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Structured Population Models in Metric Spaces

Phd Dissertation

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Author’s declaration:

Aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contens of the dissertation have been obtainet by legal means.

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Supervisors’ declaration:

The dissertation is ready to be reviewed.

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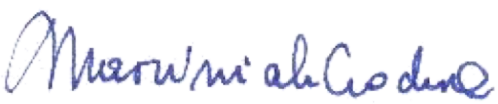

Abstract

The main goal of this thesis is the analysis of a wide class of structured population models in the space of finite, nonnegative Radon measures equipped with the flat metric. This framework allows a unified approach to a variety of problems providing them with basic well-posedness and stability results.

The first result is the existence and uniqueness of measure valued solutions to the one-sex structured population model. A nonlinear semigroup is constructed here by means of the operator splitting algorithm. This technique allows to separate the differential operator from the integral one, which leads to a significant simplification of proofs. Concerning stability, the Lipschitz continuity of solutions with respect to the model coefficients is provided. The next analytical result is the well-posedness of the age-structured two-sex population model. Existence and uniqueness of the measure valued solutions is proved by the regularization technique as well as the stability estimates. A brief discussion on a marriage function, which is the main source of the nonlinearity in this model, is carried out and an example of the marriage function fitting into the considered framework is given.

The second part of this thesis is devoted to a development of numerical methods for a particular class of one-sex structured population models. The first method is constructed through the splitting technique and corresponds with a current trend basing on a kinetic approach to the population dynamics problems. Separation of a semigroup induced by the transport operator from a semigroup induced by the nonlocal term allows to keep the solution as a sum of Dirac deltas despite of the regularizing character of the nonlocal boundary condition. As the next step, two alternative methods based on different approximations of the boundary condition are analyzed. These are the Escalator Boxcar Train algorithm and its simplification. Convergence of both methods is proved exploiting the concept of semiflows on metric spaces. Last but not least, the rate of convergence for all schemes mentioned above is provided.

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Contents

Chapter 1

Introduction

The following thesis is devoted to the mathematical analysis of a wide group of structured population models in metric spaces. Most of those models can be illustrated as an evolutionary PDE for density of individuals [31, 82] with respect to a specific structural variable which represents, for instance, age [81], size [8], phenotypic trait [17], maturity of cells [80], [34] etc. A typical functional space in which early results were obtained is the space of integrable functions or densities. The main goal of this thesis is to present a more general approach, which is based on setting these models in a space of measures. We will prove a well-posedness theory of measure valued solutions to a class of one-dimensional problems originated from population dynamics, which are represented by the following equation

$$\begin{cases} \partial_t \mu + \partial_x \left(b(t,\mu)\, \mu \right) + c(t,\mu)\, \mu &= \int_{\mathbb{R}_+} (\eta(t,\mu))(y)\, \mathrm{d}\mu(y), \quad (t,x) \in [0,T] \times \mathbb{R}_+, \\ \mu_o &\in \mathcal{M}^+(\mathbb{R}_+), \end{cases} \tag{1.1}$$

where the initial condition is a finite, nonnegative Radon measure. In particular, we strongly focus on the space of finite, nonnegative Radon measures $\mathcal{M}^+(\mathbb{R}_+)$ equipped with a flat metric [67, 85]. This framework allows a unified approach to a variety of structured population models (for particular examples see Subections 2.2.1 – 2.2.5), providing them with basic well posedness and stability results. Here, we follow and develop the approach in [43, 44] by constructing the solutions through the use of operator splitting, or fractional step method [24, 25] in metric spaces. This allows a significant shortening of the proofs compared to [43, 44], while at the same time gaining wider generality.

Apart from (1.1), which is intended for single-species populations, we consider a two-sex population model, that is a system describing two interacting sub-populations structured with respect to age, which can be directly applied to human population dynamics. We provide results concerning the well-posedness of measure valued solutions to this problem, but in contrast to the single-species case we exploit the regularization technique instead of the splitting method. The major difficulty in this model is a nonlinear "marriage operator", which accounts for the interactions between males and females. Since there is no obvious choice of the most appropriate marriage function, we adjust for the measure setting proposed in [52], which has a sufficient generality for the practical purposes.

A part of this thesis is devoted to development of numerical methods for (1.1). We design a numerical scheme resulting from a variation of the EBT method commonly used in biology (an abbreviation for the Escalator Boxcar Train method, for applications see e.g. [15, 41, 69, 84]), which is essentially analogous to particle methods originally used in physics. In contrast to the earlier achievements, we adopt the EBT method for a type of models, where the influx of new individuals occurs not only through a boundary, but possibly through the whole domain. This was possible to accomplish due to the operator splitting method, which separately copes with the transport operator and the integral operator. We prove a convergence of this generalized method in the space of finite, nonnegative Radon measures equipped with the flat metric and provide the convergence rates. For the McKendrick model [64], as a particular case of (1.1), we additionally present an alternative approach consisting in a different definition of the boundary cohort and we also provide similar results as for the splitting method. Last but not least, we confront theoretical estimates with results of numerical simulations for several test cases.

The thesis is organized as follows. The aim of Chapter 1 is to introduce the reader to the population dynamics, space of Radon measures and related topics. Thus, in Section 1.1 we present the main ideas and a brief history of structured population models. For the paper to be consistent we put a few words concerning the notation in Section 1.2. In Section 1.3 we recall basic facts about the space of Radon measures. Sections 1.4 and 1.5 are devoted to metrics originated from the optimal transportation theory, that is the 1-Wasserstein distance and the flat metric, respectively. The last section of the first chapter, Section 1.6, covers a justification of setting the structure population models in the space of Radon measures and underlines the advantages following from application of the 1-Wasserstein distance and flat metric. In Chapter 2 and Chapter 3 we present analytical results concerning a well-posedness theory of measure valued solutions. Chapter 2, which is based on the results obtained in [21] by Carrillo, Colombo, Gwiazda and Ulikowska, is devoted to the one-dimensional problem (1.1). In Chapter 3 we present results established in [75] by Ulikowska for the age-structured, two-sex population model. Chapter 4 is devoted to development of the numerical scheme for (1.1) based on the splitting technique. This chapter also contains the effects of numerical simulations for the particular test cases. Results related to the latter topic were achieved by Carrillo, Gwiazda and Ulikowska in [22]. In Chapter 5 we present two alternative numerical methods for the McKendrick model together with their convergence analysis. This issue is a subject of the recent research of Gwiazda, Jablonski, Marciniak-Czochra and Ulikowska.

1.1. Structure Population Models

The main purpose of the first population dynamics models was to describe changes in the size of a population and to investigate factors which influence its evolution. A significant step in the development of these models was made by Malthus [59], who noticed that growth of a population is proportional to its size, which gave rise to a linear ordinary differential equation of the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}P(t) = rP(t).$$

Here, $P(t)$ denotes a total number of individuals and r is a constant growth rate. Depending on the sign of r the population grows or reduces exponentially, which in many cases proves to be wrong in nature, since the exponential growth can be inhibited by environmental limitations such as lack of nutrients, space, or partners to reproduction. A modification of the model incorporating these factors was proposed in [77] by Verlust, who made the growth rate dependent on the total population size.

Both models mentioned above were developed and became more complex during the years of studies, but eventually it turned out that they are applicable only in case of homogeneous populations. In such populations vital processes (e.g. birth, death, development of individuals) do not depend on the individual's state, which is not a common phenomenon. For instance, in human population fertility and mortality depend strongly on the age of a human, in cell populations the mitosis process is often influenced by the size or maturity of a cell, the phenotypic trait of an offspring depends on its parents' trait. It is worth mentioning that a common characteristic of physiologically structured population models is that they all base on the individual's behaviour. The starting point in creating such models is thus to find a description of an individual's life history related to its survival, dynamic of a transformation, and reproduction. Then, the key point is to investigate how the mechanisms at "individual level" generate phenomena at the "population" level. The main idea of the population dynamics states that changes occurring in a certain population are in fact a sum of actions of its individuals. In other words, there is a balance law according to which a change of the "amount" of individuals at the state x is equal to decrease of these individuals due to processes of death and transformation (e.g. aging, growing, maturation, mutation) combined with their increase due to the process of birth. In the sense described above, structured population models link processes on the "individual level" and the "population level". From that reason they are often called individual-based models.

One of the first structured population models was described in [64] by McKendrick, who took into account a population's age structure. This led to the following partial differential equation

$$\begin{cases} \partial_t u(t,x) + \partial_x u(t,x) = -c(x)u(t,x), \\ u(0,t) = \int_0^{+\infty} \beta(y)u(t,y)\mathrm{d}y, \end{cases} \tag{1.2}$$

where x is the age of an individual and $u(t,\cdot)$ is a distribution of a population with respect to x. Note that (1.2) is a particular case of (1.1). Functions β and c are age-dependent birth and death rates respectively. A generalization of this model incorporating the environmental influences, which brought it far closer to solving real problems, was formulated in [45] by Gurtin and MacCamy, who considered a birth rate $\beta(x,P)$ and a death rate $c(x,P)$ as functions dependent on the total population size P. The main advantage of introducing nonlinearities of such type is the possibility of tracking not only individual behaviour but also the interactions and influences on the environment. For instance, consider a population which consumes food and thus decreases natural resources. Consumption of food makes the population grow, but on the other hand, smaller amount of nutrition results in smaller fertility and as a consequence the population is expected to decrease. This feedback loop gives a rise to a natural question about existence of stability, where both effects are balanced, however this issue is not a subject of this thesis.

The accuracy of structured population models led to their intensive development and variety of further generalizations. Size, length, maturity, level of the particular chemical substance within a cell, phenotypic trait and many others started to play a role of the structural variable. For instance, the first size-structured population model was presented by Anderson and Bell in [6]. The other direction of development consisted in considering multi-dimensional structural variables. It was initiated in [76], where a population was structured by age and size. Another trend was based on involving at least two interacting populations. One of the first attempts of including both sexes and tracking interactions between them was made by Kendall in 1949 [58]. Although the age structure was left out of consideration, this paper became a starting point for many recently used models (see [23] and [48] for extensions). A significant generalization based on incorporating the age structure was formulated for the first time in [37] by Fredrickson and reintroduced in [50] by Hoppensteadt in 70's. This subject was in the center of researchers' interest, as the results could be directly applied to the human population dynamics. Predicting how the population will grow and change its age structure is essential, since these changes affect many areas of public life, e.g., health care, education, insurance market, job market, social systems. It influences also the quality and availability of natural resources. Nowadays, such models are used in modeling sexually transmitted diseases (e.g. HIV), see [32, 40, 46, 63]. Coming back to the Fredrickson-Hoppensteadt model, it describes the evolution of males and females structured by age as well as the process of heterogeneous couples formation, which depends on the age of both parters. The latter assumption brings the model far closer to reality, as individuals at different ages usually show a tendency to different social and sexual behaviours. Although there is many more factors which influence the marriage process in the human population, in this thesis we also consider only the most important ones - sex and age, since taking all of them into account leads to the high complexity.

Summarizing, structured population models are nowadays present not only in demography, but also in other areas of natural sciences like ecology, epidemiology or biology and the structural variable is not restricted to age in general. Note that the structural variable usually denotes a characteristic of an individual not a spacial position, since most often it is assumed that the environment is homogeneous and thus the spatial position plays no role. Apart from this restriction, one can freely choose any characteristic, which has an influence on the evolution of a population and the individual's vital processes. The model (1.1) is intended to include variety of these choices. In the age-structure, two-sex population model being a subject of the Chapter 3 the structural variable if fixed. It is worth to underline that both models we consider here are fully nonlinear, which means that all vital processes depend not only on the structural variable, but also on time and a state of the whole population. For the more detailed overview of structured population models and their history we refer to e.g. [31] (other citations: [17, 28, 31, 34, 43, 44, 64, 70, 82, 81]).

1.2. Notation

We use a notation according to which $x \in \mathbb{R}^m_+ = [0, +\infty)^m$, $m \in \mathbb{N}$. We emphasize that $\mathbb{R}^m_+$ is used only for the ambient space, so it always refers to a structural variable and not to time. The choice of $\mathbb{R}^m_+$ is justified by the fact that the ambient space does not play any specific role but is adopted to include some typical models, where traits of an individual are described by nonnegative values. In this thesis we focus on one- and two-dimensional case, but results concerning the 1-Wasserstein distance and flat metric provided in § 1.4 – § 1.5 are valid for the arbitrary $m \in \mathbb{N}$.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be metric spaces. We will use the following notation.

- $\mathbf{B}(X;Y)$ is the space of functions bounded with respect to the supremum norm,
- $\mathbf{C}(X;Y)$ is the space of continuous functions,
- $\mathbf{C^k}(X;Y)$ is the space of k times differentiable functions, $k \in \mathbb{N}$,
- $\mathbf{C_b}(X;Y)$ is the space of continuous functions bounded with respect to the supremum norm,
- $\mathbf{Lip}(X,Y)$ is the space of Lipschitz functions.

In the particular case where $Y = \mathbb{R}$, we use an abbreviated notation, i.e., we omit a specification of a codomain. For instance,

- $\mathbf{C_o}(X)$ is the space of real-valued continuous functions vanishing at infinity,
- $\mathbf{L^p}(X)$ is the usual Lebesgue space,
- $\mathbf{W^{k,p}}(X)$ is the usual Sobolev space.

To simplify the notation we define

$$\|x\|_{X^n} = \|(x_1,\ldots,x_n)\|_{X^n} := \sum_{i=1}^{n} \|x_i\|_X \qquad \forall x = (x_1,\ldots,x_n) \in X^n.$$

The subscript following a symbol of a function always denotes its argument unless it is said differently, e.g. f_t denotes a value of a mapping $f : [0,T] \to Y$ at time t. To avoid misunderstandings partial derivatives are always denoted by a symbol ∂.

1.3. Space of Radon Measures

Denoted by $\mathcal{M}(\mathbb{R}^m_+)$ stands the set of finite, real Radon measures on $\mathbb{R}^m_+$. We say that a measure is finite when its total variation is bounded. We recall that for a measure μ its total variation $\|\mu\|_{TV}$ is defined as

$$\|\mu\|_{TV} = \mu^+(\mathbb{R}^m_+) + \mu^-(\mathbb{R}^m_+),$$

where μ^+ and μ^- are measures arising from the Jordan decomposition theorem. By $\mathcal{M}^+(\mathbb{R}^m_+)$ we denote the space of finite, real and nonnegative measures. By $\mathcal{M}^+_1(\mathbb{R}^m_+)$ we denote a subspace of $\mathcal{M}^+(\mathbb{R}^m_+)$ consisting of measures with the first moment integrable,

$$\mathcal{M}^+_1(\mathbb{R}^m_+) = \left\{\mu \in \mathcal{M}^+(\mathbb{R}^m_+) \ : \ \int_{\mathbb{R}^m_+} \|x\| \mathrm{d}\mu(x) < +\infty\right\},$$

where $\|\cdot\|$ is a norm in $\mathbb{R}^m$. Space of probability measures will be denoted by $\mathcal{P}(\mathbb{R}^m_+)$ and the space of probability measures with an integrable first moment by $\mathcal{P}_1(\mathbb{R}^m_+)$. By Riesz representation theorem it follows that

$$(\mathcal{M}(\mathbb{R}^m_+), \|\cdot\|_{TV}) \quad \cong \quad (\mathbf{C_o}(\mathbb{R}^m_+), \|\cdot\|_\infty)^*,$$

where $(\mathbf{C_o}(\mathbb{R}^m_+), \|\cdot\|_\infty)$ is the space of continuous functions vanishing in infinity equipped with the supremum norm. Thus, each $\mu \in \mathcal{M}(\mathbb{R}^m_+)$ can be identified with a bounded linear functional L on $(\mathbf{C_o}(\mathbb{R}^m_+), \|\cdot\|_\infty)$ such that

$$\|\mu\|_{TV} = \|L\|_{(\mathbf{C_o}(\mathbb{R}^m_+))^*}.$$

Here, $\|\cdot\|_{(\mathbf{C_o}(\mathbb{R}^m_+))^*}$ denotes the usual norm in a dual space, i.e.

$$\|L\|_{(\mathbf{C_o}(\mathbb{R}^m_+))^*} = \sup\left\{\int_{\mathbb{R}^m_+} \varphi(x)\mathrm{d}\mu(x) \;:\; \varphi \in \mathbf{C_o}(\mathbb{R}^m_+) \text{ and } \|\varphi\|_\infty \le 1\right\}.$$

Since the space of bounded linear functionals defined on a normed space is a Banach space itself (when equipped with the dual norm), we conclude that $(\mathcal{M}(\mathbb{R}^m_+), \|\cdot\|_{TV})$ is a Banach space as well. Below we recall basic definitions and facts concerning various types of convergence in the space of measures.

Definition 1.3. *We say that a sequence $\{\mu^n\}_{n\in\mathbb{N}} \subset \mathcal{M}(\mathbb{R}^m_+)$ converges weakly* to a measure $\mu \in \mathcal{M}(\mathbb{R}^m_+)$ if and only if*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^m_+} \varphi(x)\mathrm{d}(\mu^n - \mu)(x) = 0 \quad \forall \varphi \in \mathbf{C_o}(\mathbb{R}^m_+).$$

Respectively, we say that a mapping $\mu : [0,T] \to \mathcal{M}(\mathbb{R}^m_+)$ is weakly continuous if for any $\varphi \in \mathbf{C_b}(\mathbb{R}^m_+)$ a function*

$$f : [0,T] \to \mathbb{R}, \quad f(t) = \int_{\mathbb{R}^m_+} \varphi(x)\mathrm{d}\mu_t(x)$$

is continuous.

Definition 1.4. *We say that a sequence $\{\mu^n\}_{n\in\mathbb{N}} \subset \mathcal{M}(\mathbb{R}^m_+)$ converges narrowly to a measure $\mu \in \mathcal{M}(\mathbb{R}^m_+)$ if and only if*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^m_+} \varphi(x)\mathrm{d}(\mu^n - \mu)(x) = 0 \quad \forall \varphi \in \mathbf{C_b}(\mathbb{R}^m_+).$$

Respectively, we say that a mapping $\mu : [0,T] \to \mathcal{M}(\mathbb{R}^m_+)$ is narrowly continuous if for any $\varphi \in \mathbf{C_b}(\mathbb{R}^m_+)$ a function

$$f : [0,T] \to \mathbb{R}, \quad f(t) = \int_{\mathbb{R}^m_+} \varphi(x)\mathrm{d}\mu_t(x)$$

is continuous.

Remark 1.5. Narrow convergence of measures (resp. narrow continuity) is also known as weak convergence of measures (resp. weak continuity), see e.g. [11]. The weak topology on $\mathcal{M}(\mathbb{R}_+^m)$ is the topology $\sigma(\mathcal{M}(\mathbb{R}_+^m), \mathbf{C_b}(\mathbb{R}_+^m))$, that is the base of the weak topology consists of the sets

$$U_{n,\varepsilon}(\mu) = \left\{ \nu \ : \ \left| \int_{\mathbb{R}_+^m} f_i \mathrm{d}\mu - \int_{\mathbb{R}_+^m} f_i \mathrm{d}\nu \right| < \varepsilon, \ i = 1, \dots, n \right\},$$

where $\mu \in \mathcal{M}(\mathbb{R}_+^m)$, $f_i \in \mathbf{C_b}(\mathbb{R}_+^m)$ and $\varepsilon > 0$.

Definition 1.6. *We say that a sequence $\{\mu^n\}_{n\in\mathbb{N}} \subset \mathcal{M}(\mathbb{R}_+^m)$ is tight if and only if*

$$\lim_{M\to 0} \sup_{n\in\mathbb{N}} \int_{[M,+\infty)^m} \mathrm{d}\mu^n(x) = 0.$$

Proposition 1.7. For a sequence $\{\mu^n\}_{n\in\mathbb{N}} \subset \mathcal{M}(\mathbb{R}_+^m)$ it holds that

$$\{\mu^n\}_{n\in\mathbb{N}} \text{ converges weakly}^* \text{ and is tight} \quad \Leftrightarrow \quad \{\mu^n\}_{n\in\mathbb{N}} \text{ converges narrowly.}$$

Remark 1.8. The narrow convergence is more sensitive on a behaviour of measures at infinity than the weak* convergence. For instance, consider the following sequence of probability measures on $\mathbb{R}_+$

$$u^n(x) = \chi_{[n,n+1)}(x),$$

which converges weakly* to 0, but does not converge narrowly since the tightness condition is not fulfilled.

1.4. 1-Wasserstein Distance

A concept of the Wasserstein distances originates from the optimal transportation problem, which was formulated for the first time in the 18th century by Monge [66] and can be briefly described as follows. Assume that one has a pile of sand and a hole, both of the same capacity. The question which needs to be answered is how to transport all the sand into the hole. In particular, where one should send the sand from the certain location so that the minimal effort is put into this task. The choice of a proper transport plan is crucial here, since the cost of transport usually varies depending on the distance between points. Let us formulate this problem in a more rigorous manner and introduce the cost function

$$c : \mathbb{R}_+^m \times \mathbb{R}_+^m \to \mathbb{R}_+ \cup \{+\infty\},$$

which describes how much effort one has to put into transportation of the unit mass from x to y. Mathematical equivalent of the transference plan is a probability measure π on $\mathbb{R}_+^m \times \mathbb{R}_+^m$, such that $\mathrm{d}\pi(x,y)$ measures the amount of mass transferred from x into y.

Definition 1.9. *Let $\mu, \nu \in \mathcal{P}(\mathbb{R}_+^m)$ and $\Pi(\mu,\nu)$ be a set of probability measures on $\mathbb{R}_+^m \times \mathbb{R}_+^m$ with marginal distributions μ and ν, i.e.*

$$\Pi(\mu,\nu) = \left\{ \pi \in \mathcal{P}(\mathbb{R}_+^m \times \mathbb{R}_+^m) \ : \ \pi(A \times \mathbb{R}_+^m) = \mu(A), \ \pi(\mathbb{R}_+^m \times A) = \nu(A), \ A \in \mathcal{B}(\mathbb{R}_+^m) \right\}.$$

Each element of the set $\Pi(\mu,\nu)$ is called a transference plan.

The concept of transference plan can be extended in a natural way on the subset of nonnegative, finite Radon measures such that any two measures from this subset have equal masses. Henceforth, we understand $\Pi(\mu,\nu)$ in this generalized sense, unless it is said differently.

Having the cost function c and the set Π of transference plans, we aim to minimize the following cost functional

$$\mathcal{T}^c(\pi) = \int_{\mathbb{R}^m_+\times\mathbb{R}^m_+} c(x,y)\mathrm{d}\pi(\mu,\nu)(x,y), \tag{1.10}$$

where $\pi \in \Pi(\mu,\nu)$. The problem posed above is more general that the one considered by Monge in the sense that in Monge's setting there was no possibility of splitting the mass. However, it is quite common in literature to call this problem *Monge-Kantorovitch problem*, regardless of whether the mass can be split or not, and in this thesis we follow this convention as well. Below we recall a theorem which states that the problem of minimization of the functional (1.10) is equivalent to some maximization problem. For a more detailed analysis of the Monge-Kantorovitch problem we refer to [78, 79], where a broad spectrum of problems related to the optimal transportation theory and Wasserstein distances can be found.

Theorem 1.11. ***(Kantorovich - Rubinstein theorem)*** *Let X be a Polish space, d be a lower semi-continuous metric on X and $\mathbf{Lip}(X)$ denote the space of all Lipschitz functions on X. Define $\mathcal{T}^d$ as the cost of optimal transportation for the cost function $c(x,y) = d(x,y)$, that is*

$$\mathcal{T}^d(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)} \int_{X\times X} d(x,y)\mathrm{d}\pi(x,y). \tag{1.12}$$

Then,

$$\mathcal{T}^d(\mu,\nu) = \sup\left(\int_X \varphi(x)\mathrm{d}(\mu-\nu)(x) \ : \ \mathbf{Lip}(\varphi)\le 1\right). \tag{1.13}$$

If $X = \mathbb{R}^m_+$ and $d(x,y) = \|x-y\|$, then we write $\mathcal{T}$ instead of $\mathcal{T}^d$.

Definition 1.14. *Let $X = \mathbb{R}^m_+$, $d(x,y) = \|x-y\|$ and $\mathcal{T}$ be given as in Theorem* (1.11). *Then, a map*

$$W_1 : \mathcal{P}(\mathbb{R}^m_+)\times\mathcal{P}(\mathbb{R}^m_+) \to [0,+\infty], \quad W_1(\mu,\nu) = \mathcal{T}(\mu,\nu) \tag{1.15}$$

is called the 1*-Wasserstein distance.*

From the definition above it follows that $W_1(\mu,\nu)$ is equal to the cost of optimal transportation for the cost functional $c(x,y) = \|x-y\|$. Unfortunately, at this stage it can happen that W_1 is infinite for two arbitrary probability measures and thus, W_1 is not a metric in the strict sense. The common approach, which excludes a possibility that W_1 is infinite, consists in restricting $\mathcal{P}(\mathbb{R}^m_+)$ to the measures with the first moment integrable, that is to the space $\mathcal{P}_1(\mathbb{R}^m_+)$. Indeed, let $\mu,\nu \in \mathcal{P}_1(\mathbb{R}^m_+)$ and $\pi \in \Pi(\mu,\nu)$. Then,

$$\begin{aligned}\int_{\mathbb{R}^m_+\times\mathbb{R}^m_+} \|x-y\|\mathrm{d}\pi(x,y) &\le \int_{\mathbb{R}^m_+\times\mathbb{R}^m_+} \|x\|\mathrm{d}\pi(x,y) + \int_{\mathbb{R}^m_+\times\mathbb{R}^m_+} \|y\|\mathrm{d}\pi(x,y) \\ &= \int_{\mathbb{R}^m_+} \|x\|\mathrm{d}\mu(x) + \int_{\mathbb{R}^m_+} \|y\|\mathrm{d}\nu(y) < +\infty.\end{aligned}$$

Since it holds for each $\pi \in \Pi(\mu,\nu)$, it holds also for the infimum. Hence, $W_1(\mu,\nu) < +\infty$.

Remark 1.16. $(\mathcal{P}_1(\mathbb{R}_+^m), W_1)$ is a complete and separable metric space (for the proof see [78, Theorem 6.16]).

Remark 1.17. The formula (1.15) is convenient if one needs to bound the W_1 distance between μ and ν from above. Indeed, it is sufficient to calculate

$$\int_{\mathbb{R}_+^m \times \mathbb{R}_+^m} \|x - y\| \, \mathrm{d}\pi_o(x, y),$$

where π_o is an arbitrary element from $\Pi(\mu, \nu)$.

Remark 1.18. Note that W_1 in Definition 1.14 yields a finite value for all $\mu, \nu \in \mathcal{M}_1^+(\mathbb{R}_+^m)$ such that $\mu(\mathbb{R}_+^m) = \nu(\mathbb{R}_+^m) = M$. Indeed, by (1.13)

$$W_1(\mu, \nu) = \mathcal{T}(\mu, \nu) = M\mathcal{T}(\tilde{\mu}, \tilde{\nu}) = MW_1(\tilde{\mu}, \tilde{\nu}). \tag{1.19}$$

where $\tilde{\mu} = \mu/M$, $\tilde{\nu} = \nu/M$. On the other hand, in case $\mu(\mathbb{R}_+^m) \neq \nu(\mathbb{R}_+^m)$ we obtain

$$W_1(\mu, \nu) = \mathcal{T}(\mu, \nu) \geq \sup_{\{\varphi \,:\, \varphi(x) = a\}} \int_{\mathbb{R}_+^m} a \, \mathrm{d}(\mu - \nu)(x) = a(\mu(\mathbb{R}_+^m) - \nu(\mathbb{R}_+^m))$$

and thus $W_1(\mu, \nu) = +\infty$, since a is arbitrary.

In the proposition below we present a relationship between the 1-Wasserstein distance and the distance induced by the total variation norm for compactly supported probability measures on $\mathbb{R}_+^m$.

Proposition 1.20. Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R}_+^m)$ be such that $\mathrm{supp}(\mu), \mathrm{supp}(\nu) \subseteq K$, for some compact set $K \subset \mathbb{R}_+^m$. Then,

$$W_1(\mu, \nu) \leq C_K \|\mu - \nu\|_{TV},$$

where $C_K = \mathrm{diam}(K)/2$ and $\mathrm{diam}(K) = \sup_{y_1, y_2 \in K} \|y_1 - y_2\|$.

Proof of Proposition 1.20. K is a compact subset of $\mathbb{R}_+^m$, so it is also bounded and close. Therefore, there exist $y_1, y_2 \in K$ such that supremum in the definition of the diameter of a set is attained and thus $\mathrm{diam}(K) = \|y_1 - y_2\|$. Define

$$x_o = (y_1 + y_2)/2 \quad \text{and} \quad r = \mathrm{diam}(K)/2.$$

Then, $K \subseteq \bar{B}(x_o, r)$, where the latter set is a closed ball of radius r centered at x_o. Note that in (1.13) we may assume that $\|\varphi\|_\infty \leq r$. Indeed, let $\varphi \in \mathbf{Lip}(\mathbb{R}_+^m)$ be such that $\mathbf{Lip}(\varphi) \leq 1$ and consider a function

$$\tilde{\varphi}(x) = \varphi(x) + a,$$

where $a = -\varphi(x_o)$. Namely, we shift φ in such a way that $\tilde{\varphi}(x_o) = 0$. Since $\tilde{\varphi}$ is by definition a Lipschitz function with the Lipschitz constant not greater then 1, we obtain

$$|\tilde{\varphi}(x)| = |\tilde{\varphi}(x) - \tilde{\varphi}(x_o)| \leq \|x - x_o\| \leq r \qquad \forall x \in K. \tag{1.21}$$

Moreover,

$$\begin{aligned}\int_{\mathbb{R}_+^m} \varphi(x)\mathrm{d}(\mu-\nu)(x) &= \int_{\mathbb{R}_+^m} \tilde{\varphi}(x)\mathrm{d}(\mu-\nu)(x) + \int_{\mathbb{R}_+^m} a\,\mathrm{d}(\mu-\nu)(x)\\ &= \int_{\mathbb{R}_+^m} \tilde{\varphi}(x)\mathrm{d}(\mu-\nu)(x) + a(\mu(\mathbb{R}_+^m) - \nu(\mathbb{R}_+^m)),\end{aligned}$$

where the last term is equal to 0 since μ and ν are both probability measures. Inequality (1.21) implies

$$\int_{\mathbb{R}_+^m} \tilde{\varphi}(x)\mathrm{d}(\mu-\nu)(x) \leq \int_{\mathbb{R}_+^m} r\,\mathrm{d}|\mu-\nu|(x) = r\|\mu-\nu\|_{TV}$$

Taking supremum over all functions φ ends the proof. □

1.5. Flat Metric

According to Remark 1.18, W_1 distance between measures of unequal masses is infinite. Therefore, we exploit the flat metric called also the bounded Lipschitz distance (see e.g. [67, 85]), which turns out to be bounded for any two elements of $\mathcal{M}^+(\mathbb{R}_+^m)$.

Definition 1.22. *Let $\mu,\nu \in \mathcal{M}^+(\mathbb{R}_+^m)$. A flat metric $\rho_F^m : \mathcal{M}^+(\mathbb{R}_+^m) \times \mathcal{M}^+(\mathbb{R}_+^m) \to [0,+\infty)$ is defined by the following formula*

$$\rho_F^m(\mu,\nu) = \sup\left\{\int_{\mathbb{R}_+^m} \varphi\,\mathrm{d}(\mu-\nu) \;:\; \varphi \in \mathbf{C^1}(\mathbb{R}_+^m) \text{ and } \|\varphi\|_{\infty,\mathbf{Lip}} \leq 1\right\}, \tag{1.23}$$

where $\|\varphi\|_{\infty,\mathbf{Lip}} = \max\{\|\varphi\|_{\mathbf{L}^\infty}, \mathbf{Lip}(\varphi)\}$.

Henceforth, we often omit the superscript m, which should not lead to misunderstandings. Proof that ρ_F is a metric follows directly from the Definition 1.22. Moreover, ρ_F is finite for all $\mu,\nu \in \mathcal{M}^+(\mathbb{R}_+^m)$. Indeed, let $\varphi \in \mathbf{C^1}(\mathbb{R}_+^m)$ be such that $\|\varphi\|_{\infty,\mathbf{Lip}} \leq 1$. Then,

$$\int_{\mathbb{R}_+^m} \varphi(x)\mathrm{d}(\mu-\nu)(x) \leq \|\varphi\|_\infty \int_{\mathbb{R}_+^m} 1\,\mathrm{d}|\mu-\nu|(x) = \|\mu-\nu\|_{TV} \leq \|\mu\|_{TV} + \|\nu\|_{TV} < +\infty.$$

Taking sumpremum over all functions φ yields $\rho_F(\mu,\nu) < +\infty$. In particular,

$$\rho_F(\mu,\nu) \leq \|\mu-\nu\|_{TV}. \tag{1.24}$$

Note that similarly as for W_1, the flat distance between two measures can be controlled by the total variation of their difference. However, on the contrary to the previous case, a constant in the estimate does not depend on a size of the support.

Remark 1.25. For each $\lambda > 0$, it holds that

$$\rho_F(\mu,\nu) = \frac{1}{\lambda}\sup\left\{\int_{\mathbb{R}_+^m} \varphi\,\mathrm{d}(\mu-\nu) \;:\; \varphi \in \mathbf{C^1}(\mathbb{R}_+^m) \text{ and } \|\varphi\|_{\infty,\mathbf{Lip}} \leq \lambda\right\}. \tag{1.26}$$

Remark 1.27. It holds that $(\mathcal{M}^+(\mathbb{R}^m_+), \rho_F) \subset (\mathbf{W}^{1,\infty}(\mathbb{R}^m_+))^*$, that is $(\mathcal{M}^+(\mathbb{R}^m_+), \rho_F)$ is a subspace of the dual space to $\mathbf{W}^{1,\infty}(\mathbb{R}^m_+)$ endowed with the norm

$$\|f\|_{\mathbf{W}^{1,\infty}} = \max\{\|f\|_{\mathbf{L}^\infty}, \|Df\|_{\mathbf{L}^\infty}\}, \quad \text{where } \|Df\|_{\mathbf{L}^\infty} = \sum_{i=1}^{m} \|\partial_{x_i} f\|_{\mathbf{L}^\infty}.$$

It follows from the fact that the condition $\varphi \in \mathbf{C}^1(\mathbb{R}^m_+)$ in (1.23) can be substituted by $\varphi \in \mathbf{W}^{1,\infty}(\mathbb{R}^m_+)$ through a standard mollifying sequence argument applied to the test function φ, since its derivative is not involved in the value of the integral. Therefore, this metric is exactly the one induced by the dual norm of $\mathbf{W}^{1,\infty}(\mathbb{R}^m_+)$.

Lemma 1.28. $(\mathcal{M}^+(\mathbb{R}^m_+), \rho^m_F)$ *is a complete and separable metric space.*

For a proof we refer to [44, Theorem 2.6]. The proof concerns $(\mathcal{M}^+(\mathbb{R}_+), \rho_F)$, however the same technique can be applied to obtain an analogous results for $(\mathcal{M}^+(\mathbb{R}^m_+), \rho_F)$.

Remark 1.29. The space of all finite Radon measures is not complete when endowed in the flat metric. Indeed, according to Remark 1.27

$$\rho_F(\mu, \nu) = \|\mu - \nu\|_{(\mathbf{W}^{1,\infty})^*}, \quad \text{but} \quad (\mathcal{M}(\mathbb{R}^m_+), \rho_F) \ncong (\mathbf{W}^{1,\infty}(\mathbb{R}^m_+))^*.$$

Proposition 1.30. The flat metric metricizes the topology of narrow convergence on $\mathcal{M}^+(\mathbb{R}^m_+)$ (see Definition 1.4 and Remark 1.5). In other words, for any sequence $\{\mu^n\}_{n\in\mathbb{N}}$ in $\mathcal{M}^+(\mathbb{R}^m_+)$ it holds that

$$\{\mu^n\}_{n\in\mathbb{N}} \text{ converges narrowly to } \mu \quad \Leftrightarrow \quad \lim_{n\to+\infty} \rho_F(\mu^n, \mu) = 0.$$

Now, we will investigate how the flat metric is related to W_1 and estimate the flat distance between two measures being sums of Dirac deltas. We will use these facts further in Chapter 4 in order to estimate the bounded Lipschitz distance between the output of numerical simulations and the exact solution in Chapter 5.

Proposition 1.31. Let $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^m_+)$ be such that $\mathrm{supp}(\mu), \mathrm{supp}(\nu) \subseteq K$, for some compact set $K \subset \mathbb{R}^m_+$. Then,

$$C_K W_1(\mu, \nu) \le \rho_F(\mu, \nu) \le W_1(\mu, \nu),$$

where $C_K = \min\{1, 2/\mathrm{diam}(K)\}$.

Proof of Proposition 1.31. Define $r = \mathrm{diam}(K)/2$. Using analogous arguments as in the proof of Proposition 1.20 and Remark 1.25 we obtain

$$\begin{aligned} W_1(\mu,\nu) &= \sup\left\{\int_{\mathbb{R}^m_+} \varphi \, \mathrm{d}(\mu-\nu) \ : \ \varphi \in \mathbf{Lip}(\mathbb{R}^m_+) \text{ and } \mathbf{Lip}(\varphi) \le 1, \|\varphi\|_\infty \le r\right\} \\ &\le \sup\left\{\int_{\mathbb{R}^m_+} \varphi \, \mathrm{d}(\mu-\nu) \ : \ \varphi \in \mathbf{Lip}(\mathbb{R}^m_+) \text{ and } \|\varphi\|_{\infty,\mathbf{Lip}} \le \max\{1,r\}\right\} \\ &= \sup\left\{\int_{\mathbb{R}^m_+} \varphi \, \mathrm{d}(\mu-\nu) \ : \ \varphi \in \mathbf{C}^1(\mathbb{R}^m_+) \text{ and } \|\varphi\|_{\infty,\mathbf{Lip}} \le \max\{1,r\}\right\} \end{aligned}$$

$$= \max\{1, r\}\rho_F(\mu, \nu).$$

Here, the condition $\varphi \in \mathbf{Lip}(\mathbb{R}_+^m)$ was substituted by $\varphi \in \mathbf{C^1}(\mathbb{R}_+^m)$ through a standard mollifying sequence argument applied to the test function φ. Thus,

$$\min\{1, r^{-1}\}W_1 \le \rho_F(\mu, \nu).$$

On the other hand, it is clear that

$$\rho_F(\mu, \nu) \le W_1(\mu, \nu), \tag{1.32}$$

since in the definition of W_1 the supremum is taken over the bigger set of functions. Therefore, (1.32) holds even if we remove the assumption concerning the compactness of the support. Note that if $r \le 1$ (or equivalently $\operatorname{diam}(K) \le 2$), then both metrics are equal. □

Let take a closer look at both metrics and consider the following example.

Example 1.33. Let $\delta_a, \delta_b \in \mathcal{P}_1(\mathbb{R}_+)$ be Dirac deltas, $a, b \ge 0$, and calculate their distance in the total variation norm, 1-Wasserstein distance and flat metric. We obtain

$$\|\delta_a - \delta_b\|_{TV} = \int_{\mathbb{R}_+} 1 \, \mathrm{d}(\delta_a + \delta_b) = 2.$$

The only possible transference plan between both measures is given by $\pi(\delta_a, \delta_b) = \delta_{(a,b)}$ and thus

$$W_1(\delta_a, \delta_b) = |a - b|.$$

From Proposition 1.31 it follows that

$$\min\{1, 2/|a-b|\}|a - b| \le \rho_F(\delta_a, \delta_b) \le |a - b|. \tag{1.34}$$

If $|a - b| \le 2$, then $\min\{1, 2/|a - b|\} = 1$ and

$$|a - b| \le \rho_F(\delta_a, \delta_b) \le |a - b| \quad \Rightarrow \quad \rho_F(\delta_a, \delta_b) = \min\{2, |a - b|\}.$$

If $|a - b| > 2$, then $\min\{1, 2/|a - b|\} = 2/|a - b|$ and the first inequality in (1.34) yields $2 \le \rho_F(\delta_a, \delta_b)$. On the other hand, it follows from (1.24) that $\rho_F(\delta_a, \delta_b) \le 2$. Since in this case $2 = \min\{2, |a - b|\}$, we conclude that in general

$$\rho_F(\delta_a, \delta_b) = \min\{2, |a - b|\}.$$

Lemma 1.35. *Assume that* $\mu = \sum_{i=1}^J m^i \delta_{x^i}$ *and* $\nu = \sum_{i=1}^J n^i \delta_{y^i}$*, where* $J \in \mathbb{N}$*,* $x^i, y^i \in \mathbb{R}_+^m$ *and* $m^i, n^i \in \mathbb{R}_+$*. Then,*

$$\rho_F(\mu, \nu) \le \sum_{i=1}^J \left(\left| x^i - y^i \right| m^i + |m^i - n^i| \right). \tag{1.36}$$

Proof of Lemma 1.35. Note that both measures are sums of J Dirac deltas, which have possibly different locations and different masses. By triangle inequality

$$\rho_F(\mu,\nu) \le \rho_F\left(\sum_{i=1}^J m^i\delta_{x^i}, \sum_{i=1}^J m^i\delta_{y^i}\right) + \rho_F\left(\sum_{i=1}^J m^i\delta_{y^i}, \sum_{i=1}^J n^i\delta_{y^i}\right).$$

Let $\varphi \in \mathbf{C^1}(\mathbb{R}^m_+)$ be such that $\|\varphi\|_{\infty,\mathbf{Lip}} \le 1$. In order to estimate the first term we calculate

$$\int_{\mathbb{R}^m_+} \varphi(x)\, \mathrm{d}\left(\sum_{i=1}^J m^i\delta_{x^i} - \sum_{i=1}^J m^i\delta_{y^i}\right)(x) \le \sum_{i=1}^J \|\varphi(x^i) - \varphi(y^i)\| m^i \le \sum_{i=1}^J \|x^i - y^i\| m^i.$$

Regarding the second term,

$$\int_{\mathbb{R}^m_+} \varphi(x)\, \mathrm{d}\left(\sum_{i=1}^J m^i\delta_{y^i} - \sum_{i=0}^J n^i\delta_{y^i}\right)(x) \le \sum_{i=1}^J \int_{\mathbb{R}^m_+} \varphi(x)|m^i - n^i|\, \mathrm{d}\delta_{y^i}(x) \le \sum_{i=1}^J |m^i - n^i|.$$

Taking supremum over all functions φ in both inequalities ends the proof. □

1.6. Advantages of the Measure Setting for Population Dynamics Models

Setting population dynamics models in the space of measures was already suggested in late 80's in [31, Section III.5], but for a long time there was no suitable analytical tools to cope with the issue. However, since it is often necessary to describe a population's state with a measure, which is not absolutely continuous with respect to the Lebesgue measure, this approach falls within the scope of intensively developed areas of mathematics. In exemplifying the usefulness of the measure setting we will focus mainly on Dirac deltas, which are natural representation of strongly localized effects. Nevertheless, one should keep in mind that our analysis applies to Radon measures in general. The choice of the Radon measures space appears to be convenient here, since it contains in particular Dirac measures and $\mathbf{L^1}$ functions, which are typically associated with densities.

There are many reasons of setting population dynamics models in the space of measures. The most direct one is a possibility of analyzing populations, which are concentrated with respect to a structural variable. One can set up an initial data in the form of Dirac measures or analyse rigorous models, which show a tendency towards aggregation. As an example we mention selection models, which describe a long-time evolution of populations structured with respect to a phenotypic trait. The individuals compete between each other, so that the strongest survive and the other extinct on the ecological time scale, which is called the speciation process.

Another positive aspect of using Dirac measures is the consistency of this approach with experimental data. As a matter of fact, a result of an experiment or observation is usually a number of individuals, which state is within the specific range, e.g. demographical data provides the number of particular individuals within age cohorts. Thus, describing the initial data by a sum of Dirac deltas does not have to reflect our intuition about concentration of the individuals at one point. Through basing on such experimental data, a real distribution of individuals can be approximated by an appropriate $\mathbf{L^1}$

function, a sum of Dirac deltas with proper masses or a general Radon measure. The essential expectation of the approximation is as follows: the narrower ranges of the measurement are, the lower the error rate of the approximation becomes(so called empirical stability). In other words, more detailed measurement should assure that its result is getting closer to the real distribution. To illustrate this by an example, let us consider a data given by a set of numbers $\{a_n\}_{n\in\mathbb{N}}$, where each a_n is the amount of individuals within the range $[nh,(n+1)h)$, i.e. $\int_{[nh,(n+1)h)} \mathrm{d}\mu = a_n$ and μ is a distribution of a population. Then, a set of all possible states of the population is defined as follows

$$A = \left\{ \mu \in \mathcal{M}^+(\mathbb{R}_+) \ : \ \int_{[nh,(n+1)h)} \mathrm{d}\mu = a_n, \ n \in \mathbb{N} \right\}.$$

If we consider the total variation norm $\|\cdot\|_{TV}$ we conclude that the diameter of the set A does not depend on the range of the measurement h. The same result holds for the set A restricted to $\mathbf{L^1}(\mathbb{R}_+)$ functions. Indeed,

$$\mathrm{diam}_{\|\cdot\|_{TV}}(A) = \sup_{\mu,\nu\in A} \|\mu - \nu\|_{TV} \le \sup_{\mu,\nu\in A} (\mu(\mathbb{R}_+) + \nu(\mathbb{R}_+)) = 2\sum_{n\in\mathbb{N}} a_n$$

and thus $\mathrm{diam}_{\|\cdot\|_{\mathbf{L^1}}}(A \cap \mathbf{L^1}) \le 2\sum_{n\in\mathbb{N}} a_n$, becasue in the latter case the supremum is taken over a smaller set. In fact, the supremum in both cases is equal to $2\sum_{n\in\mathbb{N}} a_n$, since there exist integrable functions $u, v \in A$

$$u(x) = \sum_{n\in\mathbb{N}} \frac{2a_n}{h} \chi_{[nh,(n+1/2)h)}(x) \quad \text{and} \quad v(x) = \sum_{n\in\mathbb{N}} \frac{2a_n}{h} \chi_{[(n+1/2)h,(n+1)h)}(x)$$

such that $\|u - v\|_{TV} = \|u - v\|_{\mathbf{L^1}} = 2\sum_{n\in\mathbb{N}} a_n$. Nevertheless, empirical stability can still be assured if other metric are used. Let us calculate a diameter of a given set A in the flat metric ρ_F. To proceed, let $\mu, \nu \in A$ and define $M = \sum_{n\in\mathbb{N}} a_n$, $\tilde{\mu} = (\mu/M)$, $\tilde{\nu} = (\nu/M)$. From Remark 1.25 and (1.32) it follows that

$$\rho_F(\mu,\nu) = M\rho_F(\tilde{\mu},\tilde{\nu}) \le MW_1(\tilde{\mu},\tilde{\nu}). \tag{1.37}$$

According to Remark 1.17 it is sufficient to construct one transference plan $\pi \in \Pi(\tilde{\mu},\tilde{\nu})$ in order to estimate $W_1(\tilde{\mu},\tilde{\nu})$. To proceed with the construction define a measure

$$\zeta = \sum_{n\in\mathbb{N}} a_n \delta_{(n+1/2)h}.$$

Namely, we place the Dirac delta of mass a_n at the middle of the interval $[nh,(n+1)h)$. Consider a transference plan $\pi_1(\tilde{\mu},\zeta)$ which shifts the mass, distributed according to $\tilde{\mu}$, from the interval $[nh,(n+1)h)$ into the point $x = (n+1/2)h$. Similarly, let $\pi_2 \in \Pi(\zeta,\tilde{\nu})$ be a transference plan which spreads the mass from $x = (n+1/2)h$ onto the interval $[nh,(n+1)h)$ such that the resulting distribution is $\tilde{\nu}$. Then,

$$W_1(\tilde{\mu},\zeta) \le \int_{\mathbb{R}_+\times\mathbb{R}_+} |x-y| \,\mathrm{d}\pi_1(x,y) \quad \le \quad \int_{\mathbb{R}_+\times\mathbb{R}_+} (h/2)\,\mathrm{d}\pi_1(x,y) = h/2.$$

Similarly, $W_1(\zeta,\tilde{\nu}) \le h/2$. Thus, $W_1(\tilde{\mu},\tilde{\nu}) \le h$ and by (1.37)

$$\rho_F(\mu,\nu) \le W_1(\tilde{\mu},\tilde{\nu}) \sum_{n\in\mathbb{N}} a_n \le h \sum_{n\in\mathbb{N}} a_n.$$

The empirical stability is one of the justifications of equipping the space of measures in metrics originated from the optimal transportation theory. Below we give another two examples (see [44], Example 1.1 and Example 1.2) which provide further explanations why the total variation norm is not suitable for population dynamics models. To be more specific, we will show that working with this norm does not guarantee a continuity of solutions even in the case of the simplest transport equation.

Example 1.38. Consider the following linear problem with no boundary conditions

$$\begin{cases} \partial_t \mu + b\, \partial_x \mu &= 0, \quad \text{in } [0,T] \times \mathbb{R}, \\ \mu_o &= \delta_o, \end{cases}$$

where $\delta_o \in \mathcal{M}(\mathbb{R}_+)$ is a Dirac delta located at 0 and $b > 0$ is a constant coefficient. A distributional solution of this problem is $\mu_t = \delta_{bt}$. A simple calculation shows that μ is not continuous as a mapping from $[0,T]$ into $(\mathcal{M}(\mathbb{R}_+), \|\cdot\|_{TV})$. Indeed, let $0 \le t \le s \le T$. Then, it follows from the Example 1.33 that

$$\|\mu_t - \mu_s\|_{TV} = 2.$$

On the contrary,

$$W_1(\delta_{bt}, \delta_{bs}) \le b\,|t-s| \quad \text{and} \quad \rho_F(\delta_{bt}, \delta_{bs}) \le b \min\{2, |t-s|\},$$

which means that μ is continuous as a mapping from $[0,T]$ into $(\mathcal{P}_1(\mathbb{R}_+), W_1)$ (resp. $(\mathcal{M}^+(\mathbb{R}_+), \rho_F)$).

Example 1.39. Consider a sequence of solutions μ^n to the following linear problems with no boundary conditions

$$\begin{cases} \partial_t \mu^n + b^n\, \partial_x \mu^n &= 0, \quad \text{in } [0,T] \times \mathbb{R}, \\ \mu_o &= \delta_o, \end{cases}$$

where $\{b^n\}_{n\in\mathbb{N}}$ is a sequence of positive coefficients converging to $b > 0$. A distributional solution of each problem is given by $\mu^n_t = \delta_{b^n t}$. In this case, even though $b_n \to b$, we cannot expect that a distance between μ^n and μ calculated in terms of the total variation norm tends to zero. Similarly as in the previous example we obtain

$$\|\mu_t - \mu^n_t\|_{TV} = 2$$

and

$$W_1(\delta_{b^n t}, \delta_{bt}) \le t\,|b^n - b| \quad \text{and} \quad \rho_F(\delta_{b^n t}, \delta_{bt}) \le t \min\{2, |b^n - b|\},$$

which proves that the convergence of coefficients implies the convergence of solutions in W_1 and ρ_F, respectively.

Finally, we briefly summarize the advantages coming from exploiting the flat metric from the point of view of numerical schemes and their stability. The measure setting allows for the application of mesh-free methods based on the EBT method and particle methods, which are discussed in details in Chapter 4 and Chapter 5. Here, we just mention that these methods base goal is to solve a particular ODEs system instead of a PDE. Thus, they require a solution in the form of particles (or formally, a sum of Dirac deltas), since each ODE describe a behavior of a single particle. However, in population dynamics one often cannot avoid dealing with continuous distributions even in a case where the initial data are Dirac measures. For instance, consider two particular cases of (1.1), i.e., the McKendrick model obtained by setting

$$\eta(t,\mu)(y) = \beta(t,\mu)(y)\delta_{x=0}$$

(for details see § 2.2.1) or a basic selection model (for details see § 2.2.5)

$$\begin{cases} \partial_t \mu &= \int_{\mathbb{R}_+} \beta(y)\gamma(x,y)\mathrm{d}\mu_t(y), \quad \text{in } [0,T]\times\mathbb{R}_+, \\ \mu_o &= \delta_{x_o}. \end{cases}$$

In the first case, continuous distribution appears due to the boundary condition, while in the second one it appears immediately on the whole domain, if β and γ are regular enough. This phenomenon can be partially eliminated by using the splitting technique, which in some particular cases (e.g., McKendrick model) gives the approximation of a solution in the form of Dirac deltas as an output . It turns out that this approximation converges to a solution in the flat metric and we are able to provide the rate of convergence (see Chapter 4). As the selection model is regarded, in Chapter 2 we will prove that a solution to (1.1) is continuous in the flat metric with respect to the model coefficients. Note that if we approximate $\gamma(\cdot,y)$ in the example above by a sum of Dirac deltas, then a solution to the approximated problem covers Dirac deltas as well. Moreover, the error of such approximation measured in the flat metric can be arbitrarily small and thus, the numerical scheme based on the particle methods applies and converges to a solution.

Henceforth, we consider only the flat metric, since the Wasserstein distance between measures which have different masses is infinite. Unfortunately, this property makes it practically useless in problems coming from population dynamics, as in majority of populations conservation laws do not hold.

Chapter 2

One-sex Population Model: Well-posedness

2.1. Formulation of the Model

The main aim of this chapter is to prove a well posedness of the following Cauchy problem.

$$\begin{cases} \partial_t \mu + \partial_x \left(b(t,\mu)\,\mu \right) + c(t,\mu)\,\mu &= \int_{\mathbb{R}_+} (\eta(t,\mu))(y)\,\mathrm{d}\mu(y), \quad (t,x) \in [0,T] \times \mathbb{R}_+, \\ \mu_o &\in \mathcal{M}^+(\mathbb{R}_+), \end{cases} \tag{2.1}$$

The weak* semigroup approach to (2.1) was developed in [28], where global existence of solutions in the set of finite Radon measures was proved, together with their weak* continuous dependence on time and initial datum. A different treatment of the problem, based on the theory of nonlinear semigroups in metric spaces, was presented in [43] and [44]. An alternative construction of measure-valued solutions to these models can be obtained following ideas coming from kinetic theory [33, 19] by means of a Picard-type result for evolutions in the set of measures, see [18]. Our approach bases on the splitting techniqe in metric spaces developed in [25]. We exploit the latter technique to obtain a well posedness of the autonomous linear version of (2.1). In order to prove the well posedness of the non-autonomous linear case we approximate vital functions b, c and η by piecewise constant (with respect to time) functions. From the previous case we obtain existence of solutions on corresponding time intervals. Passing to the limit with a length of the intervals finishes the proof. The results for the nonlinear case are obtained by exploiting Banach fixed-point theorem.

We assume that $t \in [0,T]$ and $x \geq 0$ are time and the structural variable respectively. A measure μ_t, which is a value of the map $\mu : [0,T] \to \mathcal{M}^+(\mathbb{R}_+)$ at the point t, is a Radon measure describing a distribution of individuals with respect to x at time t. Coefficients b, c, η are called vital functions and below we explain their meaning. The dynamics of the transformation of the individual being at the state x_o at time t_o is given by the following ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = b(t, x(t)), \quad x(t_o) = x_o.$$

The coefficient b describes thus how fast the individual changes its state, e.g. how fast it grows or mature. In general, it represents the speed of changes of the structural variable.

In case of the age-structured models the coefficient b is always equal to 1, since the aging process can be equivalently expressed as the amount of time, which passes through the individual's life cycle. The survival chances are described by the coefficient c, which is most often interpreted as a death rate. A component of the model related to the birth process is the integral $\int_{\mathbb{R}_+} (\eta(t,\mu))(y)\mathrm{d}\mu(y)$ on the right hand side of (2.1). Formally, it is a Bochner integral with values in $\mathcal{M}^+(\mathbb{R}_+)$. Note that since $(\mathcal{M}^+(\mathbb{R}_+), \rho_F)$ is separable (Lemma 1.28), strong measurability and weak measurability are equivalent, see [44] for more details. We also refer to [35, Appendix E.5] for the basic results about Banach space valued functions. A measure $\eta(t,\mu)(y)$ can be interpreted as a distribution of possible offspring of the individual being at the state y, for each $y \in \mathbb{R}_+$. In case all new born individuals have always the same physiological state x_b, η takes the following form

$$\eta(t,\mu)(y) = \beta(t,\mu)(y)\delta_{x=x_b},$$

which leads to

$$\begin{cases} \partial_t \mu + \partial_x \left(b(t,\mu)\,\mu\right) + c(t,\mu)\,\mu &= 0, \qquad (t,x) \in [0,T] \times \mathbb{R}_+, \\ (b(t,\mu)(x_b))\, D_\lambda \mu_t(x_b^+) &= \int_{x_b}^{+\infty} \beta(t,\mu)(x)\mathrm{d}\mu_t(x), \\ \mu_o &\in \mathcal{M}^+(\mathbb{R}_+). \end{cases} \tag{2.2}$$

Here, $D_\lambda \mu_t(x_b^+)$ is the Radon-Nikodym derivative of μ_t with respect to the one dimensional Lebesgue measure λ at the point x_o. This notation is used to underline that we have in mind a value of a density of the measure μ_t at a point x_b. Note that under the particular assumptions on the coefficients this density is a continuous function and thus we can evaluate it pointwisely. For further consideration we assume without loss of generality that $x_b = 0$ (see Remark 2.12).

This chapter is organized as follows. Section 2.2 is devoted to showing that (2.1) comprehends various relevant models originated from the mathematical biology. We consider the models in their simplest version, often referring to their formulations in terms of $\mathbf{L}^1$ densities. However, they are currently studied in the framework of Radon measures, as highlighted in the references given above. In particular, in § 2.2.1 we show that (2.1) includes the McKendrick age structured population model [64] as well as the nonlinear age structured model [82, 81]. Then, § 2.2.2 deals with the linear and nonlinear size structured models for cell division presented in [31] and in [70, Chapter 4], as well as with the size structured model for evolution of phytoplankton aggregates, see [8]. Also a simple cell cycle structured population model fits in the present setting, as shown in § 2.2.3. The body size structured model [28] is considered in § 2.2.4. Finally, the selection-mutation models [4, 17, 20] are tackled in § 2.2.5. In Section 2.3 we present the analytical results, separately considering the linear autonomous case in Section 2.4, the linear non-autonomous case in Section 2.5 and the general case in Section 2.6.

2.2. Biological Models

2.2.1. Age Structured Cell Population Model with Crowding

Consider the age structured population of cells evolving due to processes of mortality and equal mitosis. Here, mitosis is understood as the birth of two new cells and the death of a mother cell. Linear models are based on the assumption, that birth and death rates are linear functions of the population density, what excludes such phenomena as crowding effects or environment limitations. Hence, it is more reasonable to consider nonlinear models, as an example we recall [81, Ex. 5.1], where the death rate depends on the population density:

$$\begin{aligned} \partial_t p(t,x) + \partial_x p(t,x) &= -\left(\beta(x) + u(x) + \tau \int_{\mathbb{R}_+} p(t,y)\mathrm{d}y\right) p(t,x), \\ p(t,0) &= 2\int_{\mathbb{R}_+} \beta(y)p(t,y)\mathrm{d}y\,. \end{aligned} \tag{2.3}$$

Here, t denotes time, x is age and $p(t,x)$ is the density of cells having age x at time t. Functions $\beta(x)$, $u(x)$ and τ are respectively the division rate, the natural mortality rate and the coefficient describing the influence of crowding effects on the evolution. Setting in (2.1)

$$\begin{aligned} \mu_t(A) &= \textstyle\int_A p(t,x)\mathrm{d}x, & b(t,\mu)(x) &= 1, \\ c(t,\mu)(x) &= \beta(x) + u(x) + \tau\mu_t(\mathbb{R}_+) \quad \text{and} & (\eta(t,\mu)(y))(A) &= 2\beta(y)\delta_{x=0} \quad \text{if } 0 \in A, \end{aligned}$$

we obtain (2.3). This model is at the basis of several studies. For instance, one may introduce a birth rate that depends on the population density

$$\beta(x) = \beta_1(x)\cdot\beta_2\left(\int_{\mathbb{R}_+} p(t,y)\mathrm{d}y\right).$$

Otherwise, one may simplify (2.3) obtaining the well-known and widely studied McKendrick age structured model [64]. Refer to [82] and the references therein for further possibilities.

2.2.2. Nonlinear Size Structured Model for Asymmetric Cell Division

For unicellular organisms, structuring population by age does not apply well, mainly because age is not the most relevant parameter that determines mitosis. Therefore, it is often more reasonable to consider size structured models, see [31, Section I.4.3, Ex. 4.3.6], for which

$$\partial_t n + \partial_x\left(V(x)n\right)n = -\left(\beta(x) + u(x)\right)n + 2\int_{\mathbb{R}_+} \beta(y)d(x,y)n(t,y)\mathrm{d}y \tag{2.4}$$

where t is time, x is size, $n(t,x)$ is a density of cells having size x at time t, $\beta(x)$ and $u(x)$ are division and mortality rate respectively. $V(x)$ is the dynamics of evolution of the individual at the state x. The integral on the right hand side describes the process of division. If division occurs, a mother cell of size y divides into two daughter cells of sizes x and $y-x$, which is expressed by the kernel d. In particular, $d(\cdot, y)$ is a probability measure on $\mathbb{R}_+$ for each fixed y. In general, the structural variable considered here does not need to be a size. It can be maturity (see § 2.2.3), which is described by the cell diameter or by the level of a chemical substance significant for the cell division process. Another biological process which fits into (2.4) is the evolution of phytoplankton aggregates (without a coagulation term), see [8]. Setting in (2.1)

$$\begin{aligned} \mu_t(A) &= \int_A n(t,x)\mathrm{d}x, & b(t,\mu)(x) &= V(x), \\ c(t,\mu)(x) &= \beta(x) + u(x) \quad \text{and} & (\eta(t,\mu)(y))(A) &= 2\int_A \beta(y)\, d(x,y)\mathrm{d}x, \end{aligned}$$

we obtain (2.4). In the linear case, with $d(x,y) = 2\delta_{x=y/2}$, we obtain the model [70, Section 4.1] describing equal mitosis. If $d(x,y) = \delta_{x=\sigma y} + \delta_{x=(1-\sigma)y}$, we obtain the general mitosis model [70, Section 4.2]. Setting $d(x,y) = \delta_{x=y} + \delta_{x=0}$, we return to the McKendrick model [64].

2.2.3. Cell Cycle Structured Population Model

This model is a special case of the one mentioned in § 2.2.2. It describes the structure of cells characterized by the position x in the cell cycle, where $0 < x_o \le x \le 1$. A new born cell has a maturity x_o and mitosis occurs only at a maturity $x = 1$. For simplicity, no mortality of cells is assumed, see [81, Ex. 2.3]. The model thus reads

$$\begin{cases} \partial_t p(t,x) + \partial_x (xp(t,x)) &= 0, \\ x_o p(t,x_0) &= 2p(t,1), \\ p(0,x) &= p_o(x). \end{cases}$$

This model is a particular case of (2.1) if one sets $\mu_t(A) = \int_A p(t,x)\mathrm{d}x$, $b(t,\mu)(x) = x$, $c(t,\mu)(x) = 0$ and

$$(\eta(t,\mu)(y))(A) = \begin{cases} 2\delta_{x=x_o}, & \text{if } x_o \in A, y = 1, \\ 0, & \text{in the opposite case.} \end{cases}$$

2.2.4. Body Size Structured Model with Possible Cannibalistic Interactions

Let us now present a model slightly more general than the one in § 2.2.2. This generalization is necessary for modeling those biological phenomenas, where the growth rate depends on the population density. As an example, we consider the following model, studied in [28, 43, 44], which describes the evolution of a body size structured population:

$$\begin{cases} \partial_t n + \partial_x \left(g(x,n)\, n \right) + h(x,n)\, n &= 0, \\ g(x_0,n)\, n(t,x_0) &= \int_{x_0}^{x_m} \beta(y,n)\, n(t,y)\, \mathrm{d}y, \\ n(0,x) &= n_o(x), \end{cases}$$

where t is time, x is the individual body size, x_o is the size of each new born individual, x_m is the maximum body size, $n(t,x)$ is the density of population having size x at time t (or rather concentration, if we allow $n(t,\cdot)$ to be a Radon measure), g describes the dynamics of individual's growth, h is a death rate and β is related to the influx of new individuals. In particular, dependence of the coefficients on n allows to model the evolution of e.g. cannibalistic populations, see [29]. Again, this is a particular case of (2.1), obtained by setting

$$\begin{aligned} \mu_t(A) &= \int_A n(t,x)\mathrm{d}x\,, & b(t,\mu)(x) &= g(x,n)\,, \\ c(t,\mu)(x) &= h(x,n) \qquad \text{and} & \left(\eta(t,\mu)(y)\right)(A) &= \beta(y,n)\delta_{x=x_o}. \end{aligned}$$

2.2.5. Selection-Mutation Models

Selection mutation models have been proposed in [4, 17, 20] to model species evolution. More precisely, one is interested in the evolution of a density of individuals $u(t,x)$ at time t with respect to an evolutionary variable $x \in \mathbb{R}_+$. For instance, one could consider x as the maturation age of a species. These models typically include a selection part due to the environment that can be modeled by logistic growth and a mutation term in which offspring are born with a slightly different trait than their parents. For instance, a typical model reads

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) &= (1-\varepsilon)\beta(x)u(t,x) - m(x,P(t))u(t,x) + \varepsilon \displaystyle\int_{\mathbb{R}_+} \beta(y)\gamma(x,y)u(t,y)\,dy, \\ u(0,x) &= u_o(x), \end{cases}$$

where $P(t) = \int_{\mathbb{R}_+} u(t,x)\mathrm{d}x$ is the total population, m is the natural mortality rate, β is fertility rate and ε gives the probability of mutation of the offspring. Finally, the mutant population is modeled by an integral operator where $\gamma(x,y)$ is the density of probability that the trait of the mutant offspring of an individual with trait y is x. Also this model is a particular case of (2.1), obtained by setting

$$\begin{aligned} \mu_t(A) &= \int_A u(t,x)\mathrm{d}x, & b(t,\mu)(x) &= 0, \\ c(t,\mu)(x) &= (1-\varepsilon)\beta(x) - m\left(x,\mu_t(\mathbb{R}_+)\right), & \left(\eta(t,\mu)(y)\right)(A) &= \varepsilon \int_A \beta(y)\gamma(x,y)\mathrm{d}x\,. \end{aligned}$$

2.3. Main Results

In this section section we present the main result of Chapter 2. Theorem (2.13) states that there exists a unique solution to (2.1), which is Lipschitz continuous as a mapping from $[0,T]$ to $(\mathcal{M}^+(\mathbb{R}_+),\rho_F)$. Moreover, we show stability of the solution with respect to the initial datum and model functions. Since the proof of the theorem is not straightforward, we deffer it to Section 2.6. We assume the following regularity of model coefficients:

$$\begin{aligned} b, c : [0,T] \times \mathcal{M}^+(\mathbb{R}_+) &\to \mathbf{W^{1,\infty}}(\mathbb{R}_+) \quad \text{and} \quad \forall_{(t,\mu)\in[0,T]\times\mathcal{M}^+(\mathbb{R}_+)} \ b(t,\mu)(0) \geq 0, \\ \eta : [0,T] \times \mathcal{M}^+(\mathbb{R}_+) &\to (\mathbf{C_b} \cap \mathbf{Lip})(\mathbb{R}_+; \mathcal{M}^+(\mathbb{R}_+)). \end{aligned}$$

The space $\mathbf{C_b}(\mathbb{R}_+; \mathcal{M}^+(\mathbb{R}_+))$ denotes the set of functions which are bounded with respect to the norm $\|\cdot\|_{(\mathbf{W^{1,\infty}})^*}$ and continuous with respect to ρ_F. The space $\mathbf{Lip}(\mathbb{R}_+; \mathcal{M}^+(\mathbb{R}_+))$ consists of all Lipschitz functions from $\mathbb{R}_+$ with values in the metric space $(\mathcal{M}^+(\mathbb{R}_+), \rho_F)$. A norm in the space $(\mathbf{C_b} \cap \mathbf{Lip})(\mathbb{R}_+; \mathcal{M}^+(\mathbb{R}_+))$ is defined as

$$\|f\|_{\mathbf{BL}} = \|f\|_{\mathbf{BC_x}} + \mathbf{Lip}(\cdot), \quad \text{where} \quad \|f\|_{\mathbf{BC_x}} = \sup_{x\in\mathbb{R}_+} \|f(x)\|_{(\mathbf{W^{1,\infty}})^*}$$

and $\mathbf{Lip}(f)$ is the Lipschitz constant of f.

Remark 2.5. It is worth to note that the space $(\mathbf{C_b} \cap \mathbf{Lip})(\mathbb{R}_+; \mathcal{M}^+(\mathbb{R}_+))$ is not a subspace of $\mathbf{W^{1,\infty}}\left(\mathbb{R}_+; (\mathbf{W^{1,\infty}}(\mathbb{R}_+))^*\right)$, although the set of Radon measures $\mathcal{M}^+(\mathbb{R}_+)$ is a nonnegative cone in $(\mathbf{W^{1,\infty}}(\mathbb{R}_+))^*$. This follows from the fact that the Rademacher's theorem, in general, does not apply to functions $f : \mathbb{R}^m \to Y$, where Y is an infinite dimensional Banach space [10]. Rademacher's theorem does not fail provided that Y has a Radon-Nikodym property, for instance if Y is a separable dual space or a reflexive space ([9, Corollary 5.12]). Nevertheless, this is not the case. Consider the function $f(x) = \delta_x$, where δ_x is a Dirac delta located at x. It is easy to check, that f is bounded with respect to the $\|\cdot\|_{(\mathbf{W^{1,\infty}})^*}$ norm and Lipschitz continuous with respect to ρ_F, since

$$\begin{aligned} \|f\|_{\mathbf{BC_x}} = \sup_{x\in\mathbb{R}_+} \|f(x)\|_{(\mathbf{W^{1,\infty}})^*} &= \sup_{x\in\mathbb{R}_+} \sup_{\{\psi : \|\psi\|_{\mathbf{W^{1,\infty}}} \leq 1\}} \int_{\mathbb{R}_+} \psi(x)\, \mathrm{d}\delta(x) = 1, \\ \rho_F(f(x_1), f(x_2)) &= \min\{2, |x_1 - x_2|\} \leq |x_1 - x_2|. \end{aligned}$$

However, $f'(x) = \delta'_x$ is not a well defined functional on $\mathbf{W^{1,\infty}}(\mathbb{R}_+)$. Indeed, the distributional derivative of δ_x is given as

$$\delta'_x(\varphi) = -\delta_x(\varphi') = -\varphi'(x) \quad \forall \varphi \in \mathbf{C_c^\infty}(\mathbb{R}_+).$$

If $\varphi \in \mathbf{W^{1,\infty}}(\mathbb{R}_+)$, then $\varphi' \in \mathbf{L^\infty}(\mathbb{R}_+)$, which is not a sufficient regularity for the expression above to be well defined.

We assume the following regularity for our model functions

$$b, c \in \mathbf{C_b^{\alpha,1}}\left([0,T] \times \mathcal{M}^+(\mathbb{R}_+);\ \mathbf{W^{1,\infty}}(\mathbb{R}_+)\right), \tag{2.6}$$

$$\eta \in \mathbf{C_b^{\alpha,1}}\left([0,T] \times \mathcal{M}^+(\mathbb{R}_+);\ (\mathbf{C_b} \cap \mathbf{Lip})(\mathbb{R}_+; \mathcal{M}^+(\mathbb{R}_+))\right). \tag{2.7}$$

We recall that the space $\mathbf{C_b^{\alpha,1}}([0,T]\times\mathcal{M}^+(\mathbb{R}_+); X)$ denotes the space of X valued functions which are bounded with respect to the $\|\cdot\|_X$ norm, Hölder continuous with exponent α with respect totime and Lipschitz continuous in ρ_F with respect to the measure variable. It is equipped with the $\|\cdot\|_{\mathbf{C_b^{\alpha,1}}}$ norm defined by

$$\|f\|_{\mathbf{C_b^{\alpha,1}}} = \|f\|_{\mathbf{BC_{t,\mu}}} + \sup_{t\in[0,T]} \mathbf{Lip}(f(t,\cdot)) + \sup_{\mu\in\mathcal{M}^+(\mathbb{R}_+)} \mathrm{H}(f(\cdot,\mu)), \tag{2.8}$$

where

$$\|f\|_{\mathbf{BC_{t,\mu}}} := \sup_{(t,\mu)\in[0,T]\times\mathcal{M}^+(\mathbb{R}_+)} \|f(t,\mu)\|_X,$$

$\mathbf{Lip}(f(t,\cdot))$ is the Lipschitz constant of $f(t,\cdot)$ and

$$\mathrm{H}(f(\cdot,\mu)) := \sup_{s_1,s_2\in[0,T]} \left(\|f(s_1,\mu)-f(s_2,\mu)\|_X/|s_1-s_2|^\alpha\right).$$

A relevant choice of functions b, c and η is the following:

$$\begin{aligned} b(t,\mu) =& \tilde{b}\left(t, \int_{\mathbb{R}_+} \beta(y)\,\mathrm{d}\mu(y)\right),\\ c(t,\mu) =& \tilde{c}\left(t, \int_{\mathbb{R}_+} \gamma(y)\,\mathrm{d}\mu(y)\right),\\ \eta(t,\mu) =& \tilde{\eta}\left(t, \int_{\mathbb{R}_+} h(y)\mathrm{d}\mu(y)\right), \end{aligned} \tag{2.9}$$

with

$$\begin{aligned} \beta,\gamma,h &\in \mathbf{W^{1,\infty}}(\mathbb{R}_+;\mathbb{R}_+),\\ \tilde{b},\tilde{c} &\in \mathbf{C_b^{\alpha,1}}\left([0,T]\times\mathbb{R}_+;\ \mathbf{W^{1,\infty}}(\mathbb{R}_+)\right),\\ \tilde{\eta} &\in \mathbf{C_b^{\alpha,1}}\left([0,T]\times\mathbb{R}_+;\ (\mathbf{C_b}\cap\mathbf{Lip})(\mathbb{R}_+;\mathcal{M}^+(\mathbb{R}_+))\right). \end{aligned}$$

As a particular example, consider the following nonlinear functions b, c and η:

$$b(t,\mu)(x) = h_b(t)\, f_b(x)\, G_b\left(\int_{\mathbb{R}_+} \varphi_b \mathrm{d}\mu\right), \quad c(t,\mu)(x) = h_c(t)\, f_c(x)\, G_c\left(\int_{\mathbb{R}_+} \varphi_c \mathrm{d}\mu\right),$$

and

$$[\eta(t,\mu)(x)]\,(A) = h_\eta(t)\, f_\eta(x)\, G_\eta\left(\int_{\mathbb{R}_+} \varphi_\eta \mathrm{d}\mu\right)\nu(A),$$

where $h_i \in \mathbf{C}^\alpha([0,T];\mathbb{R}_+)$, $f_i,\varphi_i,G_i \in \mathbf{W^{1,\infty}}(\mathbb{R}_+;\mathbb{R}_+)$, $i=b,c,\eta$, and $\nu\in\mathcal{M}^+(\mathbb{R}_+)$. Here, $\mathbf{C}^\alpha([0,T];\mathbb{R}_+)$ is the space of functions which are Hölder continuous with exponent α with respect to time. We begin our analytical study with the basic definition of solutions to (2.1).

Definition 2.10. *Given $T>0$, a function μ: $[0,T]\to\mathcal{M}^+(\mathbb{R}_+)$ is a* weak solution *to* (2.1) *on the time interval $[0,T]$ if μ is narrowly continuous with respect to time and for all $\varphi\in(\mathbf{C^1}\cap\mathbf{W^{1,\infty}})([0,T]\times\mathbb{R}_+)$, the following equality holds:*

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}_+} \left(\partial_t\varphi(t,x) + (b(t,\mu))\,(x)\,\partial_x\varphi(t,x) - (c(t,\mu))\,(x)\,\varphi(t,x)\right)\mathrm{d}\mu_t(x)\mathrm{d}t\\ &\quad+\int_0^T \int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+}\varphi(t,y)\mathrm{d}[\eta(t,\mu)(x)](y)\right)\mathrm{d}\mu_t(x)\mathrm{d}t\\ &=\int_{\mathbb{R}_+}\varphi(T,x)\,\mathrm{d}\mu_T(x) - \int_{\mathbb{R}_+}\varphi(0,x)\,\mathrm{d}\mu_o(x). \end{aligned} \tag{2.11}$$

The notion of *narrow continuity* has been stated in Definition 1.4, but we also refer to [5, § 5.1], where this concept was introduced. The integral $\int_{\mathbb{R}_+}\varphi(t,y)\mathrm{d}[\eta(t,\mu)(x)](y)$ denotes the integral of $\varphi(t,y)$ with respect to the measure $\eta(t,\mu)(x)$ in the variable y. Similarly, $\int_{\mathbb{R}_+}\varphi(T,x)\mathrm{d}\mu_T(x)$ is the integral of $\varphi(T,x)$ with respect to the measure μ_T in the variable x.

Remark 2.12. It is possible to rewrite (2.1) on the whole $\mathbb{R}$. We extend b, c, η for $x < 0$ as follows:

$$b(t,\mu)(x) = b(t,\mu)(-x), \quad c(t,\mu)(x) = c(t,\mu)(-x), \quad \eta(t,\mu)(x) = \eta(t,\mu)(-x)$$

for all $t \in [0,T]$, $\mu \in \mathcal{M}^+(\mathbb{R})$ and allow the initial measure μ_o to be a nonnegative measure on $\mathbb{R}$. In this more general setting, Definition 2.10 defines a solution on $\mathbb{R}$ by formula (2.11) for all test functions $\varphi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R})$. All results and proofs in this chapter remain valid for the problem considered on the whole $\mathbb{R}$. However, due to the fact that we are strongly focused on biological applications, we follow the approach presented in the earlier literature, where the use of $\mathbb{R}_+$ is suggested. The restriction to $\mathbb{R}_+$ is in fact a special case in the sense that for each initial datum $\mu_o \in \mathcal{M}^+(\mathbb{R}_+)$ it holds that $\mu_t \in \mathcal{M}^+(\mathbb{R}_+)$ for each $t \in [0,T]$ (see Lemma 2.66).

Theorem 2.13. *Let* (2.6)–(2.7)*,* (2.6)*, and* (2.7) *hold. Then, there exists a unique solution* $\mu \in (\mathbf{C_b} \cap \mathbf{Lip})\left([0,T]; \mathcal{M}^+(\mathbb{R}_+)\right)$ *to the full nonlinear problem* (2.1)*. Moreover,*

i) for all $0 \le t_1 \le t_2 \le T$ *there exist constants* K_1 *and* K_2*, such that*

$$\rho_F\left(\mu_{t_1}, \mu_{t_2}\right) \le K_1 \mathrm{e}^{K_2(t_2 - t_1)} \mu_o(\mathbb{R}_+)(t_2 - t_1).$$

ii) Let $\mu_o, \tilde{\mu}_o \in \mathcal{M}^+(\mathbb{R}_+)$ *and* b*,* $\tilde{b}$*,* c*,* $\tilde{c}$*,* η*,* $\tilde{\eta}$ *satisfy assumptions* (2.6) *and* (2.7)*. Let* μ *and* $\tilde{\mu}$ *solve* (2.1) *with initial datum and coefficients* (μ_o, b, c, η)*, and* $(\tilde{\mu}_o, \tilde{b}, \tilde{c}, \tilde{\eta})$ *respectively. Then, there exist constants* C_1*,* C_2 *and* C_3 *such that for all* $t \in [0,T]$

$$\rho_F\left(\mu_t, \tilde{\mu}_t\right) \le \mathrm{e}^{C_1 t} \rho_F\left(\mu_o, \tilde{\mu}_o\right) + C_2 \mathrm{e}^{C_3 t} t \left\|(b,c,\eta) - (\tilde{b}, \tilde{c}, \tilde{\eta})\right\|_{\mathbf{BC_{t,\mu}}}.$$

2.4. Linear Autonomous Case

The linear autonomous case of (2.1) reads

$$\begin{cases} \partial_t \mu + \partial_x \left(b(x)\,\mu\right) + c(x)\,\mu &= \int_{\mathbb{R}_+} \eta(y)\,\mathrm{d}\mu(y), \quad (t,x) \in [0,T] \times \mathbb{R}_+, \\ \mu_o &\in \mathcal{M}^+(\mathbb{R}_+), \end{cases} \tag{2.14}$$

where the unknown μ_t is in $\mathcal{M}^+(\mathbb{R}_+)$ for all times $t \in [0,T]$. In the present case, the assumptions (2.6)–(2.7) reduce to

$$b, c \in \mathbf{W^{1,\infty}}(\mathbb{R}_+) \text{ with } b(0) \ge 0\,, \tag{2.15}$$
$$\eta \in (\mathbf{C_b} \cap \mathbf{Lip})\left(\mathbb{R}_+; \mathcal{M}^+(\mathbb{R}_+)\right). \tag{2.16}$$

A first justification of Definition 2.10 of weak solution and of the assumptions (2.15)–(2.16) is provided by the following result.

Proposition 2.17. With the notations introduced above:

i) If $\eta(y)$ has density $g(y)$ for all $y \in \mathbb{R}_+$ with respect to the Lebesgue measure, with $g \in (\mathbf{C_b} \cap \mathbf{Lip})(\mathbb{R}_+; \mathbf{L^1}(\mathbb{R}_+;\mathbb{R}_+))$, that is

$$\int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} \varphi(x) \mathrm{d}[\eta(y)](x) \right) \mathrm{d}\mu_t(y) = \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} \varphi(x) g(x,y) \mathrm{d}x \right) \mathrm{d}\mu_t(y)$$

for all $\varphi \in \mathbf{C}(\mathbb{R}_+)$, where $g(y) \in \mathbf{L^1}(\mathbb{R}_+;\mathbb{R}_+)$ for all $y \in \mathbb{R}_+$ and $g(x,y) \coloneqq g(y)(x) \in \mathbb{R}_+$ for all $x, y \in \mathbb{R}_+$, then η satisfies (2.16).

ii) If μ_o has a density u_o with respect to the Lebesgue measure, $u_o \in (\mathbf{L^1} \cap \mathbf{C^1})(\mathbb{R}_+;\mathbb{R}_+)$, then $\mu_o \in \mathcal{M}^+(\mathbb{R}_+)$.

iii) Let (2.15) hold together with *i)* and *ii)* above. If μ has density u with respect to the Lebesgue measure, with $u \in \mathbf{Lip}([0,T]; \mathbf{L^1}(\mathbb{R}_+;\mathbb{R}_+))$ and $u(t) \in \mathbf{C^1}(\mathbb{R}_+;\mathbb{R}_+)$ for each $t \in [0,T]$, then u is weak a solution to

$$\begin{cases} \partial_t u + \partial_x (b(x) u) + c(x) u = \int_{\mathbb{R}_+} g(x,y)\, u(t,y) \mathrm{d}y, \\ u(0,x) = u_o(x), \\ u(t,0) = 0, \end{cases}$$

if and only if μ is a solution to the linear equation (2.14) in the sense of Definition 2.10.

The proof is immediate and, hence, omitted. As we have mentioned in the beginning of this chapter, in order to prove the well posedness of (2.14) we use the operator splitting algorithm, see [24] and [25, § 3.3]. To this aim, we consider separately the problems

$$\partial_t \mu + c(x)\, \mu = \int_{\mathbb{R}_+} \eta(y)\, \mathrm{d}\mu(y) \qquad \text{and} \qquad \partial_t \mu + \partial_x (b(x)\, \mu) = 0\,.$$

Remark that both problems are particular cases of (2.1), so that Definition 2.10 applies to both. Consider first the ODE part.

Lemma 2.18. *Let $c \in \mathbf{W^{1,\infty}}(\mathbb{R}_+)$ and $\eta \in (\mathbf{C_b} \cap \mathbf{Lip})(\mathbb{R}_+; \mathcal{M}^+(\mathbb{R}_+))$. Then, the Cauchy problem*

$$\partial_t \mu + c(x)\, \mu = \int_{\mathbb{R}_+} \eta(y)\, \mathrm{d}\mu(y) \tag{2.19}$$

generates a local Lipschitz semigroup $\hat{S}\colon [0,T] \times \mathcal{M}^+(\mathbb{R}_+) \to \mathcal{M}^+(\mathbb{R}_+)$, in the sense that:

i) $\hat{S}_0 = \mathbf{Id}$ *and for all $t_1, t_2 \in [0,T]$ with $t_1 + t_2 \in [0,T]$, we have $\hat{S}_{t_1} \circ \hat{S}_{t_2} = \hat{S}_{t_1+t_2}$.*

ii) For all $t \in [0,T]$ and for all $\mu_1, \mu_2 \in \mathcal{M}^+(\mathbb{R}_+)$, the following estimate holds:

$$\rho_F(\hat{S}_t \mu_1, \hat{S}_t \mu_2) \le \exp\left(3\left(\|c\|_{\mathbf{W^{1,\infty}}} + \|\eta\|_{\mathbf{BL}}\right) t\right) \rho_F(\mu_1, \mu_2)\,.$$

iii) For all $t \in [0,T]$ and for all $\mu_o \in \mathcal{M}^+(\mathbb{R}_+)$ define $\mu_t = \hat{S}_t \mu_o$. Then, the solution to the Cauchy problem (2.19) satisfies $\mu \in \mathbf{Lip}([0,T]; \mathcal{M}^+(\mathbb{R}_+))$ and the following estimate holds:

$$\rho_F(\hat{S}_t \mu_o, \mu_o) \le \left(\|c\|_{\mathbf{L^\infty}} + \|\eta\|_{\mathbf{BC_x}}\right)\, \exp\left(\left(\|c\|_{\mathbf{L^\infty}} + \|\eta\|_{\mathbf{BC_x}}\right) t\right) \mu_o(\mathbb{R}_+)\, t\,.$$

iv) Let $c_1, c_2 \in \mathbf{W^{1,\infty}}(\mathbb{R}_+)$, η_1, η_2 satisfy (2.16) *and denote by $\hat{S}^1, \hat{S}^2$ the corresponding semigroups. Then, for all $t \in [0,T]$ and $\mu_o \in \mathcal{M}^+(\mathbb{R}_+)$*

$$\rho_F(\hat{S}_t^1 \mu_o, \hat{S}_t^2 \mu_o) \le \left(\|c_1 - c_2\|_{\mathbf{L}^\infty} + \|\eta_1 - \eta_2\|_{\mathbf{BC_x}}\right) e^{\left(\|(c_1,c_2)\|_{\mathbf{L}^\infty} + \|(\eta_1,\eta_2)\|_{\mathbf{BC_x}}\right)t} \mu_o(\mathbb{R}_+)\, t$$

v) For all $\mu_o \in \mathcal{M}^+(\mathbb{R}_+)$, the orbit $t \to \hat{S}_t \mu_o$ of the semigroup is a weak solution to (2.19) *in the sense of Definition* 2.10*.*

The proof is deferred to § 2.4.1, where we exploit the dual formulation of (2.19). The analogous result about the convective part is stated below.

Lemma 2.20. *Let $b \in \mathbf{W^{1,\infty}}(\mathbb{R}_+)$ with $b(0) \ge 0$. Then, the Cauchy problem*

$$\partial_t \mu + \partial_x \left(b(x)\, \mu\right) = 0 \tag{2.21}$$

generates a local Lipschitz semigroup $\check{S}\colon [0,T] \times \mathcal{M}^+(\mathbb{R}_+) \to \mathcal{M}^+(\mathbb{R}_+)$, in the sense that

i) $\check{S}_0 = \mathbf{Id}$ and for all $t_1, t_2 \in [0,T]$ with $t_1 + t_2 \in [0,T]$, we have $\check{S}_{t_1} \circ \check{S}_{t_2} = \check{S}_{t_1+t_2}$.

ii) For all $t \in [0,T]$ and for all $\mu_1, \mu_2 \in \mathcal{M}^+(\mathbb{R}_+)$, the following estimate holds:

$$\rho_F(\check{S}_t \mu_1, \check{S}_t \mu_2) \le \exp\left(\|\partial_x b\|_{\mathbf{L}^\infty} t\right) \rho_F(\mu_1, \mu_2)\,.$$

iii) For all $t \in [0,T]$ and for all $\mu_o \in \mathcal{M}^+(\mathbb{R}_+)$ define $\mu_t = \check{S}_t \mu_o$. Then, the solution of the Cauchy problem (2.21) *satisfies $\mu \in \mathbf{Lip}([0,T];\mathcal{M}^+(\mathbb{R}_+))$ and the following estimate holds:*

$$\rho_F(\check{S}_t \mu_o, \mu_o) \le \|b\|_{\mathbf{L}^\infty}\, \mu_o(\mathbb{R}_+)\, t\,.$$

iv) Let $b_1, b_2 \in \mathbf{W^{1,\infty}}(\mathbb{R}_+)$ with $b_1(0), b_2(0) \ge 0$ and denote by $\check{S}^1, \check{S}^2$ the corresponding semigroups. Then, for all $t \in [0,T]$ and $\mu_o \in \mathcal{M}^+(\mathbb{R}_+)$

$$d(\check{S}_t^1 \mu_o, \check{S}_t^2 \mu_o) \le \|b_1 - b_2\|_{\mathbf{L}^\infty}\, \mu_o(\mathbb{R}_+)\, t\,.$$

v) For all $\mu_o \in \mathcal{M}^+(\mathbb{R}_+)$, the orbit $t \to \check{S}_t \mu_o$ of the semigroup is a weak solution to (2.21) *in the sense of Definition* 2.10*.*

The proof of the latter Lemma is deferred to § 2.4.2. To apply [25, Corollary 3.3] (see also [24, theorems 3.5 and 3.8]) which states about the convergence of the operator splitting algorithm, we need to estimate the defect of commutativity of the two semigroups $\hat{S}$, $\check{S}$ defined above.

Proposition 2.22. Let (2.15) and (2.16) hold. Let $\hat{S}$ be the semigroup defined in Lemma 2.18 and $\check{S}$ the one defined in Lemma 2.20. Then, for all $\mu \in \mathcal{M}^+(\mathbb{R}_+)$ and for all $t \in [0,T]$, the following estimate on the lack of commutativity of $\check{S}$ and $\hat{S}$ holds:

$$\rho_F(\check{S}_t \hat{S}_t \mu, \hat{S}_t \check{S}_t \mu) \le 3\, t^2\, \|b\|_{\mathbf{L}^\infty} \left(\|c\|_{\mathbf{W^{1,\infty}}} + \|\eta\|_{\mathbf{BL}}\right) \exp\left[3\left(\|c\|_{\mathbf{W^{1,\infty}}} + \|\eta\|_{\mathbf{BL}}\right) t\right]. \tag{2.23}$$

The above commutativity estimate allows us to apply the usual operator splitting technique, obtaining the following final result in the linear autonomous case. For transparency of the thesis we place the proof in § 2.4.3.

Theorem 2.24. *Let* (2.15) *and* (2.16) *hold. The operator splitting procedure applied to the semigroups* $\check{S}$ *and* $\hat{S}$ *yields a local Lipschitz semigroup* S *that enjoys the following properties:*

i) $S_0 = \mathbf{Id}$ *and for all* $t_1, t_2 \in [0,T]$ *with* $t_1 + t_2 \in [0,T]$*, we have* $S_{t_1} \circ S_{t_2} = S_{t_1+t_2}$.

ii) For all $t \in [0,T]$ *and for all* $\mu_1, \mu_2 \in \mathcal{M}^+(\mathbb{R}_+)$*, the following estimate holds:*

$$\rho_F(S_t\mu_1, S_t\mu_2) \le \exp\left[3\left(\|\partial_x b\|_{\mathbf{L}^\infty} + \|c\|_{\mathbf{W}^{1,\infty}} + \|\eta\|_{\mathbf{BL}}\right)t\right]\rho_F(\mu_1,\mu_2)\,.$$

iii) For all $t \in [0,T]$ *and for all* $\mu_o \in \mathcal{M}^+(\mathbb{R}_+)$*, define* $\mu_t = S_t\mu_o$*. Then, the solution of the Cauchy problem* (2.14) *satisfies* $\mu \in \mathbf{Lip}([0,T];\mathcal{M}^+(\mathbb{R}_+))$ *and the following estimate holds:*

$$\rho_F(S_t\mu_o,\mu_o) \le \left(\|b\|_{\mathbf{L}^\infty} + \left(\|c\|_{\mathbf{L}^\infty} + \|\eta\|_{\mathbf{BC_x}}\right)\exp\left[\left(\|c\|_{\mathbf{L}^\infty} + \|\eta\|_{\mathbf{BC_x}}\right)t\right]\right)\mu_o(\mathbb{R}_+)\,t\,.$$

iv) For $i = 1,2$*, let* b_i*,* c_i*,* η_i *satisfy assumptions* (2.15) *and* (2.16)*. Denote by* S^i *the corresponding semigroup. Then, for all* $t \in [0,T]$ *and* $\mu_o \in \mathcal{M}^+(\mathbb{R}_+)$

$$\begin{aligned}\rho_F(S^1_t\mu_o, S^2_t\mu_o) &\le \exp\left[5\left(\|(b_1,c_1)\|_{\mathbf{W}^{1,\infty}} + \|\eta_1\|_{\mathbf{BL}}\right)t\right]\\ &\quad\cdot t\ \mu_o(\mathbb{R}_+)\left(\|(b_1-b_2, c_1-c_2)\|_{\mathbf{L}^\infty} + \|\eta_1-\eta_2\|_{\mathbf{BC_x}}\right).\end{aligned}$$

v) For all $\mu_o \in \mathcal{M}^+(\mathbb{R}_+)$*, the orbit* $t \to S_t\mu_o$ *of the semigroup is a weak solution to linear autonomous problem* (2.14) *in the sense of Definition* 2.10.

vi) The following tangency condition holds: $\lim_{t\to 0+}\frac{1}{t}\,d(S_t\mu, \check{S}_t\hat{S}_t\mu) = 0$.

Above, *ii)* corresponds to the Lipschitz dependence from the initial datum, *iii)* to the time regularity of the solution, *iv)* shows the stability with respect to the defining equations and *vi)* allows for a characterization in terms of evolution equations in metric spaces, see [25].

Proof of Theorem 2.24. Points *i)*, *ii)*, *iii)* and *vi)* are consequences of the results obtained in [25, Corollary 3.3 and Lemma 3.4], see also [24, Proposition 3.2], combined with the estimates provided by Lemma 2.18 and Lemma 2.20.

Passing to *iv)*, we use [14, Theorem 2.9], to estimate the distance between $S^1_t\mu$ and $S^2_t\mu$:

$$\rho_F(S^1_t\mu, S^2_t\mu) \le \mathbf{Lip}(S^1_t)\int_0^t \liminf_{h\to 0}\frac{1}{h}d(S^1_hS^1_\tau\mu, S^2_hS^1_\tau\mu)\,\mathrm{d}\tau\,. \tag{2.25}$$

Let $\nu = S^1_\tau\mu$. Using Lemma 2.18 and Lemma 2.20 compute

$$\begin{aligned}\rho_F(S^1_h\nu, S^2_h\nu) &\le \rho_F(S^1_h\nu, \check{S}^1_h\hat{S}^1_h\nu) + \rho_F(\check{S}^1_h\hat{S}^1_h\nu, \check{S}^2_h\hat{S}^2_h\nu) + \rho_F(\check{S}^2_h\hat{S}^2_h\nu, S^2_h\nu)\\ &\le \rho_F(\check{S}^1_h\hat{S}^1_h\nu, \check{S}^1_h\hat{S}^2_h\nu) + \rho_F(\check{S}^1_h\hat{S}^2_h\nu, \check{S}^2_h\hat{S}^2_h\nu) + o(h)\\ &\le \exp(\|\partial_x b_1\|_{\mathbf{L}^\infty}h)\,\rho_F(\hat{S}^1_h\nu, \hat{S}^2_h\nu) + \|b_1-b_2\|_{\mathbf{L}^\infty}\,(\hat{S}^2_h\nu)(\mathbb{R}_+)\,h + o(h)\\ &\le \left(\|c_1-c_2\|_{\mathbf{L}^\infty} + \|\eta_1-\eta_2\|_{\mathbf{BC_x}}\right)\mathrm{e}^{\left(\|\partial_x b_1\|_{\mathbf{L}^\infty} + \|(c_1,c_2)\|_{\mathbf{L}^\infty} + \|(\eta_1,\eta_2)\|_{\mathbf{BC_x}}\right)h}\nu(\mathbb{R}_+)\,h\\ &+ \|b_1-b_2\|_{\mathbf{L}^\infty}\,\mathrm{e}^{2\left(\|c_2\|_{\mathbf{L}^\infty} + \|\eta_2\|_{\mathbf{BC_x}}\right)h}\nu(\mathbb{R}_+)\,h + o(h).\end{aligned}$$

Therefore, by the lim inf formula (2.25),

$$\rho_F(S^1_t\mu, S^2_t\mu) \le \mathbf{Lip}(S^1)\left(\|b_1 - b_2\|_{\mathbf{L}^\infty} + \|c_1 - c_2\|_{\mathbf{L}^\infty} + \|\eta_1 - \eta_2\|_{\mathbf{BC_x}}\right)\int_0^t (S^1_\tau\mu)(\mathbb{R}_+)\mathrm{d}\tau\,.$$

An estimate of $\mathbf{Lip}(S^1_t)$ is provided by *ii)*, while the latter term above is bounded using *iii)* and the definition of the metric ρ_F:

$$\begin{aligned}
\int_0^t (S^1_\tau\mu)(\mathbb{R}_+)\mathrm{d}\tau &\le \int_0^t \left|(S^1_\tau\mu)(\mathbb{R}_+) - \mu(\mathbb{R}_+)\right|\mathrm{d}\tau + t\,\mu(\mathbb{R}_+)\\
&\le \int_0^t \left[\|b_1\|_{\mathbf{L}^\infty} + \left(\|c_1\|_{\mathbf{L}^\infty} + \|\eta_1\|_{\mathbf{BC_x}}\right)\mathrm{e}^{\left(\|c_1\|_{\mathbf{L}^\infty}+\|\eta_1\|_{\mathbf{BC_x}}\right)\tau}\right]\mu(\mathbb{R}_+)\,\tau\mathrm{d}\tau + t\,\mu(\mathbb{R}_+)\\
&\le\ t\,\mu(\mathbb{R}_+)\left[\left(\|b_1\|_{\mathbf{L}^\infty} + \left(\|c_1\|_{\mathbf{L}^\infty} + \|\eta_1\|_{\mathbf{BC_x}}\right)\mathrm{e}^{\left(\|c_1\|_{\mathbf{L}^\infty}+\|\eta_1\|_{\mathbf{BC_x}}\right)t}\right)t + 1\right]\\
&\le\ t\,\mu(\mathbb{R}_+)\mathrm{e}^{\left(\|c_1\|_{\mathbf{L}^\infty}+\|\eta_1\|_{\mathbf{BC_x}}\right)t}\left(\left(\|b_1\|_{\mathbf{L}^\infty} + \|c_1\|_{\mathbf{L}^\infty} + \|\eta_1\|_{\mathbf{BC_x}}\right)t + 1\right)\\
&\le\ t\,\mu(\mathbb{R}_+)\mathrm{e}^{\left(\|c_1\|_{\mathbf{L}^\infty}+\|\eta_1\|_{\mathbf{BC_x}}\right)t}\mathrm{e}^{\left(\|b_1\|_{\mathbf{L}^\infty}+\|c_1\|_{\mathbf{L}^\infty}+\|\eta_1\|_{\mathbf{BC_x}}\right)t}.
\end{aligned}$$

Since $\mathbf{Lip}(S^1_t) = \exp\left[3\left(\|\partial_x b\|_{\mathbf{L}^\infty} + \|c\|_{\mathbf{W}^{1,\infty}} + \|\eta\|_{\mathbf{BL}}\right)t\right]$, we finally obtain

$$\mathbf{Lip}(S^1_t)\int_0^t (S^1_\tau\mu)(\mathbb{R}_+)\mathrm{d}\tau \le \exp\left[5\left(\|(b_1,c_1)\|_{\mathbf{W}^{1,\infty}} + \|\eta_1\|_{\mathbf{BL}}\right)t\right],$$

which proves point *iv)* in Theorem 2.24.

To complete the proof, we show that $t \mapsto S_t\mu$ solves the linear autonomous problem (2.14) in the sense of Definition 2.10. Fix $n \in \mathbb{N}$ and define $\varepsilon = T/n$. First, as in [24, Section 5.3], consider the following continuous operator splitting:

$$F^\varepsilon(t)\mu = \begin{cases} \check{S}_{2t-2i\varepsilon}\,(\hat{S}_\varepsilon\check{S}_\varepsilon)^i\mu & \text{for}\quad t \in [i\varepsilon, (i+1/2)\varepsilon)\,,\\ \hat{S}_{2t-(2i+1)\varepsilon}\,\check{S}_\varepsilon\,(\hat{S}_\varepsilon\check{S}_\varepsilon)^i\mu & \text{for}\quad t \in [(i+1/2)\varepsilon, (i+1)\varepsilon)\,, \end{cases}$$

where $i = 0,\dots,n-1$. This formula is, in our case, equivalent to that given by [25, Corollary 3.3]. Define $\mu^\varepsilon_t = F^\varepsilon(t)\mu_o$ for a $\mu_o \in \mathcal{M}^+(\mathbb{R}_+)$. For any $\varphi \in \mathbf{C}^\infty_\mathbf{c}([0,T]\times\mathbb{R}_+)$,

$$\begin{aligned}
&\int_{\mathbb{R}_+}\varphi(T,x)\mathrm{d}\mu^\varepsilon_T(x) - \int_{\mathbb{R}_+}\varphi(0,x)\mathrm{d}\mu_o(x) = \qquad (2.26)\\
&= \int_0^T\int_{\mathbb{R}_+}\left[\partial_t\varphi(t,x) + b(x)\partial_x\varphi(t,x) - c(x)\varphi(t,x) + \int_{\mathbb{R}_+}\varphi(t,y)\mathrm{d}[\eta(x)](y)\right]\mathrm{d}\mu^\varepsilon_t(x)\mathrm{d}t + R(\varepsilon),
\end{aligned}$$

where

$$\begin{aligned}
R(\varepsilon) \;=\; &\sum_{i=0}^{n-1}\int_{i\varepsilon}^{(i+1/2)\varepsilon}\Big[\int_{\mathbb{R}_+}\left(\partial_t\varphi(t+\varepsilon/2,x) - 2c(x)\varphi(t+\varepsilon/2,x)\right)\mathrm{d}\mu^\varepsilon_{(t+\varepsilon/2)}(x)\\
&+\int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+}\varphi(t+\varepsilon/2,y)\mathrm{d}[2\eta(x)](y)\right)\mathrm{d}\mu^\varepsilon_{(t+\varepsilon/2)}(x)\\
&-\int_{\mathbb{R}_+}\left(\partial_t\varphi(t,x) - 2c(x)\varphi(t,x) + \int_{\mathbb{R}_+}\varphi(t,y)\mathrm{d}[2\eta(x)](y)\right)\mathrm{d}\mu^\varepsilon_t(x)\Big]\mathrm{d}t\\
&+\sum_{i=0}^{n-1}\int_{(i+1/2)\varepsilon}^{(i+1)\varepsilon}\Big[\int_{\mathbb{R}_+}\left(\partial_t\varphi(t-\varepsilon/2,x) + 2b(x)\partial_x\varphi(t-\varepsilon/2,x)\right)\mathrm{d}\mu^\varepsilon_{(t-\varepsilon/2)}(x)
\end{aligned}$$

$$
\begin{aligned}
&\quad - \int_{\mathbb{R}_+} (\partial_t\varphi(t,x) + 2b(x)\partial_x\varphi(t,x))\,\mathrm{d}\mu_t^\varepsilon(x)\Big]\mathrm{d}t \\
&= \sum_{i=0}^{n-1}\int_{i\varepsilon}^{(i+1/2)\varepsilon}\Big\{\int_{\mathbb{R}_+} (\partial_t\varphi(t+\varepsilon/2,x) - \partial_t\varphi(t,x))\,\mathrm{d}\mu^\varepsilon_{(t+\varepsilon/2)}(x) && (2.27)\\
&\quad - \int_{\mathbb{R}_+} 2c(x)\,(\varphi(t+\varepsilon/2,x) - \varphi(t,x))\,\mathrm{d}\mu^\varepsilon_{(t+\varepsilon/2)}(x) && (2.28)\\
&\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} (\varphi(t+\varepsilon/2,y) - \varphi(t,x))\,\mathrm{d}[2\eta(x)](y)\mathrm{d}\mu^\varepsilon_{(t+\varepsilon/2)}(x) && (2.29)\\
&\quad + \int_{\mathbb{R}_+} (\partial_t\varphi(t,x) - 2c(x)\varphi(t,x))\,\mathrm{d}\left(\mu^\varepsilon_{(t+\varepsilon/2)} - \mu^\varepsilon_t(x)\right) && (2.30)\\
&\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \varphi(t,y)\mathrm{d}[2\eta(x)](y)\mathrm{d}\left(\mu^\varepsilon_{(t+\varepsilon/2)} - \mu^\varepsilon_t(x)\right)\Big\}\mathrm{d}t && (2.31)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=0}^{n-1}\int_{(i+1/2)\varepsilon}^{(i+1)\varepsilon}\Big\{\int_{\mathbb{R}_+} (\partial_t\varphi(t-\varepsilon/2,x) - \partial_t\varphi(t,x))\,\mathrm{d}\mu^\varepsilon_{t-\varepsilon/2}(x) && (2.32)\\
&\quad + \int_{\mathbb{R}_+} 2b(x)\,(\partial_x\varphi(t-\varepsilon/2,x) - \partial_x\varphi(t-\varepsilon/2,x))\,\mathrm{d}\mu^\varepsilon_{t-\varepsilon/2}(x) && (2.33)\\
&\quad + \int_{\mathbb{R}_+} (\partial_t\varphi(t,x) + 2b(x)\partial_x\varphi(t,x))\,\mathrm{d}(\mu^\varepsilon_{t-\varepsilon/2} - \mu^\varepsilon_t)(x)\Big\}\mathrm{d}t. && (2.34)
\end{aligned}
$$

Notice, that $t \to \mu^\varepsilon(t)$ is uniformly bounded in $\mathbf{C_b}\left([0,T],(\mathcal{M}^+(\mathbb{R}_+),\rho_F)\right)$. Due to the regularity of φ, we have the following uniform convergences:

$$
\begin{aligned}
\partial_t\varphi(t+\varepsilon/2,x) - \partial_t\varphi(t,x) &\rightrightarrows 0 \Rightarrow (2.27) \to 0,\\
2c(x)\,(\varphi(t+\varepsilon/2,x) - \varphi(t,x)) &\rightrightarrows 0 \Rightarrow (2.28) \to 0,\\
(\varphi(t+\varepsilon/2,y) - \varphi(t,y)) &\rightrightarrows 0 \Rightarrow (2.29) \to 0,\\
\partial_t\varphi(t-\varepsilon/2,x) - \partial_t\varphi(t,x) &\rightrightarrows 0 \Rightarrow (2.32) \to 0,\\
2b(x)\,(\partial_x\varphi(t-\varepsilon/2,x) - \partial_x\varphi(t,x)) &\rightrightarrows 0 \Rightarrow (2.33) \to 0,
\end{aligned}
$$

as $\varepsilon \to 0$. To show the convergence of (2.30), (2.31) and (2.34), it is sufficient to note that μ_t^ε is uniformly Lipschitz continuous, i.e. $\rho_F\left(\mu_t^\varepsilon,\mu_{t-\varepsilon}^\varepsilon\right) \le K\varepsilon$, where K is a Lipschitz constant independent on t, for instance the same as in Theorem 2.24 (see [24, Proposition 3.2]). Moreover, according to [25, Corollary 4.4 and Proposition 4.6], μ_t^ε converges uniformly with respect to time in ρ_F to $S_t\mu$. Hence, passing to the limit in (2.26) yields

$$
\begin{aligned}
&\int_{\mathbb{R}_+}\varphi(T,x)\mathrm{d}\mu_T(x) - \int_{\mathbb{R}_+}\varphi(0,x)\mathrm{d}\mu_o(x)\\
&\quad = \int_0^T\int_{\mathbb{R}_+}\Big[\partial_t\varphi(t,x) + b(x)\partial_x\varphi(t,x) - c(x)\varphi(t,x) + \int_{\mathbb{R}_+}\varphi(t,y)\mathrm{d}[\eta(x)](y)\Big]\mathrm{d}\mu_t(x)\mathrm{d}t.
\end{aligned}
$$

We need now to show that the above equality holds for all $\varphi \in (\mathbf{C^1}\cap\mathbf{W^{1,\infty}})([0,T]\times\mathbb{R}_+)$. To this aim, fix $\varphi \in (\mathbf{C^1}\cap\mathbf{W^{1,\infty}})([0,T]\times\mathbb{R}_+)$ and choose a sequence

$$
\varphi_n \in \mathbf{C_c^\infty}([0,T]\times\mathbb{R}_+) \quad \text{such that} \quad \varphi_n \to \varphi \ \text{ in } \ \mathbf{W^{1,\infty}_{\mathrm{loc}}} \quad \text{and} \quad \sup_n \|\varphi_n\|_{\mathbf{W^{1,\infty}}} < C.
$$

An application of a standard limiting procedure completes the proof of *v)*. □

2.4.1. O.D.E. (2.19) - technical details

A convenient way to deal with the problem (2.19) relies on its dual formulation

$$\begin{aligned} \partial_t\varphi - c(x)\varphi + \int_{\mathbb{R}_+} \varphi(t,y)\mathrm{d}[\eta(x)](y) &= 0 \qquad (t,x) \in [0,T]\times\mathbb{R}_+, \\ \varphi(T,x) &= \psi(x) \qquad x\in\mathbb{R}_+, \end{aligned} \tag{2.35}$$

with $\psi \in (\mathbf{C^1}\cap\mathbf{W^{1,\infty}})(\mathbb{R}_+)$ and c,η as in (2.15), (2.16). A function $\varphi_{T,\psi}\in\mathbf{C^1}([0,T]\times\mathbb{R}_+)$ is a solution to the dual problem to (2.19), if it satisfies (2.35) in the classical strong sense. In the following lemma we give some basic results concerning a solution to the equation (2.35).

Lemma 2.36. *For any $T>0$ and $\psi\in(\mathbf{C^1}\cap\mathbf{W^{1,\infty}})(\mathbb{R}_+)$ there exists a unique solution $\varphi_{T,\psi}\in\mathbf{C^1}([0,T]\times\mathbb{R}_+)$ to* (2.35)*. If $\psi\geq 0$, then $\varphi_{T,\psi}(t,x)\geq 0$ for all $t\in[0,T]$ and $x\in\mathbb{R}_+$. Moreover, for $\tau\in[0,T]$ and $x\in\mathbb{R}_+$ the following estimates hold*

$$\|\varphi_{T,\psi}(\tau,\cdot)\|_{\mathbf{L}^\infty} \leq \|\psi\|_{\mathbf{L}^\infty}\mathrm{e}^{\left(\|c\|_{\mathbf{L}^\infty}+\|\eta\|_{\mathbf{BC_x}}\right)(T-\tau)}, \tag{2.37}$$

$$\|\partial_x\varphi_{T,\psi}(\tau,\cdot)\|_{\mathbf{L}^\infty} \leq \|\psi\|_{\mathbf{W^{1,\infty}}}\mathrm{e}^{3\left(\|c\|_{\mathbf{W^{1,\infty}}}+\|\eta\|_{\mathbf{BL}}\right)(T-\tau)}, \tag{2.38}$$

$$\sup_{\tau\in[T-t,T]}|\partial_\tau\varphi_{T,\psi}(\cdot,x)| \leq \|\psi\|_{\mathbf{L}^\infty}\left(\|c\|_{\mathbf{L}^\infty}+\|\eta\|_{\mathbf{BC_x}}\right)\mathrm{e}^{\left(\|c\|_{\mathbf{L}^\infty}+\|\eta\|_{\mathbf{BC_x}}\right)t}. \tag{2.39}$$

If moreover φ_1, respectively φ_2, solves (2.35) *with terminal data ψ and parameters c_1,η_1, respectively c_2,η_2, then*

$$\begin{aligned} \|\varphi_1(\tau,\cdot)-\varphi_2(\tau,\cdot)\|_{\mathbf{L}^\infty} \leq\ & \|\psi\|_{\mathbf{W^{1,\infty}}}\left(\|c_1-c_2\|_{\mathbf{L}^\infty}+\|\eta_1-\eta_2\|_{\mathbf{BC_x}}\right)(T-\tau) \\ & \cdot\,\mathrm{e}^{\left(\|(c_1,c_2)\|_{\mathbf{L}^\infty}+\|(\eta_1,\eta_2)\|_{\mathbf{BC_x}}\right)(T-\tau)}. \end{aligned} \tag{2.40}$$

The proof of Lemma 2.36 is an immediate consequence of standard ODE estimates. A relation between (2.19) (the original problem) and (2.35) (the dual probelm) will be explained in Lemma 2.41 given below. The results stated in Lemma 2.36 are used further in the proof of Lemma 2.18 and Lemma 2.41.

Lemma 2.41. *Fix $\mu_o\in\mathcal{M}^+(\mathbb{R}_+)$. Then:*

i) Problem (2.19) *admits a unique solution $\mu\in\mathbf{Lip}\left([0,T];\mathcal{M}^+(\mathbb{R}_+)\right)$. More precisely, for all $t,\tau\in[0,T]$*

$$\rho_F\left(\mu(t,\cdot),\mu(\tau,\cdot)\right)\leq\left(\|c\|_{\mathbf{L}^\infty}+\|\eta\|_{\mathbf{BC_x}}\right)\mathrm{e}^{\left(\|c\|_{\mathbf{L}^\infty}+\|\eta\|_{\mathbf{BC_x}}\right)\cdot\max\{t,\tau\}}\mu_o(\mathbb{R}_+)|t-\tau|.$$

ii) Fix $t_1,t_2\in[0,T]$ with $t_1<t_2$. If $\mu\in\mathbf{Lip}\left([0,T];\mathcal{M}^+(\mathbb{R}_+)\right)$ solves (2.19)*, then for any $\varphi\in(\mathbf{C^1}\cap\mathbf{W^{1,\infty}})([t_1,t_2]\times\mathbb{R}_+)$ we have*

$$\begin{aligned} &\int_{t_1}^{t_2}\int_{\mathbb{R}_+}\left(\partial_t\varphi(t,x)-c(x)\,\varphi(t,x)+\int_{\mathbb{R}_+}\varphi(t,y)\mathrm{d}[\eta(x)](y)\right)\mathrm{d}\mu_t(x)\mathrm{d}t \\ &=\int_{\mathbb{R}_+}\varphi(t_2,x)\,\mathrm{d}\mu_{t_2}(x)-\int_{\mathbb{R}_+}\varphi(t_1,x)\,\mathrm{d}\mu_{t_1}(x)\,. \end{aligned} \tag{2.42}$$

iii) *If* $\mu \in \mathbf{Lip}([0,T];\mathcal{M}^+(\mathbb{R}_+))$ *solves* (2.19)*, then for any* $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+)$ *there exists a function* $\varphi_{T,\psi} \in \mathbf{C^1}([0,T]\times\mathbb{R}_+)$ *solving the dual problem* (2.35) *and such that*

$$\int_{\mathbb{R}_+} \psi(x)\mathrm{d}\mu_t(x) = \int_{\mathbb{R}_+} \varphi_{T,\psi}(T-t,x)\mathrm{d}\mu_o(x)\,. \tag{2.43}$$

iv) *For any* $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+)$*, let* $\varphi_{T,\psi} \in \mathbf{C^1}([0,T]\times\mathbb{R}_+)$ *solve the dual problem* (2.35)*. Then, the measure* $\mu \in \mathbf{Lip}([0,T];\mathcal{M}^+(\mathbb{R}_+))$ *defined by* (2.43) *solves* (2.19)*.*

v) *If* μ_o *is positive, then also* μ_t *is positive for all* $t \in [0,T]$*.*

Proof of Lemma 2.41. The proof consists of several steps. The first five steps concern claim *i)*.

1. Regularization. Let $\rho \in \mathbf{C_c^\infty}(\mathbb{R};\mathbb{R}_+)$ be such that $\int_{\mathbb{R}} \rho(x)\mathrm{d}x = 1$. For $\varepsilon > 0$ define a family of mollifiers $\rho^\varepsilon = \rho(x/\varepsilon)/\varepsilon$. We consider equation (2.19) with initial datum u_o^ε and coefficient η^ε, where

$$\begin{aligned} u_o^\varepsilon\cdot\mathcal{L}^1 &= \mu_o * \rho^\varepsilon \qquad \text{and} \qquad & u_o^\varepsilon &\in \mathbf{C_b}(\mathbb{R}_+;\mathbb{R}_+),\\ \eta^\varepsilon(y)\cdot\mathcal{L}^1 &= \eta(y) * \rho^\varepsilon \qquad \text{and} \qquad & \eta^\varepsilon(y) &\in \mathbf{C_b}(\mathbb{R}_+;\mathbb{R}_+)\,, \end{aligned}$$

with $\eta^\varepsilon \in (\mathbf{C_b}\cap\mathbf{Lip})(\mathbb{R}_+;\mathbf{C_b}(\mathbb{R}_+;\mathbb{R}_+))$. Here, the usual Lebesgue measure on $\mathbb{R}$ is denoted by $\mathcal{L}^1$. Above, the convolution on $\mathbb{R}_+$ is given by

$$(\nu * \rho^\varepsilon)(x) = \int_{\mathbb{R}_+} \rho^\varepsilon(x-\varepsilon-\xi)\mathrm{d}\nu(\xi).$$

The reason why we shifted ρ^ε by ε is that

$$\mathrm{supp}\,((\nu\star\rho)(z)) = \mathrm{supp}\left(\int_{\mathbb{R}^2}\rho(z-\zeta)\mathrm{d}\nu(\zeta)\right) \subseteq [-\varepsilon,+\infty),$$

where $\star$ is a standard convolution. Below, we denote $\eta^\varepsilon(y)(x) = \eta^\varepsilon(y,x)$. Note that

$$\|\eta^\varepsilon\|_{\mathbf{BC_x}} \le \|\eta\|_{\mathbf{BC_x}}\,,\quad \rho_F(\mu_o,u_o^\varepsilon) \overset{\varepsilon\to0}{\longrightarrow} 0 \quad\text{and}\quad \sup_{y\in\mathbb{R}_+}\rho_F\left(\eta(y),\eta^\varepsilon(y)\right) \overset{\varepsilon\to0}{\longrightarrow} 0. \tag{2.44}$$

Indeed, fix $\psi \in (\mathbf{C^1}\cap\mathbf{W^{1,\infty}})(\mathbb{R}_+)$ with $\|\psi\|_{\mathbf{W^{1,\infty}}} \le 1$. Then,

$$\begin{aligned} \int_{\mathbb{R}_+}\psi(x)\mathrm{d}(\rho^\varepsilon*\mu_o-\mu_o)(x) &= \int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+}\rho_\varepsilon(\xi-\varepsilon-x)(\psi(\xi)-\psi(x))\mathrm{d}\xi\right)\mathrm{d}\mu_o(x)\\ &\le \int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+}\rho_\varepsilon(\xi-\varepsilon-x)|x-\xi|\mathrm{d}\xi\right)\mathrm{d}\mu_o(x) \le 2\varepsilon\mu_o(\mathbb{R}_+)\,.\\ \int_{\mathbb{R}_+}\psi(x)\mathrm{d}(\rho^\varepsilon*\eta(y)-\eta(y))(x) &= \int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+}\rho_\varepsilon(\xi-\varepsilon-x)(\psi(\xi)-\psi(x))\mathrm{d}y\right)\mathrm{d}[\eta(y)](x)\\ &\le \int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+}\rho_\varepsilon(\xi-\varepsilon-x)|x-\xi|\mathrm{d}y\right)\mathrm{d}[\eta(y)](x)\\ &\le 2\varepsilon[\eta(y)](\mathbb{R}_+) \le 2\varepsilon\|\eta(y)\|_{(\mathbf{W^{1,\infty}})^*} \le 2\varepsilon\|\eta\|_{\mathbf{BC_x}}\,. \end{aligned}$$

2. Equality (2.43) **Holds in the Regular Case.** Note that

$$\begin{cases} \dfrac{\partial}{\partial t} u^\varepsilon(t,x) = -c(x) u^\varepsilon(t,x) + \displaystyle\int_{\mathbb{R}_+} \eta^\varepsilon(y,x) u^\varepsilon(t,y) \mathrm{d}y, & (t,x) \in [0,T] \times \mathbb{R}_+, \\ u^\varepsilon(0,x) = u_o^\varepsilon(x), & x \in \mathbb{R}_+, \end{cases} \tag{2.45}$$

is a Cauchy problem for an ODE in $\mathbf{L^1}(\mathbb{R}_+)$ with a globally Lipschitz right hand side. Therefore, the existence and uniqueness of a classical solution u^ε is immediate, see [17]. Integrating (2.45) we obtain that for any for any $\varphi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})([0,T] \times \mathbb{R}_+)$ and $0 \le t_1 \le t_2 \le T$,

$$\begin{aligned} &\int_{t_1}^{t_2} \int_{\mathbb{R}_+} \left(\partial_\tau \varphi(\tau,x) - c(x)\, \varphi(\tau,x) + \int_{\mathbb{R}_+} \varphi(\tau,y)\, \eta^\varepsilon(x,y) \mathrm{d}y \right) u^\varepsilon(\tau,x) \mathrm{d}x \mathrm{d}\tau \\ &= \int_{\mathbb{R}_+} \varphi(t_2,x)\, u^\varepsilon(t_2,x) \mathrm{d}x - \int_{\mathbb{R}_+} \varphi(t_1,x)\, u^\varepsilon(t_1,x) \mathrm{d}x\,. \end{aligned} \tag{2.46}$$

Let $t_1 = 0$, $t_2 = t$ and $\varphi(\tau,x) = \varphi_{T,\psi}(\tau + (T - t_2), x)$, where $\varphi_{T,\psi}$ is a solution of the dual problem (2.35). Then,

$$\int_{\mathbb{R}_+} \psi(x) u^\varepsilon(t,x) \mathrm{d}x = \int_{\mathbb{R}_+} \varphi^\varepsilon_{T,\psi}(T-t,x) u_o^\varepsilon(x) \mathrm{d}x\,, \tag{2.47}$$

which is the smooth version of (2.43).

3. Convergence of the Regularizations. Let u^{ε_m}, respectively u^{ε_n}, solve problem (2.45) with ε replaced by ε_m, respectively ε_n. Moreover, let v be the solution to

$$\begin{cases} \dfrac{\partial}{\partial t} v(t,x) = -c(x) v(t,x) + \displaystyle\int_{\mathbb{R}_+} \eta^{\varepsilon_m}(y,x) v(t,y) \mathrm{d}y, & (t,x) \in [0,T] \times \mathbb{R}_+, \\ v(0,x) = u_o^{\varepsilon_n}(x), & x \in \mathbb{R}_+. \end{cases}$$

By estimate (2.40) for dual problem and push-forward formula (2.47),

$$\rho_F\left(u^{\varepsilon_n}(t,\cdot), v(t,\cdot)\right) \le \sup_{y \in \mathbb{R}_+} \rho_F\left(\eta(y), \eta^\varepsilon(y)\right) \mathrm{e}^{(2\|c\|_{\mathbf{L}^\infty} + \|(\eta_{\varepsilon_n}, \eta_{\varepsilon_m})\|_{\mathbf{BC_x}})T} u_o^{\varepsilon_n}(\mathbb{R}_+)\, T,$$

while by estimates (2.37)–(2.38) for a dual problem and push-forward formula (2.47)

$$\rho_F\left(u^{\varepsilon_m}(t,\cdot), v(t,\cdot)\right) \le \exp\left(3(\|c\|_{\mathbf{W^{1,\infty}}} + \|\eta^{\varepsilon_m}\|_{\mathbf{BL}})\, T\right)\, \rho_F(u_o^{\varepsilon_m}, u_o^{\varepsilon_n})\,.$$

Therefore, by (2.44), $\rho_F\left(u^{\varepsilon_n}(t,\cdot), u^{\varepsilon_m}(t,\cdot)\right) \overset{n,m\to\infty}{\longrightarrow} 0$ uniformly with respect to time. By the completeness of $\mathcal{M}^+(\mathbb{R}_+)$, the sequence $u^{\varepsilon_n}(t,\cdot) \cdot \mathcal{L}^1$ converge uniformly with respect to time to a unique limit μ_t.

4. The Limit is Narrowly Lipschitz in Time. Using (2.39) and (2.44), we obtain the following uniform Lipschitz estimate for all $0 \le \tau \le t$

$$\rho_F\left(u^\varepsilon(t,\cdot), u^\varepsilon(\tau,\cdot)\right) \le \left(\|c\|_{\mathbf{L}^\infty} + \|\eta\|_{\mathbf{BC_x}}\right) \exp\left(\left(\|c\|_{\mathbf{L}^\infty} + \|\eta\|_{\mathbf{BC_x}}\right) \tau\right) u_o^\varepsilon(\mathbb{R}_+)(t-\tau)\,.$$

Hence, μ_t is also narrowly Lipschitz in time.

5. The Limit Solves (2.19)**.** We proved, that $u^\varepsilon(t,\cdot)$ converges narrowly and uniformly with respect to time to the unique limit μ_t. Notice, that $\partial_\tau\varphi(\tau,\cdot)$ and $c(x)\varphi(\tau,\cdot)$ are bounded continuous functions, while

$$\int_{\mathbb{R}_+}\varphi(\tau,y)\eta^\varepsilon(\cdot,y)\mathrm{d}y \quad \rightrightarrows \quad \int_{\mathbb{R}_+}\varphi(\tau,y)\eta(\cdot,y)\mathrm{d}y,$$

where a symbol " $\rightrightarrows$ " denotes the uniform converges. Thus, passing to the limit in the integral (2.46) completes the proof of *i)*.

6. *ii)* Holds. Let $\mu \in \mathbf{Lip}([0,T];\mathcal{M}^+(\mathbb{R}_+))$ be a solution to (2.19) in the sense of Definition 2.10 and $\varphi \in (\mathbf{C^1}\cap\mathbf{W^{1,\infty}})([0,T]\times\mathbb{R}_+)$. We prove that (2.42) holds for $t\in[t_1,t_2]$. To this end, define $\varphi^\varepsilon(t,x) = \kappa_\varepsilon(t)\,\varphi(t,x)$, where

$$\kappa_\varepsilon \in \mathbf{C_c^\infty}([t_1,t_2),[0,1]), \quad \kappa_\varepsilon(t_1)=1, \quad \lim_{\varepsilon\to0}\kappa_\varepsilon(\tau) = \chi_{[t_1,t_2)}(\tau)$$

and

$$\lim_{\varepsilon\to0}\kappa'_\varepsilon = \delta_{t=t_1} - \delta_{t=t_2} \quad \text{in} \quad \mathcal{M}^+([0,T]).$$

Use φ^ε as a test function in the definition of weak solution. Using the Lipschitz continuity of $t\to\mu_t$ and the Lebesgue Dominated Convergence Theorem we conclude

$$\begin{aligned}
&\int_{\mathbb{R}_+}\varphi(t_2,x)\,\mathrm{d}\mu_{t_2}(x) - \int_{\mathbb{R}_+}\varphi(t_1,x)\,\mathrm{d}\mu_{t_1}(x) = \lim_{\varepsilon\to0}\int_0^T\frac{\mathrm{d}}{\mathrm{d}t}\kappa^\varepsilon(t)\int_{\mathbb{R}_+}\varphi(t,x)\mathrm{d}\mu_t(x)\mathrm{d}t\\
&= \lim_{\varepsilon\to0}\int_0^T\kappa^\varepsilon(t)\int_{\mathbb{R}_+}\Big[\partial_t\varphi(t,x) - c(x)\varphi(t,x) + \int_{\mathbb{R}_+}\varphi(t,y)\mathrm{d}[\eta(x)](y)\Big]\mathrm{d}\mu_t(x)\mathrm{d}t\\
&= \int_{t_1}^{t_2}\int_{\mathbb{R}_+}\Big[\partial_t\varphi(t,x) - c(x)\varphi(t,x) + \int_{\mathbb{R}_+}\varphi(t,y)\mathrm{d}[\eta(x)](y)\Big]\mathrm{d}\mu_t(x)\mathrm{d}t.
\end{aligned}$$

7. *iii)* Holds. Equality (2.43) arises by setting in (2.42) $t_1=0$, $t_2=t$ and $\varphi(s,x) = \varphi_{T,\psi}(s+(T-t_2),x)$.

8. *iv)* Holds. We proved that there exists a unique solution to (2.19) which also fulfills (2.43). This expression characterizes μ uniquely, therefore each μ given by (2.43) is a solution to (2.19).

9. *v)* Holds. It immediately follows from the analogous property of the dual equation stated in Lemma 2.36 and push-forward formula (2.43). □

Proof of Lemma 2.18. Claims *i)* and *v)* follow from *iii)* in Lemma 2.41, since the dual problem to (2.19) is autonomous. Claim *iii)* is a consequence of *i)* in Lemma 2.41.

To prove *ii)*, choose $\psi\in\mathbf{C^1}(\mathbb{R}_+)$ with $\|\psi\|_{\mathbf{W^{1,\infty}}}\le 1$ and $\mu_1,\mu_2\in\mathcal{M}^+(\mathbb{R}_+)$. By the push-forward formula in *iii)* of Lemma 2.41 and by the estimates (2.37)–(2.38) for the dual problem, we have

$$\begin{aligned}
&\int_{\mathbb{R}_+}\psi(x)\mathrm{d}(\hat{S}_t\mu_1 - \hat{S}\mu_2)(x) = \int_{\mathbb{R}_+}\varphi_{T,\psi}(T-t,x)\mathrm{d}(\mu_1-\mu_2)(x)\\
&\le \sup\left\{\int_{\mathbb{R}_+}\varphi(x)\mathrm{d}(\mu_1-\mu_2)(x) \;:\; \varphi(x)\in\mathbf{C^1}(\mathbb{R}_+),\ \|\varphi\|_{\mathbf{W^{1,\infty}}}\le e^{3\left(\|c\|_{\mathbf{W^{1,\infty}}}+\|\eta\|_{\mathbf{BL}}\right)t}\right\}\\
&\le \sup\left\{\int_{\mathbb{R}_+}\psi(x)\mathrm{d}(\mu_1-\mu_2)(x) \;:\; \psi(x)\in\mathbf{C^1}(\mathbb{R}_+), \|\psi\|_{\mathbf{W^{1,\infty}}}\le 1\right\}e^{3\left(\|c\|_{\mathbf{W^{1,\infty}}}+\|\eta\|_{\mathbf{BL}}\right)t}
\end{aligned}$$

$$= \exp\left(3\left(\|c\|_{\mathbf{W^{1,\infty}}} + \|\eta\|_{\mathbf{BL}}\right) t\right) \rho_F(\mu_1, \mu_2).$$

Hence, *ii)* holds.

Finally, to prove *iv)*, let c_1, c_2 satisfy (2.15), η_1, η_2 satisfy (2.16) and call $\hat{S}^1, \hat{S}^2$ the corresponding semigroups. Then, using (2.40) and Lemma 2.41

$$\begin{aligned}\int_{\mathbb{R}_+} \varphi(x) \mathrm{d}(\hat{S}^1\mu - \hat{S}^2\mu) &= \int_{\mathbb{R}_+} \left(\varphi^1_{T,\psi}(T-t,\cdot) - \varphi^2_{T,\psi}(T-t,\cdot)\right) \mathrm{d}\mu(x) \\ &\le \left(\|c_1 - c_2\|_{\mathbf{L^\infty}} + \|\eta_1 - \eta_2\|_{\mathbf{BC_x}}\right) \mathrm{e}^{\left(\|(c_1,c_2)\|_{\mathbf{L^\infty}} + \|(\eta_1,\eta_2)\|_{\mathbf{BC_x}}\right)t} \mu(\mathbb{R}_+)\, t,\end{aligned}$$

which completes the proof. □

2.4.2. Transport Equation (2.21) - technical details

Proof of Lemma 2.20. Claims *i)* and *v)* are classical results, see for instance [5, Section 8.1]. Integrating along characteristics, we can explicitly write

$$\check{S}_t\mu = X(t;0,\cdot)\#\mu \quad \text{where} \quad \begin{cases} \partial_\tau X(\tau;t,x) = b\left(X(\tau;t,x)\right), \\ X(t;t,x) = x. \end{cases} \tag{2.48}$$

Hence $(\check{S}_t\mu)(A) = \mu\left(X(0;t,A)\right)$ for any measurable subset A of $\mathbb{R}_+$. By the standard theory of ODEs, we have $X\left(t_o;t,X(t;t_o,x)\right) = x$. Using the definition of the distance ρ_F, we prove *ii)* as follows:

$$\begin{aligned}&\rho_F(\check{S}_t\mu_1, \check{S}_t\mu_2) = \sup\left\{\int_{\mathbb{R}_+} \varphi(x)\mathrm{d}(\check{S}_t\mu_1 - \check{S}_t\mu_2)(x) \;:\; \varphi \in \mathbf{C^1}(\mathbb{R}_+) \text{ and } \|\varphi\|_{\mathbf{W^{1,\infty}}} \le 1\right\} \\ &= \sup\left\{\int_{\mathbb{R}_+} \varphi\left(X(0;t,x)\right)\mathrm{d}(\mu_1 - \mu_2)(x) \;:\; \varphi \in \mathbf{C^1}(\mathbb{R}_+) \text{ and } \|\varphi\|_{\mathbf{W^{1,\infty}}} \le 1\right\} \\ &\le \sup\left\{\int_{\mathbb{R}_+} \psi(x)\mathrm{d}(\mu_1 - \mu_2)(x) \;:\; \varphi \in \mathbf{C^1}(\mathbb{R}_+) \text{ and } \begin{array}{l}\|\psi\|_{\mathbf{L^\infty}} \le 1, \\ \|\partial_x\psi\|_{\mathbf{L^\infty}} \le \|\partial_x X(0;t,\cdot)\|_{\mathbf{L^\infty}}\end{array}\right\} \\ &\le \max\left\{1, \|\partial_x X(0;t,\cdot)\|_{\mathbf{L^\infty}}\right\} \rho_F(\mu_1,\mu_2) \\ &\le \exp\left(\|\partial_x b\|_{\mathbf{L^\infty}}\, t\right) \rho_F(\mu_1,\mu_2).\end{aligned}$$

where we used [70, § 6.1.2]. Concerning *iii)*, i.e. the Lipschitz continuity with respect to time,

$$\begin{aligned}\rho_F(\check{S}_t\mu, \mu) &= \sup\left\{\int_{\mathbb{R}_+} \varphi(x)\mathrm{d}(\check{S}_t\mu - \mu)(x) \;:\; \varphi \in \mathbf{C^1}(\mathbb{R}_+) \text{ and } \|\varphi\|_{\mathbf{W^{1,\infty}}} \le 1\right\} \\ &\le \sup\left\{\int_{\mathbb{R}_+} |\varphi\left(X(0;t,x)\right) - \varphi(x)|\mathrm{d}\mu(x) \;:\; \varphi \in \mathbf{C^1}(\mathbb{R}_+) \text{ and } \|\varphi\|_{\mathbf{W^{1,\infty}}} \le 1\right\} \\ &\le \|b\|_{\mathbf{L^\infty}}\, \mu(\mathbb{R}_+)\, t\,.\end{aligned}$$

Finally, to prove *iv)*, let b_1, b_2 satisfy (2.15) and call $\check{S}^1, \check{S}^2$ the corresponding semigroups. Then, with obvious notation,

$$\rho_F(\check{S}^1_t\mu, \check{S}^2_t\mu) = \sup\left\{\int_{\mathbb{R}_+} \varphi(x)\mathrm{d}(\check{S}^1_t\mu - \check{S}^2_t\mu)(x) \;:\; \varphi \in \mathbf{C1}(\mathbb{R}_+) \text{ and } \|\varphi\|_{\mathbf{W^{1,\infty}}} \le 1\right\}$$

$$
\begin{aligned}
&= \sup\left\{\int_{\mathbb{R}_+}\varphi(x)\mathrm{d}(\check S^1_t\mu)(x) - \int_{\mathbb{R}_+}\varphi(x)\mathrm{d}(\check S^2_t\mu)(x) \ : \ \varphi\in\mathbf{C}1(\mathbb{R}_+) \text{ and } \|\varphi\|_{\mathbf{W^{1,\infty}}}\le 1\right\}\\
&\le \sup\left\{\int_{\mathbb{R}_+}\left|\varphi\left(X^1(0;t,x)\right)-\varphi\left(X^2(0;t,x)\right)\right|\mathrm{d}\mu(x) \ : \ \varphi\in\mathbf{C}1(\mathbb{R}_+) \text{ and } \|\varphi\|_{\mathbf{W^{1,\infty}}}\le 1\right\}\\
&\le \|b_1-b_2\|_{\mathbf{L}^\infty}\,\mu(\mathbb{R}_+)\,t
\end{aligned}
$$

completing the proof. □

Further in the proof of Proposition 2.22 we will need results on the dual formulation of (2.21), namely

$$
\begin{aligned}
\partial_t\varphi + b(x)\,\partial_x\varphi &= 0, \qquad (t,x)\in[0,T]\times\mathbb{R}_+,\\
\varphi(T,x) &= \psi(x) \quad x\in\mathbb{R}_+,
\end{aligned}
\tag{2.49}
$$

with $\psi\in(\mathbf{C^1}\cap\mathbf{W^{1,\infty}})(\mathbb{R}_+)$ and b as in (2.15). We say that a map $\varphi_{T,\psi}\in\mathbf{C^1}([0,T]\times\mathbb{R}_+)$ solves (2.21), if (2.49) is satisfied in the classical strong sense. For completeness, we state the following results, whose proofs are found where referred.

Lemma 2.50. [43, Lemma 3.6] *Fix $\mu_o\in\mathcal{M}^+(\mathbb{R}_+)$. A map $\mu:[0,T]\to\mathcal{M}^+(\mathbb{R}_+)$ solves* (2.21) *with initial datum μ_o in the sense of Definition* 2.10 *if and only if for any $\psi\in(\mathbf{C^1}\cap\mathbf{W^{1,\infty}})(\mathbb{R}_+)$*

$$
\int_{\mathbb{R}_+}\psi(x)\mathrm{d}\mu_t(x) = \int_{\mathbb{R}_+}\varphi_{T,\psi}(T-t,x)\mathrm{d}\mu_o(x)\,, \tag{2.51}
$$

where $\varphi_{T,\psi}\in\mathbf{C^1}([0,T]\times\mathbb{R}_+)$ is the solution of the dual problem (2.49) *for any $T>0$. Moreover, if μ_o is nonnegative, then so is μ.*

Lemma 2.52. [5, Lemma 8.1.2] *Fix $t_1,t_2\in[0,T]$ with $t_1<t_2$. If $\mu\in\mathbf{Lip}([0,T];\mathcal{M}^+(\mathbb{R}_+))$ solves* (2.21)*, then for any $\varphi\in(\mathbf{C^1}\cap\mathbf{W^{1,\infty}})([t_1,t_2]\times\mathbb{R}_+)$ we have*

$$
\int_{t_1}^{t_2}\int_{\mathbb{R}_+}\left(\partial_t\varphi(t,x)+\partial_x\varphi(t,x)\,b(x)\right)\mathrm{d}\mu t(x)\mathrm{d}t = \int_{\mathbb{R}_+}\varphi(t_2,x)\mathrm{d}\mu_{t_2}(x) - \int_{\mathbb{R}_+}\varphi(t_1,x)\mathrm{d}\mu_{t_1}(x)\,. \tag{2.53}
$$

Lemma 2.54. [43, Lemma 3.5] *For any $T>0$ and $\psi\in(\mathbf{C^1}\cap\mathbf{W^{1,\infty}})(\mathbb{R}_+)$, there exists a unique solution $\varphi_{T,\psi}\in\mathbf{C^1}([0,T]\times\mathbb{R}_+)$ of* (2.49)*. Moreover, for $\tau\in[0,T]$ and $x\in\mathbb{R}_+$,*

$$
\begin{aligned}
\|\varphi_{T,\psi}(\tau,\cdot)\|_{\mathbf{L}^\infty} &\le \|\psi\|_{\mathbf{L}^\infty},\\
\|\partial_x\varphi_{T,\psi}(\tau,\cdot)\|_{\mathbf{L}^\infty} &\le \|\psi'\|_{\mathbf{L}^\infty}\mathrm{e}^{\|\partial_x b\|_{\mathbf{L}^\infty}(T-\tau)},\\
\|\partial_\tau\varphi_{T,\psi}(\cdot,x)\|_{\mathbf{L}^\infty} &\le \|\psi'\|_{\mathbf{L}^\infty}\|b\|_{\mathbf{L}^\infty}\,.
\end{aligned}
$$

2.4.3. Operator Splitting Algorithm

Proof of Proposition 2.22. Let $\psi\in(\mathbf{C}1\cap\mathbf{W^{1,\infty}})(\mathbb{R}_+)$ with $\|\psi\|_{\mathbf{W^{1,\infty}}}\le 1$ and $\mu\in\mathcal{M}^+(\mathbb{R}_+)$. Then,

$$
\int_{\mathbb{R}_+}\psi(x)\mathrm{d}(\hat S_t\check S_t\mu - \check S_t\hat S_t\mu) = \int_{\mathbb{R}_+}\hat\varphi_{T,\psi}(T-t,x)\mathrm{d}(\check S_t\mu) - \int_{\mathbb{R}_+}\check\varphi_{T,\psi}(T-t,x)\mathrm{d}(\hat S_t\mu)
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}_+} \check{\varphi}_{T,(\hat{\varphi}_{T,\psi}(T-t,\cdot))}(T-t,x)\mathrm{d}\mu - \int_{\mathbb{R}_+} \hat{\varphi}_{T,(\check{\varphi}_{T,\psi}(T-t,\cdot))}(T-t,x)\mathrm{d}\mu \\
&= \int_{\mathbb{R}_+} \big(\hat{\varphi}_{T,\psi}(T-t,X(T-t;T,x)) - \hat{\varphi}_{T,\psi(X(T-t;T,\cdot))}(T-t,x)\big)\mathrm{d}\mu \\
&\le \sup_{x\in\mathbb{R}_+} \left|\hat{\varphi}_{T,\psi}(T-t,X(T-t;T,x)) - \hat{\varphi}_{T,\psi(X(T-t;T,\cdot))}(T-t,x)\right| \mu(\mathbb{R}_+).
\end{aligned}
$$

Set $\varphi_1 = \hat{\varphi}_{T,\psi}$ and $\varphi_2 = \hat{\varphi}_{T,\psi(X(T-t;T,\cdot))}$ and consider the term in the modulus. Use the estimates for the dual problem in Lemma 2.36 and (2.48) to obtain

$$
\begin{aligned}
&\varphi_1(T-t,X(T-t;T,x)) - \varphi_2(T-t,x) \\
&\quad= \psi(X(T-t;T,x)) - \int_{T-t}^{T} c(X(T-t;T,x))\varphi_1(s,X(T-t;T,x)) \\
&\qquad+ \int_{T-t}^{T}\int_{\mathbb{R}_+} \varphi_1(s,y)\mathrm{d}[\eta(X(T-t;T,x))](y)\mathrm{d}s - \psi(X(T-t;T,x)) \\
&\qquad+ \int_{T-t}^{T} c(x)\varphi_2(s,x)\mathrm{d}s - \int_{T-t}^{T}\int_{\mathbb{R}_+} \varphi_2(s,y)\mathrm{d}[\eta(x)](y)\mathrm{d}s \pm \int_{T-t}^{T} c(x)\varphi_1(s,x)\mathrm{d}s \\
&\qquad\pm \int_{T-t}^{T} c(X(T-t;T,x))\varphi_1(s,x)\mathrm{d}s \pm \int_{T-t}^{T}\int_{\mathbb{R}_+} \varphi_1(s,y)\mathrm{d}[\eta(x)](y)\mathrm{d}s \\
&\quad= \int_{T-t}^{T} c(x)\left(\varphi_2(s,x) - \varphi_1(s,x)\right)\mathrm{d}s + \int_{T-t}^{T} \varphi_1(s,x)\left(c(x) - c\left(X(T-t;T,x)\right)\right)\mathrm{d}s \\
&\qquad+ \int_{T-t}^{T} c\left(X(T-t;T,x)\right)\left(\varphi_1(s,x) - \varphi_1\left(s,X(T-t;T,x)\right)\right)\mathrm{d}s \\
&\qquad+ \int_{T-t}^{T}\int_{\mathbb{R}_+} \left(\varphi_1(s,x) - \varphi_2(s,x)\right)\mathrm{d}[\eta(x)](y)\mathrm{d}s \\
&\qquad+ \int_{T-t}^{T}\int_{\mathbb{R}_+} \varphi_1(s,y)\mathrm{d}[\eta\left(X(T-t;T,x)\right) - \eta(x)](y)\mathrm{d}s \\
&\quad\le \|c\|_{\mathbf{L}^\infty} \int_{T-t}^{T} \sup_x |\varphi_1(s,x) - \varphi_2(s,x)|\mathrm{d}s + \|\partial_x c\|_{\mathbf{L}^\infty}\|b\|_{\mathbf{L}^\infty} t \int_{T-t}^{T} \sup_x |\varphi_1(s,x)|\mathrm{d}s \\
&\qquad+ \|c\|_{\mathbf{L}^\infty}\|b\|_{\mathbf{L}^\infty} t \int_{T-t}^{T} \sup_x |\partial_x\varphi_1(s,x)|\mathrm{d}s \\
&\qquad+ \|\eta\|_{\mathbf{BC_x}} \int_{T-t}^{T} \sup_x |\varphi_1(s,x) - \varphi_2(s,x)|\mathrm{d}s + \mathbf{Lip}(\eta)\|b\|_{\mathbf{L}^\infty} t \int_{T-t}^{T} \sup_x |\varphi_1(s,x)|\mathrm{d}s \\
&\quad:= I_1 + I_2 + I_3 + I_4 + I_5\,.
\end{aligned}
$$

Using estimate (2.37) for the dual problem, we conclude that

$$
\begin{aligned}
I_1 &\le \|c\|_{\mathbf{L}^\infty}\|\psi'\|_{\mathbf{L}^\infty}\|b\|_{\mathbf{L}^\infty} t^2\, \mathrm{e}^{(\|c\|_{\mathbf{L}^\infty}+\|\eta\|_{\mathbf{BC_x}})t}, \\
I_2 &\le \|\partial_x c\|_{\mathbf{L}^\infty}\|b\|_{\mathbf{L}^\infty} t^2\, \|\psi\|_{\mathbf{L}^\infty}\, \mathrm{e}^{(\|c\|_{\mathbf{L}^\infty}+\|\eta\|_{\mathbf{BC_x}})t}, \\
I_4 &\le \|\eta\|_{\mathbf{BC_x}}\|\psi'\|_{\mathbf{L}^\infty}\|b\|_{\mathbf{L}^\infty} t^2\, \mathrm{e}^{(\|c\|_{\mathbf{L}^\infty}+\|\eta\|_{\mathbf{BC_x}})t}, \\
I_5 &\le \mathbf{Lip}(\eta)\, \|b\|_{\mathbf{L}^\infty} t^2\, \|\psi\|_{\mathbf{L}^\infty}\, \mathrm{e}^{(\|c\|_{\mathbf{L}^\infty}+\|\eta\|_{\mathbf{BC_x}})t}.
\end{aligned}
$$

Directly from estimate (2.38) follows, that $I_3 \le \|c\|_{\mathbf{L}^\infty}\|b\|_{\mathbf{L}^\infty} t^2\, \|\psi\|_{\mathbf{W^{1,\infty}}} \mathrm{e}^{3(\|c\|_{\mathbf{W^{1,\infty}}}+\|\eta\|_{\mathbf{BL}})t}$. Hence,

$$
\int_{\mathbb{R}_+} \psi(x)\mathrm{d}(\hat{S}_t\check{S}_t\mu - \check{S}_t\hat{S}_t\mu) \le
$$

$$\leq 3t^2 \, \|\psi\|_{\mathbf{W^{1,\infty}}} \|b\|_{\mathbf{L^\infty}} \exp\left[3\left(\|c\|_{\mathbf{W^{1,\infty}}} + \|\eta\|_{\mathbf{BL}}\right) t\right] \left(\|c\|_{\mathbf{W^{1,\infty}}} + \|\eta\|_{\mathbf{BL}}\right).$$

Taking the supremum over all functions ψ finishes the proof. □

2.5. Linear Non-autonomous Case

We now assume that, for a fixed $\alpha \in (0,1]$,

$$\begin{aligned} b, c &\in \mathbf{C_b^\alpha}\big([0,T]; \mathbf{W^{1,\infty}}(\mathbb{R}_+)\big) \quad \text{with} \quad b(t)(0) \geq 0 \text{ for all } t \in [0,T], && (2.55)\\ \eta &\in \mathbf{C_b^\alpha}\left([0,T]; (\mathbf{C_b} \cap \mathbf{Lip})(\mathbb{R}_+; \mathcal{M}^+(\mathbb{R}_+))\right), && (2.56)\end{aligned}$$

and consider the following linear non-autonomous version of (2.1):

$$\begin{cases} \partial_t \mu + \partial_x \left(b(t,x)\,\mu\right) + c(t,x)\,\mu = \int_{\mathbb{R}_+} \eta(t,y)\,\mathrm{d}\mu(y), \quad (t,x) \in [0,T] \times \mathbb{R}_+, \\ \mu_o \in \mathcal{M}^+(\mathbb{R}_+), \end{cases} \tag{2.57}$$

The space $\mathbf{C_b^\alpha}([0,T]; X)$ consists of Hölder continuous, X valued functions with a norm

$$\|f\|_{\mathbf{BH}} = \|f\|_{\mathbf{BC_t}} + H(f),$$

where

$$\|f\|_{\mathbf{BC_t}} = \sup_{t \in [0,T]} \|f(t)\|_X \quad \text{and} \quad H(f) = \sup_{s_1, s_2 \in [0,T]} \frac{\|f(s_1) - f(s_2)\|_X}{|s_1 - s_2|^\alpha}.$$

Remark 2.58. We assume that b, c, η are Hölder continuous with respect to time, because the method we used in the proof of Theorem 2.59 requires this regularity. In general, uniform continuity is not sufficient in our case. However, all proofs remain valid for uniform continuous functions whose modulus of continuity ω is such that

$$\sum_{n=1}^{\infty} \omega(2^{-n}) < +\infty.$$

Theorem 2.59. *Let (2.55) and (2.56) hold. Then, the linear non-autonomous problem (2.57) generates a global process $P: [0,T]^2 \times \mathcal{M}^+(\mathbb{R}_+) \to \mathcal{M}^+(\mathbb{R}_+)$, in the sense that*

i) For all t_o, t_1, t_2, μ satisfying $0 \leq t_o \leq t_1 \leq t_2 \leq T$ and $\mu \in \mathcal{M}^+(\mathbb{R}_+)$

$$\begin{aligned} &P(t_o, t_o)\mu = \mu, \\ &P(t_1, t_o)\mu \in \mathcal{M}^+(\mathbb{R}_+), \\ &P(t_2, t_1) \circ P(t_1, t_o)\mu = P(t_2, t_o)\mu. \end{aligned}$$

ii) For all t_o, t, μ_1, μ_2 satisfying $0 \leq t_o \leq t \leq T$ and $\mu_1, \mu_2 \in \mathcal{M}^+(\mathbb{R}_+)$

$$\rho_F(P(t,t_o)\mu_1, P(t,t_o)\mu_2) \leq \mathrm{e}^{3\|(b,c,\eta)\|_{\mathbf{BC_t}}(t-t_o)}\, \rho_F(\mu_1, \mu_2).$$

iii) For all t_o, t, μ satisfying $0 \leq t_o \leq t \leq T$ and $\mu \in \mathcal{M}^+(\mathbb{R}_+)$

$$\rho_F(P(t,t_o)\mu, P(t_o,t_o)\mu) \leq \|(b,c,\eta)\|_{\mathbf{BC_t}} \mathrm{e}^{2\|(b,c,\eta)\|_{\mathbf{BC_t}}(t-t_o)} \mu(\mathbb{R}_+)\,(t - t_o).$$

iv) For $i = 1, 2$, let b_i, c_i, η_i satisfy assumptions (2.55) *and* (2.56)*. Call P^i the corresponding process. Then, for all t_o, t, μ satisfying $0 \le t_o \le t \le T$ and $\mu \in \mathcal{M}^+(\mathbb{R}_+)$, there exists a constant $C^* = C^*\left(t_o, T, \|b_1\|_{\mathbf{BC_t}}, \|c_1\|_{\mathbf{BC_t}}, \|\eta_1\|_{\mathbf{BC_t}}\right)$ such that*

$$\begin{aligned}\rho_F\left(P^1(t,t_o)\mu, P^2(t,t_o)\mu\right) &\le C^*(t-t_o)\mathrm{e}^{5\|(b_1,b_2,c_1,c_2,\eta_1,\eta_2)\|_{\mathbf{BC_t}}(t-t_o)} \\ &\quad \cdot \|(b_1,c_1,\eta_1)-(b_2,c_2,\eta_2)\|_{\mathbf{BC_t}}\mu(\mathbb{R}_+).\end{aligned}$$

v) For all $\mu \in \mathcal{M}^+(\mathbb{R}_+)$, the trajectory $t \to P(t,0)\mu$ is a weak solution to linear non-autonomous problem (2.14) *in the sense of Definition* 2.10*. This solution is unique.*

Proof of Theorem 2.59. First, we prove that there exists a process P given by *i)*. Fix $n \in \mathbb{N}$, define $t_n^i = iT/2^n$ for $i = 0, 1, \ldots, 2^n$ and approximate b, c and η as follows:

$$\begin{aligned}b_n(t,x) &= \sum_{i=0}^{2^n-1} b(t_n^i, x)\,\chi_{[t_n^i, t_n^{i+1})}(t),\\ c_n(t,x) &= \sum_{i=0}^{2^n-1} c(t_n^i, x)\,\chi_{[t_n^i, t_n^{i+1})}(t),\\ \eta_n(t,x) &= \sum_{i=0}^{2^n-1} \eta(t_n^i, x)\,\chi_{[t_n^i, t_n^{i+1})}(t).\end{aligned}$$

Call $S^{k,n}$ the semigroup constructed in Theorem 2.24 on the time interval $[t_n^k, t_n^{k+1})$. Assume $t_o \le t$, $t_o \in \left[t_n^{i_o}, t_n^{i_o+1}\right)$, $t \in [t_n^i, t_n^{i+1})$ and define the map $F^n_{t,t_o}\colon [0,T] \times \mathcal{M}^+(\mathbb{R}_+) \to \mathcal{M}^+(\mathbb{R}_+)$ by

$$F^n_{t,t_o}\mu = \begin{cases} S^{i,n}_{t-t_o}\mu & \text{if } i = i_o,\\ \left(S^{i,n}_{t-t_n^i} \circ S^{i-1,n}_{t_n^i - t_o}\right)\mu & \text{if } i = i_o+1,\\ \left(S^{i,n}_{t-t_n^i} \circ \left(\bigcirc_{j=i_o+1}^{i-1} S^{j,n}_{T/2^n}\right) \circ S^{i_o,n}_{t_n^{i_o+1}-t_o}\right)\mu & \text{otherwise.}\end{cases} \tag{2.60}$$

Here, by $\bigcirc_{i=1}^n S^i$ we denote the n-fold composition of S. We now prove that as $n \to +\infty$, F^n converges to a process P, see also [25, Definition 2.4], whose trajectories solve (2.14) in the sense of Definition 2.10. Assume first that $t_o = i_oT/2^n$ and $t = iT/2^n$ with $i > i_o$. Then, $F^n_{t,t_o}\mu = \left(\bigcirc_{j=i_o}^{i-1} S^{j,n}_{T/2^n}\right)\mu$ and

$$\begin{aligned}\rho_F(F^n_{t,t_o}\mu, F^{n+1}_{t,t_o}\mu) &= \rho_F\left(F^n_{iT/2^n,t_o}\mu,\ F^{n+1}_{2iT/2^{n+1},t_o}\mu\right) = \rho_F\left(\bigcirc_{j=i_o}^{i-1} S^{j,n}_{T/2^n}\mu, \bigcirc_{j=2i_o}^{2i-1} S^{j,n+1}_{T/2^{n+1}}\mu\right)\\ &\le \rho_F\left(S^{i-1,n}_{T/2^n}\bigcirc_{j=i_o}^{i-2} S^{j,n}_{T/2^n}\mu,\ S^{i-1,n}_{T/2^n}\bigcirc_{j=2i_o}^{2i-3} S^{j,n+1}_{T/2^{n+1}}\mu\right)\\ &\quad + \rho_F\left(S^{i-1,n}_{T/2^{n+1}}\left(S^{i-1,n}_{T/2^{n+1}}\bigcirc_{j=2i_o}^{2i-3} S^{j,n+1}_{T/2^{n+1}}\mu\right),\ S^{2i-1,n+1}_{T/2^{n+1}}\left(S^{2i-2,n+1}_{T/2^{n+1}}\bigcirc_{j=2i_o}^{2i-3} S^{j,n+1}_{T/2^{n+1}}\mu\right)\right)\\ &\le \mathrm{e}^{3\left(\|(b_n,c_n)(t_n^{i-1})\|_{\mathbf{W^{1,\infty}}}+\|\eta_n(t_n^{i-1})\|_{\mathbf{BL}}\right)T/2^n}\rho_F\left(F^n_{(i-1)T/2^n,t_o}\mu, F^{n+1}_{2(i-1)T/2^{n+1},t_o}\mu\right)\\ &\quad + \mathrm{e}^{5\left(\|(b_n,c_n)(t_n^{i-1})\|_{\mathbf{W^{1,\infty}}}+\|\eta_n(t_n^{i-1})\|_{\mathbf{BL}}\right)T/2^{n+1}}T/2^{n+1}\left(\left(\bigcirc_{j=2i_o}^{2i-2} S^{j,n+1}_{T/2^{n+1}}\mu\right)(\mathbb{R}_+)\right)\\ &\qquad \cdot \left(\|(b_n,c_n)(t_n^{i-1}) - (b_{n+1},c_{n+1})(t_{n+1}^{2i-1})\|_{\mathbf{L^\infty}} + \|\eta_n(t_n^{i-1}) - \eta_{n+1}(t_{n+1}^{2i-1})\|_{\mathbf{BC_x}}\right)\end{aligned}$$

$$
\begin{aligned}
&\le e^{3\|(b,c,\eta)\|_{\mathbf{BC_t}} T/2^n} \rho_F \left(F^n_{(i-1)T/2^n, t_o}\mu, F^{n+1}_{2(i-1)T/2^{n+1}, t_o}\mu \right) \\
&\quad + \; e^{5\|(b,c,\eta)\|_{\mathbf{BC_t}}(t-t_o)} \left(\|(b_n, c_n, \eta_n) - (b_{n+1}, c_{n+1}, \eta_{n+1})\|_{\mathbf{BC_t}} \right) T/2^{n+1} \, \mu(\mathbb{R}_+) \\
&= e^{3\|(b,c,\eta)\|_{\mathbf{BC_t}} T/2^n} \rho_F \left(F^n_{(i-1)T/2^n, t_o}\mu, F^{n+1}_{2(i-1)T/2^{n+1}, t_o}\mu \right) \\
&\quad + \; e^{5\|(b,c,\eta)\|_{\mathbf{BC_t}}(t-t_o)} \, \|(b_n, c_n, \eta_n) - (b_{n+1}, c_{n+1}, \eta_{n+1})\|_{\mathbf{BC_t}} T/2^{n+1} \, \mu(\mathbb{R}_+),
\end{aligned}
$$

where the last inequality holds due to the fact, that

$$
\left(\bigcirc_{j=2i_o}^{2i-2} S^{j,n+1}_{T/2^{n+1}} \mu \right)(\mathbb{R}_+) \le e^{2\|(b,c,\eta)\|_{\mathbf{BC_t}}(t-t_o-T/2^{n+1})} \mu(\mathbb{R}_+).
$$

Gronwall's inequality (see [44, Lemma 4.2]) allows us to obtain the estimate

$$
\begin{aligned}
\rho_F \left(F^n_{t,t_o}\mu, F^{n+1}_{t,t_o}\mu \right) \;\; &\le \;\; \frac{1}{2} \left[\frac{e^{3\|(b,c,\eta)\|_{\mathbf{BC_t}}(t-t_o)} - 1}{3\|(b,c,\eta)\|_{\mathbf{BC_t}}} \right] e^{5\|(b,c,\eta)\|_{\mathbf{BC_t}}(t-t_o)} \mu(\mathbb{R}_+) \\
&\quad \cdot \|(b_n, c_n, \eta_n) - (b_{n+1}, c_{n+1}, \eta_{n+1})\|_{\mathbf{BC_t}} \,.
\end{aligned}
$$

Since we consider a bounded time interval, there exist a constant $C^* = C^* \left(T, \|(b,c,\eta)\|_{\mathbf{BC_t}} \right)$, such that for all $t \in [t_o, T]$

$$
\frac{1}{2} \left[\frac{e^{3\|(b,c,\eta)\|_{\mathbf{BC_t}}(t-t_o)} - 1}{3\|(b,c,\eta)\|_{\mathbf{BC_t}}} \right] \le C^*(t - t_o).
$$

Therefore,

$$
\rho_F \left(F^n_{t,t_o}\mu, F^{n+1}_{t,t_o}\mu \right) \le C^* e^{5\|(b,c,\eta)\|_{\mathbf{BC_t}}(t-t_o)} \|(b_n, c_n, \eta_n) - (b_{n+1}, c_{n+1}, \eta_{n+1})\|_{\mathbf{BC_t}} \, \mu(\mathbb{R}_+)(t - t_o).
$$

Due to the assumptions about Hölder regularity of functions b, c, η we conclude that there exist constants $H(b), H(c), H(\eta)$ such that

$$
\begin{aligned}
\sup_t \|b_n(t) - b_{n+1}(t)\|_{\mathbf{L}^\infty} \quad &\le \quad \sup_t \|b_n(t) - b_{n+1}(t)\|_{\mathbf{W^{1,\infty}}} \quad \le \quad H(b) 2^{-n\alpha}, \\
\sup_t \|c_n(t) - c_{n+1}(t)\|_{\mathbf{L}^\infty} \quad &\le \quad \sup_t \|c_n(t) - c_{n+1}(t)\|_{\mathbf{W^{1,\infty}}} \quad \le \quad H(c) 2^{-n\alpha}, \\
\sup_t \|\eta_n(t) - \eta_{n+1}(t)\|_{\mathbf{BC_x}} \quad &\le \quad \sup_t \|\eta_n(t) - \eta_{n+1}(t)\|_{\mathbf{BL}} \quad \le \quad H(\eta) 2^{-n\alpha},
\end{aligned}
\tag{2.61}
$$

meaning that

$$
\|b_n - b_{n+1}\|_{\mathbf{BC_t}} + \|c_n - c_{n+1}\|_{\mathbf{BC_t}} + \|\eta_n - \eta_{n+1}\|_{\mathbf{BC_t}} \le (H(b) + H(c) + H(\eta)) 2^{-n\alpha},
$$

which implies $\sum_{n=1}^{\infty} \|(b_n, c_n, \eta_n) - (b_{n+1}, c_{n+1}, \eta_{n+1})\|_{\mathbf{BC_t}} < \infty$. Therefore,

$$
\sum_{n=m}^{k} \|(b_n, c_n, \eta_n) - (b_{n+1}, c_{n+1}, \eta_{n+1})\|_{\mathbf{BC_t}} \overset{m,k\to\infty}{\longrightarrow} 0.
$$

Thus, we conclude that for each $\mu \in \mathcal{M}^+(\mathbb{R}_+)$ the sequence $F^n_{t,t_o}\mu$ is a Cauchy sequence, which converges uniformly with respect to time to a measure $\nu \in \mathcal{M}^+(\mathbb{R}_+)$. By definition we set $P(t, t_o)\mu = \nu$. Claim *i)* follows then from the construction of F^n_{t,t_o}, since we are

dealing with linear problems.
Now assume that $t_o \in \left(t_n^{i_o-1}, t_n^{i_o}\right)$ and $t \in (t_n^i, t_n^{i+1})$, meaning that t_o and t are not grid points. Then,

$$\rho_F\left(F_{t,t_o}^n\mu, F_{t,t_o}^{n+1}\mu\right) \le \rho_F\left(\bigcirc_{j=i_o}^{i-1} S_{T/2^n}^{j,n}\mu,\ \bigcirc_{j=2i_o}^{2i-1} S_{T/2^{n+1}}^{j,n+1}\mu\right) + o\left(\frac{1}{2^n}\right),$$

which holds due to the fact, that $F_{t,t_o}^n\mu$ is Lipschitz continuous and the length of time intervals $\left(t_n^{i_o-1}, t_n^{i_o}\right)$, (t_n^i, t_n^{i+1}) is equal to $T/2^n$.

To pass from the estimates performed for n-th and $(n+1)$-th level of approximation to the estimates for arbitrary n and k we need to be able to claim that the series $\sum_{n=1}^{\infty} \|(b_n, c_n, \eta_n) - (b_{n+1}, c_{n+1}, \eta_{n+1})\|_{\mathbf{BC_t}}$ converges. In general, uniform continuity does not guarantee such convergence. However, as mentioned in Remark 2.58, it is sufficient to assume that b, c and η are uniformly continuous with modulus of continuity ω such that $\sum_{n=1}^{\infty} \omega(2^{-n}) < +\infty$.

To prove *iii)*, one can easily check, that for $t = iT/2^n$

$$\left(\bigcirc_{j=i_o}^{i-1} S_{T/2^n}^{j,n}\mu\right)(\mathbb{R}_+) \quad \le \quad \mathrm{e}^{2\|(b,c,\eta)\|_{\mathbf{BC_t}}(t-t_o)}\mu(\mathbb{R}_+).$$

Therefore,

$$\begin{aligned}\rho_F\left(\bigcirc_{j=i_o}^{i-1} S_{T/2^n}^{j,n}\mu, \mu\right) &\le \rho_F\left(S_{T/2^n}^{i-1,n}\left(\bigcirc_{j=i_o}^{i-2} S_{T/2^n}^{j,n}\mu\right), \bigcirc_{j=i_o}^{i-2} S_{T/2^n}^{j,n}\mu\right) + \rho_F\left(\bigcirc_{j=i_o}^{i-2} S_{T/2^n}^{j,n}\mu, \mu\right)\\ &\le \|(b,c,\eta)\|_{\mathbf{BC_t}}\mathrm{e}^{\|(b,c,\eta)\|_{\mathbf{BC_t}}T/2^n}\frac{T}{2^n}\left(\mathrm{e}^{2\|(b,c,\eta)\|_{\mathbf{BC_t}}((i-i_o)-1)T/2^n}\mu(\mathbb{R}_+)\right) + \rho_F\left(\bigcirc_{j=i_o}^{i-2} S_{T/2^n}^{j,n}\mu, \mu\right).\end{aligned}$$

Hence, iterating the procedure we obtain

$$\rho_F\left(\bigcirc_{j=i_o}^{i-1} S_{T/2^n}^{j,n}\mu, \mu\right) \quad \le \quad \|(b,c,\eta)\|_{\mathbf{BC_t}}\mathrm{e}^{2\|(b,c,\eta)\|_{\mathbf{BC_t}}(t-t_o)}\mu(\mathbb{R}_+)(t-t_o).$$

Passing to *ii*) and *iv)*, let b, $\tilde{b}$, c, $\tilde{c}$, η and $\tilde{\eta}$ satisfy assumptions (2.55) and (2.56). Call $S^{i,n}$ and $\tilde{S}^{i,n}$ corresponding semigroups constructed in Theorem 2.24 on the time interval $[t_n^i, t_n^{i+1})$. Define maps $F_{t,t_o}^n\mu$ and $\tilde{F}_{t,t_o}^n\nu$ as in (2.60). Assume first that $t_o = i_oT/2^n$ and $t = iT/2^n$ with $i > i_o$. Then,

$$\begin{aligned}\rho_F(F_{t,t_o}^n\mu, \tilde{F}_{t,t_o}^n\nu) &= \rho_F(F_{iT/2^n,t_o}^n\mu, \tilde{F}_{iT/2^n,t_o}^n\nu) = \rho_F\left(\bigcirc_{j=i_o}^{i-1} S_{T/2^n}^{j,n}\mu, \bigcirc_{j=i_o}^{i-1} \tilde{S}_{T/2^n}^{j,n}\nu\right)\\ &= \rho_F\left(S_{T/2^n}^{i-1,n}\bigcirc_{j=i_o}^{i-2} S_{T/2^n}^{j,n}\mu, \tilde{S}_{T/2^n}^{i-1,n}\bigcirc_{j=i_o}^{i-2} \tilde{S}_{T/2^n}^{j,n}\nu\right)\\ &\le \rho_F\left(S_{T/2^n}^{i-1,n}\bigcirc_{j=i_o}^{i-2} S_{T/2^n}^{j,n}\mu, S_{T/2^n}^{i-1,n}\bigcirc_{j=i_o}^{i-2} \tilde{S}_{T/2^n}^{j,n}\nu\right) + \rho_F\left(S_{T/2^n}^{i-1,n}\bigcirc_{j=i_o}^{i-2} \tilde{S}_{T/2^n}^{j,n}\nu, \tilde{S}_{T/2^n}^{i-1,n}\bigcirc_{j=i_o}^{i-2} \tilde{S}_{T/2^n}^{j,n}\nu\right)\\ &\le \mathrm{e}^{3\left(\|(b_n,c_n)(t_n^{i-1})\|_{\mathbf{W^{1,\infty}}} + \|\eta_n(t_n^{i-1})\|_{\mathbf{BL}}\right)T/2^n}\rho_F(F_{(i-1)T/2^n,t_o}^n\mu, \tilde{F}_{(i-1)T/2^n,t_o}^n\nu)\\ &\quad + \mathrm{e}^{5\left(\|(b_n,c_n)(t_n^{i-1})\|_{\mathbf{W^{1,\infty}}} + \|\eta_n(t_n^{i-1})\|_{\mathbf{BL}}\right)T/2^n}T/2^n\left(\left(\bigcirc_{j=i_o}^{i-2} \tilde{S}_{T/2^n}^{j,n}\nu\right)(\mathbb{R}_+)\right)\\ &\qquad \cdot\left(\|(b_n,c_n)(t_n^{i-1}) - (\tilde{b}_n,\tilde{c}_n)(t_n^{i-1})\|_{\mathbf{L^\infty}} + \|\eta_n(t_n^{i-1}) - \tilde{\eta}_n(t_n^{i-1})\|_{\mathbf{BC_x}}\right)\\ &\le \mathrm{e}^{3\|(b,c,t)\|_{\mathbf{BC_t}}T/2^n}\rho_F(F_{(i-1)T/2^n,t_o}^n\mu, \tilde{F}_{(i-1)T/2^n,t_o}^n\nu)\end{aligned}$$

$$+ \mathrm{e}^{5\|(b,\tilde{b},c,\tilde{c},\eta,\tilde{\eta})\|_{\mathbf{BC_t}}(t-t_o)} \|(b,c,\eta) - (\tilde{b},\tilde{c},\tilde{\eta})\|_{\mathbf{BC_t}} T/2^n \ \nu(\mathbb{R}_+).$$

Application of the Gronwall's inequality (see [44, Lemma 4.2]) yields

$$\rho_F(F^n_{t,t_o}\mu, \tilde{F}^n_{t,t_o}\nu) \leq \mathrm{e}^{3\|(b,c,t)\|_{\mathbf{BC_t}}(t-t_o)} \ \rho_F(\mu,\nu) \\ + \left[\frac{\mathrm{e}^{3\|(b,c,t)\|_{\mathbf{BC_t}}(t-t_o)} - 1}{3\|(b,c,t)\|_{\mathbf{BC_t}}}\right] \mathrm{e}^{5\|(b,\tilde{b},c,\tilde{c},\eta,\tilde{\eta})\|_{\mathbf{BC_t}}(t-t_o)} \nu(\mathbb{R}_+) \|(b,c,\eta) - (\tilde{b},\tilde{c},\tilde{\eta})\|_{\mathbf{BC_t}}.$$

Using analogous arguments as in the proof of *i)*, we conclude that there exists a constant C^*, such that

$$\rho_F(F^n_{t,t_o}\mu, \tilde{F}^n_{t,t_o}\nu) \leq \mathrm{e}^{3\|(b,c,\eta)\|_{\mathbf{BC}_t}(t-t_o)} \ \rho_F(\mu,\nu) \\ + C^*(t-t_o)\mathrm{e}^{5\|(b,\tilde{b},c,\tilde{c},\eta,\tilde{\eta})\|_{\mathbf{BC}_t}(t-t_o)} \nu(\mathbb{R}_+) \|(b,c,\eta) - (\tilde{b},\tilde{c},\tilde{\eta})\|_{\mathbf{BC}_t}.$$

For t_o and t, which are not grid points, we prove this inequality using the same argument as in the proof of *i)*. Therefore, passing to the limit with n ends the proof.

Passing to *v)*, let $\varphi \in (\mathbf{C1} \cap \mathbf{W^{1,\infty}})([0,T] \times \mathbb{R}_+)$ and $n \in \mathbb{N}$. From Theorem 2.24 we know that for each $i = 0, 1, \cdots, 2^{n-1}$, orbits of the semigroup $S^{i,n}$ are weak solutions of the linear non-autonomous problem (2.57) on $[t_n^{i-1}, t_n^i)$. Therefore,

$$\int_{\mathbb{R}_+} \varphi(T,x)\mathrm{d}\mu^n(T) - \int_{\mathbb{R}_+} \varphi(0,x)\mathrm{d}\mu_o \\ = \int_0^T \int_{\mathbb{R}_+} \left[\partial_t \varphi(t,x) + b_n(t,x)\partial_x\varphi(t,x) - c_n(x)\varphi(t,x) + \int_{\mathbb{R}_+} \varphi(t,y)\mathrm{d}[\eta_n(t,x)](y)\right]\mathrm{d}\mu^n_t(x)\mathrm{d}t \\ = \int_0^T \int_{\mathbb{R}_+} \left[\partial_t \varphi(t,x) + b(t,x)\partial_x\varphi(t,x) - c(x)\varphi(t,x) + \int_{\mathbb{R}_+} \varphi(t,y)\mathrm{d}[\eta(t,x)](y)\right]\mathrm{d}\mu^n_t(x)\mathrm{d}t + R^n,$$

where

$$R^n = \int_0^T \int_{\mathbb{R}_+} \big((b_n(t,x) - b(t,x))\,\partial_x\varphi(t,x) + (c(t,x) - c_n(t,x))\,\varphi(t,x)\big)\mathrm{d}\mu^n_t(x)\mathrm{d}t + \\ \int_0^T \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \varphi(t,y)\mathrm{d}[\eta_n(t,x) - \eta(t,x)](y)\mathrm{d}\mu^n_t(x)\mathrm{d}t.$$

From the previous analysis in this proof we know, that μ^n_t converges narrowly and uniformly with respect to time to the unique limit μ_t. Due to the assumptions (2.61) about Hölder regularity of functions b, c, η, we use the analogous arguments as in proof of claim *i)* in Lemma 2.41 and pass to the limit in the integral, which ends the proof.

A convenient method to prove the uniqueness of solutions to (2.57) relies on the dual formulation. Indeed, the existence of a solution to the dual problem implies the uniqueness of solutions to (2.57). More precisely, if $\varphi_{T,\psi}$ solves (2.63), then the equality

$$\int_{\mathbb{R}_+} \psi(x)\mathrm{d}\mu_t(x) = \int_{\mathbb{R}_+} \varphi_{T,\psi}(T-t,x)\mathrm{d}\mu_o(x)$$

defines the solution to (2.57). Thus, uniqueness follows from Lemma 2.62 below. □

We remark here that the cost of applying the operator splitting method is that the tangency condition *vi)* in Theorem 2.24 suffices to guarantee the uniqueness of the semigroup based on the operator splitting method, but not in general weak solutions to (2.11). It follows from the fact that with the $\mathbf{W^{1,\infty}}$ regularity of coefficients, there is a strong convergence of the operator splitting method for the original problem, but not for the dual one (in sense of the norm topology in $(\mathbf{C^1} \cap \mathbf{W^{1,\infty}})$). Therefore, a limit of the dual semigroups under operator splitting converges only weakly* in $\mathbf{W^{1,\infty}}$ and as a consequence it is not necessarily an admissible test function for (2.11). Hence, the idea of the proof of uniqueness of weak solutions from [44] could not be applied. Therefore, we directly prove below the uniqueness of solutions to (2.63) exploiting its dual formulation.

Lemma 2.62. *Define a dual problem to* (2.57)*, that is*

$$\begin{cases} \partial_t \varphi(t,x) + b(t,x)\partial_x \varphi(t,x) - c(t,x)\varphi(t,x) + \int_{\mathbb{R}_+} \varphi(t,y) \mathrm{d}[\eta(t,x)](y) &= 0, \\ \varphi(T,x) &= \psi(x). \end{cases} \tag{2.63}$$

For any $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+)$ *there is a unique* $\varphi_{T,\psi} \in \mathbf{C^1}([0,T] \times \mathbb{R}_+)$ *solving* (2.63).

Proof of Lemma 2.62. The proof consists of two steps.

1. We assume additional regularity of b. Assume that $b \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})([0,T] \times \mathbb{R}_+)$. To prove the assertion we change variables in (2.63) in order to investigate the behavior of $\varphi_{T,\psi}$ along the characteristics X, that is along the solutions to

$$\begin{aligned} \partial_\tau X(\tau;t,x) &= b\left(X(\tau;t,x)\right), \\ X(t;t,x) &= x\,. \end{aligned}$$

The following change of variables in (2.63)

$$(t,x) \to \left(t, \Theta_{(t,T)}(x)\right), \quad \text{where} \quad \Theta_{(t,T)}(x) = X(T-t;t,x), \tag{2.64}$$

yields

$$\partial_t \varphi(t,\tilde{x}) = c(t,\tilde{x})\varphi(t,\tilde{x}) - \int_{\mathbb{R}_+} \varphi(t,\tilde{y}) \mathrm{d}[\eta(t,\tilde{x})](y)$$

with $\tilde{x} = \Theta_{(t,T)}(x)$ and $\tilde{y} = \Theta_{(t,T)}(y)$. The assumption of differentiability of b with respect to x implies that $\{\Theta_{(t,T)}\}_{t \in [0,T]}$ is a family of $\mathbf{C^1}$–diffeomorphisms on $\mathbb{R}_+$. To prove the existence of a unique solution we use the Banach Fixed Point Theorem. Let $\mathbf{C}([0,T], \mathbf{C^1_b}(\mathbb{R}_+))$ be the space of continuous functions attaining values in $\mathbf{C^1_b}(\mathbb{R}_+)$, which denotes the space of bounded continuously differentiable functions on $\mathbb{R}_+$. We equip the former space with the norm

$$\|\varphi\|_{\mathbf{BC_t}} = \sup_{t \in [0,T]} \|\varphi(t)\|_{\mathbf{W^{1,\infty}}}, \quad \text{where} \quad \|\varphi(t)\|_{\mathbf{W^{1,\infty}}} = \max\left\{\|\varphi(t,\cdot)\|_{\mathbf{L^\infty}}, \|\partial_x \varphi(t,\cdot)\|_{\mathbf{L^\infty}}\right\}.$$

Let us introduce the complete metric space $\mathbf{C_b}(I, \bar{B}_{R,\psi})$, where $I = [T-\varepsilon, T]$ with ε to be chosen later on, $\psi \in \mathbf{C^1_b}(\mathbb{R}_+)$ being a fixed function and

$$\bar{B}_{R,\psi} = \left\{f \;:\; f \in \mathbf{C^1_b}(\mathbb{R}_+) \text{ and } \|f - \psi\|_{\mathbf{W^{1,\infty}}} \le R\right\}.$$

Define the operator

$$\Gamma : \mathbf{C_b}(I, \bar{B}_{R,\psi}) \to \mathbf{C_b}(I, \bar{B}_{R,\psi})$$

as follows:

$$(\Gamma\varphi)(t)(\tilde{x}) = \psi(\tilde{x}) + \int_t^T \left(c(s,\tilde{x})\varphi(s,\tilde{x})\mathrm{d}s - \int_{\mathbb{R}_+} \varphi(s,\tilde{y})\mathrm{d}[\eta(s,\tilde{x})](\tilde{y}) \right) \mathrm{d}s\,.$$

We prove that Γ is well defined, i.e., that its image is continuously differentiable with respect to x and contained in $\bar{B}_{R,\psi}$, for ε small enough. Indeed, let $\varphi \in \mathbf{C_b}(I, \bar{B}_{R,\psi})$. Then,

$$\begin{aligned}
\|(\Gamma\varphi)(t) - \psi\|_{\mathbf{L}^\infty} &\leq \varepsilon\left(\|\psi\|_{\mathbf{W^{1,\infty}}} + R\right)\left(\|c\|_{\mathbf{BC_t}} + \|\eta\|_{\mathbf{BC_t}}\right) \qquad \text{and} \\
\partial_{\tilde{x}}\left(\Gamma\varphi(t) - \psi\right) &= \int_t^T \partial_{\tilde{x}} c(s,\tilde{x})\, \varphi(s,\tilde{x})\mathrm{d}s + \int_t^T c(s,\tilde{x})\, \partial_{\tilde{x}}\varphi(s,\tilde{x})\mathrm{d}s \\
&\quad -\partial_{\tilde{x}}\left(\int_t^T \int_{\mathbb{R}_+} \varphi(s,\tilde{y})\mathrm{d}[\eta(s,\tilde{x})](\tilde{y})\mathrm{d}s \right).
\end{aligned}$$

Due to the regularity of the coefficients c and η, we conclude that the derivative of the image of the operator Γ is continuous. In order to estimate $\|\partial_x(\Gamma\varphi(t) - \psi)\|_{\mathbf{L}^\infty}$ we need to bound the latter term of the equality above. To this end we will estimate the Lipschitz constant of the function $\tilde{x} \to \int_t^T \int_{\mathbb{R}_+} \varphi(s,\tilde{y})\mathrm{d}[\eta(s,\tilde{x})](\tilde{y})\mathrm{d}s$.

$$\begin{aligned}
&\int_t^T \int_{\mathbb{R}_+} \varphi(s,\tilde{y})\mathrm{d}[\eta(s,\tilde{x}_1)](\tilde{y})\mathrm{d}s - \int_t^T \int_{\mathbb{R}_+} \varphi(s,\tilde{y})\mathrm{d}[\eta(s,\tilde{x}_2)](\tilde{y})\mathrm{d}s \\
= &\int_t^T \|\varphi(s,\cdot)\|_{\mathbf{L}^\infty} \int_{\mathbb{R}_+} \mathrm{d}\left(\eta(s,\tilde{x}_1) - \eta(s,\tilde{x}_2)\right)(\tilde{y})\mathrm{d}s \\
\leq &\ \varepsilon\left(\|\psi\|_{\mathbf{W^{1,\infty}}} + R\right) \sup_{t\in[T-\varepsilon,T]} \|\eta(t,\tilde{x}_1) - \eta(t,\tilde{x}_2)\|_{(\mathbf{W^{1,\infty}})^*} \\
\leq &\ \varepsilon\left(\|\psi\|_{\mathbf{W^{1,\infty}}} + R\right) \sup_{t\in[T-\varepsilon,T]} \mathbf{Lip}(\eta(t,\cdot))\, |\tilde{x}_1 - \tilde{x}_2| \leq \varepsilon\left(\|\psi\|_{\mathbf{W^{1,\infty}}} + R\right)\|\eta\|_{\mathbf{BC_t}}\, |\tilde{x}_1 - \tilde{x}_2|\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\partial_x(\Gamma\varphi(t) - \psi)\|_{\mathbf{L}^\infty} &\leq \varepsilon\, \|c\|_{\mathbf{BC_t}}\left(\|\psi\|_{\mathbf{W^{1,\infty}}} + R\right) + \varepsilon\, \|c\|_{\mathbf{BC_t}}\left(\|\psi\|_{\mathbf{W^{1,\infty}}} + R\right) \\
&\quad + \varepsilon\, \|\eta\|_{\mathbf{BC_t}}\left(\|\psi\|_{\mathbf{W^{1,\infty}}} + R\right) \leq 2\varepsilon\left(\|\psi\|_{\mathbf{W^{1,\infty}}} + R\right)\left(\|c\|_{\mathbf{BC_t}} + \|\eta\|_{\mathbf{BC_t}}\right).
\end{aligned}$$

We need that $\varepsilon < \nu_1$, where

$$\nu_1 = \frac{R}{2} \cdot \left[\left(\|\psi\|_{\mathbf{W^{1,\infty}}} + R\right)\left(\|c\|_{\mathbf{BC_t}} + \|\eta\|_{\mathbf{BC_t}}\right)\right]^{-1}.$$

For such ε operator Γ is well defined. Now, we prove that for ε small enough, Γ is a contraction. To this aim we estimate

$$\begin{aligned}
&\sup_{t\in[T-\varepsilon,T]} \|\Gamma\varphi^1(t) - \Gamma\varphi^2(t)\|_{\mathbf{L}^\infty} \leq \sup_{t\in[T-\varepsilon,T]} \int_t^T \|c(s,\cdot)\|_{\mathbf{L}^\infty} \cdot \|\varphi^1(s,\cdot) - \varphi^2(s,\cdot)\|_{\mathbf{L}^\infty}\mathrm{d}s \\
&\quad + \sup_{t\in[T-\varepsilon,T]} \int_t^T \left(\|\varphi^1(s,\cdot) - \varphi^2(s,\cdot)\|_{\mathbf{L}^\infty} \int_{\mathbb{R}_+} \mathrm{d}[\eta(s,\tilde{x})](y) \right)\mathrm{d}s
\end{aligned}$$

$$\leq \varepsilon \left(\|c\|_{\mathbf{BC_t}} + \|\eta\|_{\mathbf{BC_t}} \right) \|\varphi^1 - \varphi^2\|_{\mathbf{BC_t}}$$

and

$$\sup_{t\in[T-\varepsilon,T]} \|\partial_x \left(\Gamma\varphi^1(t) - \Gamma\varphi^2(t) \right)\|_{\mathbf{L}^\infty} \leq \sup_{t\in[T-\varepsilon,T]} \int_t^T \|\partial_x c(s,\cdot)\|_{\mathbf{L}^\infty} \cdot \|\varphi^1(s,\cdot) - \varphi^2(s,\cdot)\|_{\mathbf{L}^\infty} \mathrm{d}s$$
$$+ \sup_{t\in[T-\varepsilon,T]} \int_t^T \|c(s,\cdot)\|_{\mathbf{L}^\infty} \cdot \|\partial_x \left(\varphi^1(s,\cdot) - \varphi^2(s,\cdot) \right)\|_{\mathbf{L}^\infty} \mathrm{d}s$$
$$+ \sup_{t\in[T-\varepsilon,T]} \left| \partial_x \int_t^T \int_{\mathbb{R}_+} \left(\varphi^1(s,\tilde{y}) - \varphi^2(s,\tilde{y}) \right) \mathrm{d}[\eta(t,\tilde{x})](\tilde{y}) \mathrm{d}s \right|$$
$$\leq \varepsilon \left(\|c\|_{\mathbf{BC_t}} \|\varphi^1 - \varphi^2\|_{\mathbf{BC_t}} + \|c\|_{\mathbf{BC_t}} \|\varphi^1 - \varphi^2\|_{\mathbf{BC_t}} + \|\eta\|_{\mathbf{BC_t}} \|\varphi^1 - \varphi^2\|_{\mathbf{BC_t}} \right)$$
$$\leq 2\varepsilon \left(\|c\|_{\mathbf{BC_t}} + \|\eta\|_{\mathbf{BC_t}} \right) \|\varphi^1 - \varphi^2\|_{\mathbf{BC_t}}.$$

Hence, we conclude that $\varepsilon < \nu_2$, where

$$\nu_2 = \frac{1}{2} \cdot \left(\|c\|_{\mathbf{BC_t}} + \|\eta\|_{\mathbf{BC_t}} \right)^{-1}.$$

From the Banach Fixed Point Theorem it follows that, for each ψ, there exists unique solution $\varphi_{T,\psi}$ to (2.63). This solution can be extended to the whole interval $[0,T]$, as ε does not depend on time. Since Θ is a $\mathbf{C^1}$–diffeomorphism, the regularity of solutions does not diminish after changing variables back to the original ones (t,x).

2. Avoiding the additional regularity. In Step 1 we assumed that the additional regularity of b, that is $b \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})([0,T]\times\mathbb{R}_+)$. Actually, we will show now that it is sufficient to require $\mathbf{W^{1,\infty}}$ regularity. Notice that for the linear non-autonomous case (2.57) a well posedness of the expression (2.11) in the Definition 2.10 of a weak solution requires less regularity of a test function φ, that is

$$\varphi \in \mathbf{C}([0,T]\times\mathbb{R}_+) \quad \text{and} \quad D_{[1,b]}\varphi \in \mathbf{C}([0,T]\times\mathbb{R}_+),$$

where $D_{[1,b]}\varphi(t,x) = \partial_t\varphi(t,x) + b(t,x)\partial_x\varphi(t,x)$ is a directional derivative of φ along a characteristic. Such a regularity is guaranteed, if we assume that $b \in \mathbf{W^{1,\infty}}([0,T]\times\mathbb{R}_+)$, what we shall show in the following. In view of Step 1, it is sufficient to show that $D_{[1,b]}\varphi$ is a continuous function on $[0,T]\times\mathbb{R}_+$. To this end, fix $(t_o,x_o) \in [0,T]\times\mathbb{R}_+$ and chose an arbitrary sequence $\{(t_n,x_n)\}_{n\in\mathbb{N}}$ such that $(t_n,x_n) \to (t_o,x_o)$. Lipschitz continuity of b implies that $\Theta_{(t,T)}(x)$ defined in (2.64) is a homeomorphism on $\mathbb{R}_+$ for each $T>0$ and $t\in[0,T]$. Moreover, $\Theta_{(t,T)}(x)$ is a continuous map in variables t, T and x. Notice that the change of variables (2.64) yields

$$D_{[1,b]}\varphi(t,x) = \partial_t\varphi\left(t, \Theta_{(t,T)}(x)\right),$$

meaning that the directional derivative along a characteristic is transformed into the derivative with respect to the t variable. Taking into account the argumentation given above we conclude that $D_{[1,b]}\varphi(t_n,x_n) \to D_{[1,b]}\varphi(t_o,x_o)$ is directly implied by

$$\partial_t\varphi\left(t_n, \Theta_{(t_n,t_o)}(x_n)\right) \to \partial_t\varphi\left(t_o,x_o\right),$$

which holds due to the regularity of Θ and the fact that the solution φ to (2.62) after a change of variables (2.64) is a $\mathbf{C^1}([0,T],\mathbf{C^1_b}(\mathbb{R}_+))$ function. □

2.6. Nonlinear Case

This section is devoted to the general problem (2.1) and presents the proof of Theorem 2.13 stated in Section 2.3.

Proof of Theorem 2.13. Let b, c, η be functions given by (2.6)–(2.7) and $\mu_o \in \mathcal{M}^+(\mathbb{R}_+)$ be an initial measure in (2.1). Let us introduce a complete metric space $\mathbf{C_b}(I;\bar{B}_R(\mu_o))$ where $I = [0,\varepsilon]$ with ε to be chosen later on and $\bar{B}_R(\mu_o) = \{\nu \in \mathcal{M}^+(\mathbb{R}_+): \rho_F(\mu_o,\nu) \le R\}$. The space $\mathbf{C_b}(I;\bar{B}_R(\mu_o))$ is equipped with the norm given by

$$\|\mu\|_{\mathbf{BC}} = \sup_{t\in[0,T]} \|\mu_t\|_{(\mathbf{W^{1,\infty}})^*}.$$

This space is complete since $\bar{B}_R(\mu_o)$ is a closed subset of the complete metric space $\mathcal{M}^+(\mathbb{R}_+)$. We define the operator $\mathcal{T}$ on $\mathbf{C_b}(I;\bar{B}_R(\mu_o))$ as follows

$$\mathcal{T}: \mathbf{C_b}(I;\bar{B}_R(\mu_o)) \to \mathbf{C_b}(I;\bar{B}_R(\mu_o)), \qquad \text{where} \quad \mathcal{T}(\mu) = \nu_{(b,c,\eta)(\mu)}\,.$$

Here, $\nu_{(b,c,\eta)(\mu)}$ is the solution to (2.57) with coefficients $b(\cdot,\mu)$, $c(\cdot,\mu)$, $\eta(\cdot,\mu)$ and initial datum μ_o. From the assumptions on coefficients and the definition of norm $\|\cdot\|_{\mathbf{BC^{\alpha,1}}}$ (2.8) we observe

$$\begin{array}{rcll} M_b &=& \sup_{t\in[0,T],\nu\in\mathcal{M}^+(\mathbb{R}_+)} \|b(t,\nu)\|_{\mathbf{W^{1,\infty}}} & < \infty, \\ M_c &=& \sup_{t\in[0,T],\nu\in\mathcal{M}^+(\mathbb{R}_+)} \|c(t,\nu)\|_{\mathbf{W^{1,\infty}}} & < \infty, \\ M_\eta &=& \sup_{t\in[0,T],\nu\in\mathcal{M}^+(\mathbb{R}_+)} \|\eta(t,\nu)\|_{\mathbf{BL}} & < \infty. \end{array}$$

For further simplicity we introduce a constant $M = M_b + M_c + M_\eta$. First, we need to prove, that the operator $\mathcal{T}$ is well defined, meaning that its image must be a bounded continuous function taking values in $\bar{B}_R(\mu_o)$. Continuity of $\nu_{(b,c,\eta)(\mu)}$ follows from *iii)* in Theorem 2.59. Moreover, for each $t \in [0,\varepsilon]$ we have

$$\rho_F(\mathcal{T}(\mu)(t),\mu_o) \le \|(b,c,\eta)\|_{\mathbf{BC_{t,\mu}}} e^{2\|(b,c,\eta)\|_{\mathbf{BC_{t,\mu}}} t} \mu_o(\mathbb{R}_+)\, t \le M e^{2M\varepsilon} \mu_o(\mathbb{R}_+)\,\varepsilon \le R.$$

We need to assume that $\varepsilon < 1$. Then, the latter inequality holds if $M e^{2M} \mu_o(\mathbb{R}_+)\,\varepsilon \le R$, which is equivalent to

$$\varepsilon \;\le\; R\left[M e^{2M} \mu_o(\mathbb{R}_+)\right]^{-1} =: \zeta_1. \tag{2.65}$$

Now, we prove that $\mathcal{T}$ is a contraction for ε small enough. To this end, we show, that $\mathcal{T}$ is a Lipschitz operator with the Lipschitz constant smaller than 1. Here, we use *iv)* from Theorem 2.59.

$$\begin{aligned} \|\mathcal{T}(\mu) - \mathcal{T}(\nu)\|_{\mathbf{BC}} &= \sup_{t\in[0,\varepsilon]} \|\mathcal{T}(\mu)(t) - \mathcal{T}(\tilde{\mu})(t)\|_{(\mathbf{W^{1,\infty}})^*} = \sup_{t\in[0,\varepsilon]} d\left(\mathcal{T}(\mu)(t), \mathcal{T}(\tilde{\mu})(t)\right) \\ &= \sup_{t\in[0,\varepsilon]} \rho_F\left(\nu_{(b,c,\eta)(\mu)}(t), \nu_{(b,c,\eta)(\tilde{\mu})}(t)\right) \\ &\le \sup_{t\in[0,\varepsilon]} C^* t\, e^{5\left(\|(b,c,\eta)(\mu)\|_{\mathbf{BC_t}} + \|(b,c,\eta)(\tilde{\mu})\|_{\mathbf{BC_t}}\right)t} \cdot \|(b,c,\eta)(\mu) - (b,c,\eta)(\tilde{\mu})\|_{\mathbf{BC_t}} \mu_o(\mathbb{R}_+) \\ &\le \sup_{t\in[0,\varepsilon]} C^* \varepsilon\, e^{10(M_b+M_c+M_\eta)\varepsilon}\, \mu_o(\mathbb{R}_+)\; \left(\mathbf{Lip}(b) + \mathbf{Lip}(c) + \mathbf{Lip}(\eta)\right) \cdot \rho_F(\mu_t, \tilde{\mu}_t) \end{aligned}$$

$$\leq C^* \varepsilon\, \mathrm{e}^{10M\varepsilon}\, \mu_o(\mathbb{R}_+)\cdot\ (\mathbf{Lip}(b) + \mathbf{Lip}(c) + \mathbf{Lip}(\eta))\, \|\mu - \tilde{\mu}\|_{\mathbf{BC}}.$$

where $(b,c,\eta)(\mu) = (b(\cdot,\mu), c(\cdot,\mu), \eta(\cdot,\mu))$ and $\mathbf{Lip}(b) = \sup_{t\in[0,T]} \mathbf{Lip}(b(t,\cdot)) < \infty$, which holds due to assumptions on b (similarly for c and η). Lipschitz constant of $\mathcal{T}$ is smaller than 1, if the following inequality holds

$$\mathbf{Lip}(\mathcal{T}) \;=\; C^* \varepsilon\, \mathrm{e}^{10M\varepsilon}\, \mu_o(\mathbb{R}_+)\, (\mathbf{Lip}(b) + \mathbf{Lip}(c) + \mathbf{Lip}(\eta)) < 1.$$

We need to assume that $\varepsilon < 1$. Then, the latter inequality holds if

$$\begin{aligned} &C^* \varepsilon\, \mathrm{e}^{10M}\, \mu_o(\mathbb{R}_+)\, (\mathbf{Lip}(b) + \mathbf{Lip}(c) + \mathbf{Lip}(\eta)) < 1. \\ &\varepsilon < \left(C^*\, \mathrm{e}^{10M}\, \mu_o(\mathbb{R}_+)\, (\mathbf{Lip}(b) + \mathbf{Lip}(c) + \mathbf{Lip}(\eta))\right)^{-1} =: \zeta_2. \end{aligned}$$

We have just proved, that $\mathcal{T}$ is a contraction on a complete metric space $\mathbf{C_b}(I, \bar{B}_R(\mu_o))$, where $0 < \varepsilon \leq \min\{1, \zeta_1, \zeta_2\}$. From the Banach Fixed Point Theorem it follows that there exists unique μ^*, such that $\mathcal{T}(\mu^*) = \mu^*$. Hence, existence of the unique solution to (2.1) on the time interval $[0,\varepsilon]$ is proved. This solution can be extended on the whole $[0,T]$ interval, because ζ_1 and ζ_2 depend only on the model coefficients. Moreover, from *iii)* in Theorem 2.59, we conclude that solution to (2.1) is Lipschitz continuous with respect to time. The estimates in claims *i)* and *ii)* are consequences of the estimates for the linear non-autonomous case (see Theorem 2.59). □

At the end of this chapter we formalizes the content of Remark 2.12.

Lemma 2.66. *Consider equation* (2.1) *on the whole* $\mathbb{R}$. *If the initial datum* μ_o *is supported in* $\mathbb{R}_+$, *then for all* $t \in [0,T]$, *the corresponding solution* μ_t *to* (2.1) *is also a measure supported on* $\mathbb{R}_+$.

Proof of Lemma 2.66. As the first step we consider the linear autonomous case given by (2.14). In view of the construction of the solution via the operator splitting algorithm, it is sufficient to show that any value attained by the map $F^\varepsilon(t)\mu_o$ defined in the proof of Theorem 2.24, that is

$$F^\varepsilon(t)\mu_o = \begin{cases} \check{S}_{2t-2i\varepsilon}\, (\hat{S}_\varepsilon \check{S}_\varepsilon)^i \mu_o & \text{for} \quad t \in [i\varepsilon, (i+1/2)\varepsilon)\,, \\ \hat{S}_{2t-2(i+1)\varepsilon}\, \check{S}_\varepsilon\, (\hat{S}_\varepsilon \check{S}_\varepsilon)^i \mu_o & \text{for} \quad t \in [(i+1/2)\varepsilon, (i+1)\varepsilon) \end{cases}$$

is a measure on $\mathbb{R}_+$ for each $t \in [0,T]$ and $\mu_o \in \mathcal{M}^+(\mathbb{R}_+)$. Fix $\varepsilon > 0$ and $\check{\nu}, \hat{\mu} \in \mathcal{M}^+(\mathbb{R}_+)$. Formula (2.48) and the assumption $b(0) > 0$ imply that $\check{S}_\varepsilon \check{\nu} \in \mathcal{M}^+(\mathbb{R}_+)$. To prove that $\hat{S}_\varepsilon \hat{\mu} \in \mathcal{M}^+(\mathbb{R}_+)$ we use the regularized version of (2.19), that is the equation (2.45) on the whole $\mathbb{R}$

$$\begin{cases} \dfrac{\partial}{\partial t} u^\varepsilon(t,x) = -c(x) u^\varepsilon(t,x) + \displaystyle\int_{\mathbb{R}_+} \eta^\varepsilon(y,x) u^\varepsilon(t,y)\mathrm{d}y, & (t,x) \in [0,T]\times\mathbb{R}, \\ \qquad u^\varepsilon(0,x) = u_o^\varepsilon(x), & x \in \mathbb{R}, \end{cases} \tag{2.67}$$

where $u_o^\varepsilon = \hat{\nu} * \rho^\varepsilon$. Without loss of generality, ρ^ε can be chosen so that

$$\mathrm{supp}(u_o^\varepsilon) \subset (-\varepsilon, +\infty) \quad \text{and} \quad \mathrm{supp}(\eta^\varepsilon(y,\cdot)) \subset (-\varepsilon, +\infty).$$

Fix $\bar{x} \in (-\infty, -\varepsilon]$. It is straightforward that $u_o^\varepsilon(\bar{x}) = 0$ and $\left(\int_{\mathbb{R}} \eta^\varepsilon(y, \bar{x}) u^\varepsilon(t, y) \mathrm{d}y\right) = 0$. Then, (2.67) can be considered as an ODE of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} u^\varepsilon(t) = -c(\bar{x}) u^\varepsilon(t)\,, \quad u^\varepsilon(0) = u_o^\varepsilon(\bar{x}) = 0\,.$$

From the standard ODE theory it follows that $u^\varepsilon(t, \bar{x}) = 0$ for all $t \in [0, T]$, meaning that if $\mathrm{supp}(u_o^\varepsilon) \subset (-\varepsilon, +\infty)$, then $\mathrm{supp}(u^\varepsilon(t, \cdot)) \subset (-\varepsilon, +\infty)$ as well. The previous analysis shows that $d(u_t^\varepsilon, \mu_t) \to 0$ for $\varepsilon \to 0$ uniformly with respect to time, where μ_t is a solution to (2.19). Therefore, we conclude that a limit μ_t is a measure on $\mathbb{R}_+$.

Since the choice of $\check{\nu}, \hat{\mu}$ is arbitrary, we proved that $F_t^\varepsilon \mu_o \in \mathcal{M}^+(\mathbb{R}_+)\mu_o$ for all $\mu_o \in \mathcal{M}^+(\mathbb{R}_+)$. In view of Theorem 2.24, letting $\varepsilon \to 0$ in F_t^ε completes the proof in the linear autonomous case. The linear non-autonomous case and the nonlinear case follow from the construction of solutions as in the proof of Theorem 2.59 and of Theorem 2.13. $\square$

Chapter 3

Age-structured Two-sex Population Model: Well-posedness

3.1. Formulation of the Model

In this chapter we consider the age-structured two-sex population model formulated by Fredrickson in [37] and further analyzed by Hoppensteadt in [50]. This model, in full generality, consists of the following PDEs.

$$
\begin{aligned}
&(3.1.1)\begin{cases}
\partial_t \mu_t^m + \partial_x \mu_t^m + \xi_m(t,\mu_t^m,\mu_t^f)\mu_t^m = 0, & (t,x)\in[0,T]\times\mathbb{R}_+,\\
D_\lambda \mu_t^m(0^+) = \int_{\mathbb{R}_+^2} b_m(t,\mu_t^m,\mu_t^f)(z)\,\mathrm{d}\mu_t^c(z), & \\
\mu_o^m \in \mathcal{M}^+(\mathbb{R}_+), &
\end{cases}\\
&(3.1.2)\begin{cases}
\partial_t \mu_t^f + \partial_y \mu_t^f + \xi_f(t,\mu_t^m,\mu_t^f)\mu_t^f = 0, & (t,y)\in[0,T]\times\mathbb{R}_+,\\
D_\lambda \mu_t^f(0^+) = \int_{\mathbb{R}_+^2} b_f(t,\mu_t^m,\mu_t^f)(z)\,\mathrm{d}\mu_t^c(z), & \\
\mu_o^f \in \mathcal{M}^+(\mathbb{R}_+), &
\end{cases}\\
&(3.1.3)\begin{cases}
\partial_t \mu_t^c + \partial_{z_1} \mu_t^c + \partial_{z_2}\mu_t^c + \xi_c(t,\mu_t^m,\mu_t^f,\mu_t^c)\mu_t^c = \mathcal{T}(t,\mu_t^m,\mu_t^f,\mu_t^c), \\
\qquad (t,z)\in[0,T]\times\mathbb{R}_+^2,\\
\mu_t^c(\{0\}\times B) = \mu_t^c(B\times\{0\}) = 0,\\
\mu_o^c \in \mathcal{M}^+(\mathbb{R}_+^2).
\end{cases}
\end{aligned}
\tag{3.1}
$$

Equations (3.1.1) and (3.1.2) describe the dynamic of males and females. Both equations are of McKendrick-type, that is they consist of a transport equation with a growth term and a boundary term, which determine the influx of new individuals. The evolution of couples is described by a similar equation, however a source term for couples is much more complicated than the analogous process for males and females. It cannot be expressed as a boundary condition, because new couples can be formed between the individuals being at different ages. Moreover, this term is nonlinear, since it takes into account interactions between both sexes. We set the Fredrickson–Hoppensteadt model in the space of finite, nonnegative Radon measures. Additionally, we introduce a nonlinear birth rate and death rate for males and females. We consider a nonlinear disappearance rate for couples as well. Let μ_t^m be the age distribution of males at time t and μ_t^f be the age distribution of females at time t. μ_t^c denotes the distribution of couples with respect to the age of each of the spouses. In particular, for any Borel sets $B_1, B_2 \subseteq \mathbb{R}_+$, $B_3 \subseteq \mathbb{R}_+^2$, $\mu_t^m(B_1)$ is the number of males being at the age $x \in B_1$ at time t, $\mu_t^f(B_2)$ is the number of females being at the age $y \in B_2$ at time t and $\mu_t^c(B_3)$ is the number of couples formed between males of the age x and females of the age y such that $(x,y) \in B_3$. One should be aware that μ_t^m, μ_t^f contain married and unmarried individuals. μ_t^m, μ_t^f are finite, nonnegative Radon measures on $\mathbb{R}_+$ and μ_t^c is such a measure on $\mathbb{R}_+^2$. Distribution of the single males and single females is given by measures s_t^m and s_t^f respectively. Formally, s_t^m and s_t^f are the measures on $\mathbb{R}_+$, such that for each Borel set $B \in \mathcal{B}(\mathbb{R}_+)$

$$s_t^i(B) = (\mu_t^i - \sigma_t^i)(B), \quad \text{for } i = m, f,$$

holds. Measures σ_t^m and σ_t^f are one dimensional projections of μ_t^c and describe the distribution of males and females respectively, who are in the marriage at time t. More precisely, for each Borel set $B \in \mathcal{B}(\mathbb{R}_+)$

$$\sigma_t^m(B) = \mu_t^c(B \times \mathbb{R}_+) \quad \text{and} \quad \sigma_t^f(B) = \mu_t^c(\mathbb{R}_+ \times B). \tag{3.2}$$

Functions ξ_m, ξ_f are death rates for males and females respectively. ξ_c is a rate of disappearance for couples, which incorporates the phenomenon of a divorce and death of one of the spouses as well. In particular,

$$(\xi_c(t,\mathbf{u}))(x,y) = (\xi_m(t,\mathbf{v}))(x) + (\xi_f(t,\mathbf{v}))(y) + (\delta(t,\mathbf{u}))(x,y),$$

where $\mathbf{u} = (\mu_t^m, \mu_t^f, \mu_t^c), \mathbf{v} = (\mu_t^m, \mu_t^f)$ and δ is a divorce rate. Functions b_m and b_f are birth rates for males and females respectively. A source term for couples is given by the operator $\mathcal{T}$. For any Borel set $B \in \mathcal{B}(\mathbb{R}_+^2)$, $\mathcal{T}(B)$ is a measure of the set containing couples formed between males being at the age x and females being at the age y, such that $(x,y) \in B$. Influx of the new individuals is described by nonlocal boundary conditions. More precisely, $D_\lambda \mu_t^m(0^+)$ and $D_\lambda \mu_t^f(0^+)$ are Radon-Nikodym derivatives of μ_t^m and μ_t^f respectively, with respect to the one dimensional Lebesgue measure λ at the point 0. From the assumption (3.5) on the birth rates b^m, b^f and the Definition 3.9 of solution it follows that in the region $D = \{(t,x) : t < x\}$ measures μ_t^m and μ_t^f are absolutely continuous with respect to the Lebesgue measure. However, if we used the notation $\mu_t^m(0^+)$, $\mu_t^f(0^+)$ instead of $D_\lambda \mu_t^m(0^+)$, $D_\lambda \mu_t^f(0^+)$ respectively, it would suggest that we keep in mind the measure of the point 0, which is zero. The aim of using the $D_\lambda \mu_t$ symbol is to underline that the boundary condition gives a value of a density of the measure μ_t with respect to the Lebesgue measure in a point 0.

Since the moment of its creation, this model has been analysed under various assumptions in many papers. In [47] symmetry of all parameter functions with respect to males and females is required. In [52] it is assumed that new individuals are produced only in the first marriage. A specific form of the marriage function (harmonic type) is postulated in [72]. Additionally, in [47, 52, 72] one extra structural variable, namely the age of the couple, is considered. A duration of the marriage can influence a probability of a divorce or birth of an offspring, but we do not consider this case in the present thesis in order to avoid greater complexity. Well-posedness of the Fredrickson–Hoppensteadt model in the class of $\mathbf{C^1}$ functions was established under biologically relevant conditions on the parameters in [61] (see also [51]). However, in [60] it has been shown that in a long time period exponential growth of the population is observed, which is not a realistic phenomenon. Due to this fact, environmental influences were taken into account, e.g., by introducing dependency of birth and death rates on the state of the whole population in [65].

This chapter is organized as follows. In Section 3.2 we state the main theorem of this chapter. Section 3.3 is devoted to a brief discussion on the marriage function. We give an example of the marriage function, which is reasonable from the biological point of view and satisfies the model assumptions. Section 3.4 contains results concerning a linear non-autonomous case. At first, we analyse the equation describing the evolution of couples, since it is independent on the other equations in this case. By a regularization technique and estimates for a dual equation we obtain existence, uniqueness and continuous dependence of solutions with respect to time, initial datum and model coefficients. Similarly we handle the equations describing males and females dynamics. The estimates for the linear non-autonomous case are crucial in passing to a nonlinear case. This subject is held in Section 3.5, where we prove the theorem for the nonlinear case via Banach fixed-point theorem.

3.2. Main Results

In this section section we present the main result of Chapter 3. Theorem (3.10) states that there exists a unique solution to (3.1), which is Lipschitz continuous as a mapping from $[0,T]$ to $(\mathcal{M}^+(\mathbb{R}_+),\rho_F)\times(\mathcal{M}^+(\mathbb{R}_+),\rho_F)\times(\mathcal{M}^+(\mathbb{R}_+^2),\rho_F)$. Moreover, we show stability of the solution with respect to the initial datum and model functions. Since the proof of the theorem is not straightforward, we deffer it to Section 2.6. Let define product spaces

$$\mathcal{U}=\mathcal{M}^+(\mathbb{R}_+)\times\mathcal{M}^+(\mathbb{R}_+)\times\mathcal{M}^+(\mathbb{R}_+^2),\qquad \mathcal{V}=\mathcal{M}^+(\mathbb{R}_+)\times\mathcal{M}^+(\mathbb{R}_+)$$

and equip $\mathcal{U}$ in the following metric

$$\mathbf{d}(\mathbf{u},\mathbf{v})=\rho_F(\mu_1,\nu_2)+\rho_F(\mu_2,\nu_2)+\rho_F(\mu_3,\nu_3)\quad \forall\mathbf{u}=(\mu_1,\mu_2,\mu_3),\mathbf{v}=(\nu_1,\nu_2,\nu_3)\in\mathcal{U}. \tag{3.3}$$

We assume that

$$\xi_m,\xi_f:[0,T]\times\mathcal{V}\to\mathbf{W^{1,\infty}}(\mathbb{R}_+),\qquad\qquad b_m,b_f:[0,T]\times\mathcal{V}\to\mathbf{W^{1,\infty}}(\mathbb{R}_+^2),$$

$$\xi_c,\delta:[0,T]\times\mathcal{U}\to\mathbf{W^{1,\infty}}(\mathbb{R}_+^2)\quad\text{and}\quad\mathcal{T}:[0,T]\times\mathcal{U}\to\mathcal{M}^+(\mathbb{R}_+^2).$$

and require the following regularity of the model functions

$$\xi_m,\xi_f\in\mathbf{C_b^{0,1}}([0,T]\times\mathcal{V};\ \mathbf{W^{1,\infty}}(\mathbb{R}_+)) \tag{3.4}$$

$$b_m, b_f \in \mathbf{C_b^{0,1}}([0,T]\times\mathcal{V};\ \mathbf{W^{1,\infty}}(\mathbb{R}_+^2)) \tag{3.5}$$

$$\xi_c, \delta \in \mathbf{C_b^{0,1}}([0,T]\times\mathcal{U};\ \mathbf{W^{1,\infty}}(\mathbb{R}_+^2)) \tag{3.6}$$

$$\mathcal{T} \in \mathbf{C_b^{0,1}}([0,T]\times\mathcal{U};\ \mathcal{M}^+(\mathbb{R}_+^2)). \tag{3.7}$$

Here, $\mathbf{C_b^{0,1}}([0,T]\times\mathcal{V};X)$ and $\mathbf{C_b^{0,1}}([0,T]\times\mathcal{U};X)$ are spaces of X valued functions, bounded with respect to the $\|\cdot\|_X$ norm, continuous with respect to time (the first superscript) and Lipschitz continuous with respect to the measure variable (the second superscript). The norm $\|\cdot\|_{\mathbf{C_b^{0,1}}}$ in the $\mathbf{C_b^{0,1}}$ space is defined as

$$\|f\|_{\mathbf{C_b^{0,1}}} = \|f\|_{\mathbf{BC_{t,\mu}}} + \mathbf{Lip}\,(f(t,\cdot)), \quad \text{where} \quad \|f\|_{\mathbf{BC_{t,\mu}}} = \sup_{(t,\mu)\in[0,T]\times Y} \|f(t,\mu)\|_X, \tag{3.8}$$

$Y = \mathcal{V}$ (for (3.4), (3.5)) or $Y = \mathcal{U}$ (for (3.6)) and $\mathbf{Lip}\,(f(t,\cdot))$ is the Lipschitz constant of $f(t,\cdot)$. Boundedness of the operator $\mathcal{T}$ with respect to the $\|\cdot\|_X$ norm is understood as the following. For any $t\in[0,T]$, $\mu_1,\mu_2 \in \mathcal{M}^+(\mathbb{R}_+)$ and $\mu_3 \in \mathcal{M}^+(\mathbb{R}_+^2)$ there exists a constant C, such that

$$\|\mathcal{T}(t,\mu_1,\mu_2,\mu_3)\|_{(\mathbf{W^{1,\infty}})^*} \le C\Big(\|\mu_1\|_{(\mathbf{W^{1,\infty}})^*} + \|\mu_2\|_{(\mathbf{W^{1,\infty}})^*} + \|\mu_3\|_{(\mathbf{W^{1,\infty}})^*}\Big).$$

The solution to (3.1) is defined as follows.

Definition 3.9. *A triple* $\mathbf{u} = (\mu^m,\mu^f,\mu^c)$*, such that* $\mathbf{u}:[0,T]\to\mathcal{U}$ *is a weak solution to the system* (3.1) *on the time interval* $[0,T]$*, if* μ^m,μ^f,μ^c *are narrowly continuous with respect to time and for all* $(\varphi_m,\varphi_f,\varphi_c)$ *such that* $\varphi_m,\varphi_f \in (\mathbf{C^1}\cap\mathbf{W^{1,\infty}})([0,T]\times\mathbb{R}_+)$ *and* $\varphi_c \in (\mathbf{C^1}\cap\mathbf{W^{1,\infty}})([0,T]\times\mathbb{R}_+^2)$*, the following equalities hold*

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}_+}\big(\partial_t\varphi_i(t,x)+\partial_x\varphi_i(t,x)-\xi_i(t,\mu_t^m,\mu_t^f)\varphi_i(t,x)\big)\,\mathrm{d}\mu_t^i(x)\mathrm{d}t\\
&+\int_0^T\varphi_i(t,0)\int_{\mathbb{R}_+^2} b_i(t,\mu_t^m,\mu_t^f)(z)\mathrm{d}\mu_t^c(z)\mathrm{d}t\\
&=\int_{\mathbb{R}_+}\varphi_i(T,x)\mathrm{d}\mu_T^i(x)-\int_{\mathbb{R}_+}\varphi_i(0,x)\mathrm{d}\mu_o^i(x)\quad \textit{for } i=f,m
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}_+^2}\big(\partial_t\varphi_c(t,z)+\partial_x\varphi_c(t,z)+\partial_y\varphi_c(t,z)-\xi_c(t,\mu_t^m,\mu_t^f,\mu_t^c)\varphi_c(t,z)\big)\,\mathrm{d}\mu_t^c(z)\mathrm{d}t\\
&+\int_0^T\int_{\mathbb{R}_+^2}\varphi_c(t,z)\mathrm{d}\mathcal{T}(t,\mu_t^m,\mu_t^f,\mu_t^c)(z)\mathrm{d}t\\
&=\int_{\mathbb{R}_+^2}\varphi_c(T,z)\mathrm{d}\mu_T^c(z)-\int_{\mathbb{R}_+^2}\varphi_c(0,z)\mathrm{d}\mu_o^c(z).
\end{aligned}$$

Theorem 3.10. *Let* $\mathbf{u}_o = (\mu_o^m,\mu_o^f,\mu_o^c)\in\mathcal{U}$ *and* (3.4) – (3.7) *hold. Then, there exists a unique solution* $\mathbf{u}:[0,T]\to\mathcal{U}$ *to the full nonlinear problem* (3.1)*. Moreover,*

i) for all $0\le t_1\le t_2\le T$ *there exist constants* K_1 *and* K_2*, such that*

$$\mathbf{d}\,(\mathbf{u}_{t_1},\mathbf{u}_{t_2}) \le K_1 e^{K_2 t_2}(t_2-t_1),$$

where K_1*,* K_2 *depend on all model coefficients and additionally* K_1 *depends on the initial datum.*

ii) Let $\tilde{\mathbf{u}}_o \in \mathcal{U}$ *and* $\tilde{b}_m$, $\tilde{b}_f$, $\tilde{\xi}_m$, $\tilde{\xi}_f$, $\tilde{\xi}_c$, $\tilde{\mathcal{T}}$ *satisfy assumptions* (3.4) – (3.7). *Let* $\tilde{\mathbf{u}}$ *solve* (3.1) *with initial datum* $\tilde{\mathbf{u}}_o$ *and coefficients* $\tilde{b}_m, \tilde{b}_f, \tilde{\xi}_m, \tilde{\xi}_f, \tilde{\xi}_c, \tilde{\mathcal{T}}$. *Then, there exist constants* K_1, K_2 *and* K_3 *such that for all* $t \in [0, T]$

$$\mathbf{d}\left(\mathbf{u}_t, \tilde{\mathbf{u}}_t\right) \le \mathrm{e}^{K_1 t} \mathbf{d}\left(\mathbf{u}_o, \tilde{\mathbf{u}}_o\right) + K_2\, t\, \mathrm{e}^{K_3 t} \cdot \|(b - \tilde{b}, \xi - \tilde{\xi}, \mathcal{T} - \tilde{\mathcal{T}})\|_{\mathbf{BC_{t,\mu}}},$$

where $(b - \tilde{b}) = (b_m - \tilde{b}_m, b_f - \tilde{b}_f)$, $(\xi - \tilde{\xi}) = (\xi_m - \tilde{\xi}_m, \xi_f - \tilde{\xi}_f, \xi_c - \tilde{\xi}_c)$ *and* K_1, K_2, K_3 *are constant dependent on the model coefficients.*

3.3. Marriage Function

Note that (3.1.1) and (3.1.2) have the analogous structure as the equation (2.2), which is the special case of (2.1). Therefore, in this section we focus on the ingredients of the equation (3.1.3), namely on the form of the operator $\mathcal{T}$ which describes the pair formation process. This operator is usually called the "marriage function" and in this thesis we follow this convention. The marriage function is a function, which provides the number of new marriages between males and females in the particular ages in a certain moment. The major difficulty in modeling marriages is defining an exact form of the marriage function. Several functions have been proposed, e.g., male dominance, female dominance, minimum function, geometric mean, harmonic mean (see [47, 48, 53, 61, 62, 72] and references therein for more details and examples), but none of the functions mentioned above can be rigorously derived from sociological data or a microscopic description of the marriage process. Even though we cannot point out the one marriage function which should be preferred over another, there are still some general properties accepted by most of researchers, i.e., *non-negativity, heterosexuality, homogeneity, consistency, monotonicity, competition* (see [51, Section 2.5], [52] for details). The property, which raises the most serious concerns is homogeneity. The homogeneity property states that if populations of males and females increase λ times, then the number of new marriages also increases λ times. It is intuitively clear that each individual has a limited number of contacts with other individuals. However, in populations which are dense enough this fact does not influence the marriage process, which the homogeneity property is also supposed to reflect. On the other hand, it is believed that the homogeneity assumption does not hold at low densities, when the time needed for finding an appropriate mate increases significantly. Also some rigorous derivations of the marriage function lead to non-homogeneous functions ([38]).

Remark 3.11. It is more convenient to formulate and analyse the model (3.1) for the whole populations of males and females. Therefore, instead of the commonly used marriage function $\mathcal{F}$ dependent on the single males and single females distributions we use the operator $\mathcal{T}$, such that

$$\mathcal{T}(t, \mu_t^m, \mu_t^f, \mu_t^c) = \mathcal{F}(t, \mu_t^m - \sigma_t^m, \mu_t^f - \sigma_t^f), \tag{3.12}$$

where σ_t^m and σ_t^f are given by (3.2).

One of the possible choices of the marriage function, which fits into our framework is the one used in [52], that is

$$\mathcal{F}(t,s_t^m,s_t^f)=\left(\frac{\theta(x,y)h(x)g(y)}{\gamma+\int_{\mathbb{R}_+}h(z)\mathrm{d}s_t^m(z)+\int_{\mathbb{R}_+}g(\omega)\mathrm{d}s_t^f(\omega)}\right)(s_t^m\otimes s_t^f), \tag{3.13}$$

where $(s_t^m\otimes s_t^f)$ is a product measure on $\mathbb{R}_+^2$. Therefore, we can include the homogeneous and non-homogeneous case assuming that $\gamma\in[0,1]$. In particular, for $\gamma=0$ and $s_t^m,s_t^f\in\mathcal{M}^+(\mathbb{R}_+)$ the marriage function is homogeneous of degree 1, i.e., it satisfies

$$\mathcal{F}(t,\gamma s_t^m,\gamma s_t^f)=\gamma\,\mathcal{F}(t,s_t^m,s_t^f).$$

$h,g\in(\mathbf{L^1}\cap\mathbf{W^{1,\infty}})(\mathbb{R}_+;\mathbb{R}_+)$ are Lipschitz functions describing preferences distributions. More precisely, h is a function describing a distribution of preferred males on the marriage market. Function h is independent on y, which reflects that the preferences do not depend on the age of a female. Although it is not a highly realistic assumption, we use it to simplify the analysis. Function g has the analogous meaning. Function $\theta(x,y)$ describes a marriage rate between a male of the age x and a female of the age y. In our case the marriage function $\mathcal{F}$ is defined as a product measure on $\mathbb{R}_+^2$. According to the formula (3.13)

$$\mathcal{F}(t,s_t^m,s_t^f)(B_m\times B_f)=0,$$

whenever $s_t^m(B_m)=0$ or $s_t^f(B_f)=0$, which ensures that the single males and single females distribution is a nonnegative measure for each time t.

Lemma 3.14. *Operator $\mathcal{T}$ defined by* (3.12) *satisfies assumption* (3.7).

For the proof of Lemma 3.14 we need the following

Lemma 3.15. *Let $\mu,\tilde{\mu}\in\mathcal{M}^+(\mathbb{R}_+^2)$ and $\sigma,\tilde{\sigma}\in\mathcal{M}^+(\mathbb{R}_+)$ be projections of measures $\mu,\tilde{\mu}$ on $\mathbb{R}_+$, respectively, defined as in* (3.2). *Then,*

$$\rho_F(\sigma,\tilde{\sigma})\le\rho_F(\mu,\tilde{\mu}).$$

Proof of Lemma 3.15. According to the Slicing Lemma [36, Section 1.5.2], there exists a Borel set N, such that $\mu(N)=0$ and for each $x\notin N$ there exists a Radon probability measure ν_x, such that

$$\int_{\mathbb{R}_+^2}f(x,y)\mathrm{d}\mu=\int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+}f(x,y)\mathrm{d}\nu_x(y)\right)\mathrm{d}\sigma(x),$$

for each measurable and μ–integrable function f. Define a set $\mathcal{Z}\subset\mathbf{W^{1,\infty}}(\mathbb{R}_+^2)$ as

$$\mathcal{Z}=\left\{f\in\mathbf{W^{1,\infty}}(\mathbb{R}_+^2)\;:\;f(x,y)=g(x),\;g\in\mathbf{W^{1,\infty}}(\mathbb{R}_+),\;\|g\|_{\infty,\mathbf{Lip}}\le1\right\}.$$

It is straightforward, that $\|f\|_{\infty,\mathbf{Lip}}\le1$, for each $f\in\mathcal{Z}$. Moreover,

$$\int_{\mathbb{R}_+^2}f(x,y)\mathrm{d}\mu(x,y)=\int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+}g(x)\mathrm{d}\nu_x(y)\right)\mathrm{d}\sigma(y)=\int_{\mathbb{R}_+}g(x)\mathrm{d}\sigma(x).$$

Analogous equality holds for $\tilde{\mu}$ and $\tilde{\sigma}$, which implies that

$$\sup_{f\in\mathcal{Z},\|f\|_{\infty,\mathbf{Lip}}\le1}\left\{\int_{\mathbb{R}_+^2}f(x,y)\mathrm{d}(\mu-\tilde{\mu})(x,y)\right\}=\sup_{g\in\mathbf{W^{1,\infty}},\|g\|_{\infty,\mathbf{Lip}}\le1}\int_{\mathbb{R}_+}g(x)\mathrm{d}(\sigma-\tilde{\sigma})(x).$$

According to (1.26), the left hand side is not greater than $\rho_F(\mu,\tilde{\mu})$ and the right hand side is equal to $\rho_F(\sigma,\tilde{\sigma})$, which ends the proof. □

Proof of Lemma 3.14. Let $t \in [0,T]$, $(\mu_t^m, \mu_t^f, \mu_t^c), (\nu_t^m, \nu_t^f, \nu_t^c) \in \mathcal{U}$ and σ_t^{m,μ_t^c}, σ_t^{f,μ_t^c}, σ_t^{m,ν_t^c}, σ_t^{f,ν_t^c} be measures defined as in (3.2), that is

$$\sigma_t^{m,\mu}(B) = \mu(B \times \mathbb{R}_+) \quad \text{and} \quad \sigma_t^{f,\mu}(B) = \mu(\mathbb{R}_+ \times B),$$

for any $\mu \in \mathcal{M}^+(\mathbb{R}_+^2)$ and Borel set $B \in \mathcal{B}(\mathbb{R}_+)$.

$$\begin{aligned}
&\|\mathcal{T}(t,\mu_t^m,\mu_t^f,\mu_t^c)\|_{(\mathbf{W^{1,\infty}})^*} \\
&\quad= \int_{\mathbb{R}_+^2} \frac{\theta(x,y)h(x)g(y)}{\gamma + \int_{\mathbb{R}_+} h(z)\mathrm{d}s_t^m(z) + \int_{\mathbb{R}_+} g(w)\mathrm{d}s_t^f(w)} \,\mathrm{d}(s_t^m \otimes s_t^f)(x,y) \\
&\quad\le \|\theta\|_{\mathbf{L}^\infty}\|h\|_{\mathbf{L}^\infty} \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{g(y)}{\int_{\mathbb{R}_+} g(w)\mathrm{d}s_t^f(w)} \,\mathrm{d}(s_t^m \otimes s_t^f)(x,y) \\
&\quad\le \|\theta\|_{\mathbf{L}^\infty}\|h\|_{\mathbf{L}^\infty} \int_{\mathbb{R}_+} \frac{\int_{\mathbb{R}_+} g(y)\mathrm{d}s_t^f(y)}{\int_{\mathbb{R}_+} g(w)\mathrm{d}s_t^f(w)} \,\mathrm{d}s_t^m(x) \le \|\theta\|_{\mathbf{L}^\infty}\|h\|_{\mathbf{L}^\infty} s_t^m(\mathbb{R}_+) \\
&\quad\le \|\theta\|_{\mathbf{L}^\infty}\|h\|_{\mathbf{L}^\infty} \left(\mu_t^m(\mathbb{R}_+) + \mu_t^c(\mathbb{R}_+^2)\right).
\end{aligned}$$

Similarly, we obtain

$$\|\mathcal{T}(t,\mu_t^m,\mu_t^f,\mu_t^c)\|_{(\mathbf{W^{1,\infty}})^*} \le \|\theta\|_{\mathbf{L}^\infty}\|g\|_{\mathbf{L}^\infty} \left(\mu_t^f(\mathbb{R}_+) + \mu_t^c(\mathbb{R}_+^2)\right).$$

Summarizing,

$$\|\mathcal{T}(t,\mu_t^m,\mu_t^f,\mu_t^c)\|_{(\mathbf{W^{1,\infty}})^*} \le \|\theta\|_{\mathbf{L}^\infty}\|(h,g)\|_{\mathbf{L}^\infty} \left(\|\mu_t^m\|_{(\mathbf{W^{1,\infty}})^*} + \|\mu_t^f\|_{(\mathbf{W^{1,\infty}})^*} + \|\mu_t^c\|_{(\mathbf{W^{1,\infty}})^*}\right).$$

To prove that $\mathcal{T}$ is Lipschitz continuous it is sufficient to show that

$$\mathcal{F}(t,\cdot,\cdot) \in \mathbf{Lip}(\mathcal{V};\ \mathcal{M}^+(\mathbb{R}_+^2)).$$

Indeed, if this holds, then

$$\begin{aligned}
&\rho_F\left(\mathcal{T}(t,\mu_t^m,\mu_t^f,\mu_t^c), \mathcal{T}(t,\nu_t^m,\nu_t^f,\nu_t^c)\right) \\
&\quad= \rho_F\left(\mathcal{F}\left(t,\mu_t^m - \sigma_t^{m,\mu_t^c}, \mu_t^f - \sigma_t^{f,\mu_t^c}\right), \mathcal{F}\left(t,\nu_t^m - \sigma_t^{m,\nu_t^c}, \nu_t^f - \sigma_t^{f,\nu_t^c}\right)\right) \\
&\quad\le \mathbf{Lip}(\mathcal{F}(t))\left(\rho_F\left(\mu_t^m - \sigma_t^{m,\mu_t^c}, \nu_t^m - \sigma_t^{m,\nu_t^c}\right) + \rho_F\left(\mu_t^f - \sigma_t^{f,\mu_t^c}, \nu_t^f - \sigma_t^{f,\nu_t^c}\right)\right) \\
&\quad\le \mathbf{Lip}(\mathcal{F}(t))\left(\rho_F(\mu_t^m,\nu_t^m) + \rho_F(\sigma_t^{m,\mu_t^c},\sigma_t^{m,\nu_t^c}) + \rho_F(\mu_t^f,\nu_t^f) + \rho_F(\sigma_t^{f,\mu_t^c},\sigma_t^{f,\nu_t^c})\right) \\
&\quad\le 2\mathbf{Lip}(\mathcal{F}(t))\left(\rho_F(\mu_t^m,\nu_t^m) + \rho_F(\mu_t^f,\nu_t^f) + \rho_F(\mu_t^c,\nu_t^c)\right),
\end{aligned}$$

where the last inequality holds due to Lemma 3.15. Let (s_t^m, s_t^f), $(\tilde{s}_t^m, \tilde{s}_t^f) \in \mathcal{V}$ and define

$$Z = \int_{\mathbb{R}_+} h(z)\mathrm{d}s_t^m(z), \quad \tilde{Z} = \int_{\mathbb{R}_+} h(z)\mathrm{d}\tilde{s}_t^m(z), \quad W = \int_{\mathbb{R}_+} g(w)\mathrm{d}s_t^f(w).$$

Let $\varphi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+^2;\mathbb{R}_+)$ be such that $\|\varphi\|_{\infty,\mathbf{Lip}} \le 1$. Adding and subtracting the following term

$$\int_{\mathbb{R}_+^2} \varphi(x,y) \frac{\theta(x,y)h(x)g(y)}{\gamma + \int_{\mathbb{R}_+} h(z)\mathrm{d}s_t^m(z) + \int_{\mathbb{R}_+} g(w)\mathrm{d}s_t^f(w)} \,\mathrm{d}(\tilde{s}_t^m \otimes s_t^f)(x,y)$$

to the expression $\int_{\mathbb{R}_+^2} \varphi \mathrm{d}\left(\mathcal{F}(t, s_t^m, s_t^f) - \mathcal{F}(t, \tilde{s}_t^m, s_t^f)\right)$ yieds

$$
\begin{aligned}
&\int_{\mathbb{R}_+^2} \varphi(x,y) \mathrm{d}\left(\mathcal{F}(t, s_t^m, s_t^f) - \mathcal{F}(t, \tilde{s}_t^m, s_t^f)\right)(x,y) \\
&\quad = \int_{\mathbb{R}_+^2} \varphi(x,y) \, \frac{\theta(x,y)h(x)g(y)}{\gamma + Z + W} \, \mathrm{d}(s_t^m \otimes s_t^f - \tilde{s}_t^m \otimes s_t^f)(x,y) \\
&\qquad + \int_{\mathbb{R}_+^2} \varphi(x,y)\theta(x,y)h(x)g(y) \left(\frac{1}{\gamma + Z + W} - \frac{1}{\gamma + \tilde{Z} + W} \right) \mathrm{d}(\tilde{s}_t^m \otimes s_t^f)(x,y) \\
&\quad = \int_{\mathbb{R}_+} h(x) \, \frac{\int_{\mathbb{R}_+} \varphi(x,y)\theta(x,y)g(y)\mathrm{d}s_t^f(y)}{\gamma + Z + W} \, \mathrm{d}(s_t^m - \tilde{s}_t^m)(x) \\
&\qquad - \int_{\mathbb{R}_+} h(x) \, \frac{\int_{\mathbb{R}_+} h(z) \left(\int_{\mathbb{R}_+} \varphi(z,y)\theta(z,y)g(y)\mathrm{d}s_t^f(y) \right) \mathrm{d}\tilde{s}_t^m(z)}{(\gamma + Z + W)(\gamma + \tilde{Z} + W)} \, \mathrm{d}(s_t^m - \tilde{s}_t^m)(x) \\
&\quad = \int_{\mathbb{R}_+} h(x)\Phi(x)\mathrm{d}(s_t^m - \tilde{s}_t^m)(x),
\end{aligned}
$$

where

$$
\Phi(x) = \frac{\int_{\mathbb{R}_+} \varphi(x,y)\theta(x,y)g(y)\mathrm{d}s_t^f(y)}{\gamma + Z + W} - \frac{\int_{\mathbb{R}_+} h(z) \left(\int_{\mathbb{R}_+} \varphi(z,y)\theta(z,y)g(y)\mathrm{d}s_t^f(y) \right) \mathrm{d}\tilde{s}_t^m(z)}{(\gamma + Z + W)(\gamma + \tilde{Z} + W)}.
$$

One can easily check that

$$
\|\Phi\|_{\mathbf{L}^\infty} \leq 2\|\varphi\|_{\mathbf{L}^\infty}\|\theta\|_{\mathbf{L}^\infty} \quad \text{and} \quad \|\Phi'\|_{\mathbf{L}^\infty} \leq 2\|\varphi\|_{\infty,\mathbf{Lip}}\|\theta\|_{\infty,\mathbf{Lip}}.
$$

Therefore, it holds that $h\Phi \in \mathbf{W}^{1,\infty}(\mathbb{R}_+)$ and

$$
\begin{aligned}
\|h \cdot \Phi\|_{\mathbf{L}^\infty} &\leq 2\|\varphi\|_{\mathbf{L}^\infty}\|h\|_{\mathbf{L}^\infty}\|\theta\|_{\mathbf{L}^\infty} \\
\|(h \cdot \Phi)'\|_{\mathbf{L}^\infty} &\leq 4\|\varphi\|_{\infty,\mathbf{Lip}} \, \|h\|_{\infty,\mathbf{Lip}} \, \|\theta\|_{\infty,\mathbf{Lip}}.
\end{aligned}
$$

By (1.26) we conclude that

$$
\rho_F\left(\mathcal{F}(t, s_t^m, s_t^f), \mathcal{F}(t, \tilde{s}_t^m, s_t^f)\right) \leq 4\|h\|_{\infty,\mathbf{Lip}} \, \|\theta\|_{\infty,\mathbf{Lip}} \, \rho_F(s_t^m, \tilde{s}_t^m).
$$

Analogous arguments lead to the inequality

$$
\rho_F\left(\mathcal{F}(t, \tilde{s}_t^m, s_t^f), \mathcal{F}(t, \tilde{s}_t^m, \tilde{s}_t^f)\right) \leq 4\|g\|_{\infty,\mathbf{Lip}} \, \|\theta\|_{\infty,\mathbf{Lip}} \, \rho_F(s_t^f, \tilde{s}_t^f).
$$

Therefore,

$$
\rho_F\left(\mathcal{F}(t, s_t^m, s_t^f), \mathcal{F}(t, \tilde{s}_t^m, \tilde{s}_t^f)\right) \leq 4\|(h,g)\|_{\infty,\mathbf{Lip}}\|\theta\|_{\infty,\mathbf{Lip}} \left(\rho_F(s_t^m, \tilde{\mu}_t^m) + \rho_F(s_t^f, \tilde{s}_t^f)\right).
$$

□

3.4. Linear Non-autonomous Case

In this section we consider a non-autonomous version of (3.1), that is

$$
\begin{aligned}
&(3.16.1)\begin{cases}
\partial_t \mu_t^m + \partial_x \mu_t^m + \xi_m(t,x)\mu_t^m &= 0, \qquad (t,x) \in [0,T]\times\mathbb{R}_+,\\
D_\lambda \mu_t^m(0^+) &= \int_{\mathbb{R}_+^2} b_m(t,z)\mathrm{d}\mu_t^c(z),\\
\mu_o^m &\in \mathcal{M}^+(\mathbb{R}_+),
\end{cases}\\
&(3.16.2)\begin{cases}
\partial_t \mu_t^f + \partial_y \mu_t^f + \xi_f(t,y)\mu_t^f &= 0, \qquad (t,y) \in [0,T]\times\mathbb{R}_+,\\
D_\lambda \mu_t^f(0^+) &= \int_{\mathbb{R}_+^2} b_f(t,z)\mathrm{d}\mu_t^c(z),\\
\mu_o^f &\in \mathcal{M}^+(\mathbb{R}_+),
\end{cases}\\
&(3.16.3)\begin{cases}
\partial_t \mu_t^c + \partial_{z_1} \mu_t^c + \partial_{z_2} \mu_t^c + \xi_c(t,z)\mu_t^c &= \mathcal{T}(t), \quad (t,z) \in [0,T]\times\mathbb{R}_+^2,\\
\mu_t^c(\{0\}\times B) &= \mu_t^c(B\times\{0\}) = 0,\\
\mu_o^c &\in \mathcal{M}^+(\mathbb{R}_+^2).
\end{cases}
\end{aligned}
\tag{3.16}
$$

We assume that

$$\xi_m, \xi_f \in \mathbf{C_b}([0,T]\times\mathcal{V};\ \mathbf{W^{1,\infty}}(\mathbb{R}_+)) \tag{3.17}$$

$$b_m, b_f \in \mathbf{C_b}([0,T]\times\mathcal{V};\ \mathbf{W^{1,\infty}}(\mathbb{R}_+^2)) \tag{3.18}$$

$$\xi_c, \delta \in \mathbf{C_b}([0,T]\times\mathcal{U};\ \mathbf{W^{1,\infty}}(\mathbb{R}_+^2)) \tag{3.19}$$

$$\mathcal{T} \in \mathbf{C_b}([0,T]\times\mathcal{U};\ \mathcal{M}^+(\mathbb{R}_+^2)). \tag{3.20}$$

The space $\mathbf{C_b}\left([0,T];X\right)$ consists of continuous, X valued functions bounded with respect to the norm

$$\|f\|_{\mathbf{BC_t}} = \sup_{t\in[0,T]} \|f(t)\|_X.$$

To simplify the notation let define the auxiliary function

$$F^{i,\nu}(t) = \int_{\mathbb{R}_+^2} b_i(t,z)d\nu(z),$$

where the second superscript ν will be omitted if no ambiguity occurs. To deal with the non-autonomous system (3.16) we begin with the analysis of (3.16.3), which is independent on the equations (3.16.1) and (3.16.2). A convenient way to deal with (3.16.3) relies on its dual formulation, that is

$$
\begin{cases}
\partial_t \varphi(t,z) + D_z\varphi(t,z) - \xi_c(t,z)\varphi(t,z) &= 0, \qquad (t,z)\in[0,T]\times\mathbb{R}_+^2,\\
\varphi(T,z) &= \psi(z),
\end{cases}
\tag{3.21}
$$

where $D_z\varphi = (\partial_{z_1}\varphi + \partial_{z_2}\varphi)$ and $\psi \in (\mathbf{C^1}\cap\mathbf{W^{1,\infty}})(\mathbb{R}_+^2;\mathbb{R})$. Function $\varphi_{T,\psi} \in \mathbf{C^1}([0,T]\times\mathbb{R}_+^2)$ is a solution to the dual problem to (3.16.3), if it satisfies (3.21) in the classical strong sense. In the following Lemma we present some results concerning (3.21).

Lemma 3.22. *For all* $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}^2_+)$ *there exists a unique solution to* (3.21). *Moreover, the following estimates hold:*

$$\|\varphi_{T,\psi}(t,\cdot)\|_{\mathbf{L}^\infty} \le \|\psi\|_{\mathbf{L}^\infty} \mathrm{e}^{\|\xi_c\|_{\mathbf{BC_t}}(T-t)}, \tag{3.23}$$

$$\mathbf{Lip}\,(\varphi_{T,\psi}(t,\cdot)) \le \|\psi\|_{\infty,\mathbf{Lip}} \mathrm{e}^{2\|\xi_c\|_{\mathbf{BC_t}}(T-t)}, \tag{3.24}$$

$$\sup_{s\in[t,T]} \|\partial_s \varphi_{T,\psi}(s,\cdot)\|_{\mathbf{L}^\infty} \le \|\psi\|_{\mathbf{L}^\infty} \left(1 + \|\xi_c\|_{\mathbf{BC_t}}\right) \mathrm{e}^{(1+\|\xi_c\|_{\mathbf{BC_t}})(T-t)}. \tag{3.25}$$

If moreover $\tilde\varphi$ *solves* (3.21) *with terminal data* ψ *and parameter* $\tilde\xi_c$, *then*

$$\|\varphi_{T,\psi}(t,\cdot) - \tilde\varphi_{T,\psi}(t,\cdot)\|_{\mathbf{L}^\infty} \le \|\psi\|_{\mathbf{L}^\infty} \|\xi_c - \tilde\xi_c\|_{\mathbf{BC_t}} (T-t) \mathrm{e}^{2\left(1+\|(\xi_c,\tilde\xi_c)\|_{\mathbf{BC_t}}\right)(T-t)}. \tag{3.26}$$

Proof of Lemma 3.22. Let $z = (x,y) \in \mathbb{R}^2_+$, $\bar\psi(t,x,y) = \psi(x+T-t, y+T-t)$ and $\bar\xi_c(s,t,x,y) = \xi_c(s, x+s-t, y+s-t)$. We check that the function

$$\begin{aligned} \varphi(t,x,y) &= \psi(x+T-t,y+T-t)\exp\left(-\int_t^T \xi_c(s,x+s-t,y+s-t)\mathrm{d}s\right) \\ &= \bar\psi(t,x,y)\exp\left(\int_T^t \bar\xi_c(s,t,x,y)\mathrm{d}s\right) \end{aligned} \tag{3.27}$$

is a solution to (3.21). To this end let us calculate the derivatives of φ.

$$\begin{aligned} \partial_t\varphi(t,x,y) &= \partial_t\bar\psi(t,x,y)\exp\left(\int_T^t \bar\xi_c(s,t,x,y)\mathrm{d}s\right) \\ &\quad + \bar\psi(t,x,y)\exp\left(\int_T^t \bar\xi_c(s,t,x,y)\mathrm{d}s\right)\left(\int_T^t \partial_t\bar\xi_c(s,t,x,y)\mathrm{d}s + \bar\xi_c(t,t,x,y)\right) \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} \partial_x\varphi(t,x,y) &= \partial_x\bar\psi(t,x,y)\exp\left(\int_T^t \bar\xi_c(s,t,x,y)\mathrm{d}s\right) \\ &\quad + \bar\psi(t,x,y)\exp\left(\int_T^t \bar\xi_c(s,t,x,y)\mathrm{d}s\right)\int_T^t \partial_x\bar\xi_c(s,t,x,y)\mathrm{d}s. \end{aligned} \tag{3.29}$$

Derivative $\partial_y\varphi(t,x,y)$ is analogous to (3.29). From the definition of $\bar\psi$ and $\bar\xi_c$ it follows that

$$\begin{aligned} \partial_t\bar\psi(t,x,y) &= -\partial_x\psi(x+T-t,y+T-t) - \partial_y\psi(x+T-t,y+T-t) \\ &= -\partial_x\bar\psi(t,x,y) - \partial_y\bar\psi(t,x,y), \end{aligned} \tag{3.30}$$

$$\begin{aligned} \partial_t\bar\xi_c(s,t,x,y) &= -\partial_x\xi_c(s,x+s-t,y+s-t) - \partial_y\xi_c(s,x+s-t,y+s-t) \\ &= -\partial_x\bar\xi_c(s,t,x,y) - \partial_y\bar\xi_c(s,t,x,y). \end{aligned} \tag{3.31}$$

Using (3.27) – (3.31) it is straightforward that

$$\partial_t\varphi(t,x,y) + \partial_x\varphi(t,x,y) + \partial_y\varphi(t,x,y) - \xi_c(t,x,y)\varphi(t,x,y) = 0.$$

The estimate (3.23) follows directly from the definition (3.27) of the function. To prove (3.24), let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}_+^2$ and estimate the following difference.

$$
\begin{aligned}
&|\varphi(t,x_1,y_1) - \varphi(t,x_2,y_2)| \\
&\quad \leq |\varphi(t,x_1,y_1) - \varphi(t,x_2,y_1)| + |\varphi(t,x_2,y_1) - \varphi(t,x_2,y_2)| \\
&\quad \leq \left(\sup_{x\in\mathbb{R}_+} |\partial_x \varphi(t,x,y_1)| + \sup_{y\in\mathbb{R}_+} |\partial_y \varphi(t,x_2,y)|\right)(|x_1-x_2| + |y_1-y_2|) \\
&\quad = \mathbf{Lip}(\varphi(t,\cdot,\cdot))\,(|x_1-x_2| + |y_1-y_2|)\,,
\end{aligned}
$$

where $\mathbf{Lip}(\varphi(t,\cdot,\cdot)) = \sup_{x\in\mathbb{R}_+} |\partial_x\varphi(t,x,y_1)| + \sup_{y\in\mathbb{R}_+} |\partial_y \varphi(t,x_2,y)|$. By (3.29) and the analogous formula on $\partial_y\varphi$

$$
\begin{aligned}
\mathbf{Lip}(\varphi(t,\cdot,\cdot)) &\leq \|\partial_x\psi\|_{\mathbf{L}^\infty} \mathrm{e}^{\|\xi_c\|_{\mathbf{BC_t}}(T-t)} + \|\psi\|_{\mathbf{L}^\infty} \mathrm{e}^{\|\xi_c\|_{\mathbf{BC_t}}(T-t)}(T-t) \sup_{v\in[t,T]} |\partial_x \xi_c(v,\cdot,\cdot)| \\
&\quad + \|\partial_y\psi\|_{\mathbf{L}^\infty} \mathrm{e}^{\|\xi_c\|_{\mathbf{BC_t}}(T-t)} + \|\psi\|_{\mathbf{L}^\infty} \mathrm{e}^{\|\xi_c\|_{\mathbf{BC_t}}(T-t)}(T-t) \sup_{v\in[t,T]} |\partial_y \xi_c(v,\cdot,\cdot)| \\
&\leq \|\psi\|_{\infty,\mathbf{Lip}} \mathrm{e}^{\|\xi_c\|_{\mathbf{BC_t}}(T-t)} + \|\psi\|_{\infty,\mathbf{Lip}} \mathrm{e}^{\|\xi_c\|_{\mathbf{BC_t}}(T-t)}(T-t) \sup_{v\in[t,T]} \|\xi_c(v,\cdot,\cdot)\|_{\mathbf{W^{1,\infty}}} \\
&\leq \|\psi\|_{\infty,\mathbf{Lip}} \mathrm{e}^{\|\xi_c\|_{\mathbf{BC_t}}(T-t)} \left(1 + \|\xi_c\|_{\mathbf{BC_t}}(T-t)\right) \leq \|\psi\|_{\infty,\mathbf{Lip}} \mathrm{e}^{2\|\xi_c\|_{\mathbf{BC_t}}(T-t)}.
\end{aligned}
$$

The estimate on $\partial_t\varphi$ follows from (3.28), (3.30) and (3.31).

$$
\begin{aligned}
|\partial_t\varphi(t,x,y)| &\leq \left(\|\partial_x\psi\|_{\mathbf{L}^\infty} + \|\partial_y\psi\|_{\mathbf{L}^\infty}\right) \mathrm{e}^{\|\xi_c\|_{\mathbf{BC_t}}(T-t)} \\
&\quad + \|\psi\|_{\mathbf{L}^\infty} \mathrm{e}^{\|\xi_c\|_{\mathbf{BC_t}}(T-t)}(T-t)\left(\sup_{v\in[t,T]} |\partial_x\xi_c(v,\cdot,\cdot)| + \sup_{v\in[t,T]} |\partial_y \xi_c(v,\cdot,\cdot)|\right) \\
&\quad + \|\psi\|_{\mathbf{L}^\infty} \mathrm{e}^{\|\xi_c\|_{\mathbf{BC_t}}(T-t)} \|\xi_c\|_{\mathbf{BC_t}} \\
&\leq \|\psi\|_{\infty,\mathbf{Lip}} \mathrm{e}^{\|\xi_c\|_{\mathbf{BC_t}}(T-t)} + \|\psi\|_{\infty,\mathbf{Lip}} \mathrm{e}^{\|\xi_c\|_{\mathbf{BC_t}}(T-t)} \left((T-t)\|\xi_c\|_{\mathbf{BC_t}} + \|\xi_c\|_{\mathbf{BC_t}}\right) \\
&\leq \|\psi\|_{\infty,\mathbf{Lip}} \mathrm{e}^{\|\xi_c\|_{\mathbf{BC_t}}(T-t)} \left(1 + (T-t)\|\xi_c\|_{\mathbf{BC_t}} + \|\xi_c\|_{\mathbf{BC_t}}\right) \\
&\leq \|\psi\|_{\infty,\mathbf{Lip}} \mathrm{e}^{\|\xi_c\|_{\mathbf{BC_t}}(T-t)} \left(1 + \|\xi_c\|_{\mathbf{BC_t}} \mathrm{e}^{(T-t)}\right) \leq \|\psi\|_{\infty,\mathbf{Lip}} \mathrm{e}^{(1+\|\xi_c\|_{\mathbf{BC_t}})(T-t)} \left(1 + \|\xi_c\|_{\mathbf{BC_t}}\right).
\end{aligned}
$$

Let $\tilde{\varphi}$ be a solution to (3.21) with terminal data ψ and coefficient $\tilde{\xi}_c$. It follows from (3.27) that

$$
\begin{aligned}
|\varphi(t,x,y) - \tilde{\varphi}(t,x,y)| &\leq \|\psi\|_{\mathbf{L}^\infty} \cdot \left| \mathrm{e}^{\int_T^t \xi_c(s,x+s-t,y+s-t)\mathrm{d}s} - \mathrm{e}^{\int_T^t \tilde{\xi}_c(s,x+s-t,y+s-t)\mathrm{d}s} \right| \\
&\leq \|\psi\|_{\mathbf{L}^\infty} \mathrm{e}^{\|\xi_c\|_{\mathbf{BC_t}}(T-t)} \cdot \left| 1 - \exp\left(\int_T^t (\tilde{\xi}_c - \xi_c)(s,x+s-t,y+s-t)\mathrm{d}s\right)\right|.
\end{aligned}
\tag{3.32}
$$

Let us define a function

$$
g : [0,T] \to \mathbb{R}, \quad g(t) = \exp\left(\int_T^t (\tilde{\xi}_c - \xi_c)(s,x+s-t,y+s-t)\right)\mathrm{d}s.
$$

The latter term of the inequality (3.32) is equal to $|g(T) - g(t)|$. Since g is regular enough, it holds that $|g(T) - g(t)| \le \sup_{v\in[t,T]} |g'(v)|(T-t)$. Thus, we estimate the derivative of g.

$$
\begin{aligned}
g'(v) = \exp&\left(\int_T^v (\tilde{\xi}_c - \xi_c)(s, x+s-v, y+s-v)\right) \\
&\cdot\left(-\int_T^v (\partial_x + \partial_y)(\tilde{\xi}_c - \xi_c)(s, x+s-v, y+s-v)\mathrm{d}s + (\tilde{\xi}_c - \xi_c)(v, x, y)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\sup_{v\in[t,T]} |g'(v)| &\le \mathrm{e}^{\|\tilde{\xi}_c - \xi_c\|_{\mathbf{BC_t}}(T-t)} \left(\|\tilde{\xi}_c - \xi_c\|_{\mathbf{BC_t}}(T-t) + \|\tilde{\xi}_c - \xi_c\|_{\mathbf{BC_t}}\right) \\
&\le \|\xi_c - \tilde{\xi}_c\|_{\mathbf{BC_t}} \mathrm{e}^{\|\xi_c - \tilde{\xi}_c\|_{\mathbf{BC_t}}(T-t)} \left((T-t) + 1\right) \le \|\xi_c - \tilde{\xi}_c\|_{\mathbf{BC_t}} \mathrm{e}^{\left(1 + \|\xi_c - \tilde{\xi}_c\|_{\mathbf{BC_t}}\right)(T-t)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|\varphi(t,\cdot,\cdot) - \tilde{\varphi}(t,\cdot,\cdot)\|_{\mathbf{L^\infty}} &\le \|\psi\|_{\mathbf{L^\infty}} \mathrm{e}^{\|\xi_c\|_{\mathbf{BC_t}}(T-t)} \cdot \|\xi_c - \tilde{\xi}_c\|_{\mathbf{BC_t}} \mathrm{e}^{\left(1 + \|\xi_c - \tilde{\xi}_c\|_{\mathbf{BC_t}}\right)(T-t)}(T-t) \\
&\le \|\psi\|_{\mathbf{L^\infty}} \|\xi_c - \tilde{\xi}_c\|_{\mathbf{BC_t}} (T-t) \mathrm{e}^{2\left(1 + \|\xi_c\|_{\mathbf{BC_t}} + \|\tilde{\xi}_c\|_{\mathbf{BC_t}}\right)(T-t)}.
\end{aligned}
$$

□

The relation between (3.16.3) and (3.21) is explained in the following Lemma.

Lemma 3.33. *Fix* $\mu_o^c \in \mathcal{M}^+(\mathbb{R}_+^2)$ *and let* ξ_c, $\mathcal{T}$ *satisfy assumptions* (3.19), (3.20). *Then:*

i) Problem (3.16.3) admits a unique solution $\mu^c \in \mathbf{Lip}([0,T], \mathcal{M}^+(\mathbb{R}_+^2))$, *that is for all* $0 \le t_1 \le t_2 \le T$,

$$\rho_F\left(\mu_{t_2}^c, \mu_{t_1}^c\right) \le \max\{1, \mu_o^c(\mathbb{R}_+)\} \cdot K\mathrm{e}^{2Kt_2}(t_2 - t_1),$$

where $K = \left(1 + \|(\xi_c, \mathcal{T})\|_{\mathbf{BC_t}}\right)$.

ii) Let μ^c *and* ν^c *be solutions to (3.16.3) with initial datum* μ_o^c *and* ν_o^c *respectively. Then,*

$$\rho_F(\mu_t^c, \nu_t^c) \le \mathrm{e}^{2\|\xi_c\|_{\mathbf{BC_t}} t} \rho_F(\mu_o^c, \nu_o^c).$$

iii) Let $\tilde{\xi}_c, \tilde{\mathcal{T}}$ *satisfy assumptions* (3.19), (3.20) *and* $\tilde{\mu}_t^c$ *be a solution to (3.16.3) with initial datum* μ_o^c *and coefficients* $\tilde{\xi}_c, \tilde{\mathcal{T}}$. *Then,*

$$\rho_F\left(\mu_t^c, \tilde{\mu}_t^c\right) \le t\ \max\{1, \mu_o^c(\mathbb{R}_+^2)\}\ \|(\xi_c - \tilde{\xi}_c, \mathcal{T} - \tilde{\mathcal{T}})\|_{\mathbf{BC_t}} \mathrm{e}^{Kt},$$

where $K = 2\left(1 + \|(\xi_c, \tilde{\xi}_c, \mathcal{T})\|_{\mathbf{BC_t}}\right)$.

iv) Fix $t_1, t_2 \in [0,T]$ *with* $t_1 < t_2$. *If* $\mu^c \in \mathbf{Lip}\left([0,T]; \mathcal{M}^+(\mathbb{R}_+)\right)$ *solves (3.16.3), then for any* $\varphi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})([t_1,t_2] \times \mathbb{R}_+^2)$ *we have*

$$
\begin{aligned}
&\int_{t_1}^{t_2} \left(\int_{\mathbb{R}_+^2} \left(\partial_t\varphi(t,z) + D_z\varphi(t,z) - \xi_c(t,z)\varphi(t,z)\right) \mathrm{d}\mu_t^c(z) + \int_{\mathbb{R}_+^2} \varphi(t,z)\mathrm{d}[\mathcal{T}(t)](z)\right)\mathrm{d}t \\
&= \int_{\mathbb{R}_+^2} \varphi(t_2,z)\mathrm{d}\mu_{t_2}^c(z) - \int_{\mathbb{R}_+^2} \varphi(t_1,z)\mathrm{d}\mu_{t_1}^c(z).
\end{aligned}
\tag{3.34}
$$

v) If $\mu^c \in \mathbf{Lip}([0,T];\mathcal{M}^+(\mathbb{R}^2_+))$ *solves (3.16.3), then for any* $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}^2_+)$ *there exists a function* $\varphi_{T,\psi} \in \mathbf{C^1}([0,T]\times\mathbb{R}^2_+;\mathbb{R})$ *solving the dual problem* (3.21) *and such that*

$$\int_{\mathbb{R}^2_+} \psi(z)\mathrm{d}\mu^c_t(z) = \int_{\mathbb{R}^2_+} \varphi_{T,\psi}(T-t,z)\mathrm{d}\mu^c_o(z) + \int_0^t \int_{\mathbb{R}^2_+} \varphi_{T,\psi}(s+(T-t),z)\mathrm{d}[\mathcal{T}(s)](z)\mathrm{d}s. \tag{3.35}$$

vi) For any $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}^2_+;\mathbb{R})$ *let* $\varphi_{T,\psi} \in \mathbf{C^1}([0,T]\times\mathbb{R}^2_+)$ *solves the dual problem* (3.21)*. Then the measure defined by* (3.35) *solves (3.16.3).*

Proof of Lemma 3.33. i) We shall prove that problem (3.16.3) admits the unique solution. The proof is based on a regularization technique. More precisely, we regularize the initial datum μ^c_o and the coeffcient $\mathcal{T}$. This leads to a standard problem that can be solved by the method of characteristics. Then we show a convergence of the sequence of regularized solutions and prove that the limit is a solution to (3.16.3) in the sense of Definition 3.9. Let $\rho \in \mathbf{C^\infty_c}(\mathbb{R}^2;\mathbb{R}_+)$ be such that $\int_{\mathbb{R}^2}\rho(z)\mathrm{d}z = 1$ and the support of ρ is contained in the unit ball centered at the origin. For $\varepsilon > 0$ define the family of mollifiers $\rho^\varepsilon(z) = \rho(z/\varepsilon)/\varepsilon$. We define the covolution $*$ as

$$(\nu * \rho)(z) = \int_{\mathbb{R}^2_+} \rho\left((z-\boldsymbol{\varepsilon})-\zeta\right)\mathrm{d}\nu(\zeta), \quad \text{where} \quad \boldsymbol{\varepsilon} = (\varepsilon,\varepsilon).$$

The reason why we shifted ρ by $\boldsymbol{\varepsilon}$ is that $\operatorname{supp}\left((\nu \star \rho)(z) = \int_{\mathbb{R}^2}\rho(z-\zeta)\mathrm{d}\nu(\zeta)\right) \subseteq [-\varepsilon,+\infty)^2$, where $\star$ is a standard convolution. We consider (3.16.3) with initial datum $u^{c,\varepsilon}_o$ and coefficient $\mathcal{T}^\varepsilon$, where

$$\begin{aligned} u^{c,\varepsilon}_o =& (\mu^c_o * \rho^\varepsilon) \in (\mathbf{BC}^\infty \cap \mathbf{L^1})(\mathbb{R}^2_+;\mathbb{R}_+) \\ \mathcal{T}^\varepsilon(t) =& (\mathcal{T}(t) * \rho^\varepsilon) \in (\mathbf{BC}^\infty \cap \mathbf{L^1})(\mathbb{R}^2_+;\mathbb{R}_+). \end{aligned}$$

Due to assumption on $\mathcal{T}$, it holds that $\mathcal{T}^\varepsilon \in \mathbf{BC}\left([0,T];(\mathbf{BC}^\infty \cap \mathbf{L^1})(\mathbb{R}^2;\mathbb{R}_+)\right)$. It can be shown that

$$\|\mathcal{T}^\varepsilon\|_{\mathbf{BC_t}} \le \|\mathcal{T}\|_{\mathbf{BC_t}}, \quad \rho_F(\mu^c_o, u^{c,\varepsilon}_o) \xrightarrow{\varepsilon\to0} 0 \quad \text{and} \quad \sup_{t\in[0,t]} \rho_F(\mathcal{T}(t),\mathcal{T}^\varepsilon(t)) \xrightarrow{\varepsilon\to0} 0. \tag{3.36}$$

The proof of the analogous statement is contained in [21, Proof of Lemma 4.1], hence we do not present it here. Consider the equation (3.16.3) in the regular case, that is

$$\begin{cases} \partial_t u^{c,\varepsilon}(t,z) + D_z u^{c,\varepsilon}(t,z) + \xi_c(t,z)u^{c,\varepsilon}(t,z) = \mathcal{T}^\varepsilon(t), & (t,z) \in [0,T]\times\mathbb{R}^2_+, \\ u^{c,\varepsilon}(0,z) = u^{c,\varepsilon}_o(z). \end{cases} \tag{3.37}$$

The change of variables $(t,z) \xrightarrow{\Phi} (t,X(t;z))$ in (3.37), where $X(t;z)$ is a solution to

$$\frac{d}{dt}X(t;z) = 1, \quad X(0;z) = z,$$

transforms the original equation into the ODE

$$\begin{cases} \partial_t v^{c,\varepsilon}(t,y) &= -\xi_c(t,y)v^{c,\varepsilon}(t,y) + \mathcal{T}^\varepsilon(t,y), \quad (t,z) \in [0,T] \times \mathbb{R}_+^2, \\ v^{c,\varepsilon}(0,y) &= u_o^{c,\varepsilon}(y), \end{cases} \tag{3.38}$$

where $y = X(t;z)$. This equation is an ODE in $\mathbf{L^1}(\mathbb{R}_+^2)$ with globally Lipschitz right hand side. Therefore, existence and uniqueness of a classical solution $v^{c,\varepsilon} \in \mathbf{BC^1}([0,T];\mathbf{L^1}(\mathbb{R}_+^2))$ of (3.38) follows from the Banach Fixed Point Theorem. Moreover, $v^{c,\varepsilon}(t)$ is a nonnegative function on $\mathbb{R}_+^2$. The solution of (3.37) is obtained by taking the inverse transform Φ^{-1}, that is $u^{c,\varepsilon}(t,\Phi^{-1}(y)) = v^{c,\varepsilon}(t,y)$. Φ is a $\mathbf{C^1}$ diffeomorphism which implies that the regularity of solutions under the inverse transform does not change. Integrating (3.37) we obtain that for every $0 \le t_1 \le t_2 \le T$ and for any $\varphi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})([0,T] \times \mathbb{R}_+^2)$

$$\begin{aligned} &\int_{t_1}^{t_2} \int_{\mathbb{R}_+^2} \left(\partial_t \varphi(t,z) + D_z \varphi(t,z) - \xi_c(t,z)\varphi(t,z)\right) u^{c,\varepsilon}(t,z) \mathrm{d}z \mathrm{d}t \\ &= \int_{\mathbb{R}_+^2} \varphi(t_2,z) u^{c,\varepsilon}(t_2,z)\mathrm{d}z - \int_{\mathbb{R}_+^2} \varphi(t_1,z)u^{c,\varepsilon}(t_1,z)\mathrm{d}z - \int_{t_1}^{t_2} \int_{\mathbb{R}_+^2} \varphi(t,z)\mathrm{d}[\mathcal{T}^\varepsilon(t)](z)\mathrm{d}t. \end{aligned} \tag{3.39}$$

Choosing φ as a solution to the dual problem (3.21) with $T = t_2$, we obtain

$$\int_{\mathbb{R}_+^2} \psi(z) u^{c,\varepsilon}(t_2,z)\mathrm{d}z = \int_{\mathbb{R}_+^2} \varphi_{t_2,\psi}(t_1,z)u^{c,\varepsilon}(t_1,z)\mathrm{d}z + \int_{t_1}^{t_2} \int_{\mathbb{R}_+^2} \varphi_{t_2,\psi}(t,z)\mathrm{d}[\mathcal{T}^\varepsilon(t)](z)\mathrm{d}t. \tag{3.40}$$

Let u^{c,ε_m}, respectively u^{c,ε_n}, solve problem (3.37) with ε replaced by ε_m, respectively ε_n. Moreover, let v be the solution to

$$\begin{cases} \partial_t v(t,z) + D_z v(t,z) + \xi_c(t,z)v(t,z) &= \mathcal{T}^{\varepsilon_m}(t), \quad (t,z) \in [0,T] \times \mathbb{R}_+^2, \\ v(0,z) &= u_o^{c,\varepsilon_n}(z). \end{cases}$$

Let $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+^2)$ be such that $\|\psi\|_{\infty,\mathbf{Lip}} \le 1$. Using (3.40) with $t_1 = 0$ and $t_2 = t$ yields

$$\begin{aligned} \int_{\mathbb{R}_+^2} \psi(z) \left(u^{c,\varepsilon_n}(t,z) - v(t,z)\right) \mathrm{d}z &= \int_0^t \int_{\mathbb{R}_+^2} \varphi_{t,\psi}(s,z)\mathrm{d}[\mathcal{T}^{\varepsilon_n}(s) - \mathcal{T}^{\varepsilon_m}(s)](z)\mathrm{d}t \\ &\le \int_0^t \rho_F(\mathcal{T}^{\varepsilon_n}(s), \mathcal{T}^{\varepsilon_m}(s))\mathrm{d}s \\ &\le T \sup_{s\in[0,T]} \rho_F(\mathcal{T}^{\varepsilon_n}(s), \mathcal{T}^{\varepsilon_m}(s)). \end{aligned}$$

Taking supremum over all functions ψ gives

$$\rho_F\left(u^{c,\varepsilon_n}(t,\cdot), v(t,\cdot)\right) \le T \sup_{s\in[0,T]} \rho_F(\mathcal{T}^{\varepsilon_n}(s), \mathcal{T}^{\varepsilon_m}(s)).$$

Due to (3.36) $\rho_F(u^{c,\varepsilon_n}(t,\cdot), v(t,\cdot))$ converges to 0 uniformly wih respect to time. Analogously,

$$\int_{\mathbb{R}_+^2} \psi(z) \left(u^{c,\varepsilon_m}(t,z) - v(t,z)\right) \mathrm{d}z = \int_{\mathbb{R}_+^2} \varphi_{t,\psi}(0,z) \left(u_o^{c,\varepsilon_m}(z) - u_o^{c,\varepsilon_n}(z)\right) \mathrm{d}z.$$

Taking supremum over all functions ψ yields

$$\rho_F\left(u^{c,\varepsilon_m}(t,\cdot), v(t,\cdot)\right) \le e^{2\|\xi_c\|_{\mathbf{BC_t}} t}\rho_F(u_o^{c,\varepsilon_m}, u_o^{c,\varepsilon_n}) \le e^{2\|\xi_c\|_{\mathbf{BC_t}} T}\rho_F(u_o^{c,\varepsilon_m}, u_o^{c,\varepsilon_n}),$$

which holds due to estimates (3.23) and (3.24) for a dual problem. Therefore, by (3.36), $d\left(u^{\varepsilon_n}(t,\cdot), u^{\varepsilon_m}(t,\cdot)\right) \overset{n,m\to\infty}{\longrightarrow} 0$ uniformly with respect to time. Note that

$$\mathbf{BC^1}\left([0,T];\mathbf{L^1}(\mathbb{R}^2;\mathbb{R}_+)\right) \subset \mathbf{BC}\left([0,T];\left(\mathcal{M}^+(\mathbb{R}^2), \|\cdot\|_{(\mathbf{W^{1,\infty}})^*}\right)\right)$$

and metric ρ_F is equal to $(\mathbf{W^{1,\infty}})^*$ distance on the set $\mathcal{M}^+(\mathbb{R}^2_+)$. Completeness of $(\mathcal{M}^+(\mathbb{R}^2), \rho_F)$ implies that the sequence $u^{\varepsilon_n}(t,\cdot)$ converges uniformly with respect to t to the unique limit μ^c_t. Notice that $\partial_t\varphi(t,\cdot)$, $D_z\varphi(t,\cdot)$, $\xi_c(t,\cdot)\varphi(t,\cdot)$ are continuous functions bounded uniformly with respect to t. The integral $\int_{\mathbb{R}^2_+}\varphi(t,z)\mathrm{d}[\mathcal{T}^{\varepsilon_n}(t)](z)$ converges to $\int_{\mathbb{R}^2_+}\varphi(t,x)\mathrm{d}[\mathcal{T}(t)](z)$ uniformly with respect to t. Therefore, passing to the limit with u^{ε_n} and $\mathcal{T}^{\varepsilon_n}$ in (3.39) yields

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\mathbb{R}^2_+}\left(\partial_t\varphi(t,z) + D_z\varphi(t,z) - \xi^\varepsilon_c(t,z)\varphi(t,z)\right)\mathrm{d}\mu^c_t(z)\mathrm{d}t \\
&= \int_{\mathbb{R}^2_+}\varphi(t_2,z)\mathrm{d}\mu^c_{t_2}(z)\mathrm{d}z - \int_{\mathbb{R}^2_+}\varphi(t_1,z)\mathrm{d}\mu^c_{t_1}(z)\mathrm{d}z - \int_{t_1}^{t_2}\int_{\mathbb{R}^2_+}\varphi(t,z)\mathrm{d}[\mathcal{T}(t)](z)\mathrm{d}s,
\end{aligned} \tag{3.41}$$

which proves that μ^c is a solution to (3.16.3) in the sense of Definition 3.9 (for the proof of uniqueness we refer to the proof of claim **iv**). Similarly we prove that passing to the limit in (3.40) yields

$$\int_{\mathbb{R}^2_+}\psi(z)\mathrm{d}\mu^c_{t_2}(z)\mathrm{d}z = \int_{\mathbb{R}^2_+}\varphi_{t_2,\psi}(t_1,z)\mathrm{d}\mu^c_{t_1}(z)\mathrm{d}z + \int_{t_1}^{t_2}\int_{\mathbb{R}^2_+}\varphi_{t_2,\psi}(t,z)\mathrm{d}[\mathcal{T}(t)](z)\mathrm{d}t. \tag{3.42}$$

Using estimates (3.23), (3.25) and formula (3.42) for $t_1 = 0$ and $t_2 = t$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2_+}\psi(z)\mathrm{d}\left(\mu^c_t - \mu^c_o\right)(z) \\
&\quad= \int_{\mathbb{R}^2_+}\left(\varphi_{t,\psi}(0,z) - \psi(z)\right)\mathrm{d}\mu^c_o(z) + \int_0^t\int_{\mathbb{R}^2_+}\varphi_{t,\psi}(s,z)\mathrm{d}[\mathcal{T}(s)](z)\mathrm{d}s \\
&\quad\le \sup_{s\in[0,t]}\|\partial_s\varphi_{t,\psi}(s,\cdot)\|_{\mathbf{L^\infty}}\, t\,\mu^c_o(\mathbb{R}^2_+) + t\sup_{s\in[0,t]}\|\varphi_{t,\psi}(s,\cdot)\|_{\mathbf{L^\infty}}\cdot\sup_{s\in[0,t]}\|\mathcal{T}(s)\|_{(\mathbf{W^{1,\infty}})^*} \\
&\quad\le \|\psi\|_{\mathbf{L^\infty}}\left(1 + \|\xi_c\|_{\mathbf{BC_t}}\right)e^{\left(1+\|\xi_c\|_{\mathbf{BC_t}}\right)t}\, t\,\mu^c_o(\mathbb{R}^2_+) + t\,\|\psi\|_{\mathbf{L^\infty}}e^{\|\xi_c\|_{\mathbf{BC_t}}t}\,\|\mathcal{T}\|_{\mathbf{BC_t}} \\
&\quad\le \|\psi\|_{\mathbf{L^\infty}}\cdot\max\{1,\mu^c_o(\mathbb{R}_+)\}\cdot\left(1 + \|\xi_c\|_{\mathbf{BC_t}} + \|\mathcal{T}\|_{\mathbf{BC_t}}\right)e^{\left(1+\|\xi_c\|_{\mathbf{BC_t}}\right)t}\, t.
\end{aligned}$$

Taking supremum over all functions ψ such that $\varphi \in (\mathbf{C^1}\cap\mathbf{W^{1,\infty}})(\mathbb{R}^2_+)$ and $\|\psi\|_{\infty,\mathbf{Lip}} \le 1$ gives

$$\rho_F(\mu^c_t,\mu^c_o) \le \max\{1,\mu^c_o(\mathbb{R}_+)\}\cdot\left(1 + \|(\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)e^{\left(1+\|\xi_c\|_{\mathbf{BC_t}}\right)t}\, t,$$

for all $t \in [0,T]$. This allows to estimate a total mass of μ^c in time t.

$$\begin{aligned}
\mu^c_t(\mathbb{R}^2_+) &\le \rho_F(\mu^c_t,\mu^c_o) + \mu^c_o(\mathbb{R}^2_+) \\
&\le \left[\left(1 + \|(\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)e^{\left(1+\|\xi_c\|_{\mathbf{BC_t}}\right)t}\, t + 1\right]\max\{1,\mu^c_o(\mathbb{R}_+)\} \\
&\le \left[\left(1 + \|(\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)t + 1\right]e^{\left(1+\|\xi_c\|_{\mathbf{BC_t}}\right)t}\max\{1,\mu^c_o(\mathbb{R}_+)\} \\
&\le \max\{1,\mu^c_o(\mathbb{R}_+)\}\,e^{2\left(1+\|(\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)t}.
\end{aligned} \tag{3.43}$$

Using the analogous arguments as above, estimates (3.23), (3.25) and formulas (3.42), (3.43) we obtain the following Lipschitz estimate

$$\begin{aligned}\rho_F\left(\mu^c_{t_2},\mu^c_{t_1}\right) &\le \max\left\{1,\mu^c_{t_1}(\mathbb{R}_+)\right\}\cdot\left(1+\|(\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)\mathrm{e}^{\left(1+\|\xi_c\|_{\mathbf{BC_t}}\right)(t_2-t_1)}(t_2-t_1)\\ &\le \max\left\{1,\mu^c_o(\mathbb{R}_+)\right\}\cdot\left(1+\|(\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)\mathrm{e}^{2\left(1+\|(\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)t_2}(t_2-t_1).\end{aligned}$$

ii) Let $\psi\in(\mathbf{C^1}\cap\mathbf{W^{1,\infty}})(\mathbb{R}^2_+)$ such that $\|\psi\|_{\infty,\mathbf{Lip}}\le 1$. By the formula (3.42), taking supremum over all functions ψ finishes the proof due to estimates (3.23) and (3.24) for a dual problem.

iii) Let $\psi\in(\mathbf{C^1}\cap\mathbf{W^{1,\infty}})(\mathbb{R}^2_+)$ be such that $\|\psi\|_{\infty,\mathbf{Lip}}\le 1$. Let $\tilde{\varphi}_{t,\psi}$ be a solution to the dual problem given by the equation (3.21) with terminal data ψ and coefficient $\tilde{\xi}_c$. It follows from (3.42) that

$$\begin{aligned}&\int_{\mathbb{R}^2_+}\psi(z)\mathrm{d}(\mu^c_t-\tilde{\mu}^c_t)(z)\\ &\quad=\int_{\mathbb{R}^2_+}\left(\varphi_{t,\psi}(0,z)-\tilde{\varphi}_{t,\psi}(0,z)\right)\mathrm{d}\mu^c_o(z)\\ &\qquad+\int_0^t\int_{\mathbb{R}^2_+}\varphi_{t,\psi}(s,z)\mathrm{d}\mathcal{T}(s)\mathrm{d}s-\int_0^t\int_{\mathbb{R}^2_+}\tilde{\varphi}_{t,\psi}(s,z)\mathrm{d}\tilde{\mathcal{T}}(s)\\ &\quad=\int_{\mathbb{R}^2_+}\left(\varphi_{t,\psi}(0,z)-\tilde{\varphi}_{t,\psi}(0,z)\right)\mathrm{d}\mu^c_o(z)+\int_0^t\int_{\mathbb{R}^2_+}\left(\varphi_{t,\psi}(s,z)-\tilde{\varphi}_{t,\psi}(s,z)\right)\mathrm{d}[\mathcal{T}(s)](z)\mathrm{d}s\\ &\qquad+\int_0^t\int_{\mathbb{R}^2_+}\tilde{\varphi}_{t,\psi}(s,z)\mathrm{d}[\mathcal{T}(s)-\tilde{\mathcal{T}}(s)](z)\mathrm{d}s.\end{aligned}$$

Due to estimates (3.23), (3.24), (3.26) and the defnition of the flat metric (1.23) it holds that

$$\begin{aligned}&\int_{\mathbb{R}^2_+}\psi(z)\mathrm{d}(\mu^c_t-\tilde{\mu}^c_t)(z)\\ &\quad\le\ \|\xi_c-\tilde{\xi}_c\|_{\mathbf{BC_t}}\ t\mathrm{e}^{2\left(1+\|(\xi_c,\tilde{\xi}_c)\|_{\mathbf{BC_t}}\right)t}\left(\mu^c_o(\mathbb{R}^2_+)+t\sup_{s\in[0,t]}\|\mathcal{T}(s)\|_{(\mathbf{W^{1,\infty}})^*}\right)\\ &\qquad+t\sup_{s\in[0,t]}\sup\left\{\int_{\mathbb{R}^2_+}f(x)\mathrm{d}(\mathcal{T}(s)-\tilde{\mathcal{T}}(s))(x)\ :\ \|f\|_{\infty,\mathbf{Lip}}\le\mathrm{e}^{2\|\tilde{\xi}_c\|_{\mathbf{BC_t}}(t-s)}\right\}\\ &\quad\le\ \|\xi_c-\tilde{\xi}_c\|_{\mathbf{BC_t}}\ t\mathrm{e}^{2\left(1+\|(\xi_c,\tilde{\xi}_c)\|_{\mathbf{BC_t}}\right)t}\max\left\{1,\mu^c_o(\mathbb{R}^2_+)\right\}\left(1+t\,\|\mathcal{T}\|_{\mathbf{BC_t}}\right)\\ &\qquad+t\,\mathrm{e}^{2\|\tilde{\xi}_c\|_{\mathbf{BC_t}}t}\sup_{s\in[0,t]}\rho_F(\mathcal{T}(s),\tilde{\mathcal{T}}(s))\\ &\quad\le\ t\,\mathrm{e}^{2\left(1+\|(\xi_c,\tilde{\xi}_c)\|_{\mathbf{BC_t}}\right)t}\max\left\{1,\mu^c_o(\mathbb{R}^2_+)\right\}\left(1+t\,\|\mathcal{T}\|_{\mathbf{BC_t}}\right)\|(\xi_c-\tilde{\xi}_c,\mathcal{T}-\tilde{\mathcal{T}})\|_{\mathbf{BC_t}}\\ &\quad\le\ t\max\left\{1,\mu^c_o(\mathbb{R}^2_+)\right\}\mathrm{e}^{2\left(1+\|(\xi_c,\tilde{\xi}_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)t}\|(\xi_c-\tilde{\xi}_c,\mathcal{T}-\tilde{\mathcal{T}})\|_{\mathbf{BC_t}}.\end{aligned}$$

iv) Assume that $\mu^c\in\mathbf{Lip}\left([0,T];\mathcal{M}^+(\mathbb{R}^2_+)\right)$ is a solution to (3.16.3) in the sense of Definition 3.9. Fix a $\varphi\in(\mathbf{C^1}\cap\mathbf{W^{1,\infty}})([0,T]\times\mathbb{R}^2_+;\mathbb{R})$ and define $\varphi^\varepsilon(t,z)=\kappa_\varepsilon(t)\,\varphi(t,z)$, where

$$\kappa_\varepsilon\in\mathbf{C^\infty_c}\left([t_1,t_2),[0,1]\right),\quad \kappa_\varepsilon(s)=\chi_{[t_1,t_2)}(s)$$

and

$$\lim_{\varepsilon\to0}\kappa'_\varepsilon=\delta(t=t_1)-\delta(t=t_2)\ \text{in}\ \mathcal{M}^+([0,T]).$$

Use φ^ε as a test function in the definition of weak solution.

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}_+^2} \left(\partial_t \varphi^\varepsilon(t,z) + D_z\varphi^\varepsilon(t,z) - \xi_c(t,z)\varphi^\varepsilon(t,z)\right) \mathrm{d}\mu_t^c(z) \\ &\quad + \int_0^T \int_{\mathbb{R}_+^2} \varphi^\varepsilon(t,z)\mathrm{d}[\mathcal{T}(t)](z)\mathrm{d}t \\ &= \int_0^T k_\varepsilon'(t) \int_{\mathbb{R}_+^2} \varphi(t,z)\mathrm{d}\mu_t^c(z)\mathrm{d}t \\ &\quad + \int_0^T k_\varepsilon(t) \int_{\mathbb{R}_+^2} \left(\partial_t\varphi(t,z) + D_z\varphi(t,x) - \xi_c(t,z)\varphi(t,z)\right)\mathrm{d}\mu_t^c(z) \\ &\quad + \int_0^T \int_{\mathbb{R}_+^2} \varphi(t,z)\mathrm{d}[\mathcal{T}(t)](z)\mathrm{d}t. \end{aligned}$$

Passing to the limit with ε and using Dominated Convergence Theorem finishes the proof.
v) Equality follows from **iv)** by setting $t_1 = 0$, $t_2 = t$ and $\varphi(s,x) = \varphi_{T,\psi}(s + (T - t_2), x)$.
vi) We proved that there exists a solution to (3.16.3) which also fulfils (3.35). This equation characterizes μ^c uniquely, hence each μ^c given by (3.35) is a solution to (3.16.3).
□

The analysis of the problems (3.16.1) and (3.16.2) is based on their dual formulations as well. These problems are analogous, therefore we restrict ourselves to performing analysis just for one of them, that is (3.16.1). We define a dual problem to (3.16.1) as

$$\left\{ \begin{aligned} \partial_t\varphi(t,x) + \partial_x\varphi(t,x) - \xi_m(t,x)\varphi(t,x) &= 0, \quad (t,x) \in [0,T]\times\mathbb{R}_+, \\ \varphi(T,x) &= \psi(x). \end{aligned} \right. \tag{3.44}$$

Lemma 3.45. *For all $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+)$ there exists a unique solution to (3.44). Moreover, the following estimates hold:*

$$\|\varphi_{T,\psi}(t,\cdot)\|_{\mathbf{L}^\infty} \le \|\psi\|_{\mathbf{L}^\infty} \mathrm{e}^{\|\xi_m\|_{\mathbf{BC_t}}(T-t)}, \tag{3.46}$$

$$\mathbf{Lip}\left(\varphi_{T,\psi}(t,\cdot)\right) \le \|\psi\|_{\infty,\mathbf{Lip}} \mathrm{e}^{2\|\xi_m\|_{\mathbf{BC_t}}(T-t)}, \tag{3.47}$$

$$\sup_{s\in[t,T]} \|\partial_s\varphi_{T,\psi}(s,\cdot)\|_{\mathbf{L}^\infty} \le \|\psi\|_{\mathbf{L}^\infty}\left(1 + \|\xi_m\|_{\mathbf{BC_t}}\right)\mathrm{e}^{(1+\|\xi_m\|_{\mathbf{BC_t}})(T-t)}, \tag{3.48}$$

If moreover $\tilde\varphi$ solves (3.44) with terminal data ψ and parameter $\tilde\xi_m$, then

$$\|\varphi_{T,\psi}(t,\cdot) - \tilde\varphi_{T,\psi}(t,\cdot)\|_{\mathbf{L}^\infty} \le \|\psi\|_{\mathbf{L}^\infty}\|\xi_m - \tilde\xi_m\|_{\mathbf{BC_t}}(T-t)\mathrm{e}^{2\left(1+\|(\xi_m,\tilde\xi_m)\|_{\mathbf{BC_t}}\right)(T-t)}. \tag{3.49}$$

Since a proof is analogous to the proof of the Lemma 3.22, we do not present it here. The relation between (3.16.1) and (3.44) is explained by the following Lemma.

Lemma 3.50. *Fix $\mu_o^m \in \mathcal{M}^+(\mathbb{R}_+^2)$ and let ξ_m, b_m satisfy assumptions (3.17), (3.18). Let μ_t^c be a unique solution to (3.16.3) with initial datum μ_o^c and coefficients ξ_c and $\mathcal{T}$. Then:*

i) Problem (3.16.1) admits a unique solution $\mu^m \in \mathbf{Lip}([0,T],\mathcal{M}^+(\mathbb{R}_+))$, that is for all $0 \le t_1 \le t_2 \le T$,

$$\begin{aligned} &\rho_F\left(\mu_{t_2}^m, \mu_{t_1}^m\right) \\ &\le \max\left\{1, \mu_o^m(\mathbb{R}_+), \mu_o^c(\mathbb{R}_+)\right\}\left(1 + \|(\xi_m, b_m)\|_{\mathbf{BC_t}}\right)\mathrm{e}^{3\left(1+\|(\xi_m,\xi_c,b_m,\mathcal{T}\|_{\mathbf{BC_t}}\right)t_2}(t_2 - t_1). \end{aligned}$$

ii) *Let μ^m and ν^m be solutions to (3.16.1) with initial datum μ_o^m and ν_o^m respectively. Then,*

$$\rho_F(\mu_t^m, \nu_t^m) \leq e^{2\|\xi_m\|_{\mathbf{BC_t}} t} \rho_F(\mu_o^m, \nu_o^m).$$

iii) *Let $\tilde{\xi}_m, \tilde{b}_m$ satisfy assumptions* (3.17), (3.18) *and $\tilde{\mu}_t^m$ be a solution to (3.16.1) with initial datum μ_o^m coefficients $\tilde{\xi}_m, \tilde{b}_m$. Moreover, let $\tilde{\mu}_t^c$ be a unique solution to (3.16.3) with initial datum μ_o^c and coefficients $\tilde{\xi}_c$ and $\tilde{\mathcal{T}}$. Then,*

$$\begin{aligned}\rho_F\left(\mu_t^m, \tilde{\mu}_t^m\right) \leq t\ \max\{1, \mu_o^m(\mathbb{R}_+), \mu_o^c(\mathbb{R}_+^2)\} e^{2\left(1+\|(\xi_m, \tilde{\xi}_m, \xi_c, \tilde{\xi}_c, \tilde{b}_m, \mathcal{T})\|_{\mathbf{BC_t}}\right)t} \\ \cdot \|(\xi_m - \tilde{\xi}_m, b_m - \tilde{b}_m, \xi_c - \tilde{\xi}_c, \mathcal{T} - \tilde{\mathcal{T}})\|_{\mathbf{BC_t}}.\end{aligned}$$

iv) *Fix $t_1, t_2 \in [0,T]$ with $t_1 < t_2$. If $\mu^m \in \mathbf{Lip}\left([0,T]; \mathcal{M}^+(\mathbb{R}_+)\right)$ solves (3.16.1), then for any $\varphi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})([t_1,t_2] \times \mathbb{R}_+; \mathbb{R})$ we have*

$$\begin{aligned}&\int_{t_1}^{t_2} \int_{\mathbb{R}_+} \left(\partial_t \varphi(t,x) + \partial_x \varphi(t,x) - \xi_c(t,x)\varphi(t,x)\right) d\mu_t^m(x) dt \\ &= \int_{\mathbb{R}_+} \varphi(t_2,x) d\mu_{t_2}^m(x) - \int_{\mathbb{R}_+} \varphi(t_1,x) d\mu_{t_1}^m(x) - \int_{t_1}^{t_2} \varphi(t,0) F^{m,\mu^c}(t) dt.\end{aligned}$$

v) *If $\mu^m \in \mathbf{Lip}([0,T]; \mathcal{M}^+(\mathbb{R}_+))$ solves (3.16.1), then for any $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+)$ there exists a function $\varphi_{T,\psi} \in \mathbf{C^1}([0,T] \times \mathbb{R}_+)$ solving the dual problem* (3.44) *and such that*

$$\int_{\mathbb{R}_+} \psi(x) d\mu_t^m(x) = \int_{\mathbb{R}_+} \varphi_{T,\psi}(T-t,x) d\mu_o^m(x) + \int_0^t \varphi_{T,\psi}(s+(T-t),0) F^{m,\mu^c}(s) ds. \tag{3.51}$$

vi) *For any $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+)$ let $\varphi_{T,\psi} \in \mathbf{C^1}([0,T] \times \mathbb{R}_+)$ solves the dual problem* (3.44). *Then the measure defined by* (3.51) *solves (3.16.1).*

All results from this lemma are valid for the problem (3.16.2) when one changes the index m to index f.

Proof of Lemma 3.50.

i) We will show that problem (3.16.1) admits the unique solution. The proof is analogous to the proof of Lemma 3.33. Let $\rho \in \mathbf{C_c^\infty}(\mathbb{R}; \mathbb{R}_+)$ be such that $\int_{\mathbb{R}} \rho(x) dx = 1$. For $\varepsilon > 0$ define a family of mollifiers $\rho^\varepsilon(x) = \rho(x/\varepsilon)/\varepsilon$. The convolution is defined as $(\nu * \rho)(x) = \int_{\mathbb{R}_+} \rho(x - \varepsilon - \zeta) d\nu(\zeta)$. We consider (3.16.1) with initial datum $u_o^{m,\varepsilon}$, where

$$u_o^{m,\varepsilon} = (\mu_o^m * \rho^\varepsilon) \in (\mathbf{BC^\infty} \cap \mathbf{L^1})(\mathbb{R}_+; \mathbb{R}_+)$$

and

$$\rho_F(u_o^{m,\varepsilon}, \mu_o^m) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Consider the equation (3.16.1) in the regular case, that is

$$\begin{cases} \partial_t u^{m,\varepsilon}(t,x) + \partial_x u^{m,\varepsilon}(t,x) + \xi_m(t,x) u^{m,\varepsilon}(t,x) = 0, & (t,x) \in [0,T] \times \mathbb{R}_+, \\ u^{m,\varepsilon}(t,0) = F^{m,\mu^c}(t), & \\ u^{m,\varepsilon}(0,x) = u_o^{m,\varepsilon}(x), & \end{cases} \tag{3.52}$$

where $F^{m,\mu^c}(t) = \int_{\mathbb{R}^2_+} b_m(t,z)\mathrm{d}\mu^c_t(z)$. Due to assumption (3.18) on b_m and properties of μ^c_t from Lemma 3.33 it holds that $F^{m,\mu^c} \in (\mathbf{BC} \cap \mathbf{L^1})([0,T];\mathbb{R}_+)$. Moreover,

$$\sup_{s\in[0,t]} \left|F^{m,\mu^c}(s)\right| \le \|b_m\|_{\mathbf{BC_t}} \sup_{s\in[0,t]} \mu^c_s(\mathbb{R}^2_+) \le \max\{1, \mu^c_o(\mathbb{R}_+)\} \|b_m\|_{\mathbf{BC_t}} \mathrm{e}^{2\left(1+\|(\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)t},$$

where the estimate on $\mu^c_s(\mathbb{R}^2_+)$ follows from (3.43). Let $\tilde{F}^{m,\mu^c}(t) = \int_{\mathbb{R}^2_+} \tilde{b}_m(t,z)\mathrm{d}\tilde{\mu}^c_t(z)$. Then,

$$\begin{aligned} F^{m,\mu^c}(t) - \tilde{F}^{m,\mu^c}(t) &= \int_{\mathbb{R}^2_+} \left(b(t,z) - \tilde{b}(t,z)\right)\mathrm{d}\mu^c_t(z) + \int_{\mathbb{R}^2_+} \tilde{b}(t,z)\mathrm{d}\left(\mu^c_t - \tilde{\mu}^c_t\right)(z) \\ &\le \|b_m - \tilde{b}_m\|_{\mathbf{BC_t}}\, \mu^c_t(\mathbb{R}^2_+) + \|\tilde{b}_m\|_{\mathbf{BC_t}}\, \rho_F\left(\mu^c_t, \tilde{\mu}^c_t\right). \end{aligned} \tag{3.53}$$

Existence and uniqueness of solutions to (3.52) follow from the method of characteristics. This method leads to the explicit formula on the solution $u^{m,\varepsilon}(t,x)$, that is

$$u^{m,\varepsilon}(t,x) = \begin{cases} u^{m,\varepsilon}_o(x-t)\exp\left(\int_0^t \xi_m\left(s,\ s+(x-t)\right)\mathrm{d}s\right) & \text{for} \quad x \in [t,+\infty), \\ F^{m,\mu^c}(t-x)\exp\left(\int_{t-x}^t \xi_m\left(s,\ s+(x-t)\right)\mathrm{d}s\right) & \text{for} \quad x \in [0,t), \end{cases} \tag{3.54}$$

for $t \in [0,T]$ and $x \in \mathbb{R}_+$. We shall prove that $u^{m,\varepsilon}(t,\cdot)$ is a Cauchy sequence in in $(\mathcal{M}^+(\mathbb{R}_+);\rho_F)$. Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be such that $\varepsilon_n \to 0$ as $n \to +\infty$. Formula (3.54) implies that $u^{m,\varepsilon_n}(t,\cdot) \in \mathbf{L^1}(\mathbb{R}_+;\mathbb{R}_+)$, if $u^{m,\varepsilon_n}_o \in \mathbf{L^1}(\mathbb{R}_+;\mathbb{R}_+)$. Moreover, $\|u^{m,\varepsilon_n}(t,\cdot)\|_{\mathbf{L^1}}$ is uniformly bounded. It can be checked that

$$u^{m,\varepsilon_n} \in \mathbf{BC}([0,T];\mathbf{L^1}(\mathbb{R}_+;\mathbb{R}_+)) \subset \mathbf{BC}\left([0,T];\left(\mathcal{M}^+(\mathbb{R}_+), \|\cdot\|_{(\mathbf{W^{1,\infty}})^*}\right)\right).$$

Now, let $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R};\mathbb{R})$ such that $\|\psi\|_{\infty,\mathbf{Lip}} \le 1$. Then,

$$\begin{aligned} &\int_{\mathbb{R}_+} \psi(x)\left(u^{m,\varepsilon_n}(t,x) - u^{m,\varepsilon_k}(t,x)\right)\mathrm{d}x \\ &\quad= \int_t^{+\infty} \psi(x)\mathrm{e}^{\int_0^t \xi_m(s,(x-t)+s)\mathrm{d}s}\left(u^{m,\varepsilon_n}_o(x-t) - u^{m,\varepsilon_k}_o(x-t)\right)\mathrm{d}x \\ &\quad= \int_{\mathbb{R}_+} \psi(x+t)\mathrm{e}^{\int_0^t \xi_m(s,\,x+s)\mathrm{d}s}\left(u^{m,\varepsilon_n}_o(x) - u^{m,\varepsilon_k}_o(x)\right)\mathrm{d}x \\ &\quad\le \mathrm{e}^{\|\xi_m\|_{\mathbf{BC_t}}T}\rho_F(u^{m,\varepsilon_n}_o, u^{m\varepsilon_k}_o). \end{aligned}$$

Taking supremum over all functions ψ yields

$$\rho_F(u^{m,\varepsilon_n}(t,\cdot), u^{m,\varepsilon_k}(t,\cdot)) \le \mathrm{e}^{\|\xi_m\|_{\mathbf{BC_t}}T}\rho_F(u^{m,\varepsilon_n}_o, u^{m,\varepsilon_k}_o) \to 0,$$

Since metric ρ_F is equal to $(\mathbf{W^{1,\infty}})^*$ distance and $(\mathcal{M}^+(\mathbb{R}_+),\rho_F)$ is a complete metric space, there exists a limit μ^m_t for each $t \in [0,T]$. Moreover, the convergence of $\rho_F(u^{m,\varepsilon}(t,\cdot),\mu_t)$ is uniform with respect to time. We shall show that μ^m_t is a solution in the sense of Definition 3.9. Integrating (3.52) we obtain that for every $0 \le t_1 \le t_2 \le T$ and $\varphi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})([0,T]\times\mathbb{R}_+)$,

$$\begin{aligned} &\int_{t_1}^{t_2}\int_{\mathbb{R}_+} \left(\partial_t\varphi(t,x)(t,x) + \partial_x\varphi(t,x) - \xi_m(t,x)\varphi(t,x)\right)u^{m,\varepsilon}(t,x)\mathrm{d}x\mathrm{d}t \\ &\quad+ \int_{t_1}^{t_2} \varphi(t,0)F^{m,\mu^c}(t)\mathrm{d}t \\ &= \int_{\mathbb{R}_+} \varphi(t_2,x)u^{m,\varepsilon}(t_2,x)\mathrm{d}x - \int_{\mathbb{R}_+} \varphi(t_1,x)u^{m,\varepsilon}(t_1,x)\mathrm{d}x. \end{aligned} \tag{3.55}$$

In particular, choosing φ as a solution to the dual problem (3.44) with $T = t_2$ we obtain

$$\int_{\mathbb{R}_+} \psi(x) u^{m,\varepsilon}(t_2,x)\mathrm{d}x = \int_{\mathbb{R}_+} \varphi_{t_2,\psi}(t_1,x) u^{m,\varepsilon}(t_1,x)\mathrm{d}x + \int_{t_1}^{t_2} \varphi_{t_2,\psi}(t,0) F^{m,\mu^c}(t)\mathrm{d}t. \tag{3.56}$$

Notice that $\partial_t\varphi(t,\cdot)$, $\partial_x\varphi(t,\cdot)$, $\varphi(t,\cdot)$ and $\xi_m(t,\cdot)\varphi(t,\cdot)$ are continuous functions uniformly bounded with respect to t. Therefore, passing to the limit in (3.55) for $t_1 = 0$ and $t_2 = T$ yields

$$\begin{aligned}&\int_{t_1}^{t_2}\int_{\mathbb{R}_+} \left(\partial_t\varphi(t,x)(t,x) + \partial_x\varphi(t,x) - \xi_m(t,x)\varphi(t,x)\right) \mathrm{d}\mu_t^m(x)\mathrm{d}t + \int_{t_1}^{t_2} \varphi(t,0) F^{m,\mu^c}(t)\mathrm{d}t \\ &= \int_{\mathbb{R}_+} \varphi(t_2,x)\mathrm{d}\mu_t^m(x) - \int_{\mathbb{R}_+} \varphi(t_1,x)\mathrm{d}\mu_t^m(x),\end{aligned} \tag{3.57}$$

which proves that μ^m is a solution to (3.16.1). In particular, passing to the limit in (3.56) yields

$$\int_{\mathbb{R}_+} \psi(x)\mathrm{d}\mu_{t_2}^m(x)\mathrm{d}x = \int_{\mathbb{R}_+} \varphi_{t_2,\psi}(t_1,x)\mathrm{d}\mu_{t_1}^m(x)\mathrm{d}x + \int_{t_1}^{t_2} \varphi_{t_2,\psi}(t,0) F^{m,\mu^c}(t)\mathrm{d}t. \tag{3.58}$$

Using (3.46), (3.48), (3.53) and (3.58) for $t_1 = 0$ and $t_2 = t$ we obtain

$$\begin{aligned}&\int_{\mathbb{R}_+} \psi(x)\mathrm{d}\left(\mu_t^m - \mu_o^m\right)(x) \\ &\quad= \int_{\mathbb{R}_+} \left(\varphi_{t,\psi}(0,x) - \psi(x)\right)\mathrm{d}\mu_o^m(x) + \int_0^t \varphi_{t,\psi}(s,0) F^{m,\mu^c}(s)\mathrm{d}s \\ &\quad\le \sup_{s\in[0,t]} \|\partial_s\varphi_{t,\psi}(s,\cdot)\|_{\mathbf{L}^\infty}\mu_o^m(\mathbb{R}_+)\, t + \sup_{s\in[0,t]} \|\varphi_{t,\psi}(s,\cdot)\|_{\mathbf{L}^\infty} \sup_{s\in[0,t]} \left|F^{m,\mu^c}(s)\right| t \\ &\quad\le \|\psi\|_{\mathbf{L}^\infty}\left(1 + \|\xi_m\|_{\mathbf{BC_t}}\right)\mathrm{e}^{\left(1+\|\xi_m\|_{\mathbf{BC_t}}\right)t}\mu_o^m(\mathbb{R}_+)\, t \\ &\qquad + \|\psi\|_{\mathbf{L}^\infty}\mathrm{e}^{\|\xi_m\|_{\mathbf{BC_t}}t}\cdot\max\left\{1,\mu_o^c(\mathbb{R}_+)\right\}\|b_m\|_{\mathbf{BC_t}}\mathrm{e}^{2\left(1+\|(\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)t}\, t \\ &\quad\le \|\psi\|_{\infty,\mathbf{Lip}}\cdot\max\left\{1,\mu_o^m(\mathbb{R}_+),\mu_o^c(\mathbb{R}_+)\right\}\left(1 + \|(\xi_m,b_m)\|_{\mathbf{BC_t}}\right)\mathrm{e}^{2\left(1+\|(\xi_m,\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)t}\, t.\end{aligned}$$

Taking supremum over all functions ψ such that $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+)$ and $\|\psi\|_{\infty,\mathbf{Lip}} \le 1$ gives

$$\rho_F(\mu_t^m,\mu_o^m) \le \max\left\{1,\mu_o^m(\mathbb{R}_+),\mu_o^c(\mathbb{R}_+)\right\}\left(1 + \|(\xi_m,b_m)\|_{\mathbf{BC_t}}\right)\mathrm{e}^{2\left(1+\|(\xi_m,\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)t}\, t,$$

for all $t \in [0,T]$. This allows to estimate a total mass of μ^m in time t.

$$\begin{aligned}\mu_t^m(\mathbb{R}_+) &\le \max\left\{1,\mu_o^m(\mathbb{R}_+),\mu_o^c(\mathbb{R}_+)\right\}\left(1 + \|(\xi_m,b_m)\|_{\mathbf{BC_t}}\right)\mathrm{e}^{2\left(1+\|(\xi_m,\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)t}\, t + \mu_o^m(\mathbb{R}_+) \\ &\le \max\left\{1,\mu_o^m(\mathbb{R}_+),\mu_o^c(\mathbb{R}_+)\right\}\left[\left(1 + \|(\xi_m,b_m)\|_{\mathbf{BC_t}}\right)\, t + 1\right]\mathrm{e}^{2\left(1+\|(\xi_m,\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)t} \\ &\le \max\left\{1,\mu_o^m(\mathbb{R}_+),\mu_o^c(\mathbb{R}_+)\right\}\mathrm{e}^{3\left(1+\|(\xi_m,\xi_c,b_m,\mathcal{T})\|_{\mathbf{BC_t}}\right)t}.\end{aligned}$$

Using the analogous arguments as above, formulas (3.48), (3.58) and the latter inequality we obtain the following Lipschitz estimate

$$\begin{aligned}&\rho_F(\mu_{t_2}^m,\mu_{t_1}^m) \\ &\quad\le \max\left\{1,\mu_{t_1}^m(\mathbb{R}_+),\mu_{t_1}^c(\mathbb{R}_+)\right\}\left(1 + \|(\xi_m,b_m)\|_{\mathbf{BC_t}}\right)\mathrm{e}^{2\left(1+\|(\xi_m,\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)(t_2-t_1)}(t_2 - t_1) \\ &\quad\le \max\left\{1,\mu_{t_1}^m(\mathbb{R}_+),\mu_{t_1}^c(\mathbb{R}_+)\right\}\left(1 + \|(\xi_m,b_m)\|_{\mathbf{BC_t}}\right)\mathrm{e}^{3\left(1+\|(\xi_m,\xi_c,b_m,\mathcal{T})\|_{\mathbf{BC_t}}\right)t_2}(t_2 - t_1).\end{aligned}$$

ii) Let $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+)$ such that $\|\psi\|_{\infty,\mathbf{Lip}} \leq 1$. By the formula (3.58)

$$\int_{\mathbb{R}_+} \psi(x)\mathrm{d}(\mu_t^c - \nu_t^c)(x) = \int_{\mathbb{R}_+} \varphi_{t,\psi}(0,x)\mathrm{d}(\mu_o^c - \nu_o^c)(x).$$

Taking supremum over all functions ψ finishes the proof due to estimates (3.46) and (3.47) for a dual problem.
iii) Let $\psi \in (\mathbf{C^1} \cap \mathbf{W^{1,\infty}})(\mathbb{R}_+)$ be such that $\|\psi\|_{\infty,\mathbf{Lip}} \leq 1$ and $\tilde{\varphi}_{t,\psi}$ be a solution to the dual problem (3.44) with terminal data ψ and coefficient $\tilde{\xi}_m$. Let $\tilde{\mu}_t^m$ be a solution to (3.16.1) with a boundary condition given by $D_\lambda \tilde{\mu}_t^m(0^+) = \tilde{F}^{m,\mu^c}(t) = \int_{\mathbb{R}_+^2} \tilde{b}(t,z)\mathrm{d}\tilde{\mu}_t^c(z)$. Then, by formula (3.58) and estimates (3.46), (3.49), (3.53)

$$\begin{aligned}
&\int_{\mathbb{R}_+} \psi(x)\mathrm{d}\left(\mu_t^m - \tilde{\mu}_t^m\right)(x) \\
&= \int_{\mathbb{R}_+} \left(\varphi_{t,\psi}(0,x) - \tilde{\varphi}_{t,\psi}(0,x)\right)\mathrm{d}\mu_o(x) + \int_0^t \varphi_{t,\psi}(s,x)\left(F^{m,\mu^c}(s) - \tilde{F}^{m,\mu^c}(s)\right)\mathrm{d}s \\
&\leq \|\varphi_{t,\psi}(0,\cdot) - \tilde{\varphi}_{t,\psi}(0,\cdot)\|_{\mathbf{L^\infty}}\mu_o^m(\mathbb{R}_+) + t\sup_{s\in[0,t]}\|\varphi_{t,\psi}(s,\cdot)\|_{\mathbf{L^\infty}}\sup_{s\in[0,t]}\left|F^m(s) - \tilde{F}^m(s)\right| \\
&\leq \|\xi_m - \tilde{\xi}_m\|_{\mathbf{BC_t}} t\mathrm{e}^{2\left(1+\|(\xi_m,\tilde{\xi}_m)\|_{\mathbf{BC_t}}\right)t}\mu_o^m(\mathbb{R}_+) \\
&\qquad + t\mathrm{e}^{\|\xi_m\|_{\mathbf{BC_t}}t}\left(\|b_m - \tilde{b}_m\|_{\mathbf{BC_t}}\sup_{s\in[0,t]}\mu_s^c(\mathbb{R}_+^2) + \|\tilde{b}_m\|_{\mathbf{BC_t}}\sup_{s\in[0,t]}\rho_F(\mu_s^c,\tilde{\mu}_s^c)\right) \\
&\leq \|\xi_m - \tilde{\xi}_m\|_{\mathbf{BC_t}} t\mathrm{e}^{2\left(1+\|(\xi_m,\tilde{\xi}_m)\|_{\mathbf{BC_t}}\right)t}\mu_o^m(\mathbb{R}_+) \\
&\qquad + t\mathrm{e}^{\|\xi_m\|_{\mathbf{BC_t}}t}\|b_m - \tilde{b}_m\|_{\mathbf{BC_t}}\max\left\{1,\mu_o^c(\mathbb{R}_+)\right\}\mathrm{e}^{2\left(1+\|(\xi_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)t} \\
&\qquad + t\mathrm{e}^{\|\xi_m\|_{\mathbf{BC_t}}t}\|\tilde{b}_m\|_{\mathbf{BC_t}}\, t\,\max\{1,\mu_o^c(\mathbb{R}_+^2)\}\,\|(\xi_c - \tilde{\xi}_c, \mathcal{T} - \tilde{\mathcal{T}})\|_{\mathbf{BC_t}}\mathrm{e}^{2\left(1+\|(\xi_c,\tilde{\xi}_c,\mathcal{T})\|_{\mathbf{BC_t}}\right)t} \\
&\leq t\,\max\{1,\mu_o^m(\mathbb{R}_+),\mu_o^c(\mathbb{R}_+^2)\}\mathrm{e}^{2\left(1+\|(\xi_m,\tilde{\xi}_m,\xi_c,\tilde{\xi}_c,\tilde{b}_m,\mathcal{T})\|_{\mathbf{BC_t}}\right)t} \\
&\quad\cdot\qquad \|(\xi_m - \tilde{\xi}_m, b_m - \tilde{b}_m, \xi_c - \tilde{\xi}_c, \mathcal{T} - \tilde{\mathcal{T}})\|_{\mathbf{BC_t}}.
\end{aligned}$$

Taking supremum over all functions ψ finishes the proof.
iv) The proof is analogous to the proof of Lemma 3.50, claim *iv)*.
v) The equality follows from **iv)** by setting

$$t_1 = 0, t_2 = t \text{ and } \varphi(s,x) = \varphi_{T,\psi}(s + (T - t_2), x).$$

vi) We proved that there exists a solution to (3.16.1) which also fulfils (3.58). This equation characterizes μ^m uniquely, hence each μ^m given by (3.58) is a solution to (3.16.1).
□

3.5. Nonlinear Case

Before we prove the main theorem of this chapter, we show that if a solution to the nonlinear problem (3.1) exists, then it grows at most exponentially.

Lemma 3.59. *Assume that a solution to the nonlinear system* (3.1) *exists on the time interval on which the non-autonomous model functions* ξ*,* b *and* $\mathcal{T}$ *are defined. Then, there is no blow-up in finite time, that is for a fixed* $T > 0$ *and for all* $0 \le t \le T$ *it holds that*

$$\left(\mu_t^m(\mathbb{R}_+) + \mu_t^f(\mathbb{R}_+) + \mu_t^c(\mathbb{R}_+^2)\right) \le \left(\mu_o^m(\mathbb{R}_+) + \mu_o^f(\mathbb{R}_+) + \mu_o^c(\mathbb{R}_+^2)\right) \mathrm{e}^{C^* t}.$$

Proof of Lemma 3.59. Remark 3.59 follows from a boundedness of the operator $\mathcal{T}$. Due to this assumption it holds that

$$\|\mathcal{T}(t, \mu_t^m, \mu_t^f, \mu_t^c)\|_{(\mathbf{W}^{1,\infty})^*} \le C\left(\|\mu_t^m\|_{(\mathbf{W}^{1,\infty})^*} + \|\mu_t^f\|_{(\mathbf{W}^{1,\infty})^*} + \|\mu_t^c\|_{(\mathbf{W}^{1,\infty})^*}\right).$$

Let us analyse the equation describing the evolution of couples, that is

$$\partial_t \mu_t^c + \partial_{z_1}\mu_t^c + \partial_{z_2}\mu_t^c + \xi_c(t, \mu_t^m, \mu_t^f, \mu_t^c)\mu_t^c = \mathcal{T}(t, \mu_t^m, \mu_t^f, \mu_t^c).$$

The following change of variables $(t,z) \xrightarrow{\Phi} (t, X(t;z))$, where $X(t;z)$ is a solution to

$$\frac{d}{dt}X(t;z) = 1, \quad X(0;z) = z,$$

transforms the original equation into the ODE

$$\partial_t \mu_t^c + \xi_c(t, \mu_t^m, \mu_t^f, \mu_t^c)\mu_t^c = \mathcal{T}(t, \mu_t^m, \mu_t^f, \mu_t^c).$$

Let $\varphi \in \mathbf{C^1}([0,T]\times\mathbb{R}_+;\mathbb{R})$. Then, for any $t \in [0,T]$ it holds that

$$\int_0^t \left(\int_{\mathbb{R}_+^2} \left(\partial_s\varphi(s,x) - \xi_c(s, \mu_s^m, \mu_s^f, \mu_s^c)\varphi(s,x)\right) \mathrm{d}\mu_s^c(x)\mathrm{d}s + \int_{\mathbb{R}_+^2} \varphi(s,x)\mathrm{d}\mathcal{T}(s, \mu_s^m, \mu_s^f, \mu_s^c)\right)\mathrm{d}s$$
$$= \int_{\mathbb{R}_+^2} \varphi(t,x)\mathrm{d}\mu_t^c(x) - \int_{\mathbb{R}_+^2} \varphi(0,x)\mathrm{d}\mu_o^c(x).$$

In particular, for a constant function $\varphi(t,x) = 1$ we have

$$-\int_0^t \int_{\mathbb{R}_+^2} \xi_c(s, \mu_s^m, \mu_s^f, \mu_s^c)\mathrm{d}\mu_s^c(x)\mathrm{d}s + \int_0^t \int_{\mathbb{R}_+^2} \mathrm{d}\mathcal{T}(s, \mu_s^m, \mu_s^f, \mu_s^c)\mathrm{d}s = \mu_t^c(\mathbb{R}_+^2) - \mu_o^c(\mathbb{R}_+^2).$$

Therefore,

$$\begin{aligned} \mu_t^c(\mathbb{R}_+^2) &\le \mu_o^c(\mathbb{R}_+^2) + \int_0^t \left(\|\xi_c\|_{\mathbf{BC}}\, \mu_s^c(\mathbb{R}_+^2) + \mathcal{T}(\mathbb{R}_+^2)\right)\mathrm{d}s \\ &\le \mu_o^c(\mathbb{R}_+^2) + \int_0^t \|\xi_c\|_{\mathbf{BC}}\, \mu_s^c(\mathbb{R}_+^2)\mathrm{d}s + C\int_0^t \left(\mu_s^m(\mathbb{R}_+) + \mu_s^f(\mathbb{R}_+) + \mu_s^c(\mathbb{R}_+^2)\right)\mathrm{d}s. \end{aligned}$$

Analogously we show that for $i = m, f$ the following estimate holds

$$\mu_t^i(\mathbb{R}_+) \le \mu_o^i(\mathbb{R}_+) + \int_0^t \left(\|\xi_i\|_{\mathbf{BC}}\mu_s^i(\mathbb{R}_+) + \|b_i\|_{\mathbf{BC}}\mu_s^c(\mathbb{R}_+^2)\right)\mathrm{d}s.$$

Summarizing,

$$\begin{aligned} \left(\mu_t^m(\mathbb{R}_+) + \mu_t^f(\mathbb{R}_+) + \mu_t^c(\mathbb{R}_+^2)\right) &\le \left(\mu_o^m(\mathbb{R}_+) + \mu_o^f(\mathbb{R}_+) + \mu_o^c(\mathbb{R}_+^2)\right) \\ &\quad + C^*\int_0^t \left(\mu_s^m(\mathbb{R}_+) + \mu_s^f(\mathbb{R}_+) + \mu_s^c(\mathbb{R}_+^2)\right)\mathrm{d}s, \end{aligned}$$

where $C^* = C^*(\|(\xi_m, \xi_f, \xi_c, b_m, b_f)\|_{\mathbf{BC}}, \|\theta\|_{\mathbf{L}^\infty}, \|(h,g)\|_{\mathbf{L}^\infty})$. Application of the Gronwall's Lemma yields

$$\left(\mu_t^m(\mathbb{R}_+) + \mu_t^f(\mathbb{R}_+) + \mu_t^c(\mathbb{R}_+^2)\right) \le \left(\mu_o^m(\mathbb{R}_+) + \mu_o^f(\mathbb{R}_+) + \mu_o^c(\mathbb{R}_+^2)\right)\mathrm{e}^{C^* t}.$$

□

Proof of Theorem 3.10. Let $\mathbf{u}_o = (\mu_o^m, \mu_o^f, \mu_o^c) \in \mathcal{U}$ be an initial measure in (3.1) and b_m, b_f, ξ_m, ξ_f, ξ_c, $\mathcal{T}$ satisfy assumptions (3.4)–(3.7). Let us introduce a space $\mathbf{BC}(I; \bar{B}_R(\mathbf{u}_o))$, where $I = [0, \varepsilon]$ with ε to be chosen later on and

$$\bar{B}_R(\mathbf{u}_o) = \{\mathbf{v} \in \mathcal{U} \ : \ \mathbf{d}(\mathbf{u}_o, \mathbf{v}) \leq R\},$$

where $\mathbf{d}$ is defined by (3.3). We equip $\mathbf{BC}(I; \bar{B}_R(\mathbf{u}_o))$ in the following norm

$$\|\mathbf{u}\|_{\mathbf{BC}} = \sup_{t \in [0,T]} \left(\|\mu_1(t)\|_{(\mathbf{W^{1,\infty}})^*} + \|\mu_2(t)\|_{(\mathbf{W^{1,\infty}})^*} + \|\mu_3(t)\|_{(\mathbf{W^{1,\infty}})^*} \right).$$

Note that $\|\mathbf{u} - \mathbf{v}\|_{\mathbf{BC}} = \sup_{t \in [0,T]} \mathbf{d}(\mathbf{u}(t), \mathbf{v}(t))$. Moreover, the space $\left(\mathbf{BC}(I; \bar{B}_R(\mathbf{u}_o)), \|\cdot\|_{\mathbf{BC}}\right)$ is complete. We define an operator $\mathcal{Z}$ on $\mathbf{BC}(I; \bar{B}_R(\mathbf{u}_o))$ as follows

$$\mathcal{Z} : \mathbf{BC}(I; \bar{B}_R(\mathbf{u}_o)) \to \mathbf{BC}(I; \bar{B}_R(\mathbf{u}_o)), \quad \mathcal{Z}(\mathbf{u}) = \mathbf{v}_{(b,\, \xi, \mathcal{T})(\mathbf{u})}.$$

The evaluation of the operator $\mathcal{Z}$ on any element $\mathbf{u} = (\mu^m, \mu^f, \mu^c)$ is obtained within two steps. In the first step we solve the linear non-autonomous equation (3.16.3) with initial datum $\mu^c(0)$ and coefficients $\xi_c(\cdot, \mu^m, \mu^f, \mu^c)$, $\mathcal{T}(\cdot, \mu^m, \mu^f, \mu^c)$. Let denote this solution as ν^c. In the second step we solve equations (3.16.1) and (3.16.2) with initial datum $\mu^m(0)$, $\mu^f(0)$ and coefficients $b_m(\cdot, \mu^m, \mu^f)$, $\xi_m(\cdot, \mu^m, \mu^f)$ and $b_f(\cdot, \mu^m, \mu^f)$, $\xi_f(\cdot, \mu^m, \mu^f)$ respectively. We plug ν^c into the boundary terms. As a result we obtain measures ν^m and ν^f. This procedure produces a vector (ν^m, ν^f, ν^c) being the value of $\mathcal{Z}(\mathbf{u})$, which is further denoted as $\mathbf{v}_{(b,\, \xi, \mathcal{T})(\mathbf{u})}$. Now, we need to prove that the operator $\mathcal{Z}$ is well defined, meaning that its image is a bounded continuous function taking values in $\bar{B}_R(\mathbf{u}_o)$. Let $\mathbf{u} = (\mu^m, \mu^f, \mu^c) \in \mathbf{BC}(I, \bar{B}_R(\mathbf{u}_o))$ and $\mathbf{v}_{(b,\, \xi, \mathcal{T})(\mathbf{u})} = (\nu^m, \nu^f, \nu^c)$. In order to estimate $\|\mathbf{v}_{(b,\, \xi, \mathcal{T})(\mathbf{u})} - \mathbf{u_o}\|_{\mathbf{BC}}$ let us recall Lemma 3.33, claim i), that is

$$\sup_{t \in [0,\varepsilon]} \rho_F(\nu_t^c, \mu_o^c) \leq \max\{1, \mu_o^c(\mathbb{R}_+^2)\} K_c \mathrm{e}^{2K_c\varepsilon} \varepsilon, \text{ where}$$

$$\begin{aligned}
K_c &\leq 1 + \|\xi_c\|_{\mathbf{BC}} + \sup_{t \in [0,\varepsilon]} \|\mathcal{T}(t)\|_{(\mathbf{W^{1,\infty}})^*} \\
&\leq 1 + \|\xi_c\|_{\mathbf{BC}} + \sup_{t \in [0,\varepsilon]} \left(\|h\|_{\mathbf{L}^\infty} + \|g\|_{\mathbf{L}^\infty}\right) \left(\mu_t^m(\mathbb{R}_+) + \mu_t^m(\mathbb{R}_+) + \mu_t^c(\mathbb{R}_+^2)\right) \\
&\leq 1 + \|\xi_c\|_{\mathbf{BC}} + \|\theta\|_{\mathbf{L}^\infty} \|(h,g)\|_{\mathbf{L}^\infty} \\
&\qquad \cdot \sup_{t \in [0,\varepsilon]} \left(\rho_F(\mu_t^m, \mu_o^m) + \mu_o^m(\mathbb{R}_+) + \rho_F(\mu_t^f, \mu_o^f) + \mu_o^f(\mathbb{R}_+) + \rho_F(\mu_t^c, \mu_o^c) + \mu_o^c(\mathbb{R}_+^2)\right) \\
&\leq 1 + \|\xi_c\|_{\mathbf{BC}} + \|\theta\|_{\mathbf{L}^\infty} \|(h,g)\|_{\mathbf{L}^\infty} \left(R + \mu_o^m(\mathbb{R}_+) + \mu_o^f(\mathbb{R}_+) + \mu_o^c(\mathbb{R}_+^2)\right).
\end{aligned}$$

Analogously, according to Lemma 3.50,

$$\sup_{t \in [0,\varepsilon]} \rho_F(\nu_t^m, \mu_o^m) \leq \max\{1, \mu_o^m(\mathbb{R}_+), \mu_o^c(\mathbb{R}_+^2)\} \left(1 + \|(b_m, \xi_m)\|_{\mathbf{BC}}\right) \mathrm{e}^{5K_m\varepsilon} \varepsilon, \text{ where}$$

$$\begin{aligned}
K_m &\leq 1 + \|(\xi_m, \xi_c, b_m)\|_{\mathbf{BC}} + \sup_{t \in [0,\varepsilon]} \|\mathcal{T}(t)\|_{(\mathbf{W^{1,\infty}})^*} \\
&\leq 1 + \|(\xi_m, \xi_c, b_m)\|_{\mathbf{BC}} + \|\theta\|_{\mathbf{L}^\infty} \|(h,g)\|_{\mathbf{L}^\infty} \left(R + \mu_o^m(\mathbb{R}_+) + \mu_o^f(\mathbb{R}_+) + \mu_o^c(\mathbb{R}_+^2)\right).
\end{aligned}$$

The analogous estimate holds for $\sup_{t\in[0,\varepsilon]}\rho_F(\nu_t^f,\mu_o^f)$. Let define

$$M=\max\{1,\mu_o^m(\mathbb{R}_+)+\mu_o^f(\mathbb{R}_+)+\mu_o^c(\mathbb{R}_+^2)\}.$$

Additionally, assume that $\varepsilon<1$. Summarizing, we need the following inequality to be fulfilled in order to obtain that $\mathcal{Z}$ is well-defined.

$$\varepsilon M K_c \mathrm{e}^{2K_c}+\varepsilon M\left(1+\|(b_m,\xi_m)\|_{\mathbf{BC}}\right)\mathrm{e}^{5K_m}+\varepsilon M\left(1+\|(b_f,\xi_f)\|_{\mathbf{BC}}\right)\mathrm{e}^{5K_f}<R,$$

which is equivalent to

$$\varepsilon<R\mathrm{e}^{-5K^*}\left(1+\|(\xi_c,\xi_m,\xi_f,b_m,b_f)\|_{\mathbf{BC}}+\|\theta\|_{\mathbf{L}^\infty}\|(h,g)\|_{\mathbf{L}^\infty}(R+M)\right)^{-1}/3M:=\upsilon_1,$$

where $K^*=\max\{K_c,K_m,K_f\}$. Now, we prove that $\mathcal{Z}$ is a contraction for ε small enough. We will show that $\mathcal{Z}$ is a Lipschitz operator with Lipschitz constant smaller than 1.

$$\begin{aligned}
&\|\mathcal{Z}(\mathbf{u})-\mathcal{Z}(\bar{\mathbf{u}})\|_{\mathbf{BC}}\\
&\quad=\sup_{t\in[0,\varepsilon]}\mathbf{d}\left(\mathcal{Z}(\mathbf{u})(t),\mathcal{Z}(\bar{\mathbf{u}})(t)\right)=\sup_{t\in[0,\varepsilon]}\mathbf{d}\left(\mathbf{v}_{(b,\,\xi,\mathcal{T})(\mathbf{u})}(t),\mathbf{v}_{(b,\,\xi,\mathcal{T})(\bar{\mathbf{u}})}(t)\right)\\
&\quad=\sup_{t\in[0,\varepsilon]}C_3\,t\mathrm{e}^{C_4t}\Big(\|\xi_m(t,\mathbf{u})-\xi_m(t,\bar{\mathbf{u}})\|_{\infty,\mathbf{Lip}}+\|\xi_f(t,\mathbf{u})-\xi_f(t,\bar{\mathbf{u}})\|_{\infty,\mathbf{Lip}}+\\
&\qquad\quad+\|\xi_c(t,\mathbf{u})-\xi_c(t,\bar{\mathbf{u}})\|_{\infty,\mathbf{Lip}}+\|b_m(t,\mathbf{u})-b_m(t,\bar{\mathbf{u}})\|_{\infty,\mathbf{Lip}}+\\
&\qquad\quad+\|b_f(t,\mathbf{u})-b_f(t,\bar{\mathbf{u}})\|_{\infty,\mathbf{Lip}}+\|\mathcal{T}(t,\mathbf{u})-\mathcal{T}(t,\bar{\mathbf{u}})\|_{\infty,\mathbf{Lip}}\Big)\\
&\quad\le\sup_{t\in[0,\varepsilon]}C_3\,t\mathrm{e}^{C_4t}\Big(\mathbf{Lip}(\xi_m(t,\cdot))+\mathbf{Lip}(\xi_f(t,\cdot))+\mathbf{Lip}(\xi_c(t,\cdot))+\mathbf{Lip}(b_m(t,\cdot))+\\
&\qquad\quad+\mathbf{Lip}(b_f(t,\cdot))+\mathbf{Lip}(\mathcal{T}(t,\cdot))\Big)\sup_{t\in[0,T]}\mathbf{d}\left(\mathbf{u}(t),\bar{\mathbf{u}}(t)\right)\\
&\quad\le C_3\,\varepsilon\mathrm{e}^{C_4\varepsilon}L\|\mathbf{u}(t)-\bar{\mathbf{u}}(t)\|_{\mathbf{BC}},
\end{aligned}$$

where $C_3=3M$,

$$\begin{aligned}
C_4&=2\left(1+\|(\xi,\tilde{\xi},\tilde{b})\|_{\mathbf{BC_t}}+\sup_{t\in[0,\varepsilon]}\|\mathcal{T}(t)\|_{(\mathbf{W^{1,\infty}})^*}\right)\\
&\le 2\left(1+\|(\xi,\tilde{\xi},\tilde{b})\|_{\mathbf{BC_t}}+\|\theta\|_{\mathbf{L}^\infty}\|(h,g)\|_{\mathbf{L}^\infty}(R+M)\right),\\
L&=\mathbf{Lip}(\xi_m)+\mathbf{Lip}(\xi_f)+\mathbf{Lip}(\xi_c)+\mathbf{Lip}(b_m)+\mathbf{Lip}(b_f)+\mathbf{Lip}(\mathcal{T}).
\end{aligned}$$

Assumption $\varepsilon<1$ implies that the following inequality has to be fulfilled

$$\|\mathcal{Z}(\mathbf{u})-\mathcal{Z}(\bar{\mathbf{u}})\|_{\mathbf{BC}}\le C_3\,\varepsilon\mathrm{e}^{C_4}L\|\mathbf{u}(t)-\bar{\mathbf{u}}(t)\|_{\mathbf{BC}}.$$

Lipschitz constant of $\mathcal{Z}$ is smaller then 1, if $\mathbf{Lip}(\mathcal{Z})=C_3\varepsilon\mathrm{e}^{C_4}L<1$. Hence,

$$\varepsilon<\left(C_3\mathrm{e}^{C_4}L\right)^{-1}=:\upsilon_2.$$

Constants υ_1 and υ_2 are finite and independent on time. We proved that $\mathcal{Z}$ is a contraction on a complete metric space $\mathbf{BC}(I,\bar{B}_R(\mathbf{u}_o))$, where $0<\varepsilon<\min\{1,\upsilon_1,\upsilon_2\}$. From the Banach Fixed Point Theorem it follows that there exists unique $\mathbf{u}^*$, such that $\mathcal{Z}(\mathbf{u}^*)=$

$\mathbf{u}^*$. This solution can be extended on the $[\varepsilon, 2\varepsilon]$ interval, since υ_1 and υ_2 were chosen independently on time. Therefore, iterating this procedure for all intervals $[(n-1)\varepsilon, n\varepsilon]$ we obtain a solution, which is defned on the whole $[0, T]$. Moreover, the sequence of solutions to a non-autonomus system defined inductively by

$$\mathbf{u}_1 = \mathcal{Z}(\mathbf{u}_o), \quad \mathbf{u}_{n+1} = \mathcal{Z}(\mathbf{u}_n)$$

converges in $\|\cdot\|_{\mathbf{BC}}$ to $\mathbf{u}^*$. Thus, passing to the limit in the integrals (3.41), (3.57) (and in the corresponding integral for μ^f) proves that $\mathbf{u}^*$ is the solution to the system (3.1) in the sense of Definition 3.9. From *i)* in Lemma 3.50 and *i)* in Lemma 3.33 it follows that $\mathbf{u}^*$ is Lipschitz continuous with respect to time. Estimates in claims *i)* and *ii)* are consequences of estimates for a linear non-autonomous case (see Lemma 3.50 and Lemma 3.33). □

Chapter 4

Numerical Scheme Based on the Splitting Technique for the One-sex Structured Population Model

4.1. Introduction

In the present chapter we develop a numerical scheme for a particular class of one-sex structured population models. This scheme is constructed through the splitting technique applied in Chapter 2 and corresponds with a current trend basing on a kinetic approach to the population dynamics problems [1, 2, 42, 43, 44, 68, 71]. Certainly, the kinetic approach is coherent with empirical data, as the result of a measurement is usually a number of individuals, which state is within a specified range. A challenge associated with the application of kinetic theory in the population models is the non-conservative character of these problems. Depending on the model, new individuals appear either on the boundary or at the arbitrary point of the domain. Therefore, natural distances for probability measures like Wasserstain distances cannot be exploited.

One of the commonly used methods for solving (2.2), which originates from the kinetic theory, is the Escalator Boxcar Train algorithm described in [26]. A concept of this method bases on approximating the solution by a sum of Dirac measures, each one of which represents the average state and number of individuals within a specified group called a cohort. The EBT method will be further analyzed in Chapter 5 of this thesis. Another group of mesh-free methods originated from the kinetic theory are the particle methods. They are frequently used in models describing large groups of interacting particles or individuals. On the contrary to the EBT, the particle methods were originally designed for problems where conservation laws hold and have been successfully applied for solving numerically kinetic models from physics, see [49, 54, 73, 74] and references therein. In particular, they were adopted to the Euler equation in fluid mechanics [83], isentropic Euler equations [39, 13], Vlasov equation in plasma physics, Boltzmann equation, Fokker-Planck equation. Recently, they are also used in problems related to crowd dynamics, pedestrians flow [71, 68] or collective motion of large groups of agents [19].

The scheme presented further in this chapter is similar to the EBT algorithm and particle methods in the sense that the output of all methods can be understood as a description of a collective behaviour of individuals divided into cohorts. The main difference between these schemes bases on defining how new individuals appear in the system. In the particle methods the new individuals usually do not appear due to the conservative character of problems they are applied to. In the EBT method an ODE describing the dynamics of the boundary cohort is imposed. The main idea of the splitting technique is to separate the dynamics induced by a transport operator from the dynamics induced by a nonlocal term. Formally, the algorithm bases on representing a semigroup generated by the solution as a product of two semigroups related to simpler equations. Due to this separation the solution is a sum of Dirac deltas if the initial datum are, despite of the regularizing character of the nonlocal term.

Setting population dynamics problems in the measure framework is a relatively new approach and thus, formal convergence of the particle-based schemes was difficult to establish for a long period of time. One of the first steps in this direction was made in [43, 44], where existence, uniqueness and stability of solutions to (4.1)(B) in the space of Radon measures equipped with the flat metric were proved. The adequate choice of the metric allowed to overcame the non-conservative character of the problem. This result was a basis for the proof of convergence of the EBT method in [12].

In this chapter we will show how the method used for proving well posendess of (4.1) in Chapter 2 can be translated into an applicable numerical scheme and provide estimates on the convergence rate. In particular, we generalize the results obtained in [12], where the problem of the convergence rate was not raised. We discuss the problem of increasing number of Dirac measures which appear due to the process of birth and propose a procedure of the measure approximation together with the error estimate. This chapter is organized as follows. In Section 4.2 we describe the numerical method and the measure reconstruction procedure. In Section 4.3 we present the proof of convergence of the scheme together with the estimate on the convergence rate. In Section 4.4 we present results of numerical simulations for several test cases.

4.2. Particle Method

4.2.1. The Model Equation

Our aim is to present a numerical scheme for the following equation

$$\begin{aligned} \partial_t\mu + \partial_x(b(t,\mu)\mu) + c(t,\mu)\mu &= \int_{\mathbb{R}_+} (\eta(t,\mu))(y)\mathrm{d}\mu(y), \\ \mu(0) &= \mu_o, \end{aligned} \tag{4.1}$$

of which mathematical properties have been already analyzed in Chapter 2. Here, $t \in [0,T]$ and $x > 0$ denote time and a structural variable respectively, b, c, η are vital functions and μ is a Radon measure describing a distribution of individuals with respect to x. We recall that a function $b(t,\mu)$ describes a dynamics of a transformation of an individual's state. By $c(t,\mu)$ we denote a rate of evolution (growth or death rate). The integral on the right hand side accounts for an influx of the new individuals into the system. In this chapter we assume the following form of the function η

$$\eta(t,\mu)(y) = \sum_{p=1}^{r} \beta_p(t,\mu)(y)\delta_{x=\bar{x}_p(y)}, \tag{4.2}$$

which means that an individual at the state y gives rise to offsprings being at the states $\{\bar{x}_p(y)\}$, $p = 1,\dots,r$. We assume that

$$\begin{aligned} \beta_p \,:\, [0,T]\times\mathcal{M}^+(\mathbb{R}_+) &\rightarrow \mathbf{W^{1,\infty}}(\mathbb{R}_+;\mathbb{R}), && (4.3)\\ \bar{x}_p \,:\, \mathbb{R}_+ &\rightarrow \mathbb{R}_+, \quad \text{for } p = 1,\dots,r, && (4.4) \end{aligned}$$

and require the following regularity of β_p and $\bar{x}_p$.

$$\begin{aligned} \beta_p &\in \mathbf{C_b^{\alpha,1}}\left([0,T]\times\mathcal{M}^+(\mathbb{R}_+);\ \mathbf{W^{1,\infty}}(\mathbb{R}_+;\mathbb{R})\right), && (4.5)\\ \bar{x}_p &\in \mathbf{Lip}(\mathbb{R}_+;\mathbb{R}_+). && (4.6) \end{aligned}$$

Additionally, we need to assume that

$$\sum_{p=1}^{r} \|\beta_p\|_{\mathbf{C_b^{\alpha,1}}} < +\infty \quad \text{and} \quad \sum_{p=1}^{r} \mathbf{Lip}(\bar{x}_p) < +\infty. \tag{4.7}$$

Regularity of β_p and x_p imposed in (4.3) - (4.7) guarantees that η defined by (4.2) fulfills the assumption (2.7). In case all new born individuals have the same physiological state x_b, then

$$r = 1 \quad \text{and} \quad \bar{x}^1(y) = x_b \quad \forall y \in \mathbb{R}_+$$

and the integral transforms into a boundary condition (see also Chapter 2, (2.2), Chapter 5). Further in this chapter we will refer to this particular case as (4.1)(B). The assumptions on b and c are the same as in Chapter 2 (2.6).

4.2.2. Description of the Scheme

The main idea of the particle methods is to approximate a solution at each time by a sum of Dirac measures. Note that even if the initial datum in (4.1) is a sum of Dirac Deltas, the integral term possibly produces a continuous distribution at $t > 0$. This phenomenon can be avoided due to the splitting algorithm, which allows to separate the transport operator from the integral one and simulate the corresponding problems successively. This is essentially the reason why we have exploited this technique in our scheme. To proceed with a description of the method assume that the approximation of the solution at time $t_k = k\Delta t$ is provided as a sum of Dirac measures, that is

$$\mu_{t_k} = \sum_{i=1}^{M_k} m_k^i\, \delta_{x_k^i}, \quad M_k \in \mathbb{N}. \tag{4.8}$$

The procedure of calculating the approximation of the solution at time t_{k+1} is divided into three main steps. In the first step one calculates the characteristic lines for the cohorts (m^i, x^i) given by (4.8), which is equivalent to solving the following ODE's system on a time interval $[t_k, t_{k+1}]$

$$\frac{d}{ds}x^i(s) = b_k(x^i(s)), \quad x^i(t_k) = x_k^i, \quad i = 1, \ldots, M_k, \tag{4.9}$$

where

$$b_k(x) = b(t_k, \mu_{t_k})(x). \tag{4.10}$$

In other words, each Dirac Delta is transported along its characteristic to the new location x_{k+1}^i without changing its mass. The second step consists in creating new Dirac Deltas due to the influx of new individuals and recalculating the mass of each Dirac Delta. We have already mentioned in the introduction above that for each $(t, \nu) \in [0, T] \times \mathcal{M}^+(\mathbb{R}_+)$, η is given by

$$\eta(t,\nu)(y) = \sum_{p=1}^{r} \beta_p(t,\nu)(y)\, \delta_{x=\bar{x}_p(y)}. \tag{4.11}$$

From this form of η it follows that the set of possible new states x_{k+1}^l at time t_k is

$$\{x_{k+1}^l,\ l = M_k + 1, \ldots, M_{k+1}\} := \{\bar{x}_p(x_{k+1}^i),\ i = 1, \ldots, M_k,\ p = 1, \ldots, r\}.$$

Let us define

$$\begin{aligned} \mu_k^1 &= \sum_{i=1}^{M_{k+1}} m_k^i\, \delta_{x_{k+1}^i}, \\ c_k(x) &= c\left(t_k, \mu_k^1\right)(x), \qquad (4.12) \\ \eta_k(y) &= \sum_{p=1}^{r} \beta_p(t_k, \mu_k^1)(y)\, \delta_{x=\bar{x}_p(y)} \qquad (4.13) \end{aligned}$$

and for $i, j \in \{1, \ldots, M_{k+1}\}$

$$\alpha(x_{k+1}^i, x_{k+1}^j) = \begin{cases} \beta_p(t_k, \mu_k^1)(x_{k+1}^j), & \text{if } p \text{ is such that } \bar{x}_p(x_{k+1}^j) = x_{k+1}^i, \\ 0, & \text{otherwise.} \end{cases}$$

We cannot solve an ODE system for the masses directly, since new states will be created at any time $t_k < t < t_{k+1}$. Therefore, we approximate it by the following explicit Euler scheme

$$\begin{aligned} \frac{m_{k+1}^i - m_k^i}{t_{k+1} - t_k} &= -c_k(x_{k+1}^i)m_k^i + \sum_{j=1}^{M_{k+1}} \alpha_k(x_{k+1}^i, x_{k+1}^j)m_k^j, \qquad (4.14) \\ m_k^i &= 0, \text{ for } i = M_k + 1, \ldots, M_{k+1.} \end{aligned}$$

The resulting measure

$$\mu_k^2 = \sum_{i=1}^{M_{k+1}} m_{k+1}^i \delta_{x_{k+1}^i} \tag{4.15}$$

consists of $M_{k+1} \geq M_k$ Dirac Deltas. In some cases it is necessary to approximate the measure (4.15) by a smaller number of Dirac Deltas (see Subsection 4.2.3). If so, we define $\mu_{t_{k+1}} = \mathcal{R}(\mu_k^2)$, where $\mathcal{R}(\mu_k^2)$ is the result of this approximation. Otherwise we let $\mu_{t_{k+1}} = \mu_k^2$.

Remark 4.16. In the particular case where only one new state x^b is allowed we can use the continuum ODE system:

$$\begin{aligned}\frac{d}{ds}m^i(s) &= -c_k(x^i_{k+1})m^i(s), \quad \text{for } i \neq b, \\ \frac{d}{ds}m^b(s) &= -c_k(x^b)m^b(s) + \sum_{j=1}^{M_{k+1}} \alpha_k(x^b, x^j_{k+1})m^j(s),\end{aligned} \tag{4.17}$$

instead of the Euler approximation (4.14).

In the method presented above one has to deal with an increasing number of Dirac measures, which is an important issue to solve from the point of view of numerics. In the simplest case all new individuals have the same size x_b at birth and just one additional Dirac Delta is created at the boundary at each time step. Unfortunately, in many models the number of new particles increases so fast that after several steps the computational cost become unacceptable. For example, in the case of equation describing the process of cell equal mitosis, the number of Dirac Deltas is doubled at each time step. This growth forces us to approximate the numerical solution by a smaller number of Dirac measures after several iterations. We propose some different methods of this reconstruction, which are discussed in the next subsection.

4.2.3. Measure Reconstruction

Due to Definition 4.40 and Lemma 4.44 we restrict our analysis to the probability measures. Let $\mu = \sum_{i=1}^{M} m_i \delta_{x_i}$ be a probability measure with a compact support $K = [k_1, k_2]$. The aim of the reconstruction is to find a smaller number of Dirac Deltas $\bar{M} < M$ such that

$$\mathcal{R}(\mu) := \sum_{j=1}^{\bar{M}} \tilde{m}_j \delta_{\tilde{x}_j} = \text{argmin } W_1\left(\mu, \sum_{j=1}^{\bar{M}} n_j \delta_{y_j}\right), \quad \text{where } \sum_{j=1}^{\bar{M}} n_j = 1 \text{ and } n_j \geq 0, x_j \in \mathbb{R}_+.$$

This minimisation procedure is essentially a linear programming problem which, under some particular assumptions on cycles, can be solved by the simplex algorithm providing the global minimum. However, its complexity is at least cubic. From that reason we exploit less costly (linear cost in the size of the problem) methods of reconstruction, which provide the error of the order $\mathcal{O}(1/\bar{M})$. Note that the cubic cost is unacceptable in our case, since the total cost of the method is quadratic if the number of particles grows linearly with the time step.

A) Fixed-location reconstruction

The idea of the fixed-location reconstruction is to divide the support of the measure μ into $\bar{M}$ equal intervals and put a Dirac Delta with a proper mass in the middle of each interval. The mass of this Dirac Delta is equal to the mass of μ contained in this particular interval. Let $\Delta x = |K|/\bar{M}$ and define

$$\tilde{x}_j = k_1 + \left(j - \frac{1}{2}\right)\Delta x, \quad \tilde{m}_j = \begin{cases} \mu\left([\tilde{x}_j - \Delta x/2, \tilde{x}_j + \Delta x/2)\right), & \text{for } j = 1, \ldots, \bar{M} - 1, \\ \mu\left([\tilde{x}_{\bar{M}} - \Delta x/2, \tilde{x}_{\bar{M}} + \Delta x/2]\right), & \text{for } j = \bar{M}. \end{cases}$$

To estimate the error between μ and $\mathcal{R}(\mu)$ consider a transportation plan γ between both measures. Then, according to [79, Introduction]

$$W_1\left(\mu,\mathcal{R}(\mu)\right) \leq \int_{\mathbb{R}_+^2} |\, x-y\,| \mathrm{d}\gamma(x,y) \leq \int_{\mathbb{R}_+^2} \frac{\Delta x}{2} \mathrm{d}\gamma(x,y) \leq \frac{\Delta x}{2} = \frac{|K|}{2\bar{M}}. \tag{4.18}$$

The second inequality follows from the fact that each particle was shifted by a distance not greater than a half of the interval of a length Δx, while the third one is a consequence of the fact that γ is a probability measure on $\mathbb{R}_+^2$.

B) Fixed-Equal mass reconstruction

The aim of the fixed-equal mass reconstruction is to distribute Dirac Deltas of equal masses over the support of a given measure in a proper way. In our particular case, we want to reduce the number of Dirac Deltas from M to $\bar{M}$, and thus we need to explain an algorithm allowing for splitting of the Dirac Deltas into two. Then, we set

$$\tilde{m}_j = m := \frac{1}{\bar{M}}, \text{ for } j = 1,\dots,\bar{M}.$$

The scheme for determining $\tilde{x}_j$ is the following. We first look for an index n_1, such that $\sum_{i=1}^{n_1-1} m_i < m \leq \sum_{i=1}^{n_1} m_i$. We set

$$\tilde{x}_1 = \sum_{i=1}^{n_1-1} m_i x_i + m'_{n_1} x_{n_1}, \text{ where } m'_{n_1} = \frac{1}{\bar{M}} - \sum_{i=1}^{n_1-1} m_i x_i.$$

Namely, the mass located in x_{n_1} is split into two parts – the amount of mass equal to m'_{n_1} is shifted to $\tilde{x}_1$ and the rest, that is $m_{n_1} - m'_{n_1}$ stays in x_{n_1}. For simplicity, we redefine $m_{n_1} := m_{n_1} - m'_{n_1}$ and repeat the procedure described above until the last point $\tilde{x}_{\bar{M}}$ is found. Note that in each step of the procedure one changes the locations of the Dirac Deltas, of which joint mass is not greater than m. Using an analogous argument as in the previous case, we conclude that in the j-th step we commit an error not greater than $\left|x_{n_j} - x_{n_{j-1}}\right| m$, where $x_{n_o} = k_1$. Since $k_1 = x_{n_o} \leq x_{n_1} \cdots \leq x_{n_{\bar{M}}} \leq k_2$, the total error can be bounded by

$$W_1(\mu,\mathcal{R}(\mu)) \leq |K| m = \frac{|K|}{\bar{M}}. \tag{4.19}$$

Summarizing, the error of the fixed-location and fixed-equal mass reconstruction is a function of $\bar{M}$, i.e., the number of Dirac Deltas approximating the original measure. This reconstruction can be used at $t = 0$, if the initial datum in (4.1) is not a sum of Dirac Deltas or in $t > 0$ in order to deal with the problem of increasing number of Dirac Deltas, which are produced due to birth processes. Note that both reconstructions discussed above are of the order $\mathcal{O}(1/\bar{M})$.

We introduce the following notation:

- $E_I(\bar{M}_o)$ is the upper bound for the error of the initial datum reconstruction defined in terms of W_1 distance. More specifically, for a measure μ such that $M_\mu := \int_{\mathbb{R}_+} \mathrm{d}\mu(x)$, $M_\mu \neq 0$ it holds that

$$W_1\left(\frac{\mu}{M_\mu}, \frac{\mathcal{R}(\mu)}{M_\mu}\right) \leq E_I(\bar{M}_o).$$

- $E_R(\bar{M})$ is the upper bound for the error of the measure reconstruction at time $t > 0$ defined in terms of W_1 distance.

4.3. Convergence Results

The aim of this section is to obtain an estimate on the error between the numerical solution μ_t and the exact solution $\mu(t)$. Let $[0,T]$ be a time interval, N be a number of time steps, $\Delta t = T/N$ be a length of the time step. We define a time mesh $\{t_k\}_{k=0}^N$, where $t_k = k\Delta t$. Let $\bar{M}_k$, $k = 0,1,\ldots N$, be parameters of the measure reconstruction. In particular, $\bar{M}_o$ is the number of Dirac Deltas approximating the initial condition and $\bar{M}_k$ stands for the number of Dirac measures approximating the numerical solution at $t > 0$ after a reconstruction, if performed. We assume that the latter reconstruction is done once per n steps, which means that there are $\mathcal{K} = N/n$ reconstructions, each at the time t_{jn}, where $j = 1,\ldots,\mathcal{K}$. Let $\bar{M}$ be the number of Dirac Deltas after the reconstruction that will not depend on time.

Theorem 4.20. *Let μ be a solution to* (4.1) *with the initial time t_o and initial datum μ_o. Assume that μ_{t_m} is defined as in Subsection 4.2.2 and $m = jn$ for some $j \in \{1,\ldots,\mathcal{K}\}$. Then, there exist nonnegative constants C_1, C_2, C_3, C_4 such that*

$$\rho_F\left(\mu_{t_m}, \mu(t_m)\right) \le C_1\Delta t + C_2(\Delta t)^\alpha + C_3 E_I(\bar{M}_o) + C_4 E_R(\bar{M})j. \tag{4.21}$$

Remark 4.22. The error estimate (4.21) accounts for different error sources. More specifically, the error of the order $\mathcal{O}(\Delta t)$ is a consequence of using the split up algorithm. Term of the order $\mathcal{O}((\Delta t)^\alpha)$ follows from the fact that we solve (4.9), (4.14) with parameter functions independent on time, while b, c and η are in fact of $\mathbf{C}\alpha$ regularity with respect to time. Finally, E_I and E_R are the errors coming from the measure reconstruction procedure.

Proof of Theorem 4.20. The proof is divided into several steps. For simplicity, in all estimates below we will use a generic constant C without specifying its exact form.

Step 1: The auxiliary scheme. Let us define the auxiliary semi-continuous scheme, which consists in solving subsequently the following problems:

$$\begin{cases} \partial_t\mu + \partial_x(\bar{b}_k(x)\mu) &= 0, \quad \text{on} \quad [t_k, t_{k+1}] \times \mathbb{R}_+, \\ \mu(t_k) &= \bar{\mu}_k \end{cases} \tag{4.23}$$

and

$$\begin{cases} \partial_t\mu &= -\bar{\bar{c}}_k(x)\mu + \int_{\mathbb{R}_+} \bar{\bar{\eta}}_k(y)\mathrm{d}\mu(y), \quad \text{on} \quad [t_k, t_{k+1}] \times \mathbb{R}_+, \\ \mu(t_k) &= \bar{\mu}_k^1, \end{cases} \tag{4.24}$$

where $\bar{\mu}_k \in \mathcal{M}^+(\mathbb{R}_+)$, $\bar{\mu}_k^1$ is a solution to the first equation at time t_{k+1} and $\bar{b}_k, \bar{\bar{c}}_k, \bar{\bar{\eta}}_k$ are defined as

$$\begin{aligned} \bar{b}_k(x) &= b\left(t_k, \bar{\mu}_k\right)(x), \\ \bar{\bar{c}}_k(x) &= c\left(t_k, \bar{\mu}_k^1\right), \quad \bar{\bar{\eta}}_k(y) = \sum_{p=1}^r \beta_p(t_k, \bar{\mu}_k^1)(y)\, \delta_{x=\bar{x}_p(y)}. \end{aligned} \tag{4.25}$$

A solution to the second equation at time t_{k+1} is denoted by $\bar{\mu}_k^2$. The output of the one step of this scheme is defined as $\bar{\mu}_{k+1} = \mathcal{R}(\bar{\mu}_k^2)$.

Step 2: Error of the reconstruction. Since $\bar{\mu}_{k+1}$ arises from $\bar{\mu}_k^2$ through the reconstruction, masses of both measures are equal. Therefore, application of Lemma 4.44 yields

$$\rho_F(\bar{\mu}_{k+1}, \bar{\mu}_k^2) \leq \rho(\bar{\mu}_{k+1}, \bar{\mu}_k^2) = M_{\bar{\mu}_k^2} W_1\left(\frac{\bar{\mu}_{k+1}}{M_{\bar{\mu}_k^2}}, \frac{\bar{\mu}_k^2}{M_{\bar{\mu}_k^2}}\right) \leq M_{\bar{\mu}_k^2} E_R(\bar{M}), \tag{4.26}$$

where $M_{\bar{\mu}_k^2} = \bar{\mu}_{k+1}(\mathbb{R}_+) = \bar{\mu}_k^2(\mathbb{R}_+)$ and $E_R(\bar{M})$ is the error of the reconstruction introduced in Subsection 4.2.3. $E_R(\bar{M})$ depends on the reconstruction type and is equal to $|K|/(2\bar{M})$ in the case of the fixed-location reconstruction and $|K|/\bar{M}$ in the case of the fixed-equal mass reconstruction. Note that $M_{\bar{\mu}_k^2}$ can be bounded independently on k. Indeed, on each time interval $[t_k, t_{k+1}]$ mass grows at most exponentially, which follows from [21, Theorem 2.10, (i)], and reconstructions, if performed, do not change the mass. Thus, there exists a constant $C = C(T, b, c, \eta, \mu_o)$ such that

$$M_{\bar{\mu}_k^2} \leq C.$$

Step 3: Error of splitting. Let $\nu(t)$ be a solution to (4.1) on a time interval $[t_k, t_{k+1}]$ with initial datum $\bar{\mu}_k$ and parameter functions $\bar{b}_k$, $\bar{c}_k$, $\bar{\eta}_k$, where $\bar{b}_k$ is defined by (4.25),

$$\bar{c}_k(x) = c(t_k, \bar{\mu}_k), \tag{4.27}$$

$$\bar{\eta}_k(y) = \sum_{p=1}^{r} \bar{\beta}_{p,k}(y)\, \delta_{x=\bar{x}_p(y)}, \quad \text{where} \quad \bar{\beta}_{p,k}(y) = \beta_p(t_k, \bar{\mu}_k)(y). \tag{4.28}$$

According to [25, Proposition 2.7] and Proposition 2.22 the distance between $\bar{\mu}_k^2$ and $\nu(t_{k+1})$, that is the error coming from the application of the splitting algorithm can be estimated as following

$$\rho_F(\bar{\mu}_k^2, \nu(t_{k+1})) \leq C(\Delta t)^2. \tag{4.29}$$

To estimate a distance between $\nu(t_{k+1})$ and $\mu(t_{k+1})$ consider $\zeta(t)$, which is a solution to (4.1) on a time interval $[t_k, t_{k+1}]$ with initial datum $\mu(t_k)$ and coefficients $\bar{b}_k$, $\bar{c}_k$, $\bar{\eta}_k$. By triangle inequality

$$\rho_F(\nu(t_{k+1}), \mu(t_{k+1})) \leq \rho_F(\nu(t_{k+1}), \zeta(t_{k+1})) + \rho_F(\zeta(t_{k+1}), \mu(t_{k+1})).$$

The first term of the inequality above is a distance between solutions to (4.1) with different initial datum, that is $\bar{\mu}_k$ and $\mu(t_k)$ respectively. The second term is equal to a distance between solutions to (4.1) with coefficients $(\bar{b}_k, \bar{c}_k, \bar{\eta}_k)$ defined by (4.25), (4.27), (4.28) and $(b(t, \mu(t)), c(t, \mu(t)), \eta(t, \mu(t)))$. By the continuity of solutions to (4.1) with respect to the initial datum and coefficients (see Chapter 2, Theorem 2.13) we obtain

$$\rho_F(\nu(t_{k+1}), \zeta(t_{k+1})) \leq \mathrm{e}^{C\Delta t} \rho_F(\bar{\mu}_k, \mu(t_k)), \tag{4.30}$$

and

$$\rho_F(\zeta(t_{k+1}), \mu(t_{k+1})) \leq C\Delta t \mathrm{e}^{C\Delta t}\left(\|\bar{b}_k - b\|_{\mathbf{BC_t}} + \|\bar{c}_k - c\|_{\mathbf{BC_t}} + \sum_{p=1}^{r} \|\bar{\beta}_{p,k} - \beta_p\|_{\mathbf{BC_t}}\right), \tag{4.31}$$

where

$$
\begin{aligned}
\|\bar{b}_k - b\|_{\mathbf{BC_t}} &= \sup_{t\in[t_k,t_{k+1}]} \|\bar{b}_k - b(t,\mu(t))\|_{\mathbf{L}^\infty}, \\
\|\bar{c}_k - c\|_{\mathbf{BC_t}} &= \sup_{t\in[t_k,t_{k+1}]} \|\bar{c}_k - c(t,\mu(t))\|_{\mathbf{L}^\infty}, \qquad (4.32)\\
\|\bar{\beta}_{p,k} - \beta_p\|_{\mathbf{BC_t}} &= \sup_{t\in[t_k,t_{k+1}]} \|\bar{\beta}_{p,k} - \beta_p(t,\mu(t))\|_{\mathbf{L}^\infty}. \qquad (4.33)
\end{aligned}
$$

Due to the assumptions (4.3) - (4.7) and the definition of $\bar{b}_k$, $\bar{c}_k$, $\bar{\eta}_k$ we obtain

$$
\begin{aligned}
\|\bar{b}_k - b(t,\mu(t))\|_{\mathbf{L}^\infty} &\le \|b(t_k,\bar{\mu}_k) - b(t_k,\mu(t))\|_{\mathbf{L}^\infty} + \|b(t_k,\mu(t)) - b(t,\mu(t))\|_{\mathbf{L}^\infty} \\
&\le \mathbf{Lip}(b(t_k,\cdot))\ \rho_F(\bar{\mu}_k,\mu(t)) + \|b\|_{\mathbf{C_b^{\alpha,1}}}|t-t_k|^\alpha \\
&\le \mathbf{Lip}(b)\ \rho_F(\bar{\mu}_k,\mu(t)) + \|b\|_{\mathbf{C_b^{\alpha,1}}}(\Delta t)^\alpha, \qquad (4.34)
\end{aligned}
$$

where $\mathbf{Lip}(b) = \sup_{t\in[0,T]} \mathbf{Lip}(b(t,\cdot)) \le \|b\|_{\mathbf{C_b^{\alpha,1}}}$. Using Lipschitz continuity of the solution $\mu(t)$ (see Chapter 2, Theorem 2.13) we obtain

$$
\rho_F(\bar{\mu}_k,\mu(t)) \le \rho_F(\bar{\mu}_k,\mu(t_k)) + \rho_F(\mu(t_k),\mu(t)) \le \rho_F(\bar{\mu}_k,\mu(t_k)) + C\Delta t \mathrm{e}^{C\Delta t}.
$$

Substituting the latter expression into (4.34) yields

$$
\|b_k - b(t,\mu(t))\|_{\mathbf{L}^\infty} \le \|b\|_{\mathbf{C_b^{\alpha,1}}}\left(\rho_F(\bar{\mu}_k,\mu(t_k)) + C\Delta t\mathrm{e}^{C\Delta t}\right) + \|b\|_{\mathbf{C_b^{\alpha,1}}}(\Delta t)^\alpha.
$$

Bounds for (4.32) and (4.33) can be proved analogously. From the assumptions it holds that

$$
\|(b,c,\beta)\|_{\mathbf{C_b^{\alpha,1}}} = \|b\|_{\mathbf{C_b^{\alpha,1}}} + \|c\|_{\mathbf{C_b^{\alpha,1}}} + \sum_{p=1}^r \|\beta_p\|_{\mathbf{C_b^{\alpha,1}}} < +\infty,
$$

and as a consequence

$$
\begin{aligned}
&\|\bar{b}_k - b\|_{\mathbf{BC_t}} + \|\bar{c}_k - c\|_{\mathbf{BC_t}} + \sum_{p=1}^r \|\bar{\beta}_{p,k} - \beta_p\|_{\mathbf{BC_t}} \\
&\quad\le \|(b,c,\beta)\|_{\mathbf{C_b^{\alpha,1}}}\left(\rho_F(\bar{\mu}_k,\mu(t_k)) + C\Delta t\mathrm{e}^{C\Delta t}\right) + \|(b,c,\beta)\|_{\mathbf{C_b^{\alpha,1}}}(\Delta t)^\alpha \\
&\quad\le \|(b,c,\beta)\|_{\mathbf{C_b^{\alpha,1}}}\left(\rho_F(\bar{\mu}_k,\mu(t_k)) + C\Delta t\mathrm{e}^{CT}\right) + \|(b,c,\beta)\|_{\mathbf{C_b^{\alpha,1}}}(\Delta t)^\alpha \\
&\quad\le \|(b,c,\beta)\|_{\mathbf{C_b^{\alpha,1}}}\rho_F(\bar{\mu}_k,\mu(t_k)) + C\mathrm{e}^{CT}\|(b,c,\beta)\|_{\mathbf{C_b^{\alpha,1}}}\Delta t + \|(b,c,\beta)\|_{\mathbf{C_b^{\alpha,1}}}(\Delta t)^\alpha.
\end{aligned}
$$

Therefore, a form of the estimate is the following

$$
\|\bar{b}_k - b\|_{\mathbf{BC_t}} + \|\bar{c}_k - c\|_{\mathbf{BC_t}} + \sum_{p=1}^r \|\bar{\beta}_{p,k} - \beta_p\|_{\mathbf{BC_t}} \le C\left(\rho_F(\bar{\mu}_k,\mu(t_k)) + \Delta t + (\Delta t)^\alpha\right).
$$

Use of this inequality in (4.31) yields

$$
\begin{aligned}
\rho_F(\zeta(t_{k+1}),\mu(t_{k+1})) &\le C\Delta t\mathrm{e}^{C\Delta t}\left(\rho_F(\bar{\mu}_k,\mu(t_k)) + \Delta t + (\Delta t)^\alpha\right) \\
&\le C\Delta t\mathrm{e}^{C\Delta t}\rho_F(\bar{\mu}_k,\mu(t_k)) + C\mathrm{e}^{CT}(\Delta t)^2 + C\mathrm{e}^{CT}(\Delta t)^{1+\alpha},
\end{aligned}
$$

which can be rewritten as

$$\rho_F(\zeta(t_{k+1}),\mu(t_{k+1})) \le C\Delta t e^{C\Delta t}\rho_F(\bar{\mu}_k,\mu(t_k)) + C(\Delta t)^2 + C(\Delta t)^{1+\alpha},$$

for some constant C. Combining the inequality above with (4.30) leads to

$$\begin{aligned}\rho_F(\nu(t_{k+1}),\mu(t_{k+1})) &\le e^{C\Delta t}\rho_F(\bar{\mu}_k,\mu(t_k)) \\ &\quad + C\Delta t e^{C\Delta t}\rho_F(\bar{\mu}_k,\mu(t_k)) + C(\Delta t)^2 + C(\Delta t)^{1+\alpha} \\ &\le e^{C\Delta t}(1+C\Delta t)\rho_F(\bar{\mu}_k,\mu(t_k)) + C(\Delta t)^2 + C(\Delta t)^{1+\alpha} \\ &\le e^{2C\Delta t}\rho_F(\bar{\mu}_k,\mu(t_k)) + C(\Delta t)^2 + C(\Delta t)^{1+\alpha}.\end{aligned} \tag{4.35}$$

Step 4: Adding the errors. Now, let $w = jn$, $v = (j-1)n$, $j \in \{1,\dots,\mathcal{K}\}$, that is t_w and t_v are the time points in which the measure reconstruction occurs. Since for t_i such that $t_v < t_i < t_w$ it holds that $\bar{\mu}_i = \mathcal{R}(\bar{\mu}^2_{i-1}) = \bar{\mu}^2_{i-1}$, i.e., the measure reconstruction is not performed, the application of Gronwall's inequality to (4.35) yields

$$\rho_F(\bar{\mu}_w,\mu(t_w)) \le e^{C(n\Delta t)}\rho_F(\bar{\mu}_v,\mu(t_v)) + \frac{e^{C(n\Delta t)}-1}{e^{C\Delta t}-1}\left(C(\Delta t)^2 + C(\Delta t)^{1+\alpha}\right).$$

There exists $C^* = C^*(T)$ such that $e^{C(n\Delta t)} - 1 < C^*(n\Delta t)$ for each $n\Delta t \in [0,T]$. Therefore,

$$\frac{e^{C(n\Delta t)}-1}{e^{C\Delta t}-1} \le \frac{C^*(n\Delta t)}{C\Delta t} = \frac{C^*}{C}n$$

and thus,

$$\rho_F(\bar{\mu}_w,\mu(t_w)) \le e^{C(n\Delta t)}\rho_F(\bar{\mu}_v,\mu(t_v)) + n\left(C(\Delta t)^2 + C(\Delta t)^{1+\alpha}\right),$$

for some constant C. Combining this inequality with (4.26) in Step 1 and (4.29) in Step 2 of the proof yields

$$\rho_F(\bar{\mu}_w,\mu(t_w)) \le e^{C(n\Delta t)}\rho_F(\bar{\mu}_v,\mu(t_v)) + n(C(\Delta t)^2 + C(\Delta t)^{1+\alpha}) + CE_R(\bar{M}).$$

Step 5: Final estimate for the auxiliary scheme. Analogous argumentation as in the previous step of the proof results in the following estimate

$$\begin{aligned}\rho_F(\bar{\mu}_w,\mu(t_w)) &\le e^{C(jn\Delta t)}\rho_F(\mathcal{R}(\mu_o),\mu_o) + \frac{e^{C(jn\Delta t)}-1}{e^{C(n\Delta t)}-1}\left[n(C(\Delta t)^2 + C(\Delta t)^{1+\alpha}) + CE_R(\bar{M})\right] \\ &\le e^{Ct_w}CE_I(\bar{M}_o) + Cj\left[n(C(\Delta t)^2 + C(\Delta t)^{1+\alpha}) + CE_R(\bar{M})\right] \\ &\le Ce^{Ct_w}E_I(\bar{M}_o) + C^2(jn\Delta t)\left(\Delta t + (\Delta t)^\alpha\right) + C^2 jE_R(\bar{M})\end{aligned} \tag{4.36}$$

and since $jn\Delta t = t_w \le T$ the assertion is proved.

Step 6: Full error estimate. The full error estimate (4.21) takes into account the error following from the numerical approximation of the auxiliary scheme (see Subsection 4.2.2). According to [16, (515.62)] one commits error of the order Δt when solves (4.23) - (4.24) using its Euler approximation (4.9), (4.14). Therefore, the estimate (4.36) still holds. □

Remark 4.37. In this thesis we assumed that η is given as a sum of Dirac Deltas. If $\eta(t,\mu)(y)$ is not in such a form, one has to use a proper approximation by the sum of Dirac measures in order to apply the scheme we propose. One of the possible method of this approximation is through the measure reconstruction described in Subsection 4.2.3. Assume that there exists a bounded interval K such that

$$\forall_{(t,\mu)\in[0,T]\times\mathcal{M}^+(\mathbb{R}_+)}\ \forall_{y\in\mathbb{R}^+}\quad \mathrm{supp}(\eta(t,\mu)(y)) \subseteq K. \tag{4.38}$$

Fix $r \in \mathbb{N}$ and let $\{K_p\}_{p=1}^r$ be a family of intervals such that

$$\bigcup_{p=1}^{r} K_p = K, \quad K_i \cap K_j = \varnothing, \text{ for } i \neq j \quad \text{and} \quad |K_p| = \frac{|K|}{r}, \text{ where } p = 1, \ldots, r.$$

Namely, we divide K into r disjoint intervals of equal length. Denote the center of each interval by $\bar{x}_p$ and define

$$\beta_p(t,\mu)(y) = \int_{K_p} \mathrm{d}(\eta(t,\mu)(y))(x).$$

The approximation of $\eta(t,\mu)(y)$ is thus given by

$$\sum_{p=1}^{r} \beta_p(t,\mu)(y)\delta_{x=\bar{x}_p}. \tag{4.39}$$

If η is regular enough, then the assumptions on β_p and $\bar{x}_p$ (4.3) - (4.7) are fulfilled and the numerical scheme we propose applies. We would like to emphasize, that this approximation implies that the sums in (4.7) are bounded uniformly with respect to r. Moreover, since (4.38) holds, the distance between η and its approximation (4.39) expressed in terms of the proper norm can be bounded by C/r, where C does not depend on t, μ and y. Thus, the stability result from Theorem 2.13 guarantees that if r tends to $+\infty$, then the numerical solution obtained for the approxiamted η converges towards a solution to (4.1) with the function η. For all technical details we refer to Chapter 2.

4.4. Simulation Results

4.4.1. Measurement of the Error

The flat metric is defined as a supremum over a subset of bounded, Lipschitz functions and that is why its calculation is not a straightforward task. From that reason, we introduce a function ρ which is defined through the 1-Wasserstein distance. It is a convenient formulation, since [79, Section 2.2.2] provides an explicit formula on W_1 in terms of the cumulative distribution functions of measures.

Definition 4.40. *Let $\mu_1, \mu_2 \in \mathcal{M}^+(\mathbb{R}_+)$ be such that $M_{\mu_i} = \int_{\mathbb{R}_+} \mathrm{d}\mu_i \neq 0$ and $\tilde{\mu}_i = \mu_i / M_{\mu_i}$ for $i = 1, 2$. Define $\rho : \mathcal{M}^+(\mathbb{R}_+) \times \mathcal{M}^+(\mathbb{R}_+) \to \mathbb{R}_+$ as the following*

$$\rho(\mu_1,\mu_2) = \min\{M_{\mu_1}, M_{\mu_2}\}\, W_1(\tilde{\mu}_1,\tilde{\mu}_2) + |M_{\mu_1} - M_{\mu_2}|, \tag{4.41}$$

where W_1 is the 1-Wasserstein distance.

The error of the numerical solution with parameters $(\Delta t, \bar{M}_o, \bar{M})$ at time t_k is defined as

$$\mathrm{Err}(T; \Delta t, \bar{M}_o, \bar{M}) := \rho(\mu(t_k), \mu_k). \tag{4.42}$$

The order of the method q is given by

$$q := \frac{\log\left(\mathrm{Err}(T; 2\Delta t, 2\bar{M}_o, 2\bar{M})/\mathrm{Err}(T; \Delta t, \bar{M}_o, \bar{M})\right)}{\log 2}. \tag{4.43}$$

We also define $\Delta x := (|K|/\bar{M}_o)$, where K is a domain. Below we provide the lemma, which relates ρ to ρ_F.

Lemma 4.44. *Let $\mu_1, \mu_2 \in \mathcal{M}^+(\mathbb{R}_+)$ be such that $M_{\mu_i} = \int_{\mathbb{R}_+} \mathrm{d}\mu_i \neq 0$ and $\tilde{\mu}_i = \mu_i/M_{\mu_i}$ for $i = 1, 2$. Let ρ be as in the Definition 4.40. Then, there exists a constant $C_K = \frac{1}{3}\min\left\{1, \frac{2}{|K|}\right\}$, such that*

$$C_K \rho(\mu_1, \mu_2) \leq \rho_F(\mu_1, \mu_2) \leq \rho(\mu_1, \mu_2),$$

where K is the smallest interval such that $\mathrm{supp}(\mu_1), \mathrm{supp}(\mu_2) \subseteq K$ and $|K|$ is the length of the interval K. If K is unbounded we set $C_K = 0$.

Remark 4.45. For $\tilde{\mu}_1$, $\tilde{\mu}_2$ defined as in the lemma above, it holds that

$$W_1(\tilde{\mu}_1, \tilde{\mu}_2) = \int_0^1 \left|F^{-1}_{\tilde{\mu}_1}(t) - F^{-1}_{\tilde{\mu}_2}(t)\right| \mathrm{d}t = \int_{\mathbb{R}_+} |F_{\tilde{\mu}_1}(x) - F_{\tilde{\mu}_2}(x)| \mathrm{d}x,$$

which follows from [79, Section 2.2.2]. Since a cumulative distribution function F_μ does not have to be continuous or strictly increasing we set

$$F^{-1}_\mu(s) = \sup\{x \in \mathbb{R}_+ \,:\, F_\mu(x) \leq s\}, s \in [0, 1].$$

In the proof of Lemma 4.44 we will also use the following

Remark 4.46. Let $\mu \in \mathcal{M}^+(\mathbb{R}_+)$ be a probability measure and $M_1, M_2 > 0$. Then,

$$\rho_F(M_1\mu, M_2\mu) \leq |M_1 - M_2|. \tag{4.47}$$

Indeed, let $\varphi \in \mathbf{C^1}(\mathbb{R}_+)$ be such that $\|\varphi\|_{\mathbf{W^{1,\infty}}} \leq 1$. Then,

$$\int_{\mathbb{R}_+} \varphi(x) \mathrm{d}(M_1\mu - M_2\mu)(x) \leq |M_1 - M_2| \int_{\mathbb{R}_+} \|\varphi\|_{\mathbf{L}^\infty} \mathrm{d}\mu(x) \leq |M_1 - M_2|.$$

Taking supremum over all admissible functions φ proves the assertion.

Proof of Lemma 4.44. It follows from Proposition 1.31 that

$$C_K W_1(\mu, \nu) \leq \rho_F(\mu, \nu) \leq W_1(\mu, \nu),$$

where $C_K = \min\{1, 2/|K|\}$, if K is bounded, and 0 otherwise. Let μ_1, μ_2 be as in the statement of the Lemma. Then,

$$\rho_F(\mu_1, \mu_2) \;=\; M_{\mu_1}\rho_F\left(\frac{\mu_1}{M_{\mu_1}}, \frac{\mu_2}{M_{\mu_1}}\right) \leq M_{\mu_1}\rho_F\left(\frac{\mu_1}{M_{\mu_1}}, \frac{\mu_2}{M_{\mu_2}}\right) + M_{\mu_1}\rho_F\left(\frac{\mu_2}{M_{\mu_2}}, \frac{\mu_2}{M_{\mu_1}}\right)$$

$$
\begin{aligned}
&\leq M_{\mu_1}\rho_F(\tilde{\mu}_1,\tilde{\mu}_2) + M_{\mu_1}M_{\mu_2}\left|\frac{1}{M_{\mu_1}} - \frac{1}{M_{\mu_2}}\right| \\
&= M_{\mu_1}W_1(\tilde{\mu}_1,\tilde{\mu}_2) + |M_{\mu_1} - M_{\mu_2}|,
\end{aligned}
$$

where we used triangle inequality and 4.47. Analogously we obtain

$$\rho_F(\mu_1,\mu_2) \leq M_{\mu_2}W_1(\tilde{\mu}_1,\tilde{\mu}_2) + |M_{\mu_1} - M_{\mu_2}|$$

and thus,

$$\rho_F(\mu_1,\mu_2) \leq \min\{M_{\mu_1}, M_{\mu_2}\}\, W_1(\tilde{\mu}_1,\tilde{\mu}_2) + |M_{\mu_1} - M_{\mu_2}| = \rho(\mu_1,\mu_2).$$

Note that this estimate does not depend on $|K|$. Using $\varphi = \pm 1$ as a test function in the definition of the flat metric (1.23) we obtain that $|M_{\mu_1} - M_{\mu_2}| \leq \rho_F(\mu_1,\mu_2)$. Then,

$$
\begin{aligned}
\rho(\mu_1,\mu_2) &\leq M_{\mu_1}W_1(\tilde{\mu}_1,\tilde{\mu}_2) + |M_{\mu_1} - M_{\mu_2}| \\
&= M_{\mu_1}\max\{1,|K|/2\}\rho_F(\tilde{\mu}_1,\tilde{\mu}_2) + |M_{\mu_1} - M_{\mu_2}| \\
&\leq \max\{1,|K|/2\}\rho_F\left(\mu_1, \frac{M_{\mu_1}}{M_{\mu_2}}\mu_2\right) + \rho_F(\mu_1,\mu_2) \\
&\leq \max\{1,|K|/2\}\left(\rho_F(\mu_1,\mu_2) + \rho_F\left(\mu_2, \frac{M_{\mu_1}}{M_{\mu_2}}\mu_2\right)\right) + \rho_F(\mu_1,\mu_2) \\
&\leq 2\max\{1,|K|/2\}\rho_F(\mu_1,\mu_2) + \max\{1,|K|/2\}M_{\mu_2}\left|1 - \frac{M_{\mu_1}}{M_{\mu_2}}\right| \\
&\leq 3\max\{1,|K|/2\}\rho_F(\mu_1,\mu_2),
\end{aligned}
$$

which implies that

$$\frac{1}{3}\min\left\{1, \frac{2}{|K|}\right\}\rho(\mu_1,\mu_2) \leq \rho_F(\mu_1,\mu_2).$$

In case $|K| = +\infty$ we obtain a trivial inequality $0 \leq \rho_F(\mu_1,\mu_2)$. □

Next subsections are devoted to presenting results of numerical simulations for several test cases. In all examples presented here, we used the 4-th order Runge-Kutta method for solving (4.9) and the explicit Euler scheme for solving (4.14), as described in Subsection 4.2.2.

4.4.2. Example 1: McKendrick-type Equation

In this subsection we present numerical results for an equation describing the evolution of an age-structured population. We set

$$b(x) = 0.2(1-x),\;\; c(x) = 0.2,\;\; [\eta(y)](x) = 2.4(y^2 - y^3)\delta_{x=0} \;\text{ and }\; \mu_o = \chi_{[0,1]}(x)$$

and solve (4.1) for $x \in [0,1]$ (see also [7]). Our first test case is a linear problem, where the solution is given by the formula $u(t,x) = \chi_{[0,1]}(x)$. In Table 4.1 we present the relative error and the order of the scheme, where we used just one measure reconstruction in order to approximate the initial datum. In Table 4.2 we present results for the scheme with the measure reconstruction performed at $t = 0, 1, \ldots, 10$ and $\bar{M}_o = \bar{M}$.

$\Delta t = \Delta x$	$\text{Err}(10, \Delta t, \bar{M}_o, \bar{M})$	q
$1.000000 \cdot 10^{-1}$	$1.2532 \cdot 10^{-2}$	–
$5.000000 \cdot 10^{-2}$	$5.0543 \cdot 10^{-3}$	1.31006
$2.500000 \cdot 10^{-2}$	$2.2225 \cdot 10^{-3}$	1.18533
$1.250000 \cdot 10^{-2}$	$1.0349 \cdot 10^{-4}$	1.10272
$6.250000 \cdot 10^{-3}$	$4.9832 \cdot 10^{-4}$	1.05431
$3.125000 \cdot 10^{-3}$	$2.4438 \cdot 10^{-4}$	1.02796
$1.562500 \cdot 10^{-3}$	$1.2099 \cdot 10^{-4}$	1.01419
$7.812500 \cdot 10^{-5}$	$6.0198 \cdot 10^{-5}$	1.00715
$3.906250 \cdot 10^{-4}$	$3.0024 \cdot 10^{-5}$	1.00359
$1.953125 \cdot 10^{-4}$	$1.4993 \cdot 10^{-5}$	1.00180
$9.765625 \cdot 10^{-5}$	$7.4920 \cdot 10^{-6}$	1.00090

Table 4.1: (Example 1) The relative error and order of the scheme at $T = 10$. One reconstruction performed at $t = 0$, $\bar{M} = \bar{M}_o$.

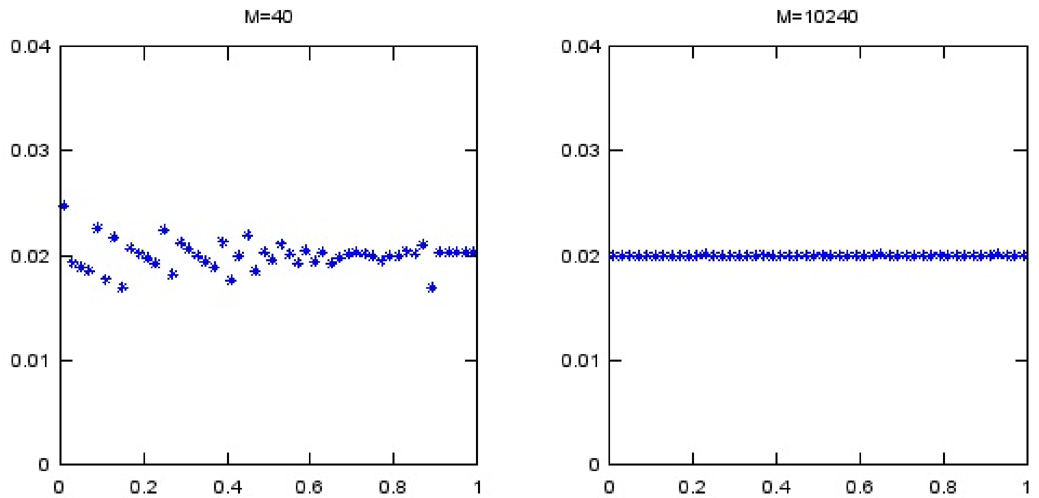

Figure 4.1: (Example 1) The numerical solution aggregated over intervals of length 0.02 at time $T = 10$ with parameters $\Delta t = \Delta x = 0.025$ (on the left hand side) and $\Delta t = \Delta x = 9.76562510^{-5}$ (on the right hand side).

On the Figure 4.1 we present numerical solutions for parameters $\Delta t = \Delta x = 0.025$ and $\Delta t = \Delta x = 9.76562510^{-5}$ respectively at $T = 10$. The solutions are aggregated over intervals of a length $h = 0.02$, that is each Dirac Delta located in $x = (j - 0.5)h$, for $j = 1, \ldots, 50$, has a mass equal to $\sum_{i \in I} m^i$, where i is such that $x^i \in [(j-1)h, jh)$.

$\Delta t = \Delta x$	$\mathrm{Err}(10, \Delta t, \bar{M}_o, \bar{M})$ (Fixed-location)	q	$\mathrm{Err}(10, \Delta t, \bar{M}_o, \bar{M})$ (Fixed-equal mass)	q
$1.000000 \cdot 10^{-1}$	$3.4657 \cdot 10^{-1}$	–	$8.8838 \cdot 10^{-2}$	–
$5.000000 \cdot 10^{-2}$	$1.1670 \cdot 10^{-1}$	1.5703	$2.9437 \cdot 10^{-2}$	1.5935
$2.500000 \cdot 10^{-2}$	$3.4080 \cdot 10^{-2}$	1.7759	$1.0879 \cdot 10^{-2}$	1.4361
$1.250000 \cdot 10^{-2}$	$1.1863 \cdot 10^{-2}$	1.5224	$4.4725 \cdot 10^{-3}$	1.2824
$6.250000 \cdot 10^{-3}$	$3.6874 \cdot 10^{-3}$	1.6858	$1.9907 \cdot 10^{-3}$	1.1678
$3.125000 \cdot 10^{-3}$	$1.6866 \cdot 10^{-3}$	1.1285	$9.3351 \cdot 10^{-4}$	1.0926
$1.562500 \cdot 10^{-3}$	$6.8067 \cdot 10^{-4}$	1.3091	$4.5131 \cdot 10^{-4}$	1.0486
$7.812500 \cdot 10^{-4}$	$3.3212 \cdot 10^{-4}$	1.0352	$2.2178 \cdot 10^{-4}$	1.0250
$3.906250 \cdot 10^{-4}$	$1.5814 \cdot 10^{-4}$	1.0705	$1.0992 \cdot 10^{-4}$	1.0127
$1.953125 \cdot 10^{-4}$	$7.4507 \cdot 10^{-5}$	1.0858	$5.4719 \cdot 10^{-5}$	1.0063
$9.765625 \cdot 10^{-5}$	$3.6414 \cdot 10^{-5}$	1.0329	$2.7299 \cdot 10^{-5}$	1.0032

Table 4.2: (Example 1) The relative error and order of the scheme at $T = 10$. Reconstruction performed at $t = 0, 1, \ldots, T$, $\bar{M} = \bar{M}_o$.

4.4.3. Example 2: Equation with a Nonlinear Growth Term

In this subsection we present results for a model, where b and η are equal to zero. We consider a nonlinear growth function c as in [27]

$$c(t,\mu)(x) = a(x) - \int_{\mathbb{R}} \alpha(x,y)\mathrm{d}\mu(y),$$

where

$$a(x) = A - x^2, \;\; A > 0 \quad \text{and} \quad \alpha(x,y) = \frac{1}{1 + (x-y)^2}.$$

According to [21, Remark 2.3, Lemma 4.8] one can consider (4.1) on the whole $\mathbb{R}$, so that the result concerning well posedness still holds. If $|x| > \sqrt{A}$, then the solution decreases exponentially to zero, since $\alpha(x,y) \geq 0$, for all $x, y \in \mathbb{R}$. This equation can describe a population structured with respect to the trait x and then its asymptotic behaviour reflects the speciation process. Typically, after a long time period only a few traits are observable, since the rest of the population got extinct. Under some assumptions, there exists a linearly stable steady solution $\bar{\mu}$ being a sum of Dirac Deltas, which is shown in [27]. The number of Dirac measures depends on the parameter A and some stationary solutions are explicit. Figures 4.2 and 4.3 present the evolution and long time behaviour of solutions for different choices of the parameter A. These results are consistent with the findings in [27]. In all cases we assumed that initial datum are given as a sum of uniformly distributed Dirac Deltas with the same mass.

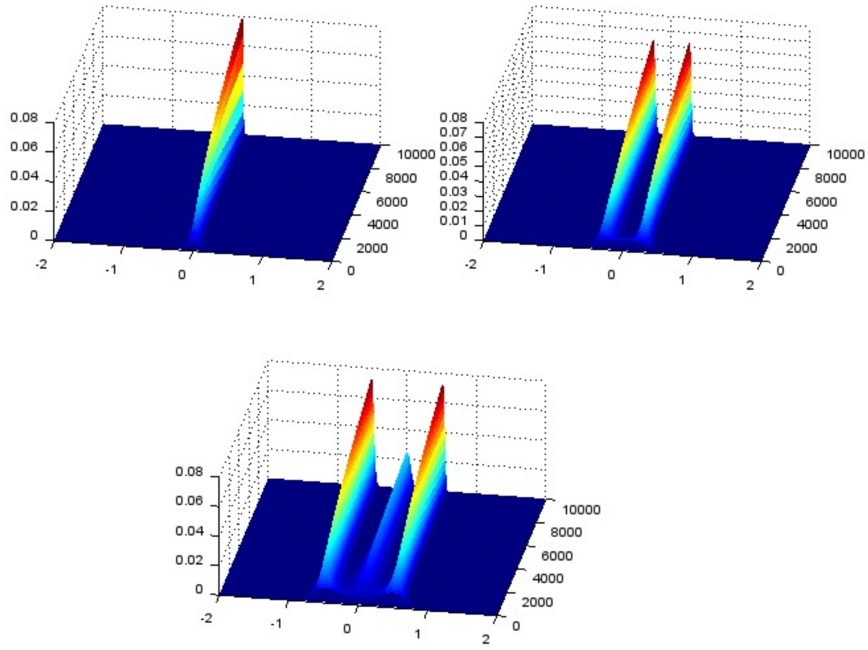

Figure 4.2: (Example 2) Long time behaviour of numerical solutions. Subsequent pictures present the evolution of a numerical solution on the time interval $[0, 10\,000]$ for $A = 0.5, 1.5$ and 2.5, respectively. For calculations we set $\Delta t = 0.1$, $\Delta x = 0.004$, $\bar{M}_o = 1000$ and $\mu_o = \sum_{i=1}^{\bar{M}_o}(1/\bar{M}_o)\delta_{x_o^i}$, where $x_o^i := -2 + (i - \frac{1}{2})\Delta x$, $i = 1, \dots, \bar{M}_o$. No measure reconstruction has been performed.

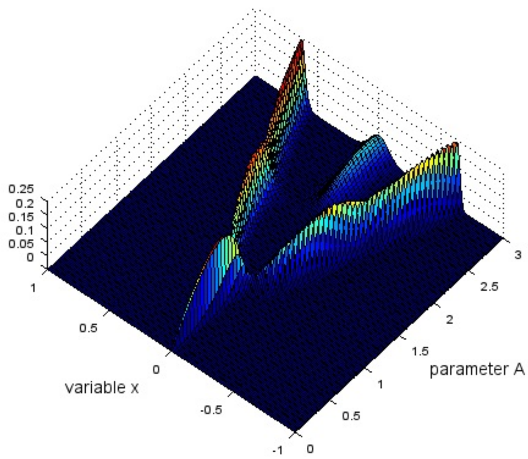

Figure 4.3: (Example 2) Long time behaviour of numerical solutions. Picture presents a numerical solution at the time $t = 10\,000$ depending on the parameter $A \in [0, 3]$. For calculations we set $\Delta t = 0.05$, $\Delta x = 0.0125$, $\bar{M}_o = 320$ and $\mu_o = \sum_{i=1}^{\bar{M}_o}(1/\bar{M}_o)\delta_{x_o^i}$, where $x_o^i := -2 + (i - \frac{1}{2})\Delta x$, $i = 1, \dots, \bar{M}_o$. No measure reconstruction has been performed.

4.4.4. Example 3: Size Structure Model with Reproduction by Equal Fission

In this subsection we shall concentrate on a size-structured cell population model, in which a cell reproducts itself by a fission into two equal parts. We assume that the cell divides after it has reached a minimal size $x_o > 0$. Therefore, there exists a minimum size whose value is $x_o/2$. Moreover, cells have to divide before they reach a maximal size, which is normalized to be equal to $x_{max} = 1$. Similarly as in [3] we set

$$x_o = \frac{1}{4},\ \ b(x) = 0.1(1-x),\ \ c(x) = 0,\ \ \eta(t,\mu)(y) = \beta(y)\delta_{x=y/2} \text{ and } u_o(x) = (1-x)(x-x_o/2)^3,$$

where

$$\beta(y) = \begin{cases} 0, & \text{for } y \in (\mathbb{R}_+ \setminus [x_o, 1]), \\ \frac{b(y)\varphi(y)}{1-\int_{x_o}^{y}\varphi(y)\mathrm{d}y}, & \text{for } y \in [x_o, 1], \end{cases}$$

and

$$\varphi(y) = \begin{cases} \frac{160}{117}\left(-\frac{2}{3} + \frac{8}{3}y\right)^3, & \text{for } y \in [x_o, (x_o+1)/2], \\ \frac{32}{117}\left(-20 + 40y + \frac{320}{3}\left(y - \frac{5}{8}\right)^2 + \right) + \frac{5120}{9}\left(y - \frac{5}{8}\right)^3\left(\frac{8}{3}y - \frac{11}{3}\right), & \text{for } y \in ((x_o+1)/2, 1]. \end{cases}$$

Figure 4.4 presents the long time behaviour of a numerical solution for the particular choice of parameters.

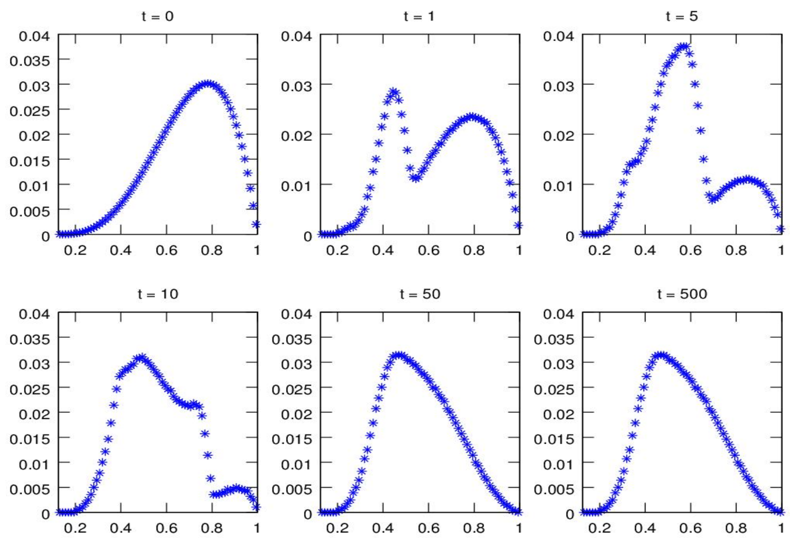

Figure 4.4: (Example 3) Numerical solution at $t = 0, 1, 5, 10, 50, 500$, calculated for $\Delta t = 0.0125$, $\bar{M}_o = \bar{M} = 2800$. Fixed-equal mass reconstruction was performed once per 4 time steps. On subsequent pictures we present the numerical solution after the fixed-location reconstruction with parameter $\bar{M} = 70$ and normalization (the mass grows exponentially).

4.4.5. Example 4: Selection - Mutation Model

The following test case concerns a simple selection-mutation model. We assume that $x \in [0,1]$ and set

$$b(x) = 0, \;\; c(\mu)(x) = (1-\varepsilon)\beta(x) - m(\mu) \quad \text{and} \quad \eta(y) = \varepsilon \sum_{p=1}^{r} \beta_p(y)\delta_{x=\bar{x}_p(y)},$$

where $\beta(x) = x(1-x)$, $m(\mu) = 1 - \exp\left\{-\int_0^1 \mathrm{d}\mu\right\}$ and r is fixed. We consider two different choices of the function η. To obtain η_1 we set

$$r = 10, \quad \bar{x}_p(y) = \bar{x}_p = \left(p - \frac{1}{2}\right)\frac{1}{r}, \quad \beta_p(y) = \frac{1}{r}, \quad \text{for } p = 1, \ldots, r.$$

In the second case we consider η_2, where

$$r = 10, \quad \bar{x}_p(y) = \begin{cases} (y-a) + \frac{a}{r}(2p-1), & \text{if } 0 \le (y-a) + \frac{a}{r}(2p-1) \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\beta_p(y) = \frac{\check{\beta}_p(y)}{\sum_{p=1}^{r} \check{\beta}_p(y)}, \quad \text{where } \check{\beta}_p(y) = \begin{cases} \exp\left(-\frac{a^2}{a^2 - (\bar{x}_p(y) - y)^2}\right), & \text{if } p \text{ is s.t. } 0 \le \bar{x}_p(y) \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Parameter a is related to the distribution of new individuals in the sense that a distance between a parent and its offspring is not greater than a. In this particular test case we set $a = 0.4$.

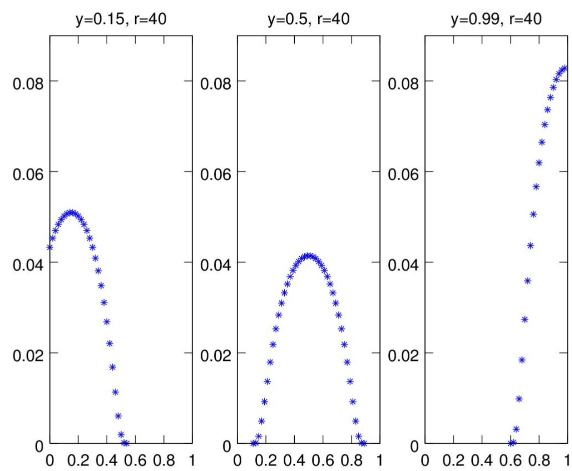

Figure 4.5: (Example 4) Subsequent figures present function η_2 for $y = 0.15$, $y = 0.5$ and $y = 0.99$, respectively, and parameters $r = 40$, $a = 0.4$.

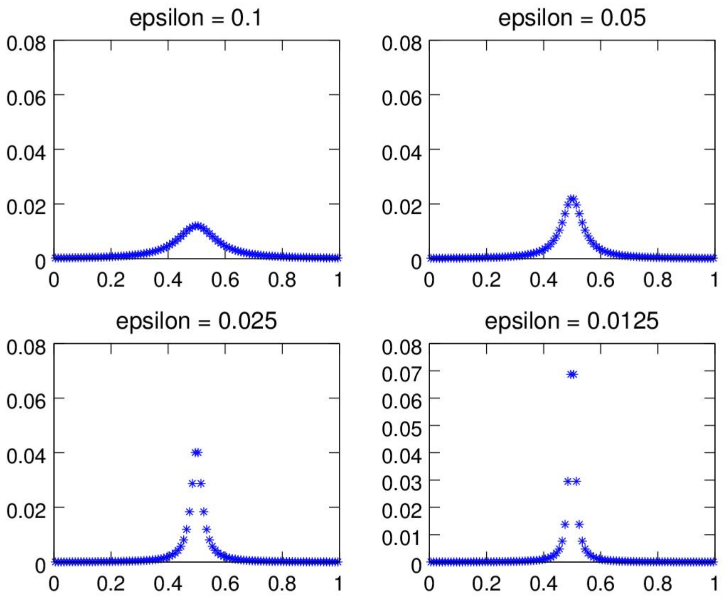

Figure 4.6: (Example 4) Steady state calculated by Newton method for different values of ε and $\eta = \eta_1$.

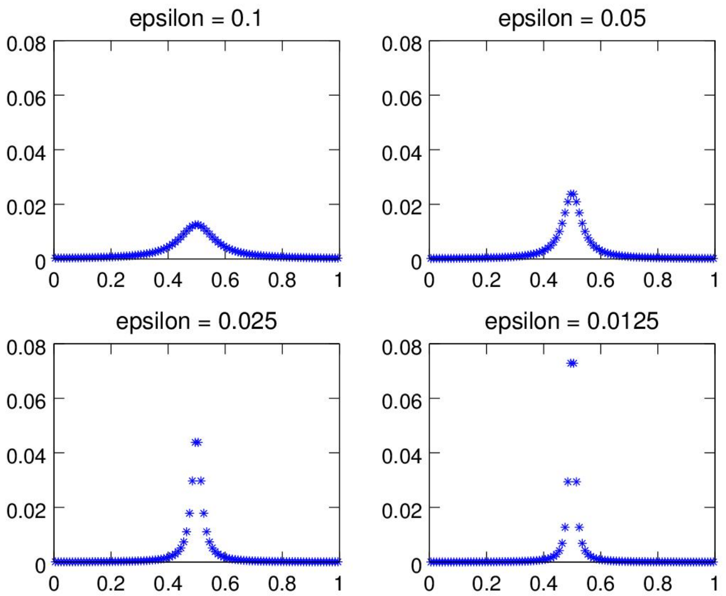

Figure 4.7: (Example 4) Numerical solution at $T = 20000$ for different values of ε and $\eta = \eta_1$. The initial datum is $\mu_o = \sum_{i=1}^{M} m_i \delta_{x_i}$, where $m_i = 1/M$ and $M = 1000$. On the figure we present the solution after the fixed-location reconstruction with parameter $\bar{M} = 100$.

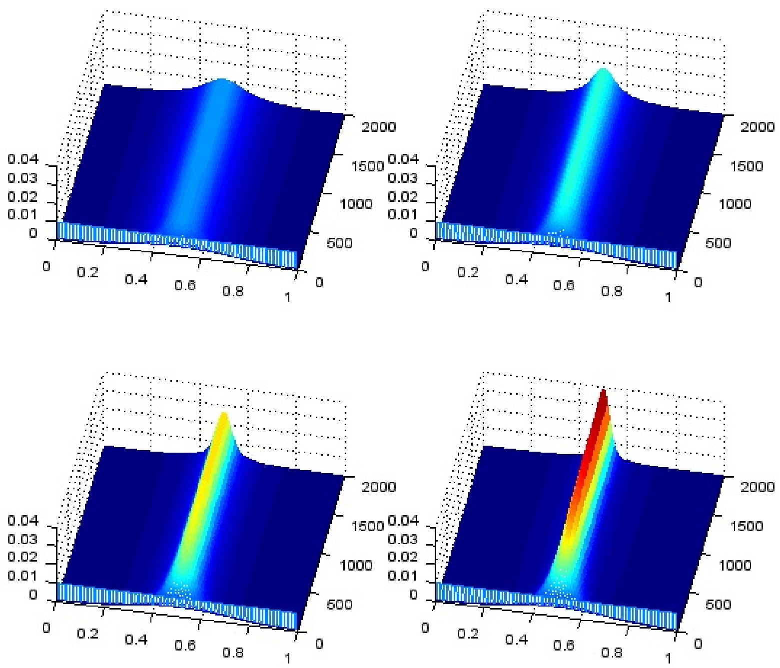

Figure 4.8: (Example 4) Long time behaviour of numerical solutions for $\eta = \eta_2$. Subsequent pictures present the evolution of a numerical solution on the time interval $[0, 2000]$ for $\varepsilon = 0.1, 0.05, 0.025$ and 0.0125, respectively. For calculations we set $\Delta t = 0.025$, $\bar{M}_o = \bar{M} = 100$ and $\mu_o = \sum_{i=1}^{\bar{M}_o}(1/\bar{M}_o)\delta_{x_o^i}$, where $x_o^i := (i - \frac{1}{2})/\bar{M}_o$. Fixed location reconstruction has been performed once per 2 time steps.

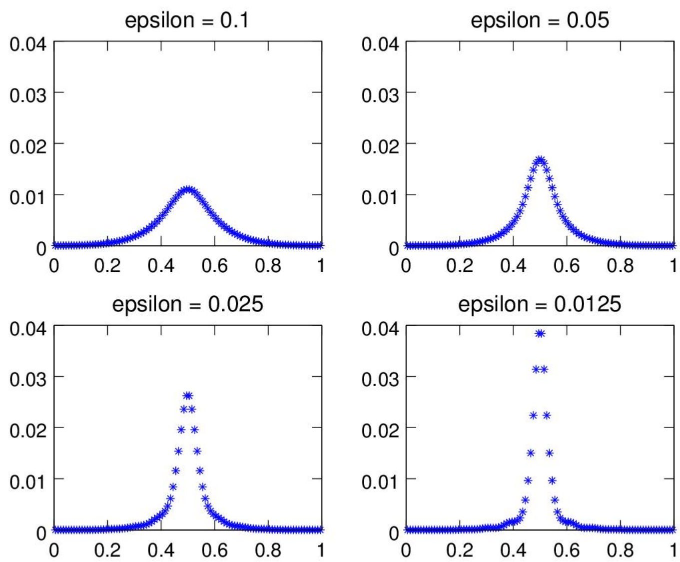

Figure 4.9: (Example 4) Subsequent pictures present the numerical solution at time $t = 2000$ for $\varepsilon = 0.1, 0.05, 0.025$ and 0.0125, respectively, and the same set of parameters as described below Figure 4.8.

Chapter 5

Alternative definitions of the boundary cohort

5.1. Introduction

In Chapter 4 we have derived a numerical scheme for the particular case of equation (2.1), which arises by assumption that the set of possible new states is finite. In the present chapter we consider a bit more specific problem, namely we set $r = 1$ in (4.2), which means that all new born individuals are in the same state regardless of the parent's state. Thus, we obtain biologically relevant and widely know model, that is the McKendrick model (see Section 2.2). We have already described difficulties which appear when one attempts to apply the particle methods to structured population models. For instance, a continuous distribution which arises due to the boundary condition has to be properly approximated. In Chapter 4 we solved this problem using the splitting technique. In this chapter we will show an alternative approach based on solving a system of ODEs, which describes the evolution of locations and masses of cohorts, that is groups of individuals which are similar to each other. In particular, the boundary cohort approximates the number of newborns and their average state. We will consider two different definitions of the boundary cohort leading to different ODEs systems. The first one, described in [26], is called the Escalator Boxcar Train scheme and the second one was introduced in the recent paper [12] as its simplification. The EBT method has been widely used in natural sciences since the moment of its formulation (see [15], [41], [69], [84]). A concept of this method bases on the observation of behaviour of the cohorts. More precisely, equations constituting the original EBT scheme arise from tracking particular moments of a population density on specific domains (see Section 5.4 for more details). An output of this method gives the information of the population distribution over the specified domains, which is usually more meaningful than the density's values in nodal points. In our framework the output is interpreted as a sum of Dirac measures $\sum_{i \in I} m^i \delta_{x_i}$, where $m^i(t)$ denotes the number of individuals within the cohort and $x^i(t)$ is the average value of the structural variable within this cohort at time t.

The aim of this chapter is to prove that algorithms described in [26] and [12] are equivalent to the scheme presented in Chapter 4 in the sense of the convergence rate.

5.2. Escalator Boxcar Train Method

5.2.1. General Description of the Method

Let us start with a brief description of the EBT method. In the first step individuals are divided into groups, which are characterized by pairs $(x^i(0), m^i(0))$. Formally, the initial distribution is approximated by a sum of Dirac measures. In our framework, one of the possible method of the approximation is through the measure reconstruction procedure described in Subsection 4.2.3. Once an individual is allocated in the particular cohort, it stays there till the moment of death. Evolution of the cohort's characteristics is described by an ODE so that $m^i(t)$ changes its value due to the process of growth or death, while $x^i(t)$ evolves according to the characteristic lines defined by a transport operator. The boundary cohort, which accounts for the influx of new individuals, additionally changes its state due to the process of birth. A single step of the EBT algorithm bases on solving this ODEs system on a sufficiently short time interval. Depending on the version of the EBT method, we solve (5.1) with a boundary condition defined by either (5.2) or (5.21). Once the system is solved the boundary cohort is internalized, that is treated henceforth as the other (internal) cohorts. The index of each cohort is increased by 1 and a new cohort with index B is created. The new boundary cohort is initially empty and located in x_b. The procedure described above is repeated on subsequent time intervals until the final time T is reached.

Henceforth, without loss of generality we assume that $x_b = 0$. Until it is said differently, the boundary cohort is always indicated by an index B and the internal cohorts are denoted by $i \in \{B+1, \dots, J\}$. For shortening a notation we denote the number of the internal cohorts by $L := J - B$.

5.2.2. EBT Method with the Original Boundary Equations: Well-posedness

The equations constituting the original EBT scheme are the following.

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{dt}} x^i(t) = b(t, \mu_t^n)(x^i(t)), & \text{for } i = B+1, \dots, J, \\ \dfrac{\mathrm{d}}{\mathrm{dt}} m^i(t) = -c(t, \mu_t^n)(x^i(t)) m^i(t), & \text{for } i = B+1, \dots, J, \end{cases} \tag{5.1}$$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{dt}} \pi^B(t) = b(t, \mu_t^n)(x_b) m^B(t) + \partial_x b(t, \mu_t^n)(x_b) \pi^B(t) \\ \qquad\qquad -c(t, \mu_t^n)(x_b) \pi^B(t), \\ \dfrac{\mathrm{d}}{\mathrm{dt}} m^B(t) = -c(t, \mu_t^n)(x_b) m^B(t) - \partial_x c(t, \mu_t^n)(x_b) \pi^B(t) \\ \qquad\qquad + \sum_{i=B}^J \beta(t, \mu_t^n)(x^i(t)) m^i(t), \end{cases} \tag{5.2}$$

where $\mu_t^n = \sum_{i=B}^J m^i(t) \delta_{x^i(t)}$ and

$$x^B(t) = \begin{cases} \frac{\pi^B(t)}{m^B(t)} + x_b, & \text{if } m^B(t) > 0, \\ x_b, & \text{otherwise.} \end{cases} \tag{5.3}$$

Here, superscript n denotes the initial number of cohorts. If no ambiguity occurs, we omit this superscript in the present section. Note that the dynamics of x^B is not given explicitly, since the boundary cohort is initially empty and thus, the center of mass is not well-defined (see Section 5.4). Instead of x^B, a quantity which represents the cumulative amount by which the individuals exceed their birth size is observed. This quantity is denoted as π^B. In particular, the equations in (5.2) were derived through series expansion around x_b, which implies that additional terms comparing to (5.1) appear.

We assume that dependence of the model functions b, c, β on the measure μ is implicit (see (2.9)), that is

$$b(t,\mu) = \hat{b}\left(t, E_{b,\mu}\right), \quad c(t,\mu) = \hat{c}\left(t, E_{c,\mu}\right) \quad \text{and} \quad \beta(t,\mu) = \hat{\beta}\left(t, E_{\beta,\mu}\right), \tag{5.4}$$

where

$$E_{b,\mu} = \int_{\mathbb{R}_+} \gamma_b(y)\,\mathrm{d}\mu(y), \;\; E_{c,\mu} = \int_{\mathbb{R}_+} \gamma_c(y)\,\mathrm{d}\mu(y), \;\; E_{\beta,\mu} = \int_{\mathbb{R}_+} \gamma_\beta(y)\,\mathrm{d}\mu(y)$$

and

$$\hat{b}, \hat{c}, \hat{\beta} \;\; \in \;\; \mathbf{C}^{\mathbf{1},\alpha}_{\mathbf{b}}\Big([0,T]\times\mathbb{R}_+;\; \mathbf{W}^{\mathbf{1},\infty}(\mathbb{R}_+)\Big), \quad \gamma_b, \gamma_c, \gamma_\beta \in \mathbf{W}^{\mathbf{1},\infty}(\mathbb{R}_+;\mathbb{R}_+). \tag{5.5}$$

We recall that $\mathbf{C}^{\mathbf{1},\alpha}_{\mathbf{b}}\left([0,T]\times\mathbb{R}_+;\; \mathbf{W}^{\mathbf{1},\infty}(\mathbb{R}_+)\right)$ denotes the space of $\mathbf{W}^{\mathbf{1},\infty}(\mathbb{R}_+)$ valued functions, bounded with respect to the $\|\cdot\|_{\mathbf{W}^{\mathbf{1},\infty}}$ norm, Hölder continuous with respect to time and Lipschitz continuous with respect to the second variable. We define

$$\|f\|_{\mathbf{C}^{\mathbf{1},\alpha}_{\mathbf{b}}} = \|f\|_{\mathbf{BC}} + \sup_{t\in[0,T]} \mathbf{Lip}(f(t,\cdot)) + \sup_{x\in\mathbb{R}_+} H(f(\cdot,x)),$$

where

$$\|f\|_{\mathbf{BC}} = \sup_{(t,x)\in[0,T]\times\mathbb{R}_+} \|f(t,x)\|_{\mathbf{W}^{\mathbf{1},\infty}},$$

$\mathbf{Lip}(f(t,\cdot))$ denotes the Lipschitz constant and $H(f(\cdot,x))$ is equal to

$$H(f(\cdot,x)) = \sup_{s_1,s_2\in[0,T]} \frac{|f(s_1,x) - f(s_2,x)|}{|s_1 - s_2|^\alpha}.$$

The space $\mathbf{W}^{\mathbf{1},\infty}$ is equipped with its usual norm, that is $\|\gamma\|_{\mathbf{W}^{\mathbf{1},\infty}} = \max\{\|\gamma\|_{L^\infty}, \|\partial_x\gamma\|_{L^\infty}\}$. We would like to mention that the assumption about the implicit dependence of the model functions on the measure variable is rather technical, but the rigorous proof of existence of solutions to (5.1) - (5.2) in a full generality would require advanced tools from the measure theory and thus, we do not consider this problem here. Even dough we stick to the assumption about the implicit dependence, the proof is not straightforward because of the specific definition of the dynamics of the boundary cohort (5.3), which implies that the right hand side is not Lipschitz continuous in general. Namely, the term which causes the difficulties is $\beta(t,\mu)(x^B(t))$ appearing in the last equation, since x^B is given as a quotient π^B/m^B and m^B is not separated from zero. Therefore, instead of looking at (5.1) - (5.2) directly we consider a modified $2(L+1)$-dimensional ODEs system

$$\frac{\mathrm{d}}{\mathrm{dt}}y(t) = F\Big(t, \Gamma(y(t))\Big), \quad y(0) = y_o, \tag{5.6}$$

where Γ is an operator which projects two first coordinates of a vector on some closed, convex set in $\mathbb{R}^2$. In the first step we prove well-posedness of (5.6). As the second step we show that $\Gamma(y(t)) = y(t)$, which implies existence and uniqueness of solutions to the original system.

Theorem 5.7. *Assume that* (5.4) - (5.5) *hold and the initial datum in* (5.1) - (5.2) *are such that* $\{x_o^i, m_o^i\}_{i=B+1}^J$ *are positive and* $\pi_o^B = m_o^B = 0$*. Then, there exists a unique solution to* (5.1) - (5.2) *on the time interval* $[0, q^*]$*, where* q^* *depends on the model functions and the dimension of the system. Moreover, for all* $0 \le s \le t \le q^*$ *it holds that*

$$\begin{aligned} &(i) \quad |x^i(t) - x^i(s)| \le L_i(t-s), \quad i = B, B+1, \ldots, J, \\ &(ii) \quad 0 \le M(t) \le M(s)\exp(C(t-s)), \text{ where } M(t) = \textstyle\sum_{i=B}^J m^i(t). \end{aligned} \tag{5.8}$$

Constants L_i *and* C *depend only on the model functions.*

Proof of Theorem 5.7.
Step 1: Well-posedness of (5.6)**.** Define a closed, convex set

$$K = \left\{(\pi^B, m^B) \;:\; 0 \le \pi^B,\ 0 \le m^B \text{ and } \pi^B \le C_K m^B\right\} \subset \mathbb{R}_+^2,$$

where C_K is a positive constant. Let Π and Γ be operators defined as the following.

$$\begin{aligned} \Pi : \mathbb{R}^2 \to K, \quad \Pi(z) = \operatorname{argmin}_{(\pi^B, m^B)\in K} \|(z_1, z_2) - (\pi^B, m^B)\|_2, \\ \Gamma : \mathbb{R}^{2(L+1)} \to K \times \mathbb{R}^2, \quad \Gamma(w) = \left(\Pi(w_1, w_2), \mathrm{Id}_{2L}(w_3, \ldots, w_{2(L+1)})\right), \end{aligned} \tag{5.9}$$

where $\mathrm{Id}_{2L} : \mathbb{R}^{2L} \to \mathbb{R}^{2L}$ is the identity operator. For any $z \in \mathbb{R}^2$ there exists a unique vector $\Pi(z)$ such that (5.9) holds, which is called the projection of z on K. A proof of this claim together with properties of Π can be found in Lemma 5.17. Since $\Pi : (\mathbb{R}^2, l_2) \to (\mathbb{R}^2, l_2)$ is non-expansive and all norms are equivalent in the n-dimensional Euclidean space, we obtain that there exists a constant $\mathbf{Lip}(\Pi)$ such that

$$\|\Pi(z) - \Pi(\tilde{z})\|_1 \le \mathbf{Lip}(\Pi)\|z - \tilde{z}\|_1 \quad \forall\ z, \tilde{z} \in \mathbb{R}^2. \tag{5.10}$$

In particular, it can be shown that $|\Pi_1(z)| \le |z(1)|$ and $|\Pi_2(z)| \le C\|z\|_1$.
Let $F : [0,T] \times (K \times \mathbb{R}_+^{2L}) \to \mathbb{R}^{2(L+1)}$ be a function defined as

$$\begin{aligned} F_1(t,y) &= b(t,\mu_t)(x_b)m^B + \partial_x b(t,\mu_t)(x_b)\pi^B - c(t,\mu_t)(x_b)\pi^B, \\ F_2(t,y) &= -c(t,\mu_t)(x_b)m^B - \partial_x c(t,\mu_t)(x_b)\pi^B + \sum_{i\in\{B,1,\ldots,L\}} \beta(t,\mu_t)(x^i)m^i, \\ F_{2i-1}(t,y) &= b(t,\mu_t)(x^i), \quad F_{2i}(t,y) = -c(t,\mu_t)(x^i)m^i \quad \text{for } i = 1, \ldots, L. \end{aligned}$$

Here, $y = (\pi^B, m^B, x^1, m^1, \ldots, x^L, m^L)$, $\mu_t = \sum_{i\in\{B,1,\ldots,L\}} m^i\delta_{x^i}$ and x^B is defined as in (5.3). Function F is continuous, locally bounded and locally Lipschitz. Continuity is straightforward because of the assumptions (5.4) - (5.5) and the definition of set K, which assures that x^B is bounded. F is locally bounded, since the following estimates hold.

$$\sup_{(t,y)\in K} |F_{2i-1}(t,y)| = \|b\|_{\mathbf{BC}} \quad \text{and} \quad \sup_{(t,y)\in K} |F_{2i}(t,y)| \le \|c\|_{\mathbf{BC}}|m^i|,$$

where $i = 2, \ldots, L$. $(\pi^B, m^B) \in K$ implies that $0 \le \pi^B \le C_K m^B$ and thus,

$$\begin{aligned}
\sup_{(t,y)\in K} |F_1(t,y)| &\le \|b\|_{\mathbf{BC}} \left|m^B\right| + C_K \|(b,c)\|_{\mathbf{BC}} \left|m^B\right|, \\
\sup_{(t,y)\in K} |F_2(t,y)| &\le \|c\|_{\mathbf{BC}} \left|m^B\right| + C_K \|c\|_{\mathbf{BC}} \left|m^B\right| + \|\beta\|_{\mathbf{BC}} \sum_{i\in\{B,1,\ldots,L\}} \left|m^i\right|.
\end{aligned}$$

Summarizing,

$$\begin{aligned}
\|F\|_\infty &= \sum_{i=1}^{2(L+1)} \sup_{(t,y)\in K} |F_i(t,y)| \\
&\le L\|b\|_{\mathbf{BC}} + \left|m^B\right| \left(\|b\|_{\mathbf{BC}} + C_K \|(b,c)\|_{\mathbf{BC}} + C_K \|c\|_{\mathbf{BC}}\right) \\
&\quad + \|(c,\beta)\|_{\mathbf{BC}} \sum_{i\in\{B,1,\ldots,L\}} \left|m^i\right| \le C_\infty \sum_{i\in\{1,B,\ldots,L\}} \left|m^i\right|, \qquad (5.11)
\end{aligned}$$

where $C_\infty = C_\infty \left(\|(b,c,\beta)\|_{\mathbf{BC}}, C_K, L\right)$. It remains to show that F is locally Lipschitz continuous with respect to y. This is the only part of this proof which requires the implicit dependency of the model functions on the measure variable. For $i \in \{2, \ldots, L\}$ we obtain the following estimates

$$\begin{aligned}
&|F_{2i-1}(t,y) - F_{2i-1}(t,\tilde{y})| \\
&\qquad \le \left|\hat{b}(t,E_{b,\mu})(x^i) - \hat{b}(t,E_{b,\mu})(\tilde{x}^i)\right| + \left|\hat{b}(t,E_{b,\mu})(\tilde{x}^i) - \hat{b}(t,E_{b,\tilde{\mu}})(\tilde{x}^i)\right| \\
&\qquad \le \|\hat{b}\|_{\mathbf{BC}} \left|x^i - \tilde{x}^i\right| + \|\hat{b}\|_{\mathbf{C_b^{1,\alpha}}} |E_{b,\mu} - E_{b,\tilde{\mu}}|,
\end{aligned}$$

$$\sum_{i=1}^{L} |F_{2i-1}(t,y) - F_{2i-1}(t,\tilde{y})| \le \|\hat{b}\|_{\mathbf{BC}} \sum_{i=1}^{L} \left|x^i - \tilde{x}^i\right| + L\, \|\hat{b}\|_{\mathbf{C_b^{1,\alpha}}} |E_{b,\mu} - E_{b,\tilde{\mu}}|,$$

$$\begin{aligned}
|F_{2i}(t,y) - F_{2i}(t,\tilde{y})| &\le \left|\hat{c}(t,E_{c,\mu})(x^i)m^i - \hat{c}(t,E_{c,\tilde{\mu}})(x^i)m^i\right| \\
&\quad + \left|\hat{c}(t,E_{c,\tilde{\mu}})(x^i)m^i - \hat{c}(t,E_{c,\tilde{\mu}})(\tilde{x}^i)m^i\right| + \left|\hat{c}(t,E_{c,\tilde{\mu}})(\tilde{x}^i)(m^i - \tilde{m}^i)\right| \\
&\le \|\hat{c}\|_{\mathbf{C_b^{1,\alpha}}} \left|m^i\right| |E_{c,\mu} - E_{c,\tilde{\mu}}| + \|\hat{c}\|_{\mathbf{BC}} \left|m^i\right| \left|x^i - \tilde{x}^i\right| + \|\hat{c}\|_{\mathbf{BC}} \left|m^i - \tilde{m}^i\right|,
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{L} |F_{2i}(t,y) - F_{2i}(t,\tilde{y})| &\le \|\hat{c}\|_{\mathbf{C_b^{1,\alpha}}} |E_{c,\mu} - E_{c,\tilde{\mu}}| \sum_{i=1}^{L} \left|m^i\right| + \|\hat{c}\|_{\mathbf{BC}} \sum_{i=1}^{L} \left|m^i\right| \left|x^i - \tilde{x}^i\right| \\
&\quad + \|\hat{c}\|_{\mathbf{BC}} \sum_{i=1}^{L} \left|m^i - \tilde{m}^i\right|,
\end{aligned}$$

$$\begin{aligned}
|F_1(t,y) - F_1(t,\tilde{y})| &\le \left|\hat{b}(t,E_{b,\mu})(x_b) - \hat{b}(t,E_{b,\tilde{\mu}})(x_b)\right| \left|m^B\right| + \left|\hat{b}(t,E_{b,\tilde{\mu}})(x_b)(m^B - \tilde{m}^B)\right| \\
&\quad + \left|\partial_x \hat{b}(t,E_{b,\mu})(x_b) - \partial_x \hat{b}(t,E_{b,\tilde{\mu}})(x_b)\right| \left|\pi^B\right| + \left|\partial_x \hat{b}(t,E_{b,\tilde{\mu}})(x_b)(\pi^B - \tilde{\pi}^B)\right| \\
&\quad + \left|\hat{c}(t,E_{c,\mu})(x_b) - \hat{c}(t,E_{c,\tilde{\mu}})(x_b)\right| \left|\pi^B\right| + \left|\hat{c}(t,E_{c,\tilde{\mu}})(x_b)(\pi^B - \tilde{\pi}^B)\right| \\
&\le \|\hat{b}\|_{\mathbf{C_b^{1,\alpha}}} \left|m^B\right| |E_{b,\mu} - E_{b,\tilde{\mu}}| + \|\hat{b}\|_{\mathbf{BC}} \left|m^B - \tilde{m}^B\right| + \|\hat{b}\|_{\mathbf{C_b^{1,\alpha}}} \left|\pi^B\right| |E_{b,\mu} - E_{b,\tilde{\mu}}| \\
&\quad + \|\hat{b}\|_{\mathbf{BC}} \left|\pi^B - \tilde{\pi}^B\right| + \|\hat{c}\|_{\mathbf{C_b^{1,\alpha}}} \left|\pi^B\right| |E_{c,\mu} - E_{c,\tilde{\mu}}| + \|\hat{c}\|_{\mathbf{BC}} \left|\pi^B - \tilde{\pi}^B\right|
\end{aligned}$$

$$\leq \|\hat{b}\|_{\mathbf{C^{1,\alpha}_b}}(1+C_K)\big|m^B\big||E_{b,\mu}-E_{b,\tilde{\mu}}|+C_K\|c\|_{\mathbf{C^{1,\alpha}_b}}\big|m^B\big||E_{c,\mu}-E_{c,\tilde{\mu}}| + \|\hat{b}\|_{\mathbf{BC}}\big|m^B-\tilde{m}^B\big|+\|(\hat{b},\hat{c})\|_{\mathbf{BC}}\big|\pi^B-\tilde{\pi}^B\big|,$$

$$\begin{aligned}
|F_2(t,y)-F_2(t,\tilde{y})| &\leq |\hat{c}(t,E_{c,\mu})(x_b)-\hat{c}(t,E_{c,\tilde{\mu}})(x_b)|\big|m^B\big|+\big|\hat{c}(t,E_{c,\tilde{\mu}})(x_b)(m^B-\tilde{m}^B)\big| \\
&+|\partial_x\hat{c}(t,E_{c,\mu})(x_b)-\partial_x\hat{c}(t,E_{c,\tilde{\mu}})(x_b)|\big|\pi^B\big|+\big|\partial_x\hat{c}(t,E_{c,\tilde{\mu}})(x_b)(\pi^B-\tilde{\pi}^B)\big| \\
&+|\hat{\beta}(t,E_{\beta,\mu})(x^B)-\hat{\beta}(t,E_{\beta,\tilde{\mu}})(x^B)|\big|m^B\big|+|\hat{\beta}(t,E_{\beta,\tilde{\mu}})(x^B)m^B-\hat{\beta}(t,E_{\beta,\tilde{\mu}})(\tilde{x}^B)\tilde{m}^B| \\
&+\sum_{i=1}^{L}|\hat{\beta}(t,E_{\beta,\mu})(x^i)-\hat{\beta}(t,E_{\beta,\tilde{\mu}})(x^i)|\big|m^i\big|+\sum_{i=1}^{L}|\hat{\beta}(t,E_{\beta,\tilde{\mu}})(x^i)-\hat{\beta}(t,E_{\beta,\tilde{\mu}})(\tilde{x}^i)|\big|m^i\big| \\
&+\sum_{i=1}^{L}|\hat{\beta}(t,E_{\beta,\tilde{\mu}})(\tilde{x}^i)(m^i-\tilde{m}^i)| \\
&\leq \|\hat{c}\|_{\mathbf{C^{1,\alpha}_b}}\big|m^B\big||E_{c,\mu}-E_{c,\tilde{\mu}}|+\|\hat{c}\|_{\mathbf{BC}}\big|m^B-\tilde{m}^B\big|+\|\hat{c}\|_{\mathbf{C^{1,\alpha}_b}}C_K\big|m^B\big||E_{c,\mu}-E_{c,\tilde{\mu}}| \\
&+\|\hat{c}\|_{\mathbf{BC}}\big|\pi^B-\tilde{\pi}^B\big|+\|\hat{b}\|_{\mathbf{C^{1,\alpha}_b}}\big|m^B\big||E_{b,\mu}-E_{b,\tilde{\mu}}| \\
&+C(\|\hat{\beta}\|_{\mathbf{BC}},C_K)\left(\big|\pi^B-\tilde{\pi}^B\big|+\big|m^B-\tilde{m}^B\big|\right)+\|\hat{\beta}\|_{\mathbf{C^{1,\alpha}_b}}|E_{\beta,\mu}-E_{\beta,\tilde{\mu}}|\sum_{i=1}^{L}\big|m^i\big| \\
&+\|\hat{\beta}\|_{\mathbf{BC}}\sum_{i=1}^{L}\left(\big|m^i\big|\big|x^i-\tilde{x}^i\big|+\big|m^i-\tilde{m}^i\big|\right).
\end{aligned}$$

In the reasoning above we have estimated $|\hat{\beta}(t,E_{\beta,\tilde{\mu}})(x^B)m^B-\hat{\beta}(t,E_{\beta,\tilde{\mu}})(\tilde{x}^B)\tilde{m}^B|$ using Lemma 5.18, which assures that $(\pi^B,m^B)\to\left(\hat{\beta}(t,E_{\beta,\tilde{\mu}})(x^B)\right)m^B$ is Lipschitz continuous with the Lipschitz constant $C(\|\hat{\beta}\|_{\mathbf{BC}},C_K)=(2+C_K)\|\hat{\beta}\|_{\mathbf{BC}}$ on the set K. Summing up all the estimates yields

$$\begin{aligned}
&\sum_{i=1}^{2(L+1)}|F_i(t,y)-F_i(t,\tilde{y})| \qquad (5.12)\\
&\leq\left(\|\hat{b}\|_{\mathbf{BC}}+\|\hat{c}\|_{\mathbf{BC}}\sum_{i=1}^{L}\big|m^i\big|+\|\hat{\beta}\|_{\mathbf{BC}}\sum_{i=1}^{L}\big|m^i\big|\right)\sum_{i=1}^{L}\big|x^i-\tilde{x}^i\big| \\
&+\left(\|(\hat{b},\hat{c})\|_{\mathbf{BC}}+\|\hat{c}\|_{\mathbf{BC}}+C(\|\hat{\beta}\|_{\mathbf{BC}},C_K)\right)\big|\pi^B-\tilde{\pi}^B\big| \\
&+\left(\|\hat{c}\|_{\mathbf{BC}}+\|\hat{\beta}\|_{\mathbf{BC}}+C(\|\hat{\beta}\|_{\mathbf{BC}},C_K)\right)\sum_{i=1}^{L}\big|m^i-\tilde{m}^i\big|+\left(\|\hat{b}\|_{\mathbf{BC}}+\|\hat{c}\|_{\mathbf{BC}}\right)\big|m^B-\tilde{m}^B\big| \\
&+\left(L\|\hat{b}\|_{\mathbf{C^{1,\alpha}_b}}+\|\hat{b}\|_{\mathbf{C^{1,\alpha}_b}}(1+C_K)\big|m^B\big|+\|\hat{b}\|_{\mathbf{C^{1,\alpha}_b}}\big|m^B\big|\right)|E_{b,\mu}-E_{b,\tilde{\mu}}| \\
&+\left(\|\hat{c}\|_{\mathbf{C^{1,\alpha}_b}}\sum_{i=1}^{L}\big|m^i\big|+2C_K\|\hat{c}\|_{\mathbf{C^{1,\alpha}_b}}\big|m^B\big|+\|\hat{c}\|_{\mathbf{C^{1,\alpha}_b}}\right)|E_{c,\mu}-E_{c,\tilde{\mu}}| \\
&+\left(\|\hat{\beta}\|_{\mathbf{C^{1,\alpha}_b}}\sum_{i=1}^{L}\big|m^i\big|\right)|E_{\beta,\mu}-E_{\beta,\tilde{\mu}}| \\
&\leq C^*\left(\big|\pi^B-\tilde{\pi}^B\big|+\sum_{i=1}^{L}\big|x^i-\tilde{x}^i\big|+\sum_{i\in\{B,1,\ldots,L\}}\big|m^i-\tilde{m}^i\big|\right),
\end{aligned}$$

where

$$C^*=C^*\left(\|(\gamma_b,\gamma_c,\gamma_\beta\|_{\mathbf{W^{1,\infty}}},\|(\hat{b},\hat{c},\hat{\beta})\|_{\mathbf{C^{1,\alpha}_b}},C_K,L\right)\max\Big\{1,\sum_{i\in\{B,1,\ldots,L\}}\big|m^i\big|\Big\}. \qquad (5.13)$$

Let $y_o = (\pi_o^B, m_o^B, x^1, m^1, \dots, x^L, m^L) \in \mathbb{R}^{2(L+1)}$ be the initial datum in (5.6). Let $R_i > 0$ be such that $(y_o(i) - R_i) > 0$ for $i \in \{3, \dots, 2(L+1)\}$ and $R_1, R_2 > 0$ are arbitrary. Define

$$\bar{B}_R(y_o) = \left\{ y \in \mathbb{R}^{2(L+1)} \; : \; |y(i) - y_o(i)| \le R_i \right\}$$

and the operator

$$\mathcal{F} : [0,T] \times \bar{B}_R(y_o) \to \mathbb{R}^{2(L+1)}, \quad \mathcal{F}(t,y) = (F \circ \Pi)(t,y).$$

$\mathcal{F}$ is bounded and Lipschitz on $[0,T] \times \bar{B}_R(y_o)$, which follows from the corresponding properties of Π and F. Thus, there exists a unique solution to (5.6) on the time interval $[0, \varepsilon_o]$, where

$$\varepsilon_o = \frac{C_1}{C_2 + (|\pi_o^B| + |m_o^B|) + \sum_{i=1}^L |m_o^i|}.$$

The from of ε_o is a consequence of estimates (5.10), (5.11) and (5.13). Note that the length of the existence interval depends essentially on the joint mass of the initial datum. Thus, in order to be able to extend a solution on $[0,T]$ we need to show that the expression in the denominator is bounded for all $t \in [0,T]$. We will prove that this quantity grows at most exponentially in time and thus, it is bounded for all finite times. It is clear that for $i = 1, \dots, L$ it holds that

$$\left| m^i(t) \right| \le \left| m_o^i \right| \mathrm{e}^{Ct} \quad \Rightarrow \quad \sum_{i=1}^L \left| m^i(t) \right| \le \mathrm{e}^{Ct} \sum_{i=1}^L \left| m_o^i \right|.$$

For π^B and m^B we obtain that

$$\left| \pi^B(t) \right| \le \left| \pi_o^B \right| + \int_0^t \Big(\|(b,c)\|_{\mathbf{BC}} \left| \Pi_1(\pi^B(s), m^B(s)) \right| + \|b\|_{\mathbf{BC}} \left| \Pi_2(\pi^B(s), m^B(s)) \right| \Big) \, \mathrm{d}s$$

and

$$\begin{aligned} \left| m^B(t) \right| \; &\le \; \left| m_o^B \right| + \int_0^t \Big(\|c\|_{\mathbf{BC}} \left| \Pi_1(\pi^B(s), m^B(s)) \right| + \|(c,\beta)\|_{\mathbf{BC}} \left| \Pi_2(\pi^B(s), m^B(s)) \right| \Big) \, \mathrm{d}s \\ &+ \; \int_0^t \|\beta\|_{\mathbf{BC}} \Big(\mathrm{e}^{Ct} \sum_{i=1}^L \left| m_o^i \right| \Big) \, \mathrm{d}s. \end{aligned}$$

Summing up the expressions above yields

$$\begin{aligned} \left| \pi^B(t) \right| + \left| m^B(t) \right| \; &\le \; \left| \pi_o^B \right| + \left| m_o^B \right| + t \, \|\beta\|_{\mathbf{BC}} \mathrm{e}^{Ct} \sum_{i=1}^L \left| m_o^i \right| \\ &+ \; \|(b,c,\beta)\|_{\mathbf{BC}} C \int_0^t \Big(\left| \pi^B(s) \right| + \left| m^B(s) \right| \Big) \, \mathrm{d}s, \end{aligned}$$

which, by the Gronwall's inequality implies

$$\begin{aligned} \left| \pi^B(t) \right| + \left| m^B(t) \right| \; &\le \; \Big(\left| \pi_o^B \right| + \left| m_o^B \right| + t \, \|\beta\|_{\mathbf{BC}} \mathrm{e}^{Ct} \sum_{i=1}^L \left| m_o^i \right| \Big) \mathrm{e}^{Ct} \\ &\le \; \mathrm{e}^{2Ct} \Big(\left| \pi_o^B \right| + \left| m_o^B \right| + t \, \|\beta\|_{\mathbf{BC}} \sum_{i=1}^L \left| m_o^i \right| \Big) \end{aligned}$$

$$\le \quad e^{2Ct}\Big(|\pi_o^B| + |m_o^B| + \sum_{i=1}^{L} |m_o^i|\Big)(1 + t\,\|\beta\|_{\mathbf{BC}})$$

$$\le \quad e^{2Ct} e^{\|\beta\|_{BC} t}\Big(|\pi_o^B| + |m_o^B| + \sum_{i=1}^{L} |m_o^i|\Big).$$

Step 2: Non-negativity of solutions and well-posedness of (5.1) **-** (5.2)**.** We will prove that there exists q^* such that $y(t) = \Pi(y(t))$ for all $t \in [0, q^*]$. It is clear that for a positive initial datum solutions to (5.1) remain positive.
(1) non-negativity of m^B:
Note that $t \to \frac{d}{dt} m^B(t)$ is continuous and $\frac{d}{dt} m^B(0) > 0$. Therefore, there exists t_1 such that $\frac{d}{dt} m^B(t_1) > 0$ on $(0, t_1)$ and thus, $m^B(t) > 0$ on $(0, t^*]$ for some t^*.
(2) non-negativity of π^B:
If there exists $t_2 \in (0, t^*]$ such that $\pi^B(t_2) > 0$, then π^B remains nonnegative on the whole interval $[t_2, t^*]$, which holds due to the fact that $\pi^B(\cdot)$ is continuous and

$$\pi^B(\bar{t}) = 0 \quad \Rightarrow \quad \Pi(\pi^B(\bar{t})) = 0 \quad \Rightarrow \quad \frac{d}{dt}\pi^B(\bar{t}) > 0.$$

Positivity of $\pi^B(t)$ will be shown by a contradiction argument. Assume that there exists $t_3 \le t^*$ such that $\pi^B(t_3) < 0$. Then, it follows from the reasoning above that $\pi^B(t) < 0$ for all $t \in (0, t_3]$ and as a consequnce $\Pi_1(\pi^B(t), m^B(t)) = 0$. We recall that

$$\begin{aligned} \frac{d}{dt}\pi^B(t) &= b(t,\mu_t)(x_b)\Pi_2(\pi^B(t), m^B(t)) \\ &\quad + \Big(\partial_x b(t,\mu_t)(x_b) - \partial_x c(t,\mu_t)(x_b)\Big)\Pi_1(\pi^B(t), m^B(t)). \end{aligned}$$

However, $b(t,\mu_t)(x_b)\Pi_2(\pi^B(t), m^B(t)) = b(t,\mu_t)(x_b) m^B(t) > 0$ and the latter term of the equality is equal to zero. Therefore, we conclude that $\frac{d}{dt}\pi^B(t) > 0$ on $(0, t_3]$, which leads to the contradiction, since we assumed that $\pi^B(t) < 0$ for all $t \in (0, t_3]$ and $\pi^B(0) = 0$.
(3) $(\pi^B(t), m^B(t)) \in K$:
To prove this claim we recall [12, Lemma 17] stating that there exist positive constants C_K and q^*, depending only on the model functions, such that

$$0 \le x^B(t) \le C_K t \qquad \text{for all } t \in [0, q^*]. \tag{5.14}$$

The inequality is fulfilled for $t = 0$, as $m^B(0) = 0$ and as a consequence $x^B(0) = x_b = 0$. Since $m^B(t) > 0$ on $t \in (0, t^*]$, $x^B(t)$ is defined as a quotient $m^B(t)/\pi^B(t)$ and thus, is differentiable. Then, the first inequality holds due to the fact that $\pi^B(t)$ and $m^B(t)$ are positive on some non-degenerate interval. To obtain the second inequality we calculate $\frac{d}{dt} x^B(t)$.

$$\begin{aligned} \frac{d}{dt} x^B(t) &= \frac{d}{dt}\left(\frac{\pi^B(t)}{m^B(t)}\right) = \frac{\frac{d}{dt}\pi^B(t)}{m^B(t)} - \frac{\pi^B(t)\frac{d}{dt}m^B(t)}{(m^B(t))^2} \\ &= b(t,\mu_t)(x_b) + \partial_x b(t,\mu_t)(x_b)\frac{\pi^B(t)}{m^B(t)} + \partial_x c(t,\mu_t)(x_b)\left(\frac{\pi^B(t)}{m^B(t)}\right)^2 \\ &\quad - \frac{\pi^B(t)}{(m^B(t))^2}\sum_{i=B}^{J} \beta(t,\mu_t)(x^i(t)) m^i(t) \end{aligned} \tag{5.15}$$

$$\leq \quad b(t,\mu_t)(x_b) + \partial_x b(t,\mu_t)(x_b)x^B(t) + \partial_x c(t,\mu_t)(x_b)\left(x^B(t)\right)^2.$$

From [12] it follows that there exists $q^* > 0$ such that $\frac{\mathrm{d}}{\mathrm{dt}}x^B(t) \leq C_K$ for $t \in (0,q^*]$. Hence, $x^B(t) \leq C_K t$ on $[0,q^*]$. In particular, this estimate implies continuity of x^B in $t = 0$. We know that $\frac{\mathrm{d}}{\mathrm{dt}}x^B(t)$ is continuous and bounded from above for $t \in (0,q^*]$. Therefore, in order to show that x^B is Lipschitz continuous it is sufficient to prove that $\lim_{t\to 0^+}\frac{\mathrm{d}}{\mathrm{dt}}x^B(t)$ is bounded. Note that

$$\left|\partial_x b(t,\mu_t)(x_b)\frac{\pi^B(t)}{m^B(t)} + \partial_x c(t,\mu_t)(x_b)\left(\frac{\pi^B(t)}{m^B(t)}\right)^2\right| \leq \|b\|_{\mathbf{BC}}C_K t + \|c\|_{\mathbf{BC}}C_K^2 t^2 \xrightarrow{t\to 0^+} 0$$

and

$$\begin{aligned}
0 \quad &\leq \quad \lim_{t\to 0^+}\frac{\pi^B(t)}{(m^B(t))^2}\sum_{i=B}^J \beta(t,\mu_t)(x^i(t))m^i(t) \\
&= \quad \lim_{t\to 0^+}\frac{1}{m^B(t)}\left(\frac{\pi^B(t)}{m^B(t)}\right)\sum_{i=B}^J \beta(t,\mu_t)(x^i(t))m^i(t) \\
&\leq \quad C_K \lim_{t\to 0^+}\left(\frac{t}{m^B(t)}\right)\sum_{i=B}^J \beta(t,\mu_t)(x^i(t))m^i(t) = C_K,
\end{aligned}$$

where we calculated the limit of $t/m^B(t)$ using d'Hospital rule:

$$\lim_{t\to 0^+}\frac{t}{m^B(t)} = \frac{1}{\lim_{t\to 0^+}\frac{\mathrm{d}}{\mathrm{dt}}m^B(t)} = \frac{1}{\sum_{i=B}^J \beta(t,\mu_t)(x^i(t))m^i(t)}.$$

Therefore,

$$b(0,\mu_o)(x_b) - C_K \leq \lim_{t\to 0^+}\frac{\mathrm{d}}{\mathrm{dt}}x^B(t) \leq b(0,\mu_o)(x_b).$$

Step 3: Proof of claims (i) and (ii). Integration of (5.1) leads to

$$x^i(t) = x^i(s) + \int_s^t b(\tau,\mu_\tau)(x^i(\tau))\mathrm{d}\tau \quad \Rightarrow \quad \left|x^i(t) - x^i(s)\right| \leq \|b\|_{\mathbf{BC}}|t-s| \quad \Rightarrow \quad L_i = \|b\|_{\mathbf{BC}},$$

for $i = B,\ldots,J$. It follows from the reasoning presented in the previous step that x^B is Lipschitz continuous on $[0,q^*]$ with the Lipschtitz constant L_B, which depends only on the model functions. Integrating and summing up the equations describing the dynamics of m^i yields

$$\begin{aligned}
\sum_{i=B}^J m^i(t) \quad &= \quad \sum_{i=B}^J m^i(s) - \int_s^t c(\tau,\mu_\tau)(x_b)m^B(\tau) + c(\tau,\mu_\tau)(x^i(\tau))\sum_{i=B+1}^J m^i(\tau)\,\mathrm{d}\tau \\
&+ \quad \int_s^t \sum_{i=B}^J \beta(\tau,\mu_\tau)(x^i(\tau))m^i(\tau)\mathrm{d}\tau + \int_s^t (\partial_x b(\tau,\mu_\tau)(x_b) - c(\tau,\mu_\tau)(x_b))\pi^B(\tau)\mathrm{d}\tau \\
&\leq \quad \sum_{i=B}^J m^i(s) + \|(c,\beta)\|_{\mathbf{BC}}\int_s^t \sum_{i=B}^J m^i(\tau)\mathrm{d}\tau + \|(b,c)\|_{\mathbf{BC}}C_K\int_s^t m^B(\tau)\mathrm{d}\tau \\
&\leq \quad \sum_{i=B}^J m^i(s) + C\int_s^t \sum_{i=B}^J m^i(\tau)\mathrm{d}\tau,
\end{aligned}$$

where $C = C(\|(b,c,\beta)\|_{\mathbf{BC}}, C_K)$. Application of the Gronwall's inequality proves the assertion (ii).

□

Remark 5.16. Note that to obtain non-negativity of $\frac{\mathrm{d}}{\mathrm{dt}} m^B(t)$ in the proof above we had to assume that $\beta(t,\mu_t)(x^i)$ is strictly positive for at least one index $i \in \{1,\dots,L\}$. This is a reasonable assumption, since otherwise the boundary cohort would not be created due to the lack of any new individuals. To obtain (5.14) using [12, Lemma 17] we additionally have to assume that $\partial_x b(t,\mu_t)(x_b)$ and $\partial_x c(t,\mu_t)(x_b)$ are nonnegative.

5.2.3. EBT Method with the Original Boundary Equations: Technical Details

In this subsection we present technical lemmas necessary for the for the proof of well-posedness of (5.1) - (5.2).

Lemma 5.17. *Let $K \subset \mathbb{R}^n$ be a non-empty, closed, convex set and $y \in \mathbb{R}^n$. Then, there exists a unique $P(y) \in K$ such that for all $x \in K$ it holds that $P(y) = \operatorname{argmin}\|y-x\|_2$. Moreover, P is non-expansive meaning that*

$$\|P(y) - P(z)\|_2 \le \|y - z\|_2.$$

Proof of Lemma 5.17. Let $m = \inf_{x\in K}\|y-x\|_2$, $R = 2m$ and $K_R = K \cap \bar{B}(y,R)$. K_R is closed and bounded, which implies that it is also compact. It is also a non-empty set. Function $f(x) := \|y-x\|_2$ is continuous which, by Weierstrass theorem, implies that there exists $P(y) \in K_R \subset K$ such that $m = \|y - P(y)\|_2$. Concerning the uniqueness, let $y \in \mathbb{R}^n$ and assume that $x_1, x_2 \in K$ are such that $m = \|y-x_1\|_2 = \|y-x_2\|_2$. By the parallelogram law we have that

$$\|x_1 - x_2\|_2^2 = \|x_1 - y\|_2^2 + \|x_2 - y\|_2^2 - 2\Big\langle x_1 - y, x_2 - y\Big\rangle.$$

But also

$$4\left\|\frac{x_1 + x_2}{y}\right\|_2^2 = \|(x_1 - y) + (x_2 - y)\|_2^2 = \|x_1 - y\|_2^2 + \|x_2 - y\|_2^2 + 2\Big\langle x_1 - y, x_2 - y\Big\rangle.$$

Therefore,

$$\|x_1 - x_2\|_2^2 = 2\|x_1 - x_2\|_2^2 + 2\|x_2 - y\|_2^2 - 4\left\|\frac{x_1 + x_2}{2} - y\right\|_2^2.$$

Let $\delta := \|x_1 - y\|_2$. Since K is convex, $(x_1+x_2)/2 \in K$ and $\|(x_1+x_2)/2 - y\|_2 \ge \delta$. Therefore,

$$\|x_1 - x_2\|_2^2 \le 2\delta^2 + 2\delta^2 - 4\delta^2 = 0 \quad \Rightarrow \quad x_1 = x_2.$$

To prove that P is non-expansive let $x \in K$ and $\lambda \in [0,1]$. We have that

$$\begin{aligned}\|P(y) - y\|_2^2 &\le \|y - (\lambda x + (1-\lambda)P(y)\|_2^2 = \|y - P(y) - \lambda(x - P(y))\|_2^2 \\ &= \|y - P(y)\|_2^2 + \lambda^2\|x - P(y)\|_2^2 - 2\lambda\Big\langle y - P(y), x - P(y)\Big\rangle,\end{aligned}$$

which implies that

$$\Big\langle y - P(y), x - P(y)\Big\rangle \le \frac{\lambda}{2}\|x - P(y)\|_2^2 \quad \Rightarrow \quad \Big\langle y - P(y), x - P(y)\Big\rangle \le 0 \quad \forall x \in K.$$

Using this inequality we obtain that for any $z, y \in \mathbb{R}^n$ it holds that

$$\left\langle z - P(z), P(y) - P(z) \right\rangle \leq 0 \quad \text{and} \quad \left\langle y - P(y), P(z) - P(y) \right\rangle \leq 0$$

and therefore

$$\left\langle z - y + P(y) - P(z), P(y) - P(z) \right\rangle \leq 0.$$

As a consequence we obtain

$$\left\langle P(z) - P(y) \right\rangle^2 \leq \left\langle P(z) - P(y), z - y \right\rangle \leq \|P(z) - P(y)\|_2 \|z - y\|_2,$$

which ends the proof. $\square$

Lemma 5.18. *Let $f : [0, +\infty) \to \mathbb{R}$ be a bounded Lipschitz function and V be a convex set defined as the following*

$$V = \left\{ (x, y) \in \mathbb{R}^2 \,:\, 0 \leq x,\ 0 \leq y \text{ and } x \leq C_V y \right\},$$

where C_V is some positive constant. Define

$$G : V \to \mathbb{R}, \quad G(x, y) = \begin{cases} y f\left(\frac{x}{y}\right), & \text{for } y \neq 0, \\ 0, & \text{for } y = 0. \end{cases}$$

Then, G is Lipschitz and $\mathbf{Lip}(G) = (2 + C_V)\|f\|_{W^{1,\infty}}$.

Proof of Lemma 5.18. Function $u(x, y) = \frac{x}{y}$ is differentiable for $y \neq 0$. Due to the Rademacher theorem f is differentiable almost everywhere, so is G. Let us estimate the gradient of G

$$|\partial_x G(x, y)| = y \left| f'\left(\frac{x}{y}\right) \right| \frac{1}{y} \leq \mathbf{Lip}(f)$$

and

$$|\partial_y G(x, y)| \leq \left| f\left(\frac{x}{y}\right) \right| + y \left| f'\left(\frac{x}{y}\right) \right| \frac{x}{y^2} \leq \|f\|_{L^\infty} + \mathbf{Lip}(f) C_V \leq (1 + C_V)\|\gamma\|_{W^{1,\infty}}.$$

Partial derivatives of G exist a.e and are bounded. The mean value theorem implies that G is Lipschitz on $(V \backslash \{(x, y : y = 0)\})$ with the Lipschitz constant $\mathbf{Lip}(G) = (2 + C_V)\|f\|_{W^{1,\infty}}$. Now, let $(x, y), (\tilde{x}, \tilde{y}) \in V$ be such that $y > 0$ and $\tilde{y} = 0$. Then,

$$|G(x, y) - G(\tilde{x}, \tilde{y})| = \left| y f\left(\frac{x}{y}\right) \right| = \left| (y - \tilde{y}) f\left(\frac{x}{y}\right) \right| \leq \|f\|_{L^\infty} \left(|x - \tilde{x}| + |y - \tilde{y}| \right).$$

Thus, we obtained that G is Lipschitz on V with the Lipschitz constant equal to $(2 + C_V)\|f\|_{W^{1,\infty}}$. $\square$

Lemma 5.19. *Let* $\mu = \sum_{i\in\{B,1,\dots,L\}} m^i\delta_{x^i}$, $\tilde{\mu} = \sum_{i\in\{B,1,\dots,L\}} \tilde{m}^i\delta_{\tilde{x}^i}$, $\gamma \in W^{1,\infty}(\mathbb{R}_+;\mathbb{R}_+)$ *and* K *be a closed, compact set*

$$K = \left\{(\pi^B, m^B) \,:\, 0 \le \pi^B,\ 0 \le m^B \ \text{ and } \ \pi^B \le C_K m^B\right\}.$$

Assume that x^B *is defined as in* (5.3) *and* $(\pi^B, m^B) \in K$. *Define* $E_{\gamma,\mu} = \int_{\mathbb{R}_+} \gamma(y)\, d\mu(y)$. *Then, there exists a constant* $C = C(\gamma, m^i, C_K)$ *such that*

$$|E_{\gamma,\mu} - E_{\gamma,\tilde{\mu}}| \le C\Big(\left|\pi^B - \tilde{\pi}^B\right| + \sum_{i=1}^{L}\left|x^i - \tilde{x}^i\right| + \sum_{i\in\{B,1,\dots,L\}}\left|m^i - \tilde{m}^i\right|\Big).$$

Proof of Lemma 5.19. Define a function $G : K \to \mathbb{R}^2$, $G(\pi^B, m^B) = m^B\gamma\left(\frac{\pi^B}{m^B}\right)$, if $m^B \ne 0$, and $G(\pi^B, m^B) = 0$ otherwise. Then,

$$\begin{aligned} E_{\gamma,\mu} - E_{\gamma,\tilde{\mu}} &= \int_{\mathbb{R}_+} \gamma(y) d\Big(\sum_{i\in\{B,1,\dots,L\}} m^i\delta_{x^i} - \sum_{i\in\{B,1,\dots,L\}} \tilde{m}^i\delta_{\tilde{x}^i}\Big) \\ &= \sum_{i=1}^{L}\left(m^i\gamma(x^i) - \tilde{m}^i\gamma(\tilde{x}^i)\right) + \left(G(\pi^B, m^B) - G(\tilde{\pi}^B, \tilde{m}^B)\right). \end{aligned} \tag{5.20}$$

The first term can be estimated in a standard way, that is,

$$\begin{aligned} \sum_{i=1}^{L}\left|m^i\gamma(x^i) - \tilde{m}^i\gamma(\tilde{x}^i)\right| &= \sum_{i=1}^{L}\left|m^i(\gamma(x^i) - \gamma(\tilde{x}^i))\right| + \sum_{i=1}^{L}\left|\gamma(x^i)\left(m^i - \tilde{m}^i\right)\right| \\ &\le \mathbf{Lip}(\gamma)\sum_{i=1}^{L}\left|m^i\right|\left|x^i - \tilde{x}^i\right| + \|\gamma\|_{L^\infty}\sum_{i=1}^{L}\left|m^i - \tilde{m}^i\right| \\ &\le \|\gamma\|_{W^{1,\infty}} \max\Big\{1, \sum_{i=1}^{L}\left|m^i\right|\Big\}\sum_{i=1}^{L}\left(\left|x^i - \tilde{x}^i\right| + \left|m^i - \tilde{m}^i\right|\right). \end{aligned}$$

The second therm can be estimated using Lemma 5.18.

$$\left|G(\pi^B, m^B) - G(\tilde{\pi}^B, \tilde{m}^B)\right| \le (2 + C_K)\|\gamma\|_{W^{1,\infty}}\left(\left|\pi^B - \tilde{\pi}^B\right| + \left|m^B - \tilde{m}^B\right|\right).$$

Summarizing, we have that the assertion of the lemma holds with the constant C equal to

$$C = \|\gamma\|_{W^{1,\infty}}\left(3 + C_K + \sum_{i=1}^{L}\left|m^i\right|\right).$$

□

5.2.4. EBT Method with the Simplified Boundary Equations: Well-posedness

The main difference between the original and simplified version of the EBT method is a definition of the boundary cohort. In the simplified scheme the dynamics of the boundary cohort is defined through the following ODEs.

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{dt}}x^B(t) = b(t, \mu_t^n)(x^B(t)), \\ \dfrac{\mathrm{d}}{\mathrm{dt}}m^B(t) = -c(t, \mu_t^n)(x^B(t))m^B(t) + \sum_{i=B}^{J}\beta(t, \mu_t^n)(x^i(t))m^i(t). \end{cases} \tag{5.21}$$

On the contrary to (5.2) we evaluate functions b, c and β at $x^B(t)$. Moreover, the dynamics of x^B is directly described by equation (5.21). In this case we consider explicit dependence of the model functions on the measure variable, which leads to the following assumptions.

$$b, c, \beta \in \mathbf{C_b^{1,\alpha}}\left([0,T] \times \mathcal{M}^+(\mathbb{R}_+); \mathbf{W^{1,\infty}}(\mathbb{R}_+)\right). \tag{5.22}$$

We recall that $\mathbf{C_b^{1,\alpha}}([0,T] \times \mathcal{M}^+(\mathbb{R}_+);\ \mathbf{W^{1,\infty}}(\mathbb{R}_+))$ denotes the space of $\mathbf{W^{1,\infty}}(\mathbb{R}_+)$ valued functions, bounded with respect to the $\|\cdot\|_{\mathbf{W^{1,\infty}}}$ norm, Hölder continuous with respect to time and Lipschitz continuous with respect to the measure variable. We equip this space in the norm

$$\|f\|_{\mathbf{C_b^{1,\alpha}}} = \|f\|_{\mathbf{BC}} + \sup_{\mu \in \mathcal{M}^+(\mathbb{R}_+)} H(f(\cdot,\mu)) + \sup_{t \in [0,T]} \mathbf{Lip}(f(t,\cdot)),$$

where $\|f\|_{\mathbf{BC}} = \sup_{(t,\mu)\in[0,T]\times\mathcal{M}^+(\mathbb{R}_+)} \|f(t,\mu)\|_{\mathbf{W^{1,\infty}}}$.

Theorem 5.23. *Assume that* (5.22) *hold and the initial datum in* (5.1), (5.21) *are such that* $\{x_o^i, m_o^i\}_{i=B+1}^J$ *are positive and* $x_o^B = x_b = 0$, $m_o^B = 0$. *Then, there exists a unique solution to* (5.1), (5.21) *on the time interval* $[0,T]$. *Moreover, for all* $0 \le s \le t \le T$ *it holds that*

$$\begin{aligned} &(i)\quad |x^i(t) - x^i(s)| \le L_i(t-s), \quad i = B, B+1, \dots, J,\\ &(ii)\quad 0 \le M(t) \le M(s)\exp(C(t-s)), \ \text{ where } \ M(t) = \textstyle\sum_{i=B}^J m^i(t). \end{aligned} \tag{5.24}$$

Constants L_i *and* C *depend only on the model functions.*

Proof of Theorem 5.23. To prove existence and uniqueness of solutions we essentially need to show that the right hand side of (5.1), (5.21) is locally Lipschitz with respect to (x^i, m^i). A proof of this claim can be conducted analogously as the corresponding part of the proof of Theorem 5.7, therefore we do not repeat it here. The only difference is that in the estimate (5.12) a distance $\rho_F(\mu,\tilde{\mu})$ appears instead of the terms of the type $|E_{f,\mu} - E_{f,\tilde{\mu}}|$. According to Lemma 1.35

$$\begin{aligned} \rho_F(\mu,\tilde{\mu}) &\le \sum_{i=B}^J \left(|m^i||x^i - \tilde{x}^i| + |m^i - \tilde{m}^i|\right) \\ &\le \max\left\{1, \sum_{i=1}^L |m_i|\right\} \sum_{i=B}^J \left(|x^i - \tilde{x}^i| + |m^i - \tilde{m}^i|\right), \end{aligned} \tag{5.25}$$

which proves that the right hand side is locally Lipschitz. Non-negativity of solutions is straightforward assuming that $b(t,\mu)(x_b) \ge 0$ and $\beta(t,\mu)(\cdot) > 0$ (see Remark 5.16). Proof of claims (i) and (ii) is analogous to the corresponding part of the proof of Theorem 5.7. □

5.3. Convergence of the Algorithms

Definition 5.26. *Let* (E,ρ) *be a metric space. A family of bounded operators* $S : E \times [0,\delta] \times [0,T] \to E$ *is called a Lipschitz semiflow if the following conditions are satisfied,* *i)* $S(0;\tau) = I$ *for* $0 \le \tau \le T$,

ii) $S(t+s;\tau) = S(t;\tau+s)S(s;\tau)$ *for* $0 \le \tau, s, t \le T$ *such that* $s \le t$ *and* $0 \le \tau + s + t \le T$,

iii) $\rho\big(S(t;\tau)\mu, S(s;\tau)\nu\big) \le L\left(\rho(\mu,\nu) + |t-s|\right)$ *for* $0 \le s, t \le T$.

To estimate the distance between μ_t^n and the trajectory of semiflow S starting at μ_0 we use the following

Proposition 5.27. Let $S : E \times [0,\delta] \times [0,T] \to E$ be a Lipschitz semiflow. For every Lipschitz continuous map $\mu : [0,T] \to E$ the following estimate holds,

$$\rho\big(\mu_t, S(t;0)\mu_0\big) \le L \int_{[0,t]} \liminf_{h \downarrow 0} \frac{\rho\big(\mu_{\tau+h}, S(h;\tau)\mu_\tau\big)}{h} \mathrm{d}\tau, \tag{5.28}$$

where ρ is a corresponding metric.

The proof of Proposition 5.27 is the analogue of the proof of Theorem 2.9 in [14]. To apply Proposition 5.27 we need to show that a map $t \to \mu_t^n = \sum_{i=B}^J m^i(t)\delta_{x^i(t)}$ is Lipschitz continuous in the flat metric.

Theorem 5.29. *Let* $\mu_t^n = \sum_{i=B}^J m^i(t)\delta_{x^i(t)}$, *where* $\{x^i, m^i\}_{i=B}^J$ *is a solution to* (5.1) *with a boundary cohort defined as either in* (5.2) *or* (5.21). *Then,* $\mu^n : [0,T] \to (\mathcal{M}^+(\mathbb{R}_+), \rho_F)$ *is Lipschitz continuous.*

Proof of Theorem 5.29. Let $0 \le s \le t \le T$ be such that $|t-s| \le q^*$, where q^* is the length of the interval of existence of solutions to (5.1) with a boundary cohort defined as either in (5.2) or (5.21). Without loss of generality we may assume that there is no internalization process on (s,t). According to Lemma 1.35 (or formula (5.25)) we obtain

$$\begin{aligned}
\rho_F(\mu_t^n - \mu_s^n) &\le \sum_{i=B}^J \left(m^i(s)\left|x^i(t) - x^i(s)\right| + \left|m^i(t) - m^i(s)\right|\right) \\
&\le (t-s)\sum_{i=B}^J \Big(m^i(s)\mathbf{Lip}(x^i) + \mathbf{Lip}(m^i)\Big) \\
&\le (t-s)\max\{1, C\}\left(\sum_{i=B}^J m^i(s) + \sum_{i=B}^J \mathbf{Lip}(m^i)\right).
\end{aligned}$$

From Theorem 5.7 and Theorem 5.23) we know that $C = \max_i\{\mathbf{Lip}(x^i)\} < +\infty$ and $\sum_{i=B}^J m^i$ grows at most exponentially in time, so it is bounded on each finite time interval. For the simplified version of the EBT scheme we obtain

$$\sum_{i=B}^J \mathbf{Lip}(m^i) \le \sum_{i=B}^J \sup_t \left|\frac{\mathrm{d}}{\mathrm{dt}} m^i(t)\right| \le \sum_{i=B}^J \|(c,\beta)\|_{\mathbf{BC}} m^i(t) \le \|(c,\beta)\|_{\mathbf{BC}} \sum_{i=B}^J m^i(t) < +\infty.$$

Analogous estimate (with a different constant) holds for the original EBT scheme, since $|\partial_x c(t,\mu)(x_b)\pi^B(t)|$ is bounded by $\|c\|_{\mathbf{BC}} C_K m^B(t)$. Note that the Lipschitz constant of μ^n does not depend on the dimension of the corresponding system of ODEs. □

Theorem 5.30. *Let μ be a solution to* (2.2)*, that is*

$$\begin{cases} \partial_t \mu + \partial_x (b(t,\mu)\,\mu) + c(t,\mu)\,\mu &= 0, \qquad (t,x) \in [0,T] \times \mathbb{R}_+, \\ (b(t,\mu)(x_b))\, D_\lambda \mu_t(x_b^+) &= \int_{x_b}^{+\infty} \beta(t,\mu)(x) \mathrm{d}\mu_t(x), \\ \mu_o &\in \mathcal{M}^+(\mathbb{R}_+). \end{cases} \tag{5.31}$$

Let $\sum_{i=B}^J m_o^i \delta_{x_o^i}$ be an approximation of μ_o by a sum of Dirac deltas such that

$$\rho_F\left(\mu_o, \sum_{i=B}^J m_o^i \delta_{x_o^i}\right) \le C_1 \Delta x, \tag{5.32}$$

where $\Delta x = \max_{i=B+1,\dots,J} |x_o^i - x_o^{i-1}|$ and C_1 is a constant depending on $\mu_o(\mathbb{R}_+)$ and the approximation method. Let μ^n be the output of the EBT algorithm either for the original definition of the boundary cohort (5.2) *or the simplified one* (5.21) *with the initial condition given by $\sum_{i=B}^J m_o^i \delta_{x_o^i}$. Then, there exists a constant C_2 such that*

$$\rho_F(\mu_t, \mu_n^t) \le C_1 \Delta x + C_2 \Delta t.$$

Proof of Theorem 5.30. Let Δt be the length of the time step such that $0 < \Delta t \le q^*$, where $[0,q^*]$ is the interval of existence of the unique, nonnegative solutions to the original EBT scheme (5.1) - (5.2) such that (5.14) holds. If we consider the simplified version of the EBT algorithm (5.1), (5.21), then q^* can be chosen arbitrarily, since solutions to the latter system are unique and nonnegative for all times. Without loss of generality we assume that $\tau \in [0,\Delta t)$ and consider the interval $[\tau, \tau+h] \subset [0,\Delta t]$. Assume for now that τ is not the internalization moment. If h is sufficiently small, we can assume that there is no internalization procedure on $(\tau, \tau+h]$. According to the formula from Proposition 5.27 we need to estimate a distance between $\mu^n_{\tau+h}$ and $S(t;\tau)\mu^n_\tau$ such that

i) $\mu^n_{\tau+h} = \sum_{i=B}^J m^j(\tau+h)\delta_{x^j(\tau+h)}$, where $\{x^i(\cdot), m^i(\cdot)\}_{i=B}^J$ is the output of the EBT algorithm,

ii) $S(h;\tau)\mu^n_\tau$ is the solution at time $\tau+h$ to (2.2) with the initial time τ and initial datum $\mu^n_\tau = \sum_{i=B}^J m^j(\tau)\delta_{x^j(\tau)}$. In order to shorten the notation we denote $\mu_t := S(t-\tau;\tau)\mu^n_\tau$.

μ_t is a measure consisting of L Dirac deltas, denoted henceforth as $n^i(\tau+t)\delta_{y^i(\tau+t)}$, and the density $f(t,\cdot)$, which arises due to the boundary condition. The support of $f(t,\cdot)$ is contained in $[x_b, y^{abs}(t)]$, where $y^{abs}(\cdot)$ denotes the location of the characteristic line starting from x_b at time τ. Let $n^{abs}(t) = \int_{x_b}^{y^{abs}(t)} f(t,x)\mathrm{d}x$ denote the total mass of $f(t,\cdot)$. Using proper test functions in the definition of weak solution [21, Definition 2.2] we obtain

$$y^i(\tau+h) = x^i(\tau) + \int_\tau^{\tau+h} b(t,\mu_t)(y^i(t))\mathrm{d}t, \tag{5.33}$$

$$n^i(\tau+h) = m^i(\tau) - \int_\tau^{\tau+h} c(t,\mu_t)(y^i(t))n^i(t)\mathrm{d}t, \tag{5.34}$$

and

$$n^{abs}(\tau+h) = n^{abs}(\tau) + \int_\tau^{\tau+h}\left(\int_{x_b}^{y^{abs}(t)} -c(t,\mu_t)(x)\mathrm{d}\mu_t(x) + \int_{x_b}^{+\infty}\beta(t,\mu_t)(x)\mathrm{d}\mu_t(x)\right)\mathrm{d}t$$
$$= \int_\tau^{\tau+h}\int_0^{y^{abs}(t)}\left(-c(t,\mu_t)(x)+\beta(t,\mu_t)(x)\right)\mathrm{d}\mu_t(x) + \sum_{i=B}^{J}\int_\tau^{\tau+h}\beta(t,\mu_t)(y^i(t))n^i(t)\mathrm{d}t$$
$$= \mathcal{O}(h^2) + \sum_{i=B}^{J}\int_\tau^{\tau+h}\beta(t,\mu_t)(y^i(t))n^i(t)\mathrm{d}t.$$

The term of the order h^2 appears due to the fact that $|y^{abs}(t)| \le \|b\|_{\mathbf{BC}}h$, functions c, β are bounded and $\mu_t(\mathbb{R}_+)$ is uniformly bounded on $[0,T]$. We need to define a new measure ζ, which is created by shifting the mass $n^{abs}(\tau+h)$ distributed on $[x_b, y^{abs}(h)]$ to the closest Dirac delta, that is to $n^B(\tau+h)\delta_{y^B(\tau+h)}$. Thus, we obtain

$$\zeta = \sum_{i=B}^{J} p^i(\tau+h)\delta_{y^i(\tau+h)},$$

where

$$\begin{aligned} p^i(\tau+h) &= n^i(\tau+h), \qquad \text{for } i = B+1,\dots,J,\\ p^B(\tau+h) &= n^B(\tau+h) + n^{abs}(\tau+h). \end{aligned}$$

We will estimate $\rho_F(\mu_t,\zeta)$ and $\rho_F(\zeta,\mu^n_{\tau+h})$ in order to use the triangle inequality. Concerning the first term,

$$\begin{aligned} \rho_F(\mu_t,\zeta) &= \rho_F(n^{abs}\left(\tau+h)\delta_{y^B(\tau+h)}, f(t,\cdot)\right) \le |y^B(\tau+h)|n^{abs}(\tau+h)\\ &\le \|b\|_{\mathbf{BC}}\Delta t\left(\|\beta\|_{\mathbf{BC}}\int_\tau^{\tau+h}\sum_{i=B}^{J} n^i(t)\,\mathrm{d}t + \mathcal{O}(h^2)\right)\\ &\le \|b\|_{\mathbf{BC}}\Delta t\left(\|\beta\|_{\mathbf{BC}}hC + \mathcal{O}(h^2)\right) = \Delta t(\mathcal{O}(h)+\mathcal{O}(h^2)). \end{aligned} \tag{5.35}$$

In the estimate above we used the fact that the characteristic $y^B(\cdot)$ is Lipschitz continuous with the Lipschitz constant equal to $\|b\|_{\mathbf{BC}}$ and the mass in uniformly bounded on $[0,T]$, that is $\sum_{i=B}^{J} n^i(t) \le C$. In order to estimate $\rho_F(\zeta,\mu^n_{\tau+h})$ we use the formula provided by (5.25). Let us start with terms of the form $|x^i - y^i|$. For $i \in \{B+1,\dots,J\}$ it holds that

$$|x^i(\tau+h) - y^i(\tau+h)| \le \int_\tau^{\tau+h}|b(t,\mu^n_t)(x^i(t)) - b(t,\mu_t)(y^i(t))|\,\mathrm{d}t$$
$$\le \int_\tau^{\tau+h}|b(t,\mu^n_t)(x^i(t)) - b(t,\mu_t)(x^i(t))|\,\mathrm{d}t + \int_\tau^{\tau+h}|b(t,\mu_t)(x^i(t)) - b(t,\mu_t)(y^i(t))|\,\mathrm{d}t$$
$$\le \|b\|_{\mathbf{C^{1,\alpha}_b}}\int_\tau^{\tau+h}\rho_F(\mu^n_t,\mu_t)\,\mathrm{d}t + \|b\|_{\mathbf{BC}}\int_\tau^{\tau+h}|x^i(t)-y^i(t)|\,\mathrm{d}t$$
$$\le \|b\|_{\mathbf{C^{1,\alpha}_b}}\int_\tau^{\tau+h}(\mathbf{Lip}_\tau(\mu^n)h + \rho_F(\mu^n_\tau,\mu_\tau) + \mathbf{Lip}(\mu)h)\,\mathrm{d}t$$
$$+ \|b\|_{\mathbf{BC}}\int_\tau^{\tau+h}\left(\mathbf{Lip}_\tau(x^i)h + \left|x^i(\tau)-y^i(\tau)\right| + \mathbf{Lip}(y^i)h\right)\mathrm{d}t \le Ch^2,$$

which holds due to the fact that $\rho_F(\mu_\tau^n, \mu_\tau) = 0$ and $|x^i(\tau) - y^i(\tau)| = 0$. For the simplified version of the EBT scheme we have the analogous estimate for $i = B$, that is

$$\left|x^B(\tau + h) - y^B(\tau + h)\right| \leq Ch^2.$$

Multiplying $|x^i(\tau + h) - y^i(\tau + h)|$ by $m^i(\tau + h)$ and summing up over $i = B, \ldots, J$ yields

$$\sum_{i=B}^{J} m^i(\tau + h)|x^i(\tau + h) - y^i(\tau + h)| = \mathcal{O}(h^2), \tag{5.36}$$

since $\sum_{i=B}^{J} m^i(\tau + h) \leq C$. For the original EBT algorithm we need to estimate the entire expression $m^B(\tau + h)|x^B(\tau + h) - y^B(\tau + h)|$. First, let us calculate the derivative of $x^B(t)m^B(t)$.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}(x^B(t)m^B(t)) = x^B(t)\frac{\mathrm{d}}{\mathrm{dt}}m^B(t) + m^B(t)\frac{\mathrm{d}}{\mathrm{dt}}x^B(t)\\
&\quad = -c(t, \mu_t^n)(x_b)m^B(t)x^B(t) - \partial_x c(t, \mu_t^n)(x_b)\pi^B(t)x^B(t)\\
&\qquad + \Big(\sum_{i=B}^{J} \beta(t, \mu_t^n)(x^i(t))m^i(t)\Big)x^B(t)\\
&\qquad + b(t, \mu_t^n)(x_b)m^B(t) + \partial_x b(t, \mu_t^n)(x_b)\frac{\pi^B(t)}{m^B(t)}m^B(t)\\
&\qquad + \partial_x c(t, \mu_t^n)(x_b)\left(\frac{\pi^B(t)}{m^B(t)}\right)^2 m^B(t) - \frac{\pi^B(t)}{m^B(t)}\sum_{i=B}^{J} \beta(t, \mu_t^n)(x^i(t))m^i(t).
\end{aligned}$$

Integrating the expression above and subtracting (5.33) yields

$$\begin{aligned}
&m^B(\tau + h)\left|x^B(\tau + h) - y^B(\tau + h)\right|\\
&\leq \int_\tau^{\tau+h} m^B(t)\left|b(t, \mu_t^n)(x_b) - b(t, \mu_t)(y^B(t))\right|\mathrm{d}t\\
&\quad + \int_\tau^{\tau+h} \Big(|c(t, \mu_t^n)(x_b)| + C_K|\partial_x c(t, \mu_t^n)(x_b)|\Big)m^B(t)x^B(t)\mathrm{d}t\\
&\quad + \int_\tau^{\tau+h} \sup_x |\beta(t, \mu_t^n)(x)|\Big(\sum_{i=B}^{J} m^i(t)\Big)x^B(t)\mathrm{d}t\\
&\quad + \int_\tau^{\tau+h} \frac{\pi^B(t)}{m^B(t)}\left(|\partial_x b(t, \mu_t^n)(x_b)|m^B(t) + \sup_x |\beta(t, \mu_t^n)(x)|\sum_{i=B}^{J} m^i(t)\right)\mathrm{d}t\\
&\quad + \int_\tau^{\tau+h} |\partial_x c(t, \mu_t^n)(x_b)|\left(\frac{\pi^B(t)}{m^B(t)}\right)^2 m^B(t)\mathrm{d}t\\
&\leq \|b\|_{\mathbf{C_b^{1,\alpha}}} \int_\tau^{\tau+h} m^B(t)\left(\rho_F(\mu_t^n, \mu_t) + \left|x_b - y^B(t)\right|\right)\mathrm{d}t\\
&\quad + \|c\|_{\mathbf{BC}}(1 + C_K)\int_\tau^{\tau+h} m^B(t)x^B(t)\mathrm{d}t\\
&\quad + \|\beta\|_{\mathbf{BC}}\int_\tau^{\tau+h} M(t)x^B(t)\mathrm{d}t + C_K\|(b, \beta)\|_{\mathbf{BC}}\int_\tau^{\tau+h} M(t)t\,\mathrm{d}t\\
&\quad + \|c\|_{\mathbf{BC}}C_K^2\int_\tau^{\tau+h} m^B(t)t^2\mathrm{d}t.
\end{aligned}$$

Since

$$m^B \le M(t) \le C, \quad x^B(t) \le C_K \Delta t, \quad \left|x_b - y^B(t)\right| = \left|y^B(t)\right| \le \|b\|_{\mathbf{BC}} \Delta t$$

and

$$\rho_F(\mu_t^n, \mu_t) \le \rho_F(\mu_t^n, \mu_\tau^n) + \rho_F(\mu_\tau^n, \mu_\tau) + \rho_F(\mu_\tau, \mu_t) \le (\mathbf{Lip}(\mu) + \mathbf{Lip}(\mu^n))\, h,$$

we obtain

$$\begin{aligned} & m^B(\tau+h)\left|x^B(\tau+h) - y^B(\tau+h)\right| \\ & \quad \le C\Big(\|(b,c,\beta)\|_{\mathbf{C_b^{1,\alpha}}}, C_K, \sum_{i=B}^{J} \mu_o(\mathbb{R}_+), \mathbf{Lip}(\mu), \mathbf{Lip}(\mu^n)\Big)\left(\Delta t h + h^2\right) = \Delta t\, \mathcal{O}(h) + \mathcal{O}(h^2). \end{aligned}$$

Thus,

$$\sum_{i=B}^{J} m^i(\tau+h)|x^i(\tau+h) - y^i(\tau+h)| = \Delta t\, \mathcal{O}(h) + \mathcal{O}(h^2). \tag{5.37}$$

Concerning the term $\sum_{i=B}^{J} |m^i - n^i|$, at first we assume that μ^n is the output of the simplified version of the EBT scheme. Then,

$$\begin{aligned} \sum_{i=B}^{J} \left|m^i(\tau+h) - p^i(\tau+h)\right| & \le \sum_{i=B}^{J} \int_\tau^{\tau+h} \left|c(t,\mu_t^n)(x^i(t)) m^i(t) - c(t,\mu_t)(y^i(t)) n^i(t)\right| \mathrm{d}t \\ & \quad + \sum_{i=B}^{J} \int_\tau^{\tau+h} \left|\beta(t,\mu_t^n)(x^i(t)) m^i(t) - \beta(t,\mu_t)(y^i(t)) n^i(t)\right| \mathrm{d}t + \mathcal{O}(h^2) \\ & \le \sum_{i=B}^{J} \int_\tau^{\tau+h} \left(\left|c(t,\mu_t^n)(x^i(t))\right| + \left|\beta(t,\mu_t^n)(x^i(t))\right|\right) |m^i(t) - n^i(t)| \mathrm{d}t \\ & \quad + \sum_{i=B}^{J} \int_\tau^{\tau+h} n^i(t) |c(t,\mu_t^n)(x^i(t)) - c(t,\mu_t^n)(y^i(t))|\, \mathrm{d}t \\ & \quad + \sum_{i=B}^{J} \int_\tau^{\tau+h} n^i(t) |c(t,\mu_t^n)(y^i(t)) - c(t,\mu_t)(y^i(t))|\, \mathrm{d}t \\ & \quad + \sum_{i=B}^{J} \int_\tau^{\tau+h} n^i(t) |\beta(t,\mu_t^n)(x^i(t)) - \beta(t,\mu_t^n)(y^i(t))|\, \mathrm{d}t \\ & \quad + \sum_{i=B}^{J} \int_\tau^{\tau+h} n^i(t) |\beta(t,\mu_t^n)(y^i(t)) - \beta(t,\mu_t)(y^i(t))|\, \mathrm{d}t + \mathcal{O}(h^2) \\ & \le \|(c,\beta)\|_{\mathbf{BC}} \sum_{i=B}^{J} \int_\tau^{\tau+h} |m^i(t) - n^i(t)|\, \mathrm{d}t + \|(c,\beta)\|_{\mathbf{BC}} \sum_{i=B}^{J} \int_\tau^{\tau+h} n^i(t) |x^i(t) - y^i(t)|\, \mathrm{d}t \\ & \quad + \|(c,\beta)\|_{\mathbf{C_b^{1,\alpha}}} \sum_{i=B}^{J} \int_\tau^{\tau+h} n^i(t) \rho_F(\mu_t^n, \mu_t)\, \mathrm{d}t + \mathcal{O}(h^2) \\ & \le \|c\|_{\mathbf{BC}} \sum_{i=B}^{J} \int_\tau^{\tau+h} \left(\mathbf{Lip}(m^i) h + \left|m^i(\tau) - n^i(\tau)\right| + \mathbf{Lip}(n^i) h\right) \mathrm{d}t \\ & \quad + \|(c,\beta)\|_{\mathbf{BC}} \sum_{i=B}^{J} \int_\tau^{\tau+h} n^i(t) \left(\mathbf{Lip}(x^i) h + \left|x^i(\tau) - y^i(\tau)\right| + \mathbf{Lip}(y^i) h\right) \mathrm{d}t \\ & \quad + \|(c,\beta)\|_{\mathbf{C_b^{1,\alpha}}} \sum_{i=B}^{J} \int_\tau^{\tau+h} n^i(t) \left(\mathbf{Lip}(\mu^n) h + \rho_F(\mu_\tau^n, \mu_\tau) + \mathbf{Lip}(\mu) h\right)\, \mathrm{d}t + \mathcal{O}(h^2) \end{aligned}$$

$$\leq \|c\|_{\mathbf{BC}} h^2 \sum_{i=B}^{J} \big(\mathbf{Lip}(m^i) + \mathbf{Lip}(n^i)\big) +$$

$$+ \|(c,\beta)\|_{\mathbf{C_b^{1,\alpha}}} h^2 \left(2\|b\|_{\mathbf{BC}} + \mathbf{Lip}(\mu^n) + \mathbf{Lip}(\mu)\right) \sum_{i=B}^{J} n^i(t) + \mathcal{O}(h^2) = \mathcal{O}(h^2). \quad (5.38)$$

For the original EBT scheme we obtain the similar result, since $|\partial_x c(t,\mu)(x_b)\pi^B(t)|$ grows at most linearly on $[0,q^*]$, that is

$$\left|\partial_x c(t,\mu)(x_b)\pi^B(t)\right| \leq \left|\partial_x c(t,\mu)(x_b) C_K m^B(t) x^B(t)\right| \leq \|c\|_{\mathbf{BC}} C_K \left|m^B(t)\right| \Delta t \leq C\Delta t$$

and $|x_b - y^B(t)| \leq \|b\|_{\mathbf{BC}} \Delta t$. Hence,

$$\sum_{i=B}^{J} \left|m^i(\tau+h) - p^i(\tau+h)\right| = \Delta t\, \mathcal{O}(h) + \mathcal{O}(h^2). \quad (5.39)$$

Combining either (5.35), (5.36) and (5.38) or (5.35), (5.37) and (5.39) we obtain

$$\mathrm{liminf}_{h\to 0^+} \frac{1}{h} \rho_F(\mu_t^n, \mu_t^n) \leq \mathrm{liminf}_{h\to 0^+} \frac{1}{h} \left[\Delta t(\mathcal{O}(h) + \mathcal{O}(h^2)) + \mathcal{O}(h^2)\right] = C\Delta t,$$

which, by Proposition 5.27 implies that

$$\rho_F(\mu_t^n, \mu_t) \leq CL\Delta t = C_2\Delta t.$$

The entire argumentation remains valid if τ is the internalization moment. The only difference is that in the latter case $n^B(t) = 0$ on $[\tau, \tau+h]$, which does not influence the final estimate. Note that in the proof we assumed that the initial datum μ_o is given as a sum of Dirac deltas. If this is not the case, then the additional term $C_1\Delta x$ appears in the final estimate (see (5.32)). □

5.4. Derivation of the Original EBT Method

To make our analysis possibly most transparent we focus on the linear case and assume that a solution $u(t,\cdot)$ is a compactly supported and integrable function, which leads to the following problem

$$\begin{aligned} \partial_t u(t,x) + \partial_x(b(t,x)u(t,x)) + c(t,x)u(t,x) &= 0 \\ b(t,x_b)u(t,x_b) &= \int_{x_b}^{+\infty} \beta(t,y)u(t,u)\mathrm{d}y. \end{aligned} \quad (5.40)$$

We also require higher regularity of the model functions b, c and β, that is

$$b, c, \beta : [0,T] \to \mathbf{C^2}([x_b, +\infty)).$$

$\mathbf{C^2}$-regularity is imposed in order to apply the first order Taylor approximation. Let $\{\Omega_i(0)\}_{i=B}^{J}$ be a collection of pairwise disjoint intervals

$$\Omega_i(0) = [l_i(0), l_{i+1}(0))$$

such that

$$\mathrm{supp}(u_o) \subset \bigcup_{i=B}^{J} \Omega_i(0),$$

where u_o is the initial distribution of individuals. These sets determine how the initial population in divided into the cohorts. We recall that the subscript $i = B+1,\ldots,J$ indicates the internal cohorts, while $i = B$ denotes the boundary cohort. Characteristics of the cohort, namely the average value of the structural variable x and the number of individuals within the cohort, change in time. The boundaries between cohorts evolve due to the following ODEs

$$\frac{\mathrm{d}}{\mathrm{dt}} l_i(t) = b(t, l_i(t)) \quad \text{for} \quad i = B,\ldots,J.$$

A lower bound of the boundary cohort is constant in time, that is $l_B(t) = x_b$. Set $\Omega_i(t) := [l_i(t), l_{i+1}(t))$ denotes a range of the i-th cohort at time t in the sense that all individuals characterized by a structural variable $x \in \Omega_i(t)$ are identified with each other. Our aim is to observe how the number of individuals

$$m^i(t) = \int_{\Omega_i(t)} u(t,x)\mathrm{d}x$$

and the average value of the structural variable within the cohort

$$x^i(t) = \frac{1}{m^i(t)} \int_{\Omega_i(t)} x u(t,x)\,\mathrm{d}x \quad \text{for} \ \ i \in \{B,\ldots,J\} \tag{5.41}$$

change in time. Since the boundary cohort is initially empty, we define

$$\pi^B(t) = \int_{\Omega_B(t)} (x - x_b) u(t,x)\mathrm{d}x \quad \text{and} \quad x^B(t) = \begin{cases} x_b + \frac{\pi^B(t)}{m^B(t)}, & \text{for } \ m^B(t) \neq 0 \\ x_b, & \text{otherwise.} \end{cases}$$

Let us differentiate m^i, x^i and π^B and use (5.40). For $i = B+1,\ldots,J$ we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}} m^i(t) &= \int_{\Omega_i(t)} \partial_t u(t,x)\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{dt}} l_{i+1}(t) u(t, l_{i+1}(t)) - \frac{\mathrm{d}}{\mathrm{dt}} l_i(t) u(t, l_i(t)) \\
&= \int_{\Omega_i(t)} \partial_t u(t,x)\mathrm{d}x + b(t, l_{i+1}(t)) u(t, l_{i+1}(t)) - b(t, l_i(t)) u(t, l_i(t)) \\
&= \int_{\Omega_i(t)} \partial_t u(t,x)\mathrm{d}x + \int_{\Omega_i(t)} \partial_x \left(b(t,x)u(t,x)\right) = -\int_{\Omega_i(t)} c(t,x)u(t,x)\mathrm{d}x.
\end{aligned}$$

For the boundary cohort it holds that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}} m^B(t) &= \int_{\Omega_B(t)} \partial_t u(t,x)\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{dt}} l_1(t) u(t, l_1(t)) \\
&= \int_{\Omega_B(t)} \partial_t u(t,x)\mathrm{d}x + b(t, l_1(t)) u(t, l_1(t)) - b(t, x_b) u(t, x_b) + b(t, x_b) u(t, x_b) \\
&= \int_{\Omega_B(t)} \partial_t u(t,x) + \partial_x (b(t,x)u(t,x))\mathrm{d}x + \int_{x_b}^{+\infty} \beta(t,y) u(t,y)\mathrm{d}y \\
&= -\int_{\Omega_B(t)} c(t,x)u(t,x)\mathrm{d}x + \int_{x_b}^{+\infty} \beta(t,u) u(t,y)\mathrm{d}y.
\end{aligned}$$

To find the dynamics of x_i for $i \in \{B+1,\dots,J\}$ we check how the first moment $y(t) = \int_{\Omega_i(t)} xu(t,x)\mathrm{d}x$ evolves.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}y_i(t) &= \int_{\Omega_i(t)} x\,\partial_t u(t,x)\mathrm{d}x + b(t,l_{i+1}(t))u(t,l_{i+1}(t))\,l_{i+1}(t) - b(t,l_i(t))u(t,l_i(t))\,l_i(t)\\
&= \int_{\Omega_i(t)} x\,\partial_t u(t,x)\mathrm{d}x + \int_{\Omega_i(t)} \partial_x\Big(x(b(t,x)u(t,x)\Big)\mathrm{d}x\\
&= \int_{\Omega_i(t)} x\,\partial_t u(t,x)\mathrm{d}x + \int_{\Omega_i(t)} x\partial_x(b(t,x)u(t,x))\mathrm{d}x + \int_{\Omega_i(t)} b(t,x)u(t,x)\mathrm{d}x\\
&= -\int_{\Omega_i(t)} xc(t,x)u(t,x)\mathrm{d}x + \int_{\Omega_i(t)} b(t,x)u(t,x)\mathrm{d}x,
\end{aligned}$$

where we used the equality

$$\int_{\Omega_i(t)} \partial_x\Big(x(b(t,x)u(t,x)\Big)\mathrm{d}x = \int_{\Omega_i(t)} b(t,x)u(t,x)\mathrm{d}x + \int_{\Omega_i(t)} x\partial_x(b(t,x)u(t,x))\mathrm{d}x.$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}x^i(t) &= \frac{\frac{\mathrm{d}}{\mathrm{dt}}y^i(t)}{m^i(t)} - \frac{y^i(t)\frac{\mathrm{d}}{\mathrm{dt}}m^i(t)}{m^i(t)^2} = \frac{\frac{\mathrm{d}}{\mathrm{dt}}y^i(t)}{m^i(t)} - x^i(t)\frac{\mathrm{d}}{\mathrm{dt}}m^i(t)\\
&= -\frac{1}{m^i(t)}\int_{\Omega_i(t)} xc(t,x)u(t,x)\mathrm{d}x + \frac{1}{m^i(t)}\int_{\Omega_i(t)} b(t,x)u(t,x)\mathrm{d}x\\
&\quad + \frac{1}{m^i(t)}\int_{\Omega_i(t)} x^i(t)c(t,x)u(t,x)\mathrm{d}x\\
&= \frac{1}{m^i(t)}\int_{\Omega_i(t)} (x^i(t)-x)c(t,x)u(t,x)\mathrm{d}x + \frac{1}{m^i(t)}\int_{\Omega_i(t)} b(t,x)u(t,x)\mathrm{d}x.
\end{aligned}$$

Finally, for the boundary cohort we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\pi^B(t) &= \frac{\mathrm{d}}{\mathrm{dt}}\left(\int_{\Omega_B(t)} xu(t,x)\mathrm{d}x\right) - x_b\frac{\mathrm{d}}{\mathrm{dt}}m^B(t)\\
&= \int_{\Omega_B(t)} x\,\partial_t u(t,x)\mathrm{d}x + b(t,l_1(t))u(t,l_1(t))\,l_1(t) \pm b(t,x_b)u(t,x_b)\,x_b - x_b\frac{\mathrm{d}}{\mathrm{dt}}m^B(t)\\
&= -\int_{\Omega_B(t)} xc(t,x)u(t,x)\mathrm{d}x + \int_{\Omega_B(t)} b(t,x)u(t,x)\mathrm{d}x + x_b\int_{x_b}^{+\infty}\beta(t,y)u(t,y)\mathrm{d}y\\
&\quad + x_b\int_{\Omega_B(t)} c(t,x)u(t,x) - x_b\int_{x_b}^{+\infty}\beta(t,u)u(t,y)\mathrm{d}y\\
&= \int_{\Omega_B(t)} (x_b - x)c(t,x)u(t,x)\mathrm{d}x + \int_{\Omega_B(t)} b(t,x)u(t,x)\mathrm{d}x.
\end{aligned}$$

Approximation: To obtain a closed form of the scheme we need to approximate b, c and β. Note that due to the definition (5.41)

$$\begin{aligned}
\int_{\Omega_i(t)} (x^i(t)-x)u(t,x)\mathrm{d}x &= x^i(t)\int_{\Omega_i(t)} u(t,x)\mathrm{d}x - \int_{\Omega_i(t)} xu(t,x)\mathrm{d}x \qquad (5.42)\\
&= x^i(t)m^i(t) - x^i(t)m^i(t) = 0.
\end{aligned}$$

Moreover, for $f \in \mathbf{C^2}(\mathbb{R}_+)$ it holds that

$$\int_{\Omega(t)} f(x)u(t,x)\mathrm{d}x = \sum_{i=B}^{J}\int_{\Omega_i(t)} f(x)u(t,x)\mathrm{d}x$$

$$
\begin{aligned}
&= \sum_{i=B}^{J} \int_{\Omega_i(t)} f(x^i(t)) u(t,x) \mathrm{dx} + \sum_{i=B}^{J} \int_{\Omega_i(t)} \frac{\mathrm{d}}{\mathrm{dx}} f(x^i(t))(x - x^i(t)) u(t,x) \mathrm{dx} \\
&\quad + \sum_{i=B}^{J} \int_{\Omega_i(t)} \mathcal{O}(\left|x - x^i(t)\right|^2) u(t,x) \mathrm{dx} \\
&= \sum_{i=B}^{J} f(x^i(t)) m^i(t) + \|u(t,\cdot)\|_{\mathbf{L^1}} \mathcal{O}(t^2).
\end{aligned}
$$

Therefore, the first order approximation of f leads to

$$
\int_{\Omega(t)} f(x) u(t,x) \mathrm{dx} \quad = \quad \sum_{i=B}^{J} f(x^i(t)) m^i(t). \tag{5.43}
$$

Application of (5.43) and neglecting terms of the second (and higher) order yields

$$
\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{dt}} m^i(t) &= -c(t, x^i(t)) m^i(t), \\
\dfrac{\mathrm{d}}{\mathrm{dt}} x^i(t) &= b(t, x^i(t)).
\end{cases}
$$

For the boundary cohort we expand $c(x,t)$ around x_b, which implies that the first order term does not disappear. Thus,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}} m^B(t) &= -\int_{\Omega_B(t)} c(t,x) u(t,x) \mathrm{dx} + \int_{x_b}^{+\infty} \beta(t,y) u(t,y) \mathrm{d}y \\
&= -\int_{\Omega_B(t)} c(t,x_b) u(t,x) \mathrm{dx} - \int_{\Omega_B(t)} \partial_x c(t,x_b)(x - x_b) u(t,x) \mathrm{dx} + \sum_{i=B}^{J} \beta(t, x^i(t)) m^i(t) \\
&= -c(t,x_b) m^B(t) - \partial_x c(t,x_b) \pi^B(t) + \sum_{i=B}^{J} \beta(t, x^i(t)) m^i(t)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}} \pi^B(t) &= \int_{\Omega_B(t)} (x_b - x) c(t,x) u(t,x) \mathrm{dx} + \int_{\Omega_B(t)} b(t,x) u(t,x) \mathrm{dx} \\
&= x_b \int_{\Omega_B(t)} c(t,x_b) u(t,x) \mathrm{dx} + x_b \int_{\Omega_B(t)} \partial_x c(t,x_b)(x - x_b) u(t,x) \mathrm{dx} \\
&\quad - \int_{\Omega_B(t)} x_b c(t,x_b) u(t,x) \mathrm{dx} - \int_{\Omega_B(t)} \Big(c(t,x_b) + x_b \partial_x c(t,x_b) \mathrm{dx} \Big)(x - x_b) + \\
&\quad + \int_{\Omega_B(t)} b(t,x_b) u(t,x) \mathrm{dx} + \int_{\Omega_B(t)} \partial_x b(t,x_b)(x - x_b) u(t,x) \mathrm{dx} \\
&= b(t,x_b) m^i(t) + \partial_x b(t,x_b) \pi^B(t) - c(t,x_b) \pi^B(t).
\end{aligned}
$$

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