$$

and there exist $c > 0$, $\nu > 0$ and $\boldsymbol{\xi}_0 \in \mathbb{R}^d$ such that

$$M(x,\boldsymbol{\xi}) \geqslant c|\boldsymbol{\xi}|^{1+\nu} \tag{VII.1.4}$$

for a.a. $x \in \Omega$ and for $\boldsymbol{\xi} \in \mathbb{R}^d$, $|\boldsymbol{\xi}| \geqslant |\boldsymbol{\xi}_0|$. Let us notice that if the function $g_M \in L^\infty(\Omega)$ in the definition of the Δ_2-condition for M^* (see (III.1.15)), then (VII.1.4) is a consequence of the assumption (VII.1.3) (see Proposition III.2.12). However, no growth restriction is made on the $\mathcal{N}-$function M itself.

An example of an operator $\boldsymbol{a}$ satisfying our assumptions with an $\mathcal{N}-$function M which does not satisfy the Δ_2-condition is as follows:

$$\boldsymbol{a}(x,\boldsymbol{\xi}) = a_1(x)\xi_1 \exp(a_1(x)\xi_1)^2 + a_2(x)\xi_2 \exp(a_2(x)\xi_1)^2,$$

$$M(x,\boldsymbol{\xi}) = \frac{1}{2}\left(\exp(a_1(x)\xi_1^2) + \exp(a_1(x)\xi_2^2)\right),$$

where $a_1, a_2 : \Omega \to \mathbb{R}$ are measurable functions strictly greater than zero and $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2$.

As to the nonlinearity β in the problem (E, f) we assume that $\beta : \Omega \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \varnothing$ is a set-valued mapping such that, for almost every $x \in \Omega$, $\beta(x, \cdot) : \mathbb{R} \to 2^{\mathbb{R}} \backslash \varnothing$ is a maximal monotone operator with $0 \in \beta(x, 0)$. Moreover, we assume that

$$\beta^0(\cdot, l) \in L^1(\Omega) \tag{VII.1.5}$$

for each $l \in \mathbb{R}$, where β^0 denotes the minimal selection of the graph of β. Namely $\beta_0(x, l)$ is the minimal in the norm element of $\beta(x, l)$,

$$\beta_0(x, l) = \inf\{|r| \mid r \in \mathbb{R} \text{ and } r \in \beta(x, l)\}$$

There already exists a vast literature on problems of this type. Most of the literature has been devoted to the study of the case where the vector field $\boldsymbol{a}$ satisfies a polynomial growth (and coerciveness) condition. A model example of this type is the homogeneous Dirichlet boundary value problem for the p-Laplacian $\Delta_p(u) = \operatorname{div}_x(|\nabla u|^{p-2}\nabla u)$, i.e. the equation

$$\beta(x, u) - \Delta_p(u) - \operatorname{div}_x \boldsymbol{F}(u) \ni f.$$

It is well-known, even in this particular case, that for L^1-data a weak solution may not exist in general or may not be unique. In order to obtain well-posedness for this type of problems the notion of renormalized solution has been introduced by DiPerna and Lions for the Boltzmann equation in [**44**] and by Murat [**95**], and Boccardo [**25**] for elliptic equations with integrable data. The existence of renormalized solutions to corresponding parabolic problem was considered by Blanchard et al. [**21, 23**]. At the same time for nonlinear elliptic problems with the right-hand side in L^1 the equivalent notion of entropy solutions have been developed independently by Bénilan et al. in [**15**]. During the last two decades these solution concepts have been adapted to the study of various problems of partial differential equations. We refer to [**3**], [**5**]-[**9**], [**16**], [**20**]-[**25**], [**30, 37, 84, 104**] among others.

More general problems involving vector fields satisfying variable growth and coerciveness condition of type

$$\boldsymbol{a}(x,\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geqslant \lambda|\boldsymbol{\xi}|^{p(x)} - c(x)$$
$$|\boldsymbol{a}(x,\boldsymbol{\xi})| \leqslant d(x) + \mu|\boldsymbol{\xi}|^{p(x)-1}$$

for a.a. $x \in \Omega$, for every $\boldsymbol{\xi} \in \mathbb{R}^d$, where $\lambda, \mu > 0$, $p : \Omega \to \mathbb{R}$ is a measurable variable exponent with $1 < p^- < p(x) < p^+ < \infty$ for a.a. $x \in \Omega$, $c \in L^1(\Omega)$, $d \in L^{p'(x)}(\Omega)$ have already been considered. For results on existence of renormalized solutions of elliptic problems of type (E, f) with $\boldsymbol{a}(\cdot, \cdot)$ satisfying a variable growth condition we refer to [**27, 129**] (for related results see also [**11, 12, 112**]). Note that vector fields satisfying this type of variable exponent growth and coerciveness condition fall into the scope of our study (with $M(x,\boldsymbol{\xi}) = c_1|\boldsymbol{\xi}|^{p(x)}$, $M^*(x,\boldsymbol{\xi}) = c_2|\boldsymbol{\xi}|^{p'(x)}$, where $p'(x) = p(x)/(p(x)-1)$, $c_1 = (1/p(x))(q(x))^{p(x)}$, $c_2 = 1/(p'(x)(q(x))^{p'(x)})$, $q : \Omega \to \mathbb{R}$ is measurable and $0 < q^- < q(x) < q^+ < \infty$). However, our setting is more general as we do not impose a growth restriction on M. Let us note that the functional setting for this type of problems involves variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ which, for the range of exponents the authors considered, are separable, reflexive Banach spaces and thus standard monotonicity methods, adapted to the renormalized case, can be used in this case. The $L^{p(x)}$-spaces, in general, are not stable by convolution and smooth functions may fail to be dense in $W^{1,p(x)}(\Omega)$ (at least if $p(\cdot)$ is not log-Hölder continuous). This fact does not lead to further difficulties in the study of the above-mentioned works as the authors settle the problem in the energy space $W_0^{1,p(x)}(\Omega)$ which, by definition, is the norm closure of $\mathcal{D}(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Anisotropic effects were considered in problems of type (E, f) with constant exponents in **[13, 24]** and with variable exponents in [**100**] (see also [**86**]), where the existence of a renormalized solution was provided with $\beta = 0, F = 0$. It was assumed that the vector field $\boldsymbol{a}(x, \boldsymbol{\xi}) = (a_1(x, \xi_1), \dots, a_d(x, \xi_d))$ with components $a_i : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following coerciveness and growth assumptions

$$a_i(x, r)r \geqslant \lambda |r|^{p_i(x)}$$
$$|a_i(x, r)| \leqslant d_i(x) + \mu |r|^{p_i(x)-1}$$

for a.a. $x \in \Omega$, for every $r \in \mathbb{R}$, where $\lambda, \mu > 0$, $p_i : \overline{\Omega} \to \mathbb{R}$, $i = 1, \dots, d$ are continuous variable exponents with $1 < p_i^- < p_i(x) < p_i^+ < d$ for all $x \in \overline{\Omega}$, $d_i \in L^{p'(x)}(\Omega)$. Moreover, the p_i^-, p_i^+, $i = 1, \dots, d$ satisfy some restrictive compatibility conditions. Choosing the $\mathcal{N}-$function $M(x, \boldsymbol{\xi}) = \sum_{i=1}^d |\xi_i|^{p_i(x)}$ the two conditions above can be rewritten in the form of our general growth assumption (**A2**). Therefore our setting also includes and extends the anisotropic case. Let us note that the functional setting in the above mentioned papers involves the anisotropic Sobolev spaces $W_0^{1,\boldsymbol{p}}(\Omega)$ and the anisotropic variable exponent Sobolev space $W_0^{1,\boldsymbol{p}(x)}(\Omega)$, $\boldsymbol{p} = (p_1, \dots, p_d)$, respectively. According to the restrictions on the exponents p_i, made by the authors, these Banach spaces are separable and reflexive, and the elliptic operator acts as a bounded monotone operator on this space into its dual. Therefore classical variational theory can be applied to prove existence of weak solutions in this case for, say, bounded data f. Moreover existence of renormalized solutions can be proved by approximation, using truncation techniques and Minty's monotonicity trick adapted to the renormalized setting.

Problems of type (E, f) involving vector fields with nonpolynomial (for instance, exponential) growth have also already been considered in the literature. Typically, the growth condition is expressed by a classical isotropic $\mathcal{N}-$function $M : \mathbb{R}_+ \to \mathbb{R}_+$, not depending on the space variable x and only depending on the modulus $|\boldsymbol{\xi}|$ of the vector $\boldsymbol{\xi}$, as, for example, in **[2, 18]**. The functional setting in these works involves the classical Orlicz spaces $L_M(\Omega)$ and Orlicz-Sobolev spaces $W^1L_M(\Omega)$ which fail to be reflexive if M and M^* do not satisfy the Δ_2-condition (see Chapter III or [**1**]). In this case, existence of approximate solutions follows from the theory of monotone operators in Orlicz-Sobolev spaces as developed by Gossez and Mustonen in [**69**]. The arguments used to prove the convergence of such approximate solutions to a renormalized solution of (E, f) are based on an approximation property in Orlicz-Sobolev spaces proved by Gossez in [**68**, Theorem 4]. The author shows that it is possible to approximate the gradient of an $W_0^1 L_M(\Omega)$-function in modular convergence by a sequence of gradients of smooth functions, compactly supported in Ω.

The setting considered in this chapter includes and generalizes variable exponent, anisotropic and classical Orlicz settings (at least in the case when the latter is built on an $\mathcal{N}$–function M whose complementary function M^* satisfies the Δ_2−condition). The function M which describes the growth condition of the vector field $\boldsymbol{a}$ is a *generalized* $\mathcal{N}$−function. The corresponding generalized Orlicz spaces $L_M(\Omega;\mathbb{R}^d)$, often called Orlicz-Musielak spaces (see [**96**]) have been introduced in [**117, 118**]. Let us recall that in general, if M and M^* do not satisfy the Δ_2−condition these spaces fail to be separable or reflexive. In the setting of generalized Orlicz spaces, due to the x-dependence of the $\mathcal{N}$−function, a result similar to Gossez [**68**] can not be achieved. As in the case of generalized Lebesgue spaces convolution with a smooth compactly supported kernel may fail to be a bounded operator.

Our techniques to overcome these difficulties are inspired by previous chapters and former works [**29, 72, 75, 131, 133**]. The authors considered equations involving vector fields satisfying general non-standard growth conditions of type (**A2**) with a generalized $\mathcal{N}$−function $M(x, \boldsymbol{\xi})$. All these works are motivated by fluid dynamics.

Gwiazda et al. in [**74**] studied a steady and in [**72**] a dynamic model for non-Newtonian fluids under an additional strict monotonicity assumption on the vector field. The authors used Young measure techniques in place of a monotonicity method. The additional assumption of strict monotonicity allows to conclude that the measure-valued solution is a Dirac delta and hence a weak solution. A similar method is used in the variable exponent setting in [**6**].

A version of the Minty-Browder trick adapted to the setting of generalized Orlicz spaces was introduced in [**131**] by Wróblewska-Kamińska (and later see [**75, 133**] and Chapter IV) in the framework of non-Newtonian fluids. As we do not assume strict monotonicity of $\boldsymbol{a}(\cdot,\cdot)$, we have to employ the generalized monotonicity method of [**131**] (see also Chapter IV. Using the Galerkin method with smooth basis functions we can thereby prove existence of a weak solution u_ε of some approximate problem $(E_\varepsilon, f_\varepsilon)$ with $f_\varepsilon \in L^\infty(\Omega)$. In a second step we show that a subsequence of the approximate solutions u_ε converges to a renormalized solution of problem (E, f). In this step we combine truncation techniques and the generalized monotonicity method of [**131**]. Thereby, it is possible to overcome a difficulty that arises from the possible lack of reflexivity of $L_M(\Omega;\mathbb{R}^d)$ and which consists in passing to the limit in expressions of the form $\int_\Omega f_\varepsilon(x) \cdot g(x)\,\mathrm{d}x$ when $g \in L_M(\Omega;\mathbb{R}^d)$ and the sequence $\{f_\varepsilon\}_{\varepsilon>0}$ only converges weak-(*) in $L_{M^*}(\Omega;\mathbb{R}^d)$ to some function f.

The chapter is organized as follows: in Section VII.2 we introduce the notions of weak and also renormalized solution for problem (E, f). Our main result, existence of a renormalized solution to (E, f) for any L^1-data f, and the results on uniqueness of renormalized solutions and on existence of weak solutions, are collected in Section VII.3. The proof of existence of renomalized solution is in Section VII.4, the uniqueness is shown in Section VII.5 and existence of a weak solution proved in Section VII.6.

VII.2. Notation

VII.2.1. The energy space. Let us introduce the linear space

$$V := \{\varphi \in L^1_{\mathrm{loc}}(\Omega) \mid \exists \{\varphi_j\}_{j=1}^{\infty} \subset \mathcal{D}(\Omega) \text{ such that } \nabla\varphi_j \overset{*}{\rightharpoonup} \nabla\varphi \text{ in } L_M(\Omega; \mathbb{R}^d) \text{ as } j \to \infty\}.$$

V endowed with the norm

$$\|\varphi\|_V = \|\nabla\varphi\|_{M,\Omega}, \ \varphi \in V$$

is a Banach space. Moreover for $\nu > 0$

$$V \hookrightarrow \{\varphi \in W_0^{1,1+\nu}(\Omega) \mid \nabla\varphi \in L_M(\Omega; \mathbb{R}^d)\}$$

where $\hookrightarrow$ denotes continuous embedding. If $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function such that $h(0) = 0$ and $u \in V$, then also $h(u) \in V$. Note that if M^* satisfies the Δ_2-condition and if $g \in L^\infty(\Omega)$ and $\varphi \in \mathcal{L}_{M^*}(\Omega; \mathbb{R}^d)$, it follows that $g\varphi \in \mathcal{L}_{M^*}(\Omega; \mathbb{R}^d)$.

VII.2.2. Notation. For any $u : \Omega \to \mathbb{R}$ and $k \geqslant 0$, we denote $\{|u| \leqslant (<, >, \geqslant, =)k\}$ for the set $\{x \in \Omega \ : \ |u(x)| \leqslant (<, >, \geqslant, =)k\}$. For $r \in \mathbb{R}$ by $\mathrm{sign}_0(r)$ we mean the usual (single-valued) sign function, $\mathrm{sign}_0^+(r) = 1$ if $r > 0$ and $\mathrm{sign}_0^+(r) = 0$ if $r \leqslant 0$. Let $h_l(r) : \mathbb{R} \to \mathbb{R}$ be defined by

$$h_l(r) = \min((l + 1 - |r|)^+, 1) \tag{VII.2.1}$$

for each $r \in \mathbb{R}$ and $l > 0$. For any given $k > 0$, we define the truncation function $T_k : \mathbb{R} \to \mathbb{R}$ by

$$T_k(r) := \begin{cases} -k & \text{if } r \leqslant -k \\ r & \text{if } |r| < k \\ k & \text{if } r \geqslant k. \end{cases}$$

VII.2.3. Weak solutions.

Definition VII.2.1. A weak solution to (E, f) is a pair of functions $(u, b) \in V \times L^1(\Omega)$ satisfying $b(x) \in \beta(x, u(x))$ a.e. in Ω such that $\boldsymbol{a}(x, \nabla u) \in L_{M^*}(\Omega; \mathbb{R}^d)$, $\boldsymbol{F}(u) \in L_{M^*}(\Omega; \mathbb{R}^d)$ and

$$b - \mathrm{div}(\boldsymbol{a}(\cdot, \nabla u) + \boldsymbol{F}(u)) = f \quad \text{in } \mathcal{D}'(\Omega). \tag{VII.2.2}$$

Corollary VII.2.2. *If (u, b) is a weak solution to (E, f) and additionally $u \in L^\infty(\Omega)$, then $\boldsymbol{F}(u) \in L^\infty(\Omega; \mathbb{R}^d)$ and consequently $\boldsymbol{F}(u) \in L_{M^*}(\Omega; \mathbb{R}^d)$. If moreover M satisfies the Δ_2-condition, then the growth assumption on $\boldsymbol{a}(x, \nabla u)$ implies that $\boldsymbol{a}(\cdot, \nabla u) \in L_{M^*}(\Omega; \mathbb{R}^d)$.*

Indeed, from (VII.1.1) it follows that

$$\frac{c_a}{2}\boldsymbol{a}(x,\nabla u)\frac{2}{c_a}\nabla u \geqslant c_a\{M^*(x,\boldsymbol{a}(x,\nabla u)) + M(x,\nabla u)\} - a_0(x) \tag{VII.2.3}$$

for $c_a \in (0,1]$ and $a_0 \in L^1(\Omega)$ nonnegative. Now, using the Fenchel-Young inequality (III.1.9) to estimate the left-hand side of (VII.2.3) we arrive at

$$M^*(x,\frac{c_a}{2}\boldsymbol{a}(x,\nabla u)) + M(x,\frac{2}{c_a}\nabla u) + a_0(x) \geqslant c_a\{M^*(x,\boldsymbol{a}(x,\nabla u)) + M(x,\nabla u)\}. \tag{VII.2.4}$$

Now, since M^* is convex, $M^*(x,0)=0$ and $0 < c_a < 1$, from (VII.2.4) we obtain

$$\frac{2}{c_a}\left(M(x,\frac{2}{c_a}\nabla u) + a_0(x)\right) \geqslant M^*(x,\boldsymbol{a}(x,\nabla u)). \tag{VII.2.5}$$

If M satisfies the Δ_2-condition, then $\nabla u \in L_M(\Omega;\mathbb{R}^d) = \mathcal{L}_M(\Omega;\mathbb{R}^d) = E_M(\Omega;\mathbb{R}^d)$ implies $\frac{2}{c_a}\nabla u \in \mathcal{L}_M(\Omega;\mathbb{R}^d)$ and the assertion follows by integrating (VII.2.5). In general, $u \in V \cap L^\infty(\Omega)$ does not imply that

$$\int_\Omega M(x,\frac{2}{c_a}\nabla u)\,\mathrm{d}x < \infty.$$

VII.2.4. Renormalized solutions.

Definition VII.2.3. A renormalized solution to (E,f) is a function u satisfying the following conditions:

(R1): $u:\Omega\to\mathbb{R}$ is measurable, $b \in L^1(\Omega)$ and $b \in \beta(x,u(x))$ for a.a. $x\in\Omega$.

(R2): For each $k>0$, $T_k(u)\in V$, $\boldsymbol{a}(x,\nabla T_k(u)) \in L_{M^*}(\Omega;\mathbb{R}^d)$ and

$$\int_\Omega bh(u)\varphi\,\mathrm{d}x + \int_\Omega (\boldsymbol{a}(x,\nabla u) + \boldsymbol{F}(u))\cdot\nabla(h(u)\varphi)\,\mathrm{d}x = \int_\Omega fh(u)\varphi\,\mathrm{d}x \tag{VII.2.6}$$

holds for all $h \in C_c^1(\mathbb{R})$ and all $\varphi \in V\cap L^\infty(\Omega)$.

(R3): $\int_{\{l<|u|<l+1\}} \boldsymbol{a}(x,\nabla u)\cdot\nabla u\,\mathrm{d}x \to 0$ as $l\to\infty$.

Remark VII.2.4. Since u is only measurable, ∇u may not be defined as an element of $\mathcal{D}'(\Omega)$. However, it is possible to define a generalized gradient ∇u in the following sense: There exists a measurable function $v:\Omega\to\mathbb{R}^d$, such that $v = \nabla T_k(u)$ on $\{|u|<k\}$ for all $k>0$. Therefore all the terms in (VII.2.6) are well-defined (see **[15]** for more details).

Remark VII.2.5. If (u,b) is a renormalized solution to (E,f), then we get

$$\boldsymbol{a}(x,\nabla T_k(u))\cdot\nabla T_k(u) \in L^1(\Omega)$$

for all $k>0$ by applying the generalized Hölder inequality. If M satisfies the Δ_2–condition, $T_k(u)\in V$ implies $\nabla T_k(u) \in L_M(\Omega;\mathbb{R}^d) = \mathcal{L}_M(\Omega;\mathbb{R}^d) = E_M(\Omega;\mathbb{R}^d)$ and using the same arguments as in Corollary VII.2.2 it follows that

$$\boldsymbol{a}(x,\nabla T_k(u)) \in L_{M^*}(\Omega;\mathbb{R}^d). \tag{VII.2.7}$$

Hence if M satisfies the Δ_2–condition, the assumption (VII.2.7) in Definition VII.2.3 can be dropped.

Remark VII.2.6. If (u, b) is a renormalized solution to (E, f) such that $u \in L^\infty(\Omega)$, it is a direct consequence of Definition VII.2.3 that u is in V and since (VII.2.6) holds with the formal choice $h \equiv 1$, (u, b) is a weak solution.

Indeed, let $\varphi \in \mathcal{D}$ and choose $h_l(u)\varphi$ as a test function in (VII.2.6). Since $u \in L^\infty(\Omega)$, we can pass to the limit with $l \to \infty$ and find that u solves (E, f) in the sense of distributions.

VII.3. Main results

Our results are stated as follows: In this section we will state existence and uniqueness of renormalized solutions to (E, f) in the two following theorems. In Proposition VII.3.3 we give conditions on a_0 and f such that the renormalized solution to (E, f) is a weak solution. In the next sections of this chapter we will present the proofs.

Theorem VII.3.1. *Let M be an $\mathcal{N}$–function satisfying condition* (VII.1.4) *and let a complementary function M^* to M satisfy the Δ_2–condition. Moreover, let $\boldsymbol{a}$ satisfy conditions* **(A1) - (A3)** *and $\boldsymbol{F}$ be locally Lipschitz. Let β be a maximal monotone operator with $0 \in \beta(x, 0)$ and with minimal selection β^0 satisfying assumption* (VII.1.5)*. Then for any $f \in L^1(\Omega)$ there exists at least one renormalized solution u to the problem (E, f).*

Theorem VII.3.2. *Let assumptions of Theorem VII.3.1 be satisfied. Moreover, let $\beta : \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ be such that $\beta(x, \cdot)$ is strictly monotone for almost every $x \in \Omega$. For $f \in L^1(\Omega)$ let (u, b), $(\tilde{u}, \tilde{b})$ be renormalized solutions to (E, f). Then $u = \tilde{u}$ and $b = \tilde{b}$.*

Proposition VII.3.3. *Let assumptions of Theorem VII.3.1 be satisfied and let (u, b) be a renormalized solution to (E, f). Moreover, assume that* **(A2)** *is satisfied with $a_0 \in L^\infty(\Omega)$ and the right-hand side f is in $L^d(\Omega)$. Then $u \in V \cap L^\infty(\Omega)$ and thus, in particular, u is a weak solution to (E, f).*

VII.4. Proof of Theorem VII.3.1 - Existence

The following section will be devoted to the proof of Theorem VII.3.1 and we will divide it into several steps.

VII.4.1. $(E_\varepsilon, f_\varepsilon)$ - approximation of the problem (E, f). First we introduce the approximate problem to (E, f), namely

$$(E_\varepsilon, f_\varepsilon) \qquad \begin{aligned} T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon))) - \operatorname{div}\big(\boldsymbol{a}(x, \nabla u_\varepsilon) + \boldsymbol{F}(T_{1/\varepsilon}(u_\varepsilon))\big) &= T_{1/\varepsilon}(f) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

where for each $\varepsilon \in (0, 1]$, $\beta_\varepsilon : \Omega \times \mathbb{R} \to \mathbb{R}$ denotes the Moreau-Yosida approximation [1] (see [**28**]) of β in the second variable. In particular $\beta_\varepsilon(\cdot, T_{1/\varepsilon}(\cdot))$ is a single-valued,

[1] $\beta^\varepsilon(x, u) = \frac{\mathrm{d}}{\mathrm{d}u} J^\varepsilon(x, u)$ where J^ε is locally Lipschitz, with Lipschitz coefficient $1/\varepsilon$ and $J(x, u) = \int_0^u \beta(x, t)\,\mathrm{d}t$, moreover $\beta^\varepsilon(\cdot, k) \to \beta(\cdot, k)$ a.e. in Ω and for all k.

monotone (with respect to the second variable, for a.a. $x \in \Omega$) Carathéodory function.

VII.4.2. Existence of solutions to the problem $(E_\varepsilon, f_\varepsilon)$ - Galerkin approximation. We will show that there exists at least one weak solution u_ε to our approximate problem $(E_\varepsilon, f_\varepsilon)$ with $f_\varepsilon = T_{1/\varepsilon}(f) \in L^\infty(\Omega)$ in the sense of Definition VII.2.1.

We start with the Galerkin approximation. Let $\{\omega_i\}_{i=1}^\infty$ be a basis built by the eigenfunctions of the Laplace operator with zero Dirichlet boundary conditions. Let us look for an approximate solution of the form

$$u_\varepsilon^n := \sum_{i=1}^n c_i^n \omega_i \quad \text{for } n \in \mathbb{N} \tag{VII.4.1}$$

with $c_i^n \in \mathbb{R}$ such that

(VII.4.2)

$$\int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon^n)))\omega_i \,\mathrm{d}x + \int_\Omega \big(\boldsymbol{a}(x, \nabla u_\varepsilon^n) + \boldsymbol{F}(T_{1/\varepsilon}(u_\varepsilon^n))\big) \cdot \nabla \omega_i \,\mathrm{d}x = \int_\Omega T_{1/\varepsilon}(f)\omega_i \,\mathrm{d}x$$

for $i = 1, \ldots, n$. Multiplying (VII.4.2) by c_i^k and summing over $i = 1, ..., j$ with $j \leqslant n$ we obtain

(VII.4.3)

$$\int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon^n)))u_\varepsilon^j \,\mathrm{d}x + \int_\Omega \big(\boldsymbol{a}(x, \nabla u_\varepsilon^n) + \boldsymbol{F}(T_{1/\varepsilon}(u_\varepsilon^n))\big) \cdot \nabla u_\varepsilon^j \,\mathrm{d}x = \int_\Omega T_{1/\varepsilon}(f)u_\varepsilon^j \,\mathrm{d}x.$$

The existence of such an approximate solution to the Galerkin approximation u_ε^n can be obtained by the lemma about zeros of a vector field [**48**, Chapter 9]. Since $\boldsymbol{F}(T_{1/\varepsilon}(\cdot))$ is a Lipschitz function, applying the Stokes theorem it follows that for $j = n$ the term

$$\int_\Omega \big(\boldsymbol{F}(T_{1/\varepsilon}(u_\varepsilon^n))\big) \cdot \nabla u_\varepsilon^n \,\mathrm{d}x = 0.$$

Hence for $j = n$ we have

$$\int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon^n)))u_\varepsilon^n \,\mathrm{d}x + \int_\Omega \boldsymbol{a}(x, \nabla u_\varepsilon^n) \cdot \nabla u_\varepsilon^n \,\mathrm{d}x = \int_\Omega T_{1/\varepsilon}(f)u_\varepsilon^n \,\mathrm{d}x. \tag{VII.4.4}$$

We want to estimate the right-hand side of (VII.4.4). Employing the Poincaré inequality, assumption (VII.1.4) and the Young inequality we infer

$$\begin{aligned}\int_\Omega T_{1/\varepsilon}(f)u_\varepsilon^n \,\mathrm{d}x &\leqslant c_d \|T_{1/\varepsilon}(f)\|_{L^\infty} \|\nabla u_\varepsilon^n\|_{L^1} \\ &\leqslant \gamma(c_d, c_a)\|T_{1/\varepsilon}(f)\|_{L^\infty} + \frac{c_a}{2}\left(\int_\Omega M(x, \nabla u_\varepsilon^n)\,\mathrm{d}x + c\right)\end{aligned} \tag{VII.4.5}$$

where $c_d > 0$ is the constant from the Poincaré inequality and $\gamma(c_d, c_a) > 0$, $c > 0$ are constants independent of $n > 0$. Combining (VII.4.5) with (VII.4.4), using the coercivity condition (VII.1.1) on $\boldsymbol{a}(\cdot,\cdot)$ and neglecting the nonnegative term $T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon^n)))u_\varepsilon^n$ gives

$$\text{(VII.4.6)} \quad \begin{aligned} &\frac{c_a}{2}\int_\Omega M(x, \nabla u_\varepsilon^n)\,\mathrm{d}x + c_a \int_\Omega M^*(x, \boldsymbol{a}(x, \nabla u_\varepsilon^n))\,\mathrm{d}x \\ &\qquad \leqslant \gamma(c_d, c_a)\|T_{1/\varepsilon}(f)\|_{L^\infty} + \frac{c_a c}{2} + \int_\Omega a_0(x)\,\mathrm{d}x. \end{aligned}$$

Consequently, passing to a subsequence if necessary, from (VII.4.6) we obtain

$$\text{(VII.4.7)} \qquad \nabla u_\varepsilon^n \overset{*}{\rightharpoonup} \nabla u_\varepsilon \text{ weakly–(*) in } L_M(\Omega; \mathbb{R}^d)$$

and

$$\text{(VII.4.8)} \qquad \boldsymbol{a}(x, \nabla u_\varepsilon^n) \overset{*}{\rightharpoonup} \boldsymbol{\alpha} \text{ weakly–(*) in } L_{M^*}(\Omega; \mathbb{R}^d) \text{ for some } \boldsymbol{\alpha} \in L_{M^*}(\Omega; \mathbb{R}^d).$$

The condition (VII.1.4) provides that $\{\nabla u_\varepsilon^n\}_{n=1}^\infty$ is uniformly bounded in the space $L^{1+\nu}(\Omega; \mathbb{R}^d)$, hence by the Poincaré inequality the sequence $\{u_\varepsilon^n\}_{n=1}^\infty$ is uniformly bounded in $W_0^{1,1+\nu}(\Omega)$. Therefore

$$\text{(VII.4.9)} \qquad \nabla u_\varepsilon^n \rightharpoonup \nabla u_\varepsilon \text{ weakly in } L^{1+\nu}(\Omega; \mathbb{R}^d),$$

$$\text{(VII.4.10)} \qquad u_\varepsilon^n \to u_\varepsilon \text{ strongly in } L^{1+\nu}(\Omega)$$

and

$$\text{(VII.4.11)} \qquad u_\varepsilon^n \to u_\varepsilon \text{ a.e. in } \Omega.$$

Let us notice that for a fixed $\varepsilon \in (0, 1]$ and almost all $x \in \Omega$ the function $\beta_\varepsilon(x, \cdot)$ is a Carathéodory function and we have that

$$|\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon^n))| \leqslant \max(\beta^0(x, 1/\varepsilon), -\beta^0(x, -1/\varepsilon)) \quad \text{a.e. in } \Omega$$

where, according to (VII.1.5), β^0 is integrable. Then this together with (VII.4.11) and the Lebesgue dominated convergence theorem provide

$$\text{(VII.4.12)} \qquad T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon^n))) \to T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon))) \text{ strongly in } L^1(\Omega).$$

Since $\boldsymbol{F}(\cdot)$ is continuous we obtain

$$\text{(VII.4.13)} \qquad \boldsymbol{F}(T_{1/\varepsilon}(u_\varepsilon^n)) \to \boldsymbol{F}(T_{1/\varepsilon}(u_\varepsilon)) \text{ a.e. in } \Omega.$$

As $\boldsymbol{F}(T_{1/\varepsilon}(u_\varepsilon^n))$ is uniformly bounded with respect to $k > 0$, i.e.

$$\text{(VII.4.14)} \qquad \|\boldsymbol{F}(T_{1/\varepsilon}(u_\varepsilon^n))\|_{L^\infty(\Omega;\mathbb{R}^d)} \leqslant \sup_{\tau \in [-1/\varepsilon, 1/\varepsilon]} |\boldsymbol{F}(\tau)| < c$$

where the constant $c > 0$ is independent of $n \in \mathbb{N}$ and as Ω is bounded, (VII.4.11) and the continuity of $\boldsymbol{F}(\cdot)$ together with the Lebesgue dominated convergence theorem provide that

$$\text{(VII.4.15)} \qquad \boldsymbol{F}(T_{1/\varepsilon}(u_\varepsilon^n)) \to \boldsymbol{F}(T_{1/\varepsilon}(u_\varepsilon)) \text{ strongly in } L^1(\Omega; \mathbb{R}^d) \text{ as } n \to \infty.$$

Recall that if M is a generalized $\mathcal{N}-$function, then M^* is also an $\mathcal{N}-$function. This, (VII.4.6) and assumption (VII.1.3)$_2$ allow us to apply Lemma III.2.2 to M^* and conclude the uniform integrability of $\{\boldsymbol{a}(\cdot,\nabla u_\varepsilon^n)\}_{n=1}^\infty$. Hence according to the Dunford-Pettis theorem we have the weak precompactness of the sequence $\{\boldsymbol{a}(x,\nabla u_\varepsilon^n)\}_{n=1}^\infty$ in $L^1(\Omega;\mathbb{R}^d)$. Therefore $\boldsymbol{\alpha}\in L^1(\Omega;\mathbb{R}^d)$ and passing to a subsequence when necessary

$$\boldsymbol{a}(\cdot,\nabla u_\varepsilon^n)\rightharpoonup\boldsymbol{\alpha}\quad\text{weakly in}\quad L^1(\Omega;\mathbb{R}^d)\text{ as }n\to\infty. \tag{VII.4.16}$$

Using (VII.4.12), (VII.4.15), (VII.4.16) and letting $n\to\infty$ in (VII.4.3) gives
(VII.4.17)

$$\int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x,T_{1/\varepsilon}(u_\varepsilon)))u_\varepsilon^j\,\mathrm{d}x+\int_\Omega\big(\boldsymbol{\alpha}+\boldsymbol{F}(T_{1/\varepsilon}(u_\varepsilon))\big)\cdot\nabla u_\varepsilon^j\,\mathrm{d}x=\int_\Omega T_{1/\varepsilon}(f)u_\varepsilon^j\,\mathrm{d}x.$$

Since (VII.4.17) is also satisfied for all test functions from the basis $\{\omega_i\}_{i=1}^\infty$, density arguments give us that u_ε and $\boldsymbol{\alpha}$ satisfy

$$T_{1/\varepsilon}(\beta_\varepsilon(x,T_{1/\varepsilon}(u_\varepsilon)))-\operatorname{div}\big(\boldsymbol{\alpha}+\boldsymbol{F}(T_{1/\varepsilon}(u_\varepsilon))\big)=T_{1/\varepsilon}(f)\qquad\text{in }\mathcal{D}'(\Omega).$$

The last step is to identify the vector $\boldsymbol{\alpha}$. Let us notice that the convective term on the left-hand side of (VII.4.17) vanishes when $j\to\infty$ by the Stokes theorem. Since M,M^* are convex and nonnegative functions, the weak lower semicontinuity of M and M^* together with (VII.4.6) imply that $\boldsymbol{\alpha}\in\mathcal{L}_{M^*}(\Omega;\mathbb{R}^d)$, $\nabla u_\varepsilon\in\mathcal{L}_M(\Omega;\mathbb{R}^d)$ respectively. Since M^* satisfies the Δ_2-condition it follows that $L_{M^*}(\Omega;\mathbb{R}^d)=\mathcal{L}_{M^*}(\Omega;\mathbb{R}^d)=E_{M^*}(\Omega;\mathbb{R}^d)$ is a separable space. Therefore, $\boldsymbol{\alpha}+\boldsymbol{F}(T_{1/\varepsilon}(u_\varepsilon))\in E_{M^*}(\Omega;\mathbb{R}^d)$ and since $(E_{M^*}(\Omega;\mathbb{R}^d))^*=L_M(\Omega;\mathbb{R}^d)$, using (VII.4.7) and (VII.4.10) we can pass to the limit with $j\to\infty$ in (VII.4.17) and obtain

$$\int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x,T_{1/\varepsilon}(u_\varepsilon)))u_\varepsilon\,\mathrm{d}x+\int_\Omega\boldsymbol{\alpha}\cdot\nabla u_\varepsilon\,\mathrm{d}x=\int_\Omega T_{1/\varepsilon}(f)u_\varepsilon\,\mathrm{d}x. \tag{VII.4.18}$$

Now we apply the monotonicity trick for non reflexive spaces to obtain

$$\boldsymbol{\alpha}=\boldsymbol{a}(x,\nabla u_\varepsilon)\text{ a.e. in }\Omega.$$

First note that for $\boldsymbol{\zeta}\in L^\infty(\Omega;\mathbb{R}^d)$ it follows that $\boldsymbol{a}(x,\boldsymbol{\zeta})\in\mathcal{L}_{M^*}(\Omega;\mathbb{R}^d)$. Indeed, with the same arguments as in Corollary VII.2.2 it follows that

$$\int_\Omega M^*(x,\boldsymbol{a}(x,\boldsymbol{\zeta}))\,\mathrm{d}x\leqslant\frac{2}{c_a}\int_\Omega M(x,\frac{2}{c_a}\boldsymbol{\zeta})+a_0(x)\,\mathrm{d}x \tag{VII.4.19}$$

and for $\boldsymbol{\zeta}\in L^\infty(\Omega;\mathbb{R}^d)$ the integral on the right-hand side of (VII.4.19) is finite. Passing to a subsequence if necessary, for $n\to\infty$ from (VII.4.4) we get

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega\boldsymbol{a}(x,\nabla u_\varepsilon^n)\cdot\nabla u_\varepsilon^n\,\mathrm{d}x&=\lim_{n\to\infty}\left(\int_\Omega T_{1/\varepsilon}(f)u_\varepsilon^n\,\mathrm{d}x-\int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x,T_{1/\varepsilon}(u_\varepsilon^n)))u_\varepsilon^n\,\mathrm{d}x\right)\\&=\int_\Omega T_{1/\varepsilon}(f)u_\varepsilon\,\mathrm{d}x-\int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x,T_{1/\varepsilon}(u_\varepsilon)))u_\varepsilon\,\mathrm{d}x\end{aligned}$$

which together with (VII.4.18) provides

$$\lim_{k\to\infty}\int_\Omega\boldsymbol{a}(x,\nabla u_\varepsilon^n)\cdot\nabla u_\varepsilon^n\,\mathrm{d}x=\int_\Omega\boldsymbol{\alpha}\cdot\nabla u_\varepsilon\,\mathrm{d}x. \tag{VII.4.20}$$

Since $\boldsymbol{a}(x,\cdot)$ is monotone

$$(\boldsymbol{a}(x,\boldsymbol{\zeta}) - \boldsymbol{a}(x,\nabla u_\varepsilon^n)) \cdot (\boldsymbol{\zeta} - \nabla u_\varepsilon^n) \geqslant 0 \tag{VII.4.21}$$

a.e. in Ω and for all $\boldsymbol{\zeta} \in L^\infty(\Omega;\mathbb{R}^d)$. Integrating (VII.4.21), using $\boldsymbol{a}(x,\boldsymbol{\zeta}) \in \mathcal{L}_{M^*}(\Omega;\mathbb{R}^d) = E_{M^*}(\Omega;\mathbb{R}^d)$ and (VII.4.20) to pass to the limit with $n \to \infty$ we obtain

$$\int_\Omega (\boldsymbol{a}(x,\boldsymbol{\zeta}) - \boldsymbol{\alpha}) \cdot (\boldsymbol{\zeta} - \nabla u_\varepsilon)\,\mathrm{d}x \geqslant 0. \tag{VII.4.22}$$

For $l > 0$ let

$$\Omega_l := \{x \in \Omega : |\nabla u_\varepsilon(x)| \leqslant l \quad \text{a.e. in } \Omega\}.$$

Now let $0 < j < i$ be arbitrary, $\boldsymbol{z} \in L^\infty(\Omega;\mathbb{R}^d)$ and $h > 0$. Inserting

$$\boldsymbol{\zeta} = (\nabla u_\varepsilon)\mathbb{1}_{\Omega_i} + h\boldsymbol{z}\mathbb{1}_{\Omega_j},$$

into (VII.4.22) we get

$$-\int_{\Omega\setminus\Omega_i} (\boldsymbol{a}(x,0) - \boldsymbol{\alpha}) \cdot \nabla u_\varepsilon\,\mathrm{d}x + h\int_{\Omega_j} (\boldsymbol{a}(x,\nabla u_\varepsilon + h\boldsymbol{z}) - \boldsymbol{\alpha}) \cdot \boldsymbol{z}\,\mathrm{d}x \geqslant 0. \tag{VII.4.23}$$

Note that by (VII.1.1) $M^*(x,\boldsymbol{a}(x,0)) \leqslant a_0(x)$ a.e. in Ω and from the Fenchel-Young inequality (III.1.9) it follows that

$$\int_\Omega |\boldsymbol{a}(x,0) \cdot \nabla u_\varepsilon|\,\mathrm{d}x \leqslant \int_\Omega a_0(x) + M(x,\nabla u_\varepsilon)\,\mathrm{d}x. \tag{VII.4.24}$$

Since $\nabla u_\varepsilon \in \mathcal{L}_M(\Omega;\mathbb{R}^d)$ the right-hand side of (VII.4.24) is finite and consequently

$$\boldsymbol{a}(x,0) \cdot \nabla u_\varepsilon \in L^1(\Omega).$$

As $\boldsymbol{\alpha} \in \mathcal{L}_{M^*}(\Omega;\mathbb{R}^d)$ and $\nabla u_\varepsilon \in \mathcal{L}_M(\Omega;\mathbb{R}^d)$ it follows immediately by (III.1.9) that $\boldsymbol{\alpha} \cdot \nabla u_\varepsilon$ is in $L^1(\Omega)$. Therefore, by the Lebesgue dominated convergence theorem, the first term on the left-hand of (VII.4.23) vanishes for $i \to \infty$. Passing to the limit with $i \to \infty$ in (VII.4.23) and dividing by h we get

$$\int_{\Omega_j} (\boldsymbol{a}(x,\nabla u_\varepsilon + h\boldsymbol{z}) - \boldsymbol{\alpha}) \cdot \boldsymbol{z}\,\mathrm{d}x \geqslant 0.$$

Note that $\boldsymbol{a}(x,\nabla u_\varepsilon + h\boldsymbol{z}) \to \boldsymbol{a}(x,\nabla u_\varepsilon)$ a.e. in Ω_j when $h \to 0$. Moreover, for $0 < h < 1$

$$\int_{\Omega_j} M^*(x,\boldsymbol{a}(x,\nabla u_\varepsilon + h\boldsymbol{z}))\,\mathrm{d}x \leqslant \frac{2}{c_a} \sup_{0<h<1} \int_{\Omega_j} M(x,\frac{2}{c_a}(\nabla u_\varepsilon + h\boldsymbol{z})) + a_0(x)\,\mathrm{d}x \tag{VII.4.25}$$

and the right-hand side of (VII.4.25) is bounded since $\nabla u_\varepsilon + h\boldsymbol{z}$ is uniformly (in h) bounded in $L^\infty(\Omega_j;\mathbb{R}^d)$ and according to (III.1.4) $M(x,\frac{2}{c_a}(\nabla u_\varepsilon + h\boldsymbol{z}))$ is bounded. Hence it follows from Lemma III.2.2 that $\{\boldsymbol{a}(x,\nabla u_\varepsilon + h\boldsymbol{z})\}_h$ is uniformly integrable. Note that $|\Omega_j| < \infty$, hence by the Vitali lemma it follows that

$$\boldsymbol{a}(x,\nabla u_\varepsilon + h\boldsymbol{z}) \to \boldsymbol{a}(x,\nabla u_\varepsilon) \quad \text{in } L^1(\Omega_j;\mathbb{R}^d)$$

for $h \to 0^+$ and therefore

$$\lim_{h\to 0}\int_{\Omega_j}(\boldsymbol{a}(x,\nabla u_\varepsilon + h\boldsymbol{z}) - \boldsymbol{\alpha})\cdot \boldsymbol{z}\,\mathrm{d}x = \int_{\Omega_j}(\boldsymbol{a}(x,\nabla u_\varepsilon) - \boldsymbol{\alpha})\cdot \boldsymbol{z}\,\mathrm{d}x.$$

Consequently,

$$\int_{\Omega_j}(\boldsymbol{a}(x,\nabla u_\varepsilon) - \boldsymbol{\alpha})\cdot \boldsymbol{z}\,\mathrm{d}x \geqslant 0$$

for all $\boldsymbol{z} \in L^\infty(\Omega;\mathbb{R}^d)$. Substituting

$$\boldsymbol{z} = \begin{cases} -\frac{\boldsymbol{a}(x,\nabla u_\varepsilon)-\boldsymbol{\alpha}}{|\boldsymbol{a}(x,\nabla u_\varepsilon)-\boldsymbol{\alpha}|} & \text{if } \boldsymbol{a}(x,\nabla u_\varepsilon) - \boldsymbol{\alpha} \neq 0 \\ 0 & \text{if } \boldsymbol{a}(x,\nabla u_\varepsilon) - \boldsymbol{\alpha} = 0 \end{cases}$$

into the above, we obtain

$$\int_{\Omega_j}|\boldsymbol{a}(x,\nabla u_\varepsilon) - \boldsymbol{\alpha}|\,\mathrm{d}x \leqslant 0.$$

Hence

$$\boldsymbol{a}(x,\nabla u_\varepsilon) = \boldsymbol{\alpha} \quad \text{a.e. in } \Omega_j. \tag{VII.4.26}$$

Since j is arbitrary (VII.4.26) holds a.e. in Ω.

VII.4.3. A priori estimates.

Lemma VII.4.1. *For $0 < \varepsilon \leqslant 1$ and $f \in L^1(\Omega)$ let $u_\varepsilon \in V$ be a weak solution to $(E_\varepsilon, f_\varepsilon)$. Then*

$$\int_\Omega M(x,\nabla T_k(u_\varepsilon))\,\mathrm{d}x \leqslant k\|f\|_{L^1(\Omega)} + \|a_0\|_{L^1(\Omega)} \tag{VII.4.27}$$

and

$$\int_\Omega M^*(x,\boldsymbol{a}(x,\nabla T_k(u_\varepsilon)))\,\mathrm{d}x \leqslant k\|f\|_{L^1(\Omega)} + \|a_0\|_{L^1(\Omega)} \tag{VII.4.28}$$

holds for any $k > 0$. Moreover, for any $l > 0$,

$$\int_{\{l<|u_\varepsilon|<l+1\}}\boldsymbol{a}(x,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon\,\mathrm{d}x \leqslant \int_{\{l<|u_\varepsilon|\}}|f|\,\mathrm{d}x \tag{VII.4.29}$$

holds for all $\varepsilon \in (0,1]$.

Remark VII.4.2. Using Lemma III.2.2, (VII.1.4), (VII.4.27), (VII.4.28) and (VII.1.3)$_2$ we deduce that the sequences
(VII.4.30)

$$\{\boldsymbol{a}(x,\nabla T_k(u_\varepsilon))\}_{\varepsilon>0},\ \{\nabla T_k(u_\varepsilon)\}_{\varepsilon>0} \text{ are uniformly integrable in } L^1(\Omega;\mathbb{R}^d)$$

w.r.t. $\varepsilon > 0$ for any fixed $k \in \mathbb{N}$.

PROOF. Testing in $(E_\varepsilon, f_\varepsilon)$ by $T_k(u_\varepsilon)$ yields

$$\int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon)))T_k(u_\varepsilon)\,\mathrm{d}x + \int_\Omega \big(\boldsymbol{a}(x, \nabla T_k(u_\varepsilon)) + \boldsymbol{F}(T_{1/\varepsilon}(u_\varepsilon))\big)\cdot \nabla T_k(u_\varepsilon)\,\mathrm{d}x$$
$$= \int_\Omega T_{1/\varepsilon}(f)T_k(u_\varepsilon)\,\mathrm{d}x.$$

As the first term on the left-hand side is nonnegative and the integral over the convection term vanishes, by (VII.1.1) and the Hölder inequality we get

$$c_a \int_\Omega \left(M^*(x, \boldsymbol{a}(x, \nabla T_k(u_\varepsilon))) + M(x, \nabla T_k(u_\varepsilon))\right)\,\mathrm{d}x \leqslant k\|f\|_{L^1(\Omega)} + \|a_0\|_{L^1(\Omega)},$$

where $c_a \in (0, 1]$, and therefore (VII.4.27) and (VII.4.28) holds.

Let us define $g_l : \mathbb{R} \to \mathbb{R}$ by

$$g_l(r) := T_{l+1}(r) - T_l(r) = \begin{cases} -1 & \text{if } r \leqslant -(l+1) \\ r + l & \text{if } -(l+1) < r \leqslant -l \\ 0 & \text{if } |r| < l \\ r - l & \text{if } l \leqslant r < l+1 \\ 1 & \text{if } l + 1 \leqslant r. \end{cases}$$

Using $g_l(u_\varepsilon)$ as a test function in the problem $(E_\varepsilon, f_\varepsilon)$ we obtain

$$\int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon)))g_l(u_\varepsilon)\,\mathrm{d}x + \int_\Omega \big[\boldsymbol{a}(x, \nabla u_\varepsilon) + \boldsymbol{F}(T_{1/\varepsilon}(u_\varepsilon))\big]\cdot \nabla g_l(u_\varepsilon)\,\mathrm{d}x$$
$$= \int_\Omega T_{1/\varepsilon}(f)g_l(u_\varepsilon)\,\mathrm{d}x.$$

As the first term on the left-hand side is nonnegative and the convection term vanishes, we find that

(VII.4.31) $$\int_{\{l<|u_\varepsilon|<l+1\}} \boldsymbol{a}(x, \nabla T_{l+1}(u_\varepsilon))\cdot\nabla T_{l+1}(u_\varepsilon)\,\mathrm{d}x \leqslant \int_{\{l<|u_\varepsilon|\}} |f|\,\mathrm{d}x.$$

Let us notice that (VII.4.29) is equivalent to (VII.4.31). □

Corollary VII.4.3. *There exists a function* $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\lim_{r\to 0^+} \gamma(r) = 0$ *and*

(VII.4.32) $$\int_{\{l<|u_\varepsilon|<l+1\}} \boldsymbol{a}(x, \nabla u_\varepsilon)\cdot\nabla u_\varepsilon\,\mathrm{d}x \leqslant \gamma(Cl^{-\nu})$$

for any $\varepsilon \in (0, 1]$*, where* C *is independent of* ε *and* l*. Moreover*

(VII.4.33) $$|\{|u_\varepsilon| \geqslant l\}| \leqslant l^{-\nu}C$$

holds for $C(\nu, d, f)$ *independently of* ε*.*

PROOF. Let us concentrate on (VII.4.33). Note that

$$|\{|u_\varepsilon| \geqslant l\}| = |\{|T_l(u_\varepsilon)| \geqslant l\}|,$$

then by the Chebyshev, the Poincaré inequality and (VII.1.4), (VII.4.27) we obtain

$$\begin{aligned}|\{|u_\varepsilon| \geqslant l\}| &\leqslant \int_\Omega \frac{|T_l(u_\varepsilon)|^{1+\nu}}{l^{1+\nu}}\,\mathrm{d}x \\ &\leqslant C(\nu,d)l^{-(1+\nu)}\int_\Omega |\nabla T_l(u_\varepsilon)|^{1+\nu}\,\mathrm{d}x \leqslant C(\nu,d)(\|f\|_{L^1(\Omega)} + \|a_0\|_{L^1(\Omega)})l^{-\nu}\end{aligned}$$

Since $f \in L^1(\Omega)$, there exists $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{r\to 0^+} \gamma(r) = 0$ and for any subset E of Ω holds $\int_E |f| + |a_0|\,\mathrm{d}x \leqslant \gamma(|E|)$. Hence (VII.4.31) provides (VII.4.32). □

VII.4.4. Convergence results. The a priori estimates in Lemma VII.4.1 and Corollary VII.4.3 imply the following convergences as $\varepsilon \to 0$:

Proposition VII.4.4. *For $\varepsilon \in (0,1]$ and $f \in L^1(\Omega)$ let $u_\varepsilon \in V$ be a weak solution of $(E_\varepsilon, f_\varepsilon)$. Then there exists a Lebesgue measurable function $u : \Omega \to \mathbb{R}$ with $T_k(u) \in W_0^{1,1+\nu}(\Omega)$, $\nabla T_k(u) \in L_M(\Omega;\mathbb{R}^d)$ such that for a subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$*

(VII.4.34) $$u_\varepsilon \to u \text{ a.e. in } \Omega,$$

where

(VII.4.35) $$|\{|u| > l\}| \leqslant Cl^{-\nu}.$$

for any $l > 0$. Moreover,

(VII.4.36) $$T_k(u_\varepsilon) \to T_k(u) \text{ strongly in } L^p(\Omega) \text{ for } p \in [1,\infty) \text{ and a.e. in } \Omega,$$

(VII.4.37) $$\nabla T_k(u_\varepsilon) \rightharpoonup \nabla T_k(u) \text{ weakly in } L^{1+\nu}(\Omega;\mathbb{R}^d),$$

(VII.4.38) $$\nabla T_k(u_\varepsilon) \overset{*}{\rightharpoonup} \nabla T_k(u) \text{ weakly–(*) in } L_M(\Omega;\mathbb{R}^d),$$

for any $k \in \mathbb{N}$ and

(VII.4.39) $$\boldsymbol{a}(x,\nabla T_k(u_\varepsilon)) \overset{*}{\rightharpoonup} \boldsymbol{a}(x,\nabla T_k(u)) \text{ weakly–(*) in } L_{M^*}(\Omega;\mathbb{R}^d).$$

for any $k \in \mathbb{N}$.

Proof. Applying directly Lemma VII.4.1 and (VII.1.4) together with the Sobolev embedding theorem we obtain (VII.4.36), (VII.4.37), (VII.4.38). Moreover there exists $\boldsymbol{\alpha}_k \in L_{M^*}(\Omega;\mathbb{R}^d)$ such that

(VII.4.40) $$\boldsymbol{a}(x,\nabla T_k(u_\varepsilon)) \overset{*}{\rightharpoonup} \boldsymbol{\alpha}_k \text{ weakly–(*) in } L_{M^*}(\Omega;\mathbb{R}^d) \text{ as } \varepsilon \to 0.$$

In (VII.4.36) we choose by the diagonal method a subsequence such that the convergence in (VII.4.36) holds for any $k \in \mathbb{N}$ (ε_i is still indicated by ε). Obviously the same subsequence can be taken in (VII.4.37), (VII.4.38) and (VII.4.40).

Since (VII.4.36) holds for any $k \in \mathbb{N}$ we obtain (VII.4.34) where u is the Lebesgue measurable function which may take values $\pm\infty$. By (VII.4.34)

$$\liminf_{\varepsilon\to 0} |\{|u_\varepsilon| > l\}| \geqslant |\{|u| > l\}|$$

and using (VII.4.33) we obtain (VII.4.35).

We intend to show now that

$$\boldsymbol{\alpha}_k = \boldsymbol{a}(x, \nabla T_k(u)) \tag{VII.4.41}$$

a.e. in Ω. The proof of (VII.4.41) is divided into several steps.
Step 1. Let us introduce the auxiliary sequence which we can choose from the Galerkin approximation of $(E_\varepsilon, f_\varepsilon)$ as follows: $u_\delta = u^n_{\varepsilon(n)}$ with $\delta = \delta(n) = \frac{1}{n} > 0$ such that $T_k(u_\delta) \in W_0^{1,\infty}(\Omega)$ for each δ and

$$u_\delta \to u \text{ a.e. in } \Omega, \tag{VII.4.42}$$

$$\nabla T_k(u_\delta) \overset{*}{\rightharpoonup} \nabla T_k(u) \text{ weakly–(*) in } L_M(\Omega; \mathbb{R}^d), \tag{VII.4.43}$$

$$\nabla T_k(u_\delta) \rightharpoonup \nabla T_k(u) \text{ weakly in } L^{1+\nu}(\Omega; \mathbb{R}^d), \tag{VII.4.44}$$

Step 2. In order to obtain (VII.4.41) we show

$$\limsup_{\varepsilon\to 0} \int_\Omega \boldsymbol{a}(x, \nabla T_k(u_\varepsilon)) \cdot \nabla T_k(u_\varepsilon) \,\mathrm{d}x \leqslant \int_\Omega \boldsymbol{\alpha}_k \cdot \nabla T_k(u) \,\mathrm{d}x. \tag{VII.4.45}$$

To this end we fix $k, l > 0$, take $\varphi = h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u_\delta))$ as a test function in $(E_\varepsilon, f_\varepsilon)$ and obtain:

$$\begin{aligned}
&\int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon))) \left[h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u_\delta))\right] \mathrm{d}x \\
&+ \int_\Omega \boldsymbol{a}(x, \nabla u_\varepsilon) \cdot \nabla \left[h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u_\delta))\right] \mathrm{d}x \\
&+ \int_\Omega \boldsymbol{F}(T_{1/\varepsilon}(u_\varepsilon)) \cdot \nabla \left[h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u_\delta))\right] \mathrm{d}x \\
&= \int_\Omega T_{1/\varepsilon}(f) \left[h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u_\delta))\right] \mathrm{d}x.
\end{aligned} \tag{VII.4.46}$$

We denote (VII.4.46) by

$$I^0_{\varepsilon,\delta} + I^1_{\varepsilon,\delta} + I^2_{\varepsilon,\delta} = I^3_{\varepsilon,\delta}.$$

First we focus on easier terms - $I^0_{\varepsilon,\delta}$, $I^2_{\varepsilon,\delta}$ and $I^3_{\varepsilon,\delta}$. As

$$I^0_{\varepsilon,\delta} = \int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x, T_{l+1}(u_\varepsilon))) \left[h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u_\delta))\right] \mathrm{d}x$$

for $\varepsilon > 0$ small enough, using (VII.4.36), (VII.1.5), the Lebesgue dominated convergence theorem and the property (VII.4.42) we get

$$\lim_{\delta\to 0} \lim_{\varepsilon\to 0} I^0_{\varepsilon,\delta} = 0.$$

Let us write

$$I^2_{\varepsilon,\delta} = I^{2,1}_{\varepsilon,\delta} + I^{2,2}_{\varepsilon,\delta},$$

where

$$\begin{aligned} I^{2,1}_{\varepsilon,\delta} &= \int_\Omega \boldsymbol{F}(T_{1/\varepsilon}(u_\varepsilon)) \cdot \nabla(T_k(u_\varepsilon) - T_k(u_\delta))h_l(u_\varepsilon)\,\mathrm{d}x, \\ I^{2,2}_{\varepsilon,\delta} &= \int_\Omega \boldsymbol{F}(T_{1/\varepsilon}(u_\varepsilon)) \cdot \nabla u_\varepsilon h_l'(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u_\delta))\,\mathrm{d}x. \end{aligned}$$

For $\varepsilon > 0$ small enough we have

$$I^{2,1}_{\varepsilon,\delta} = \int_\Omega \boldsymbol{F}(T_{l+1}(u_\varepsilon)) \cdot \nabla(T_k(u_\varepsilon) - T_k(u_\delta))h_l(u_\varepsilon)\,\mathrm{d}x,$$

therefore by (VII.4.36), (VII.4.37) and (VII.4.44) it follows that

$$\lim_{\delta\to0}\lim_{\varepsilon\to0} I^{2,1}_{\varepsilon,\delta} = 0.$$

Now let us write

$$I^{2,2}_{\varepsilon,\delta} = \int_\Omega \operatorname{div}\left(\int_0^{T_{l+1}(u_\varepsilon)} \boldsymbol{F}(r)h_l'(r)\ \mathrm{d}r\right)(T_k(u_\varepsilon) - T_k(u_\delta))\,\mathrm{d}x,$$

hence from Gauss-Green theorem for Sobolev functions it follows that

$$I^{2,2}_{\varepsilon,\delta} = -\int_\Omega \int_0^{T_{l+1}(u_\varepsilon)} \boldsymbol{F}(r)h_l'(r)\ \mathrm{d}r \cdot \nabla(T_k(u_\varepsilon) - T_k(u_\delta))\,\mathrm{d}x,$$

and therefore we also get

$$\lim_{\delta\to0}\lim_{\varepsilon\to0} I^{2,3}_{\varepsilon,\delta} = 0$$

from (VII.4.36), (VII.4.37) and (VII.4.43).
Moreover, since

$$|h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u_\delta))| \leqslant 2k$$

and $|T_{1/\varepsilon}(f)| \leqslant |f|$ a.e. in Ω, by (VII.4.36), the Lebesgue dominated convergence theorem and (VII.4.42) it follows that

$$\lim_{\delta\to0}\lim_{\varepsilon\to0} I^{3}_{\varepsilon,\delta} = 0.$$

Finally we concentrate on the most difficult term $I^1_{\varepsilon,\delta}$.

$$\begin{aligned} I^1_{\varepsilon,\delta} = I^{1,1}_{\varepsilon,\delta} + I^{1,2}_{\varepsilon,\delta} &= \int_\Omega \boldsymbol{a}(x,\nabla u_\varepsilon) \cdot \nabla h_l(u_\varepsilon)\,[(T_k(u_\varepsilon) - T_k(u_\delta))]\ \mathrm{d}x \\ &+ \int_\Omega \boldsymbol{a}(x,\nabla u_\varepsilon) \cdot h_l(u_\varepsilon)\nabla[T_k(u_\varepsilon) - T_k(u_\delta)]\,\mathrm{d}x \end{aligned}$$

Applying (VII.4.32) we infer

$$
\begin{aligned}
&\sup_{\delta}\sup_{\varepsilon\in(0,1]}|I^{1,1}_{\varepsilon,\delta}|\\
&=\sup_{\delta}\sup_{\varepsilon\in(0,1]}\int_{\{l<|u_\varepsilon|<l+1\}}\boldsymbol{a}(x,\nabla T_{l+1}(u_\varepsilon))\cdot\nabla T_{l+1}(u_\varepsilon)|\,[(T_k(u_\varepsilon)-T_k(u_\delta))]\,|\,\mathrm{d}x\\
&\leqslant\sup_{\delta}\sup_{\varepsilon\in(0,1]}2k\int_{\{l<|u_\varepsilon|<l+1\}}\boldsymbol{a}(x,\nabla T_{l+1}(u_\varepsilon))\cdot\nabla T_{l+1}(u_\varepsilon)\,\mathrm{d}x\\
&\leqslant 2k\gamma(Cl^{-\nu})
\end{aligned}
$$

therefore

$$
\lim_{l\to\infty}\sup_{\delta}\sup_{\varepsilon\in(0,1]}|I^{1,1}_{\varepsilon,\delta}|=0. \tag{VII.4.47}
$$

Then the above considerations for (VII.4.46) provide
(VII.4.48)

$$
\limsup_{l\to\infty}\limsup_{\delta\to0}\limsup_{\varepsilon\to0}\int_\Omega\boldsymbol{a}(x,\nabla T_k(u_\varepsilon))\cdot h_l(u_\varepsilon)\nabla(T_k(u_\varepsilon)-T_k(u_\delta))\,\mathrm{d}x\leqslant0.
$$

Note that for $l>k$
(VII.4.49)

$$
\begin{aligned}
&\int_\Omega\boldsymbol{a}(x,\nabla T_k(u_\varepsilon))\cdot\nabla(T_k(u_\varepsilon)-T_k(u_\delta))\,\mathrm{d}x-\int_{\{|u_\varepsilon|>l\}}h_l(u_\varepsilon)\boldsymbol{a}(x,0)\cdot\nabla T_k(u_\delta)\,\mathrm{d}x\\
&=\int_\Omega\boldsymbol{a}(x,\nabla T_k(u_\varepsilon))\cdot h_l(u_\varepsilon)\nabla(T_k(u_\varepsilon)-T_k(u_\delta))\,\mathrm{d}x.
\end{aligned}
$$

Let us now concentrate on the second term of (VII.4.49) and notice that

$$
\mathbb{1}_{\{|u_\varepsilon|>l\}}\overset{*}{\rightharpoonup}\chi\text{ weakly–* in }L^\infty(\Omega),
$$

where $\chi\in L^\infty(\Omega)$ and $\chi\in\operatorname{sign}^+(|u_\varepsilon|-l)$ a.e. in Ω. As (VII.4.34) holds and h_l is bounded, $\boldsymbol{a}(x,0)\in L_{M^*}(\Omega;\mathbb{R}^d)=E_{M^*}(\Omega;\mathbb{R}^d)$ and, for fixed δ, $\nabla T_k(u_\delta)\in L^\infty(\Omega;\mathbb{R}^d)$, we obtain

$$
\lim_{\varepsilon\to0}\int_{\{|u_\varepsilon|>l\}}h_l(u_\varepsilon)\boldsymbol{a}(x,0)\cdot\nabla T_k(u_\delta)\,\mathrm{d}x=\int_\Omega\chi h_l(u)a(x,0)\cdot\nabla T_k(u_\delta)\,\mathrm{d}x.
$$

Then by (VII.4.43) and since $\chi h_l(u)a(x,0)\in E_{M^*}(\Omega;\mathbb{R}^d)$ we get

$$
\lim_{\delta\to0}\int_\Omega\chi h_l(u)\boldsymbol{a}(x,0)\cdot\nabla T_k(u_\delta)\,\mathrm{d}x=\int_\Omega\chi h_l(u)\boldsymbol{a}(x,0)\cdot\nabla T_k(u)\,\mathrm{d}x.
$$

As $\chi=0$ on the set $\{|u|<l\}$, the right-hand side in the above vanishes.

Since $\nabla T_k(u_\delta)\in L^\infty(\Omega;\mathbb{R}^d)$, we can now combine (VII.4.48) with (VII.4.49) and pass to the limit with $\varepsilon\to0$ and next with $\delta\to0$ in order to obtain (VII.4.45).

Step 3. Since $\boldsymbol{a}(x,\cdot)$ is monotone we have

$$\begin{aligned}&\int_\Omega \boldsymbol{a}(x,\nabla T_k(u_\varepsilon))\cdot\nabla T_k(u_\varepsilon)\,\mathrm{d}x\\ &\qquad\geqslant \int_\Omega \boldsymbol{a}(x,\nabla T_k(u_\varepsilon))\cdot\boldsymbol{\zeta}\,\mathrm{d}x+\int_\Omega \boldsymbol{a}(x,\boldsymbol{\zeta})\cdot(\nabla T_k(u_\varepsilon)-\boldsymbol{\zeta})\,\mathrm{d}x\end{aligned} \tag{VII.4.50}$$

for $\boldsymbol{\zeta}\in L^\infty(\Omega;\mathbb{R}^d)$. Note that $\boldsymbol{a}(x,\boldsymbol{\zeta})\in E_{M^*}(\Omega;\mathbb{R}^d)$.

Letting $\varepsilon\to 0$ in (VII.4.50) and using (VII.4.40), (VII.4.38) and (VII.4.45) we achieve

$$\int_\Omega (\boldsymbol{a}(x,\boldsymbol{\zeta})-\boldsymbol{\alpha}_k)\cdot(\boldsymbol{\zeta}-\nabla T_k(u))\,\mathrm{d}x\geqslant 0. \tag{VII.4.51}$$

Then in the same way as in the previous section we will use the monotonicity trick in order to obtain that

$$\boldsymbol{\alpha}_k=\boldsymbol{a}(x,\nabla T_k(u))\quad \text{a.e. in } \Omega.$$

□

Remark VII.4.5. If $\boldsymbol{a}(x,\boldsymbol{\xi})$ is strictly monotone, from (VII.4.48) and (VII.4.49) we can deduce the convergence of $\nabla T_k(u_\varepsilon)$ to $\nabla T_k(u)$ a.e. on Ω for $\varepsilon\to 0$. More precisely, by the above considerations it can be shown the a.e. convergence

$$\boldsymbol{a}(x,\nabla T_k(u_\varepsilon))-\boldsymbol{a}(x,\nabla T_k(u))\cdot\nabla(T_k(u_\varepsilon)-T_k(u))\to 0 \tag{VII.4.52}$$

when $\varepsilon\to 0$. For more details we refer the reader to the proof of Lemma 3.2 in [**72**] (based on Young measures) or to the proof of Lemma 4.1 in [**131**] (based on classical arguments as in [**36**]).

Moreover, proceeding step by step as in [**72**, Lemma 3.2] or [**131**, Lemma 4.1], in the strictly monotone case, it can be shown that

$$\nabla T_k(u_\varepsilon)\xrightarrow{M}\nabla T_k(u) \text{ in modular in } L_M(\Omega;\mathbb{R}^d)$$

and

$$\boldsymbol{a}(x,\nabla T_k(u_\varepsilon))\xrightarrow{M^*}\nabla\boldsymbol{a}(x,T_k(u)) \text{ in modular in } L_{M^*}(\Omega;\mathbb{R}^d).$$

VII.4.5. Renormalized solutions to (E,f) with $f\in L^1$. Now we will show existence of the renormalized solution and finish the proof of Theorem VII.3.1. From the Galerkin approximation of $(E_\varepsilon,f_\varepsilon)$ again we can choose a sequence $u_\delta=u^n_{\varepsilon(n)}$ with $\delta=\delta(n)=\frac{1}{n}>0$ such that

$$u_\delta\to u \text{ a.e. in } \Omega, \tag{VII.4.53}$$

$$\nabla T_k(u_\delta)\xrightarrow{*}\nabla T_k(u) \text{ weakly–(*) in } L_M(\Omega;\mathbb{R}^d), \tag{VII.4.54}$$

$$\nabla h(u_\delta)\xrightarrow{*}\nabla h(u) \text{ weakly–(*) in } L_M(\Omega;\mathbb{R}^d) \tag{VII.4.55}$$

for all $h\in C^1_c(\Omega)$ as $\delta\to 0$.

Testing

$$T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon))) - \operatorname{div}\big(\boldsymbol{a}(x, \nabla u_\varepsilon) + \boldsymbol{F}(T_{1/\varepsilon}(u_\varepsilon))\big) = T_{1/\varepsilon}(f)$$

by $h_l(u_\varepsilon)h(u_\delta)\phi$, where $\phi \in W_0^{1,\infty}(\Omega)$, $h \in C_c^1(\Omega)$ and h_l is defined by (VII.2.1) we get

$$\int_\Omega T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon)))h_l(u_\varepsilon)h(u_\delta)\phi \,\mathrm{d}x + \int_\Omega \boldsymbol{a}(x, \nabla u_\varepsilon) \cdot \nabla\left[h_l(u_\varepsilon)h(u_\delta)\phi\right] \mathrm{d}x$$
$$+ \int_\Omega \boldsymbol{F}(T_{1/\varepsilon}(u_\varepsilon)) \cdot \nabla\left[h_l(u_\varepsilon)h(u_\delta)\phi\right] \mathrm{d}x = \int_\Omega T_{1/\varepsilon}(f)\left[h_l(u_\varepsilon)h(u_\delta)\phi\right] \mathrm{d}x$$

and we denote the above the above equality by

$$I^0_{\varepsilon,\delta,l} + I^1_{\varepsilon,\delta,l} + I^2_{\varepsilon,\delta,l} = I^3_{\varepsilon,\delta,l}.$$

Note that in $I^0_{\varepsilon,\delta,l}$ the term u_ε can be replaced by $T_{l+1}(u_\varepsilon)$. For fixed l, the sequence $\{(\beta_\varepsilon(x, T_{l+1}(u_\varepsilon))\}_{\varepsilon>0}$ is a.e. bounded in Ω by $\max(\beta^0(x, l+1), -\beta^0(x, -l-1))$ and, by (VII.1.5), this function is in $L^1(\Omega)$. It follows that there exists b_l such that

(VII.4.56) $$\beta_\varepsilon(\cdot, (T_{l+1}(u_\varepsilon))) \rightharpoonup b_l \text{ weakly in } L^1(\Omega) \text{ for fixed } l \in \mathbb{R}.$$

Moreover we also have

$$T_{1/\varepsilon}(\beta_\varepsilon(\cdot, T_{l+1}(u_\varepsilon))) \rightharpoonup b_l \text{ weakly in } L^1(\Omega) \text{ for fixed } l \in \mathbb{R}.$$

Note that $h_l(u_\varepsilon)h(u_\delta)\phi$ is bounded uniformly (with respect to $\varepsilon > 0$) in $L^\infty(\Omega)$, hence using (VII.4.34) and the Egorov theorem applied to $\{h_l(u_\varepsilon)\}_{\varepsilon>0}$, combining this with uniform integrability of $\{T_{1/\varepsilon}(\beta_\varepsilon(x, T_{1/\varepsilon}(u_\varepsilon)))h_l(u_\varepsilon)h(u_\delta)\phi\}_{\varepsilon>0}$, we obtain

$$\lim_{\varepsilon\to 0} I^0_{\varepsilon,\delta,l} = \int_\Omega b_l h_l(u)h(u_\delta)\phi \,\mathrm{d}x =: I^0_{\delta,l}.$$

Now the Lebesgue dominated convergence theorem provides

$$\lim_{\delta\to 0} I^0_{\delta,l} = \int_\Omega b_l h_l(u)h(u)\phi \,\mathrm{d}x := I^0_l.$$

Since there exists $m > 0$ such that h has compact support in $[-m, m]$, for all $l > m$ we obtain

$$I^0_l = \int_\Omega b_l h(u)\phi \,\mathrm{d}x.$$

We continue the investigation of $\lim\limits_{l\to\infty} I^0_l$ in Section VII.4.6.

Observe that

(VII.4.57)
$$I^1_{\varepsilon,\delta,l} = \int_\Omega \boldsymbol{a}(x, \nabla T_{l+1}(u_\varepsilon)) \cdot \nabla h_l(u_\varepsilon)h(u_\delta)\phi \,\mathrm{d}x$$
$$+ \int_\Omega \boldsymbol{a}(x, \nabla T_{l+1}(u_\varepsilon))h_l(u_\varepsilon) \cdot \nabla[h(u_\delta)\phi] \,\mathrm{d}x =: I^{1,1}_{\varepsilon,\delta,l} + I^{1,2}_{\varepsilon,\delta,l},$$

where

(VII.4.58)
$$\sup_{\varepsilon\in(0,1]} |I^{1,1}_{\varepsilon,\delta,l}| \leqslant \|h\|_{L^\infty(\Omega)}\|\phi\|_{L^\infty(\Omega)} \sup_{\varepsilon\in(0,1]} \int_{\{l<|u_\varepsilon|<l+1\}} |\boldsymbol{a}(x, \nabla T_{l+1}(u_\varepsilon)) \cdot \nabla T_{l+1}(u_\varepsilon)| \,\mathrm{d}x.$$

Using Corollary VII.4.3 from (VII.4.58) it follows that

$$\lim_{l\to\infty}\sup_{\delta}\limsup_{\varepsilon\to 0}|I^{1,1}_{\varepsilon,\delta,l}| = 0. \tag{VII.4.59}$$

By (VII.4.28), (VII.4.39) and Lemma III.2.2 it follows that $\boldsymbol{a}(x,\nabla T_{l+1}(u_\varepsilon)) \rightharpoonup \boldsymbol{a}(x,\nabla T_{l+1}(u))$ in $L^1(\Omega;\mathbb{R}^d)$. Moreover, $h_l(u_\varepsilon)\to h_l(u)$ a.e. in Ω, $|h_l(u_\varepsilon)|\leqslant 1$ and $\nabla(h(u_\delta)\phi)\in L^\infty(\Omega;\mathbb{R}^d)$. Applying the Egorov theorem to $\{h_l(u_\varepsilon)\}_{\varepsilon>0}$ and using the uniform integrability of the sequence $\{\boldsymbol{a}(x,\nabla T_{l+1}(u_\varepsilon))h_l(u_\varepsilon)\cdot\nabla[h(u_\delta)\phi]\}_{\varepsilon>0}$ it follows that

$$\lim_{\varepsilon\to 0} I^{1,2}_{\varepsilon,\delta,l} = \int_\Omega \boldsymbol{a}(x,\nabla T_{l+1}(u))h_l(u)\nabla(h(u_\delta)\phi)\,\mathrm{d}x =: I^{1,2}_{\delta,l} \tag{VII.4.60}$$

Since $\boldsymbol{a}(x,\nabla T_{l+1}(u))h_l(u)\in E_{M^*}(\Omega;\mathbb{R}^d)$, using (VII.4.55) we can pass to the limit with $\delta\to 0$ and obtain

$$\lim_{\delta\to 0} I^{1,2}_{\delta,l} = \int_\Omega \boldsymbol{a}(x,\nabla T_{l+1}(u))h_l(u)\nabla(h(u)\phi)\,\mathrm{d}x =: I^{1,2}_{l} \tag{VII.4.61}$$

For $l>m$, where m is such that supp $h\subset[-m,m]$, from (VII.4.61) we get

$$I^{1,2}_l = I^{1,2} = \int_\Omega \boldsymbol{a}(x,\nabla u)\nabla(h(u)\phi)\,\mathrm{d}x.$$

For ε such that $1/\varepsilon\geqslant l+1$ we have

$$\begin{aligned} I^2_{\varepsilon,\delta,l} &= \int_\Omega \boldsymbol{F}(T_{l+1}(u_\varepsilon))\cdot\nabla h_l(u_\varepsilon)h(u_\delta)\phi\,\mathrm{d}x \\ &+ \int_\Omega \boldsymbol{F}(T_{l+1}(u_\varepsilon))h_l(u_\varepsilon)\cdot\nabla[h(u_\delta)\phi]\,\mathrm{d}x =: I^{2,1}_{\varepsilon,\delta,l} + I^{2,2}_{\varepsilon,\delta,l}. \end{aligned} \tag{VII.4.62}$$

Since $\nabla T_{l+1}(u_\varepsilon)\rightharpoonup\nabla T_{l+1}(u)$ weakly in $L^{1+\nu}(\Omega;\mathbb{R}^d)$ and as $\boldsymbol{F}(T_{l+1}(u_\varepsilon))h_l'(u_\varepsilon)\to\boldsymbol{F}(T_{l+1}(u))h_l'(u)$ in $L^p(\Omega;\mathbb{R}^d)$ for $p=(1+\nu)'$ we have

$$\begin{aligned} \lim_{\varepsilon\to 0} I^{2,1}_{\varepsilon,\delta,l} &= \lim_{\varepsilon\to 0}\int_\Omega \boldsymbol{F}(T_{l+1}(u_\varepsilon))h_l'(u_\varepsilon)\nabla T_{l+1}(u_\varepsilon)h(u_\delta)\phi\,\mathrm{d}x \\ &= \int_\Omega \boldsymbol{F}(T_{l+1}(u))h_l'(u)\nabla T_{l+1}(u)h(u_\delta)\phi\,\mathrm{d}x. \end{aligned}$$

By Lebesgue dominated convergence theorem

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} I^{2,1}_{\varepsilon,\delta,l} = \int_\Omega \boldsymbol{F}(T_{l+1}(u))h_l'(u)\nabla T_{l+1}(u)h(u)\phi\,\mathrm{d}x. \tag{VII.4.63}$$

Choosing $m>0$ such that $\operatorname{supp} h\subset[-m,m]$, T_{l+1} can be replaced by T_m in (VII.4.63) and since $h_l'(u)=h_l'(T_m(u))=0$ for $l+1>m$ it follows that

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} I^{2,1}_{\varepsilon,\delta,l} = 0 \text{ for } l>m-1.$$

Since $\boldsymbol{F}(T_{l+1}(\cdot))h_l(\cdot)$ is uniformly bounded, the a.e. convergence of $\{u_\varepsilon\}_{\varepsilon>0}$ and the Vitali lemma provide that $\boldsymbol{F}(T_{l+1}(u_\varepsilon))h_l(u_\varepsilon)\to\boldsymbol{F}(T_{l+1}(u))h_l(u)$ in $L^p(\Omega;\mathbb{R}^d)$ for any $p\in[1,\infty)$, thus

$$\lim_{\varepsilon\to 0} I^{2,2}_{\varepsilon,\delta,l} = \int_\Omega \boldsymbol{F}(T_{l+1}(u))h_l(u)\cdot\nabla[h(u_\delta)\phi]\,\mathrm{d}x.$$

As $\nabla[h(u_\delta)\phi] \overset{*}{\rightharpoonup} \nabla[h(u)\phi]$ in $L_M(\Omega;\mathbb{R}^d)$ and $\boldsymbol{F}$ is locally Lipschitz continuous, we find that

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} I^{2,2}_{\varepsilon,\delta,l} = \int_\Omega \boldsymbol{F}(T_{l+1}(u))h_l(u)\cdot\nabla[h(u)\phi]\,\mathrm{d}x. \tag{VII.4.64}$$

Again, for $m > 0$ such that $\operatorname{supp} h \subset [-m,m]$, T_{l+1} can be replaced by T_m in (VII.4.64) and $h_l(u) = h_l(T_m(u)) = 1$ for $l > m$. Rewriting (VII.4.64) we obtain

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} I^{2,2}_{\varepsilon,\delta,l} = \int_\Omega \boldsymbol{F}(u)\cdot\nabla[h(u)\phi]\,\mathrm{d}x \text{ for } l > m.$$

Applying the Lebesgue dominated convergence theorem we get

$$\lim_{l\to\infty}\lim_{\delta\to 0}\lim_{\varepsilon\to 0} I^{3}_{\varepsilon,\delta,l} = \lim_{l\to\infty}\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_\Omega T_{1/\varepsilon}(f)h_l(u_\varepsilon)h(u_\delta)\phi\,\mathrm{d}x = \int_\Omega fh(u)\phi\,\mathrm{d}x.$$

VII.4.6. Subdifferential argument. Since $\beta(x,\cdot)$ is maximal monotone for almost all $x \in \Omega$, there exists $j : \Omega\times\mathbb{R}\to\mathbb{R}$, such that

$$\beta(x,r) = \partial_r j(x,r) \text{ for all } r\in\mathbb{R}, \text{ a.e. in } \Omega.$$

For $0 < \varepsilon \leqslant 1$ let us define $j_\varepsilon : \Omega\times\mathbb{R}\to\mathbb{R}$ by

$$j_\varepsilon(x,r) = \inf_{s\in\mathbb{R}}\{j(x,s) + \frac{1}{2\varepsilon}|r-s|^2\}.$$

According to **[28]**, j_ε has the following properties:

i.) j_ε is a Carathéodory function.

ii.) For any $0 < \varepsilon \leqslant 1$, $j_\varepsilon(x,r)$ is convex and differentiable with respect to $r\in\mathbb{R}$, moreover

$$\partial_r j_\varepsilon(x,r) = \beta_\varepsilon(x,r) \text{ for all } r\in\mathbb{R} \text{ and any } 0<\varepsilon\leqslant 1 \text{ and a.e. in } \Omega.$$

iii.) $j_\varepsilon(x,r)\uparrow j(x,r)$ pointwise in $\mathbb{R}$ as $\varepsilon\to 0$ and a.e. in Ω.

From *ii.)* it follows that

$$j_\varepsilon(x,r) \geqslant j_\varepsilon(x,T_{1/\varepsilon}(u_\varepsilon)) + (r - T_{1/\varepsilon}(u_\varepsilon))\beta_\varepsilon(x,T_{1/\varepsilon}(u_\varepsilon)) \tag{VII.4.65}$$

holds for all $r\in\mathbb{R}$ and almost everywhere in Ω. Let $E\subset\Omega$ be an arbitrary measurable set and $\mathbb{1}_E$ its characteristic function. We fix $\varepsilon_0 > 0$. Multiplying (VII.4.65) by $h_l(u_\varepsilon)\mathbb{1}_E$, integrating over Ω and using *iii.)*, we obtain

(VII.4.66)

$$\int_E j(x,r)h_l(u_\varepsilon)\,\mathrm{d}x \geqslant \int_E j_{\varepsilon_0}(x,T_{l+1}(u_\varepsilon))h_l(u_\varepsilon) + (r - T_{l+1}(u_\varepsilon))h_l(u_\varepsilon)\beta_\varepsilon(x,T_{1/\varepsilon}(u_\varepsilon))\,\mathrm{d}x$$

for all $r\in\mathbb{R}$ and all $0<\varepsilon<\min(\varepsilon_0,\frac{1}{l})$. Passing to the limit with $\varepsilon\to 0$, and then with $\varepsilon_0\to 0$ in (VII.4.66) we obtain from (VII.4.66) and by (VII.4.56)

$$j(x,r) \geqslant j(x,u) + b_l(r-u) \tag{VII.4.67}$$

for all $r \in \mathbb{R}$ almost everywhere in $\{|u| \leqslant l\}$ and therefore $b_l \in \beta(x,u)$ a.e. in $\{|u| \leqslant l\}$. Note that $b_l = b_m$ a.e. on $\{|u| \leqslant m\}$ for all $l \geqslant m > 0$. Moreover u is measurable and finite a.e. in Ω. Thus the function $b : \Omega \to \mathbb{R}$ defined by $b = b_l$ on $\{|u| \leqslant l\}$ is well-defined and measurable with $b \in \beta(x,u)$ a.e. in Ω. Next, we use $h_l(u_\varepsilon)\frac{1}{k}T_k(u_\varepsilon)$ as a test function in $(E_\varepsilon, f_\varepsilon)$. Applying Corollary VII.4.3 to the diffusion term, the Stokes theorem to the convection term and neglecting nonnegative terms we can pass to the limit with $\varepsilon \to 0$ and obtain

$$\int_\Omega b_l \frac{1}{k} T_k(u) h_l(u)\,\mathrm{d}x \leqslant \int_\Omega |f|\,\mathrm{d}x. \tag{VII.4.68}$$

According to (VII.4.67), $b_l \mathrm{sign}_0(u_\varepsilon) h_l(u) = |b_l| h_l(u)$ a.e. in Ω. Moreover, $|b_l|h_l(u) \to |b|$ a.e. in Ω for $l \to \infty$. Therefore, passing to the limit with $k \to \infty$ in (VII.4.68) and using the Fatou lemma we find

$$\int_\Omega |b|\,\mathrm{d}x \leqslant \|f\|_{L^1(\Omega)}, \tag{VII.4.69}$$

and $b \in L^1(\Omega)$.

VII.4.7. Conclusion of Theorem VII.3.1. Gathering all convergence results from Subsection VII.4.5 it follows finally that u satisfies

(VII.4.70)
$$\int_\Omega (b_l h(u)\phi + (\boldsymbol{a}(x,\nabla u) + \boldsymbol{F}(u))\nabla(h(u)\phi))\,\mathrm{d}x + \limsup_{\delta\to 0}\limsup_{\varepsilon\to 0} I^{1,1}_{\varepsilon,\delta,l} = \int_\Omega f h(u)\phi\,\mathrm{d}x$$

for all $l > m-1 > 0$, $\phi \in W^{1,\infty}_0(\Omega)$ and $h \in C^1_c(\mathbb{R})$ such that $\operatorname{supp} h \subset [-m,m]$, where $I^{1,1}$ is defined in (VII.4.57). Thanks to (VII.4.59) and (VII.4.69) we can pass to the limit in (VII.4.70) and obtain (VII.2.6) for all $\phi \in W^{1,\infty}_0(\Omega)$ and arbitrary $h \in C^1_c(\mathbb{R})$. Moreover, from (VII.4.34) and (VII.4.35) it follows that (u,b) satisfies **(R1)**. From (VII.4.38) and (VII.4.39) we have $T_k(u) \in V \cap L^\infty(\Omega)$ and $\boldsymbol{a}(x,\nabla T_k(u)) \in L_{M^*}(\Omega;\mathbb{R}^d)$ for all $k > 0$. Using that the gradients of functions in V can be approximated by smooth functions in the weak-$*$ topology of $L_M(\Omega;\mathbb{R}^d)$ we finally arrive at

$$\int_\Omega b h(u)\phi\,\mathrm{d}x + (\boldsymbol{a}(x,\nabla u) + \boldsymbol{F}(u))\nabla(h(u)\phi)\,\mathrm{d}x = \int_\Omega f h(u)\phi\,\mathrm{d}x \tag{VII.4.71}$$

for all $\phi \in V \cap L^\infty(\Omega)$ and $h \in C^1_c(\mathbb{R})$, hence (u,b) satisfies (**R2**). Finally, from (VII.4.32) with classical arguments we obtain (**R3**) and the proof of Theorem VII.3.1 is complete.

Remark VII.4.6. The assumption that the function $\boldsymbol{F}$ is locally Lipschitz continuous is not crucial. In the proof of Theorem VII.3.1 only the continuity of $\boldsymbol{F}$ is needed. However, the uniqueness of renormalized solutions is an open problem if $\boldsymbol{F}$ is only continuous. If $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{\xi})$ does not depend on the space variable x and F is only continuous, uniqueness can be proved by the method of doubling variables.

VII.5. Proof of Theorem VII.3.2 - Uniqueness

We will need the following

Lemma VII.5.1. *For* $f, \tilde{f} \in L^1(\Omega)$ *let* (u,b), $(\tilde{u},\tilde{b})$ *be the renormalized solutions to* (E,f) *and* $(E,\tilde{f})$ *respectively. Then*

$$\int_\Omega (b-\tilde{b})\mathrm{sign}_0^+(u-\tilde{u})\,\mathrm{d}x \leqslant \int_\Omega (f-\tilde{f})\mathrm{sign}_0^+(u-\tilde{u})\,\mathrm{d}x. \tag{VII.5.1}$$

Proof. The proof follows the same lines as in the classical L^p and $L^{p(\cdot)}$ setting (see [**129**]). For $\delta > 0$, let H_δ^+ be a Lipschitz approximation of the sign_0^+ -function. Since (u,b), $(\tilde{u},\tilde{b})$ are renormalized solutions, it follows that $T_{l+1}(u), T_{l+1}(\tilde{u}) \in V \cap L^\infty(\Omega)$ for all $l > 0$. Hence $H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u}))$ is in $V \cap L^\infty(\Omega)$ for all $\delta, l > 0$ and therefore is an admissible test function. Now, we choose $H_\delta^+(T_{l+1}(u) - T_{l+1}(\tilde{u}))$ as a test function in the renormalized formulation with $h = h_l$ for (u,b) and for $(\tilde{u},\tilde{b})$ respectively. Subtracting the resulting equalities, we obtain

$$I_{l,\delta}^1 + I_{l,\delta}^2 + I_{l,\delta}^3 + I_{l,\delta}^4 + I_{l,\delta}^5 = I_{l,\delta}^6, \tag{VII.5.2}$$

and

$$\begin{aligned}
I_{l,\delta}^1 &= \int_\Omega \big(bh_l(u) - \tilde{b}h_l(\tilde{u})\big)H_\delta^+\big(T_{l+1}(u) - T_{l+1}(\tilde{u})\big)\,\mathrm{d}x,\\
I_{l,\delta}^2 &= \int_\Omega \big(h_l'(u)\boldsymbol{a}(x,\nabla u)\cdot\nabla u - h_l'(\tilde{u})\boldsymbol{a}(x,\nabla\tilde{u})\cdot\nabla\tilde{u}\big)H_\delta^+\big(T_{l+1}(u) - T_{l+1}(\tilde{u})\big)\,\mathrm{d}x,\\
I_{l,\delta}^3 &= \frac{1}{\delta}\int_K \big(h_l(u)\boldsymbol{a}(x,\nabla u) - h_l(\tilde{u})\boldsymbol{a}(x,\nabla\tilde{u})\big)\cdot\nabla\big(T_{l+1}(u) - T_{l+1}(\tilde{u})\big)\,\mathrm{d}x,\\
I_{l,\delta}^4 &= \int_\Omega \big(h_l'(u)\boldsymbol{F}(u)\cdot\nabla u - h_l'(\tilde{u})\boldsymbol{F}(\tilde{u})\cdot\nabla\tilde{u}\big)H_\delta^+\big(T_{l+1}(u) - T_{l+1}(\tilde{u})\big)\,\mathrm{d}x,\\
I_{l,\delta}^5 &= \frac{1}{\delta}\int_K \big(h_l(u)\boldsymbol{F}(u) - h_l(\tilde{u})\boldsymbol{F}(\tilde{u})\big)\cdot\nabla\big(T_{l+1}(u) - T_{l+1}(\tilde{u})\big)\,\mathrm{d}x,\\
I_{l,\delta}^6 &= \int_\Omega \big(fh_l(u) - \tilde{f}h_l(\tilde{u})\big)H_\delta^+\big(T_{l+1}(u) - T_{l+1}(\tilde{u})\big)\,\mathrm{d}x,
\end{aligned}$$

where $K := \{0 < T_{l+1}(u) - T_{l+1}(\tilde{u}) < \delta\}$. Using the same arguments as in [**129**], i.e. neglecting the nonnegative part of $I_{l,\delta}^3$ and using that $\boldsymbol{F}$ is locally Lipschitz continuous, we can pass to the limit with $\delta \to 0$. Using the energy dissipation condition **(R3)** we can pass to the limit with $l \to \infty$ and obtain (VII.5.1). □

Now we are in the position to give the proof of Theorem VII.3.2:
Assuming $f = \tilde{f}$, from Lemma VII.5.1 we get

$$\int_\Omega (b-\tilde{b})\mathrm{sign}_0^+(u-\tilde{u})\,\mathrm{d}x \leqslant 0, \tag{VII.5.3}$$

hence $(b-\tilde{b})\mathrm{sign}_0^+(u-\tilde{u}) = 0$ almost everywhere in Ω. Now, let us write $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 := \{x \in \Omega :\ \mathrm{sign}_0^+(u(x) - \tilde{u}(x)) = 0\}$, $\Omega_2 := \{x \in \Omega :\ (b(x) - \tilde{b}(x)) = 0\}$. Since $r \mapsto \beta(x,r)$ is strictly increasing for a.e. $x \in \Omega$, we can define the function

$\beta_x^{-1} : \mathbb{R} \to \mathbb{R}$ such that $\beta_x^{-1}(r) = s$ for all $(r, s) \in \mathbb{R}^2$ satisfying that $r \in \beta(x, s)$ for almost every $x \in \Omega$. For a.e. $x \in \Omega_2$ we have $b(x) = \tilde{b}(x)$, hence $u(x) = \beta_x^{-1}(b(x)) = \beta_x^{-1}(\tilde{b}(x)) = \tilde{u}(x)$. Therefore, $u(x) = \tilde{u}(x)$ a.e. in Ω_2 and $\operatorname{sign}_0^+(u - \tilde{u}) = 0$ a.e. in Ω. Interchanging the roles of u and $\tilde{u}$ and repeating the arguments, we get $\operatorname{sign}_0^+(\tilde{u} - u) = 0$ a.e. in Ω and we finally arrive at $u = \tilde{u}$ a.e. in Ω. Now, we write the renormalized formulation for (u, b) and $(\tilde{u}, \tilde{b})$ respectively. Subtracting the resulting equalities, we obtain

$$\int_\Omega (b - \tilde{b}) h(u) \varphi \,\mathrm{d}x = 0$$

for all $h \in C_c^1(\mathbb{R})$ and all $\varphi \in \mathcal{D}(\Omega)$. Choosing $h(u) = h_l(u)$ and passing to the limit with $l \to \infty$ we obtain that $b = \tilde{b}$ a.e. in Ω.

VII.6. Proof of Proposition VII.3.3 - Weak solutions

The proof of Proposition VII.3.3 follows along the same lines as in [**129**].

From Remark VII.2.6 it follows that it suffices to prove $u \in L^\infty(\Omega)$:

Note that for $\varepsilon, k > 0$, $h_l(u)\frac{1}{\varepsilon}T_\varepsilon(u - T_k(u))$ is an admissible test function in (VII.2.6). Neglecting positive terms and passing to the limit with $l \to \infty$, we use (VII.1.1) to obtain

(VII.6.1)
$$\frac{1}{\varepsilon}\int_{\{k<|u|<k+\varepsilon\}} c_a M(x, \nabla u)\,\mathrm{d}x \leqslant \left(\|f\|_d \,(\phi(k))^{(d-1)/d} + \frac{\phi(k) - \phi(k+\varepsilon)}{\varepsilon}\|a_0\|_\infty \right),$$

where $\phi(k) := |\{|u| > k\}|$ for $k > 0$. Now we apply similar arguments as in [**17**]. Continuous embedding of $W_0^{1,1}(\Omega)$ into $L^{d/(d-1)}(\Omega)$ and the Hölder inequality provide that

(VII.6.2)
$$\frac{1}{\varepsilon C_d}\|T_\varepsilon(u - T_k(u))\|_{\frac{d}{d-1}} \leqslant \left(\frac{\phi(k) - \phi(k+\varepsilon)}{\varepsilon}\right)^{\frac{1}{(1+\nu)'}} \left(\frac{1}{\varepsilon}\int_{\{k<|u|<k+\varepsilon\}} |\nabla u|^{1+\nu}\,\mathrm{d}x\right)^{\frac{1}{1+\nu}},$$

where $C_d > 0$ is the constant coming from the Sobolev embedding. From (VII.1.4) it follows that

$$\text{(VII.6.3)} \qquad \frac{1}{\varepsilon}\int_{\{k<|u|<k+\varepsilon\}} |\nabla u|^{1+\nu}\,\mathrm{d}x \leqslant \frac{1}{c c_a \varepsilon}\int_{\{k<|u|<k+\varepsilon\}} c_a M(x, \nabla u)\,\mathrm{d}x,$$

hence from (VII.6.1), (VII.6.2) and (VII.6.3) we deduce

$$\frac{1}{\varepsilon C_d}\|T_\varepsilon(u - T_k(u))\|_{\frac{d}{d-1}} \leqslant$$

(VII.6.4)
$$\left(\frac{\phi(k) - \phi(k+\varepsilon)}{\varepsilon}\right)^{\frac{1}{(1+\nu)'}} \left(\frac{1}{c c_a}\left(\|f\|_d \,(\phi(k))^{(d-1)/d} + \frac{\phi(k) - \phi(k+\varepsilon)}{\varepsilon}\|a_0\|_\infty \right)\right)^{\frac{1}{1+\nu}}.$$

From (VII.6.4) and Young's inequality with $\alpha > 0$ it follows that
(VII.6.5)
$$\frac{1}{C_d C}(\phi(k+\varepsilon))^{(d-1)/d} - \frac{\alpha^{1+\nu}}{(1+\nu)Ccc_a}\left(\|f\|_d\,(\phi(k))^{(d-1)/d}\right) - \frac{\phi(k)-\phi(k+\varepsilon)}{\varepsilon} \leqslant 0,$$
where
$$C := \left(\frac{1}{\alpha^{(1+\nu)'}(1+\nu)'} + \frac{\|a_0\|_\infty}{cc_a}\frac{\alpha^{1+\nu}}{1+\nu}\right) > 0.$$
The mapping $(0,\infty) \ni k \mapsto \phi(k)$ is non-increasing and therefore of bounded variation, hence it is differentiable almost everywhere on $(0,\infty)$ with $\phi' \in L^1_{\text{loc}}(0,\infty)$. Since it is also continuous from the right, we can pass to the limit with $\varepsilon \to 0$ in (VII.6.5) to find

(VII.6.6)
$$C''(\phi(k))^{(d-1)/d} + \phi'(k) \leqslant 0$$
for almost every $k > 0$ and $\alpha > 0$ chosen so small that
$$C'' := \left(\frac{1}{C_d C} - \frac{\alpha^{1+\nu}}{(1+\nu)Ccc_a}\|f\|_d\right) > 0.$$
Now, the conclusion of the proof follows by contradiction. We assume that $\phi(k) > 0$ for each $k > 0$. For $k > 0$ fixed, we choose $k_0 < k$. Multiplying (VII.6.6) by $\frac{1}{d}(\phi(k))^{-(d-1/d)}$ it follows that

(VII.6.7)
$$\frac{1}{d}C'' + \frac{d}{ds}\left((\phi(s))^{(1/d)}\right) \leqslant 0$$
for almost all $s \in (k_0, k)$. The left hand side of (VII.6.7) is in $L^1(k_0, k)$, hence we integrate (VII.6.7) over (k_0, k). Moreover, since ϕ is non-increasing, integrating (VII.6.7) over the interval (k_0, k) we get

(VII.6.8)
$$(\phi(k))^{1/d} \leqslant \phi(k_0)^{1/d} + \frac{1}{d}C''(k_0 - k).$$
Thanks to the second term on the right-hand side of (VII.6.8), we conclude that there exists $k_1 > k_0$ such that $(\phi(k))^{1/d} \leqslant 0$ for all $k > k_1 > k_0$. Therefore $\phi(k) = 0$ for all $k > k_1 > k_0$ and the assertion follows.

Bibliography

[1] R. A. Adams and J. J. A. Fournier. *Sobolev spaces.* Academic Press, 2003.

[2] L. Aharouch, J. Bennouna and A. Touzani. Existence of renormalized solution of some elliptic problems in Orlicz spaces. *Rev. Mat. Complut.* 22 (2009), no. 1, 91–110.

[3] N. Alibaud, B. Andreianov and M. Bendahmane. Renormalized solutions of the fractional Laplace equation. *C. R. Acad. Sci., Paris Sér. A* 348(2010), no.13-14, 759–762.

[4] H. Amann and J. Escher. Analysis. II. Birkhäuser Verlag, Basel, 2008.

[5] K. Ammar and P. Wittbold. Existence of renormalized solutions of degenerate elliptic-parabolic problems. *Proc. Royal Soc. Edinburgh* 133A (2003), no.3, 477–496.

[6] B. Andreianov, M. Bendahmane and S. Ouaro. Structural stability for variable exponent elliptic problems, I: the $p(x)$-Laplacian kind problems. *Nonlinear Anal. TMA* 73 (2010), no. 1, 2–24.

[7] F. Andreu, J.M. Mazón, S. Segura de León and J. Toledo. Existence and uniqueness for a degenerate parabolic equation with L1-data. Trans. Amer. Math. Soc. 351 (1999), no. 1, 285–306.

[8] F. Andreu, N. Igbida, J.M. Mazón and J. Toledo. L^1 existence and uniqueness results for quasi-linear elliptic equations with nonlinear boundary conditions. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 24 (2007), 61–89.

[9] F. Andreu, N. Igbida, J.M. Mazón and J. Toledo. Renormalized solutions for degenerate elliptic-parabolic problems with nonlinear dynamical boundary conditions and L1-data. *J. Diff. Equ.* 244 (2008), no. 11, 2764–2803.

[10] S. N. Antontsev, A. V. Kazhikhov and V.N. Monakhov. *Boundary value problems in mechanics of nonhomogeneus fluids.* Studies in Mathematics and its Applications (translated from Russian), Vol. 22, North-Holland. Amsterdam, 1990.

[11] M. Bendahmane and P. Wittbold. Renormalized solutions for nonlinear elliptic equations with variable exponents and L1-data. *Nonlinear Analysis TMA* 70 (2009), no.2, 567–583

[12] M. Bendahmane, P. Wittbold and A. Zimmermann. Renormalized solutions for a nonlinear parabolic equation with variable exponents and L1-data. *J. Diff. Equ.* 249 (2010), no. 6, 1483–1515.

[13] M. Bendahmane and K. Karlsen. Nonlinear Anisotropic Elliptic and Parabolic Equations in $\mathbb{R}^N$ with Advection and Lower Order Terms and Locally Integrable Data. *Pot. Anal.* 22 (2005), 207–227.

[14] C. Bennett, R. Sharpley. *Interpolation of Operators.* Academic Press, 1988.

[15] Ph. Bénilan, L. Boccardo, Th. Gallouët, R. Gariepy, M. Pierre and J.L. Vazquez. An L^1-existence theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* 22 (1995), 241–273.

[16] Ph. Bénilan, J. Carrillo and P. Wittbold. Renormalized entropy solutions of scalar conservation laws. *Ann. Scuola Norm. Sup. Pisa Cl. Sci* 29 (2000), 313–329.

[17] P. Bénilan, P. Wittbold. On mild and weak solutions of elliptic-parabolic problems. *Adv. In Diff. Equ.* 1 (1996), 1053–1073.

[18] A. Benkirane and J. Bennouna. Existence of entropy solutions for some elliptic problems involving derivatives of nonlinear terms in Orlicz spaces. *Abstr. Appl. Anal.* 7 (2002), no.2, 85–102.

[19] J. Bergh and J. Löfström. *Interpolation spaces: an introduction.* Springer–Verlag, Berlin, Heilderberg, New York, 1976.

[20] D. Blanchard and F. Murat. Renormalised solutions of nonlinear parabolic problems with L^1-data: existence and uniqueness. *Proc. Roy. Soc. Edinburgh* 127A (1997), 1137–1152.

[21] D. Blanchard, F. Murat, H. Radwane. Existence and uniqueness of renormalized solution for a fairly general class of nonlinear parabolic problems. *J. Differential Equations.* 177 (2001), no. 2, 331–374.

[22] D. Blanchard and A. Porretta. Stefan problems with nonlinear diffusion and convection. *J. Diff. Eq.* 210 (2005), no.2, 383–428.

[23] D. Blanchard, H. Radwane. Renormalized solutions for a class of nonlinear evolution problems. *J. Math. Pures Appl.* (9) 77 (1998), no. 2, 117–151.

[24] L. Boccardo, T. Gallouët and P. Marcellini. Anisotropic equations in L1. *Diff. Int. Equ.* 9 (1996), no. 1, 209–212.

[25] L. Boccardo, D. Giachetti, J.I. Diaz and F. Murat. Existence of a solution for a weaker form of a nonlinear elliptic equation, in "Recent Advances in Nonlinear Elliptic and Parabolic Problems", Pitman Research Notes in Mathematics Series, Vol. 208, pp. 229–246, Longman Sci. Tech., Harlow, 1989.

[26] L. Boccardo, F. Murat and J. P. Puel. Existence of Boundary Solutions for Non Linear Elliptic Unilatered Problems. *Annali di Matematica Pura ed Applicata.* 152(1):183–196,1988.

[27] B.K. Bonzi and S. Ouaro. Entropy solutions for a doubly nonlinear ellitic problem with variable exponent. *J. Math. Anal. and Appl.* 370 (2010), 392–405.

[28] H. Brézis. *Opérateurs Maximaux Monotones.* North-Holland, Amsterdam, 1973.

[29] M. Bulíček, P. Gwiazda, A. Świerczewska-Gwiazda and K.R. Rajagopal. On flows of fluids described by an implicit constitutive equation characterized by a maximal monotone graph. London Math. Soc. Lecture Note, Cambridge Univ. Press, Cambridge. 2012.

[30] J. Carrillo and P. Wittbold. Uniqueness of renormalized solutions of degenerate elliptic-parabolic problems. *J. Diff. Equ.* 156 (1999) no.1, 93–121.

[31] O. N. Cavatorta and R. D. Tonini. *Dimensionless velocity profiles and parameter maps for non-Newtonian fluids, International Communications in Heat and Mass Transfer.* 14 (1987), 359 – 369.

[32] A. Cianchi. Strong and weak type inequalities for some classical operators in Orlicz spaces. *J. London Math. Soc.* 60(2):187–202, 1997.

[33] A. Cianchi, *A fully anisotropic Sobolev inequality, Pacific J. Math.*, 196 (2000), 283–295.

[34] A. Cianchi, *Optimal Orlicz-Sobolev embeddings, Rev. Mat. Iberoam.*, 20 (2004), 427–474.

[35] M. Costabel, A. McIntosh. On Bogovskii and regularized Poincaré integral operators for de Rham complexes on Lipshitz domains. *Mathematische Zeitschrift.* 265 (2010),no. 2, 297–320.

[36] G. Dal Maso, F. Murat. Almost everywhere convergence of gradients of solutions to nonlinear elliptic systems. *Nonlinear Anal. TMA* 31 (1998), no. 3-4, 405–412.

[37] G. Dal Maso, F. Murat, L. Orsina and A. Prignet. Renormalized solutions of elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* 28 (1999), 741–809.

[38] R. Danchin, P.B. Mucha. The divergence equation in rough spaces. *Journal of Mathematical Analysis and Applications.* 386 (2012) 10-31.

[39] B. Desjardins and M.J. Esteban. Existence of weak solutions for the motion of rigid bodies in a viscous fluid. *Arch. Rational Mech. Anal.*, 146:59–71, 1999.

[40] B. Desjardins and M.J. Esteban. On weak solutions for fluid-rigid structure interaction: Compressible and incompressible models. *Commun. Partial Differential Equations*, 25:1399–1413, 2000.

[41] L. Diening, J. Málek and M. Steinhauer. On Lipschitz truncations of Sobolev functions (with variable exponent) and their selected applications. ESAIM. *Control, Optimisation and Calculus of Variations.* 14(2)211-232, 2008.

[42] L. Diening, M. Ružička, and K. Schumacher, *A decomposition technique for John domains, Ann. Acad. Sci. Fenn. Math.* 35 (2010), 87–114.

[43] L. Diening, M. Ružička and J. Wolf. Existence of weak solutions for unsteady motion of generalized Newtonian fluids. *Ann. Scuola Norm. Sup. Pisa.* IX:1-46, 2010.

[44] R. DiPerna and P.L. Lions. On the Cauchy problem for Boltzmann equations: Global existence and weak stability. *Ann. of Math.* 130 (1989), 321–366.

[45] R.J. DiPerna and P.L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Inventionse mathematicae.* 98:511-547, 1989.

[46] T. Donaldson. Inhomogeneous Orlicz-Sobolev spaces and nonlinear parabolic initial value problems. *J. Differential Equations.* 16:201-256, 1974.

[47] N. Dunford and J. Schwartz. *Linear operators.* Interscience Publishers, New York, 1958.

[48] L.C. Evans. *Partial Differential Equations.* American Mathematical Society, 1998.

[49] R.G. Egres Jr, Y.S. Lee, J.E. Kirkwood, K.M. Kirkwood, E.D. Wetzl, N.J. Wagner. "Liquid armor": Protective fabrics utilising shear thickening fluids. *Proceedings of the 4th International Conference of Safety and Protective Fabrics*, Pittsburg, PA, October, 26-27, 2004.

[50] A. Elmahi and D. Meskine. Parabolic equations in Orlicz spaces. *J. London Math. Soc. (2).* 72(2):410-428, 2005.

[51] E.Emmrich, A.Wróblewska. Convergence of a Full Discretization of Quasilinear Parabolic Equations in Isotropic and Anisotropic Orlicz Spaces. Prepreprint PhD Programme: Mathematical Methods in Natural Sciences, Nr 2011- 006, 2011.

[52] R. Erban. On the existence of solutions to the Navier -Stokes equations of two -dimensional compressible flow. *Mathematical Methods in the Applied Sciences.* **26**:489-517, 2003.

[53] H. J. Eyring, *Viscosity, plasticity, and diffusion as examples of absolute reaction rates, J. Chemical Physics.* **4** (1936), 283–291.

[54] E. Feireisl and A. Novotný. *Singular limits in thermodynamics of viscous fluids.* Birkhauser, Basel, 2009.

[55] E. Feireisl. On the motion of rigid bodies in a viscous compressible fluid. *Arch. Rational Mech. Anal.*, **167**:281–308, 2003.

[56] E. Feireisl, M. Hillairet and Š. Nečasová. On the motion of several rigid bodies in an incompressible non-Newtonian fluid. *Nonlinearity.* **21**:1349-1366, 2008.

[57] E. Fernández–Cara, F. Guillén–Gonzalez and R. R. Ortega. Some theoretical results for viscoplastic and dilatant fluids with variable density. *Nonlinear Anal.* 28(6):1079-1100, 1997.

[58] J. Frehse, J. Málek, M. Ružička. Large data existence results for unsteady flows of inhomogeneus heat-conducting incompressible fluids. *Communication in Partial Differential Equations.* 35(10):1891–1919, 2010.

[59] J. Frehse and M. Ružička. Non-homogenous generalized Newtonian fluids. *Mathematische Zeitschrift.* 260:355-375, 2008.

[60] J. Frehse, J. Málek and M. Steinhauer. On analysis of steady flows of fluids with shear-dependent viscosity based on the Lipschitz truncation method. SIAM *Journal on Mathematical Analysis*, 34(5):1064-1083, 2003.

[61] M. Fuchs. Korn inequalities in Orlicz spaces. *Irish Math. Soc. Bulletin*, 65: 5-9, 2010.

[62] M. Fuchs and M. Bildhauer. *Compact embeddings of the space of functions with bounded logarithmic deformation,* Preprint Nr. 276, Universitaet des Saarlandes (2010).

[63] M. Fuchs and G. Seregin. *Variational methods for fluids of Prandtl-Eyring type and plastic materials with logarithmic hardening, Math. Methods Appl. Sci.* 22 (1999), 317–351.

[64] M. Fuchs and G. Seregin. *Variational methods for problems from plasticity theory and for generalized Newtonian fluids, Ann. Univ. Sarav., Ser. Math.* **10** (1999), 1–283.

[65] H. Gajewski, K. Gröger and K. Zacharias. *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen.* Akademie-Verlag, Berlin, 1974.

[66] G. P. Galdi. On the steady self-propelled motion of a body in a viscous incompressible fluid. *Arch. Rat. Mech. Anal.*, **148**:53–88, 1999.

[67] G. P. Galdi. On the motion of a rigid body in a viscous fluid: A mathematical analysis with applications. *Handbook of Mathematical Fluid Dynamics, Vol. I, Elsevier Sci, Amsterdam*, 2002.

[68] J.-P. Gossez. Some approximation properties in Orlicz-Sobolev spaces. *Studia Math.* 74 (1982), 17–24.

[69] J.-P. Gossez and V. Mustonen. Variational inequalities in Orlicz-Sobolev spaces. *Nonlinear Analysis TMA* 11 (1987), no. 3, 379–392.

[70] F. Gulilén–González. Density-dependent incompressible fluids with non-Newtonian viscosity. *Czechoslovak Math. J.* 54(3):637–656, 2004.

[71] J. Gustavsson and J. Peetre. Interpolation of Orlicz spaces. *Studia Mathematica.* 60:33–59, 1977.

[72] P. Gwiazda and A. Świerczewska–Gwiazda. On non-Newtonian fluids with the property of rapid thickening under different stimulus. *Math. Models Methods Appl. Sci.*, 7(18):1073-1092, 2008.

[73] P. Gwiazda and A. Świerczewska–Gwiazda. *Parabolic equations in anisotropic orlicz spaces with general N-functions,* Parabolic Problems. The Herbert Amann Festschrift, Progress in Nonlinear Differential Equations and Their Applications, **60** (2011), 301–311.

[74] P. Gwiazda and A. Świerczewska–Gwiazda. On steady non-Newtonian fluids with growth conditions in generalized Orlicz spaces. *Topological Methods in Nonlinear Analysis*, 32(1):103-114, 2008.

[75] P. Gwiazda, A. Świerczewska-Gwiazda, and A. Wróblewska. Monotonicity methods in generalized Orlicz spaces for a class of non-Newtonian fluids. *Math. Methods Appl. Sci.* 33 (2010), 125–137.

[76] P. Gwiazda, A. Świerczewska–Gwiazda and A. Wróblewska. Generalized Stokes system in Orlicz spaces *Discrete and Continuous Dynamical Systems - A.* 32 (2012), Issue 6, 2125–2146.

[77] P. Gwiazda, A. Świerczewska-Gwiazda, A. Wróblewska. Turbulent flow of rapidly thickening fluids. Proceedings of Polish-Japanese Days , GAKUTO Internat. Ser. Math. Sci. Appl., Vol.32 (2010), 307-325.

[78] P. Gwiazda, A. Świerczewska-Gwiazda, A. Warzyński, A. Wróblewska. Well-posedness for non-Newtonian flows with general growth conditions, *Banach Center Publications*, 86 (2009), 115–128.

[79] P. Gwiazda, P. Wittbold, A. Wróblewska, A. Zimmermann. Renormalized solutions of nonlinear elliptic problems in generalized Orlicz spaces. *Journal of Differential Equations* 253, 635 - 666, 2012.

[80] K. H. Hoffmann and V. N. Starovoitov. On a motion of a solid body in a viscous fluid. Two dimensional case. *Adv. Math. Sci. Appl.*, 9:633–648, 1999.

[81] J.M. Houghton, B.A. Schiffman, D.P. Kalman, E.D. Wetzel, N.J. Wagner. Hypodermic Needle Puncture of Shear Thicening Fluid (STF)-Treated Fabrics. *Proceedings of SAMPE.* Baltimore MD, 2007.

[82] J. Hron, C. Le Roux, J. Málek, and K. Rajagopal. Flows of incompressible fluids subject to Navier's slip on the boundary. *Comput. Math. Appl.* 56 (2008), 2128–2143.

[83] K. Hutter. Theoretical glaciology: material science of ice and the mechanics of glaciers and ice sheets. Dordrecht, etc., D. Reidel Publishing Co.; Tokyo, Terra Scientific Publishing Co., 1983.

[84] N. Igbida, K. Sbihi, P. Wittbold. Renormalized solution for Stefan type problems: existence and uniqueness. *NoDEA Nonlinear Differential Equations Appl.* 17 (2010), no. 1, 69–93.

[85] N.V. Judakov. The solvability of the problem of the motion of a rigid body in a viscous incompressible fluid. (Russian) *Dinamika Splošn. Sredy Vyp.* , 18: , 249–253, 1974.

[86] B. Kone, S. Ouaro and S. Traore. Weak solutions for anisotropic elliptic equations with variable exponents. *Electronic J. Diff. Equ.* 2009 (2009), no. 144, 1–11.

[87] A. Kufner, O. John and S. Fučik. *Function spaces.* Noordhoff International Publishing. Prague: Publishing House of the Czechoslovak Academy of Sciences, 1977.

[88] O. A. Ladyzhenskaya. New equations for the description of motion of viscous incompressible fluids and solvability in the large of boundary value problems for them. *Proc. Stek. Inst. Math.* 102:95–118, 1967.

[89] O. A. Ladyzhenskaya. The Mathematical Theory of Viscous Incompressible Flow. 2nd edn, Gordon and Breach, New York, 1969.

[90] Young S. Lee, E. D. Wetzel and N. J. Wagner. The ballistic impact characteristics of Kevlar woven fabrics impregnated with a colloidal shear thickening fluid. *Journal of Materials Science.* 38(13):2825-2833, 2004.

[91] P. L. Lions. *Mathematical topics in fluid mechanics. Vol. 1. Incompressible models.* Oxford Lecture Series in Mathematics and its Applications, Oxford Science Publications, New York, 1996.

[92] J. Málek, J. Nečas and M. Ružička. On the non-Newtonian incompressible fluids. *Math. Models Methods Appl. Sci.*, 3(1):35–63, 1993.

[93] J. Málek, J. Nečas, M. Rokyta and M. Ružička. *Weak and measure-valued solutions to evolutionary PDEs.* Chapman & Hall, London, 1996.

[94] J. Málek, J. Nečas, M. Rokyta and M. Ružička. Existence and regularity of solutions and the stability of the test state for fluids with shear dependent viscosity. *Math. Models and Methods in Applied Sciences.* 5:789–812, 1995.

[95] F. Murat. Soluciones renormalizadas de EDP elipticas no lineales. Publ. Laboratoire d'Analyse Numérique, Univ. Paris 6, R 93023 (1993).

[96] J. Musielak. *Orlicz spaces and modular spaces* volume 1034 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1983.

[97] V. Mustonen and M. Tienari. On monotone-like mappings in Orlicz-Sobolev spaces. *Math. Bohem.*, 124(2-3):255–271, 1999.

[98] Š. Nečasová. On the motion of several rigid bodies in an incompressible non-Newtonian and heat-conducting fluid. *Ann Univ Ferrara.* **55**:325-352, 2009.

[99] A. Novotný and I. Straškraba. *Introduction to the mathematical theory of compressible flow.* Oxford Lecture Series in Mathematics and its Applications 27. Oxford University Press., 2004.

[100] S. Ouaro. Well-posedness results for anisotropic nonlinear elliptic equations with variable exponent and L^1-data. *Cubo* 12 (2010), no. 1, 133-148.

[101] M. Patel and M. G. Timol. Numerical treatment of Powell–Eyring fluid flow using method of satisfaction of asymptotic boundary conditions (MSABC). *Appl. Numer. Math.* 59 (2009), 2584–2592.

[102] W. Pompe. Existence theorems in the viscoplasticity theory (Diss.). PhD thesis, Berichte aus der Mathematik. Shaker: Aachen; Darmstadt: TU Darmstadt, Fachbereich Mathematik (Diss.). 46 p., 2003.

[103] R. E. Powell and H. Eyring. *Mechanisms for the relation theory of viscosity,* Nature, **154** (1994), 427–428.

[104] A. Prignet. Conditions aux limites non homogènes pour des problèmes elliptiques avec second membre mesure. (French) [Nonhomogeneous boundary conditions for elliptic problems with right-hand-side measures] *Ann. Fac. Sci. Toulouse Math.* 6 (1997), no. 2, 297–318.

[105] M.M. Rao and Z.D. Ren. *Theory of Orlicz spaces.* Monographs and Textbooks in Pure and Applied Mathematics, 146. Marcel Dekker, Inc., New York, 1991.

[106] A. M. Robertson. *Review of relevant continuum mechanics,* in "Hemodynamical flows", Oberwolfach Semin., Birkhäuser, Basel, **37** (2008), 1–62.

[107] A. M. Robertson, A. Sequeira, and M. V. Kameneva. *Hemorheology.* in "Hemodynamical flows", Oberwolfach Semin., Birkhäuser, Basel, **37** (2008), 63–120.

[108] T. Roubiček. *Nonlinear Partial Differential Equations with Applications.* Birkhäuser, Basel, 2005.

[109] R. T. Rockaffellar. *Convex analysis.* Princton University Press. 1970.

[110] W. Rudin *Real and Complex Analysis, Third Edition.* McGraw-Hill, Inc., New York, 1987.

[111] M. Ružička. *Electrorheological fluids: modeling and mathematical theory*, volume 1748 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2000.

[112] M. Sanchón and J.M. Urbano. Entropy solutions for the $p(x)$-Laplace equation. *Trans. Amer. Math. Soc.* 361 (2009), no. 12, 6387–6405.

[113] J.A. San Martin, V. Starovoitov and M. Tucsnak. Global weak solutions for the two dimensional motion of several rigid bodies in an incompressible viscous fluid. *Arch. Rational Mech. Anal.*, **161**:93–112, 2002.

[114] D. Serre. Chute libre d'un solide dans un fluid visquex incompressible. Existence. *Jap. J. Appl. Math.*, **4**:99–110, 1987.

[115] J. Simon. Compact Sets in the Space $L^p(0,T;B)$. *Ann. Mat Pura Appl., VI Serie.* 146:65-96, 1987.

[116] V. Sirohi, M. G. Timol and N. L. Kalthia, *Powell-Eyring model flow near an accelerated plate,* Fluid Dynamics Research, **2** (1987) 193–204.

[117] M. S. Skaff. Vector valued Orlicz spaces generalized N-Functions, I. *Pacific Journal of Mathematics.* 28(1):193–206, 1969

[118] M. S. Skaff. Vector valued Orlicz spaces, II. *Pacific Journal of Mathematics.* 28(2):413–430, 1969.

[119] H. Sohr. *The Navier-Stokes equations: An elementary functional analytic approach.* Birkhäuser Verlag, Basel, 2001.

[120] V.N. Starovoitov. Behavior of a rigid body in an incompressible viscous fluid near boundary. *In International Series of Numerical Mathematics*, **147**:313–327, 2003.

[121] D.W. Stroock. Weyl's Lemma, one of many. London Mathematical Society Lecture Note Series. No. 354. pp. 164-173. Cambridge University Press. 2008.

[122] M. J. Strauss. *Variations of Korn's and Sobolev's inequalities,* Partial Diff. Equ., Berkeley 1971, Proc. Sympos. Pure Math. **23** (1973), 207–214.

[123] R. Temam and G. Strang. *Functions of bounded deformation,* Arch. Rational Mech. Anal., **75** (1980/81), 7–21.

[124] R. Temam, A. Miranville. *Mathematical Modelinfg in Continuum Mechanics.* Cambridge University Press, 2005.

[125] H. Triebel. *Interpolation theory, function spaces, differential operators.* VEB Deutscher Verlag der Wissenschaften. Berlin, 1978.

[126] B. Turett. Fenchel-Orlicz spaces. *Dissertationes Math. (Rozprawy Mat.)*, 181:55, 1980.

[127] R. Vodák. The problem $\nabla \cdot \mathbf{v} = f$ and singular integrals on Orlicz spaces. *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.* 41 (2002), 161–173.

[128] D. Werner. "Funktionalanalysis," Springer-Lehrbuch, 2011

[129] P. Wittbold and A. Zimmermann. Existence and uniqueness of renormalized solutions to nonlinear elliptic equations with variable exponents and L1-data. *Nonlinear Analysis TMA* 72 (2010), no.6, 2990–3008

[130] J. Wolf. Existence of weak solutions to the equations of non-stationary motion of non-newtonian fluids with shear rate dependent viscosity. *J. Math. Fluid Dynamics.* 8:1–35, 2006.

[131] A. Wróblewska. Steady flow of non-Newtonian fluids - monotonicity methods in generalized Orlicz spaces. *Nonlinear Analysis.* 72:4136-4147, 2010.

[132] A. Wróblewska. Steady flow of non-Newtonian fluids with growth conditions in Orlicz spaces. Diploma theses at Faculty of Mathematics, Informatics and Mechanics University of Warsaw, 2008.

[133] A. Wróblewska. Existence results for unsteady flows of nonhomogeneous non-Newtonian incompressible fluids - monotonicity methods in generalized Orlicz spaces. Preprints of PhD Programme: Mathematical Methods in Natural Sciences, no. 2011 - 015, 2011.

[134] A. Wróblewska. Local pressure methods in Orlicz spaces for the motions of rigid bodies in an non-Newtonian fluid with general growth conditions. accepted to *Discrete and Continuous Dynamical Systems - S*, 2012.

[135] A. Wróblewska-Kamińska. Existence result to the motion of several rigid bodies in an incompressible non-Newtonian fluid with growth conditions in Orlicz spaces, Prepreprint PhD Programme: Mathematical Methods in Natural Sciences, Nr 2011 - 024, 2012.

[136] C. S. Yeh and K. C. Chen. A thermodynamic model for magnetorheological fluids. *Continnum Mech. Thermodyn.* 9:273–291, 1997.

[137] H. Yoon and A. Ghajar. *A note on the Powell-Eyring fluid model.* International Communications in Heat and Mass Transfer, **14** (1987), 381–390.