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Deterministic and statistical solutions of micropolar fluid equations

PhD dissertation

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Author's declaration:
aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Abstract

We investigate two-dimensional micropolar fluid flows in bounded domains. In a first main part of the thesis we study the nonautonomous system and analyse a long time bahaviour of the solutions in the frame of the theory of pullback attractors. Using a recent method based on a notion of the Kuratowski measure of noncompactness of a bounded set we prove that the pullback attractors in the Sobolev spaces H^1 and H^2 exist. Deriving a new estimate on solutions we show this existence under a certain integrability condition on external forces and moments, that is a weaker assumption than the one considered so far.

In a second main part of the thesis we investigate the two-dimensional micropolar fluid flows in view of the statistical solutions theory. We prove that in the case of an autonomous system, stationary statistical solutions coincide with invariant measures for the considered system of equations. When external forces and moments depend on time, we derive a family of measures $\{\mu_t\}$ that forms a nonstationary statistical solution. We prove that the supports of the measures belong to the time sections of the pullback attractor.

Key Words

nonautonomous micropolar fluid equations, global in time solutions, long time behaviour, pullback attractor, fractal dimension, stationary statistical solutions, nonstationary statistical solutions, Reynolds equations

AMS Mathematics Subject Classification

35Q35, 35B41, 76D03, 76D06, 76F20

Streszczenie

Rozprawa poświęcona jest badaniu dwuwymiarowych przepływów płynu mikropolarnego w ograniczonych obszarach. W pierwszej z głównych części pracy zajmujemy się nieautonomicznym układem równań płynu mikropolarnego. Analizujemy zachowanie jego rozwiązań dla dużych czasów posługując się teorią atraktorów cofniętych. Pokazujemy istnienie wspomnianych atraktorów w przestrzeniach H^1 oraz H^2, wykorzystując rozwinięte w ostatnim czasie metody opierające się na pojęciu miary niezwartości zbiorów ograniczonych Kuratowskiego. Otrzymując nowe oszacowanie na rozwiązania dla rozważanego układu równań, dowodzimy istnienia atraktora przy pewnym warunku całkowym nałożonym na zewnętrzne siły i momenty sił. Założenie to jest słabsze niż te, które rozważane były do tej pory.
Druga część pracy poświęcona jest dwuwymiarowemu przepływowi płynu mikropolarnego z punktu widzenia teorii rozwiązań statystycznych. Dowodzimy, iż w przypadku układu autonomicznego, rozwiązania stacjonarne są miarami niezmienniczymi dla układu. Kiedy natomiast zewnętrzne siły i momenty sił są funkcjami czasu, konstruujemy rodzinę miar $\{\mu_t\}$, która tworzy niestacjonarne rozwiązanie statystyczne. Pokazujemy, że nośniki tych miar zawarte są w cięciach atraktora cofniętego.

Słowa kluczowe

nieautonomiczny układ równań płynu mikropolarnego, globalne w czasie rozwiązania, zachowanie dla dużych czasów, atraktor cofnięty, wymiar fraktalny, stacjonarne rozwiązania statystyczne, niestacjonarne rozwiązania statystyczne, równania Reynoldsa

Klasyfikacja tematyczna według AMS

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Contents

Chapter 1

Introduction

The aim of the thesis is to study a long time behaviour of solutions of micropolar fluid flows in two-dimensional domains. We use the theory of attractors to investigate the dynamical system that is generated by the set of equations.

1.1 Micropolar fluid model

The micropolar fluid model that is considered throughout this work is a generalization of the Navier-Stokes system of equations. It takes into account the microstructure of the fluid by which we mean the geometry and microrotation of particles. Such a model reflects experimental data better than the classical Navier-Stokes when the microchannel flows are considered, eg. [38]. Also in the case when the fluids consist of short rigid cylindrical elements (like polymeric fluids or blood, eg. [34]), the micropolar fluid model is investigated. In [35] this model is also used to describe the granular flows behaviour.

Throughout the work we study the model proposed by Eringen in [18] that is described in three dimensions by the following system of equations [18], [26], [27]

$$\frac{\partial u}{\partial t} - (\nu + \nu_r)\triangle u + (u \cdot \nabla)u + \nabla p = 2\nu_r rot\ \omega + f(t), \tag{1.1}$$

$$div\ u = 0, \tag{1.2}$$

$$\frac{\partial \omega}{\partial t} - \alpha\triangle\omega - \beta\nabla div\ \omega + (u \cdot \nabla)\omega + 4\nu_r\omega = 2\nu_r rot\ u + g(t). \tag{1.3}$$

Here, $u = (u_1, u_2, u_3)$ is the velocity field, p is the pressure and $\omega = (\omega_1, \omega_2, \omega_3)$ is the microrotation field interpreted as the angular velocity field of rotation of particles of the fluid. Moreover, $f = (f_1, f_2, f_3)$, $g = (g_1, g_2, g_3)$ are external forces and moments, respectively. Positive constants ν, ν_r, α and β denote viscosity coefficients. In particular, ν is a Newtonian viscosity coefficient and is equal to the converse of the Reynolds number. The constant ν_r is called the microrotation viscosity.

In our thesis, we consider a micropolar fluid filling a bounded region $\Omega \subset \mathbb{R}^2$.
Such a motion of the fluid can be interpreted as a motion in a cross section $x_3 = const$ of the

three-dimensional domain $\Omega \times (-\infty, +\infty)$ when $u_3 = 0$ and the axes of rotation of particles are parallel to the x_3-axis. Moreover, $f = (f_1, f_2, 0)$, $g = (0, 0, g_3)$ and the pressure $p = p(x_1, x_2)$. Hence, in such a simplified situation we have $u = (u_1, u_2)$, $\omega = \omega_3$, and

$$rot\ u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \qquad div\ u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \qquad rot\ \omega = (\frac{\partial \omega}{\partial x_2}, -\frac{\partial \omega}{\partial x_1}).$$

Let us also notice that $\beta \nabla div\ \omega = 0$, since ω_3 does not depend on x_3. Therefore, we work with the following set of equations

$$\frac{\partial u}{\partial t} - (\nu + \nu_r)\triangle u + (u \cdot \nabla)u + \nabla p = 2\nu_r rot\ \omega + f(t), \tag{1.4}$$

$$div\ u = 0, \tag{1.5}$$

$$\frac{\partial \omega}{\partial t} - \alpha \triangle \omega + (u \cdot \nabla)\omega + 4\nu_r \omega = 2\nu_r rot\ u + g(t). \tag{1.6}$$

We shall consider an arbitrary bounded and sufficiently regular domain $\Omega \subset \mathbb{R}^2$. In general we will be interested in homogenous Dirichlet boundary conditions.

1.2 Deterministic solutions and the attractors theory

Our work is divided into two main parts. Chapters 1-3 consist of the introduction of the problem. The first main part (Chapters 4, 5 and 6) is devoted to study long time behaviour of the solutions of the system of micropolar fluid equations. To this end we use the theory of attractors.
Let us briefly introduce the notion of different types of attractors (global, uniform and pullback ones).

Global attractors. When we deal with the *autonomous* system of equations, the long time behaviour of the solutions can be described using the theory of *global attractors* (e.g. [48], [41], [15] with applications in fluid dynamics e.g. in [8], [27], [20], [42], [37]). We shall recall the definition and the theorem on the existence of the mentioned type of attractors.

Let X be a Banach space. We assume there exists a semigroup of operators $\{S(t)\}_{t\geq 0}$ acting on X. Then we define the global attractor $\widehat{A} \subset X$ for the semigroup as follows

Definition 1.2.1. *A subset $\widehat{A} \subset H$ is called a global attractor for the semigroup $\{S(t)\}_{t\geq 0}$ if it satisfies the following conditions:*
1) $\widehat{A}$ is compact and invariant,

$$S(t)\widehat{A} = \widehat{A}$$

for any $t \in \mathbb{R}_+$.
2) It attracts all bounded subsets $D \subset X$, namely

$$dist(S(t)D, \widehat{A}) \to 0$$

for any bounded set $D \subset X$ when $t \to \infty$.

Before we recall the theorem on existence of a global attractor we will need the following definitions

Definition 1.2.2. *A subset $B \subset X$ is called an absorbing set if for every bounded subset $D \subset X$ there exists a $t_1(D)$ such that $S(t)D \subset B$ for all $t \geq t_1(D)$.*

Definition 1.2.3. *The ω-limit set of B in X is the set*

$$\omega(B) = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} S(s)B}.$$

Definition 1.2.4. *We say that the semigroup of operators $\{S(t)\}_{t\geq 0}$ is uniformly compact for large t if for every bounded set D there exists a $t_0 = t_0(D)$ such that*

$$\overline{\bigcup_{t\geq t_0} S(t)D}$$

is a compact set in X.

Theorem 1.2.1. *(See [48]) Let X be a Banach space and let $\{S(t)\}_{t\geq 0}$ be a semigroup of continuous operators in X.*
Let us assume that there exists a bounded absorbing set B and that the semigroup of operators is uniformly compact for large t.
Then the ω-limit set of B is a global attractor: $\widehat{A} = \omega(B)$.

Moreover, if the mapping $t \to S(t)u_0$ is continuous as the mapping from $\mathbb{R}_+$ to X for every $u_0 \in X$, the global attractor $\widehat{A}$ is connected.

Uniform attractors When the *nonautonomous* system of equations is considered, a notion of a *uniform attractor* can be introduced (e.g. [36]).

Let X and Σ be two Banach spaces. We assume the family of operators $\{U_\sigma(t,\tau)\}_{t\geq\tau}$ acting on X satisfies for any $\sigma \in \Sigma$

$$U_\sigma(\tau,\tau) = Id$$

$$U_\sigma(t,s)U_\sigma(s,\tau) = U_\sigma(t,\tau)$$

for all $t \geq s \geq \tau$.
Moreover, let $\{\mathcal{T}(s)\}_{s\geq 0}$ be a semigroup of continuous operators in Σ and let the following property hold

$$U_{\mathcal{T}(s)\sigma}(t,\tau) = U_\sigma(t+s,\tau+s). \tag{2.7}$$

Definition 1.2.5. *A closed set $\widehat{A}_\Sigma \subset X$ is called a uniform attractor for $\{U_\sigma(t,\tau)\}$ if*
1) $\widehat{A}_\Sigma$ is uniformly attracting, that is, for any bounded set $B \subset X$ and all $\tau \in \mathbb{R}$

$$\lim_{t\to\infty} \sup_{\sigma\in\Sigma} dist(U_\sigma(t,\tau)B, \widehat{A}_\Sigma) = 0.$$

2) $\widehat{A}_\Sigma$ is contained in any other closed and uniformly attracting set.

Definition 1.2.6. *We say that $\{U_\sigma(t,\tau)\}_{t\geq\tau\geq 0,\sigma\in\Sigma}$ is uniformly asymptotically compact if for any $\{u_{0_j}\}_{j\in\mathbb{N}}$ bounded in X, any $\{\sigma_j\}_{j\in\mathbb{N}}\subset\Sigma$ and any $\{t_j\}_{j\in\mathbb{N}}\subset\mathbb{R}_+$ such that $t_j\to\infty$ it follows that*

$$\{U_{\sigma_j}(t_j,0)u_{0_j}\}_{j\in\mathbb{N}}$$

is precompact in X.

In order to show that the uniform attractor for a system exists, one extends the problem to the previously discussed autonomous case. One defines a semigroup on the space $X\times\Sigma$ as

$$S(t)(u,\sigma)=(U_\sigma(t,0)u,\mathcal{T}(t)\sigma) \tag{2.8}$$

for any $t\geq 0$ and $(u,\sigma)\in X\times\Sigma$. Then, a theory of global attractors developed for autonomous dynamical systems is used and the following theorem on existence of a uniform attractor holds.

Theorem 1.2.2. *(See [36]) Let X and Σ be two Banach spaces. We assume Σ is bounded. Let $\{U_\sigma(t,\tau)\}_{t\geq\tau\geq 0,\sigma\in\Sigma}$ be a $(X\times\Sigma,X)$ be a continuous process that possesses a bounded unniformly absorbing set B. Moreover, let $\{U_\sigma(t,\tau)\}_{t\geq\tau\geq 0,\sigma\in\Sigma}$ be uniformly asymtotically compact. We assume $\{\mathcal{T}(s)\}_{s\geq 0}$ is an asymtotically compact semigroup of continuous operators in Σ. Moreover, we assume (2.7) holds.*

Then $\{U_\sigma(t,\tau)\}_{t\geq\tau\geq 0,\sigma\in\Sigma}$ possesses a uniform attractor

$$\widehat{A}_1=\bigcap_{s\geq 0}\overline{\bigcup_{\sigma\in\Sigma}\bigcup_{t\geq s}U_\sigma(t,0)B}.$$

Pullback attractors When considering *nonautonomous* dynamical systems, the theory of *pullback* attractors seems to be a more natural generalization of the theory of global attractors than the concept of uniform attractors.

Let us consider an evolutionary process U (a process U - for short) on a metric space X, i.e., a family $\{U(t,\tau):\ -\infty<\tau\leq t<+\infty\}$ of mappings $U(t,\tau):X\to X$, such that

$$U(\tau,\tau)x=x,$$

and

$$U(t,\tau)=U(t,r)U(r,\tau)\quad\text{for all } \tau\leq r\leq t.$$

Let $\mathcal{D}$ be a nonempty class of parameterized sets $\widehat{D}=\{D(t);\ t\in I\!R\}\subset\mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the family of all nonempty subsets of X.

Definition 1.2.7. *Let X be a metric space. A family $\widehat{A}=\{A(t):\ t\in I\!R\}\subset\mathcal{P}(X)$ is said to be a pullback $\mathcal{D}$-attractor for the process $U(\cdot,\cdot)$ in X if*

1. *$A(t)$ is compact for every $t\in I\!R$,*
2. *$\widehat{A}$ is pullback $\mathcal{D}$-attracting, i.e.,*

$$\lim_{\tau\to-\infty}\operatorname{dist}(U(t,\tau)D(\tau),A(t))=0\quad\textit{for all }\ \widehat{D}\in\mathcal{D}\ \textit{and all}\ \ t\in I\!R,$$

3. *$\widehat{A}$ is invariant, i.e., $U(t,\tau)A(\tau) = A(t)$ for $-\infty < \tau \le t < +\infty$.*

The three conditions in the above definition are a generalization of the ones in the Definition 1.2.1 of a global attractor. Let us notice, the pullback attractor satisfies an invariance property unlike a uniform attractor. Moreover, the concept of a pullback attractor allows to consider *random* dynamical systems when formulated in the language of cocycles. It also allows the nonautonomous term to be quite an arbitrary (not necessarily boundend) in suitable norms, function of time, cf. e.g., [6], [51] (and also [22] for the random case).

Before we formulate a theorem on existence of pullback attractors, we introduce the definitions.

Definition 1.2.8. *Let X be a metric space. It is said that $\widehat{B} \in \mathcal{D}$ is pullback $\mathcal{D}$-absorbing for the process $U(\cdot,\cdot)$ if for any $t \in I\!R$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(t,\widehat{D}) \le t$ such that*

$$U(t,\tau)D(\tau) \subset B(t) \quad \text{for all } \tau \le \tau_0(t,\widehat{D}).$$

Definition 1.2.9. *Let X be a metric space. The process U is said to be pullback $\mathcal{D}-$asymptotically compact if for any $t \in \mathbb{R}$, any family of sets $\widehat{D} \in \mathcal{D}$, any sequence $\tau_n \to -\infty$ and any sequence $x_n \in D(\tau_n)$ where $D(\tau_n) \in \widehat{D}$ the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.*

Theorem 1.2.3. *(See [6]) Let X be a metric space. Let $U(t,\tau)$ be a process in X satisfying the following conditions:*
(i) $U(t,\tau)$ is continuous in X.
(ii) There exists a family $\widehat{B} \in \mathcal{D}$ of pullback $\mathcal{D}-$absorbing sets in X.
(iii) $U(t,\tau)$ is pullback $\mathcal{D}-$asymptotically compact.
Then the family of sets $\widehat{A}$ defined by

$$A(t) = \Lambda(\widehat{B},t) \quad for\ t \in \mathbb{R}$$

satisfies the following

$$A(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{S}} \Lambda(\widehat{D},t)} \quad for\ t \in \mathbb{R}$$

and is a global pullback $\mathcal{D}-$attractor for the process U.

Attractors for micropolar fluid equations The development of the theory of attractors in hydrodynamics started from investigating the dynamical system generated by the Navier-Stokes equations in two-dimensional bounded domains by Ladyzenska in [31]. Next, the efforts were done to look into a broader class of problems, e.g. in unbounded two-dimensional domains. The important article introducing the energy equation method comes from R.Rosa ([42]).

Let us briefly recall the results on long time behaviour of micropolar fluids in the frame of attractors.

The first article devoted to global attractors for a two-dimensional micropolar fluid flow in a bounded domain comes from G.Łukaszewicz ([27]). Let us cite the main theorem from the article

Theorem 1.2.4. *Let Ω be an open bounded set in $\mathbb{R}^2$ with a boundary of class C^2. Let the exterior fields be independent of time, with $(f,g) \in H \times L^2$. Then there exists a unique global attractor $\widehat{A}$ for the semigroup $\{S(t)\}_{t\geq 0}$ in $H \times L_2$ associated with the system of equations of micropolar fluids. The attractor $\widehat{A}$ is bounded in $V \times H_0^1$, compact and connected in $H \times L_2$. It attracts bounded sets in $H \times L_2$.*

As concerning the global attractors theory for the micropolar fluid flows in bounded domains in $\mathbb{R}^2$ with homogeneous Dirichlet boundary conditions, we have the following result on the regularity of the attractor coming from the paper by J.Chen, Z.M.Chen, B.Dong ([8]).

Theorem 1.2.5. *Suppose $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$. Then, for $(f,g) \in H \times L_2$, the micropolar fluid equations admit a global attractor $\widehat{A}$ in the following sense:*
(i) $\widehat{A}$ is compact in $D(A) \times H^2$,
(ii) $\widehat{A}$ is invariant: $S(t)\widehat{A} = \widehat{A}$,
(iii) $\lim_{t\to\infty} \sup_{u_0 \in B} \inf_{\varphi \in \widehat{A}} ||S(t)u_0 - \varphi||_{D(A)\times H^2} = 0$ for any bounded in $H \times L_2$ set B.

Let us also mention an important paper on a global attractor dimension estimate by G.Łukaszewicz and M.Boukrouche ([3]). In the article the boundary-driven two-dimensional micropolar fluid flow with free boundary is considered. The authors prove a new version of Lieb-Thirring inequality with constants depending explicitly on the geometry of a domain.

When considering the micropolar fluid flow in unbounded domains of $\mathbb{R}^2$, we have the article of G. Łukaszewicz, W.Sadowski ([29]). In their paper, authors generalize the energy method of R.Rosa to nonautonomous magneto-micropolar fluid equations in unbounded domains. The existence of the uniform attractor is proved.

Uniform attractors for micropolar fluids were also considered by J.Chen, Z-M.Chen, B-Q.Dong in [13].

At last, pullback attractors for micropolar fluid flows in two dimensions were investigated. In the article [11] the system equipped with non-homogeneous boundary conditions is considered. The author proves existence of the L_2-pullback attractor when external forces $f(t) \in L^2_{loc}(\mathbb{R}, V')$ satisfy the condition

$$\int_{-\infty}^{t} e^{\sigma s} ||f(s)||^2_{V'} \infty$$

for some constant σ, and boundary condition $\varphi \in L^\infty(\partial\Omega)$. The domain Ω is Lipschitz and bounded.

When the forces and moments are translation bounded with respect to L_2 in the sense of the following definition

Definition 1.2.10. *A function $\varphi(s)$ is translation bounded in $L^2_{loc}(\mathbb{R}, H \times L^2)$ if*

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} |\varphi(s)|^2 ds < \infty,$$

we have the result obtained by J.Chen, B-Q.Dong, Z-M.Chen in [12]

Theorem 1.2.6. *If the external forces and moments (f,g) are translation bounded in $L^2_{loc}(\mathbb{R}, H \times L^2)$, then there exists a H^1 pullback attractor for the problem (1.4)-(1.6) with homogeneous Dirichlet boundary conditions.*

1.3 Statistical solutions

The second main part of the thesis (Chapters 7 and 8) is devoted to statistical description of the two-dimensional
micropolar fluid flow.

When we consider a turbulent flow, namely a flow associated with a large Reynolds number, sometimes the statistical study of the motion of the fluid becomes more appropriate then considering individual solutions. In 1894 O.Reynolds began the statistical approach to hydrodynamics by writing the equations describing the evolution of the mean values of the velocity of the fluid ([40]). However, his equations were not formally derived nor mathematically consistent. First rigorous statistical description comes form E.Hopf ([21]) who formulated the equation on evolution of the characteristic functional of certain family of measures. The theorem on existence of solutions of the Hopf equation was proved later by C.Foias ([19]). The fluid considered in the mentioned articles was modelled by the Navier-Sokes equations.

The Reynolds equations for micropolar fluid system were considered in papers by G.Ahmadi, e.g. [1]. Though, the equations for the mean velocity and mean angular velocity were not formulated in a precise mathematical way there.

There are two ways of defining the statistical solutions. The historically first one comes from C.Foias ([19]) and is called a spatial statistical solution. It consists of measures μ_t indicated with the time variable t. Given the statistical distribution of initial data μ_0, each of the measures μ_t is a statistical distribution at the moment $t > 0$.
The other concept of statistical solutions comes from M.Vishik ([50]) and is a measure supported on the set of solutions of the system in such a way that its restriction to the time $t = 0$ coincides with the initial measure μ_0. It is called a spacetime statistical solution.
C.Foias and M.Vishik developed the theory of the statistical solutions for Navier-Stokes equations in two and three-dimensional domains.

1.4 Results of the thesis

In the first part of our work, we investigate a long time behaviour of the deterministic solutions of nonautonomous micropolar fluid system of equations. In Chapter 4 we consider a heat convection problem in a two-dimensional domain when temperature of a lower part of a boundary is assumed to change in time. We prove the existence of the L_2-pullback attractor and estimate its fractal dimension using a new version of Lieb-Thirring inequality ([3]).
This result is published as the article [47].

Next, in Chapter 5 we concentrate on the existence of a pullback attractor for the nonautonomous micropolar fluid equations in a bounded domain, with homogeneous Dirichlet boundary conditions. We consider initial conditions belonging to Sobolev space H^1. A similar problem was already studied in [12] when external forces and moments were assumed to be translation bounded in the sense of the Definition 1.2.10. We show that under weaker assumptions on f and g, namely

$$\int_{-\infty}^{t} e^{\lambda s}\{||f(s)||_{L_2}^2 + ||g(s)||_{L_2}^2\}ds < \infty \quad \text{for every} \quad t \in I\!R,$$

for some constant λ, the pullback attractor in H^1 exists.
The result is published as [30].

Then, in Chapter 6 we continue studying the nonautonomous two-dimensional micropolar fluid problem and assume the initial conditions belong to H^2. Under minimal assumptions on forces and moments we show the pullback attractor in H^2 exists. We prove that the mentioned H^2-attractor is also a unique minimal pullback attractor in the space H^1.
To our knowledge the problem of H^2-pullback attractor for micropolar fluid equations hasn't been studied yet.

In the second part of the thesis we investigate the statistical solutions for the two-dimensional micropolar fluid system in a bounded domain and with homogeneous Dirichlet boundary conditions. We generalise some of the results obtained for the Navier-Stokes equations by C.Foias ([19]), M.Vishik ([50]) and G.Łukaszewicz ([28]).
In Chapter 7 we prove that any measure defined on the phase space is a stationary statistical solution if and only if it is an invariant measure. We also show that its support is contained in a global attractor.
Then, we derive Reynolds equations for the autonomous system of micropolar fluid equations in two dimensions.

Chapter 8 is devoted to the nonautonomous problem. We define a nonstationary solution as a measure on the whole trajectories, in the way proposed by M.Vishik ([50]). Then we derive from it a family of measures and show it satisfies the definition by C.Foias ([19]). We also prove that all the measures have their support on the pullback attractor if only the initial measure does.

Chapter 2

Setting of the problem, the existence theorems

In this chapter we introduce the set of micropolar fluid equations that we shall work with throughout the thesis. We give the definitions of function spaces and recall the results concerning existence of solutions, as well as cite the estimates that we use in the sequel.

2.1 Notations and function spaces

We use the following system of equations proposed by A.C.Eringen and considered by G.Łukaszewicz in case of two-dimensional domains ([18], [26], [27]),

$$\frac{\partial u}{\partial t} - (\nu + \nu_r)\triangle u + (u \cdot \nabla)u + \nabla p = 2\nu_r rot\ \omega + f(t), \tag{2.1}$$

$$div\ u = 0, \tag{2.2}$$

$$\frac{\partial \omega}{\partial t} - \alpha\triangle\omega + (u \cdot \nabla)\omega + 4\nu_r\omega = 2\nu_r rot\ u + g(t), \tag{2.3}$$

where $u = (u_1(x,t), u_2(x,t))$, $\omega = \omega(x,t)$, $f(t) = f(x,t)$, $g(t) = g(x,t)$ for $x \in \Omega \subset \mathbb{R}^2$. Moreover,

$$rot\ u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \qquad div\ u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \qquad rot\ \omega = (\frac{\partial \omega}{\partial x_2}, -\frac{\partial \omega}{\partial x_1}).$$

We study a flow of a micropolar fluid in a bounded domain $\Omega \subset \mathbb{R}^2$ with a C^2 boundary.
We assume that functions u and ω satisfy homogeneous Dirichlet boundary conditions, namely

$$u = 0, \quad \omega = 0 \quad \text{on} \ \ \partial\Omega \times [\tau, \infty) \tag{2.4}$$

for some $\tau \in \mathbb{R}$.
The initial conditions will be denoted by

$$u(x,\tau) = u_0(x), \quad \omega(x,\tau) = \omega_0(x). \tag{2.5}$$

Now, we define precisely function spaces that u, ω, f and g belong to. Chapter 4 is an exception and has its own notations, since it is devoted to a bit different problem of the heat convection in a micropolar fluid.

We denote the usual functional space $L^2(\Omega)$ by L^2 with a scalar product indicated by $(\cdot,\cdot)$.
By H^m for $m = 1, 2$, we mean the Sobolev spaces $H^m(\Omega)$ of functions having weak derivatives up to order m that are square integrable in Ω, with the norm

$$||u||_{H^m} = \left(\sum_{|\alpha|\leq m} \int_\Omega |D^\alpha u(x)|^2 dx \right)^{1/2}.$$

The space H_0^1 denotes the Sobolev space $H_0^1(\Omega)$ that is a closure of the set of smooth functions with a compact support $(C_0^\infty(\Omega))$ in the norm

$$||u||_{H_0^1} = \left(\int_\Omega |\nabla u(x)|^2 dx \right)^{1/2}.$$

We define the space

$$\tilde{V} = \{u \in C_0^\infty(\Omega)^2 : u = (u_1, u_2), div\ u = 0\}.$$

Then, H and V indicate the functional spaces

$$H \ = \text{closure of } \tilde{V} \text{ in } L^2 \times L^2,$$

and

$$V \ = \text{closure of } \tilde{V} \text{in } H_0^1 \times H_0^1,$$

respectively.

By $L^p(0,T;X)$ we mean the space of strongly measurable functions on the interval $(0,T)$ with values in a Banach space X.
The space $C([0,T];X)$ denotes the space of continuous functions on $[0,T]$ with values in a Banach space X.

2.2 Operators and inequalities

First, let us define the Stokes operator denoted by A.
Let Ω be a bounded set with the boundary of class C^2. Let P be an orthogonal projector $P : L^2 \to H$. The operator A is defined as

$$A = -P\triangle. \tag{2.6}$$

The domain of the operator is equal to

$$D(A) = \{u \in H : Au \in H\} = H^2 \cap V. \tag{2.7}$$

It can be shown that the Stokes operator $A : D(A) \to H$ is positive and self-adjoint. Its eigenfunctions form an orthonormal basis of the space H and the sequence of its eigenvalues $\{\lambda_i\}_{i=1}^\infty$ is positive and tends to infinity

$$0 < \lambda_1 \leq \lambda_2 \leq ... \to \infty.$$

Now, let us introduce the notion of trilinear forms $b(u,v,w)$ and $b_1(u,\omega,\psi)$.

$$b(u,v,w)=\int_\Omega[(u\cdot\nabla)v]wdx=\sum_{i,j=1}^{2}\int_\Omega u_i\frac{\partial v_j}{\partial x_i}w_jdx,$$

where $u,\ v,\ w\in V$,

$$b_1(u,\omega,\psi)=\int_\Omega[(u\cdot\nabla)\omega]\psi dx=\sum_{i=1}^{2}\int_\Omega u_i\frac{\partial \omega}{\partial x_i}\psi dx,$$

where $u\in V$ and $\omega,\psi\in H_0^1$.

It is easy to see that the forms have the property

$$\begin{aligned} b(u,v,w) &= -b(u,w,v),\\ b_1(u,\omega,\psi) &= -b_1(u,\psi,\omega), \end{aligned}$$

for functions belonging to appropriate spaces as in the definition of b and b_1. The equalities above imply the following ones

$$\begin{aligned} b(u,v,v) &= 0,\\ b_1(u,\omega,\omega) &= 0. \end{aligned} \tag{2.8}$$

In the sequel, we shall use some estimates b and b_1 that are valid in two-dimensions. We gather them in the lemma below.

Lemma 2.2.1. *([48]) Let b and b_1 be the trilinear forms defined as above. Then*

$$|b(u,v,w)|\le 2||u||_H^{1/2}||u||_V^{1/2}||v||_V||w||_H^{1/2}||w||_V^{1/2},$$

for $u,v,w\in V$,

$$|b(u,v,Aw)|\le\sqrt{2}||u||_{L_4}||u||_V^{1/2}||v||_V^{1/2}||Av||_H^{1/2}||Aw||_H,$$

and

$$|b(u,v,Aw)|\le 2||u||_H^{1/2}||u||_V^{1/2}||v||_V^{1/2}||Av||_H^{1/2}||Aw||_H,$$

for $u\in V,\ v,w\in D(A)$,

$$|b(u,v,w)|\le 2||u||_H^{1/2}||Au||_H^{1/2}||v||_V||w||_H,$$

for $u\in D(A),\ v\in V$ and $w\in H$.

$$|b_1(u,\omega,\psi)|\le 2||u||_H^{1/2}||u||_V^{1/2}||\omega||_{H_0^1}||\psi||_{L_2}^{1/2}||\psi||_{H_0^1}^{1/2},$$

for $u\in V,\ \omega,\psi\in H_0^1$,

$$|b_1(u,\omega,\psi)|\le 2||u||_H^{1/2}||Au||_H^{1/2}||\omega||_{H_0^1}||\psi||_{L_2},$$

for $u\in D(A),\ \omega,\in H_0^1$, and $\psi\in L_2$,

$$|b_1(u,\omega,-\triangle\psi)|\le 2||u||_H^{1/2}||Au||_H^{1/2}||\omega||_{H_0^1}||\triangle\psi||_{L_2},$$

for $u\in D(A),\ \omega,\in H_0^1$, and $\psi\in H^2$.

We associate operators $B : V \times V \to V'$ and $B_1 : V \times H_0^1 \to H^{-1}$ with the trilinear forms b and b_1, respectively.

$$(B(u,v),w) = b(u,v,w),$$
$$(B_1(u,\omega,\psi) = b_1(u,\omega,\psi).$$

2.3 Existence results and estimates

Before we recall the known results on the existence and regularity of the solutions of the equations (2.1)-(2.5), we formulate the definition of the weak solution for the problem.

Definition 2.3.1. *Let $f \in L^2(\tau,T;H)$ and $g \in L^2(\tau,T;L^2)$ for some $\tau \in \mathbb{R}$ and each $T > \tau$. Moreover, $u_0 \in H$ and $\omega_0 \in L^2$.*
A pair of functions (u,ω) is called a weak solution of the problem (2.1)-(2.5), if

$$u \in C([\tau,T];H) \cap L^2(\tau,T;V) \quad \text{for each } T > \tau, \tag{2.9}$$

$$\omega \in C([\tau,T];L^2) \cap L^2(\tau,T;H_0^1) \quad \text{for each } T > \tau, \tag{2.10}$$

where u and ω satisfy the initial conditions $u(\tau) = u_0$, $\omega(\tau) = \omega_0$ and

$$\frac{d}{dt}(u(t),\varphi) + (\nu+\nu_r)(\nabla u(t),\nabla\varphi) + b(u(t),u(t),\varphi) = 2\nu_r(rot\omega(t),\varphi) + (f(t),\varphi), \tag{2.11}$$

for any $\varphi \in V$ and

$$\begin{aligned}\frac{d}{dt}(\omega(t),\psi) + \alpha(\nabla\omega(t),\nabla\psi) + b_1(u(t),\omega(t),\psi) + 4\nu_r(\omega(t),\psi)\\ = 2\nu_r(rot\ u(t),\psi) + (g(t),\psi),\end{aligned} \tag{2.12}$$

for any $\psi \in H_0^1$.

Now, let us formulate the theorem on existence of weak solutions in the sense of the above definition.
We have, c.f., [27], [53]

Theorem 2.3.1. *Let $u_0 \in H$ and $\omega_0 \in L^2$, $f \in L^2(\tau,T,H)$ and $g \in L^2(\tau,T,L^2)$ for some $\tau \in \mathbb{R}$ and all $T > \tau$.*
Then the problem (2.1)-(2.5) possesses a unique global in time weak solution (u,ω) in the sense of the Definition 2.3.1.

Moreover, for all s,T such that $\tau < s < T$ we also have

$$u \in L^2(s,T;H^2) \quad \text{and} \quad \omega \in L^2(s,T;H^2). \tag{2.13}$$

For each $t > \tau$ the mapping $(u_0,\omega_0) \to (u(t),\omega(t))$ is continuous as a mapping in $H \times L^2$.

If we assume that $u_0 \in V$ and $\omega_0 \in H_0^1$, the weak solution (u, ω) additionaly satisfy

$$u \in C([\tau, T); V) \cap L^2(\tau, T; V \cap H^2), \tag{2.14}$$

$$\omega \in C([\tau, T); H_0^1) \cap L^2(\tau, T; H_0^1 \cap H^2) \tag{2.15}$$

for all $T > \tau$.

We also recall the result on the weak solutions of our problem when $u_0 \in D(A)$ and $\omega_0 \in (H^2 \cap H_0^1)$. We have (as a corollary from Lemma 2.1 in [39])

Theorem 2.3.2. *Let $u_0 \in D(A)$, $\omega_0 \in H^2 \cap H_0^1$, and $f \in W^{1,2}(\tau, T, H)$, $g \in W^{1,2}(\tau, T, L^2)$ for some $\tau \in \mathbb{R}$ and any $T > \tau$. Then the weak solution (u, ω) of the problem (2.1)-(2.5) satisfies*

$$Au \in L^\infty(\tau, T; H) \cap L_2(\tau, T, V),$$

$$\triangle\omega \in L^\infty(\tau, T; L_2) \cap L_2(\tau, T; H_0^1),$$

$$u_{tt},\ Au_t \in L^2(\tau, T, V'),$$
$$\omega_{tt},\ \triangle\omega_t \in L^2(\tau, T, H^{-1}).$$

for any $\tau > T$.
Moreover,

$$u \in C^1(\tau, T, H),\ \ \omega \in C^1(\tau, T, L^2) \tag{2.16}$$

for any $T > \tau$.

Now, we shall the energy estimates on the weak solutions of the problem (2.1)-(2.5) derived in [27].

Lemma 2.3.1. *Let us assume $u_0 \in H$ and $\omega_0 \in L^2$, $f \in L^2(\tau, T, H)$ and $g \in L^2(\tau, T, L^2)$ for some $\tau \in \mathbb{R}$ and all $T > 0$.*
Then the following first energy estimate holds

$$\frac{d}{ds}(||u(s)||_H^2 + ||\omega(s)||_{L_2}^2) + 2k_1(||u(s)||_V^2 + ||\omega(s)||_{H_0^1}^2) \le 2((f, u) + (g, \omega)). \tag{2.17}$$

and

$$\frac{d}{ds}(||u(s)||_H^2 + ||\omega(s)||_{L_2}^2) + k_1(||u(s)||_V^2 + ||\omega(s)||_{H_0^1}^2) \le k_3(||f(s)||_H^2 + ||g(s)||_{L_2}^2), \tag{2.18}$$

where $k_1 = min\{\nu, \alpha\}$ and $k_3 = max\{\frac{1}{\nu\lambda_1}, \frac{1}{\alpha\eta_1}\}$ (λ_1 and η_1 are first eigenvalues of Stokes and $-\triangle$ operators, respectively).

Moreover,

$$\begin{aligned} ||u(t)||_{L_2}^2 + ||\omega(t)||_{L_2}^2 &\le e^{-k_2 t}(||u_0||_{L_2}^2 + ||\omega_0||_{L_2}^2) \\ &+ k_3 e^{-k_2 t}\int_\tau^t e^{-k_2(t-s)}(||f(s)||_{L_2}^2 + ||g(s)||_{L_2}^2)ds, \end{aligned} \tag{2.19}$$

where $k_2 = min\{\nu\lambda_1, \alpha\eta_1\}$ *and* $t > \tau$.

The second energy inequality holds

$$\frac{d}{ds}(||u(s)||_V^2 + ||\omega(s)||_{H_0^1}^2) + \nu||Au(s)||_H^2 + \frac{\alpha}{2}||\triangle\omega(s)||_{L_2}^2 \\ \leq H(s)(||u(s)||_V^2 + ||\omega(s)||_{H_0^1}^2 + F(s)), \tag{2.20}$$

where

$$H(s) = C(||u(s)||_H^2||u(s)||_V^2 + ||u(s)||_H^2||\omega(s)||_{L_2}^2 + 1)$$

and

$$F(s) = \frac{4}{\nu}||f(s)||_H^2 + \frac{2}{\alpha}||g(s)||_{L_2}^2,$$

and the constant C *depends on* $|\Omega|$, n, α, ν, ν_r.

Furthermore,

$$\frac{d}{ds}(||u(s)||_V^2 + ||\omega(s)||_{H_0^1}^2) \leq H(s)(||u(s)||_V^2 + ||\omega(s)||_{H_0^1}^2) + F(s). \tag{2.21}$$

Moreover, we shall need the energy-type inequalities on the time derivatives of u and ω.

Lemma 2.3.2. *Let* $u_0 \in H$ *and* $\omega_0 \in L^2$, $f \in L^2(\tau, T, H)$ *and* $g \in L^2(\tau, T, L^2)$ *for some* $\tau \in \mathbb{R}$ *and all* $T > \tau$.
Then the weak solution for the problem (2.1)-(2.5) satisfies

$$\begin{aligned} ||u_t(t)||_H^2 + ||\omega_t(t)||_{L_2}^2 &\leq c(||u(\tau)||_{D(A)}^4 + ||\omega(\tau)||_{H^2\cap H_0^1}^4 + ||f_t(t)||_H^2 + ||g_t(t)||_{L_2}^2 \\ &+ ||f(\tau)||_H^2 + ||g(\tau)||_{L_2}^2 + 1)e^{\int_\tau^t H_1(s)ds}, \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} (\nu+\nu_r)\int_0^t ||u_t||_V^2 &+ \alpha\int_0^t ||\omega_t||_{H_0^1}^2 \\ &\leq ||u_t(\tau)||_H^2 + ||\omega_t(\tau)||_{L_2}^2 + \int_\tau^t H_1(s)(||u_t||_H^2 + ||\omega_t||_{L_2}^2)ds, \end{aligned} \tag{2.23}$$

where

$$H_1(s) = ||u||_V^2 + ||\omega||_{H_0^1}^2 + 1,$$

and c is a constant dependent on the values charactarizing the flow.

Proof. Differentiating (1.4) and (1.6) and multiplying it by u_t and ω_t, respectively we get

$$\frac{d}{dt}||u_t||_H^2 + 2(\nu+\nu_r)||u_t||_V^2 + 2b(u_t,u,u_t) + 2b(u,u_t,u_t) = 4\nu_r(rot\ \omega_t, u_t) + (f_t, u_t), \tag{2.24}$$

and

$$\frac{d}{dt}||\omega_t||_{L_2}^2 + 2\alpha||\omega_t||_{H_0^1}^2 + 2b_1(u_t,\omega,\omega_t) + 2b_1(u,\omega_t,\omega_t) + 8\nu_r||\omega_t||_{L_2}^2 \\ = 4\nu_r(rot\ u_t, \omega_t) + (g_t, \omega_t). \tag{2.25}$$

Estimating trilinear forms and using Young inequality, we arrive at

$$\begin{aligned} \frac{d}{dt}(||u_t||_H^2 + ||\omega_t||_{L_2}^2) &+ (\nu+\nu_r)||u_t||_V^2 + \alpha||\omega_t||_{H_0^1}^2 \\ &\leq c\left(H_1(t)(||u_t||_H^2 + ||\omega_t||_{L_2}^2) + ||f_t||_H^2 + ||g_t||_{H_0^1}^2\right) \end{aligned} \tag{2.26}$$

which implies

$$\frac{d}{dt}(||u_t||_H^2 + ||\omega_t||_{L_2}^2) \leq c\left(H_1(t)(||u_t||_H^2 + ||\omega_t||_{L_2}^2) + ||f_t||_H^2 + ||g_t||_{H_0^1}^2\right), \tag{2.27}$$

where

$$H_1(s) = ||u||_V^2 + ||\omega||_{H_0^1}^2 + 1$$

and c is a constant dependent on the values charactarizing the flow.

Applying Gronwall inequality to (2.27) we obtain

$$||u_t||_H^2 + ||\omega_t||_{L_2}^2 \leq c\left(||u_t(\tau)||_H^2 + ||\omega_t(\tau)||_{L_2}^2 + ||f_t||_H^2 + ||g_t||_{L_2}^2\right) e^{\int_\tau^t H_1(s)ds}. \tag{2.28}$$

Moreover,

$$||u_t(\tau)||_H + ||\omega_t(\tau)||_{L_2} \leq c(||f(\tau)||_H + ||g(\tau)||_{L_2} + ||u(\tau)||_{D(A)}^2 + ||\omega(\tau)||_{H^2\cap H_0^1}^2 + 1). \tag{2.29}$$

These two together lead to (2.22).
In order to obtain (2.23), it suffices to integrate (2.26) over time. □

Chapter 3

Pullback attractors

3.1 Introduction to the notion of the pullback attractor

In Chapters 4, 5 and 6 our aim is to study the long-time behaviour of weak solutions of micropolar fluid equations by using the theory of pullback attractors. This theory is a natural generalization of the theory of global attractors developed to study *autonomous* dynamical systems (c.f., e.g., [48], [41]). It allows to consider a number of different problems of *nonautonomous* dynamical systems and *random* dynamical systems (including some stochastic differential equations) in the same framework of a cocycle formalism (c.f., e.g., [4], [7], [43], [16], [17]). In the case of nonautonomous differential equations theory of pullback attractors has an advantage over the theory of uniform attractors (c.f., e.g., [14], [29], [53]) allowing the nonautonomous term to be quite an arbitrary, neither bounded nor uniformly bounded in suitable norms, function of time, cf., e.g., [6], [51] (and also [22] for the random case).

3.2 Basic definitions and abstract results

In this section we recall some basic notions and then formulate a general result about existence of pullback attractors.

Let us consider an evolutionary process U (a process U - for short) on a metric space X, i.e., a family $\{U(t,\tau):\ -\infty < \tau \leq t < +\infty\}$ of mappings $U(t,\tau): X \to X$, such that

$$U(\tau,\tau)x = x,$$

and

$$U(t,\tau) = U(t,r)U(r,\tau) \quad \text{for all } \tau \leq r \leq t.$$

Let $\mathcal{D}$ be a nonempty class of parameterized sets $\widehat{D} = \{D(t);\ t \in I\!R\} \subset \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the family of all nonempty subsets of X.

First, we recall what we mean by a pullback attractor for the process U.

Definition 3.2.1. *Let X be a metric space. A family $\widehat{A} = \{A(t):\ t \in I\!R\} \subset \mathcal{P}(X)$ is said to be a pullback $\mathcal{D}$-attractor for the process $U(\cdot,\cdot)$ in X if*

1. *$A(t)$ is compact for every $t \in I\!R$,*

2. $\widehat{A}$ *is pullback* $\mathcal{D}$*-attracting, i.e.,*

$$\lim_{\tau\to-\infty} \operatorname{dist}(U(t,\tau)D(\tau), A(t)) = 0 \quad \textit{for all} \ \ \widehat{D} \in \mathcal{D} \ \textit{and all} \ \ t \in I\!R,$$

where $\operatorname{dist}(Y,Z) = \sup_{y\in Y} \inf_{z\in Z} ||y-z||_X$.

3. $\widehat{A}$ *is invariant, i.e.,* $U(t,\tau)A(\tau) = A(t)$ *for* $-\infty < \tau \le t < +\infty$.

Moreover, we call $\widehat{A}$ *minimal* if for every family $\widehat{C} = \{C(t);\ t \in I\!R\} \subset \mathcal{P}(X)$ of closed sets such that $\lim_{\tau\to-\infty} \operatorname{dist}(U(t,\tau)B(\tau), C(t)) = 0$, it is $A(t) \subset C(t)$.

In this section we shall present two theorems on existence of pullback attractors. Before we do this, we need to recall certain definitions. Let us mention some of the definitions hold in an arbitrary metric space and some require X to be a Banach space.

Definition 3.2.2. *Let* X *be a metric space. It is said that* $\widehat{B} \in \mathcal{D}$ *is pullback* $\mathcal{D}$*-absorbing for the process* $U(\cdot,\cdot)$ *if for any* $t \in I\!R$ *and any* $\widehat{D} \in \mathcal{D}$*, there exists a* $\tau_0(t,\widehat{D}) \le t$ *such that*

$$U(t,\tau)D(\tau) \subset B(t) \quad \textit{for all}\ \tau \le \tau_0(t,\widehat{D}).$$

Definition 3.2.3. *Let* X *be a metric space. The process* U *is said to be pullback* $\mathcal{D}-$*asymptoticaly compact if for any* $t \in \mathbb{R}$*, any family of sets* $\widehat{D} \in \mathcal{D}$*, any sequence* $\tau_n \to -\infty$ *and any sequence* $x_n \in D(\tau_n)$ *where* $D(\tau_n) \in \widehat{D}$ *the sequence* $\{U(t,\tau_n)x_n\}$ *is relatively compact in* X*.*

Definition 3.2.4. *Let* X *be a metric space. For every* $\widehat{D} \in \mathcal{D}$ *we can define the omega-limit set of* $\widehat{D}$ *by*

$$\Lambda(\widehat{D},t) = \bigcap_{s\le t} \overline{(\bigcup_{\tau\le s} U(t,\tau)D(\tau))}.$$

Definition 3.2.5. *Let* X *be a metric space. Let* B *be a nonempty bounded set in* X*. The Kuratowski measure of noncompactness of the set* B *[23] is defined by*

$$\alpha(B) = \inf\{\delta :\ B \ \textit{admits a finite cover by sets of diameter} \ \le \delta\}.$$

Definition 3.2.6. *Let* X *be a Banach space. A process* $U(t,\tau)$ *is said to be norm-to-weak continuous on* X *if for all* $t,\tau \in I\!R$ *with* $t \ge \tau$ *and for every sequence* $(x_n) \in X$*,*

$$x_n \to x \quad \textit{strongly in } X \Longrightarrow U(t,\tau)x_n \to U(t,\tau)x \quad \textit{weakly in } X.$$

Definition 3.2.7. *Let* X *be a Banach space. A process* $U(t,\tau)$ *satisfies the pullback* $\mathcal{D}$*-flattening condition if for any* $t \in I\!R$*,* $\widehat{D} \in \mathcal{D}$ *and* $\varepsilon > 0$*, there exists* $\tau_0 = \tau_0(\widehat{D},t,\varepsilon)$ *and a finite dimensional subspace* X_1 *of* X *such that for a bounded projector* $P : X \to X_1$*,*

$$P\left(\bigcup_{\tau\le\tau_0} U(t,\tau)D(\tau)\right) \quad \text{is bounded in} \quad X$$

and

$$(I-P)\left(\bigcup_{\tau\le\tau_0} U(t,\tau)D(\tau))\right) \subset B(0,\varepsilon) \subset X,$$

where $B(0,\varepsilon)$ is a ball in X, centered at 0 and with radius ε.

Now, we state two theorems on existence of pullback attractors. The first one holds in any metric space X, while the second one uses the stronger assumption on the space, namely X is a uniformly convex Banach space.

Theorem 3.2.1. *[6] Let X be a metric space. Let $U(t,\tau)$ be a process in X satisfying the following conditions:*
(i) $U(t,\tau)$ is continuous in X.
(ii) There exists a family $\widehat{B}\in\mathcal{D}$ of pullback $\mathcal{D}-$absorbing sets in X.
(iii) $U(t,\tau)$ is pullback $\mathcal{D}-$asymptotically compact.
Then the family of sets $\widehat{A}$ defined by

$$A(t) = \Lambda(\widehat{B},t) \;\; for\; t\in\mathbb{R}$$

satisfies the following

$$A(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{S}} \Lambda(\widehat{D},t)} \;\; for\; t\in\mathbb{R}$$

and is a global pullback $\mathcal{D}-$attractor for the process U. Moreover, it is minimal.

Theorem 3.2.2. *[25] Let X be a uniformly convex Banach space. Let $U(t,\tau)$ be a process in X satisfying the following conditions:*

(i) $U(t,\tau)$ is norm-to-weak continuous in X.

(ii) There exists a family $\widehat{B}$ of pullback $\mathcal{D}$-absorbing sets in X.

(iii) $U(t,\tau)$ is pullback $\mathcal{D}$-limit-set compact.

Then there exists a minimal pullback $\mathcal{D}$-attractor $\widehat{A}$ in X given by

$$A(t) = \omega(\widehat{B},t) = \bigcap_{s\le t}\overline{\bigcup_{\tau\le s} U(t,\tau)B(\tau)}. \tag{3.1}$$

In the case of a uniformly Banach space, the following result is very useful to check the (i) condition of Theorem 3.2.2.

Theorem 3.2.3. *[43] Let X, Y be two Banach spaces, X^*, Y^* be respectively their dual spaces. Assume that X is dense in Y, the injection $i: X\to Y$ is continuous, its adjoint $i^*: Y^*\to X^*$ is dense, and U is a norm-to-weak continuous process on Y. Then U is a norm-to-weak continuous process on X if and only if for any $\tau\in I\!R$, $\tau\ge t$, $U(\tau,t)$ maps compact sets of X to bounded sets of X.*

The next theorem gives a condition which is equivalent to condition (ii) in Theorem 3.2.2.

Theorem 3.2.4. *[52], [25] Let $U(t,\tau)$ be a process in a uniformly convex Banach space X. Then the following conditions are equivalent:*

(a) $U(t,\tau)$ is pullback $\mathcal{D}$-limit-set compact.

(b) $U(t,\tau)$ satisfies the pullback $\mathcal{D}$-flattening condition.

Theorem 3.2.1 is a general theorem on existence of pullback attractors and holds in any metric space. However, its assumptions may be hard to check. Since, in the following chapters we are interested in the existence of pullback attractors in Hilbert spaces, Theorem 3.2.2 together with the results stated in Theorems 3.2.3 and 3.2.4 will be more useful.
In the case of the uniformly convex Banach space X the strong assumption on the continuity of the process can be weakened to the obviously less demanding ”norm-to-weak” continuity. Moreover, Theorem 3.2.3 gives a technically easier condition to check.
When working in a uniformly convex Banach space, we can also replace the assumption on pullback asymptotical compactness with the pullback flattening condition. Let us mention, the last one requires deriving estimates similar to the ones in the proof of the existence of the pullback absorbing sets.

Chapter 4

Pullback attractor in L_2 for a heat convection problem

In this chapter, we are interested in the behaviour of the fluid layer filling the region between two rigid surfaces. The fluid is heated from below. We assume the temperature on the upper surface is constant and on the lower one it can change in time. We shall observe how these changes influence the behaviour of the fluid. We also show how the geometry of the domain influences the dimension of the attractor. To describe this phenomena we consider equations describing the model of the micropolar fluid and the equation describing heat convection ([27], [18], [20]).

A similar problem was already dealt in [46], where the temperature of the lower surface was constant and the geometry of the domain was different. In the situation we work with throughout this chapter, a non-autonomous system of PDE arises and that is why we apply the theory of the pullback attractors.

In the beginning we recall the equations describing the problem (since we add the heat equation here). Next, the notation and functional setting is introduced. Section 4.3 is devoted to showing the existence and uniqueness of global in time solutions. That allows to define the process associated with the equations. In Section 4.4 we prove that there exists a pullback attractor, the fractal dimension of which is estimated in Section 4.5. In estimating the dimension of the attractor a version of the Lieb-Thirring inequality taken from [3] is used.

4.1 Formulation of the problem

Heat convection in a micropolar fluid in $\mathbb{R}^3$ is described by the following system of equations (following [27], [18], [20]):

$$u_t - (\nu + \nu_r)\triangle u + (u \cdot \nabla)u + \nabla p = 2\nu_r rot\, \omega + e_2(T - T_1)$$

$$div\, u = 0$$

$$\omega_t - \alpha\triangle\omega - \beta\nabla div\, \omega + (u \cdot \nabla)\omega + 4\nu_r\omega = 2\nu_r rot\, u$$

$$T_t + (u \cdot \nabla)T - \kappa\triangle T = 0$$

where $u = (u_1, u_2, u_3)$ is (as previously) the velocity field, p is the pressure and $\omega = (\omega_1, \omega_2, \omega_3)$ is the microrotation field interpreted as the angular velocity field of rotation of particles of the

fluid.
T describes the temperature of the fluid and T_1 is the temperature on the top surface. The constant κ represents thermometric conductivity. A vector e_2 is equal to $(0, 1)$.

We shall consider a micropolar fluid filling the region $\Omega \subset \mathbb{R}^2$ (with an interpretation that was introduced in the Introduction already).
Hence, in such a simplified situation we have $u = (u_1, u_2)$, $\omega = \omega_3$ and

$$u_t - (\nu + \nu_r)\triangle u + (u \cdot \nabla)u + \nabla p = 2\nu_r rot\ \omega + e_2(T - T_1) \tag{4.1}$$

$$div\ u = 0$$

$$\omega_t - \alpha\triangle\omega + (u \cdot \nabla)\omega + 4\nu_r\omega = 2\nu_r rot\ u \tag{4.2}$$

$$T_t + (u \cdot \nabla)T - \kappa\triangle T = 0 \tag{4.3}$$

in the domain $\Omega = \{x = (x_1, x_2) :\ 0 < x_1 < L,\ 0 < x_2 < h(x_1)\}$ where h is a positive, smooth, L-periodic function in x_1.

Let us also denote $\partial\Omega = \Gamma_0 \cup \Gamma_L \cup \Gamma_1$, where Γ_0 is the bottom, Γ_1 is the top and Γ_L- the lateral part of $\partial\Omega$. We consider the problem described by (4.1)-(4.3) with the following boundary conditions on Γ_0 and Γ_1

$$u = 0,\ \ \omega = 0,\ \ T = T_0(t)\ \ \ on\ \ \Gamma_0,$$

$$u \cdot n = 0,\ \ n \cdot \sigma(u, p) \cdot \tau = 0,\ \ \omega = 0,\ \ T = 0\ \ on\ \ \Gamma_1 \tag{4.4}$$

where τ and n are, respectively, tangential and normal components of the unit outward normal vector to the boundary. Tensor described by $\sigma(u, p)$ is the Cauchy stress tensor defined as

$$\sigma_{ij}(u, p) = (\nu + \nu_r)(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) - p\delta_{ij}.$$

The function $T_0(t)$ is a positive function of time.

On Γ_L we have periodic boundary conditions, namely

$$p|_{x_1=0} = p|_{x_1=L},\ \ u|_{x_1=0} = u|_{x_1=L},\ \ \omega|_{x_1=0} = \omega|_{x_1=L},\ \ T|_{x_1=0} = T|_{x_1=L},$$

$$u_{x_1}|_{x_1=0} = u_{x_1}|_{x_1=L},\ \ \omega_{x_1}|_{x_1=0} = \omega_{x_1}|_{x_1=L},\ \ T_{x_1}|_{x_1=0} = T_{x_1}|_{x_1=L}, \tag{4.5}$$

We consider the problem with the initial conditions

$$u(x, \tau) = u_{int}(x), \qquad \omega(x, \tau) = \omega_{int}(x), \qquad T(x, \tau) = T_{int}(x) \tag{4.6}$$

for $x \in \Omega$.

For our purpose it is more convenient to reformulate the above problem by homogenizing the boundary condition for T. Therefore we will introduce a smooth background function $\theta(x_2, t)$ in the way that will be described later. Then we can decompose T as

$$T(x_1, x_2, t) = \theta(x_2, t) + \tilde{T}(x_1, x_2, t)$$

where $\theta(0, t) = T_0(t)$ and $\theta(h(x_1), t) = 0$. As a consequence $\tilde{T}$ has homogeneous boundary conditions on Γ_0 and Γ_1, that is $\tilde{T}(x, t) = 0$ when $x \in \Gamma_0 \cup \Gamma_1$.

Introducing the above procedure our problem is described by the following set of equations

$$u_t - (\nu + \nu_r)\triangle u + (u \cdot \nabla)u + \nabla p = 2\nu_r rot\ \omega + e_2\theta + e_2\tilde{T} \tag{4.7}$$

$$div\ u = 0$$

$$\omega_t - \alpha\triangle\omega + (u \cdot \nabla)\omega + 4\nu_r\omega = 2\nu_r rot\ u \tag{4.8}$$

$$\tilde{T}_t + (u \cdot \nabla)\tilde{T} - \kappa\triangle\tilde{T} + u_2\theta_{x_2} = \kappa\theta_{x_2x_2} - \theta_t \tag{4.9}$$

with the following boundary conditions

$$u = 0, \ \ \omega = 0, \ \ \tilde{T} = 0 \ \ on \ \ \Gamma_0,$$

$$u \cdot n = 0, \ \ n \cdot \sigma(u,p) \cdot \tau = 0, \ \ \omega = 0, \ \ \tilde{T} = 0 \ \ on \ \ \Gamma_1 \tag{4.10}$$

and initial conditions

$$u(x,\tau) = u_{int}(x), \qquad \omega(x,\tau) = \omega_{int}(x),$$

$$\tilde{T}(x,\tau) = T_{int}(x) - \theta(x_2,\tau), \tag{4.11}$$

where $x \in \Omega$.

4.2 Functional setting and weak solutions of the problem

In this section we introduce the functional setting of the equations (4.7)-(4.9) and formulate the definition of weak solutions of the considered problem. We shall use the same letters describing the spaces or constants as in the general case (and which mean clearly something different) but we hope it will not cause any misunderstandings.

Let

$\tilde{V} = \{u \in C^\infty(\Omega)^2 : div\ u = 0\ in\ \Omega;\ u_{|\Gamma_0} = 0,\ u{\cdot}n_{|\Gamma_1} = 0\ u|_{x_1=0} = u|_{x_1=L},\ \ u_{x_1}|_{x_1=0} = u_{x_1}|_{x_1=L}\}$,

$\tilde{V}_2 = \{\omega \in C^\infty(\Omega) : \omega_{|\Gamma_0\cup\Gamma_1} = 0,\ \omega|_{x_1=0} = \omega|_{x_1=L},\ \omega_{x_1}|_{x_1=0} = \omega_{x_1}|_{x_1=L}\}$,

Now we introduce the spaces H, H^0 and V, H^1 as follows

$H =$ closure of $\tilde{V}$ in $L^2(\Omega)^2$, $H^0 =$ closure of $\tilde{H}^1$ in $L^2(\Omega)$,

$V =$ closure of $\tilde{V}$ in $H^1(\Omega)^2$, $H^1 =$ closure of $\tilde{V}_2$ in $H^1(\Omega)$.

The spaces H and H^0 are Hilbert spaces with the scalar products defined as

$$(u,v) = \int_\Omega u(x)v(x)dx, \qquad u,v \in H\ or\ H^0.$$

Also V and H^1 are Hilbert spaces with the following scalar products

$$((u,v)) = \int_\Omega u(x)v(x)dx + \int_\Omega \nabla u(x) \cdot \nabla v(x)dx, \qquad u,v \in V\ or\ H^1.$$

Let us notice that because of the homogeneous on Γ_0 boundary conditions we have the following Poincaré inequalities:

$$||u||_{L_2} \leq \frac{1}{\sqrt{\lambda_1}}||\nabla u||_{L_2} \quad for\ u \in V, \tag{4.12}$$

$$||\omega||_{L_2} \leq \frac{1}{\sqrt{\lambda}}||\nabla \omega||_{L_2} \quad for\ \omega \in H^1, \tag{4.13}$$

$$||\tilde{T}||_{L_2} \leq \frac{1}{\sqrt{\lambda}}||\nabla \tilde{T}||_{L_2} \quad for\ \tilde{T} \in H^1. \tag{4.14}$$

Hence, the expressions $||\nabla u||_{L_2}$, $||\nabla \omega||_{L_2}$ and $||\nabla \tilde{T}||_{L_2}$ are equivalent to the norms generated by the scalar products in V and H^1, respectively.

Let us introduce the following notation: $\mathcal{H} = H \times H^0 \times H^0$ and $\mathcal{V} = V \times H^1 \times H^1$.

Now, we introduce the definition of the weak solution of our problem that we shall work with throughout the rest of the chapter devoted to the heat convection problem for the micropolar fluid equation.

Definition 4.2.1. *Let $u_{int} \in H$, $\omega_{int} \in H^0$ and $(T_{int} - \theta(\tau)) \in H^0$. By a weak solution of the problem (4.7)-(4.11) we mean a triple of functions $(u, \omega, \tilde{T})$ such that:*

$$u \in C([\tau, T]; H) \cap L^2(\tau, T; V), \qquad T > \tau,$$

$$\omega \in C([\tau, T]; H^0) \cap L^2(\tau, T; H^1), \qquad T > \tau,$$

$$\tilde{T} \in C([\tau, T]; H^0) \cap L^2(\tau, T; H^1), \qquad T > \tau,$$

satisfying $u(\tau) = u_{int}$, $\omega(\tau) = \omega_{int}$, $\tilde{T}(\tau) = T_{int} - \theta(\tau)$ and such that

$$\frac{d}{dt}(u(t), \phi) + (\nu + \nu_r)(\nabla u(t), \nabla \phi) + b_1(u(t), u(t), \phi) = 2\nu_r(rot\ \omega(t), \phi) + (e_2\theta(t), \phi) + (e_2\tilde{T}(t), \phi) \tag{4.15}$$

for $\phi \in V$,

$$\frac{d}{dt}(\omega(t), \psi) + \alpha(\nabla \omega(t), \nabla \psi) + b_2(u(t), \omega(t), \psi) + 4\nu_r(\omega(t), \psi) = 2\nu_r(rot\ u(t), \psi) \tag{4.16}$$

for $\psi \in H^1$,

$$\frac{d}{dt}(\tilde{T}(t), \chi) + b_2(u(t), \tilde{T}(t), \chi) + \kappa(\nabla \tilde{T}(t), \nabla \chi) = -(u_2(t)\theta(t)_{x_2}, \chi) - \kappa(\theta(t)_{x_2}, \chi_{x_2}) - (\theta(t)_t, \chi) \tag{4.17}$$

for $\chi \in H^1$.

4.3 Existence of the solutions of the problem

In order to show that the pullback attractor for our problem exists we need to know that the set of equations (4.7)-(4.11) have a unique and global in time weak solution.

We have the following existence theorem.

Theorem 4.3.1. *Let $T_0(t)$ be a bounded locally Lipschitz positive continuous function. Then there exists a unique weak solution of the problem in the sense of Definition 4.1. Furthermore, the mapping $(u_{int}, \omega_{int}, \tilde{T}_{int}) \to (u(t), \omega(t), \tilde{T}(t))$ is continuous in $\mathcal{H}$.*

Proof: First we will look more carefully at the background function $\theta(x_2, t)$.

Lemma 4.3.1. *For all $t \in \mathbb{R}$ there exists a smooth extension $\theta(x_2, t)$ of the boundary condition $T_0(t)$. The extension is such that*

$$| b_2(u_2, \theta, \tilde{T}) | \leq \sqrt{k_1 \kappa} \, ||u||_V \cdot ||\tilde{T}||_{H^1} \tag{4.18}$$

where $k_1 = min\{2\alpha, \frac{1}{2}\nu\}$.

Proof. Let $h_0 = \min_{0 \leq x_1 \leq L} h(x_1)$ and

$$\epsilon(t) = \begin{cases} 2 & if \quad | T_0(t) | \leq \sqrt{k_1 \kappa}/(2h_0), \\ \sqrt{k_1 \kappa}/(h_0 \mid T_0(t) \mid) & if \quad | T_0(t) | \geq \sqrt{k_1 \kappa}/(2h_0). \end{cases}$$

Let us take a smooth nonincreasing function $\rho : [0, \infty) \to [0, 1]$ such that

$$\rho(0) = 1, \quad supp\, \rho \subset [0, 1/2], \quad max \mid \rho'(s) \mid \leq \sqrt{8}.$$

Now, we can introduce our extension

$$\theta(x_2, t) = T_0(t) \rho(x_2 / (h_0 \epsilon(t))).$$

It is easy to see that such a function is L-periodic in the e_1 direction and satisfies the following boundary conditions: $\theta(0, t) = T_0(t)$ and $\theta(h(x_1), t) = 0$.
Next we will show that (4.18) is valid. We have

$$| b_2(u_2, \theta e_1, \tilde{T}) | = | b_2(\frac{u_2}{x_2}, \tilde{T}, x_2 \theta e_1) | \leq ||\frac{u}{x_2}||_{L_2} ||\nabla \tilde{T}||_{L_2} ||x_2 \theta(x_2, t)||_{L^\infty(\Omega)}.$$

Due to the Hardy inequality

$$||\frac{u}{x_2}||_{L_2}^2 \leq 4 \int_0^L \int_0^{h(x_1)} | \frac{\partial u(x_1, x_2)}{\partial x_2} |^2 \, dx_2 dx_1.$$

From the way of constructing our extension we can see that

$$||x_2 \theta(x_2, t)||_{L^\infty(\Omega)} \leq \frac{h_0 \epsilon}{2} \mid T_0(t) \mid \leq \frac{\sqrt{k_1 \kappa}}{2}$$

which finishes the proof of the lemma. □

In order to prove Theorem 4.1 we will show a priori estimates for suitable norms of functions u, ω and $\tilde{T}$.

A priori estimates for u and ω : Let us take the scalar product in H of (4.7) with u. We get

$$\frac{d}{dt}||u||^2_H + 2(\nu+\nu_r)||u||^2_V - 4\nu_r(rot\ \omega, u) - 2(e_2\tilde{T}, u) = 2(e_2\theta, u) \tag{4.19}$$

since $b_1(u,u,u)=0$.

We can estimate the expression $4\nu_r(rot\ \omega, u)$ in the following way

$$4\nu_r(rot\ \omega, u) = 4\nu_r(\omega, rot\ u) \le 4\nu_r||\omega||^2_{L_2} + \nu_r||rot\ u||^2_{L_2} = 4\nu_r||\omega||^2_{L_2} + \nu_r||u||^2_V.$$

Next, using the Young inequality we arrive at

$$\frac{d}{dt}||u||^2_{L_2} + (2\nu+\nu_r)||u||^2_V \le 4\nu_r||\omega||^2_{L_2} + \frac{1}{\nu\lambda_1}||\tilde{T}||^2_{L_2} + \lambda_1\nu||u||^2_{L_2} + \frac{2}{\lambda_1\nu}||\theta||^2_{L_2} + \frac{\lambda_1\nu}{2}||u||^2_{L_2}$$

and in view of the Poincaré inequality we get

$$\frac{d}{dt}||u||^2_{L_2} + (\frac{1}{2}\nu+\nu_r)||u||^2_V \le 4\nu_r||\omega||^2_{L_2} + \frac{1}{\nu\lambda_1}||\tilde{T}||^2_{L_2} + \frac{2}{\lambda_1\nu}||\theta||^2_{L_2}. \tag{4.20}$$

Now, let us consider the scalar product in H^0 of (4.8) and ω.

$$\frac{d}{dt}||\omega||^2_{L_2} + 2\alpha||\omega||^2_{H^1} + 8\nu_r||\omega||^2_{L_2} = 4\nu_r(rot\ u, \omega). \tag{4.21}$$

After estimating certain expressions in a similar way as above we come to

$$\frac{d}{dt}||\omega||^2_{L_2} + 2\alpha||\omega||^2_{H^1} + 4\nu_r||\omega||^2_{L_2} \le \nu_r||u||^2_V. \tag{4.22}$$

Adding (4.20) and (4.22) we arrive at

$$\frac{d}{dt}(||u||^2_{L_2} + ||\omega||^2_{L_2}) + \frac{\nu}{2}||u||^2_V + 2\alpha||\omega||^2_{H^1} \le \frac{1}{\nu\lambda_1}||\tilde{T}||^2_{L_2} + \frac{2}{\nu\lambda_1}||\theta||^2_{L_2}.$$

Denoting $k_1 = min\{2\alpha, \frac{1}{2}\nu\}$ we have

$$\frac{d}{dt}(||u||^2_{L_2} + ||\omega||^2_{L_2}) + k_1(||u||^2_V + ||\omega||^2_{H^1}) \le \frac{1}{\nu\lambda_1}||\tilde{T}||^2_{L_2} + \frac{2}{\nu\lambda_1}||\theta||^2_{L_2}. \tag{4.23}$$

A priori estimates for $\tilde{T}$: Now, let us take the scalar product in H^0 of (4.9) and $\tilde{T}$. It gives us

$$\frac{d}{dt}||\tilde{T}||^2_{L_2} + 2\kappa||\tilde{T}||^2_{H^1} = -2\kappa(\theta_{x_2}, \tilde{T}_{x_2}) - 2(\theta_t, \tilde{T}) - 2b_2(u_2, \theta, \tilde{T}). \tag{4.24}$$

Now, we need to estimate the terms on the right-hand side of the above inequality. We proceed in the following way. First, we have

$$2 \mid \kappa(\theta_{x_2}, \tilde{T}_{x_2}) \mid \le 2\kappa||\theta||_{H^1}\cdot||\tilde{T}||_{H^1} \le 4\kappa||\theta||^2_{H^1} + \frac{1}{4}\kappa||\tilde{T}||^2_{H^1}$$

and

$$2 \mid (\theta_t, \tilde{T}) \mid \leq 2||\theta_t||_{L_2} \cdot ||\tilde{T}||_{L_2} \leq ||\theta_t||_{L_2} \frac{1}{\sqrt{\lambda}} ||\tilde{T}||_{H^1} \leq \frac{4\kappa}{\lambda} ||\theta_t||^2_{L_2} + \frac{1}{4}\kappa ||\tilde{T}||^2_V.$$

To estimate the last term in (4.24) we use Lemma 4.1 which gives us

$$2 \mid b_2(u_2, \theta, \tilde{T}) \mid \leq 2\sqrt{k_1 \kappa}\; ||u||_V \cdot ||\tilde{T}||_{H^1} \leq \frac{3}{4} k_1 ||u||^2_V + \frac{4}{3}\kappa ||\tilde{T}||^2_{H^1}.$$

Therefore we arrive at

$$\frac{d}{dt} ||\tilde{T}||^2_{L_2} + \frac{1}{6}\kappa ||\tilde{T}||^2_{H^1} \leq 4\kappa ||\theta||^2_{H^1} + \frac{4\kappa}{\lambda} ||\theta_t||^2_{L_2} + \frac{3}{4} k_1 ||u||^2_V. \tag{4.25}$$

Adding (4.23), (4.24) we have

$$\frac{d}{dt}(||u||^2_{L_2} + ||\omega||^2_{L_2} + ||\tilde{T}||^2_{L_2}) + k_2(||u||^2_V + ||\omega||^2_{H^1} + ||\tilde{T}||^2_{H^1}) \tag{4.26}$$

$$\leq \frac{1}{\nu\lambda_1} ||\tilde{T}||^2_{L_2} + \frac{2}{\nu\lambda_1} ||\theta||^2_{L_2} + 4\kappa ||\theta||^2_{H^1} + \frac{4\kappa}{\lambda} ||\theta_t||^2_{L_2}$$

where $k_2 = min\{\frac{\kappa}{6}, k_1\}$.
We still need to estimate the H^0 norm of a function $\tilde{T}$. Therefore we will prove the following lemma.

Lemma 4.3.2. *Let u and T satisfy* (4.1)*,*(4.3)*. Let also*

$$0 \leq T(x, \tau) \leq T_0(\tau) \tag{4.27}$$

for a.e. $x \in \Omega$, then

$$0 \leq T(x, t) \leq M$$

for a.e. $x \in \Omega$, a.e. $t \geq \tau$, where $M = \sup_{s \in [\tau, t]} T_0(s)$.
If (4.27) is not assumed, we have:

$$T(\cdot, t) = \widehat{T}(\cdot, t) + \bar{T}(\cdot, t)$$

where $\bar{T}(\cdot, t) \to 0$ when $t \to \infty$ exponentially in the $L^2(\Omega)$ norm and $0 \leq \widehat{T} \leq M$.

Proof. The proof is based on a maximum principle and is very similar to the one in [20]. □

Corollary 4.3.1. *Lemma 4.2 provides an estimate on $||\tilde{T}||_{L_2}$*

$$||\tilde{T}||^2_{L_2} \leq 2||T||^2_{L_2} + 2||\theta||^2_{L_2} \leq 2\mu(\Omega)M + 2||\theta||^2_{L_2} + \delta(t)$$

where $\delta(t)$ decays exponentially in the $L^2(\Omega)$ norm as $t \to \infty$.

The above Corollary and the inequality (4.26) give

$$\frac{d}{dt}(||u||_{L_2}^2 + ||\omega||_{L_2}^2 + ||\tilde{T}||_{L_2}^2) + k_2(||u||_V^2 + ||\omega||_{H^1}^2 + ||\tilde{T}||_{H^1}^2) \tag{4.28}$$

$$\leq \frac{2\mu(\Omega)}{\nu\lambda_1}M + \frac{4}{\nu\lambda_1}||\theta||_{L_2}^2 + 4\kappa||\theta||_{H^1}^2 + \frac{4\kappa}{\lambda}||\theta_t||_{L_2}^2 + \frac{1}{\nu\lambda_1}\delta(t).$$

Now, we need the following Lemma to estimate the terms on the right-hand side of the above inequality.

Lemma 4.3.3. *Let $\theta(x_2, t)$ be the function defined as in the proof of Lemma 4.1. Then we have*

$$||\theta(x_2, t)||_{L_2}^2 \leq Lh_0 T_0^2(t), \tag{4.29}$$

$$||\theta_{x_2}(x_2, t)||_{L_2}^2 \leq 4L\frac{T_0^2(t)}{h_0\epsilon(t)} \tag{4.30}$$

$$||\theta_t(x_2, t)||_{L_2}^2 \leq Lh_0\epsilon(t)(\mid T_0'(t) \mid^2 + 16(\frac{\epsilon'(t)}{\epsilon(t)})^2 T_0^2(t)) \tag{4.31}$$

for almost all $t \in \mathbb{R}$.

Proof. From the definition of θ we see that $\mid \theta(x_2, t) \mid \leq \mid T_0(t) \mid$ and *supp* $\theta(x_2, t)e_1 \subset [0, L] \times [0, h_0]$. Therefore we have (4.29). Let us also notice that

$$\int_\Omega \mid \theta_{x_2}(x_2, t) \mid^2 dx = \mid \frac{T_0(t)}{h_0\epsilon(t)} \mid^2 \int_\Omega \mid \rho'(\frac{x_2}{h_0\epsilon(t)}) \mid^2 dx$$

$$= \mid \frac{T_0(t)}{h_0\epsilon(t)} \mid^2 L \int_0^{\frac{h_0\epsilon(t)}{2}} \mid \rho'(\frac{x_2}{h_0\epsilon(t)}) \mid^2 dx_2 \leq 4L\frac{T_0^2(t)}{h_0\epsilon(t)}$$

which gives (4.30). Finaly

$$\mid \theta_t \mid \leq \mid T_0'(t) \mid \rho(\frac{x_2}{h_0\epsilon(t)}) + \mid T_0(t) \mid \frac{x_2 \mid \epsilon'(t) \mid}{h_0\epsilon^2(t)} \mid \rho'(\frac{x_2}{h_0\epsilon(t)}) \mid .$$

Hence we have

$$\int_\Omega \mid \theta_t(x_2, t) \mid^2 dx \leq 2L \int_0^{\frac{h_0\epsilon(t)}{2}} [\mid T_0'(t) \mid^2 + 16(\frac{\epsilon'(t)}{\epsilon(t)})^2 \mid T_0(t) \mid^2] dx_2$$

from which we arrive at (4.31). □

Now, in view of (4.28) and the previous Lemma we get

$$\frac{d}{dt}(||u||_{L_2}^2 + ||\omega||_{L_2}^2 + ||\tilde{T}||_{L_2}^2) + k_2(||u||_V^2 + ||\omega||_{H^1}^2 + ||\tilde{T}||_{H^1}^2) \leq f(t) \tag{4.32}$$

for almost all t, where

$$f(t) = C(k_1, \kappa, \Omega)(1 + M + T_0^3(t) + \mid T_0'(t) \mid^2).$$

Moreover, in view of the Poincaré inequalities we have

$$\frac{d}{dt}(||u||^2_{L_2} + ||\omega||^2_{L_2} + ||\tilde{T}||^2_{L_2}) + 2\sigma(||u||^2_{L_2} + ||\omega||^2_{L_2} + ||\tilde{T}||^2_{L_2}) \leq f(t) \tag{4.33}$$

where $2\sigma = min\{k_2\lambda_1, k_2\lambda\}$. Therefore, in particular we have

$$(||u(t)||^2_{L_2} + ||\omega(t)||^2_{L_2} + ||\tilde{T}(t)||^2_{L_2}) \leq e^{-\sigma(t-\tau)}(||u(\tau)||^2_{L_2} + ||\omega(\tau)||^2_{L_2} + ||\tilde{T}(\tau)||^2_{L_2}) + \int_\tau^t e^{\sigma(s-t)} f(s) ds. \tag{4.34}$$

The remaining part of the proof of existence is based mainly on the energy inequality (4.32) and uses standard Galerkin approximations and compactness method.

Uniqueness of the solutions To show that the solutions are unique and depend continuously on the initial conditions we assume $(u_1, \omega_1, \tilde{T}_1)$ and $(u_2, \omega_2, \tilde{T}_2)$ are two solutions of our problem. Let us denote $(\bar{u}, \bar{\omega}, \bar{\tilde{T}}) = (u_1 - u_2, \omega_1 - \omega_2, \tilde{T}_1 - \tilde{T}_2)$. Writing the weak form of our problem for $(\bar{u}, \bar{\omega}, \bar{\tilde{T}})$ and taking $(\phi, \psi, \chi) = (\bar{u}, \bar{\omega}, \bar{\tilde{T}})$ we arrive at

$$\frac{d}{dt}||\bar{u}(t)||^2_{L_2} + 2(\nu + \nu_r)||\bar{u}(t)||^2_V + 2b_1(\bar{u}(t), u_2(t), \bar{u}(t)) = 4\nu_r(rot\ \bar{\omega}(t), \bar{u}(t)) + 2(e_2\bar{\tilde{T}}(t), \bar{\tilde{T}}(t))$$

$$\frac{d}{dt}||\bar{\omega}(t)||^2_{L_2} + 2\alpha||\bar{\omega}(t)||^2_{H^1} + 2b_2(\bar{u}(t), \omega_2, \bar{\omega}(t)) + 8\nu_r||\bar{\omega}(t)||^2_{L_2} = 4\nu_r(rot\ \bar{u}(t), \bar{\omega}(t))$$

$$\frac{d}{dt}||\bar{\tilde{T}}(t)||^2_{L_2} + 2b_2(\bar{u}(t), \tilde{T}_2(t), \bar{\tilde{T}}(t)) + 2\kappa||\bar{\tilde{T}}(t)||^2_{L_2} = -2(\bar{u}(t)e_2\theta_{,x_2}, \bar{\tilde{T}}(t))$$

We shall use the Ladyzenska inequality ([49])

$$||v||_{L^4} \leq C_1(\Omega)||v||^{1/2}_{L_2}||v||^{1/2}_{H^1}$$

which is valid for all $v \in H^1$ to get the following

$$|\ b_1(u, v, w)\ | \leq C(\Omega)||u||^{1/2}_{L_2}||u||^{1/2}_V||v||_V\ ||w||^{1/2}_{L_2}||w||^{1/2}_V,$$

where $u,\ v,\ w \in V$, and

$$|\ b_2(u, v, w)\ | \leq C(\Omega)||u||^{1/2}_{L_2}||u||^{1/2}_V||v||_{H^1}\ ||w||^{1/2}_{L_2}||w||^{1/2}_{H^1},$$

for $u \in V,\ v,\ w \in H^1$.

Making estimates very similar to the ones already done we arrive at

$$\frac{d}{dt}(||\bar{u}(t)||^2_{L_2} + ||\bar{\omega}(t)||^2_{L_2} + ||\bar{\tilde{T}}(t)||^2_{L_2}) + \frac{\sigma}{4}(||\bar{u}(t)||^2_{L_2} + ||\bar{\omega}(t)||^2_{L_2} + ||\bar{\tilde{T}}(t)||^2_{L_2})$$

$$\leq C(\Omega, k_2)(||u_2(t)||^2_V + ||\omega_2(t)||^2_{H^1} + ||\tilde{T}_2(t)||^2_{H^1})(||\bar{u}(t)||^2_{L_2} + ||\bar{\omega}(t)||^2_{L_2} + ||\bar{\tilde{T}}(t)||^2_{L_2}).$$

Since (4.32) assures us about the local integrability of $(||u_2(t)||_V^2 + ||\omega_2(t)||_{H^1}^2 + ||\tilde{T}_2(t)||_{H^1}^2)$, we use the Gronwall lemma to get

$$(||\bar{u}(t)||_{L_2}^2 + ||\bar{\omega}(t)||_{L_2}^2 + ||\bar{\tilde{T}}(t)||_{L_2}^2) \leq (||\bar{u}(\tau)||_{L_2}^2 + ||\bar{\omega}(\tau)||_{L_2}^2 + ||\bar{\tilde{T}}(\tau)||_{L_2}^2) \\ \cdot \exp\{-\int_\tau^t (\frac{k_2}{4} - C(\Omega, k_2)(||u_2(t)||_V^2 + ||\omega_2(t)||_{H^1}^2 + ||\tilde{T}_2(t)||_{H^1}^2)ds\} \tag{4.35}$$

Let us notice that the above inequality gives us both the uniqueness of the solutions and the continuous dependence on the initial data.

4.4 Existence of a pullback attractor for the problem

In view of the previous section we can define the process associated with our problem (4.7)-(4.11) as:

$$U(t,\tau)v_0 = v(t;\tau,v_0) \;\; for \; \tau \leq t \tag{4.36}$$

where $v_0 = (u_0, \omega_0, \tilde{T}_0) \in \mathcal{H}$ and $v(t;\tau,v_0) = (u(t), \omega(t), \tilde{T}(t))$ is the solution of the problem in the sense of Definition 4.1.
The property $U(t,\tau)v_0 = U(t,s)U(s,\tau)v_0$ for $\tau \leq s \leq t$ and $v_0 \in \mathcal{H}$ follows from the uniqueness of the weak solution.
Later, we will need the following property of the process U.

Proposition 4.4.1. *Let $\{v_{0_n}\} \rightharpoonup v_0$ weakly in $\mathcal{H}$. Then*

$$U(t,\tau)v_{0_n} \rightharpoonup U(t,\tau)v_0 \;\; in \; \mathcal{H}, \quad for \; all \; t \geq \tau.$$

Proof. The proof is very similar to the one in [29], so we omit it. □

Since in this section we want to prove the existence of a pullback attractor for our problem in the space $\mathcal{H}$, we will use the Theorem 3.2.1. Before we do this, we need to define what family of parametrized sets $\widehat{D}$ we shall use.

Let us consider the set $\mathcal{R}_\sigma$ of functions $r : \mathbb{R} \to (0, \infty)$ satisfying

$$\lim_{t \to -\infty} e^{\sigma t} r^2(t) = 0.$$

Now, let $\widehat{D}$ be the family of parametrized sets

$$\widehat{D} = \{D(t); t \in \mathbb{R}\} \subset \mathcal{P}(\mathcal{H})$$

that satisfy

$$D(t) \subset \widehat{B}(0, r_{\widehat{D}}(t))$$

for some $r_{\widehat{D}} \in \mathcal{R}_\sigma$.
Let $\mathcal{D}_\sigma$ denote the class of all such families.

Then we have the following theorem on existence of a pullback attractor for the process (4.36).

Theorem 4.4.1. *Le $T_0(t)$ be the bounded, locally Lipschitz continuous function satisfying*

$$\int_{-\infty}^{t} e^{\sigma s} \mid T_0'(s) \mid^2 ds < +\infty \tag{4.37}$$

for all $t \in \mathbb{R}$.
Then there exists a unique pullback $\mathcal{D}_\sigma-$attractor for the process U defined by (4.36).

Proof. Let us notice that in view of the Theorem 4.3.1 the mapping $U(t,\tau) : \mathcal{H} \to \mathcal{H}$ is continuous, which is a first assumption for the process in Theorem 3.2.1.
Let us fix the family of sets $\widehat{D} \in \mathcal{D}_\sigma$.

In view of the energy estimate (4.34), for all $v_0 \in D(\tau)$ and all $t \geq \tau$ the following inequality holds

$$||U(t,\tau)v_0||^2_{\mathcal{H}} \leq e^{-\sigma(t-\tau)} r^2_{\bar{D}}(\tau) + e^{-\sigma t} \int_{-\infty}^{t} e^{\sigma s} f(s)^2 ds \tag{4.38}$$

which right-hand side is finite due to (4.37).

If we denote

$$R^2_\sigma(t) = 2e^{-\sigma t} \int_{-\infty}^{t} e^{\sigma s} f(s)^2 ds$$

we obtain a family of balls

$$B_\sigma(t) = \{v \in \mathcal{H} : \; ||v||_{\mathcal{H}} \leq R_\sigma(t)\}$$

where $B_\sigma \subset D_\sigma$ and B_σ is pullback D_σ-absorbing for the process $U(t,\tau)$.

In order to prove the existence of the pullback $\mathcal{D}_\sigma$-attractor for the process U, we need to show that the process U is pullback $\mathcal{D}_\sigma$-asymptotically compact.

Let a family of sets $\widehat{D} \in \mathcal{D}_\sigma$, $t \in \mathbb{R}$ and sequences $\tau_n \to -\infty$ and $v_{0_n} \in D(\tau_n)$ be fixed. We will show that having a sequence $\{U(t,\tau_n)v_{0_n}\}$ we are able to choose a subsequence $\{U(t,\tau_{n_l})v_{0_{n_l}}\}$ which converges in $\mathcal{H}$.

We have already shown that the family $\widehat{B}_\sigma$ is pullback $\mathcal{D}_\sigma$-absorbing which means that for each integer $k \geq 0$ there exists a $\tau_{\bar{D}}(k) \leq t-k$ such that $U(t-k,\tau)D(\tau) \subset B_\sigma(t-k)$ for every $\tau \leq \tau_{\bar{D}}(k)$.
Then, by the diagonal procedure we can extract a subsequence $\{(\tau_{n_l}, v_{0_{n_l}})\} \subset \{(\tau_n, v_{0_n})\}$ and a sequence $\{z_k : k \geq 0\} \subset \mathcal{H}$ such that $z_k \in B_\sigma(t-k)$ for all $k \geq 0$ and

$$U(t-k,\tau_{n_l})v_{0_{n_l}} \rightharpoonup z_k \quad \textit{in } \mathcal{H}. \tag{4.39}$$

Taking into account Proposition 4.4.1 we conclude that

$$z_0 = weak \lim_{n_l\to\infty} U(t,\tau_{n_l})v_{0_{n_l}} = weak \lim_{n_l\to\infty} U(t,t-k)U(t-k,\tau_{n_l})v_{0_{n_l}}$$

$$= U(t,t-k) \; weak \lim_{n_l\to\infty} U(t-k,\tau_{n_l})v_{0_{n_l}}$$

where the *weak* lim is a weak limit. We arrive at

$$U(t,t-k)z_k = z_0 \quad \textit{for all } k \geq 0. \tag{4.40}$$

Then we have

$$||z_0||_{\mathcal{H}} \le \liminf_{n_l\to\infty} ||U(t,\tau_{n_l})v_{0_{n_l}}||_{\mathcal{H}} \tag{4.41}$$

because of the lower semi-continuity of the norm.

To prove that $U(t,\tau_{n_l})v_{0_{n_l}}$ converges strongly to z_0 in $\mathcal{H}$, we have to show that

$$||z_0||_{\mathcal{H}} = \lim_{n_l\to\infty} ||U(t,\tau_{n_l})v_{0_{n_l}}||_{\mathcal{H}} \tag{4.42}$$

remembering about the weak convergence that we already have. Taking into account (4.41) we only need to prove the following

$$||z_0||_{\mathcal{H}} \ge \limsup_{n_l\to\infty} ||U(t,\tau_{n_l})v_{0_{n_l}}||_{\mathcal{H}}. \tag{4.43}$$

We shall use the energy equation method introduced in the context of the Navier-Stokes equations by R.Rosa in [42] and then generalized in [36] and [5] to nonautonomous systems.
Let us notice that adding (4.19), (4.21) and (4.24) we can write the result in the following way

$$\begin{aligned}\frac{d}{dt}(||u(t)||^2_{L_2} + ||\omega(t)||^2_{L_2} + ||\tilde{T}(t)||^2_{L_2}) + \sigma(||u(t)||^2_{L_2} + ||\omega(t)||^2_{L_2} + ||\tilde{T}(t)||^2_{L_2})\\ = (F, v(s)) - D(v(s), v(s))\end{aligned}$$

where $v = (u, \omega, \tilde{T})$ and

$$\begin{aligned}D(v,v) = 2(\nu+\nu_r)||u||^2_V + 2\alpha||\omega||^2_{H^1} + 2\kappa||\tilde{T}||^2_{H^1} - \sigma(||u||^2_{L_2} + ||\omega||^2_{L_2} + ||\tilde{T}||^2_{L_2})\\ +8\nu_r||\omega||^2_{L_2} - 4\nu_r(rot\omega, u) - 4\nu_r(rotu, \omega) - 2(e_2\tilde{T}, u) - 2(u_2\phi_{|x_2}, \tilde{T}),\end{aligned} \tag{4.44}$$

and

$$(F, v) = 2(e_2\theta, u) - 2\kappa(\theta_{|x_2}, \tilde{T}_{|x_2}) - 2(\theta_{|t}, \tilde{T}). \tag{4.45}$$

Therefore we have

$$\begin{aligned}||u(t)||^2_{L_2} + ||\omega(t)||^2_{L_2} + ||\tilde{T}(t)||^2_{L_2} = e^{\sigma(\tau-t)}(||u(\tau)||^2_{L_2} + ||\omega(\tau)||^2_{L_2} + ||\tilde{T}(\tau)||^2_{L_2})\\ + \int_\tau^t e^{\sigma(s-t)}((F, v(s)) - D(v(s), v(s))).\end{aligned} \tag{4.46}$$

Let us notice that because of the semigroup property of the considered process we have

$$\begin{aligned}&||U(t,\tau_{n_l})v_{0_{n_l}}||^2_{\mathcal{H}} = ||U(t,t-k)U(t-k,\tau_{n_l})v_{0_{n_l}}||^2_{\mathcal{H}}\\ &= e^{\sigma(\tau-t)}||U(t-k,\tau_{n_l})v_{0_{n_l}}||^2_{\mathcal{H}} + \int_{t-k}^t e^{\sigma(s-t)}(F(s), U(s,t-k)U(t-k,\tau_{n_l})v_{0_{n_l}})ds\\ &\quad - \int_{t-k}^t e^{\sigma(s-t)} D(U(s,t-k)U(t-k,\tau_{n_l})v_{0_{n_l}}, U(s,t-k)U(t-k,\tau_{n_l})v_{0_{n_l}})ds.\end{aligned}$$

Now, we need to estimate the upper limit of each of the three expressions on the right.

The first term can be estimated using (4.38). We have

$$e^{\sigma(\tau-t)}||U(t-k,\tau_{n_l})v_{0_{n_l}}||^2_{\mathcal{H}} \leq e^{-\sigma k} R^2_\sigma(t-k).$$

Now, if we remember that $F(s) \in L^2_{loc}(\mathbb{R}, \mathcal{V}')$ we can see that in particular

$$e^{\sigma(s-t)}F(s) \in L^2_{loc}(t-k,t;\mathcal{V}').$$

Therefore taking into account (4.39) we get

$$\lim_{n_l\to\infty}\int_{t-k}^t e^{\sigma(s-t)}(F(s),U(s,t-k)U(t-k,\tau_{n_l})v_{0_{n_l}})ds$$
$$= \int_{t-k}^t e^{\sigma(s-t)}(F(s),U(s,t-k)z_k)ds.$$

In order to estimate the upper limit of the remaining expression we will look at the second integral on the right as at the functional

$$I[v(s)] = \int_{t-k}^t e^{\sigma(s-t)}D(v(s),v(s))ds.$$

Looking at (4.44) we can see the functional I is convex. Moreover, it is bounded from below which is easy to see if we reconstruct the estimates that led to (4.33). More precisely, we have

$$D(v,v) \geq \sigma(||u||^2_V + ||\omega||^2_{H^1} + ||\tilde{T}||^2_{H^1}) - \frac{1}{\nu\lambda_1}||\tilde{T}||^2_{L_2}.$$

Therefore the functional is weakly lower semicontinuous on the space $L^2(t-k,t;\mathcal{V})$.

Summing up, we come to

$$\limsup_{n_l\to\infty}||U(t,\tau_{n_l})v_{0_{n_l}}||^2_{\mathcal{H}} \leq e^{-\sigma k}R^2_\sigma(t-k)$$
$$+\int_{t-k}^t e^{\sigma(s-t)}(F(s),U(s,t-k)z_k)ds - \int_{t-k}^t e^{\sigma(s-t)}D(U(s,t-k)z_k,U(s,t-k)z_k)ds.$$

On the other hand, in view of (4.42) and (4.46) we have

$$||z_0||^2_{\mathcal{H}} = ||V(t,t-k)z_k||^2_{\mathcal{H}} = ||z_k||^2_{\mathcal{H}}e^{-\sigma k}$$
$$+\int_{t-k}^t e^{\sigma(s-t)}(F(s),V(s,t-k)z_k)ds - \int_{t-k}^t e^{\sigma(s-t)}D(U(s,t-k)z_k,U(s,t-k)z_k)ds.$$

Therefore we see that

$$\limsup_{n_l\to\infty}||U(t,\tau_{n_l})v_{0_{n_l}}||^2_{\mathcal{H}} \leq e^{-\sigma k}R^2_\sigma(t-k) + ||z_0||^2_{\mathcal{H}} - ||z_k||^2_{\mathcal{H}}e^{-\sigma k}$$
$$\leq e^{-\sigma k}R^2_\sigma(t-k) + ||z_0||^2_{\mathcal{H}}.$$

The last inequality is valid for all $k \geq 0$. Hence, letting k go to infinity and taking into account that

$$e^{-\sigma k} R_\sigma^2(t-k) = 2e^{-\sigma t} \int_{-\infty}^{t-k} e^{\sigma s} f^2(s) ds \to 0$$

as $k \to +\infty$, we arrive at (4.43) which finishes the proof on existence of the pullback attractor for our problem.

□

4.5 Fractal dimension of the pullback attractor

In order to estimate the fractal dimension of the pullback attractor for our problem we use the method of Lyapunov exponents introduced in [9] and generalized for non-autonomous PDE's in [24] (Theorem 2.4).

First, we recall the definition of the fractal dimension for a compact subset of a separable real Hilbert space X.

Definition 4.5.1. *Let X be a separable real Hilbert space. Let $K \subset X$ be compact. Then the fractal dimension of K is equal*

$$d_F(K) = - \lim_{\epsilon \to 0} \frac{logN(K, \epsilon)}{log\epsilon},$$

where by $N(K, \epsilon)$ we denote the minimal number of open balls in X with radium ϵ that are necessary to cover K.

Now, we state the main theorem of this chapter.
Let a family of sets $\widehat{A} = \{A(t) : t \in \mathbb{R}\}$ denote the pullback attractor for the considered problem. Then we have the following

Theorem 4.5.1. *Let there exist $T^* \in \mathbb{R}$ such that*

$$|| \, T_0'(t) \, ||_{L_\infty(-\infty, T^*)} < \infty.$$

Then for every $t \in \mathbb{R}$ the fractal dimension of the set $A(t)$ is bounded.

Proof. First, we sketch the idea of the proof.

The method of Lyapunov exponents bases on the fact, that we investigate the evolution of m-dimensional volumes. We look for the minimal number m_0 such that the m_0-dimensional volume decays to zero with time.
Hence, we consider a linearized form of our system of equations and introduce the expressions q_m that depend on the trace of the linearized operator (the precise definition of q_m will be formulated within the proof).
We shall estimate from above the value of each q_m by the expression depending on m in order to be able to choose the smallest m_0 for which q_m is not positive.

To start the proof, in view of the theory we use, we have to look at the linearized form of (4.7)-(4.9).

Let us denote $v = (w, z, \psi)$ and introduce the following notation:

$\cdot$ a is a bilinear form defined on $\mathcal{V} \times \mathcal{V}$ as
$a(v_1, v_2) = (\nu + \nu_r)(\nabla w_1, \nabla w_2) + \alpha(\nabla z_1, \nabla z_2) + \kappa(\nabla \psi_1, \nabla \psi_2)$,

$\cdot$ R is bilinear, defined on $\mathcal{V} \times \mathcal{V}$ and
$R(v_1, v_2) = -2\nu_r(rot\ z_1, w_2) - 2\nu_r(rot\ w_1, z_2) + 4\nu_r(z_1, z_2) - (e_2\psi_1, w_2) - (e_2 w_1, \psi_2)$,

$\cdot$ B is a trilinear form defined on $\mathcal{V} \times \mathcal{V} \times \mathcal{V}$ in this way:
$B(v_1, v_2, v_3) = b_1(w_1, w_2, w_3) + b_2(w_1, z_2, z_3) + b_2(w_1, \psi_2, \psi_3)$.

Now we will introduce operators associated with the above forms.

A continuous bilinear operator $\bar{B}$ acting from $\mathcal{V} \times \mathcal{V}$ to $\mathcal{V}'$ will be defined as

$$< \bar{B}(v_1, v_2), \chi >= B(v_1, v_2, \chi) \quad for \ \ v_1, v_2, \chi \in \mathcal{V}.$$

In a similar way we can define linear operators $\bar{R}$ and $\bar{A}$

$$< \bar{R}(v), \chi >= R(v, \chi),$$

$$< \bar{A}v, \chi >= a(v, \chi) \quad for \ \ v, \chi \in \mathcal{V}.$$

Let us also define the operator $\bar{F}_\theta$

$$< \bar{F}_\theta, \chi >= (e_2\theta, \chi_1) - \kappa(\nabla\theta, \nabla\chi_3) - (\theta_{|t}, \chi_3) - b_2(\chi_1 e_2, \theta, \chi_3).$$

Now, we consider the linearized problem described by the equation

$$\frac{dv}{ds} = G'(U(s,\tau)v_0, s) \tag{4.47}$$

$$v(\tau) = v_0$$

where

$$G'(U(s,\tau)v_0, s) = -\bar{A}(v(s)) - \bar{B}(U(s,\tau)v_0, v(s)) - \bar{B}(v(s), U(s,\tau)v_0)$$
$$-\bar{R}(v(s)) + \bar{F}_{\theta(s)}.$$

In order to estimate the dimension of the attractor let $v_0, v_0^1, ..., v_0^m \in \mathcal{H}$ and τ be fixed. Let the functions $\phi_1(s), ...\phi_m(s)$, $s \geq \tau$ (where $\phi_i = (w_i, z_i, \psi_i)$) be an orthonormal basis of the subspace of $\mathcal{H}$ spanned by the solutions of the problem (4.47) , more precisely by the functions $v(s; \tau, v_0, v_0^1), ..., v(s; \tau, v_0, v_0^m)$.

Now, what we have to estimate is the value of

$$< G'(U(s,\tau)v_0)\phi_i, \phi_i >= -a(\phi_i, \phi_i) - B(\phi_i, U(s,\tau), \phi_i) - R(\phi_i, \phi_i)+ < \bar{F}_\theta, \phi_i >$$

where

$$a(\phi_i, \phi_i) = (\nu + \nu_r)||w_i||_V^2 + \alpha||z_i||_{H^1}^2 + \kappa||\psi_i||_{H^1}^2,$$

$$
\begin{aligned}
R(\phi_i,\phi_i) &= -2\nu_r(rot\ z_i,w_i) - 2\nu_r(rot\ w_i,z_i) + 4\nu_r||z_i||^2_{L_2} - (e_2w_i,\psi_i)\\
&= -2\nu_r(rot\ z_i,w_i) - 2\nu_r(rot\ w_i,z_i) + 4\nu_r||z_i||^2_{L_2} - \int_\Omega \psi_i(w_i)_2dx,
\end{aligned}
$$

$$
B(\phi_i,U(s,\tau)v_0,\phi_i) = \int_\Omega((w_i\cdot\nabla)u)w_i + \int_\Omega((w_i\cdot\nabla)\tilde{T})\psi_i + \int_\Omega((w_i\cdot\nabla)\omega)z_i,
$$

$$
< \bar{F}_\theta,\phi_i >= (e_2\theta,w_i) - \kappa(\nabla\theta,\nabla\psi_i) - (\theta_{|t},\psi_i) - b_2((w_i)_2,\theta,\psi_i).
$$

As in the proof of existence of solutions to our problem we derive the estimate

$$
a(\phi_i,\phi_i) + R(\phi_i,\phi_i) \geq \nu||w_i||^2_V + \alpha||z_i||^2_{H^1} + \kappa||\psi_i||^2_{H^1} - \int_\Omega \psi_i(w_i)_2dx.
$$

It is easy to observe we can estimate the last term on the right-hand side of the above as follows

$$
\int_\Omega \psi_i(w_i)_2dx \leq \frac{1}{2}(||\psi_i||^2_{L_2} + ||w_i||^2_{L_2}) \leq \frac{1}{2}(||\psi_i||^2_{L_2} + ||w_i||^2_{L_2} + ||z_i||_{L_2}) = \frac{1}{2}||\phi_i||^2_{\mathcal{H}} = \frac{1}{2}.
$$

Therefore

$$
a(\phi_i,\phi_i) + R(\phi_i,\phi_i) \geq \nu||w_i||^2_V + \alpha||z_i||^2_{H^1} + \kappa||\psi_i||^2_{H^1} - \frac{1}{2}.
$$

We also need to look at the integrals that arise in the form B.
Using the Schwartz inequality we obtain

$$
|\sum_{i=1}^m b_1(w_i,u,w_i)| = |\int_\Omega \sum_{i=1}^m (w_i\cdot\nabla)u)w_i| \leq \int_\Omega |\nabla u|\rho_1(x)dx
$$

where $\rho_1(x) = \sum_{i=1}^m | w_i(x) |^2$.

In a similar way we estimate the second integral

$$
|\sum_{i=1}^m b_2(w_i,\omega,z_i)| = |\int_\Omega \sum_{i=1}^m (w_i\cdot\nabla)\omega)z_i| \leq \int_\Omega |\nabla\omega|\rho_1(x)^{1/2}\rho_2(x)^{1/2}dx
$$

where $\rho_2(x) = \sum_{i=1}^m | z_i(x) |^2$ and the third integral

$$
|\sum_{i=1}^m b_2(w_i,\tilde{T},\psi_i)| = |\int_\Omega \sum_{i=1}^m (w_i\cdot\nabla)\tilde{T})\psi_i| \leq \int_\Omega |\nabla\tilde{T}|\rho_1(x)^{1/2}\rho_3(x)^{1/2}dx
$$

where $\rho_3(x) = \sum_{i=1}^m | \psi_i(x) |^2$.

Using the Cauchy inequality and setting $\rho = \rho_1 + \rho_2 + \rho_3$ we get

$$
\begin{aligned}
|\sum_{i=1}^m B(\phi_i,U(s,\tau)v_0,\phi_i)| &\leq \sqrt{3}\int \rho(|\nabla u|^2 + |\nabla\omega|^2 + |\nabla\tilde{T}|^2)^{1/2}\\
&\leq \sqrt{3}||\rho||_{L_2}(||u||_V + ||\omega||_{H^1} + ||\tilde{T}||_{H^1}).
\end{aligned}
$$

Now, we will use the Lieb-Thirring inequality in order to estimate the term $||\rho||_{L_2}$. Before, we shall recall the theorem after [3].

Theorem 4.5.2. *(Lieb-Thirring inequality) Let $\phi_j \in H$ ($\phi_j \in H^0$) be an orthonormal family in $L^2(\Omega)$. Then the following inequality holds:*

$$\int_\Omega (\sum_{j=1}^m \phi_j^2)^2 \leq c_1 \sum_{j=1}^m \int_\Omega \mid \nabla\phi_j \mid^2 + c_2 m + c_3$$

where $c_1 = l_1(1 + max_{0\leq x_1 \leq L} \mid h'(x_1) \mid^2)$, $c_2 = l_2((1/L^2) + 1/h_0^2))$

$$c_3 = l_3 \int_\Omega (\frac{h'(x_1)}{h(x_1)})^4 (1 + h'(x_1)^4) dx$$

and l_1, l_2, l_3 are absolute constants.

In view of the above theorem we get

$$\int \rho(x)^2 dx \leq 3c_1 \sum_{i=1}^m (||w_i||_V^2 + ||z_i||_{H^1}^2 + ||\psi_i||_{H^1}^2) + 3c_2 m + 3c_3. \tag{4.48}$$

Next, we also have to estimate the last term $< \bar{F}_\theta, \phi_i >$.

$$\begin{aligned}
< \bar{F}_\theta, \phi_i >&\leq ||\theta||_{L_2} \cdot ||w_i||_{L_2} + \kappa||\theta||_{H^1} \cdot ||\psi_i||_{H^1} + ||\theta_t||_{L_2} \cdot ||\psi_i||_{L_2} + 2\sqrt{k_1\kappa}||w_i||_{H^1} \cdot ||\psi_i||_V \\
&\leq \frac{1}{\sqrt{\lambda_1}}||\theta||_{L_2} \cdot ||w_i||_V + \kappa||\theta||_{H^1} \cdot ||\psi_i||_{H^1} + \frac{1}{\sqrt{\lambda_1}}||\theta_t||_{L_2} \cdot ||\psi_i||_{H^1} + 2\sqrt{k_1\kappa}||w_i||_V \cdot ||\psi_i||_{H^1} \\
&\leq \frac{8}{\lambda_1\kappa}||\theta||_{L_2}^2 + \frac{\kappa}{32}||w_i||_V^2 + 8\kappa||\theta||_{H^1}^2 + + \frac{8}{\lambda_1\kappa}||\theta_t||_{L_2}^2 + 16k_1||w_i||_V^2 + \frac{3\kappa}{32}||\psi_i||_{H^1}^2.
\end{aligned}$$

Now, we use the Young inequality to derive the following

$$\begin{aligned}
&||\rho||_{L_2}(||u||_V + ||\omega||_{H^1} + ||\tilde{T}||_{H^1}) \\
&\leq (c_1)^{1/2}[(\sum_{i=1}^m ||w_i||_V^2)^{1/2} + (\sum_{i=1}^m ||z_i||_{H^1}^2)^{1/2} + (\sum_{i=1}^m ||\psi_i||_{H^1}^2)^{1/2}](||u||_V + ||\omega||_{H^1} + ||\tilde{T}||_{H^1}) \\
&\qquad + (c_2 m)^{1/2}(||u||_V + ||\omega||_{H^1} + ||\tilde{T}||_{H^1}) + (c_3)^{1/2}(||u||_V + ||\omega||_{H^1} + ||\tilde{T}||_{H^1}) \\
&\leq \frac{3\nu}{4}\sum_{i=1}^m ||w_i||_V^2 + \frac{3\alpha}{4}\sum_{i=1}^m ||z_i||_{H^1}^2 + \frac{3\kappa}{4}\sum_{i=1}^m ||\psi_i||_{H^1}^2 + \\
&\qquad + [6c(\frac{1}{\nu} + \frac{1}{\alpha} + \frac{1}{\kappa}) + 2](||u||_V^2 + ||\omega||_{H^1}^2 + ||\tilde{T}||_{H^1}^2) + \frac{3}{2}(c_2 m + c_3).
\end{aligned}$$

Summing up, we have

$$Tr_m(G'(V(s,\tau)v_0, s)) \leq -\sum_{i=1}^m (\frac{\nu}{4}||w_i||_V^2 + \frac{\alpha}{4}||z_i||_{H^1}^2 + \frac{\kappa}{8}||\psi_i||_{H^1}^2)$$

$$+C(\Omega,\kappa,\nu,\alpha)(1+\sup_{s\geq\tau}T_0^2(s)+\mid T_0\mid^3+\mid T_0'\mid^2)$$
$$+[6c(\frac{1}{\nu}+\frac{1}{\alpha}+\frac{1}{\kappa})+2](||u||_V^2+||\omega||_{H^1}^2+||\tilde{T}||_{H^1}^2)+\frac{3}{2}(c_2m+c_3)+\frac{1}{2}m.$$

Since $\int_\Omega \rho(x,t)dx = m$, using the Schwartz inequality we have

$$m^2\leq \mu(\Omega)||\rho||_{L_2}^2.$$

Moreover, (4.48) gives

$$\sum_{j=1}^m ||\phi_j||_V^2 \geq (\frac{||\rho||_{L_2}^2}{c_1}-\frac{c_2m}{c_1}-\frac{c_3}{c_1})\geq(\frac{m^2}{c_1\mu(\Omega)}-\frac{c_2m}{c_1}-\frac{c_3}{c_1}).$$

Therefore setting $k_3 = min\{\frac{\nu}{4},\frac{\alpha}{4},\frac{\kappa}{8}\}$ we arrive at

$$Tr_m(G'(U(s,\tau)v_0,s))\leq -k_3\frac{m^2}{c_1\mu(\Omega)}+C(\kappa,\nu,\alpha,\Omega)(1+2M+\mid T_0\mid^3+\mid T_0'\mid^2)$$
$$+[6c(\frac{1}{\nu}+\frac{1}{\alpha}+\frac{1}{\kappa})+2](||u||_V^2+||\omega||_{H^1}^2+||\tilde{T}||_{H^1}^2)+(\frac{3}{2}c_2+\frac{1}{2}+k_3\frac{c_2}{c_1})m+\frac{3}{2}c_3+k_3\frac{c_3}{c_1}. \quad (4.49)$$

Moreover, in view of the inequality (4.32) we have

$$\int_\tau^t(||U(s,\tau)v_0||_{\mathcal{V}}^2 ds\leq \frac{||v_0||_{\mathcal{H}}^2}{k_2}+\frac{1}{k_2}\int_\tau^t f(s)ds. \quad (4.50)$$

Now, let us recall the following numbers:

$$q_m(T)=sup_{v_0\in\widehat{A}(\tau-T)}sup\{\frac{1}{T}\int_{\tau-T}^{\tau}Tr(G'(U(s,\tau-T)v_0,s))ds\},$$

$$\tilde{q}_m = lim\ sup_{\,T\to\infty}q_m(T).$$

Therefore taking into account (4.49) and (4.50), we obtain

$$\tilde{q}_m\leq -k_3\frac{m^2}{c_1\mu(\Omega)}+(\frac{3}{2}c_2+\frac{1}{2}+k_3\frac{c_2}{c_1})m+\frac{3}{2}c_3+k_3\frac{c_3}{c_1}$$
$$+[6c(\frac{1}{\nu}+\frac{1}{\alpha}+\frac{1}{\kappa})+2]\frac{2}{k_2}\limsup_{T\to\infty}\frac{1}{T}\int_{\tau-T}^{\tau}f(s)ds$$

Let $M=\| f \|_{L^\infty(\infty,T^*)}$. We also denote for the sake of simplicity

$$b_1=\frac{k_3}{c_1\mu(\Omega)},\quad 2b_2=(\frac{3}{2}c_2+\frac{1}{2}+k_3\frac{c_2}{c_1})$$

and

$$b_3=\frac{2M}{k_2}[6c(\frac{1}{\nu}+\frac{1}{\alpha}+\frac{1}{\kappa})+2].$$

With this notation we have

$$\tilde{q}_m \leq -b_1 m^2 + 2b_2 m + b_3.$$

Using the Young inequality we arrive at $\tilde{q}_m \leq -\frac{1}{2}m^2 + \frac{2b_2^2}{b_1} + b_3$. In view of Lemma VI, 2.2 in [48] and the Theorem 2.4 in [24] if m is an integer such that

$$m - 1 < \frac{2}{b_1}(2b_2^2 + b_1 b_3)^{1/2} \leq m$$

then the fractal dimension of $A(\tau)$ for all $\tau \leq T^*$ is less or equal to $2m$.
Moreover taking into account that the mapping $U(t,\tau)$ is Lipschitz on $A(\tau)$ for all $t \geq \tau$ ((4.35)) and recalling Proposition 13.9 from [41] we see that in fact the fractal dimension of $A(\tau)$ is bounded by the same expression for all $\tau \in \mathbb{R}$. □

Chapter 5

Pulback attractor in H^1

In this chapter we consider the nonautonomous micropolar fluid model with external forces and moments that depend on time. We study the flow in an open and bounded subset Ω of $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. We recall the system of equations we work with.

$$\frac{\partial u}{\partial t} - (\nu + \nu_r)\triangle u + (u \cdot \nabla)u + \nabla p = 2\nu_r rot\ \omega + f(t), \tag{5.1}$$

$$div\ u = 0, \tag{5.2}$$

$$\frac{\partial \omega}{\partial t} - \alpha\triangle\omega + (u \cdot \nabla)\omega + 4\nu_r\omega = 2\nu_r rot\ u + g(t). \tag{5.3}$$

We assume that homogeneous Dirichlet boundary conditions hold

$$u_{|\partial\Omega} = 0, \quad \omega_{|\partial\Omega} = 0, \tag{5.4}$$

and denote the initial conditions as

$$u(\tau) = u_0, \quad \omega(\tau) = \omega_0, \tag{5.5}$$

for $\tau \in \mathbb{R}$.

Our aim is to study the long-time behaviour of weak solutions of problem (5.1)-(5.5) when initial conditions u_0 and ω_0 belong to the Sobolev space H^1, namely we assume

$$u_0 \in V, \quad \omega_0 \in H_0^1. \tag{5.6}$$

As concerns the micropolar fluid model, H^1 pullback attractors for a flow in a smooth bounded two-dimensional domain were considered in [12], where existence of the H^1-pullback attractor was proved for *translation bounded*, with respect to L^2-topology, external forces and moments. In their proof the authors used the methods and abstract results developed recently in [6] and [43].

We shall prove the existence of a unique minimal pullback H^1-attractor, for possibly *nonuniform* with respect to time suitable norms of forces and moments, satisfying only a certain integrability condition which is less restrictive then the conditions regarded in [12]. To attain our goal we use
(1) a recent method introduced in [52], [43] to study existence of pullback attractors in Banach

spaces based on the notion of the Kuratowski measure of noncompactness of a bounded set (this method is in turn a generalization of that introduced in [32] to study autonomous dynamical systems),
(2) its further generalization in [25] to a more general setting introduced in [5],
(3) an application of the Gronwall-like lemma (Lemma 5.2.2) to the second energy inequality for higher Fourier modes of the solution.
We recalled the methods in Chapter 3 of our thesis.

5.1 Notation

Since we work with different boundary conditions than the ones in the previous section, let us recall again the standard notation for the function spaces that we shall use in the sequel. We denote by L^2 and H_0^1 the usual functional spaces $L^2(\Omega)$ and $H_0^1(\Omega)$, with scalar products

$$(u,v) = \int_\Omega u(x)v(x)dx \quad \text{and} \quad ((u,v)) = \int_\Omega \nabla u(x)\nabla v(x)dx.$$

Let $\tilde{V} = \{u \in C_0^\infty(\Omega)^2 : u = (u_1,u_2), div\ u = 0\}$. Then we define the spaces H and V as H = closure of $\tilde{V}$ in $L^2 \times L^2$, V = closure of $\tilde{V}$ in $H_0^1 \times H_0^1$.

In view of the Theorem 2.3.1, we can define a process $\{U(t,\tau), t \geq \tau\}$ in $V \times H_0^1$ as

$$U(t,\tau)(u_0,\omega_0) = (u(t;\tau,u_0,\omega_0), \omega(t;\tau,u_0,\omega_0)), \quad t \geq \tau, \tag{5.7}$$

where $(u(t;\tau,u_0,\omega_0), \omega(t;\tau,u_0,\omega_0))$ is the weak solution of problem (5.1)-(5.5) in the sense of Definition 2.3.1.

5.2 Lemmas and estimates

Before we state the main result of this Chapter, we prove some lemmas we shall use in the sequel.

Let A be the Stokes operator in the space H. Then there exists a sequence $0 < \lambda_1 \leq \lambda_2 ...$, $\lambda_j \to \infty$ as $j \to \infty$, of eigenvalues of A and an orthonormal basis $v_1, v_2, ...$ in H such that $Av_j = \lambda_j v_j$ for $j = 1, 2, ...$
Let $H_m = span\{v_1, ..., v_m\}$, $P_m : H \to H$ be an orthogonal projector onto H_m, and let $Q_m = I - P_m$.

Similarly we consider the operator $-\triangle$ in L^2 with homogeneous boundary conditions. Let $0 < \eta_1 \leq \eta_2 ...$, $\eta_j \to \infty$ as $j \to \infty$, be the sequence of eigenvalues of $-\triangle$ and let $w_1, w_2, ...$ be an orthonormal basis in L^2 such that $-\triangle w_j = \eta_j w_j$ for $j = 1, 2, ...$
Let $L_m^2 = span\{w_1, ..., w_m\}$, $P'_m : L^2 \to L^2$ be an orthogonal projector onto L_m^2, and let $Q'_m = I - P'_m$.

Lemma 5.2.1. *Let $\tau \in I\!R$ and let $(u(t), \omega(t))$, $t > \tau$ be the weak solution of problem (5.1)-(5.5) with $f \in L^2_{loc}(I\!R, H)$ and $g \in L^2_{loc}(I\!R, L^2)$. Then the following inequality holds for $s > \tau$,*

$$\frac{d}{ds}(||Q_m u(s)||_V^2 + ||Q'_m \omega(s)||_{H_0^1}^2) + \beta_{m+1}(||Q_m u(s)||_V^2 + ||Q'_m \omega(s)||_{H_0^1}^2)$$
$$\leq H(s)(||Q_m u(s)||_V^2 + ||Q'_m \omega(s)||_{H_0^1}^2) + F(s), \tag{5.8}$$

where

$$H(s) = C(||u(s)||_H^2 ||u(s)||_V^2 + ||u(s)||_H^2 ||\omega(s)||_{H_0^1}^2 + 1), \tag{5.9}$$

$$F(s) = \frac{4}{\nu}||f(s)||_H^2 + \frac{2}{\alpha}||g(s)||_{L_2}^2, \tag{5.10}$$

with $\beta_{m+1} = min\{\nu\lambda_{m+1}, \frac{\alpha}{2}\eta_{m+1}\}$.

Proof. In order to prove the desired inequality, we take the scalar products of (5.1) and (5.3) with $A\ Q_m u$ and $-\triangle Q'_m \omega$, respectively, and proceed similarly as in [27] using the Poincaré inequality. □

Now, we formulate and prove the Gronwall-like lemma that will be an important part of the proof of the main result of this chapter.

Lemma 5.2.2. *Let for some* $\lambda > 0$, $\tau \in I\!R$, *and for* $s > \tau$

$$y'(s) + \lambda y(s) \leq g(s)y(s) + h(s) \tag{5.11}$$

where the functions y, y', h, g *are assumed to be locally integrable and* y, h, g *nonnegative on the interval* $t < s < t + r$, *for some* $t \geq \tau$ *and* $r > 0$. *Then*

$$y(t+r) \leq e^{-\lambda r}\{\frac{1}{r}\int_t^{t+r} y(s)e^{\lambda(s-t)}ds + \int_t^{t+r} h(s)e^{\lambda(s-t)}ds\} \exp(\int_t^{t+r} g(s)ds).$$

Proof. Let $t < s < t + r$. We multiply both sides of (5.11) by $e^{\lambda(s-t)}$ and obtain

$$\frac{d}{ds}(e^{\lambda(s-t)}y(s)) \leq e^{\lambda(s-t)}y(s)g(s) + h(s)e^{\lambda(s-t)}. \tag{5.12}$$

Let $u(s) = e^{\lambda(s-t)}y(s)$ and $\mathcal{H}(s) = e^{\lambda(s-t)}h(s)$. Then we can apply the uniform Gronwall lemma to the obtained inequality to get

$$u(t+r) \leq \{\frac{1}{r}\int_t^{t+r} u(s)ds + \int_t^{t+r} \mathcal{H}(s)ds\}\ \exp(\int_t^{t+r} g(s)ds). \tag{5.13}$$

Coming back to the functions $y(s)$ and $h(s)$, we obtain the desired inequality. □

5.3 Theorem on existence of the H^1 pullback attractor

Let $\mathcal{D}$ be the class of all families $\{D(t) : t \in I\!R\}$ of nonempty subsets of $V \times H_0^1(\Omega)$ such that

$$\lim_{t\to-\infty} e^{\lambda t}[D(t)]^+ = 0, \tag{5.14}$$

where $[D(t)]^+ = \sup\{||u||_H^2 + ||\omega||_{L^2}^2;\ (u,\omega) \in D(t)\}$, and $\lambda > 0$ is given.

Our main result reads

Theorem 5.3.1. *Let*

$$\int_{-\infty}^{t} e^{\lambda s}\{||f(s)||_H^2 + ||g(s)||_{L_2}^2\}ds < \infty \quad \text{for every} \quad t \in I\!R, \tag{5.15}$$

where $\lambda = \min\{\nu\lambda_1, \alpha\eta_1\}$ (ν and α are the viscosities as in (5.1) and in (5.3)), and λ_1, η_1 are the the first eigenvalues of the Stokes and minus Laplacian operators, respectively).
Then the process $U(t,\tau)$ associated with problem (5.1)-(5.5) possesses a unique minimal pullback $\mathcal{D}$-attractor $\widehat{A} = \{A(t) : t \in I\!R\}$ in $V \times H_0^1$.

Proof. We prove the theorem by checking the conditions of the abstract Theorem 3.2.2 on existence of the pullback attractor in a uniformly convex Banach space.

(i) *The process $U(t,\tau)$ defined in (5.7) is norm-to weak continuous in the space $\mathcal{V} = V \times H_0^1$.*

The process $U(t,\tau)$ is continuous in the space $H \times L^2$ in view of the Theorem 2.3.1. Then, due to the Theorem 3.2.3 it suffices to show that $U(t,\tau)$ maps compact sets in $\mathcal{V}$ to bounded sets in $\mathcal{V}$ for all $\tau \in \mathbb{R}$ and all $t \geq \tau$.

From the second energy inequality (2.21) and the Gronwall lemma we obtain,

$$\begin{aligned} ||u(t)||_V^2 + ||\omega(t)||_{H_0^1}^2 &\leq \{||u(\tau)||_V^2 + ||\omega(\tau)||_{H_0^1}^2\}\exp(\int_\tau^t H(s)ds) \\ &+ \{\int_\tau^t F(\eta)\exp(-\int_\tau^\eta H(s)ds)d\eta\}\ \exp(\int_\tau^t H(s)ds). \end{aligned} \tag{5.16}$$

As the functions H and F are locally integrable, from (5.16) it follows that $U(t,\tau)$ maps bounded sets in $\mathcal{V}$ (and, in particular, compact ones) to bounded sets in $\mathcal{V}$ for all $\tau \in \mathbb{R}$ and all $t \geq \tau$. To check the local integrability of H we use the first energy inequality (2.18) to get

$$\int_\tau^t (||u(s)||_V^2 + ||\omega(s)||_{H_0^1}^2)ds \leq \frac{1}{k_1}\{(||u(\tau)||_H^2 + ||\omega(\tau)||_{L_2}^2) + k_3\int_\tau^t (||f(s)||_H^2 + ||g(s)||_{L_2}^2)ds\}.$$

Then, as $u \in C([\tau,t];H)$, $\omega \in C([\tau,t];L^2)$, $f \in L^2_{loc}(I\!R,H)$ and $g \in L^2_{loc}(I\!R,L^2)$, there exists a constant $\tilde{C}$ depending on t and τ such that

$$\begin{aligned} \int_\tau^t H(s)ds &= C\int_\tau^t \{||u(s)||_H^2||u(s)||_V^2 + ||u(s)||_H^2||\omega(s)||_{H_0^1}^2 + 1\}ds \\ &\leq \tilde{C}\{\frac{1}{k_1}(||u(\tau)||_H^2 + ||\omega(\tau)||_{L_2}^2 + k_3\int_\tau^t (||f(s)||_H^2 + ||g(s)||_{L_2}^2)ds)\}^2 \\ &+ C(t-\tau) < \infty. \end{aligned} \tag{5.17}$$

(ii) *There exists a family $\widehat{B}$ of pullback $\mathcal{D}$-absorbing sets in $\mathcal{V}$.*

We use the second energy inequality (2.21) and uniform Gronwall lemma to get

$$\begin{aligned}||u(t+r)||_V^2 \quad &+ \quad ||\omega(t+r)||_{H_0^1}^2 \\ &\leq \{\frac{1}{r}\int_t^{t+r}(||u(s)||_V^2+||\omega(s)||_{H_0^1}^2)ds+\int_t^{t+r}F(s)ds\}\exp\{\int_t^{t+r}H(s)ds\}\end{aligned} \tag{5.18}$$

for every $t \geq \tau$. From the first energy inequality (2.18) and the Gronwall lemma we have

$$\begin{aligned}||u(t)||_H^2+||\omega(t)||_{L_2}^2 &\leq e^{-k_2(t-\tau)}\{(||u(\tau)||_H^2+||\omega(\tau)||_{L_2}^2)\} \\ &+ k_3e^{-k_2(t-\tau)}\int_\tau^t e^{-k_2(t-s)}(||f(s)||_H^2+||g(s)||_{H_0^1}^2)ds \\ &\leq c_0e^{-k_2t}\int_{-\infty}^t e^{k_2s}(||f(s)||_H^2+||g(s)||_{L_2}^2)ds,\end{aligned} \tag{5.19}$$

uniformly with respect to all initial conditions $u(\tau)\in D(\tau)$ for all $\tau \leq \tau_0(t,\widehat{D})$, with $c_0=c_0(|\Omega|,k_3)$ and $k_2=\min\{\nu\lambda_1,\alpha\eta_1\}$. Observe that

$$\int_t^{t+r}|\xi(s)|^2ds \leq e^{-k_2t}\int_{-\infty}^{t+r}e^{k_2s}|\xi(s)|^2ds \tag{5.20}$$

for all functions ξ for which the right hand side is finite.
Thus, under the assumptions of Theorem 5.3.1 that we prove, from the first energy inequality again, (5.19) and (5.20) we get

$$\begin{aligned}\int_t^{t+r}(||u(s)||_V^2+||\omega(s)||_{H_0^1}^2)ds &\leq \frac{1}{k_1}\{(||u(t)||_H^2+||\omega(t)||_{L_2}^2)+\int_t^{t+r}(||f(s)||_H^2+||g(s)||_{L_2}^2)ds\} \\ &\leq c_1e^{-k_2t}\int_{-\infty}^{t+r}e^{k_2s}(||f(s)||_H^2+||g(s)||_{L_2}^2)ds,\end{aligned} \tag{5.21}$$

uniformly with respect to all initial conditions $u(\tau)\in D(\tau)$ for all $\tau \leq \tau_0(t,\widehat{D})$,
with $c_1=c_1(|\Omega|,r)$. Applying (5.21) to (5.18) we conclude that

$$\begin{aligned}||u(t+r)||_V^2+||\omega(t+r)||_{H_0^1}^2 &\leq c_2\,e^{-k_2t}\int_{-\infty}^{t+r}e^{k_2s}(||f(s)||_H^2+||g(s)||_{L_2}^2)ds \\ &\cdot\exp\{c_2e^{-k_2t}\int_{-\infty}^{t+r}e^{k_2s}(||f(s)||_H^2+||g(s)||_{L_2}^2)ds\}^2,\end{aligned} \tag{5.22}$$

uniformly with respect to all initial conditions $u(\tau)\in D(\tau)$ for all $\tau \leq \tau_0(t,\widehat{D})$, with $c_2=c_2(|\Omega|,r)$. This proves existence of the family of $\mathcal{D}$-absorbing sets in $\mathcal{V}$.

(iii) *$U(t,\tau)$ is pullback $\mathcal{D}$-limit-set compact.*

In view of Theorem 3.2.4 it suffices to prove that $U(t,\tau)$ satisfies the pullback $\mathcal{D}$-flattening condition. Using inequality (5.8) together with Lemma 5.2.2 with $y_m(t)=||Q_mu(t)||^2+||Q_m'\omega(t)||^2$ we obtain

$$y_m(t+r)\leq e^{-\beta_{m+1}r}\int_t^{t+r}\{\frac{1}{r}y_m(s)+F(s)\}e^{\beta_{m+1}(s-t)}ds\exp(\int_t^{t+r}H(s)ds). \tag{5.23}$$

We have to prove that for an arbitrary small $\epsilon > 0$ there exists m such that the right hand side is not greater then ϵ, uniformly with respect to all initial conditions $u(\tau) \in D(\tau)$ for all $\tau \leq \tau_0(t, \widehat{D})$. From inequalities (5.19) and (5.21) it follows that the right hand side of (5.23) is uniformly bounded with respect to m and with respect to all initial conditions $u(\tau) \in D(\tau)$ for all $\tau \leq \tau_0(t, \widehat{D})$. Now, we shall prove that with increasing m the right hand side of (5.23) decreases to zero, again uniformly with respect to initial conditions. To this end, consider the expressions

$$e^{-\beta_{m+1}r} \int_t^{t+r-\delta} \{\frac{1}{r} y_m(s) + F(s)\} e^{\beta_{m+1}(s-t)} ds$$

and

$$e^{-\beta_{m+1}r} \int_{t+r-\delta}^{t+r} \{\frac{1}{r} y_m(s) + F(s)\} e^{\beta_{m+1}(s-t)} ds.$$

It is easily seen that the second one converges to zero as $\delta \to 0$, uniformly with respect to m, and the first one converges to zero with $m \to \infty$ for any fixed $\delta \in (0, r)$, in both cases the convergence is uniform with respect to all initial conditions $u(\tau) \in D(\tau)$ for all $\tau \leq \tau_0(t, \widehat{D})$. This finishes the proof of property (iii) and thus of the existence of the pullback attractor. $\square$

Chapter 6

Pullback attractor in H^2

In this chapter, we continue working on the theory of pullback attractors for the following micropolar fluid equations with homogeneous Dirichlet boundary conditions

$$\frac{\partial u}{\partial t} - (\nu + \nu_r)\triangle u + (u \cdot \nabla)u + \nabla p = 2\nu_r rot\ \omega + f(t), \tag{6.1}$$

$$div\ u = 0, \tag{6.2}$$

$$\frac{\partial \omega}{\partial t} - \alpha\triangle\omega + (u \cdot \nabla)\omega + 4\nu_r\omega = 2\nu_r rot\ u + g(t), \tag{6.3}$$

$$u_{|\partial\Omega} = 0, \quad \omega_{|\partial\Omega} = 0, \tag{6.4}$$

$$u(\tau) = u_0, \quad \omega(\tau) = \omega_0, \tag{6.5}$$

where $\tau \in I\!R$.
We assume higher regularity of initial conditions, namely

$$u_0 \in D(A), \quad \omega_0 \in H^2 \cap H^1_0. \tag{6.6}$$

We shall show that under this assumption the pullback attractor in the space H^2 exists.

The Theorem 2.3.2 asserts that in view of (6.6), we can define a process

$$U(t,\tau) : D(A) \times H^2(\Omega) \cap H^1_0(\Omega) \to D(A) \times H^2(\Omega) \cap H^1_0(\Omega), \quad t \geq \tau$$

in the usual way,

$$U(t,\tau)(u_0,\omega_0) = (u(t;\tau,u_0,\omega_0), \omega(t;\tau,u_0,\omega_0)), \quad t \geq \tau, \tag{6.7}$$

where $(u(t;\tau,u_0,\omega_0), \omega(t;\tau,u_0,\omega_0))$ is the weak solution of problem (6.1)-(6.5) when (6.6) holds.

6.1 Useful lemmas

Before we state the main theorem on the existence of the H^2-pullback attractor for the dynamical system associated with (6.1)-(6.5), we need to make some observations and recall estimates on the solutions.

Lemma 6.1.1. *The operators $B : D(A) \times D(A) \to L_2$ and $B_1 : D(A) \times H^2 \cap H_0^1 \to L_2$ are compact.*

Proof. We shall show the result only for the operator B, since the compactness of the other one requires the same arguments.
Let us take a sequence $(u_k, u_k) \in D(A) \times D(A)$ that is bounded in $D(A) \times D(A)$. Since $D(A) \subset\subset W^{1,4}(\Omega)$ in view of the Rellich-Kondrashov Theorem, we can extract a subsequence that converges in $W^{1,4}(\Omega)$.
But since

$$||B(u_k, u_k)||_{L_2} \leq c||u_k||_{L_4}||u_k||_{W^{1,4}},$$

a Cauchy subsequence $B(u_{k_n}, u_{k_n})$ in L_2 exists. □

Now, we cite after ([39]) the estimate on the norms $||Au||_{L_2}$ and $||\triangle\omega||_{L_2}$.

Lemma 6.1.2. *Let (u, ω) be the weak solution for the problem (6.1)-(6.6). Then*

$$\begin{aligned} ||Au||_{L_2} \quad &+ \quad ||-\triangle\omega||_{L_2} \\ &\leq \quad c(||u_t||_{L_2} + ||\omega_t||_{L_2} + ||f||_{L_2} + ||g||_{L_2} + ||u||_V^3 + ||\omega||_{H_0^1}^3 + ||u||_V^2 + ||\omega||_{H_0^1}^2), \end{aligned} \tag{6.8}$$

where the constant c depends on $|\Omega|$, n and the constants characterizing the fluid.

Next, we formulate energy-type inequalities for the time derivatives of the weak solution, namely functions u_t and ω_t.

Lemma 6.1.3. *Let (u, ω) be the weak solution for the problem (6.1)-(6.6). Then the following inequality holds*

$$\begin{aligned} ||u_t(t+r)||_{L_2}^2 + ||\omega_t(t+r)||_{L_2}^2 \quad &\leq \quad \frac{1}{r}\int_t^{t+r}(||u_t(s)||_{L_2}^2 + ||\omega_t(s)||_{L_2}^2)ds\; e^{c_1\int_t^{t+r} H_1(s)ds} \\ &+ \quad c_1\int_t^{t+r}(||f_t(s)||_{L_2}^2 + ||g_t(s)||_{L_2}^2)ds\; e^{c_1\int_t^{t+r} H_1(s)ds}. \end{aligned} \tag{6.9}$$

where c_1 is the constant dependent on ν, $,\nu_r$ α, constants from Poincaré inequality, and

$$H_1(s) = ||u||_V^2 + ||\omega||_{H_0^1}^2 + 1.$$

Proof. We differentiate the equations (6.1) and (6.3) with respect to t and take the scalar product with u_t and ω_t, respectively. We arrive at

$$\frac{d}{dt}||u_t||_H^2 + 2(\nu + \nu_r)||u_t||_V^2 + 2b(u_t, u, u_t) + 2b(u, u_t, u_t) = 4\nu_r(rot\ \omega_t, u_t) + (f_t, u_t),$$

and

$$\frac{d}{dt}||\omega_t||_{L^2}^2 + 2\alpha||\omega_t||_{H_0^1}^2 + 2b_1(u_t, \omega, \omega_t) + 2b_1(u, \omega_t, \omega_t) + 8\nu_r||\omega_t||_{L^2}^2 = 4\nu_r(rot\ u_t, \omega_t) + (g_t, \omega_t).$$

Then, using Young inequality, estimates for the trilinear forms, and Poincaré inequality,

$$2b(u_t,u,u_t) \leq c||u_t||_V^2||u||_H^{1/2}||u||_V^{1/2} \leq \frac{\nu+\nu_r}{2}||u_t||_V^2 + c\,(\nu+\nu_r)\,||u_t||_V^2||u||_H||u||_V,$$

$$\begin{aligned} 2b_1(u_t,\omega,\omega_t) &\leq c||u_t||_V||\omega||_{L^2}^{1/2}||\omega||_{H_0^1}^{1/2}||\omega_t||_{H_0^1} \\ &\leq \frac{\alpha}{2}(||u_t||_V^2+||\omega_t||_{H_0^1}^2) + c\,(\alpha)\,(||u_t||_V^2+||\omega_t||_{H_0^1}^2)||\omega||_{H_0^1}^2 \end{aligned}$$

after standard calculations, we get

$$\begin{aligned} \frac{d}{dt}(||u_t(s)||_{L_2}^2+||\omega_t(s)||_{L_2}^2) &\leq -min\{\nu+\nu_r,\alpha\}(||u_t(s)||_{H_0^1}^2+||\omega_t(s)||_{H_0^1}^2) \\ &+ c_1\left\{H_1(s)(||u_t(s)||_{H_0^1}^2+||\omega_t(s)||_{H_0^1}^2)+(||f_t(s)||_{L_2}^2+||g_t(s)||_{L_2}^2)\right\}. \end{aligned} \quad (6.10)$$

Therefore, the following holds

$$\frac{d}{dt}(||u_t(s)||_{L_2}^2+||\omega_t(s)||_{L_2}^2) \leq c_1\left\{H_1(s)(||u_t(s)||_{H_0^1}^2+||\omega_t(s)||_{H_0^1}^2)+(||f_t(s)||_{L_2}^2+||g_t(s)||_{L_2}^2)\right\}.$$

Using the uniform Gronwall lemma we arrive at the desired inequality. □

Lemma 6.1.4. *Let (u,ω) be the weak solution for the problem (6.1)-(6.6). Then the following inequality holds*

$$\begin{aligned} \int_t^{t+r}||u_t(s)||_{L_2}^2+||\omega_t(s)||_{L_2}^2 &\leq c\int_t^{t+r}(||f(s)||_{L_2}^2+||g(s)||_{L_2}^2+||Au(s)||_{L_2}^2+||\triangle\omega(s)||_{L_2}^2 \\ &+ ||u(s)||_{H_0^1}^2+||\omega(s)||_{H_0^1}^2+H(s)(||u||_V^2+||\omega||_{H_0^1})ds. \end{aligned} \quad (6.11)$$

Proof. We multiply the equations (6.1) and (6.3) by u_t and ω_t, respectively. We have

$$||u_t||_H^2 = -(Au,u_t) - b(u,u,u_t) + (rot\ u,u_t) + (f,u_t),$$

and

$$||\omega_t||_{L_2}^2 = (\triangle\omega,\omega_t) - b_1(u,\omega,\omega_t) - 4\nu_r(\omega,\omega_t) + 2\nu_r(rot\ u,\omega_t) + (g,\omega_t).$$

Applying trilinear forms estimates

$$|b(u,u,u_t))| \leq c||u||_H^{1/2}||Au||_H^{1/2}||u||_V||u_t||_H \leq c\left(||Au||_H+||u||_H||u||_V^2\right)||u_t||_H,$$

$$|b_1(u,\omega,\omega_t)| \leq c||u||_H^{1/2}||Au||_H^{1/2}||\omega||_{H_0^1}||\omega_t||_{L_2} \leq c\left(||Au||_H+||u||_H||\omega||_{H_0^1}^2\right)||\omega_t||_{L_2},$$

and Schwartz inequality, we arrive at

$$\begin{aligned} &||u_t||_H+||\omega_t||_{L_2} \\ \leq\ &c(||Au||_H+||\triangle\omega||_{L_2}+||u||_V+||\omega||_{H_0^1}+||f||_H+||g||_{L_2}+||u||_H(||u||_V^2+||\omega||_{H_0^1}^2)). \end{aligned}$$

The above impilies (6.11) □

Lemma 6.1.5. *Let (u, ω) be the weak solution for the problem (6.1)-(6.6). Then*

$$||AQ_m u||_{L_2} \leq c(\nu_r, \nu)[||Q_m u_t||_{L_2} + ||Q_m B(u,u)||_{L_2} + ||Q_m \omega||_{H_0^1} + ||Q_m f||_{L_2}] \tag{6.12}$$

and

$$\begin{aligned} ||\triangle Q_m \omega||_{L_2} &\leq c(\alpha, \nu)[||Q_m \omega_t||_{L_2} + ||Q_m B_1(u,\omega)||_{L_2} \\ &+ ||Q_m \omega||_{H_0^1} + ||Q_m u||_{H_0^1} + ||Q_m g||_{L_2}]. \end{aligned} \tag{6.13}$$

Moreover,

$$\begin{aligned} &||Q_m u_t(t+r)||^2_{L_2} + ||Q_m \omega_t(t+r)||^2_{L_2} \\ &\leq e^{(-\beta_{m+1}+c)\,r} \int_t^{t+r} \{\frac{1}{r}||Q_m u_t(s)||^2_{L_2} + ||Q_m \omega_t(s)||^2_{L_2} + G(s)\} e^{\beta_m+1(s-t)} ds \end{aligned} \tag{6.14}$$

where

$$\begin{aligned} G(s) &= c(\nu, \nu_r, \alpha) \cdot (||u_t||^2_{L_2} + ||\omega_t||^2_{L_2} + ||f_t||^2_{L_2} + ||g_t||^2_{L_2} \\ &+ ||u_t||^2_{L_2}||u||^2_{H_0^1} + ||u||^2_{L_2}||u_t||^2_V + ||\omega_t||^2_{L_2}||u||^2_{H_0^1} + ||u||^2_{L_2}||\omega_t||^2_{H_0^1}). \end{aligned}$$

Proof. In order to prove first two estimates, we multiply (6.1) and (6.3) in L_2 by $AQ_m u$ and $-\triangle Q_m \omega$, respectively.
We arrive at

$$(u_t, AQ_m u) + \nu||AQ_m u||^2_{L_2} + b(u,u,AQ_m u) = 2\nu_r(rot\omega, AQ_m u) + (f, AQ_m u),$$

and

$$\begin{aligned} (\omega_t, -\triangle Q_m \omega) + \alpha|| - \triangle Q_m \omega||^2_{L_2} + b_1(u, \omega, -\triangle Q_m \omega) + 4\nu_r(\omega, -\triangle Q_m \omega) \\ = 2\nu_r(rotu, -\triangle Q_m \omega) + (g, -\triangle Q_m \omega). \end{aligned}$$

Since $(v, Q_m w) = (Q_m v, Q_m w)$ for any $v, w \in H_0^1$, we can rewrite the above as

$$(Q_m u_t, AQ_m u) + \nu||AQ_m u||^2_{L_2} + (Q_m B(u,u), AQ_m u) = 2\nu_r(Q_m rot\omega, AQ_m u) + (Q_m f, AQ_m u),$$

and

$$(Q_m \omega_t, -\triangle Q_m \omega) + \alpha|| - \triangle Q_m \omega||^2_{L_2} + (Q_m B_1(u,\omega), AQ_m \omega) + 4\nu_r(Q_m rot\omega, -\triangle Q_m \omega)$$

$$= 2\nu_r(Q_m rotu, -\triangle Q_m \omega) + (Q_m g, -\triangle Q_m \omega).$$

Using simply the Schwartz inequlaity, we arrive at the desired inequalities.

In order to prove the third estimate, we differentiate (6.1) and (6.3) and multiply them by $Q_m u_t$ and $Q_m \omega_t$, respectively. We have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \quad & (||Q_m u_t||^2_{L_2} + ||Q_m \omega_t||^2_{L_2}) + (\nu + \nu_r)||Q_m u_t||^2_{H_0^1} + \alpha||Q_m \omega_t||^2_{H_0^1} \\ + \quad & 4\nu_r||Q_m \omega_t||^2_{L_2} + b(u_t, u, Q_m u_t) + b(u, u_t, Q_m u_t) + b_1(u_t, \omega, Q_m \omega_t) + b_1(u, \omega_t, Q_m \omega_t) \\ \leq \quad & 2\nu_r||rot\ \omega_t||_{L_2}||Q_m u_t||_{L_2} + ||rot\ u_t||_{L_2}||Q_m \omega_t||_{L_2} + ||f_t||_{L_2}||Q_m u_t||_{L_2} + ||g_t||_{L_2}||Q_m \omega_t||_{L_2}. \end{aligned}$$

We estimate the trilinear forms

$$\begin{aligned}|b(u_t,u,Q_mu_t)| &\leq ||u_t||_{L_2}^{1/2}||u_t||_V^{1/2}||u||_{L_2}^{1/2}||u||_V^{1/2}||Q_mu_t||_V\\ &\leq \frac{1}{\nu+\nu_r}||u_t||_{L_2}\cdot||u_t||_V\cdot||u||_{L_2}\cdot||u||_V+\frac{\nu+\nu_r}{4}||Q_mu_t||_V^2\\ &\leq \frac{1}{2(\nu+\nu_r)}||u_t||_{L_2}^2||u||_V^2+\frac{1}{2(\nu+\nu_r)}||u||_{L_2}^2||u_t||_V^2+\frac{\nu+\nu_r}{4}||Q_mu_t||_V^2,\end{aligned}$$

and similarly

$$|b(u,u_t,Q_mu_t)|\leq \frac{1}{2(\nu+\nu_r)}||u_t||_{L_2}^2||u||_V^2+\frac{1}{2(\nu+\nu_r)}||u||_{L_2}^2||u_t||_V^2+\frac{\nu+\nu_r}{4}||Q_mu_t||_V^2.$$

In the same manner we estimate the forms b_1,

$$|b_1(u_t,\omega,Q_m\omega_t)|\leq \frac{1}{2\alpha}||u_t||_{L_2}^2||\omega||_{H_0^1}^2+\frac{1}{2\alpha}||\omega||_{L_2}^2||u_t||_V^2+\frac{\alpha}{4}||Q_m\omega_t||_{H_0^1}^2,$$

and

$$|b_1(u,\omega_t,Q_m\omega_t)|\leq \frac{1}{2\alpha}||u||_{L_2}^2||\omega_t||_{H_0^1}^2+\frac{1}{2\alpha}||\omega_t||_{L_2}^2||u||_V^2+\frac{\alpha}{4}||Q_m\omega_t||_{H_0^1}^2.$$

Using the above estimates and the Poincaré inequality, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(||Q_mu_t||_{L_2}^2+||Q_m\omega_t||_{L_2}^2) &+ c(\nu,\nu_r,\alpha)\beta_{m+1}(||Q_mu_t||_{L_2}^2+||Q_m\omega_t||_{L_2}^2)\\ &\leq c(\nu,\nu_r,\alpha)\cdot((||Q_mu_t||_{L_2}^2+||Q_m\omega_t||_{L_2}^2)+G(s)),\end{aligned}$$

where

$$\begin{aligned}G(s) &= c(\nu,\nu_r,\alpha)\cdot(||u_t||_{L_2}^2+||\omega_t||_{L_2}^2+||f_t||_{L_2}^2+||g_t||_{L_2}^2\\ &+ ||u_t||_{L_2}^2||u||_{H_0^1}^2+||u||_{L_2}^2||u_t||_V^2+||\omega_t||_{L_2}^2||u||_{H_0^1}^2+||u||_{L_2}^2||\omega_t||_{H_0^1}^2).\end{aligned}$$

Now, using the Gronwall-type inequality from the Lemma 5.2.2, we get:

$$\begin{aligned}&||Q_mu_t(t+r)||_{L_2}^2+||Q_m\omega_t(t+r)||_{L_2}^2\\ &\leq e^{(-\beta_{m+1}+c)\,r}\int_t^{t+r}\{\frac{1}{r}||Q_mu_t(s)||_{L_2}^2+||Q_m\omega_t(s)||_{L_2}^2+G(s)\}e^{\beta_m+1(s-t)}ds.\end{aligned}$$

□

6.2 Existence of the pullback attractor

Let $\mathcal{D}$ be the class of all families $\{D(t):t\in I\!R\}$ of nonempty subsets of $D(A)\times(H^2\cap H_0^1)$ such that

$$\lim_{t\to-\infty}e^{\lambda t}[D(t)]^+=0, \tag{6.15}$$

where $[D(t)]^+=\sup\{||u||_H^2+||\omega||_{L^2}^2;\ (u,\omega)\in D(t)\}$, and $\lambda>0$ is given.

We prove the following

Theorem 6.2.1. *Let*

$$\int_{-\infty}^{t} e^{\lambda s}\{||f(s)||_H^2 + ||f_t(s)||_H^2 + ||g(s)||_{L_2}^2 + ||g_t(s)||_{L_2}\}ds < \infty \quad \text{for every} \quad t \in I\!R, \qquad (6.16)$$

where $\lambda = \min\{\nu\lambda_1, \alpha\eta_1\}$ (ν and α are the viscosities, and λ_1, η_1 are the first eigenvalues of the Stokes and minus Laplacian operators, respectively).

Then the process $U(t, \tau)$ defined in (6.7) possesses a unique and minimal pullback $\mathcal{D}$-attractor $\widehat{A} = \{A(t) : t \in I\!R\}$ in $D(A) \times (H^2 \cap H_0^1)$.

Proof. In order to prove the result, we use Theorem 3.2.2. Therefore, we need to check three conditions for the process U.

(i) *The process $U(t, \tau)$ defined in (6.7) is norm-to weak continuous in the space $D(A) \times (H^2 \cap H_0^1)$.*

To prove this condition, we use Theorem 3.2.3.
In part (i) of the proof of Theorem 5.3.1 we have already shown that the process $U(t, \tau)$ is norm-to-weak continuous in the space $V \times H_0^1$. Therefore, in view of Theorem 3.2.3, we need to check that the process $U(t, \tau)$ maps compact sets in $D(A) \times (H^2 \cap H_0^1)$ to bounded ones in this space.
To this end, we prove that for any $\tau \in \mathbb{R}$ and $t > \tau$ the norms $||Au(t)||_{L_2}$, $||\triangle\omega(t)||_{L_2}$ are bounded uniformly with respect to initial conditions $u(\tau)$ and $\omega(\tau)$ belonging to a compact set.

In view of the estimate (6.8), we need to show the boundedness of all the expressions on the right-hand side of it, namely $||u_t||_{L_2}$, $||\omega_t||_{L_2}$ and $||u||_{H_0^1}$, $||\omega||_{H_0^1}$.

Inequality (2.22) derived in Lemma 2.3.2 assures about uniform boundedness of the norms $||u_t||_{L_2}$ and $||\omega_t||_{L_2}$.

As concerning the norms $||u||_{H_0^1}$, $||\omega||_{H_0^1}$, we have derived the following inequalities in the proof of Theorem 5.3.1

$$\begin{aligned} ||u(t)||_V^2 + ||\omega(t)||_{H_0^1}^2 &\leq \{||u(\tau)||_V^2 + ||\omega(\tau)||_{H_0^1}^2\} \exp(\int_\tau^t H(s)ds) \\ &+ \{\int_\tau^t F(\eta) \exp(-\int_\tau^\eta H(s)ds)d\eta\} \ \exp(\int_\tau^t H(s)ds) \end{aligned}$$

and

$$\begin{aligned} \int_\tau^t H(s)ds &\leq \tilde{C}\{\frac{1}{k_1}(||u(\tau)||_H^2 + ||\omega(\tau)||_{L_2}^2 + k_3 \int_\tau^t (||f(s)||_H^2 + ||g(s)||_{L_2}^2)ds)\}^2 \\ &+ C(t-\tau) < \infty. \end{aligned}$$

That finishes first part of the proof.

(ii) *There exists a family $\widehat{B}$ of pullback $\mathcal{D}$-absorbing sets in $D(A) \times (H_0^1 \cap H^2)$.*

As in the proof of the previous property, we start from the inequality (6.8). Hence, in order to estimate the expressions $||Au(t + r)||_{L_2}$ and $||\triangle\omega(t + r)||_{L_2}$, we need to derive appropriate estimates for all the terms on the right-hand side ($||u_t(t+r)||_{L_2}$, $||\omega_t(t+r)||_{L_2}$ and $||u(t+r)||_{H_0^1}$, $||\omega(t+r)||_{H_0^1}$).

In the proof of the Theorem 5.3.1 we have already derived the following inequality:

$$\begin{aligned} ||u(t+r)||_V^2 + ||\omega(t+r)||_{H_0^1}^2 &\leq c_2 e^{-k_2 t}\int_{-\infty}^{t+r} e^{k_2 s}(||f(s)||_H^2 + ||g(s)||_{L_2}^2)ds \\ &\cdot \ exp\{c_2 e^{-k_2 t}\int_{-\infty}^{t+r} e^{k_2 s}(||f(s)||_H^2 + ||g(s)||_{L_2}^2)ds\}^2, \end{aligned}$$

uniformly with respect to all initial conditions $u(\tau) \in D(\tau)$ for all $\tau \leq \tau_0(t, \widehat{D})$.

Therefore, we need to derive estimates on the expression $||u_t(t+r)||_{L_2} + ||\omega_t(t+r)||_{L_2}$. The inequality (6.9) reads

$$\begin{aligned} ||u_t(t+r)||_{L_2}^2 + ||\omega_t(t+r)||_{L_2}^2 &\leq \frac{1}{r}\int_t^{t+r}(||u_t(s)||_{L_2}^2 + ||\omega_t(s)||_{L_2}^2)ds\ e^{c_1\int_t^{t+r}H_1(s)ds} \\ &+ \ c_1\int_t^{t+r}(||f_t(s)||_{L_2}^2 + ||g_t(s)||_{L_2}^2)ds\ e^{c_1\int_t^{t+r}H_1(s)ds}. \end{aligned}$$

The integral $\int_t^{t+r} H_1(s)ds$ has been already estimated in the proof of Theorem 5.3.1 as

$$\int_t^{t+r} H_1(s)ds \leq c_2 e^{-k_2 t}\int_{-\infty}^{t+r} e^{k_2 s}(||f(s)||_{L_2}^2 + ||g(s)||_{L_2}^2)ds. \tag{6.17}$$

We have also the following inequality

$$\int_t^{t+r}(||f_t(s)||_{L_2}^2 + ||g_t(s)||_{L_2}^2)ds \leq e^{-k_2 t}\int_{-\infty}^{t+r} e^{k_2 s}(||f_t(s)||_{L_2}^2 + ||g_t(s)||_{L_2}^2)ds.$$

To estimate the integral $\int_t^{t+r} ||u_t(s)||_{L_2}^2 + ||\omega_t(s)||_{L_2}^2$, we recall Lemma 6.1.4

$$\begin{aligned} \int_t^{t+r} ||u_t(s)||_{L_2}^2 + ||\omega_t(s)||_{L_2}^2 &\leq c\int_t^{t+r}(||f(s)||_{L_2}^2 + ||g(s)||_{L_2}^2 + ||Au(s)||_{L_2}^2 + ||\triangle\omega(s)||_{L_2}^2 \\ &+ \ ||u(s)||_{H_0^1}^2 + ||\omega(s)||_{H_0^1}^2 + H(s)(||u||_V^2 + ||\omega||_{H_0^1}))ds. \end{aligned}$$

Hence, we need to look at the integral $\int_t^{t+r}(||Au||_{L_2} + ||\triangle\omega||_{L_2}^2)$ (since the other expressions have been already estimated).

The second energy inequality (2.20), after inegrating it over time, gives us

$$\begin{aligned} ||u(t)||_{H_0^1}^2 + ||\omega(t)||_{H_0^1}^2 &+ \ c(\nu,\alpha)\int_\tau^t(||Au(s)||_{L_2}^2 + ||\triangle\omega(s)||_{L_2}^2)ds \\ &\leq \ ||u(\tau)||_{H_0^1}^2 + ||\omega(\tau)||_{H_0^1}^2 + \int_\tau^t[H(s)(||u(s)||_{H_0^1}^2 + ||\omega(s)||_{H_0^1}^2) + F(s)]ds. \end{aligned}$$

which implies

$$\begin{aligned}\int_t^{t+r} & (||Au(s)||_{L_2}^2 + ||\triangle\omega(s)||_{L_2}^2)ds \\ \leq & \quad c(\nu,\alpha)[||u(t))||_{H_0^1}^2 + ||\omega(t)||_{H_0^1}^2 + \int_t^{t+r}(H(s)(||u||_{H_0^1}^2 + ||\omega||_{H_0^1}^2) + F(s))ds] \\ \leq & \quad c(\nu,\alpha)(||u(t)||_{H_0^1}^2 + ||\omega(t)||_{H_0^1}^2 \\ & + \int_t^{t+r}(||u||_{L_2}^2||u||_{H_0^1}^2 + ||u||_{L_2}^2||\omega||_{H_0^1}^2 + 1)(||u||_{H_0^1}^2 + ||\omega||_{H_0^1}^2) + F(s))ds).\end{aligned}$$

Therefore, we need to look carefully at the terms of the type $\int_t^{t+r} ||u||_{L_2}^2||u||_{H_0^1}^4$, since the rest of the expressions has been already estimated in the proof of the Theorem 5.3.1.

Let us recall, we already proved that

$$\begin{aligned}||u(t+r)||_V^2 + ||\omega(t+r)||_{H_0^1}^2 \quad \leq & \quad c_2 e^{-k_2 t}\int_{-\infty}^{t+r} e^{k_2 s}(||f(s)||_{L_2}^2 + ||g(s)||_{L_2}^2)ds \\ \cdot & \quad \exp\{e^{-k_2 t}\int_{-\infty}^{t+r} e^{k_2 s}(||f(s)||_{L_2}^2 + ||g(s)||_{L_2}^2)ds\}^2.\end{aligned}$$

For the sake of simplicity, we define the function

$$M(t) = \int_{-\infty}^{t} e^{k_2 s}(||f(s)||_{L_2}^2 + ||g(s)||_{L_2}^2)ds.$$

Since $M(t)$ is an integral of the expression that is positive, it is in fact a nondecreasing function. We will use this observation below. First, let us rewrite the above inequality as

$$||u(s)||_V^2 + ||\omega(s)||_{H_0^1}^2 \leq e^{-k_2(s-r)}M(s)\cdot\exp(e^{-2k_2(s-r)}M(s)^2).$$

The right-hand side, for any $s \in [t, t+r]$ can be estimated by

$$e^{-k_2(s-r)}M(t+r)\cdot\max\{\exp(M(t+r)^2); \exp[e^{-2k_2(t-r)}M(t+r)^2]\}.$$

The above shows that the norms $||u(s)||_V$ and $||\omega(s)||_{H_0^1}$ are integrable on $[t, t+r]$ with any power (hence, in particular with power 6) and the integral is clearly estimated independently of the choice of $\tau < t$.

Bringing together all the estimates above, we estimated the norm $||Au||_{L_2} + ||\triangle\omega||_{L_2}$ by the function dependent on t, r and the expression
$\int_{-\infty}^t e^{\lambda s}\{||f(s)||_H^2 + ||f_t(s)||_H^2 + ||g(s)||_{L_2}^2 + ||g_t(s)||_{L_2}\}ds$, uniformly with respect to initial conditions $u(\tau) \in D(\tau)$ for all $\tau \leq \tau_0(t, \widehat{D})$. Let us call this function k.

$$||Au||_{L_2} + ||\triangle\omega||_{L_2} \leq k\left(t, r, \int_{-\infty}^t e^{\lambda s}\{||f(s)||_H^2 + ||f_t(s)||_H^2 + ||g(s)||_{L_2}^2 + ||g_t(s)||_{L_2}\}ds\right). \quad (6.18)$$

(iii) $U(t,\tau)$ *is pullback $\mathcal{D}$-limit-set compact.*

Due to the Theorem 3.2.4 it is sufficient to show that the flattening condition holds. Hence, we need to look at the norms $||AQ_m u||_{L_2}$ and $||-\triangle Q_m\omega||_{L_2}$, where the operator Q_m was already

defined in the proof of the Theorem 5.3.1 in point *(iii)*.
Due to Lemma 6.1.5 we have the following estimates on these norms

$$||AQ_m u||_{L_2} \leq c(\nu_r,\nu)[||Q_m u_t||_{L_2} + ||Q_m B(u,u)||_{L_2} + ||Q_m\omega||_{H_0^1} + ||Q_m f||_{L_2}],$$

and

$$||\triangle Q_m\omega||_{L_2} \leq c(\alpha,\nu)[||Q_m\omega_t||_{L_2} + ||Q_m B_1(u,\omega)||_{L_2} + ||Q_m\omega||_{H_0^1} + ||Q_m u||_{H_0^1} + ||Q_m f||_{L_2}].$$

Let us start with estimating $||Q_m u_t||_{L_2}$ and $||Q_m\omega_t||_{L_2}$.
In Lemma 6.1.5 we have already derived the inequality

$$\begin{aligned}&||Q_m u_t(t+r)||_{L_2}^2 + ||Q_m\omega_t(t+r)||_{L_2}^2\\ &\leq e^{(-\beta_{m+1}+c)\,r}\int_t^{t+r}\{\frac{1}{r}||Q_m u_t(s)||_{L_2}^2 + ||Q_m\omega_t(s)||_{L_2}^2 + G(s)\}e^{\beta_m+1(s-t)}ds.\end{aligned}$$

where

$$\begin{aligned}G(s) &= c(\nu,\nu_r,\alpha)\cdot(||u_t||_{L_2}^2 + ||\omega_t||_{L_2}^2 + ||f_t||_{L_2}^2 + ||g_t||_{L_2}^2\\ &+ ||u_t||_{L_2}^2||u||_{H_0^1}^2 + ||u||_{L_2}^2||u_t||_{H_0^1}^2 + ||\omega_t||_{L_2}^2||u||_{H_0^1}^2 + ||u||_{L_2}^2||\omega_t||_{H_0^1}^2).\end{aligned}$$

Now, it suffices to notice that in view of (2.22) and (2.23), as well as (6.17), the integral is bounded uniformly with respect to m and initial conditions $u(\tau) \in D(\tau)$ for all $\tau \leq \tau_0(t,\widehat{D})$.
Next, we proceed exactly in the same way as in the proof of point (iii) of Theorem 5.3.1.

To show that the flattening condition holds, we only need to estimate the remaining terms $||Q_m B(u,u)||_{L_2}$ and $||Q_m B_1(u,\omega)||_{L_2}$. In Lemma 6.1.1 we proved that the operators B and B_1 are compact.
Therefore, for any $\epsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that for $m > m_0$ the terms $||Q_m B(u,u)||_{L_2}$ and $||Q_m B_1(u,\omega)||_{L_2}$ are smaller then ϵ uniformly for all (u,ω) in some bounded set in $D(A)\times H^2\cap H_0^1$.
Moreover, we have already proved in (ii) that process U possesses a family of pullback absorbing sets in $D(A)\times H^2\cap H_0^1$ (that are bounded). Hence, the proof of the flattening condition is completed.

□

We have just proved that under some, not very strong assumptions on the external forces and moments, the system of micropolar fluid equations possesses the pullback attractor in H^2. The natural question arises - is this H^2 pullback attractor the same object as the one in H^1.

Theorem 6.2.2. *Let $\mathcal{D}_1$, $\mathcal{D}_2$ be classes of all families $\{D_1(t) : t \in \mathbb{R}\}$ and $\{D_2(t) : t \in \mathbb{R}\}$ of nonempty subsets of $V\times H_0^1$ and $D(A)\times(H^2\cap H_0^1)$, respectively, such that*

$$\lim_{t\to-\infty} e^{\lambda t}[D_i(t)]^+ = 0,$$

where $[D_i(t)]^+ = \sup\{||u||_H^2 + ||\omega||_{L^2}^2;\ (u,\omega)\in D_i(t)\}$, $i=1,2$.
Let $\widehat{A}_1 = \{A_1(t)\}$ and $\widehat{A}_2 = \{A_2(t)\}$ denote the $\mathcal{D}_1$ and $\mathcal{D}_2-$pullback attractor for (6.1)-(6.5) in H^1 and in H^2, respectively.
Then $\widehat{A}_2 = \widehat{A}_1$.

Proof. Clearly, we need to show that for any $t \in \mathbb{R}$ we have $A_1(t) \subseteq \{A_2(t) : t \in \mathbb{R}\}$.
Hence, let us take an element $v \in A_1(t)$. We shall prove it belongs to $\widehat{A}_2$.

The invariance property for the pullback attractor holds: $U(t,\tau)A_1(\tau) = A_1(t)$ for all $\tau < t$. Hence, there exists a sequence $w_n \in A(\tau_n)$ such that

$$U(t,\tau_n)w_n = v.$$

And we also have

$$U(t,s)U(s,\tau_n)w_n = v \tag{6.19}$$

for all $\tau_n < s < t$.
From the proof of the point (ii) of the Theorem 6.2.1 we conclude that for any $s \in \mathbb{R}$ there exists a $\tau_0(s)$ such that $\{U(s,\tau_n)w_n\}_{\tau_n<s,\tau_n<\tau_0(s)}$ form a bounded set in $D(A) \times H^2$. All bounded sets in $D(A) \times H^2$ are attracted by the pullback attractor $\widehat{A_2}$. Hence, in view of (6.19), function v in fact belongs to $\widehat{A_2}$.
The inclusion $\widehat{A_2} \subset \widehat{A_1}$ is a corollary from proof of the point (ii) of the Theorem 6.2.1. □

Chapter 7

Stationary statistical solutions

This chapter is devoted to analysing statistical solutions for the equations

$$\frac{\partial u}{\partial t} - (\nu + \nu_r)\triangle u + (u \cdot \nabla)u + \nabla p = 2\nu_r rot\ \omega + f, \tag{7.1}$$

$$div\ u = 0, \tag{7.2}$$

$$\frac{\partial \omega}{\partial t} - \alpha\triangle\omega + (u \cdot \nabla)\omega + 4\nu_r\omega = 2\nu_r rot\ u + g. \tag{7.3}$$

We assume that homogeneous Dirichlet boundary conditions hold

$$u_{|\partial\Omega} = 0, \quad \omega_{|\partial\Omega} = 0, \tag{7.4}$$

and denote the initial conditions as

$$u(\tau) = u_0, \quad \omega(\tau) = \omega_0, \tag{7.5}$$

for $\tau \in I\!R$.
The domain Ω is a bounded subset of $\mathbb{R}^2$ with a smooth boundary. We assume $f \in H$ and $g \in L_2$.

Instead of looking at the system of equations with the initial data given by functions u_0 and ω_0, as we did so far, we look at the system equipped with the initial measure μ_0. Therefore, instead of looking at single solutions (as in a deterministic case), we will look at averaged expressions. If so, we have two natural way of averaging - over the space or over the time.

In the case of space averaging, we shall be interested in the evolution in time of the following type of expressions

$$\int_{\mathcal{H}} \Phi(u,\omega)d\mu_t(u,\omega) \tag{7.6}$$

for a certain functional Φ.
We prove the existence of the family of measures μ_t defined on $\mathcal{H}$ such that the Louville-type equation holds

$$\frac{d}{dt}\int_{\mathcal{H}} \Phi(u,\omega)d\mu_t(u,\omega) = \int_{\mathcal{H}} (F(u,\omega), \Phi'(u,\omega))d\mu_t(u,\omega). \tag{7.7}$$

where $(F(u,\omega),\phi)$ for any $\phi = (\phi_1;\phi_2) \in \mathcal{V}$ is a two-dimensional vector function. If we denote $F(u,\omega) = (F_1;F_2)$, then

$$(F_1(u,\omega),\phi_1) = -(\nu+\nu_r)(\nabla u,\nabla\phi_1) - b(u,u,\phi_1) + 2\nu_r(rot\omega,\phi_1) + (f,\phi_1), \tag{7.8}$$
$$(F_2(u,\omega),\phi_2) = -\alpha(\nabla\omega,\nabla\phi_2) - b_1(u,\omega,\phi_2) - 4\nu_r(\omega,\phi_2) + 2\nu_r(rot\ u,\phi_2) + (g,\phi_2).$$

The weak formulation of the equations (7.1)-(7.3) can be written as

$$(u_t,\phi_1) = (F_1(u,\omega),\phi_1), \tag{7.9}$$
$$(\omega_t,\phi_2) = (F_2(u,\omega),\phi_2).$$

The equation (7.7) comes naturally if we formally differentiate with respect to time the integral (7.6).

In the case of time avarages, we would like to look at

$$lim_{T\to\infty}\frac{1}{T}\int_0^T \Phi(U(t,0)(u_0,\omega_0))dt.$$

Since the classical limit may not exist, we use the notion of a generalized Banach limit.

In the sequel, we also see how space and time averages are connected with each other.
The plan for the following two chapters is as follows. We start with the problem (7.1)-(7.4) in the case when the forces on the right-hand side do not depend on time. We formulate the definition of stationary statistical solution and prove certain facts about this object.
Then, in the next chapter, we investigate the case when the external forces and moments depend on time. We also see how the statistical solutions are connected with the global and pullback attractors, respectively.

7.1 Stationary statistical solutions

Here, we are interested in the case when the forces and moments acting on the fluid do not depend on time. Then, in the statistical equilibrum, the expression (7.6) should not change with time. Hence, in the Louville-type equation (7.7), the left-hand side is equal to zero and the family of measures μ_t reduces to one probability measure μ. That is one of the points in the definition of the stationary statistical solutions. Before we formulate the whole definition, we define the class of functionals $\Phi(u,\omega)$.

Definition 7.1.1. *We will denote by $\mathcal{T}$ the class of real valued functionals $\Phi = \Phi(u,\omega)$ on $\mathcal{H}$ that are bounded on bounded subsets of $\mathcal{H}$ and such that*

1) For any $(u,\omega) \in \mathcal{V}$, the Frechet derivative $\Phi'(u,\omega)$ taken in $\mathcal{H}$ along $\mathcal{V}$ exists. Namely for each $(u,\omega) \in \mathcal{V}$ there exists $\Phi'(u,\omega) \in \mathcal{H}$ such that

$$\frac{\mid \Phi(u+v_1,\omega+v_2) - \Phi(u,\omega) - (\Phi'(u,\omega),v) \mid}{||v||_{\mathcal{V}}} \to 0$$

for every $v = (v_1,v_2) \in \mathcal{V}$, such that $||v||_{\mathcal{V}} \to 0$.

2) $\Phi'(u,\omega) \in \mathcal{V}$ for all $(u,\omega) \in \mathcal{V}$ and the mapping $(u,\omega) \to \Phi'(u,\omega)$ is continuous and bounded as a function from $\mathcal{V}$ into $\mathcal{V}$.

Definition 7.1.2. *A stationary statistical solution of the micropolar fluid equations is a probability Borel measure μ defined on $\mathcal{H}$ such that*

(i)

$$\int_{\mathcal{H}} (\| u \|_V^2 + \| \omega \|_{H_0^1}^2) d\mu(u,\omega) < \infty,$$

(ii)

$$\int_{\mathcal{H}} (F(u,\omega), \Phi'(u,\omega)) d\mu(u,\omega) = 0$$

for all test functionals $\Phi \in \mathcal{T}$, with F defined in (7.8),

(iii)

$$\int_{E_1 \leq \|u\|_H^2 + \|\omega\|_{L_2}^2 < E_2} (k_1(\| u \|_V^2 + \| \omega \|_{H_0^1}^2) - (f,u) - (g,\omega)) d\mu(u,\omega) \leq 0$$

for all $E_1, E_2 : 0 \leq E_1 < E_2 \leq \infty$ where $k_1 = min\{\nu, \alpha\}$.

Actually, the existence of the statistical solutions is immediate since we know that in two dimensions a weak solution of the stationary problem exists. In fact, we prove the following proposition

Proposition 7.1.1. *Let (u_*, ω_*) denote the stationary weak solution of the problem (7.1)- (7.5). Then a stationary statistical solution for (7.1)- (7.5) exists, namely $\mu(u,\omega) = \delta(u - u_*, \omega - \omega_*)$, where δ denotes the Dirac delta function.*

Proof. We shall show $\mu(u,\omega) = \delta(u - u_*, \omega - \omega_*)$ is a stationary statistical solution.

In fact,

$$\int_{\mathcal{H}} (\| u \|_V^2 + \| \omega \|_{H_0^1}^2) d\delta(u - u_*, \omega - \omega_*) = \| u_* \|_V^2 + \| \omega_* \|_{H_0^1}^2$$

which is finite due to the energy inequality (2.18).

The Louville-type equations holds, since

$$\int_{\mathcal{H}} (F(u,\omega), \Phi'(u,\omega)) d\delta(u - u_*, \omega - \omega_*) = (F(u_*,\omega_*), \Phi'(u_*,\omega_*))$$

and

$$\frac{d}{dt}\Phi(u_*(t), \omega_*(t)) = (\Phi'(u_*(t), \omega_*(t)), (u_{*_t}(t), \omega_{*_t}(t)) = (\Phi'(u_*(t), \omega_*(t)), F(u_*(t), \omega_*(t))) = 0.$$

since for the stationary solution (u_*, ω_*) the expression $F(u_*, \omega_*)$ is equal to zero.

Moreover,

$$\int_{E_1 \leq \|u\|_H^2 + \|\omega\|_{L_2}^2 < E_2} (k_1(\| u \|_V^2 + \| \omega \|_{H_0^1}^2) - (f,u) - (g,\omega)) d\delta(u - u_*, \omega - \omega_*)$$
$$= \chi_{\{E_1 \leq \|u_*\|_H^2 + \|\omega_*\|_{L_2}^2 < E_2\}} (k_1(\| u_* \|_V^2 + \| \omega_* \|_{H_0^1}^2) - (f,u_*) - (g,\omega_*)) \leq 0$$

because (2.17) holds. The function χ is a characteristic function.

□

Nonetheless, more important thing than simply having such a solution is a possibility to associate it with time-average measures and investigate its other properties.
To do so, we need some more definitions.

Definition 7.1.3. *A probability measure on $\mathcal{H}$ is called an invariant measure for the semigroup $\{S(t)\}_{t\geq 0}$ if*

$$\mu(E) = \mu(S(t)^{-1}E)$$

for all $t \geq 0$ and all measurable sets $E \subset \mathcal{H}$.

Definition 7.1.4. *A generalized limit (denoted as $LIM_{T\to\infty}$) is a linear functional defined on $\mathcal{B}([0,\infty))$ (bounded real valued functions on $[0,\infty)$). that satisfies*
1) $LIM_{T\to\infty} g(T) \geq 0$ for all $g \in \mathcal{B}([0,\infty))$ such that $g(s) \geq 0$ for all $s \geq 0$;
2) $LIM_{T\to\infty} g(T) = lim_{T\to\infty} g(T)$ for all $g \in \mathcal{B}([0,\infty))$ such that the classical limit exists.

In our further considerations, we are interested in a certain form of the function g from the above definition, namely

$$g(T) = \frac{1}{T}\int_0^T \psi(S(t)(u_0,\omega_0))dt \tag{7.10}$$

where $\psi \in \mathcal{C}(\mathcal{H})$ and $(u_0,\omega_0) \in \mathcal{H}$.
In order to consider generalized limit $LIM_{T\to\infty}$ of these expressions, we need to show that the averages (7.10) belong to the space $\mathcal{B}([0,\infty))$

It was shown in [27] that there exists a closed and bounded absorbing set $\mathcal{B} \in \mathcal{V}$ for the semigroup $\{S(t)\}_{t\geq 0}$. Hence, recalling the Rellich-Kondrashow theorem on the compact embedding, $\mathcal{B}$ is a compact set in $\mathcal{H}$.
Moreover, form the definition of an absorbing set, there exists $t_0 \geq 0$ such that $S(t)(u_0,\omega_0) \in \mathcal{B}$ for $t \geq t_0$. Let us notice that these two facts imply that $\{S(t)(u_0,\omega_0)\}_{t\geq 0}$ is compact into $\mathcal{H}$ (recalling that the map $t \to (u(t),\omega(t))$ is continuous as a map in $\mathcal{H}$). Hence, the map $t \to \psi(S(t)(u_0,\omega_0))$ is bounded and continuous (since $\psi \in \mathcal{C}(\mathcal{H})$) as a map from $\mathbb{R}_+$ into $\mathcal{H}$. Thus, $\frac{1}{T}\int_0^T \psi(S(t)(u_0,\omega_0))dt$ belongs to $\mathcal{B}([0,\infty))$.

Now, we introduce the concept of the time-average measure of the solution. To this end, let us fix any functional $LIM_{T\to\infty}$ satisfying Definition 7.1.4.

Definition 7.1.5. *Let $(u_0,\omega_0) \in \mathcal{H}$.*
A probability measure μ on $\mathcal{H}$ such that

$$LIM_{T\to\infty}\frac{1}{T}\int_0^T \phi(S(t)(u_0,\omega_0))dt = \int_{\mathcal{H}} \phi(u,\omega)d\mu(u,\omega) \tag{7.11}$$

for all $\phi \in \mathcal{C}(\mathcal{H})$, is called a time-average measure of the solution through (u_0,ω_0).

As we can see, the measure depends on the choice of the functional, hence in general we do not have the uniqueness of the time-average measure.
Now, we will use Tietze Extension Theorem and Kakutani-Riesz Representation Theorem to show that such a measure exists indeed. The last mentioned theorem reads

Theorem 7.1.1 (Kakutani-Riesz Representation Theorem)**.** *Let Ψ be a positive linear continuous functional on the space of real-valued continuous functions $C(S)$, where S is a compact Hausdorff space. Then there exists a Borel regular measure μ on S such that*

$$\Psi(f) = \int_S f d\mu$$

for all $f \in C(S)$.
Moreover, $||\Psi|| = |\mu|(S)$.

Proposition 7.1.2. *For any initial condition $(u_0, \omega_0) \in \mathcal{H}$ and any generalized limit $LIM_{T\to\infty}$, there exists a time-average measure $\mu(u, \omega)$ through (u_0, ω_0).*

Proof. Let us choose $u_0 \in H$ and $\omega_0 \in L^2$. Let us also fix a functional $LIM_{T\to\infty}$.
We have already shown that for any $\psi \in C(\mathcal{H})$, the expression $LIM_{T\to\infty}\frac{1}{T}\int_0^T \psi(S(t)(u_0, \omega_0))dt$ is well defined.
For the sake of simplicity, let

$$L(\psi) = LIM_{T\to\infty}\frac{1}{T}\int_0^T \psi(S(t)(u_0, \omega_0))dt.$$

From the definition of Banach generalized limit $L(\psi)$ is a linear functional and it can be shown that for any function $\psi \in C(\mathcal{H})$, $L(\psi)$ depends in fact only on the restriction of ψ to the absorbing set $\mathcal{B}$.
Moreover, using the Tietze Extension Theorem, we can see that functions $\psi|_{\mathcal{B}}$ (for $\psi \in \mathcal{C}(\mathcal{H})$) form all the space $\mathcal{C}(\mathcal{B})$.
Hence, we can define a fuctional $G(\psi_{|\mathcal{B}})$ on $C(\mathcal{B})$ as $G(\psi_{|\mathcal{B}}) = L(\psi)$. Since $\mathcal{B}$ is compact, we can use the Kakutani-Riesz Representation Theorem to see that there exists a measure μ on $\mathcal{B}$ such that

$$G(\psi_{|\mathcal{B}}) = \int_{\mathcal{B}} \psi(u, \omega)d\mu(u, \omega)$$

for any $\psi \in C(\mathcal{B})$.
Next, we can extend this measure to the whole $\mathcal{H}$ by zero outside the set $\mathcal{B}$. That means $\mu(\mathcal{H} - \mathcal{B}) = 0$ and clearly (7.11) holds. □

Now, we formulate an invariance property of the time-average measures.

Proposition 7.1.3. *Any time-average measure is invariant for the micropolar fluid semigroup $\{S(t)\}_{t\geq 0}$.*

In order to prove the proposition it has to be shown that

$$\int \psi(S(\tau)(u, \omega))d\mu = \int \psi(u, \omega)d\mu$$

for all $\tau > 0$ and $\psi \in \mathcal{C}(\mathcal{H})$. Then by the density argument we can pass to the above equation holding for $\psi \in L^1(\mu)$). Since the details of the proof are quite elementary, we will omit them (calculations are similar to that done in [20]).

Now, we have the following result on the support of any invariant measure for our problem

Lemma 7.1.1. *Let μ be an invariant probability measure for $\{S(t)\}_{t\geq 0}$ and let $\mathcal{B}$ be an absorbing set for the semigroup. Then $\mu(\mathcal{H}-\mathcal{B})=0$.*

Proof. The measure μ is invariant, form which we know that for all $t>0$ the following holds: $\mu(\mathcal{B})=\mu(S(t)^{-1}\mathcal{B})$. Now, if we take a sequence of balls $B_n=\{(u,\omega)\in\mathcal{H}: ||(u,\omega)||_{\mathcal{H}}\leq n\}$, there exist a sequence of times $\{t_n\}_{n=1}^{\infty}$ such that $S(t_n)B_n\subset\mathcal{B}$ ($\mathcal{B}$ is absorbing for the semigroup). Let us observe that

$$\mu(B_n)\leq\mu(S(t_n)^{-1}\mathcal{B})=\mu(\mathcal{B})\leq 1.$$

Since $\mathcal{H}=\bigcup_{n=1}^{\infty}B_n$, the lemma holds. □

Theorem 7.1.2. *A probability measure μ that is invariant for the semigroup $\{S(t)\}_{t\geq 0}$ has its support in the global attractor $\widehat{A}$.*

Proof. The global attractor can be defined as $\widehat{A}=\bigcap_{n\in\mathbb{N}}S(t_n)\mathcal{B}$ where $\mathcal{B}$ is a compact absorbing set for the problem. There exists a time $t_{\mathcal{B}}>0$ such that for all $t\geq t_{\mathcal{B}}$ we have $S(t)\mathcal{B}\subset\mathcal{B}$. Since $S(t)$ is a semigroup and $\mathcal{B}$ is an absorbing set, we have

$$S\left((n+1)t_{\mathcal{B}}\right)\mathcal{B}\subset S(nt_{\mathcal{B}})\mathcal{B}$$

for all $n\in\mathbb{N}$. Hence,

$$\mu(\widehat{A})=lim_{n\to\infty}\mu(S(nt_{\mathcal{B}})\mathcal{B}).$$

All the sets on the right are absorbing and therefore they are of measure 1 (due to Lemma 7.1.1) which finishes the proof. □

Now, we formulate and prove a result concerning a relationship between the stationary solutions and invariant measures. We shall see that in fact, in our two-dimensional problem these concepts are equivalent.

Theorem 7.1.3. *Let μ be a probability measure on $\mathcal{H}$. Then μ is invariant for $\{S(t)\}_{t\geq 0}$ if and only if it is a stationary statistical solution of (7.1)-(7.4).*

Proof:

Invariant measure ⇒ statistical solution

We check all three conditions from Definition 7.1.2.

(i) This one is obvious if we recall Lemma 7.1.1 stating that the measure is supported on the absorbing set $\mathcal{B}$ which belongs to the space $\mathcal{V}$.

(iii) In order to check the third condition from Definition 7.1.2, let us notice that Lemma 7.1.1 assures that the expression $(k_1(||\ u\ ||_V^2+||\ \omega\ ||_{H_0^1}^2)-(f,u)-(g,\omega))$ is integrable with respect to the measure μ.

Fist, we show that the following inequality holds

$$\int_{\mathcal{H}}\rho'(||u||_H^2+||\omega||_{L_2}^2)(k_1(||u||_V^2+||\omega||_{H_0^1}^2)-(f,u)-(g,\omega))d\mu(u,\omega)\leq 0$$

for all bounded functions $\rho\in C^1([0,\infty))$ such that $\rho'(s)\geq 0$.

Since this measure is invariant for the semigroup $\{S(t)\}$, denoting $(u(t),\omega(t)) = S(t)(u,\omega)$, we have

$$\int_{\mathcal{H}} \rho'(||u||_H^2 + ||\omega||_{L_2}^2)(k_1(||u||_V^2 + ||\omega||_{H_0^1}^2) - (f,u) - (g,\omega))d\mu(u,\omega)$$
$$= \int_{\mathcal{H}} \rho'(||u(t)||_H^2 + ||\omega(t)||_{L_2}^2)(k_1(||u(t)||_V^2 + ||\omega(t)||_{H_0^1}^2) - (f,u(t)) - (g,\omega(t))d\mu(u,\omega)$$

for all $t \geq 0$.
Integrating it in respect to time over interval $(0,T)$ and dividing by T (the right-hand side is integrable over t due to the Theorem 2.3.1), and finally using the Fubini Theorem, we arrive at

$$\int_{\mathcal{H}} \rho'(||u||_H^2 + ||\omega||_{L_2}^2)(k_1(||u||_V^2 + ||\omega||_{H_0^1}^2) - (f,u) - (g,\omega))d\mu(u,\omega)$$
$$= \int_{\mathcal{H}} \frac{1}{T}\int_0^T \rho'(||u(t)||_H^2 + ||\omega(t)||_{L_2}^2)(k_1(||u(t)||_V^2 + ||\omega(t)||_{H_0^1}^2) - (f,u(t)) - (g,\omega(t)))dtd\mu(u,\omega).$$

Recalling the energy inequality (2.17), we can estimate the right-hand side as

$$\int_{\mathcal{H}} \rho'(||u||_H^2 + ||\omega||_{L_2}^2)(k_1(||u||_V^2 + ||\omega||_{H_0^1}^2) - (f,u) - (g,\omega))d\mu(u,\omega)$$
$$\leq \int_{\mathcal{H}} \frac{1}{T}\int_0^T [-\frac{d}{dt}\rho(||u(t)||_H^2 + ||\omega(t)||_{L_2}^2)]dtd\mu(u,\omega)$$
$$= \int_{\mathcal{H}} \frac{1}{T}[\rho(||u(0)||_H^2 + ||\omega(0)||_{L_2}^2) - \rho(||u(T)||_H^2 + ||\omega(T)||_{L_2}^2)]d\mu(u,\omega).$$

Taking into account that the map $t \to S(t)(u,\omega)$ is bounded in $\mathcal{H}$ and letting $T \to \infty$, we arrive at

$$\int_{\mathcal{H}} \rho'(||u||_H^2 + ||\omega||_{L_2}^2)((k_1(||u||_V^2 + ||\omega||_{H_0^1}^2) - (f,u) - (g,\omega))d\mu(u,\omega) \leq 0$$

since ρ is a bounded function.
The characteristic functions can be approximated in L^1 by the derivatives of functions ρ. Hence, the third point of Definition 3.3 is proved.

(ii) Now, we show that $\int_{\mathcal{H}}(F(u,\omega),\Phi'(u,\omega))d\mu(u,\omega) = 0$ for all test functionals $\Phi \in \mathcal{T}$. It is easier to prove it if Φ is bounded in $\mathcal{H}$. Hence, we define a bounded functional Φ_m, for which we prove that the above equality holds, and then pass with m go to infinity.

Let us fix a test functional $\Phi \in \mathcal{T}$. We consider Φ_m defined as

$$\Phi_m(u,\omega) = \Phi(P_m(u,\omega)),$$

where P_m is the orthogonal projector $P_m : \mathcal{H} \to H_m \times (L^2)_m$.
The space H_m is spanned by the first m eigenvectors $w_1, ...w_m$ of the Stokes operator.
The space $(L^2)^m$ is spanned by the first m eigenvectors $\rho_1, ...\rho_m$ of $-\triangle$ in L^2 with the domain

$H^2 \cap H^1_0$.
More precisely, if $u = \sum_{k=1}^{\infty} \widehat{u_k} w_k$ and $\omega = \sum_{k=1}^{\infty} \widehat{\omega_k} \rho_k$, then

$$P_m(u,\omega) = (\sum_{k=1}^{m} \widehat{u_k} w_k, \sum_{k=1}^{m} \widehat{\omega_k} \rho_k).$$

In the sequel we will need a projection operator that acts only on a single function u or ω. We shall use the same notation then ($P_m u = \sum_{k=1}^{m} \widehat{u_k} w_k$ and $P_m \omega = \sum_{k=1}^{m} \widehat{\omega_k} \rho_k$) not to multiply notation.

Since Φ is bounded on a bounded subsets of $\mathcal{H}$, Φ_m is in fact bounded on $\mathcal{H}$.
Let us notice that the Frechet derivative of Φ_m can be written as $\Phi'_m(u,\omega) = \Phi'(P_m(u,\omega))P_m$. Hence Φ_m is a C^1 functional on $\mathcal{H}$. Moreover, $t \to S(t)(u,\omega)$ is $\mathcal{C}^1$ as the map from $[0,\infty)$ into $\mathcal{H}$ if we only take $u \in D(A)$ and $\omega \in H^2 \cap H^1_0$.
Therefore we can write the time derivative of $\Phi_m(S(t)(u,\omega))$ for $u \in D(A)$ and $\omega \in H^2 \cap H^1_0$ as

$$\frac{d}{dt}(\Phi_m(S(t)(u,\omega))) = (F(S(t)(u,\omega)), \Phi'_m(S(t)(u,\omega))) \tag{7.12}$$

for all $t \geq 0$.
The measure $\mu(u,\omega)$ is invariant, that is

$$\int (F(u,\omega), \Phi'_m(u,\omega)) d\mu(u,\omega) = \int (F(S(t)(u,\omega), \Phi'_m(S(t)(u,\omega))) d\mu(u,\omega)$$

and when we integrate the above with respect to time and divide by T, we get

$$\int (F(u,\omega), \Phi'_m(u,\omega)) d\mu(u,\omega) = \int \int_0^T \frac{1}{T}(F(S(t)(u,\omega), \Phi'_m(S(t)(u,\omega))) dt d\mu(u,\omega).$$

Now, we can use (7.12) and Lemma 3.1 to see that

$$\int (F(u,\omega), \Phi'_m(u,\omega)) d\mu(u,\omega) = \int \frac{1}{T}(\Phi_m(S(T)(u,\omega)) - \Phi_m(u,\omega)) d\mu(u,\omega).$$

Letting $T \to \infty$, we arrive at

$$\int (F(u,\omega), \Phi'_m(u,\omega)) d\mu(u,\omega) = 0$$

since Φ_m is bounded.

Now, we pass with m to infinity. To this end, use the Lebesgue dominated convergence theorem. First, we rewrite the left-hand side of the above equality:

$$0 = \int (F(u,\omega), \Phi'_m(u,\omega)) d\mu(u,\omega) = \int (P_m F(u,\omega), \Phi'(P_m(u,\omega))) d\mu(u,\omega)$$

$$= \int (F(u,\omega), P_m \Phi'(P_m(u,\omega))) d\mu(u,\omega).$$

Let us notice that $P_m(u,\omega) \to (u,\omega)$ in $\mathcal{V}$ for all $(u,\omega) \in \mathcal{V}$. Hence, in view of the Lemma 3.1 we have the following convergence $\mu - a.e.$

$$(F(u,\omega), P_m\Phi'(P_m(u,\omega)) \to (F(u,\omega), \Phi'(u,\omega)).$$

Moreover, Φ' is bounded (since Φ belongs to $\mathcal{T}$ and hence Φ' is bounded as a function from $\mathcal{V}$ to $\mathcal{V}$). Therefore, we have

$$(F(u,\omega), P_m\Phi'(P_m(u,\omega))) \le c(\| u \|_V + \| \omega \|_{H_0^1} + \| u \|_V^2 + \| \omega \|_{H_0^1}^2 + \|f\|_H + \|g\|_{L_2})$$

where the constat c depends on Φ.
Now, the Lebesgue Dominated Convergence Theorem gives

$$\int (F(u,\omega), \Phi'(u,\omega))d\mu(u,\omega) = lim_{m\to\infty} \int (F(u,\omega)P_m\Phi'(P_m(u,\omega)))d\mu(u,\omega) = 0$$

which finishes the first part of the proof.

Statistical solution $\Rightarrow$ invariant measure
We will show that the following equality is satisfied for any $\Phi \in \mathcal{T}$

$$\int \Phi(S(t)(u,\omega))d\mu(u,\omega) = \int \Phi(u,\omega)d\mu(u,\omega)$$

for all $t \ge 0$.
We do it using Galerkin approximations. That is, first, we will prove that

$$\int \Phi(S_m(T)(u,\omega))d\mu(u,\omega) - \int \Phi(P_m(u,\omega)d\mu(u,\omega) \to 0,$$

where $S_m(t)(u,\omega) = (u_m(t), \omega_m(t))$ is the Galerkin approximation of the solutions of (7.1)-(7.5)).

Let us take a test functional $\Phi \in \mathcal{T}$. Then for any $(u,\omega) \in \mathcal{H}$ and all $t \in \mathbb{R}$, $S_m(t)(u,\omega) \in \mathcal{V}$. Hence, we can differentiate the expression $\Phi(S_m(t)(u,\omega))$ and we get

$$\frac{d}{dt}\Phi(S_m(t)(u,\omega)) = (P_mF(S_m(t)(u,\omega)), \Phi'(S_m(t)(u,\omega))).$$

Integrating the above over $\mathcal{H}$ and over time (which can be done due to Lemma 3.1) we arrive at

$$\begin{aligned} \int \Phi(S_m(T)(u,\omega)d\mu(u,\omega) &= \int \Phi(P_m(u,\omega))d\mu(u,\omega) \\ &+ \int_0^T \int (P_mF(S_m(t)(u,\omega)), \Phi'(S_m(t)(u,\omega)))d\mu(u,\omega)dt. \end{aligned} \tag{7.13}$$

Now, our goal is to show that the last integral vanishes as m tends to infinity. But first of all we would like to "replace" somehow $P_mF(S_m(t)(u,\omega))$ by $P_mF(P_m(u,\omega))$.

Let us look more carefully at the derivative in (7.13) (the measure μ is a stationary solution for our problem, hence we know the Louville-type equality holds).
First, let us define the following functional

$$\Phi_m(t,(u,\omega)) = \Phi(S_m(t)(u,\omega))$$

for all $(u,\omega) \in \mathcal{H}$ and all $t \in \mathbb{R}$. Then, if we fix arbitrary $t \in \mathbb{R}$, $\Phi_m(t,(u,\omega))$ is a test functional with a Frechet derivative

$$(\Phi_m)'_{(u,\omega)}(t,(u,\omega)) = (D_{(u,\omega)}(S_m(t)(u,\omega)))^*(\Phi'(S_m(t)(u,\omega)))$$

where $(D_{(u,\omega)}(S_m(t)(u,\omega))^*$ is an adjoint to $D_{(u,\omega)}(S_m(t)(u,\omega))$ and is a bounded operator on $\mathcal{H}$. The element (u,ω) belongs to $\mathcal{H}$.
(The operator $(u,\omega) \to S_m(t)(u,\omega)$ is Frechet differentiable as an operator in $\mathcal{H}$ and its derivative in every $(u,\omega) \in \mathcal{H}$ is a bounded operator in $\mathcal{H}$).
Now, it can be shown that

$$(P_m F(S_m(t)(u,\omega)), \Phi'(S_m(t)(u,\omega))) = (P_m F(P_m(u,\omega)), (\Phi_m)'_{(u,\omega)}(t,(u,\omega)).$$

The proof is the same as in [20] so we will not rewrite it. The idea is to show that both sides of the above are equal to $(\Phi_m)'_t(t, P_m(u,\omega))$. Let us only mention that the group property of $\{S_m(t)\}_{t\in\mathbb{R}}$ is used in this place.
Hence, in place of the last integral in (7.13) we can put

$$\int_0^T \int (P_m F(P_m(u,\omega)), (\Phi_m)'_{(u,\omega)}(t,(u,\omega)))d\mu(u,\omega)dt. \tag{7.14}$$

Now, we will show that (7.14) tends to 0. We use the following observation: recalling that $(u,\omega) \to \Phi_m(t,(u,\omega))$ is a test functional from the set $\mathcal{T}$ for any fixed $t \in \mathbb{R}$,

$$\int (F(u,\omega), (\Phi_m)'_{(u,\omega)}(t,(u,\omega)))d\mu(u,\omega) = 0,$$

since μ is a stationary solution. Therefore

$$\begin{aligned} &\int_0^T \int (P_m F(P_m(u,\omega)), (\Phi_m)'_{(u,\omega)}(t,(u,\omega)))d\mu(u,\omega)dt \\ &= \int_0^T \int \{(P_m F(P_m(u,\omega)) - F(u,\omega), (\Phi_m)'_{(u,\omega)}(t,(u,\omega)))\}d\mu(u,\omega)dt. \end{aligned}$$

Due to the above equality, we shall look at the last integral instead of (7.14).

First of all, $\Phi_m(t,(u,\omega))$ depends only on $P_m(u,\omega)$. Hence, $(\Phi_m)'_{(u,\omega)}(t,(u,\omega)) = P_m(\Phi_m)'_{(u,\omega)}(t,(u,\omega))$ and

$$\begin{aligned} \int \Phi(S_m(T)(u,\omega))d\mu(u,\omega) &= \int \Phi(P_m(u,\omega)d\mu(u,\omega) \\ &+ \int_0^T \int (P_m F(P_m(u,\omega)) - P_m F(u,\omega), (\Phi_m)'_{(u,\omega)}(t,(u,\omega)))d\mu(u,\omega)dt. \end{aligned}$$

The only terms that do not vanish in the expression $P_m F(P_m(u,\omega)) - P_m F(u,\omega)$ are the ones connected with the bilinear operator $\mathcal{B} = (B, B_1)$. More precisely,

$$P_m F(P_m(u,\omega)) - P_m F(u,\omega) = P_m \mathcal{B}((u,\omega),(u,\omega)) - P_m \mathcal{B}(P_m(u,\omega), P_m(u,\omega)).$$

Therefore, we need to show that

$$\int_0^T \int (\mathcal{B}(P_m(u,\omega), P_m(u,\omega)) - \mathcal{B}((u,\omega),(u,\omega)), (\Phi_m)'_{(u,\omega)}(t,(u,\omega)))d\mu(u,\omega)dt \tag{7.15}$$

vanishes as m goes to infinity.

First, $(\Phi_m)'_{(u,\omega)}(t,(u,\omega))$ is an operator with values in $\mathcal{H}$. It will be more convenient for the moment to look at $(\Phi_m)'_{(u,\omega)}(t,(u,\omega))$ as at the object with two components, namely $(\Phi_m)'_{(u,\omega)}(t,(u,\omega)) = (\Theta_u, \Theta_\omega)$. Then we have

$$\begin{aligned}
&| \, (\mathcal{B}(u,\omega) - \mathcal{B}(P_m(u,\omega)), (\Phi_m)'_{(u,\omega)}(t,(u,\omega))) \, | \\
&\leq |b(u,u,\Theta_u) - b(P_m u, P_m u, \Theta_u) \, | + | \, b_1(u,\omega,\Theta_\omega) - b_1(P_m u, P_m\omega, \Theta_\omega) \, | \, .
\end{aligned}$$

Now, we estimate the trilinear forms b and b_1 as

$$\begin{aligned}
&| \, b_1(u,\omega,\Theta_\omega) - b_1(P_m u, P_m\omega, \Theta_\omega) \, | \\
&\qquad \leq c_1 ||u - P_m u||_H^{1/2} \, || \, u - P_m u \, ||_V^{1/2} \, ||\omega||_{L_2}^{1/2} \, || \, \omega \, ||_{H_0^1}^{1/2} || \, \Theta_\omega \, ||_{H_0^1} \\
&\qquad +c_1 ||P_m u||_H^{1/2} \, || \, P_m u \, ||_V^{1/2} \, ||\omega - P_m\omega||_{L_2}^{1/2} \, || \, \omega - P_m\omega \, ||_{H_0^1}^{1/2} || \, \Theta_\omega \, ||_{H_0^1} \\
&\qquad \leq c_1 ||u - P_m u||_H^{1/2} \, || \, u - P_m u \, ||_V^{1/2} \, ||\omega||_{L_2}^{1/2} \, || \, \omega \, ||_{H_0^1}^{1/2} || \, \Theta_\omega \, ||_{H_0^1} \\
&\qquad +c_1 ||u||_H^{1/2} \, || \, u \, ||_V^{1/2} \, ||\omega - P_m\omega||_{L_2}^{1/2} \, || \, \omega - P_m\omega \, ||_{H_0^1}^{1/2} || \, \Theta_\omega \, ||_{H_0^1}
\end{aligned}$$

and in the similar way

$$\begin{aligned}
&| \, b(u,u,\Theta_u) - b(P_m u, P_m u, \Theta_u) \, | \\
&\qquad \leq 2c_1 ||u - P_m u||_H^{1/2} \, || \, u - P_m u \, ||_V^{1/2} \, ||u||_H^{1/2} \, || \, u \, ||_V^{1/2} || \, \Theta_u \, ||_V \, .
\end{aligned}$$

Using the Poincaré inequality

$$||u - P_m u||_H^2 \leq \frac{1}{\lambda_m} ||u - P_m u \, ||_V^2$$

and

$$||\omega - P_m\omega||_{L_2}^2 \leq \frac{1}{\sigma_m} \, || \, \omega - P_m\omega \, ||_{H_0^1}^2$$

we arrive at

$$\begin{aligned}
&| \, (\mathcal{B}(u,\omega) - \mathcal{B}(P_m(u,\omega)), (\Phi_m)'_{(u,\omega)}(t,(u,\omega))) \, | \\
&\qquad \leq c_1 \frac{1}{\lambda_m^{1/4}} \, || \, u - P_m u \, ||_V \;\; ||u||_H^{1/2} \, || \, u \, ||_V^{1/2} || \, \Theta_u \, ||_V \\
&\qquad +c_1 \frac{1}{\sigma_m^{1/4}} \, || \, u - P_m u \, ||_V \;\; ||\omega||_H^{1/2} \, || \, \omega \, ||_{H_0^1}^{1/2} || \, \Theta_\omega \, ||_{H_0^1} \\
&\qquad +c_1 \frac{1}{\sigma_m^{1/4}} \, || \, \omega - P_m\omega \, ||_{H_0^1} \;\; ||u||_H^{1/2} \, || \, u \, ||_V^{1/2} || \, \Theta_\omega \, ||_{H_0^1} \, .
\end{aligned}$$

and

$$| (\mathcal{B}(u,\omega) - \mathcal{B}(P_m(u,\omega)), (\Phi_m)'_{(u,\omega)}(t,(u,\omega))) | \leq c_1 \frac{1}{\lambda_1^{1/4}\lambda_m^{1/4}} \| u \|_V^2 \| \Theta_u \|_V \tag{7.16}$$
$$+ c_1 \frac{1}{\sigma_1^{1/4}\sigma_m^{1/4}} \| u \|_V \cdot \| \omega \|_{H_0^1} \cdot \| \Theta_\omega \|_{H_0^1} + c_1 \frac{1}{\lambda_1^{1/4}\sigma_m^{1/4}} \| u \|_V \cdot \| \omega \|_{H_0^1} \cdot \| \Theta_\omega \|_{H_0^1} .$$

In order to show that the integral (7.15) tends to zero when m goes to infinity, it suffices to estimate the norm of $||(\Phi_m)'_{(u,\omega)}(t,(u,\omega)||_{\mathcal{V}}$ by a function of t that is integrable on the interval $(0,T)$. Indeed, λ_m and σ_m go to infinity with increasing m, and the expression $\int_{\mathcal{H}} ||(u,\omega)||_{\mathcal{V}} d\mu$ is bounded since we assumed μ is a stationary solution.

First, we define the operator $\widehat{A}$ as follows:

$$\widehat{A}(u,\omega) = (Au, -\triangle\omega)$$

where A is the Stokes operator. Moreover, we have $||(u,\omega)||_{\mathcal{V}} = ||\widehat{A}^{1/2}(u,\omega)||_{\mathcal{H}}$ for all $(u,\omega) \in \mathcal{H}$.

Taking into account the above notation, in view of the Riesz representation theorem we have

$$||(\Phi_m)'_{(u,\omega)}||_{\mathcal{V}} = ||\widehat{A}^{1/2}(\Phi_m)'_{(u,\omega)}||_{\mathcal{H}} = sup_{v\in\mathcal{H},|v|\leq 1}(\widehat{A}^{1/2}(\Phi_m)'_{(u,\omega)}, v). \tag{7.17}$$

The space $\mathcal{H}$ is dense in $\mathcal{V}$ and $\mathcal{A}^{1/2}$ is a symetric operator on $\mathcal{H}$. Moreover,

$$(\Phi_m)'_{(u,\omega)}(t,(u,\omega)) = (D_{(u,\omega)}(S_m(t)(u,\omega)))^*(\Phi'(S_m(t)(u,\omega))).$$

Therefore, the last expression in (7.17) can be estimated in the following way

$$\begin{aligned}
& sup_{v\in\mathcal{H},||v||_{\mathcal{H}}\leq 1}(\widehat{A}^{1/2}(\Phi_m)'_{(u,\omega)}, v) \\
= \; & sup_{v\in\mathcal{V},||v||_{\mathcal{H}}\leq 1}\left((D_{(u,\omega)}(S_m(t)(u,\omega)))^*(\Phi'(S_m(t)(u,\omega))\right), \widehat{A}^{1/2}v) \\
= \; & sup_{v\in\mathcal{V},||v||_{\mathcal{H}}\leq 1}((\Phi'(S_m(t)(u,\omega))), D_{(u,\omega)}(S_m(t)(u,\omega))\widehat{A}^{1/2}v) \\
= \; & sup_{v\in\mathcal{V},||v||_{\mathcal{H}}\leq 1}(\widehat{A}^{1/2}(\Phi'(S_m(t)(u,\omega))), \widehat{A}^{-1/2}D_{(u,\omega)}(S_m(t)(u,\omega))\widehat{A}^{1/2}v) \\
\leq \; & sup_{v\in\mathcal{V},||v||_{\mathcal{H}}\leq 1}||\Phi'(S_m(t)(u,\omega))||_{\mathcal{V}} \, ||\widehat{A}^{-1/2}D_{(u,\omega)}(S_m(t)(u,\omega))\widehat{A}^{1/2}v)||_{\mathcal{H}}.
\end{aligned}$$

The functional Φ belongs to the set of test functionals $\mathcal{T}$. Hence, Φ' is bounded on $\mathcal{V}$. Therefore, we need to estimate only

$$sup_{v\in\mathcal{V},||v||_{\mathcal{H}}\leq 1}||\widehat{A}^{-1/2}D_{(u,\omega)}(S_m(t)(u,\omega))\widehat{A}^{1/2}v||_{\mathcal{H}},$$

or equivalently

$$sup_{v\in\mathcal{H},||\widehat{A}^{-1/2}v||_{\mathcal{H}}\leq 1}||D_{(u,\omega)}(S_m(t)(u,\omega))v||_{\mathcal{H}}.$$

To this end, we consider the operator $D_{(u,\omega)}S_m(t)(u,\omega)$.

First, for the sake of simplicity, we denote $\overline{u_m} = (u_m, \omega_m)$.
Let $v = (v_u, v_\omega)$ be an element of $\mathcal{H}$. Then the function $v_m(t) = (D_{(u,\omega)} S_m(t)(u,\omega))v = (v_u^m, v_\omega^m)$ is the solution of the equations (written in the abstract form)

$$\begin{aligned} \frac{d}{dt} v_u^m &= -(\nu + \nu_r) A v_u^m - P_m B(u_m, v_u^m) - P_m B(v_u^m, u_m) + 2\nu_r rot\ v_\omega^m, \\ \frac{d}{dt} v_\omega^m &= \alpha \triangle v_\omega^m - P_m B_1(u_m, v_\omega^m) - P_m(v_u^m, \omega_m) + 4\nu_r v_\omega^m + 2\nu_r rot\ v_u^m. \end{aligned} \tag{7.18}$$

for all $t \in \mathbb{R}$, where $v_m(0) = v$.

Let us take a scalar product in $\mathcal{H}$ of (7.18) with $\widehat{A}^{-1} v_m$. We arrrive at

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} ||\widehat{A}^{-1/2} v_m||_{\mathcal{H}}^2 &= -(\nu + \nu_r)||v_u^m||_H^2 - \alpha ||v_\omega^m||_{L_2}^2 - b(u_m, v_u^m, A^{-1} v_u^m) \\ &- b(v_u^m, u_m, A^{-1} v_u^m) - b_1(u_m, v_\omega^m, (-\triangle v_\omega^m)) - b_1(v_u^m, \omega_m, (-\triangle v_\omega^m)) \\ &+ 2\nu_r (rot\ v_\omega^m, A^{-1} v_u^m) + 4\nu_r (v_\omega^m, (-\triangle v_\omega^m)) + 2\nu_r (rot\ v_u^m, (-\triangle v_\omega^m)). \end{aligned} \tag{7.19}$$

We estimate the trilinear forms using the interpolation inequality $||u||_V^2 \leq c(\Omega, n)||u||_H ||Au||_H$ as

$$\begin{aligned} |b(u_m, A^{-1} v_u^m, v_u^m) &+ b(v_u^m, A^{-1} v_u^m, u_m)| \\ &\leq c\, ||v_u^m||_H ||A^{-1} v_u^m||_V ||u_m||_H^{1/2} ||u_m||_V^{1/2} \\ &\leq c\, ||v_u^m||_H ||A^{1/2} A^{-1} v_m||_H^{1/2} ||A A^{-1} v_m||_H^{1/2} ||u_m||_H^{1/2} ||u_m||_H^{1/2} \\ &\leq c\, ||v_u^m||_H^{3/2} ||A^{-1/2} v_u^m||_H^{1/2} ||u_m||_H^{1/2} ||u_m||_V^{1/2} \\ &\leq \frac{k_1}{2} ||v_u^m||_H^2 + (\frac{c}{k_1})^3 ||A^{-1/2} v_u^m||_H^2 ||u_m||_H^2 ||u_m||_V^2. \end{aligned}$$

We proceed in a similar way with b_1 trilinear forms. Next, we use the results obtained in [27] in order to estimate rest of the expressions on the right-hand side of (7.19). We have

$$\begin{aligned} (\nu + \nu_r)||v_u^m||_H^2 + \alpha ||v_\omega^m||_{L_2}^2 + 2\nu_r (rot\ v_\omega^m, A^{-1} v_u^m) + 4\nu_r (v_\omega^m, (-\triangle v_\omega^m)) + 2\nu_r (rot\ v_u^m, (-\triangle v_\omega^m)) \\ \geq k_1 ||\widehat{A}^{-1/2} v_m, \widehat{A}^{-1/2} v_m||_{\mathcal{V}} = k_1 ||\widehat{A}^{-1/2} v_m, \widehat{A}^{-1/2} v_m||_{\mathcal{H}}. \end{aligned}$$

Summing up, we arrived at

$$\frac{d}{dt} ||\widehat{A}^{-1/2} v_m||_{\mathcal{H}}^2 \leq C ||\widehat{A}^{-1/2} v_m||_{\mathcal{H}}^2 ||\overline{u_m}||_{\mathcal{H}}^2 ||\overline{u_m}||_{\mathcal{V}}^2.$$

Using Gronwall lemma, we get

$$||\widehat{A}^{-1/2} v_m(t)||_{\mathcal{H}}^2 \leq ||\widehat{A}^{-1/2} v_m(0)||_{\mathcal{H}}^2 \exp(C \int_0^t ||\overline{u_m}(s)||_{\mathcal{H}}^2 ||\overline{u_m}(s)||_{\mathcal{V}}^2 ds) \tag{7.20}$$

for all $t \geq 0$.

Now, we need some a priori estimates already derived in [27]. We have

$$k_1 \int_0^t ||\overline{u_m}(s)||_{\mathcal{V}}^2 ds \leq ||(u,\omega)(0)||_{\mathcal{H}}^2 + k_3(||f||_H^2 + ||g||_{L_2}^2)t$$

where $k_1 = min(\nu, \alpha)$ and $k_3 = max(\frac{1}{\nu\lambda}, \frac{1}{\alpha\sigma})$,
and

$$||\overline{u_m}(s)||_{\mathcal{H}}^2 \leq ||(u,\omega)(0)||_{\mathcal{H}}^2 + \frac{k_3}{k_2}(||f||_H^2 + ||g||_{L_2}^2), \tag{7.21}$$

where $k_2 = min(\nu\lambda, \alpha\sigma)$.

Using the above inequalities the estimate (7.20) can be rewritten as

$$||\widehat{A}^{-1/2} v_m(t)||_{\mathcal{H}}^2 \leq ||\widehat{A}^{-1/2} v_m(0)||_{\mathcal{H}}^2 \exp(C_1(1 + ||(u,\omega)||_{\mathcal{H}}^2)(t + ||(u,\omega)||_{\mathcal{H}}^2))$$

for all $t \geq 0$, where C_1 depends on $||f||_H$ and $||g||_{L_2}$.

Since our aim is to show that the integral (7.15) vanishes when we go with m to infinity, et us notice that the integral is taken with respect to the measure μ. Therefore we can concentrate on functions $(u,\omega) \in supp\ \mu$. By the Theorem 3.1 the norm $||(u,\omega)||_{\mathcal{H}}$ is bounded in this case. We have

$$||\widehat{A}^{-1/2}(D_{(u,\omega)} S_m(t)(u,\omega)v||_{\mathcal{H}}^2 \leq ||\widehat{A}^{-1/2} v||_{\mathcal{H}}^2 \exp(C_2(1+t)).$$

Hence, the integral (7.15) can be estimated in view of (7.16) as

$$\begin{aligned} &| \ (B(u,\omega) - B(P_m(u,\omega)), (\Phi_m)'_{(u,\omega)}(t,(u,\omega))) \ | \\ &\leq c_1\left(\frac{1}{\lambda_1^{1/4}\lambda_m^{1/4}} + \frac{1}{\sigma_1^{1/4}\sigma_m^{1/4}} + \frac{1}{\lambda_1^{1/4}\sigma_m^{1/4}}\right)||(u,\omega)||_{\mathcal{V}}^2 c_2 \ \exp(C_2(1+t)) \end{aligned}$$

$\mu-$almost everywhere in $\mathcal{H}$.

Therefore,

$$\begin{aligned} &\int_0^T \int (\mathcal{B}(P_m(u,\omega), P_m(u,\omega)) - \mathcal{B}((u,\omega),(u,\omega)), (\Phi_m)'_{(u,\omega)}(t,(u,\omega))) d\mu(u,\omega) dt \\ &\leq c_1 c_2\left(\frac{1}{\lambda_1^{1/4}\lambda_m^{1/4}} + \frac{1}{\sigma_1^{1/4}\sigma_m^{1/4}} + \frac{1}{\lambda_1^{1/4}\sigma_m^{1/4}}\right) \int ||(u,\omega)||_{\mathcal{V}}^2 d\mu(u,\omega) \int_0^T \exp(C_2(1+t)) dt \\ &\leq C_T\left(\frac{1}{\lambda_1^{1/4}\lambda_m^{1/4}} + \frac{1}{\sigma_1^{1/4}\sigma_m^{1/4}} + \frac{1}{\lambda_1^{1/4}\sigma_m^{1/4}}\right), \end{aligned}$$

since μ is a statistical solution.

Hence, for any fixed $T > 0$,

$$\int_0^T \int (\mathcal{B}(P_m(u,\omega), P_m(u,\omega)) - \mathcal{B}((u,\omega),(u,\omega)), (\Phi_m)'_{(u,\omega)}(t,(u,\omega))) d\mu(u,\omega) dt \to 0$$

as m goes to infinity.

As a consequence,

$$\int \Phi(S_m(T)(u,\omega))d\mu(u,\omega) - \int \Phi(P_m(u,\omega)d\mu(u,\omega) \to 0.$$

Moreover, we have

$$\int \Phi(S_m(T)(u,\omega))d\mu(u,\omega) \to \int \Phi(S(T)(u,\omega))d\mu(u,\omega)$$

since $\Phi(S_m(T)(u,\omega))$ is uniformly bounded $\mu-$almost everywhere on $\mathcal{H}$ in view of (7.21) and the fact that $S_m(T)(u,\omega) \to S(T)(u,\omega)$.

We also have

$$\int \Phi(P_m(u,\omega))d\mu(u,\omega) \to \int \Phi(u,\omega)d\mu(u,\omega).$$

Therefore,

$$\int \Phi(S(T)(u,\omega))d\mu(u,\omega) = \int \Phi(u,\omega)d\mu(u,\omega)$$

for any $T > 0$, which finishes the proof of the theorem.

7.2 Reynolds equations

In the description of the turbulent flow we are interested in the evolution of mean velocity and angular velocity of the micropolar fluid. There is a procedure used in physics that is called an averaging technique that gives the equations for such a mean motion. The idea of it is presented in [1]. The functions u and ω are decomposed into their mean and fluctuating part. The average of the last one is equal to zero. But in the mentioned article it has not been precised what this average means. Now, we will derive the Reynolds equations for the mean flow (denoted as $\overline{u}$, $\overline{\omega}$) in a precise way, using time-averages introduced above.

$$-(\nu+\nu_r)A\overline{u} + B_1(\overline{u},\overline{u}) + \overline{B(u',u')} = 2\nu_r rot(\overline{\omega}) + f, \tag{7.22}$$

$$div\ \overline{u} = 0,$$

$$-\alpha\triangle\overline{\omega} + B_2(\overline{u},\overline{\omega}) + \overline{B(u',\omega')} + 4\nu_r\overline{\omega} = 2\nu_r rot(\overline{u}) + g. \tag{7.23}$$

We take the scalar product in H of (7.1) and $v \in V$,

$$\frac{d}{dt}(u(t),v) - (\nu+\nu_r)((u(t),v)) + (B_1(u(t),u(t)),v) = 2\nu_r(rot\ \omega(t),v) + (f,v)$$

and the scalar product in L^2 of (7.3) and $w \in H^1_0$

$$\frac{d}{dt}(\omega(t),w) - \alpha((\omega(t),w(t)) + (B_2(u(t),\omega(t)),w) + 4\nu_r(\omega(t),w) = 2\nu_r(rot\ u(t),w) + (g,w).$$

Now, we fix a certain Banach generalized limit funtional $LIM_{t\to\infty}$. If we take the time average of the above equations and then apply the functional we chose, we obtain

$$\begin{aligned}
&LIM_{T\to\infty}\frac{1}{T}(u(T),v) - LIM_{T\to\infty}\frac{1}{T}(u(0),v)\\
&+ (\nu+\nu_r)LIM_{T\to\infty}\frac{1}{T}\int_0^T((u(t),v))dt + LIM_{T\to\infty}\frac{1}{T}\int_0^T(B_1(u(t),u(t),v)dt\\
&= \; 2\nu_r LIM_{T\to\infty}\frac{1}{T}\int_0^T(rot\;\omega(t),v)dt + LIM_{T\to\infty}\frac{1}{T}\int_0^T(f,v)dt,
\end{aligned}$$

and

$$\begin{aligned}
&LIM_{T\to\infty}\frac{1}{T}(\omega(T),w) - LIM_{T\to\infty}\frac{1}{T}(\omega(0),w)\\
&+ \alpha LIM_{T\to\infty}\frac{1}{T}\int_0^T((\omega(t),w(t))dt + LIM_{T\to\infty}\frac{1}{T}\int_0^T(B_2(u(t),\omega(t),w)dt\\
&+ 4\nu_r LIM_{T\to\infty}\frac{1}{T}(\omega(t),w)dt = \; 2\nu_r LIM_{T\to\infty}\frac{1}{T}(rot\;u(t),w)dt + LIM_{T\to\infty}\frac{1}{T}\int_0^T(g,w)dt.
\end{aligned}$$

The first two terms in both equations are equal to zero. In fact, from the first energy inequality (2.18) and Gronwall lemma, we have

$$|\;(u(T),v)\;|\leq ||u(T)||_{L_2}||v||_{L_2} \leq [(||u_0||_{L_2} + ||\omega_0||_{L_2}) + k_3(||f||_{L_2} + ||g||_{L_2})]||v||_{L_2}.$$

Therefore, dividing both sides of the above by T and letting T go to infinity, we arrive at zero. The similar estimate for the terms with ω hold.

Now, we define functions $\overline{u}$ and $\overline{\omega}$.
First of all, let us notice that the map

$$v \to LIM_{T\to\infty}\frac{1}{T}\int_0^T(u(s),v)ds$$

is a linear bounded finctional on the space H. Hence, by Riesz representation theorem, there exists an element in H, which we shall denote by $\overline{u}$, such that

$$(\overline{u},v) = LIM_{T\to\infty}\frac{1}{T}\int_0^T(u(s),v)ds \quad \text{for every } v\in H.$$

Similary, we have an element $\overline{\omega}$ in L^2 that satisfies

$$(\overline{\omega},w) = LIM_{T\to\infty}\frac{1}{T}\int_0^T(\omega(s),w)ds \quad \text{for all } w\in L^2.$$

Now, we need to check that both elements $\overline{u}$ and $\overline{\omega}$ belong to V and H_0^1, respectively (since we would like to know we can apply weak Stokes operator and a weak laplacian to these elements).

Since v is a function in the space V, in particular the following holds

$$(\overline{u},v_{x_i}) = LIM_{T\to\infty}\frac{1}{T}\int_0^T(u(s),v_{x_i})ds \quad \text{for every } v\in V,\; i=1,2.$$

We also have

$$LIM_{T\to\infty}\frac{1}{T}\int_0^T (u(s), v_{x_i})ds = LIM_{T\to\infty}\frac{1}{T}\int_0^T (u(s)_{x_i}, v)ds.$$

Hence, we can consider the following bounded functional on H

$$v \to LIM_{T\to\infty}\frac{1}{T}\int_0^T (u(s)_{x_i}, v)ds.$$

Due to the Riesz Theorem, there exists an element $\overline{\nabla u} = (\overline{u_{x_1}}, \overline{u_{x_2}})$ such that

$$(\overline{u_{x_i}}, v) = LIM_{T\to\infty}\frac{1}{T}\int_0^T (u(s)_{x_i}, v)ds \quad \text{for every } v \in H, \;\; i = 1, 2.$$

Hence, bringing the above together, we have

$$(\overline{u_{x_i}}, v) = (\overline{u}, v_{x_i})$$

for any $v \in V$.
Therefore, $\overline{u_{x_i}} = \overline{u}_{x_i}$ in the weak sence. The same holds for the function $\overline{\omega}$.

We proved that $\overline{u}$ belongs to V and $\overline{\omega}$ is an element of H_0^1. We can apply the weak Stokes operator to $\overline{u}$ and the weak laplasian to $\overline{\omega}$. We have

$$(A\overline{u}, v) = (\overline{u}, Av) = LIM_{T\to\infty}\frac{1}{T}\int_0^T (u(s), Av)ds = LIM_{T\to\infty}\frac{1}{T}\int_0^T ((u(s), v))ds$$

and

$$(\triangle\overline{\omega}, w) = (\overline{\omega}, \triangle w) = LIM_{T\to\infty}\frac{1}{T}\int_0^T (\omega(s), \triangle w)ds = LIM_{T\to\infty}\frac{1}{T}\int_0^T ((\omega(s), w)).$$

Now, let us look at the terms containing trilinear forms.

$$\begin{aligned} LIM_{T\to\infty}\frac{1}{T}\int_0^T (B(u(s), u(s)), v)ds &= LIM_{T\to\infty}\frac{1}{T}\int_0^T (B(\overline{u} + u(s) - \overline{u}, \overline{u} + u(s) - \overline{u}), v)ds \\ &= LIM_{T\to\infty}\frac{1}{T}\int_0^T [(B(\overline{u}, \overline{u}), v) + (B(\overline{u}, u(s) - \overline{u}), v) \\ &+ (B(u(s) - \overline{u}, \overline{u}), v) + (B(u(s) - \overline{u}, u(s) - \overline{u}), v)]ds. \end{aligned}$$

First of the terms on the right hand-side is in fact equal to $(B(\overline{u}, \overline{u}), v)$:

$$LIM_{T\to\infty}\frac{1}{T}\int_0^T (B(\overline{u}, \overline{u}), v)ds = (B(\overline{u}, \overline{u}), v).$$

Now, since the form b is trilinear, the second and the third expression above dissapear. In fact, we could rewrite the forms as a sums to see that

$$LIM_{T\to\infty}\frac{1}{T}\int_0^T (B(\overline{u},u(s)),v)ds = LIM_{T\to\infty}\frac{1}{T}\int_0^T (B(\overline{u},\overline{u}),v)ds.$$

Moreover, the functional

$$v \to LIM_{T\to\infty}\int_0^T (B(u(s)-\overline{u},u(s)-\overline{u}),v)ds$$

is linear and bounded on V. Hence, using the Riesz representation theorem, there exists an element $\overline{B(u(s)-\overline{u},u(s)-\overline{u})}$ belonging to V', such that

$$LIM_{T\to\infty}\int_0^T (B(u(s)-\overline{u},u(s)-\overline{u}),v)ds = (\overline{B(u(s)-\overline{u},u(s)-\overline{u})},v).$$

At last, we have

$$LIM_{T\to\infty}\frac{1}{T}\int_0^T (rot\ \omega(s),v)ds = LIM_{T\to\infty}\frac{1}{T}\int_0^T (\omega(s),rot\ v)ds = (\overline{\omega},rot\ v) = (rot\ \overline{\omega},v).$$

Summing up, we obtained the following

$$(\nu+\nu_r)((\overline{u},v)) + (B_1(\overline{u},\overline{u}),v) + (\overline{B(u',u')},v) = 2\nu_r(rot(\overline{\omega}),v) + (f,v) \tag{7.24}$$

$$\alpha((\overline{\omega},w)) + (B_2(\overline{u},\overline{\omega}),w) + (\overline{B(u',u')},w) + 4\nu_r(\overline{\omega},w) = 2\nu_r(rot(\overline{u}),w) + (g,w) \tag{7.25}$$

for every $v \in V$ and $w \in H_0^1$, which are the stationary Reynolds equations (7.22)-(7.23).

Chapter 8

Nonstationary Statistical Solutions

In this chapter we study the micropolar fluid system with external forces and moments depending on time.

$$\frac{\partial u}{\partial t} - (\nu + \nu_r)\triangle u + (u \cdot \nabla)u + \nabla p = 2\nu_r rot\ \omega + f(t), \tag{8.1}$$

$$div\ u = 0, \tag{8.2}$$

$$\frac{\partial \omega}{\partial t} - \alpha\triangle\omega + (u \cdot \nabla)\omega + 4\nu_r\omega = 2\nu_r rot\ u + g(t). \tag{8.3}$$

We assume that homogeneous Dirichlet boundary conditions hold

$$u_{|\partial\Omega} = 0, \quad \omega_{|\partial\Omega} = 0, \tag{8.4}$$

and denote the initial conditions as

$$u(\tau) = u_0, \quad \omega(\tau) = \omega_0, \tag{8.5}$$

for $\tau \in I\!R$. We assume $u_0 \in H$ and $\omega_0 \in L_2$.
The domain Ω is a bounded subset of $\mathbb{R}^2$ with a smooth boundary.

8.1 Definition of nonstationary solutions

There are two known to the author ways of introducing the idea of nonstationary statistical solutions. First one comes from C. Foias ([19]) and the other one was introduced by M.Vishik ([50]). We shall sketch the idea of both of them first.

As in the previous chapter, we look at the system of micropolar fluid equations (8.1)-(8.4) with the initial probability measure μ defined on the space $H \times L_2$ instead of individual initial conditions. The nonstationary solution is a probability measure defined on the space of weak solutions of the equations.

The idea proposed by M.Vishik is to define a space-time statistical solution on the whole trajectories, in such a way that it coincides with the initial measure. In two dimensions, where the existence of a unique weak solution of (8.1)-(8.5) is known, the construction of the measure is simple. But the same idea works in the case of three dimensions, where we do not know if the

system of micropolar fluid equations has global in time solutions. Then it is more technically complicated. One has to come through Galerkin approximations of the measure and use the Prokhorov Theorem.
Since our whole work is devoted to the two-dimensional problem, we shall not study the three-dimensional case here.

The other way of defining a statistical solution is proposed by C.Foias. One introduces a family of probability measures indexed with time and defined on the time sections of trajectories. One constructs the measures using Banach generalized limit.

In this chapter, we use the idea of M. Vishik first. We define a space-time statistical solution. Then we show that we can derive a family of measures from it. Next, using generalized limits we construct the family of measures with the support belonging in the pullback attractor.

Before we give the definition of the nonstationary statistical solution, we precise on what space the probability measure will be defined.
We consider the space to which the solutions of the system (8.1)-(8.5) belong to if $f \in L^2_{loc}(R,H)$ and $g \in L^2_{loc}(R,L_2)$

$$Z = \{(u,\omega) : u \in L_2(0,T;V) \cap C([0,T];H), \omega \in L_2(0,T;H^1_0) \cap C([0,T];L_2)\}.$$

We assume that the initial measure μ has finite energy, that is

$$\int_{H\times L_2} (||u_0||^2_H + ||\omega_0||^2_{L_2})\mu(d(u_0,\omega_0)) < \infty.$$

Definition 8.1.1. *The nonstationary space-time statistical solution for the problem (8.1)-(8.4), corresponding to an initial measure μ is a probability measure P defined on the space Z such that the following conditions hold:*

(i) The measure P is supported on the solutions of problem. Namely, there exists a closed set $W \subset Z$ that is a Borel set in $Z : W \in \mathcal{B}(Z)$, $P(W) = 1$ and W consists of the solutions of the system of equations.

(ii) The measure P correponds to the initial measure in the following way

$$P(\gamma_0^{-1}\nu_0) = \mu(\nu_0)$$

for all $\nu_0 \in \mathcal{B}(H \times L_2)$.
The expression $\gamma_0^{-1}\nu_0$ denotes the trajectory, e.g. $\{(u(t),\omega(t)) : (u(t),\omega(t)) \in Z, \gamma_0(u,\omega) \in \nu_0\}$.

(iii) The energy inequality holds:

$$\int (||u||^2_{L_2(0,T;V)} + ||\omega||^2_{L_2(0,T;H^1_0)} + ||u(t)||^2_H + ||\omega(t)||^2_{L_2})P(d(u,\omega))$$

$$\leq C(\int (||u_0||^2_H + ||\omega_0||^2_{L_2})d\mu(u_0,\omega_0) + 1)$$

for all $t > 0$, where C depends on f and g and the constants characterizing the fluid.

8.2 Existence and uniqueness of the statistical solutions

First, we introduce some notation. Since the system of micropolar fluid equations in two dimensions (8.1)-(8.4) has a unique solution, we can define an operator S

$$S : (H \times L_2) \to Z$$

as

$$S(u(0), \omega(0)) = \{(u(t), \omega(t)); 0 \le t \le T\}.$$

We define the measure P for any $w \in \mathcal{B}(Z)$ as

$$P(w) = \mu(S^{-1}w).$$

Now, we show that P is in fact a nonstationary statistical solution in the sense of Definition 8.1.1.
First of all, let us notice that points (i) and (ii) are obvious form the way measure P was defined. Hence, we only need to check that the energy inequality holds.
To this end, we recall the energy inequality for the weak solutions (2.18) which after integrating it over time gives

$$\begin{aligned} ||u(t)||_H^2 + ||\omega(t)||_{L_2}^2 \quad &+ \quad k_1 \int_0^t (||u||_V^2 + ||\omega||_{H_0^1}^2) ds \\ &\le \quad k_3 \int_0^t (||f||_{L_2}^2 + ||g||_{L_2}^2) ds + ||||u(0)||_H^2 + ||\omega(0)||_{L_2}^2. \end{aligned}$$

Since the measure P is defined on the weak solutions of the system (8.1)-(8.4), we can integrate we above inequality with respect to P to arrive at the desired inequality (iii).

Let us also notice that uniqueness of the weak solutions for our problem imply that the measure P is uniquely defined.

8.3 Family of measures derived from the measure P

Let P be the measure obtained in the previous section and defined on the space

$$Z = \{(u, \omega) : u \in L_2(0, T; V) \cap C([0, T]; H), \omega \in L_2(0, T; H_0^1) \cap C([0, T]; L_2)\}.$$

Now, we shall derive a family of measures $\{\mu_t\}_{t \in [0,T]}$ by fixing $t \in [0, T]$. Indeed, if (u, ω) is a weak solution of (8.1)-(8.5), it clearly belongs to the space Z. We can fix the time t by defining the operator $\gamma_t : Z \to H \times L_2$ as follows:

$$\gamma_t u = u(t)$$

Hence, if we take $w \in \mathcal{B}(H \times L_2)$ and choose any $t \in [0, T]$, we define μ_t in a natural way as

$$\mu_t(w) = P(\gamma_t^{-1} w). \tag{8.6}$$

We shall show that the family of measures derived above is a nonstationary statistical solution in the sense of the definition inspired by the idea of C.Foias:

Definition 8.3.1. *A family $\{\mu_t\}_{t\in[0,T]}$ of probability measures defined on $H \times L_2$ is called a nonstationary statistical solution of micropolar fluid equations with initial data $\mu_0(u,\omega)$ if*

(i) the Louville equation

$$\int \Phi(u,\omega)d\mu_t(u,\omega) = \int \Phi(u,\omega)d\mu_0(u,\omega) + \int_0^t \int (F(s,u,\omega),\Phi'(u,\omega))d\mu_s(u,\omega)ds. \tag{8.7}$$

holds for all cylindrical test funcionals $\Phi \in \mathcal{T}$ depending on a finite number of components of u and ω :

$$\Phi(u,\omega) = \phi((u,\omega),g_1),...,((u,\omega),g_k))$$

where ϕ is a C^1 scalar function with compact support and g_i are elements of $V \times H_0^1$,

(ii) the energy inequality holds

$$\begin{aligned} &\int (||u(t)||_H^2 + ||\omega(t)||_{L_2}^2)d\mu_t(u,\omega) + \int_0^t \int (||u(s)||_V^2 + ||\omega(s)||_{H_0^1}^2)d\mu_s(u,\omega)ds \\ &\quad \le C\int_0^t \int ((f(s),u(s)) + (g(s),\omega(s))d\mu_s(u,\omega)ds + \int (||u(0)||_H^2 + ||\omega(0)||_{L_2}^2)d\mu_0(u,\omega). \end{aligned} \tag{8.8}$$

(iii) the function

$$t \to \int \varphi(u,\omega)d\mu_t(u,\omega)$$

is measurable on $[0,T]$ for any bounded and continuous real-valued function defined on $H \times L_2$. Moreover, the function

$$t \to \int (||u||_{L_2}^2 + ||\omega||_{L_2}^2)d\mu_t(u,\omega)$$

belongs to $L^\infty(0,T)$ and

$$t \to \int (||u||_V^2 + ||\omega||_{H_0^1}^2)d\mu_t(u,\omega)$$

belongs to $L^1(0,T)$.

We shall prove the following lemmas.

Lemma 8.3.1. *The family of measures $\{\mu_t\}_{t\in[0,T]}$ derived in (8.6) satifies the energy inequality (8.8).*

Proof. Since we already have an energy inequality for the measure P, we only need to notice that the following holds

$$\int (||u(t)||_H^2 + ||\omega(t)||_{L_2}^2)P(d(u,\omega)) = \int (||u||_H^2 + ||\omega||_{L_2}^2)\mu_t(d(u,\omega))$$

when we change $u(t)$ for u and $\omega(t)$ for ω.
Moreover, since the function $||u(\tau)||_V^2 + ||\omega(\tau)||_{H_0^1}^2$ is measurable on $[0,T] \times Z$, we can use Fubini Theorem to get

$$\begin{aligned} \int (\int_0^t (||u(\tau)||_V^2 + ||\omega(\tau)||_{H_1}^2)d\tau)P(d(u,\omega)) &= \int_0^t \int (||u(\tau)||_V^2 + ||\omega(\tau)||_{H_1}^2))P(d(u,\omega)d\tau \\ &= \int_0^t \int (||u||_V^2 + ||\omega||_{H_1}^2))\mu_t(d(u,\omega)d\tau. \end{aligned}$$

□

Lemma 8.3.2. *The Louville equation (8.7) holds for the family of measures* $\{\mu_t\}_{t\in[0,T]}$.

Proof. Let Φ be a functional satisfying assumptions of the Definition 8.3.1.
If we take $\Phi(u(t), \omega(t))$ and differentiate it with respect to t, we get

$$\frac{d\Phi(u(t), \omega(t))}{dt} = (\Phi'(u(t), \omega(t)), F(t, u(t), \omega(t))$$

We integrate the above with respect to t to arrive at

$$\Phi(u(t), \omega(t)) - \Phi(u(0), \omega(0)) = \int_0^t (\Phi'(u(s), \omega(s)), F(t, u(s), \omega(s))ds.$$

Since

$$\Phi(u, \omega) = \phi((u, \omega), g_1), ..., ((u, \omega), g_k))$$

and ϕ is a scalar C^1 function, all the expressions above are $P-$integrable. Hence, we integrate the above with respect to the measure P and arrive at

$$\int (\Phi(u(t), \omega(t)) - \Phi(u(0), \omega(0))dP = \int \int_0^t (\Phi'(u(s), \omega(s)), F(t, u(s), \omega(s))dsdP.$$

Changing variables $(u(t), \omega(t))$ for (u, ω) and get

$$\int (\Phi(u(t), \omega(t))dP = \int \Phi(u, \omega) d\mu_t(u, \omega).$$

Using Fubini Theorem, we get the desired Louville equation. □

8.4 Family of measures $\{\mu_t\}_{t\in\mathbb{R}}$ and the pullback attractor.

From now on let us assume that the forces and moments satisfy

$$\int_{-\infty}^t e^{\lambda s}\{||f(s)||_H^2 + ||g(s)||_{L_2}^2\}ds < \infty \quad \text{for every} \quad t \in I\!R, \tag{8.9}$$

where $\lambda = \min\{\nu\lambda_1, \alpha\eta_1\}$ (ν and α are the viscosities as in (8.1) and in (8.3), and λ_1, η_1 are the first eigenvalues of the Stokes and minus Laplacian operators, respectively).

We shall show that the nonstationary statistical solution is a family of measures having their support on time sections of a pullback attractor $\{A(t)\}_{t\in\mathbb{R}}$ if only the initial measure μ_τ has its support belonging to $A(\tau)$.

First, let us generalize the definition (8.6) of the family of measures $\{\mu_t\}_{t\in[0,T]}$ for all $t \in \mathbb{R}$.
We can rewrite (8.6) as

$$\mu_t(w) = P(\gamma_{\tau,t}^{-1}(w)), \tag{8.10}$$

where the operator $\gamma_{\tau,t}$ takes the trajectory starting at time τ and fixes the value of the solution at time t.
If we fix the measure μ_τ as an initial measure (given at time τ), we have

$$\mu_\tau(w) = P(\gamma_{\tau,\tau}^{-1} w) \tag{8.11}$$

Let us substitute $w = U(t,\tau)v$ for some $v \in \mathcal{B}(H \times L_2)$ in (8.10) (where $t > \tau$ and $U(t,\tau)$ is a process associated with (8.1)-(8.5)). We derive

$$\mu_t(U(t,\tau)v) = P(\gamma_{\tau,t}^{-1}(U(t,\tau)v)) = P(\gamma_{\tau,\tau}^{-1}v).$$

Therefore

$$\mu_t(U(t,\tau)v) = \mu_\tau(v) \tag{8.12}$$

for any $v \in \mathcal{B}(H \times L_2)$ and $t > \tau$.
The pullback attractor $\{A(t)\}_{t\in\mathbb{R}}$ has the following invariance property

$$U(t,\tau)A(\tau) = A(t)$$

for any $t > \tau$.

Let us assume that μ_τ for some $\tau \in \mathbb{R}$ has its support belonging to $A(\tau)$. Hence, $\mu_\tau(A(\tau)) = 1$. In view of (8.12), we have

$$\mu_t(A(t)) = \mu_t(U(t,\tau)A(\tau)) = \mu_\tau(A(\tau)) = 1.$$

8.5 Nonstationary statistical solutions - the other concept

Now, we shall construct a nonstationary statistical solution as a family of measures $\{\mu_t\}_{t\in\mathbb{R}}$ in the space $H \times L_2$ using the idea of C.Foias.
The main Theorem of this section reads

Theorem 8.5.1. *Let (u_0, ω_0) be any element belonging to the space $H \times L_2$.
Let $LIM_{T\to\infty}$ be an arbitrary but fixed Banach generalized limit.
Then there exists a family $\{\mu_t\}_{t\in\mathbb{R}}$ of probability measures defined on $H \times L_2$ with support on the time sections of the pullback attractor $\{A(t)\}_{t\in\mathbb{R}}$.*

The measures μ_t satisfy

$$LIM_{\tau\to-\infty}\frac{1}{t-\tau}\int_\tau^t \varphi(U(t,s)(u_0,\omega_0))ds = \int_{A(t)} \varphi(u,\omega)d\mu_t(u,\omega). \tag{8.13}$$

*for any $\varphi \in C(H \times L_2)$.
Moreover,*

$$\mu_t(E) = \mu_\tau(U(t,\tau)^{-1}E)$$

for any $t \geq \tau$ and every Borel set $E \in \mathcal{B}(H \times L_2)$.

We prove some needed lemmas.
The first thing we look at is if the left side of (8.13) is well defined . Since the functional $LIM_{T\to\infty}$ acts on bounded real-valued functions, we need to check if the expression

$$\frac{1}{t-\tau}\int_\tau^t \varphi(U(t,s)(u_0,\omega_0))$$

belongs to the space $\mathcal{B}([0,\infty))$.
First, we need to now, that the following lemma holds

Lemma 8.5.1. *Let $t \in \mathbb{R}$. Then there exists an absorbing compact set $B(t)$ in the space $H \times L_2$.*

Proof. In order to show that the family of compact in $H \times L_2$ absorbing sets exists, we recall that

$$\begin{aligned} ||u(t)||_V^2 + ||\omega(t)||_{H_0^1}^2 &\leq ce^{k_2(t-\tau)} \int_{-\infty}^t e^{k_2 s}(||f(s)||_H^2 + ||g(s)||_{L_2}^2)ds \\ &\cdot \quad exp\{ce^{-k_2(t-\tau)} \int_{-\infty}^t s^{k_2 s}(||f(s)||_H^2 + ||g(s)||_{L_2}^2)ds\}. \end{aligned}$$

Since Rellich- Kondrashov Theorem states that $V \times H_0^1$ is compactly imbedded in $H \times L_2$, the proof of the lemma is finished. □

Now, we prove that the left-hand side of (8.13) is well defined.

Lemma 8.5.2. *Let $(u_0, \omega_0) \in H \times L_2$ and $t \in \mathbb{R}$. Then the function*

$$\tau \rightarrow \varphi(U(t,\tau)(u_0,\omega_0))$$

is continuous and bounded on $(-\infty, t]$.

Proof. Let us recall the estimate from [27] (the inequality (2.34) in the cited article):

$$\begin{aligned} &||u_1(t) - u_2(t)||_H^2 + ||\omega_1(t) - \omega_2(t)||_{L_2}^2 \\ &\quad \leq (||u_1(\tau) - u_2(\tau)||_H^2 + ||\omega_1(\tau) - \omega_2(\tau)||_{L_2}^2) exp(\frac{17}{k_1} \int_\tau^t (||u_1(s)||_V^2 + ||\omega_1(s)||_{H_0^1}^2)ds \end{aligned}$$

for any $\tau < t$.
The functions (u_1, ω_1) and (u_2, ω_2) are two weak solutions of the problem (8.1)-(8.5).

We rewrite the above inequality using the notation of the process since that will be more convenient here. We take $U(t,s)U(s,\tau)(u_0,\omega_0)$ which is equal to $U(t,\tau)(u_0,\omega_0)$ as one solution and simply $U(t,s)(u_0,\omega_0)$ as the other one (for any $s < \tau < t$). We arrive at

$$\begin{aligned} &||U(t,s)(u_0,\omega_0) - U(t,\tau)(u_0,\omega_0)||_{H\times L_2}^2 \\ &\qquad \leq (||(u_0,\omega_0) - U(s,\tau)(u_0,\omega_0)||_{H\times L_2}^2 exp(C \int_\tau^t (||U(r,\tau)(u_0,\omega_0)||_{V\times H_0^1}^2 dr. \end{aligned}$$

Due to the continuity of solutions in the space $H \times L_2$, the first expression is small when $|s - \tau|$ is small. Moreover, the expression under the exponent is bounded for any fixed t and $\tau < t$.

In view of the previous lemma, there exists a family of compact absorbing sets. Hence, for sufficiently small τ, namely $\tau < \tau_0$ for some $\tau_0 \in \mathbb{R}$, the function

$$\tau \rightarrow \varphi(U(t,\tau)(u_0,\omega_0))$$

is bounded, since $U(t,\tau)(u_0,\omega_0)$ belongs to a compact set.
The interval $[\tau_0, \tau]$ is compact, hence the above function is also bounded on it. That finishes the proof of the lemma. □

Let us notice that Lemma 8.5.1 gives the information that the left-hand side of (8.13) depends in fact on the values of φ on the compact absorbing set. This allows us to use the Kakutani-Riesz

Theorem from which the existence of the family of measures results.
We finish the proof of the Theorem 8.5.1 here, since the remaining part of it has been done in [28] using abstract notation and the results of the lemmas above.

Now, we show the family of measures from Theorem 8.5.1 satisfies the Definition 8.3.1.
We shall prove it in the following lemmas.

Lemma 8.5.3. *The Louville equation holds.*

$$\begin{aligned}\int_{A(t)} \Phi(u,\omega)d\mu_t(u,\omega) &= \int_{A(\tau)} \Phi(u,\omega)d\mu_\tau(u,\omega)\\ &+ \int_\tau^t \int_{A(s)} (F(s,u,\omega), \Phi'(u(s),\omega(s)))d\mu_s(u,\omega)ds,\end{aligned}$$

where Φ is a cylindrical test functional depending on a finite number of components of u and ω :

$$\Phi(u,\omega) = \phi(((u,\omega),g_1),...,((u,\omega),g_k))$$

for ϕ being a C^1 scalar function with compact support and g_i denoting elements of $V \times H^1_0$.

We omit the proof since it is identical as in [28] (and done in abstract notation there).
We only need to notice that the function

$$s \to (F(s,u,\omega), \Phi'(u(s),\omega(s))),$$

where $\tau < s < t$, is integrable on $[\tau,t]$.
The solutions of (8.1)-(8.5) satisfy $u(s) \in L_2(\tau,t;D(A))$ and $\omega(s) \in L_2(\tau,t;H^2 \cap H^1_0)$.
Therefore the expression $F(s,u(s),\omega(s))$ is integrable over time in $[\tau,t]$. Hence for any $g_i \in \mathcal{V}$ the functions $(F(s,u(s),\omega(s)),g_i)$ and also $(F(s,u,\omega),\Phi'(u(s),\omega(s)))$ are integrable.

Lemma 8.5.4. *The energy inequality holds*

$$\begin{aligned}\int_{A(t)} (||u(t)||^2_H + ||\omega(t)||^2_{L_2})d\mu_t(u,\omega) &+ \int_\tau^t \int_{A(s)} (||u(s)||^2_V + ||\omega(s)||^2_{H^1_0})d\mu_s(u,\omega)ds\\ &\leq C\int_\tau^t \int_{A(s)} ((f(s),u(s)) + (g(s),\omega(s))d\mu_s(u,\omega)ds\\ &+ \int_{A(\tau)} (||u(\tau)||^2_H + ||\omega(\tau)||^2_{L_2})d\mu_\tau(u,\omega).\end{aligned}$$

Proof. We use the previous lemma to derive the desired energy inequality. We need to choose the appropriate test function Φ.

Let us notice that functional of the form $\Phi(u,\omega) = ||u||^2_H + ||\omega||^2_{L_2}$ would be the most convenient. Since it depends on infinite number of components of u and ω, first we modify it and take into consideration $\Phi_m(u,\omega) = ||u_m||^2_H + ||\omega_m||^2_{L_2}$. We have $\Phi'_m(u,\omega) = [2u_m, 2\omega_m]$.

What we get, is

$$\int_{A(t)} (||u_m(t)||_H^2 + ||\omega_m(t)||_{L_2}^2) d\mu_t(u,\omega)$$
$$+c\int_\tau^t \int_{A(s)} (||u_m(s)||_V^2 + ||\omega_m(s)||_{H_0^1}^2) d\mu_s(u,\omega) ds - c_1 \int_\tau^t \int_{A(s)} (||u(s)||_V^2 + ||\omega(s)||_{H_0^1}^2) d\mu_s(u,\omega) ds$$
$$\leq C\int_\tau^t \int_{A(s)} ((f(s), u_m(s)) + (g(s), \omega_m(s)) d\mu_s(u,\omega) ds + \int_{A(\tau)} (||u_m(\tau)||_H^2 + ||\omega_m(\tau)||_{L_2}^2) d\mu_\tau(u,\omega),$$

where constants c and c_1 depend on the parameters characterizing the fluid. The third integral comes from the estimates on the trilinear forms. Since the calculations are identical to the ones obtained when deriving the energy estimates for the solutions, we do not present them here.

Due to the Lebesgue dominated convergence theorem, we can pass to the limit with m and get the desired energy inequality. □

One of the results of the Theorem 8.5.1 is the invariance property of the family of measures, namely

$$\mu_t(E) = \mu_\tau(U(t,\tau)^{-1}E)$$

for any $t > \tau$. We obtained the same property in the case of the measures μ_t derived from the nonstationary statistical solution P.
Since the process $U(\tau, t)$ associated with the problem (8.1)-(8.5) is uniquely defined, the above property implies uniqueness of the family of measures when an initial measure is given.

Chapter 9

Conclusions and final remarks

The aim of the thesis was to develop a part of the qualitative theory of micropolar fluid equations. We used the theory of attractors and the notion of statistical solutions.

In the first main part of the thesis we concentrated on two-dimensional flows in bounded domains when external forces and moments acting on the fluid depend on time. The long time behaviour of the dynamical system in frame of the theory of pullback attractors was studied. The natural direction of further research is to investigate important three-dimensional flows as well as flows in unbounded domains. It is also interesting to develop the knowledge on the dimension and structure of pullback attractors and the way these qualities depend on the parameters of the flow.

As concerns the statistical solutions for micropolar fluid equations, again the three-dimensional case still needs to be investigated. Moreover, it is interesting to study further properties of measures μ_t and to continue the approach to the mathematical theory of turbulence in micropolar fluid flows.

Bibliography

[1] G.Ahmadi, *Turbulent shear flow of micropolar fluids,* Int. J. Engng Sci. Vol.13 (1975), 959-964

[2] M.Boukrouche, G.Łukaszewicz, and J.Real, *On pullback attractors for a class of two-dimensional turbulent shear flows*, Internat. J. Engrg. Sci Vol 44 (2006), 830-844.

[3] M.Boukrouche, G.Łukaszewicz, *Attractor dimension estimate for plane shear flow of micropolar fluid with free boundary*, Math. Methods Appl. Sci., Vol.28 (2005), 1673-1694.

[4] T.Caraballo, P.E.Kloeden and J.Real, *Pullback and forward attractors for a damped wave equation with delays*, Stochastics and Dynamics Vol.4 (3) (2004), 405-423.

[5] T.Caraballo, G.Łukaszewicz, and J.Real, *Pullback attractors for non-autonomous 2D-Navier-Stokes equations in some unbounded domains*, CR. Acad. Sci. Paris, Ser. 1, Vol.342 (2006), 263-268.

[6] T.Caraballo, G.Łukaszewicz, and J.Real, *Pullback attractors for asymptotically compact non-autonomous dynamical systems*, Nonlinear Analysis, Vol.64 (2006), 484-498.

[7] T.Caraballo, J.Real, *Attractors for 2D-Navier-Stokes models with delays*, J. Diff. Eqns. 205(2004), 270-296.

[8] J.Chen, Z-M.Chen, and B-Q.Dong, *Existence of H^2-global attractors of two-dimensional micropolar fluid flows*, J.Math. Anal. Appl., Vol.322 (2006), 512-522.

[9] P.Constantin, C.Foias, R.Temam, *Attractors representing turbulent flows*, Mem. Amer. Math. Soc 53 (1984).

[10] B-Q.Dong, Z-M.Chen, *Global attractors of two-dimensional micropolar fluid flows in some unbounded domains*, Applied Mathematics and Computation, Vol.182 (2006), 610-620.

[11] G.Chen, *Pullback atrractor for non-homogeneous micropolar fluid flows in non-smooth domains*, Nonlinear Analysis: Real World Applications, 10 (2009), 3018-3027.

[12] J.Chen, B-Q.Dong, and Z-M.Chen, *Pullback attractors of non-autonomous micropolar fluid flows*, J.Math. Anal. Appl., Vol.336 (2007), 1384–1394.

[13] J.Chen, Z-M.Chen and B-Q.Dong, *Uniform attractors of non-homogeneous micropolar fluid flows in non-smooth domains*, Nonlinearity, Vol.20 (2007), 1619-1635.

[14] V.V.Chepyzhov, M.I.Vishik, *Attractors for Equations of Mathematical Physics*, AMS Colloquium Publications, Vol.48, AMS, Providence, RI, 2002.

[15] J.W.Cholewa, T.Dłotko, *Global attractors in abstract parabolic problems*, Cambridge University Press, 2000.

[16] H.Crauel, F.Flandoli, *Attractors for random dynamical systems*, Prob. Theory Relat. Fields, Vol.100 (1994), 365–393.

[17] H.Crauel, A.Debussche & F.Flandoli, *Random attractors*, Journal of Dynamics and Differential Equations, Vol.9, no.2, (1997), 307–341.

[18] A.C.Eringen, *Theory of micropolar fluids*, J. Math. Mech. Vol.16 (1966), 1–18.

[19] C.Foias, *Statistical Study of Navier-Stokes Equations I*, Rend. Sem. Math. Univ. Padova, Vol 48 (1973), 219-348.

[20] C.Foias, O.Manley, R.Temam, *Attractors for the Bénard Problem: existence and physical bounds on their fractal dimension*, Nonlinear Analysis TMA 11(8) (1987) 939-967.

[21] E.Hopf, *Statistical Hydrodynamics and functional calculus*, J.Rat.Mech.Anal. 1 (1952), 87-123.

[22] P.E.Kloeden, J.Langa, *Flattening, squeezing and the existence of random attractors*, Proc. Royal. Soc. London Ser. A, 463 (2007), 163-181.

[23] K.Kuratowski, *Sur les espaces complets*, Fund.Math., Vol.15 (1930), 301-309.

[24] J.A.Langa, G.Łukaszewicz, J.Real, *Finite fractal dimension of pullback attractor for non-autonomous 2-D Navier-Stokes in some unbounded domains*, Nonlinear Analysis TMA 66 (2007) 735-749.

[25] Y.Li, C.K.Zhong, *Pullback attractors for the norm-to-weak continuous process and application to the nonautonomous reaction-diffusion equations*, Applied Mathematics and Computation, vol. 190, (2007), 1020–1029.

[26] G.Łukaszewicz, *Micropolar Fluids. Theory and Applications*, Birkhäuser, Boston, 1999.

[27] G.Łukaszewicz, *Long time behavior of 2D Micropolar Fluid Flows*, Math. Comp. Mod. Vol.34 (2001), 487–509.

[28] G.Łukaszewicz, *Pullback attractors and statistical solutions for 2-D Navier-Stokes equations*, Discrete and Continuous Dynamical Systems 9 (3-4) (2008), 643-659.

[29] G.Łukaszewicz, W.Sadowski, *Uniform attractor for 2D magneto-micropolar fluid flow in some unbounded domains*, Z.Angew.Math.Phys., Vol.55 (2004), 247-257.

[30] G.Łukaszewicz, A.Tarasińska, *H1-pullback attractors for nonautonomous micropolar fluid equations in a bounded domain*, Nonlinear Analysis: Theory, Methods and Applications, Vol 71 (2009) 782-788.

[31] O.A.Ladyzenska, *On the dynamical system generated by the Navier-Stokes eqautions*, J.Soviet Math. 3, 1972.

[32] Q.F.Ma, S.H.Wang, and C.K.Zhong, *Necessary and sufficient conditions for the existence of global attractors for semigroups and applications*, Indiana Univ. Math. J., Vol.51 (2002), 1541–1557.

[33] K.Matsuura, *Exponential attractors for 2D magneto-micropolar fluid flow in a bounded domain*, Discrete Contin. Dyn. Syst.,Series A, (2005), 634-641.

[34] Kh.S.Mekheimer, M.A.El Kot, *The micropolar fluid model for blood flow through a taped artery with a stenosis*, Acta Mechanica Sinica, Vol.24, No. 6 (2008).

[35] N.Mitarai, H.Hayakawa, H.Nakanishi, *Collisional Granular Flow as a Micropolar Fluid*, Phys. Rev. Lett. 88 (2002).

[36] I.Moise, R.Rosa, X.Wang, *Attractors for noncompact nonautonomous systems via energy equations*, Discrete and Continous Dynamical Systems 10 (2004) 473-496.

[37] P.B.Mucha, W.Sadowski, *Long time behavior of a flow in infinite pipe conforming to slip boundary conditions*, Mathematical Methods in Applied Sciences 28, No.15 (2005), 1867-1880

[38] I.Papautsky, J.Brazzle, T.Ameel, B.Frazier *Laminar fluid behavior in microchannels using micropolar fluid theory*, Sensors and actuators A: Physical 73 (1999) 101-108.

[39] J.Ren , X.Ma, *Asymtotic smoothing effects of solutions of two-dimensional micropolar fluid flows* , Applied Mathematics and Computation 204 (2008), 363-367.

[40] O.Reynolds *On the dynamical theory of incompressible viscous fluids and the determination of the criterion*, Phil. Trans. Royal Soc. London, 186 (1894), 123-161.

[41] J.Robinson, *Infinite-Dimensional Dynamical Systems*, Cambridge University Press, 2001.

[42] R.Rosa, *The global attractor for the 2D Navier-Stokes flow on some unbounded domains*, Nonlinear Analysis TMA 32(1) (1998)(1) 71-85.

[43] H.Song, H.Wu, *Pullback attractors of nonautonomous reaction-diffusion equations*, J.Math. Anal. Appl. Vol.325 (2007), 1200–1215.

[44] P.Szopa *On existence and regularity of solutions for 2-D micropolar fluid flow with periodic boundary conditions*, Math. Methods Appl. Sci. 30 (2007), 331-346.

[45] P.Szopa *Finite-dimensionality of 2-D micropolar fluid flow with periodic boundary*, Applicationes Mathematicae 34(3) (2007) 309-339.

[46] A.Tarasińska, *Global attractor for heat convection problem in a micropolar fluid*, Math. Methods Appl. Sci., Vol.29 (2006), 1215-1236.

[47] A.Tarasińska, *Pullback Attractor for Heat Convection Problem in a Micropolar Fluid*, Nonlinear Analysis: Real World Applications, in press, available on-line since 18.03.2009.

[48] R.Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, 2nd. ed., Springer-Verlag, New York, 1997.

[49] R.Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis, Society for Industrial and Applied Mathematics*, Philadelphia, PA, 1983.

[50] M.Visik, A.Fursikov, *Matematiceskie zadaci statistic. gidromechaniky*,

[51] Y.Wang, C.Zhong *On the existence of pullback attractors for non-autonomous reaction-diffusion equations*, Dynamical Systems An International Journal, Online Publication Date: 01 March 2008, DOI: 10.1080/14689360701611821.

[52] Y.J.Wang, C.K.Zhou, and S.F.Zhou, *Pullback attractors of nonautonomous dynamical systems*, Discrete Cont. Dyn.Syst., Vol.16 (2006), 587–614.

[53] C.Zhao, S.Zhou and X.Lian, H_0^1*-uniform attractor and asymptotic smoothing effect of solutions for a nonautonomous micropolar fluid flow in 2D unbounded domains*, Nonlinear Analysis, Real World Applications, Vol.9 (2008), 608-627.