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DG categories and derived categories of coherent sheaves

PhD dissertation

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aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Abstract

In this thesis we investigate derived categories of coherent sheaves on smooth projective varieties and their behaviour under birational morphisms.

We give description of the bounded derived category of coherent sheaves $\mathcal{D}^b(X)$ on a smooth projective variety X provided it admits a full exceptional collection. We prove that if such a collection σ exists the category $\mathcal{D}^b(X)$ is equivalent to the homotopy category of DG modules over some finitely dimensional, directed DG category $\widetilde{\mathcal{C}_\sigma}$.

Then we address the problem of calculating the DG category $\widetilde{\mathcal{C}_\sigma}$. We propose three methods. The first one is based on mutating the given collection σ to a strong one. The second one uses the language of A_∞-categories and Massey products. More precisely, we use the fact that the DG category $\widetilde{\mathcal{C}_\sigma}$ can be calculated via a structure of A_∞-algebra on the endomorphism algebra of object of σ. We prove that higher multiplication $m_n(f_n, \ldots, f_1)$ in this A_∞-algebra is, up to a sign, given by n-tuple Massey products $\mu_n(f_n, \ldots, f_1)$ provided the second set is non-empty.

The third method is based on universal extensions, which originate in representation theory of quasi-hereditary algebras. This method works for full exceptional collections with second and higher Ext-groups between objects vanishing. It constructs from the given exceptional collection σ, a tilting object $\widetilde{E}$ and uses its endomorphism algebra to calculate the DG category $\widetilde{\mathcal{C}_\sigma}$.

Comparing methods using Massey products and universal extensions we describe some new cohomological operations, the so called *relative Massey products.*

Smooth rational surfaces provide a list of examples where full exceptional collections with vanishing higher Ext-groups exist. In particular, on smooth toric surfaces we have canonical full exceptional collections satisfying this condition. We use universal extensions to calculate the corresponding DG categories explicitly.

We notice that most of calculations done for toric, or more generally rational, surfaces depend on the birational morphism from the surface to its minimal model. In more general contex we investigate the relation between $\mathcal{D}^b(X)$ and $\mathcal{D}^b(Y)$ for a birational morphism $f\colon X \to Y$ of smooth projective surfaces. We show that in this case $\mathcal{D}^b(X)$ admits a semi-orthogonal decomposition $\mathcal{D}^b(X) = \langle \mathcal{C}_f, \mathcal{D}^b(Y) \rangle$ and the category $\mathcal{C}_f$ is uniquely determined by the exceptional divisor of the map f. More precisely, all objects in $\mathcal{C}_f$ are scheme-theoretically supported on the discrepancy divisor of f. Finally, the natural abelian subcategory $\mathrm{Coh}(X) \cap \mathcal{C}_f$ is a highest weight category with duality.

Higher weight categories appear in representation theory of semisimple Lie algebras as blocks of the BGG category $\mathcal{O}$. We show that for a family of morphisms $f\colon X \to Y$ of smooth projective surfaces the abelian category $\mathrm{Coh}(X) \cap \mathcal{C}_f$ is indeed equivalent to a singular block $\mathcal{O}_{\omega_1}$ of the BGG category $\mathcal{O}$ for $\mathrm{sl}(n, \mathbb{C})$.

Key words: derived category, exceptional collection, differential graded category,

A_∞-category, quiver, Massey product, birational morphism, Lie algebra, category $\mathcal{O}$.

AMS MSC 2010 classification: 14F05, 16E45, 14E05, 14J26, 14M25, 17B10, 17B45.

Streszczenie

Tematem tej pracy są kategorie pochodne snopów koherentnych na gładkich rozmaitościach rzutowych oraz ich zachowanie przy morfizmach biwymiernych.

W pracy opisujemy ograniczoną kategorię pochodną snopów koherentnych $\mathcal{D}^b(X)$ na gładkiej rozmaitości rzutowej X, gdy na X istnieje pełna wyjątkowa kolekcja σ. Dowodzimy, że w tym przypadku kategoria $\mathcal{D}^b(X)$ jest równoważna kategorii homotopii DG modułów nad pewną skończenie wymiarową, skierowaną DG kategorią $\widetilde{\mathcal{C}_\sigma}$.

Następnie przedstawiamy trzy metody pozwalające na obliczenie DG kategorii $\widetilde{\mathcal{C}_\sigma}$. Pierwsza z nich bazuje na mutacji danej kolekcji do silnej czyli takiej, w której wszystkie wyższe grupy Ext są zerowe. Druga opiera się na iloczynach Massey'a oraz języku A_∞-kategorii. Używa ona faktu, że DG kategoria $\widetilde{\mathcal{C}_\sigma}$ może być obliczona przy pomocy struktury A_∞-algebry na algebrze endomorfizmów obiektów kolekcji σ. Dowodzimy, że wartość wyższego mnożenia $m_n(f_n, \ldots, f_1)$ w tej A_∞-algebrze jest, z dokładnością do znaku, wyznaczona przez n-krotny iloczyn Massey'a $\mu_n(f_n, \ldots, f_1)$, jeżeli tylko ten zbiór jest niepusty.

Trzecia metoda wykorzystuje uniwersalne rozszerzenia. Metoda ta pozwala na obliczenie DG kategorii $\widetilde{\mathcal{C}_\sigma}$ dla kolekcji σ, w której znikają grupy $\mathrm{Ext}^{\geq 2}(T_i, T_j)$. Z takiej kolekcji σ można skonstruować obiekt przechylający $\widetilde{E}$ (ang. tilting object). Algebra endomorfizmów $\widetilde{E}$ pozwala na obliczenie kategorii $\widetilde{\mathcal{C}_\sigma}$.

Porównując drugą i trzecią metodę znajdujemy nową operację kohomologiczną, którą nazywamy relatywnym iloczynem Massey'a.

Pełne wyjątkowe kolekcje z zerowymi grupami $\mathrm{Ext}^{\geq 2}(T_i, T_j)$ istnieją na gładkich powierzchniach wymiernych. W szczególności na gładkich powierzchniach torycznych można w kanoniczny sposób zdefiniować pełne wyjątkowe kolekcje wiązek liniwych spełniające warunek $\mathrm{Ext}^{\geq 2}(\mathcal{L}_i, \mathcal{L}_j) = 0$. Przy pomocy uniwersalnych rozszerzeń w jawny sposób obliczamy odpowiadające im DG kołczany.

W kolejnym rozdziale zauważamy, że obliczając DG kołczan dla powierzchni torycznej, w głównej mierze bierzemy pod uwagę morfizm z tej powierzchni do jej modelu minimalnego. Przyjmując bardziej ogólny punkt widzenia, rozpatrujemy więc biwymierny morfizm $f \colon X \to Y$ gładkich powierzchni rzutowych i badamy, jaką zadaje on relację pomiędzy kategoriami $\mathcal{D}^b(X)$ i $\mathcal{D}^b(Y)$. Okazuje się, że odwzorowanie f zadaje rozkład półortogonalny $\mathcal{D}^b(X) = \langle \mathcal{C}_f, \mathcal{D}^b(Y) \rangle$. Kategoria $\mathcal{C}_f$ jest jednoznacznie wyznaczona przez dywizor wyjątkowy f. Dodatkowo, wszystkie obiekty $\mathcal{C}_f$ mają schemato-teoretyczny nośnik zawarty w dywizorze dyskrepancji odwzorowania f, a naturalna podkategoria abelowa $\mathrm{Coh}(X) \cap \mathcal{C}_f \subset \mathcal{C}_f$ jest kategorią najwyższych wag.

Kategorie najwyższych wag pojawiają się w teorii reprezentacji półprostych algebr Liego jako bloki kategorii $\mathcal{O}$ Bernsteina - Gelfanda - Gelfanda. W ostatnim rozdziale pokazujemy, że dla pewnej rodziny morfizmów $f \colon X \to Y$ gładkich powierzchni rzutowych

kategoria $\mathrm{Coh}(X) \cap \mathcal{C}_f$ jest rzeczywiście równoważna blokowi $\mathcal{O}_{\omega_1}$ kategorii $\mathcal{O}$ dla algebry Liego $\mathrm{sl}(n, \mathbb{C})$.

Słowa kluczowe: kategoria pochodna, wyjątkowa kolekcja, DG kategoria, A_∞-kategoria, kołczan, iloczyn Massey'a, morfizm biwymierny, algebra Liego, kategoria $\mathcal{O}$.

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Contents

Introduction

Theory of derived categories, developed by Grothendieck and Verdier, [63], appeared in algebraic geometry as a convenient language to formulate coherent duality theory, [22]. They also provided the setting for higher cohomology groups of sheaves. Shortly afterwards, other derived functors, such as higher direct images, entered the stage. However, for many years the language of derived categories and homological algebra remained only a formal tool applied to the study of schemes and sheaves on them.

The situation changed with the paper of Mukai [52] which presented a geometrically motivated equivalence between derived categories of non-isomorphic abelian varieties. Other examples of equivalences for K3 surfaces were later constructed by Orlov, [55]. These discoveries raised the question of the invariance of the derived categories of coherent sheaves under natural operations on varieties.

In [13] Bondal and Orlov proved that the bounded derived category of coherent sheaves, $\mathcal{D}^b(X)$ determines the smooth variety X if the canonical divisor K_X of X is either ample or anti-ample. For other varieties derived categories turn out to be a less rigid but still comprehensible invariant.

First properties of $\mathcal{D}^b(X)$. Full exceptional collections and semi-orthogonal decompositions

The work of Bernstein Gelfand Gelfan, [6], and Beilinson's spectral sequence, [3] provided the first evidence that the category $\mathcal{D}^b(\mathbb{P}^n)$ can be understood in terms of finitely many objects and morphisms between them. More precisely, from these papers it follows that $\langle \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \ldots, \mathcal{O}_{\mathbb{P}^n}(n)\rangle$ is a full strong exceptional collection in $\mathcal{D}^b(\mathbb{P}^n)$.

We give a precise definition of an exceptional collection in Section 1.2. For now, let us just say that a collection $\sigma = \langle T_1, \ldots, T_n\rangle$ is a full strong exceptional collection on a smooth variety X if

$$\mathrm{Ext}^l(T_i, T_j) = 0 \text{ for } l \neq 0, \quad \dim \mathrm{Hom}(T_i, T_j) = \begin{cases} 0 & \text{for } i > j, \\ 1 & \text{for } i = j \end{cases}$$

and for every object E in $\mathcal{D}^b(X)$ there exist objects $0 = E_n, E_{n-1}, \ldots, E_0 = E$ and maps $E_i \to E_{i-1}$ with cones of the form $\bigoplus T_i[l_j]$ for some $l_j \in \mathbb{Z}$.

Due to a theorem by Bondal, [9, Theorem 6.2], if a variety X has a full strong exceptional collection then $\mathcal{D}^b(X)$ is equivalent to a bounded derived category of finitely dimensional modules over a finitely dimensional non-commutative algebra A. Moreover, the algebra A can be presented as a path algebra of a finite quiver with relations.

Beilinson's discovery was followed by papers of Kapranov, [32], [33], [34], in which full exceptional collections on quadrics, Grassmannians and homogeneous spaces. Later, full exceptional collections were also constructed on other varieties, such as Grassmannians of isotropic lines, [42], and rational surfaces, [26].

Conjectural full exceptional collections on isotropic Grassmannians were recently constructed by Kuznetsov and Polishchuk in [43]. What still remains to be proved is the fact that these collections are full. This question turns out to be the most difficult part of the theory, especially in the light of a recent discovery of existence of phantom categories, which are triangulated categories with all numerical invariants equal to zero (see [2], [8], [21]).

In the above examples collections are no longer strong and the above mentioned Bondal's theorem cannot be applied. Its generalisation, due to Bondal and Kapranov [10, Theorem 1], states that when X has a full exceptional collection σ then all the information about the category $\mathcal{D}^b(X)$ is captured by a certain DG category $\mathcal{C}_\sigma$ with finitely many objects. However, the spaces of morphisms in $\mathcal{C}_\sigma$ can a priori be infinitely dimensional.

In Chapter 1 we prove the following theorem:

Theorem A (Theorem 1.4.2)**.** *Let X be a smooth projective variety and let $\sigma = \langle \mathcal{E}_1, \ldots, \mathcal{E}_n \rangle$ be a full exceptional collection on X. Then, there exists an ordered, finite DG category $\widetilde{\mathcal{C}_\sigma}$ such that $\mathcal{D}^b(X)$ is equivalent to $\mathcal{D}^b(\widetilde{\mathcal{C}_\sigma})$.*

The fact that $\widetilde{\mathcal{C}_\sigma}$ is an ordered finite DG category implies that $\widetilde{\mathcal{C}_\sigma}$ can be presented as a path algebra of a finite DG quiver with relations Q.

However, it is not at all obvious how to calculate this DG quiver Q. In Chapter 2 we present three methods to determine such a quiver. The first one, described in Section 2.1, uses mutation of exceptional collections. The second one, presented in Section 2.2, uses the language of A_∞-categories and Massey products. Finally, the third one, given in Section 2.3, relies on the construction of universal extensions, which originated in the study of representation theory of quasi-hereditary algebras.

In Chapter 2 we prove that Massey products μ_n can be used to calculate an A_∞-structure on a complex in an enhanced triangulated category.

Theorem B (Theorem 2.2.12)**.** *Let $\mathcal{C}$ be a pretriangulated DG category and let*

$$T^0 \xrightarrow{\partial^0} T^1 \to \ldots \to T^{n-1} \xrightarrow{\partial^{n-1}} T^n$$

be a complex in $H^0(\mathcal{C})$. *Assume that* $\mu_n(\partial^{n-1},\ldots,\partial^0) \neq \emptyset$ *and choose* $f \in \mu_n(\partial^{n-1},\ldots,\partial^0)$. *Then, there exists a minimal* A_∞*-structure on* $H(\mathcal{C})$ *such that* $m_n(\partial^{n-1},\ldots,\partial^0) = -f$ *and* $m_k(\partial^{i+k-1},\ldots,\partial^i) = 0$ *for* $i \in \{0,\ldots,n-1\}$ *and* k *such that* $i+k \leq n-1$.

This A_∞-structure can be later easily translated into a DG structure. In the case when there are not many morphisms between objects of an exceptional collection this information can be sufficient to compute the DG quiver Q.

Using of universal extensions we obtain objects $E_1^n,\ldots,E_n^n$ from an exceptional collection $\langle E_1,\ldots,E_n\rangle$.

Theorem C (Theorem 2.3.1)**.** *Let* $\sigma = \langle E_1,\ldots,E_n\rangle$ *be a full exceptional collection on a smooth projective variety* X *such that* $Ext^i(E_j,E_k) = 0$ *for* $i \neq 0,1$ *and any* k, j. *Then, the DG quiver* Q *is determined by the endomorphism algebra of the tilting generator,* $Hom_X(\bigoplus_i E_i^n, \bigoplus E_i^n)$.

The assumptions of Theorem C are satisfied by a big family of exceptional collections on smooth toric surfaces. In Chapter 3 we explicitly calculate the corresponding DG quivers.

Derived categories and birational geometry

Derived categories are used in the study of birational geometry of varieties. One of the first results in this direction was a proof of existence of crepant resolutions in dimension three purely in the language of derived categories, see [16]. Recently, in [47] Lunts and Kuznetsov proved existence of a non-commutative crepant resolution for irregular singularities.

In [12] Bondal and Orlov suggested that the minimal model program could be understood as a process of minimalisation of derived category of coherent sheaves in a given birational class. They also stated a conjecture that for a variety Y with rational singularities and a proper birational morphism $\pi\colon X \to Y$ the functor $R^{\bullet}\pi_*$ is simply a quotient of $\mathcal{D}^b(X)$ by the kernel of $R^{\bullet}\pi_*$. Finally, the authors conjectured that for any generalized flip $X \dashrightarrow X^+$ there exists a fully faithful functor $D^b(X^+) \to D^b(X)$.

Evidence for this conjecture were provided by Kawamata, [35]. It was proved for flops in dimension three by Bridgeland, [14]. In his paper Bridgeland constructed X^+ as a moduli space of some objects in $\mathcal{D}^b(X)$. Moreover, he speculates that for a small contraction $f\colon X \to Y$ with $-K_X$ being f-ample, the moduli space of appropriate objects of $\mathcal{D}^b(X)$ should be isomorphic to the flip W of X and there should be a fully faithful functor $D(W) \to D(X)$. However, the theory of derived categories for singular spaces is not sufficiently well developed to prove this conjecture. The first steps in the direction of developing the theory for singular varieties were taken by Kawamata in [35] and Abramovich and Chen in [1].

For a birational morphism $f\colon X \to Y$ of smooth projective surfaces Toda in [61] constructed Y as a moduli space of some stable objects in $\mathcal{D}^b(X)$. In particular, he obtained the minimal model of X in this way.

The calculations conducted in Chapter 3 turn out to depend mostly on the birational morphism from a smooth rational surface to its minimal model. Therefore, in Chapter 4 we focus our attention on a birational morphism $f\colon X \to Y$ of smooth projective surfaces. We investigate the relation between $\mathcal{D}^b(X)$ and $\mathcal{D}^b(Y)$ given by the map f.

We associate with f a category $\mathcal{C}_f \subset \mathcal{D}^b(X)$. Then, we prove the following theorem:

Theorem D (Theorem 4.4.4)**.** *Let $f\colon X \to Y$ be a birational morphism of smooth projective surfaces and let $E = \sum E_i$ be the exceptional divisor of f. It has a non-reduced scheme structure $\widetilde{E} = \sum a_i E_i$ given by the discrepancy of f, $K_X = f^*K_Y + \widetilde{E}$ and let $\iota\colon \widetilde{E} \hookrightarrow X$ denote the closed embedding.*

Let $\mathcal{C}_f = \{\mathcal{E} \in \mathcal{D}^b(X) \,|\, R^{\bullet}f_*(\mathcal{E}) = 0\}$ be a full subcategory of $\mathcal{D}^b(X)$. Then any object $\mathcal{E} \in \mathcal{C}_f$ is of the form $\iota_*\widetilde{\mathcal{E}}$ for some $\widetilde{\mathcal{E}} \in \mathcal{D}^b(\widetilde{E})$.

Thus, the category $\mathcal{C}_f$ is closely related to the exceptional divisor of the map f.

We continue the study of $\mathcal{C}_f$ and prove in Proposition 4.5.4 that its natural abelian subcategory $\mathrm{Coh}(X) \cap \mathcal{C}_f \subset \mathcal{C}_f$ has a structure of a highest weight category with duality.

Highest weight categories appeared first in the study of representation theory of semisimple Lie algebras. In Chapter 5 we prove that for some birational morphisms $f\colon X \to Y$ the abelian category $\mathrm{Coh}(X) \cap \mathcal{C}_f$ is indeed equivalent to a category of modules over $\mathrm{sl}(n, k)$.

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Notation

In the following we work over an algebraically closed field k of characteristic zero. All the varieties are considered as algebraic varieties in the sense of [23].

Chapter 1

DG categories and exceptional collections

The main result of this chapter is Theorem 1.4.2 about existence of a finite DG quiver of any exceptional collection on a smooth projective variety X. This theorem provides description of the bounded derived category of coherent sheaves $\mathcal{D}^b(X) = \mathcal{D}^b(\mathrm{Coh}(X))$, provided a full exceptional collection exists. In this case, $\mathcal{D}^b(X)$ is equivalent to the homotopy category of DG modules over a DG algebra $\widetilde{\mathcal{C}}_\sigma$. This DG algebra is finitely dimensional over k and can be presented as a path algebra of an ordered DG quiver. Thus Theorem 1.4.2 generalises [9, Theorem 6.2].

Theorem 1.4.2 follows from existence of a DG enhancement of $\mathcal{D}^b(X)$, see [10, Theorem 1], and Theorem 1.4.1. Let $\mathcal{C}$ be a DG category with finitely many objects and such that $H(\mathcal{C})$ is ordered and has finitely dimensional Hom-spaces. Theorem 1.4.1 allows us to substitute $\mathcal{C}$ with a derived Morita-equivalent DG category $\widetilde{\mathcal{C}}$ which itself is ordered and has finitely dimensional Hom-spaces. Note that by the result of Lunts and Orlov, [48, Theorem 9.9], the DG enhancement of $\mathcal{D}^b(X)$ is unique.

The proof of Theorem 1.4.1 uses A_∞-categories and existence of a minimal model of an A_∞-category. It relies on existence of the universal enveloping DG category of an A_∞-category and explicit description of this DG category by bar and cobar construction given in [45, Section 2].

For the convenience of reader in Section 1.1 we recall all basic definitions from homological algebra in the thesis. In Section 1.2 we define after [9] exceptional collections, the associated quiver and description of $\mathcal{D}^b(X)$ it gives. In Section 1.2.1 we define DG categories and we recall the crucial result of Bondal and Kapranov (see [10, Theorem 1]). In Section 1.3 we define after [38] and [45] A_∞-categories and we describe the construction of [45] of the universal DG category associated to an A_∞-category. Finally, in Section 1.3.2 the category of A_∞-modules is introduced after [38, 45]. All of the above results and definitions are used in Section 1.4 to prove Theorems 1.4.1 and 1.4.2.

1.1 Preliminary definitions

We recall same basic homological algebra definitions following [20], [50] and [64].

We work only with *k-linear categories*, i.e. with categories in which the morphism space between any two objects has a structure of a k-vector space.

Definition 1.1.1. *A category $\mathcal{A}$ is* preadditive *if*

(A1) For any pair of objects $A_1, A_2 \in Ob(\mathcal{A})$ the set $Hom_{\mathcal{A}}(A_1, A_2)$ has a structure of an abelian group, i.e. of a $\mathbb{Z}$-module.

(A2) For any three objects $A_1, A_2, A_3 \in Ob(\mathcal{A})$ the composition map

$$Hom_{\mathcal{A}}(A_2, A_3) \otimes_{\mathbb{Z}} Hom_{\mathcal{A}}(A_1, A_2) \to Hom_{\mathcal{A}}(A_1, A_3)$$

is a homomorphism of abelian groups.

Definition 1.1.2. *A preadditive category $\mathcal{A}$ is called* additive *if*

(A3) There exists a zero object $0_{\mathcal{A}} \in Ob(\mathcal{A})$ which is both initial and terminal in $\mathcal{A}$.

(A4) For any pair of objects $A_1, A_2 \in Ob(\mathcal{A})$ the direct sum $A_1 \oplus A_2$ and the direct product $A_1 \times A_2$ exist and they coincide, i.e. $A_1 \oplus A_2 = A_1 \times A_2$.

The axioms $(A1)$ - $(A4)$ allow us to define in any additive category finite products and coproducts. In particular, for objects $A_1, A_2 \in \mathrm{Ob}(\mathcal{A})$ and any map $f \in \mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$ we define the *kernel* of f, $\mathrm{Ker}\, f$, as the equaliser of f and the zero map $0_{A_1,A_2} \in \mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$. From the definition of the equaliser it follows that there is a canonical morphism $\iota\colon \mathrm{Ker}\, f \to A_1$. Analogously, we define the *cokernel* of f, $\mathrm{Coker}\, f$, as the coequaliser of f and the zero map $0_{A_1,A_2} \in \mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$. Again, it follows from the definition that there is a canonical morphism $\pi\colon A_2 \to \mathrm{Coker}\, f$.

Definition 1.1.3. *An additive category $\mathcal{A}$ is* abelian *if*

(A5) For any $A_1, A_2 \in Ob(\mathcal{A})$ and any $f \in Hom_{\mathcal{A}}(A_1, A_2)$ there exists a sequence

$$K \xrightarrow{\iota} A_1 \xrightarrow{p} I \xrightarrow{i} A_2 \xrightarrow{\pi} K'$$

such that

(i) $i \circ p = f$,

(ii) $(K, \iota\colon X \to A_1)$ is the kernel of f, $(K', \pi\colon A_2 \to K')$ is the cokernel of f,

(iii) $(I, p\colon A_1 \to I)$ is the cokernel of ι and $(I, i\colon I \to A_2)$ is the kernel of π.

The basic example of an abelian category is the category $\mathcal{A}b$ of abelian groups.

Morphisms
$$X^\bullet = \ldots \xrightarrow{\partial^{n-2}} X^{n-1} \xrightarrow{\partial^{n-1}} X^n \xrightarrow{\partial^n} \ldots$$
in an abelian category $\mathcal{A}$ form a *cochain complex* if the composition $\partial^n \circ \partial^{n-1} = 0$ for all $n \in \mathbb{Z}$. The n-th *cohomology group* of a cochain complex $X^\bullet$ is defined as

$$H^n(X^\bullet) = \operatorname{Coker} a^{n-1} = \operatorname{Ker} b^n$$

where the maps a^{n-1}, b^n fit into the commutative diagram

$$\begin{tikzcd}
 & \operatorname{Coker} \partial^{n-1} \arrow[dr, "b^n"] & \\
X^{n-1} \arrow[r, "\partial^{n-1}"] \arrow[dr, "a^{n-1}"'] & X^n \arrow[u] \arrow[r, "\partial^n"] & X^{n+1} \\
 & \operatorname{Ker} \partial^n \arrow[u] &
\end{tikzcd}$$

A cochain complex $(X^\bullet, \partial^\bullet)$ is called *exact* if all its cohomology groups are zero.

An additive functor $F\colon \mathcal{A} \to \mathcal{A}'$ between abelian categories is exact if for any short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$ the sequence

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is exact in $\mathcal{A}'$.

Let us recall that an object $I \in \operatorname{Ob}(\mathcal{A})$ is *injective* if the functor $\operatorname{Hom}_{\mathcal{A}}(-, I)\colon \mathcal{A} \to \mathcal{A}b$ is exact.

Finally, we say that an abelian category $\mathcal{A}$ *has enough injectives* if any object $A \in \operatorname{Ob}(\mathcal{A})$ can be embedded into an injective object of $\mathcal{A}$.

$$0 \to A \to I$$

An injective object I of $\mathcal{A}$ is an *injective envelope* of A if $A \hookrightarrow I$ is an *essential extension*, i.e. any $J \subset I$ such that $J \cap A = 0$ is zero.

We will be mostly interested in derived categories of abelian categories. Such categories are triangulated in the following sense:

Definition 1.1.4. *A* triangulated category $\mathcal{T}$ *is an additive category equipped with an automorphism $T\colon \mathcal{T} \to \mathcal{T}$ called a translation functor and a family of distinguished triangles of the form*
$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$$
satisfying the following axioms:

(TR1) *Every morphism* $u\colon A \to B$ *can be embedded into a distinguished triangle. If* $A = B$ *and* $C = 0$ *then the triangle*

$$A \xrightarrow{\mathrm{id}_A} A \xrightarrow{0} 0 \xrightarrow{0} T(A)$$

is distinguished. If a triangle (u, v, w) *is isomorphic to a distinguished triangle* (u', v', w')

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "\simeq"] & B \arrow[r, "v"] \arrow[d, "\simeq"] & C \arrow[r, "w"] \arrow[d, "\simeq"] & T(A) \arrow[d, "\simeq"] \\
A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & T(A')
\end{tikzcd}$$

then the triangle (u, v, w) *is also distinguished.*

(TR2) *If*

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$$

is a distinguished triangle then so is

$$B \xrightarrow{v} C \xrightarrow{w} T(A) \xrightarrow{-T(u)} T(B)$$

(TR3) *Any morphisms* $f\colon A \to A'$, $g\colon B \to B'$ *such that* $g \circ u = u' \circ f$ *can be completed to a map of distinguished triangles*

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, dashed, "h"] & T(A) \arrow[d, "T(f)"] \\
A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & T(A')
\end{tikzcd}$$

(TR4) *Every diagram of the form*

$$\begin{tikzcd}
A' \arrow[dr, "{[1]}"] \arrow[dd, "{[1]}"'] & & C \arrow[ll] \\
& B \arrow[ur] \arrow[dl] \arrow[u, phantom, "\star"] \arrow[d, phantom, "\star"] \arrow[l, phantom, "\circlearrowleft"] \arrow[r, phantom, "\circlearrowleft"] & \\
C' \arrow[rr, "{[1]}"'] & & A \arrow[ul] \arrow[uu]
\end{tikzcd}$$

where triangles marked with $\star$ *are distinguished and the ones marked with* $\circlearrowleft$ *commute, can be completed to an octahedron diagram, i.e. there exists an object* B' *which fits into a diagram*

$$\begin{tikzcd}
A' \arrow[dd, "{[1]}"'] & & C \arrow[ll] \arrow[dl] \\
& B' \arrow[ul] \arrow[dr, "{[1]}"] \arrow[u, phantom, "\circlearrowleft"] \arrow[d, phantom, "\circlearrowleft"] \arrow[l, phantom, "\star"] \arrow[r, phantom, "\star"] & \\
C' \arrow[ur] \arrow[rr, "{[1]}"'] & & A \arrow[uu]
\end{tikzcd}$$

Finally, we require that the composite morphisms $B \to B'$, through C and C', coincide and that the composite morphisms $B' \to B$, through A and A' coincide.

The value of the functor T on an element $t \in \mathcal{T}$ is often denoted by $t[1]$. Analogously, $t[n] = T^n(t)$ and $t[-n] = (T^{-1})^n(t)$.

The first example of a triangulated category is the homotopy category $K^b(\mathcal{A})$ of bounded complexes over an abelian category $\mathcal{A}$. The category $K^b(\mathcal{A})$ is constructed from an additive category $\mathrm{Kom}_0^b(\mathcal{A})$ of bounded complexes of objects of $\mathcal{A}$. Objects of both $\mathrm{Kom}_0^b(\mathcal{A})$ and $K^b(\mathcal{A})$ are cochain complexes of objects of $\mathcal{A}$ with all but finitely many objects equal to zero. Morphisms in the category $\mathrm{Kom}_0^b(\mathcal{A})$ are morphisms of complexes, i.e. collections of morphisms $f^i \colon K^i \to L^i$ such that $\partial^{i+1}_{L^\bullet} f^i - f^{i+1}\partial^i_{K^\bullet} = 0$. Morphisms in the homotopy category $K^b(\mathcal{A})$ are morphisms in the category $\mathrm{Kom}_0^b(\mathcal{A})$ modulo the homotopy equivalence. Recall that a morphism $f^i \colon K^i \to L^i$ is *homotopic to zero* if there exists a homotopy $h^i \colon K^i \to L^{i-1}$ such that $f^i = h^i \partial^{i-1}_{K^\bullet} + \partial^i_{L^\bullet} h^{i-1}$.

Analogously, one can consider categories $\mathrm{Kom}_0^+(\mathcal{A})$, $\mathrm{Kom}_0^-(\mathcal{A})$ and $\mathrm{Kom}_0(\mathcal{A})$ of respectively bounded from below, bounded from above and unbounded complexes of objects of $\mathcal{A}$. Dividing the spaces of morphisms in these categories by the relation of homotopy equivalence we obtain categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K(\mathcal{A})$.

A morphism $f^\bullet \colon K^\bullet \to L^\bullet$ in the category $\mathrm{Kom}_0^b(\mathcal{A})$ ($\mathrm{Kom}_0^+(\mathcal{A})$, $\mathrm{Kom}_0^-(\mathcal{A})$, $\mathrm{Kom}_0(\mathcal{A})$ respectively) is a *quasi-isomorphism* if it induces the identity morphism on the cohomology groups $H^i(K^\bullet) \simeq H^i(L^\bullet)$. Analogously, we define quasi-isomorphism in $K^b(\mathcal{A})$, $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K(\mathcal{A})$.

To every abelian category $\mathcal{A}$ we can associate its bounded derived category $\mathcal{D}^b(\mathcal{A})$ which is a triangulated category. It can be characterised by the following universal property. There exists a functor $Q \colon \mathrm{Kom}_0^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{A})$ such that $Q(f)$ is an isomorphism if f is a quasi-isomorphism and any functor $F \colon \mathrm{Kom}_0^b(\mathcal{A}) \to \mathcal{D}$ which transforms quasi-isomorphisms into isomorphisms can be uniquely factorised through $Q \colon \mathrm{Kom}_0^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{A})$. Analogously, we define bounded from below $\mathcal{D}^+(\mathcal{A})$, bounded from above $\mathcal{D}^-(\mathcal{A})$ and unbounded $\mathcal{D}(\mathcal{A})$ derived category of $\mathcal{A}$. In general, $\mathcal{D}^b(\mathcal{A})$ is a localisation of $K^b(\mathcal{A})$ with respect to the class of quasi-isomorphisms, i.e. to obtain $\mathcal{D}^b(\mathcal{A})$ we formally invert all quasi-isomorphisms in $K^b(A)$. An analogous construction for $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K(\mathcal{A})$ gives $\mathcal{D}^+(A)$, $\mathcal{D}^-(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ respectively.

If the category $\mathcal{A}$ has enough injectives then every object A has an *injective resolution.* The first element in the resolution, I^0, is the injective object such that there exists a monomorphism $A \hookrightarrow I_0$. Element I^n is constructed as the injective object into which the cokernel of map $I^{n-2} \to I^{n-1}$ embeds, i.e. we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\iota_A"] & I^0 \arrow[rr, "\partial^0"] \arrow[dr] \arrow[rr, phantom, "\circlearrowleft"', shift right=4ex] & & I^1 \arrow[rr, "\partial^1"] \arrow[dr] \arrow[rr, phantom, "\circlearrowleft"', shift right=4ex] & & I^2 \\
& & & \operatorname{Coker} \iota_A \arrow[ur] \arrow[dr] & & \operatorname{Coker} \partial^0 \arrow[ur] \arrow[dr] & \\
& & 0 \arrow[ur] & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

The cochain complex $0 \to I^0 \to I^1 \to \dots$ has only zeroth cohomology group $H^0(I^\bullet) = A$.

Let $\mathcal{A}$ be an abelian category with enough injectives and let $\mathrm{Inj}_{\mathcal{A}} \subset \mathcal{A}$ be a full subcategory with the class of objects equal to the class of injective objects in $\mathcal{A}$. Then the bounded from below derived category of $\mathcal{A}$, $\mathcal{D}^+(\mathcal{A})$ is the homotopy category $K + (\mathrm{Inj}_{\mathcal{A}})$ of complexes of injective objects of $\mathcal{A}$ bounded from below. Injective resolutions of objects of $\mathcal{A}$ define an additive functor $\mathcal{A} \to \mathcal{D}^+(\mathcal{A})$.

Let X be a smooth projective variety defined over k. We can associate to it a k-linear abelian category $\mathrm{Coh}(X)$ of coherent sheaves on X. This category does not have injective objects. However, it is a full subcategory of the category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X. The category $\mathrm{QCoh}(X)$ has enough injectives and the construction described above gives $\mathcal{D}^+(\mathrm{QCoh}(X))$, the bounded from below derived category of quasi-coherent sheaves on X. The bounded derived category of coherent sheaves on X, denoted by $\mathcal{D}^b(X)$ is then the full subcategory of $\mathcal{D}^+(\mathrm{QCoh}(X)) = K^+(\mathrm{Inj}_{\mathrm{QCoh}(X)})$ consisting of complexes having only finitely many non-zero and coherent cohomology sheaves.

1.2 Exceptional collections

Full strong exceptional collections provide an easy description of triangulated categories. We recall definitions and the resulting description of $\mathcal{D}^b(X)$ after [9].

Recall, that an object $T \in \mathcal{T}$ of a triangulated category is *exceptional* if $\mathrm{Hom}_{\mathcal{T}}(T, T) = k$ and $\mathrm{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for $i \neq 0$. An ordered collection $\sigma = \langle T_1, \dots, T_n \rangle$ of exceptional objects is called an *exceptional collection* if $\mathrm{Hom}_{\mathcal{T}}(T_i, T_j[l]) = 0$ for $i > j$ and all l. An exceptional collection σ is *strong* if we also have $\mathrm{Hom}_{\mathcal{T}}(T_i, T_j[l]) = 0$ for $l \neq 0$ and all i, j. Finally, the collection σ is *full* if the smallest strictly full subcategory of $\mathcal{T}$ containing $T_1, \dots, T_n$ is equal to $\mathcal{T}$. Recall, that a subcategory $\mathcal{T}' \subset \mathcal{T}$ is strictly full if it is full and if for any $T \in \mathcal{T}'$ and S is isomorphic to T in $\mathcal{T}$ we have $S \in \mathcal{T}'$.

If an Ext-finite triangulated category $\mathcal{T}$ has a full strong exceptional collection $\sigma = \langle T_1, \dots, T_n \rangle$ then $\mathcal{T}$ is equivalent to the bounded derived category of right modules over an algebra $A = \mathrm{End}_{\mathcal{T}}(\bigoplus T_i)$. Equivalence is given by the functor

$$R\,\mathrm{Hom}(-, \bigoplus T_i) \colon \mathcal{T} \to \mathcal{D}^b(\text{mod-}A).$$

The endomorphism algebra A of $\bigoplus T_i$ can be represented as the path algebra of a quiver with relations Q. The vertices of the quiver correspond to objects $T_1, \ldots, T_n$ of σ. The arrows between vertices correspond to a chosen basis of the Hom spaces. Finally, relations in the quiver are obtained from composition of homomorphisms in $\mathcal{T}$.

For a given quiver Q with n vertices its path algebra $\overline{k[Q]}$ is a k-algebra with basis consisting of all paths in the quiver Q together with paths ε_i of length zero at every vertex. The composition of two paths is equal to the path obtained by concatenation or zero if the head of one path does not coincide with the tail of another. If the quiver Q has finitely many arrows and no oriented cycles (as in the case of a quiver of a full strong exceptional collection on a smooth projective variety) then the path algebra $\overline{k[Q]}$ is a finitely generated non-commutative algebra with orthogonal projectors $\varepsilon_1, \ldots \varepsilon_n$.

Relations in the quiver Q are elements of $\overline{k[Q]}$ which are finite sums of paths having the same head and tail. Relations generate an ideal S of $\overline{k[Q]}$ and the path algebra of a quiver with relations is by definition $k[Q] = \overline{k[Q]}/S$.

The first example of a full strong exceptional collection is $\langle \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2) \rangle$ in $\mathcal{D}^b(\mathbb{P}^2)$. Its quiver Q has three vertices. As $\mathrm{Hom}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1)) \simeq \mathrm{Hom}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2)) \simeq k^3$ there are three arrows between adjacent vertices. Finally, the composition map

$$\mathrm{Hom}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2)) \otimes \mathrm{Hom}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1)) \to \mathrm{Hom}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(2))$$

is surjective and hence the quiver Q has the form

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}^2} \arrow[r, shift left=2, "\alpha_1"] \arrow[r, "\alpha_2"] \arrow[r, shift right=2, "\alpha_3"'] & \mathcal{O}_{\mathbb{P}^2}(1) \arrow[r, shift left=2, "\beta_1"] \arrow[r, "\beta_2"] \arrow[r, shift right=2, "\beta_3"'] & \mathcal{O}_{\mathbb{P}^2}(2) \end{tikzcd}$$

Relations in this quiver are given by

$$\beta_2\alpha_1 - \beta_1\alpha_2 = 0, \qquad \beta_3\alpha_1 - \beta_1\alpha_3 = 0, \qquad \beta_3\alpha_2 - \beta_2\alpha_3 = 0.$$

Let Q be a quiver with relations. As $\varepsilon_1 + \ldots + \varepsilon_n = 1$ any right $k[Q]$-module V can be decomposed as a direct sum $V = \bigoplus V_i$ with $V_i = V \cdot \varepsilon_i$. The structure of $k[Q]$-module gives also for every arrow a in Q with head $h(a)$ and tail $t(a)$ a linear map $\widetilde{a}\colon V_{h(a)} \to V_{t(a)}$. Composing these linear maps we obtain a map $\widetilde{p}\colon V_{h(p)} \to V_{t(p)}$ for any path in the quiver. Also, for any element $p_1 + \ldots + p_s$ of the ideal S generated by relations the map $\widetilde{p}_1 + \ldots + \widetilde{p}_s$ is zero.

On the other hand, a set $V_1, \ldots, V_n$ of k-vector spaces and linear maps $\widetilde{a}\colon V_{h(a)} \to V_{t(a)}$ for every arrow a in Q define a $k[Q]$ module structure on $V = \bigoplus V_i$ if for any $p_1 + \ldots + p_s \in S$ the linear map $\widetilde{p}_1 + \ldots + \widetilde{p}_s$ is zero.

One can also think of the path algebra $k[Q]$ as of a k-linear category $\mathcal{A}_Q$ with n objects. The space $\mathrm{Hom}_{\mathcal{A}_Q}(i, j)$ has a k-basis consisting of paths between corresponding vertices and the composition between them is given by multiplication in $k[Q]$. Then the category $\mathcal{A}_Q$ is pre-additive and right modules over $k[Q]$ correspond to contravariant functors from $\mathcal{A}_Q$ to the category Vect_k of k-vector spaces.

Already mentioned full strong exceptional collection $\langle \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2)\rangle$ on $\mathbb{P}^2$, and more generally, $\langle \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \ldots, \mathcal{O}_{\mathbb{P}^n}(n)\rangle$ on $\mathbb{P}^n$ were the first examples of full strong exceptional collections. The fact that they are full follows from Beilinson's spectral sequence, [3] and also [6]. In [54] Orlov described behaviour of derived categories of coherent sheaves under blow ups in smooth centres. In particular, [54, Theorem 4.3] shows that if $Z \subset X$ is smooth and both Z and X have full exceptional collections then so does $Y = \mathrm{Bl}_Z(X)$. However, even if the collections on X and Z are strong the resulting collection on Y does not need to be strong.

One of the simplest examples when such a situation occurs is that of a smooth surface X obtained from $\mathbb{P}^2$ by blowing up a point and then blowing up a point on the exceptional divisor $E_1 \subset X_1$. Let $X \xrightarrow{f} X_1 \xrightarrow{g} \mathbb{P}^2$ be the blow ups of the points, let $E_1 \subset X_1$ be the exceptional divisor of g and $E_2 \subset X_2$ the exceptional divisor of f.

We have already seen that $\langle \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2)\rangle$ is a full strong exceptional collection on $\mathbb{P}^2$. Orlov's theorem says that $\langle \mathcal{O}_{E_1}(E_1)[-1], g^*\mathcal{O}_{\mathbb{P}^2}, g^*\mathcal{O}_{\mathbb{P}^2}(1), g^*\mathcal{O}_{\mathbb{P}^2}(2)\rangle$ is a full exceptional collection on X_1. It is easy to check that it is also strong. By the abuse of notation E_1 also denotes the strict transform of E_1 to X. Then, again by Orlov's theorem, $\langle \mathcal{O}_{E_2}(E_2)[-1], \mathcal{O}_{E_1+E_2}(E_1+E_2)[-1], f^*g^*\mathcal{O}_{\mathbb{P}^2}, f^*g^*\mathcal{O}_{\mathbb{P}^2}(1), f^*g^*\mathcal{O}_{\mathbb{P}^2}(2)\rangle$ is a full exceptional collection on X. However, as

$$\mathrm{Hom}_X(\mathcal{O}_{E_2}(E_2)[-1], \mathcal{O}_{E_1+E_2}(E_1+E_2)[-1]) = k = \mathrm{Ext}^1_X(\mathcal{O}_{E_2}(E_2)[-1], \mathcal{O}_{E_1+E_2}(E_1+E_2)[-1])$$

the collection is not strong.

1.2.1 DG categories

In order to generalise the description of $\mathcal{D}^b(X)$ given by a full strong exceptional collection σ to the case of an arbitrary full exceptional collection Bondal and Kapranov used in [10] the techniques of differential graded categories.

Definition 1.2.1. *A* differential graded category *(or a DG category) is a preadditive category $\mathcal{C}$ in which $\mathrm{Hom}^\bullet_{\mathcal{C}}(A, B)$ are endowed with a $\mathbb{Z}$-grading and a differential ∂ of degree one. The composition of morphisms*

$$\mathrm{Hom}^\bullet_{\mathcal{C}}(B, C) \otimes \mathrm{Hom}^\bullet_{\mathcal{C}}(A, B) \to \mathrm{Hom}^\bullet_{\mathcal{C}}(A, C)$$

is a morphism of complexes and for any object C of $\mathcal{C}$ the identity morphism id_C is a closed morphism of degree zero.

For an element x in a graded vector space the grading of x is denoted by $|x|$. The morphisms of degree i in $\mathcal{C}$ are denoted by $\mathrm{Hom}^i_{\mathcal{C}}(A, B)$.

A DG category $\mathcal{C}$ is *ordered* if there exists a partial order $\preceq$ on the set of objects of $\mathcal{C}$ such that $\mathrm{Hom}^{\bullet}_{\mathcal{C}}(A, B) = 0$ for $B \preceq A$. It is called *finite* if the set of objects of $\mathcal{C}$ is finite and for any objects C_1, C_2 of $\mathcal{C}$ the vector space $\mathrm{Hom}^{\bullet}_{\mathcal{C}}(C_1, C_2)$ is finite dimensional.

We associate to a DG category $\mathcal{C}$ the following categories. The graded category $\mathcal{C}^{\mathrm{gr}}$ has the same objects as the category $\mathcal{C}$ while for any pair of objects C_1, C_2 the space $\mathrm{Hom}_{\mathcal{C}^{\mathrm{gr}}}(C_1, C_2)$ is a graded vector space obtained from $\mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$ by forgetting the differential. The *homotopy category* $H(\mathcal{C})$ also has the same objects as $\mathcal{C}$ and a space morphisms between any pair $\mathrm{Hom}_{H(\mathcal{C})}(C_1, C_2)$ is the graded vector space $H^{\bullet}(\mathrm{Hom}_{\mathcal{C}}(C_1, C_2))$. Finally, $H^0(\mathcal{C})$ is a preadditive category with the same objects as $\mathcal{C}$ and $\mathrm{Hom}_{H^0(\mathcal{C})}(C_1, C_2) = H^0(\mathrm{Hom}_{\mathcal{C}}(C_1, C_2))$ for any pair of objects C_1, C_2.

A morphism $s\colon C \to C'$ in $\mathcal{C}$ is a *homotopy equivalence* if the induced map on cohomology $H(s)$ is an equivalence. Then we say that C and C' are *homotopy equivalent*.

A DG functor $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ between two DG categories is an additive functor such that for any pair of objects C_1, C_2 of $\mathcal{C}_1$ the induced map $\mathrm{Hom}_{\mathcal{C}_1}(C_1, C_2) \to \mathrm{Hom}_{\mathcal{C}_2}(F(C_1), F(C_2))$ is a morphism of complexes. Such a functor F induces functors $F^{\mathrm{gr}}\colon \mathcal{C}_1^{\mathrm{gr}} \to \mathcal{C}_2^{\mathrm{gr}}$, $H(F)\colon H(\mathcal{C}_1) \to H(\mathcal{C}_2)$ and $H^0(F)\colon H^0(\mathcal{C}_1) \to H^0(\mathcal{C}_2)$.

We construct the DG category $\text{DG-Fun}(\mathcal{C}_1, \mathcal{C}_2)$ of DG functors in the following way. Objects of this category are covariant DG functors $F, G\colon \mathcal{C}_1 \to \mathcal{C}_2$ and we define morphism $\mathrm{Hom}^k_{\text{DG-Fun}(\mathcal{C}_1,\mathcal{C}_2)}(F, G)$ as the set of natural transformations $t\colon F^{\mathrm{gr}} \to G^{\mathrm{gr}}[k]$ (i.e. for $C \in \mathcal{C}_1$ we have $t_C \in \mathcal{C}_2^k(F(C), G(C))$). The differential ∂ is defined pointwise; for $t_C\colon F(C) \to G[k](C)$ we have $(\partial(t))_C = \partial(t_C)\colon F(C) \to G[k-1](C)$.

The category of contravariant DG functors is denoted by $\text{DG-Fun}^{\mathrm{o}}(\mathcal{C}, \mathcal{C}')$. A DG functor $F\colon \mathcal{C} \to \mathcal{C}'$ is called a *quasi-equivalence* if for any $C_1, C_2 \in \mathrm{Ob}\,\mathcal{C}$ the map $\mathrm{Hom}^{\bullet}_{\mathcal{C}}(C_1, C_2) \to \mathrm{Hom}^{\bullet}_{\mathcal{C}'}(F(C_1), F(C_2))$ is a quasi-isomorphism and the map $H(F)\colon H(\mathcal{C}) \to H(\mathcal{C}')$ is an equivalence of categories. A quasi-equivalence inducing a bijection on the set of objects of $\mathcal{C}$ and $\mathcal{C}'$ is a *quasi-isomorphism*.

The category DGVect_k of complexes of vector spaces with morphisms $f^{\bullet}\colon V^{\bullet} \to W^{\bullet}$ for which there exists $l \in \mathbb{Z}$, such that $f^{\bullet}(V^i) \subset W^{i+l}$ and a differential

$$\partial(f)^i = \partial_W f^{i+1} - (-1)^l f^{i+1} \partial_V$$

is a DG category.

In general, if $\mathcal{A}$ is an abelian category then the category $\mathrm{Kom}^+(\mathcal{A})$ of cochain complexes of objects of $\mathcal{A}$ is a DG category with morphisms and differentials defined as in the category DGVect_k. If the category $\mathcal{A}$ has enough injectives then one can also consider the DG category $\mathrm{Kom}^+(\mathrm{Inj}_{\mathcal{A}})$ of bounded from below cochain complexes of injective objects in $\mathcal{A}$. Its homotopy category is the category $K^+(\mathrm{Inj}_{\mathcal{A}})$ defined before. Hence, if an abelian category $\mathcal{A}$ has enough injectives we have $\mathcal{D}^+(\mathcal{A}) \simeq H^0(\mathrm{Kom}^+(\mathrm{Inj}_{\mathcal{A}}))$. It turns out that the DG category $\mathrm{Kom}^+(\mathrm{Inj}_{\mathcal{A}})$ has more structure, i.e. it is pretriangulated and the equivalence $\mathcal{D}^+(\mathcal{A}) \simeq H^0(\mathrm{Kom}^+(\mathrm{Inj}_{\mathcal{A}}))$ is the first example of a DG enhancement of a triangulated category.

In order to define a DG enhancement of a triangulated category in full generality we need to introduce twisted complexes.

Analogously to modules over a quiver, we define a *right DG module* M over a DG category C to be an element of $\text{DG-Fun}^{\circ}(\mathcal{C}, \mathrm{DGVect}_k)$. For simplicity we denote this category by $\mathcal{M}\text{od-}\mathcal{C}$. Following [36, Section 4] we define a derived category $\mathcal{D}(\mathcal{C})$ as a localization of $H^0(\mathcal{M}\text{od-}\mathcal{C})$ with respect to the class of quasi-equivalences.

Recall after [47] that an object A of an additive category $\mathcal{A}$ is *compact* if $\mathrm{Hom}_{\mathcal{A}}(A, -)$ commutes with arbitrary direct sums. Let us denote by $\mathcal{D}^b(\mathcal{A})$ the subcategory of $\mathcal{D}(\mathcal{A})$ consisting of compact objects. Then the Yoneda embedding gives a functor $h\colon \mathcal{C} \to \mathcal{M}\text{od-}\mathcal{C}$ which assigns to every $C \in \mathcal{C}$ a module $h_C = \mathrm{Hom}_{\mathcal{C}}(-, C)$. If a module M is quasi-isomorphic to h_C for some $C \in \mathcal{C}$ then it is *quasi-representable*.

A *quasi-functor* between DG categories $\mathcal{C}$ and $\mathcal{C}'$ is a functor $F\colon \mathcal{C} \to \mathcal{M}\text{od-}\mathcal{C}'$ whose essential image consists of quasi-representable functors, (see [48, Section 1]). Because the homotopy category of quasi-representable functors over $\mathcal{C}'$ is equivalent to the homotopy category of $\mathcal{C}'$, a quasi-functor F gives a functor $H^0(F)\colon H^0(\mathcal{C}) \to H^0(\mathcal{C}')$.

For a DG category $\mathcal{C}$ we define the category $\widehat{\mathcal{C}}$ of formal shifts. The objects of $\widehat{\mathcal{C}}$ are $C[n]$ where $C \in \mathcal{C}$ and $n \in \mathbb{N}$. For elements $C_1[m]$ and $C_2[n]$ of $\widehat{\mathcal{C}}$ we put $\mathrm{Hom}^l_{\widehat{\mathcal{C}}}(C_1[m], C_2[n]) = \mathrm{Hom}^{l+n-m}_{\mathcal{C}}(C_1, C_2)$. For appropriate sign convention see [11, Section 3].

Let B, C be objects of a DG category $\mathcal{C}$ and let $f \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ be a closed morphism. Assume that $B[1]$ is also an object of $\mathcal{C}$, i.e., there exists an object B' and closed morphisms $t\colon B \to B'$, $t'\colon B' \to B$ of degree 1 and -1 respectively such that $t't = \mathrm{id}_B$ and $tt' = \mathrm{id}_{B'}$. An object $D$of $\mathcal{C}$ is called a *cone* of f if there exist morphisms

$$B' \xrightarrow{i} D \xrightarrow{p} B', \qquad\qquad C \xrightarrow{j} D \xrightarrow{s} C,$$

such that

$$pi = \mathrm{id}_{B'}, \quad sj = \mathrm{id}_C, \quad si = 0, \quad pj = 0, \quad ip + js = \mathrm{id}_D.$$

[11, Lemma 3.8] says that a cone of closed degree zero morphism is uniquely defined up to a DG isomorphism.

To every DG category $\mathcal{C}$ one can formally add cones of closed morphisms by considering the category $\mathcal{C}^{\text{pre-tr}}$ of one-sided twisted complexes over $\mathcal{C}$.

Definition 1.2.2. *A* one-sided twisted complex *over a DG category $\mathcal{C}$ is a collection $\{(C_i),\, q_{i,j}\colon C_i \to C_j\}$ where C_i's are objects of $\mathcal{C}$, zero for all but finitely many i, and $q_{i,j} \in \mathrm{Hom}^{i-j+1}_{\mathcal{C}}(C_i, C_j)$ satisfy $q_{i,j} = 0$ for $i \geq j$ and $(-1)^j \partial q_{i,j} + \sum_{l=1}^{j-i-1} q_{i+l,j} q_{i,i+l} = 0$. For brevity, a twisted complex will be denoted by $(C_i,\, q_{i,j})$.*

One-sided twisted complexes over $\mathcal{C}$ form a DG category $\mathcal{C}^{pre\text{-}tr}$ with morphism spaces equal to

$$\operatorname{Hom}^p_{\mathcal{C}^{pre\text{-}tr}}((C_i, q_{i,j}),\ (D_i, r_{i,j})) = \bigoplus_{l-s=p-q} \operatorname{Hom}^q_{\mathcal{C}}(C_s, D_l)$$

and differential

$$\partial_{\mathcal{C}^{pre\text{-}tr}}(\gamma_{k,l}) = (-1)^l \partial_{\mathcal{C}}(\gamma_{s,l}) + \sum r_{l,m} \circ \gamma_{s,l} - (-1)^p \gamma_{s,l} \circ q_{n,s}$$

for $\gamma_{s,l} \in \operatorname{Hom}^q_{\mathcal{C}}(C_s, D_l)$.

The zeroth homotopy category of $\mathcal{C}^{pre\text{-}tr}$ is denoted $\mathcal{C}^{tr}$.

Let $C = (C_i, q_{i,j})$ be an object of $\mathcal{C}^{\text{pre-tr}}$. The shift of C is defined as $C[1] = (D_i, r_{i,j})$, where $D_i = C_{i+1}$ and $r_{i,j} = -q_{i+1,j+1}$. Clearly, the category $\mathcal{C}^{\text{pre-tr}}$ is closed under shifts.

Let $f\colon (C_i, q_{i,j}) \to (D_i, r_{i,j})$ be a closed morphism of degree zero in $\mathcal{C}^{\text{pre-tr}}$. Assume that $C_i = 0$ for $i < i_0$ or $i > i_n$ and $D_i = 0$ for $i < j_0$ or $i > j_l$. *The cone* of f is a twisted complex $\operatorname{Cone}(f) = (E_i, s_{i,j})$, where

$$E_i = \begin{cases} D_i & \text{for } i \in \{j_0, \dots, j_l\} \\ C_{i+i_n+1-j_0}[j_0 - i_n] & \text{for } i \in \{j_0 - i_n + i_0 - 1, \dots, j_0 - 1\} \end{cases}$$

$$s_{i,j} = \begin{cases} r_{i,j} & \text{for } i, j \geq j_0, \\ (-1)^{i_n - j_0 + 1} q_{i,j} & \text{for } i, j < j_0, \\ (-1)^{(i_n+j_0)(i+i_n)} f_{i+i_n+1-j_0,j} & \text{for } i < j_0 \leq j. \end{cases}$$

The *convolution* functor $\operatorname{Tot}\colon (\mathcal{C}^{\text{pre-tr}})^{\text{pre-tr}} \to \mathcal{C}^{\text{pre-tr}}$ establishes a quasi-equivalence between $(\mathcal{C}^{\text{pre-tr}})^{\text{pre-tr}}$ and $\mathcal{C}^{\text{pre-tr}}$. The cone described above is the first example of a convolution.

The DG category $\mathcal{C}$ is *pretriangulated* if the embedding $H^0(\mathcal{C}) \to \mathcal{C}^{tr}$ is an equivalence. The category $H^0(\mathcal{C})$ for a pretriangulated category is triangulated.

A triangulated category $\mathcal{T}$ is *enhanced* if it has an *enhancement* – a pretriangulated DG category $\mathcal{C}$ such that $\mathcal{T}$ is equivalent to $H^0(\mathcal{C})$.

Recall that a preadditive category $\mathcal{A}$ is *Karoubian* if every idempotent morphism $p\colon A \to A$ (i.e. such that $p^2 = p$) has a kernel. Note that then every idempotent admits an image and p splits, that is $A = \operatorname{Ker} p \oplus \operatorname{Im} p$.

The category $\mathcal{C}^{\text{tr}}$ needs not to be Karoubian. As showed in [11, Section 3.4] the category $D^b(\mathcal{C})$ is the Karoubisation of $\mathcal{C}^{\text{tr}}$.

As it has already been mentioned, a standard example of an enhanced triangulated category is the derived category $D^+(\mathcal{A})$ of an abelian category $\mathcal{A}$ with enough injectives. Its enhancement is the category of complexes of injective sheaves $\operatorname{Kom}(\operatorname{Inj}_{\mathcal{A}})$.

With the above definitions we are ready to state the following theorem.

Theorem 1.2.3. *([10, Theorem 1]) Let $\widetilde{\mathcal{C}}$ be a pretriangulated category, $E_1, \ldots, E_n$ objects of $\widetilde{\mathcal{C}}$ and let $\mathcal{C} \subset \widetilde{\mathcal{C}}$ be the full DG subcategory on the objects E_i. Then the smallest triangulated subcategory of $H^0(\widetilde{\mathcal{C}})$ containing $E_1, \ldots, E_n$ is equivalent to $\mathcal{C}^{tr}$ as a triangulated category.*

Therefore a full exceptional collection $\sigma = \langle T_1, \ldots, T_n \rangle$ in an enhanced Ext-finite triangulated category $\mathcal{T}$ leads to an equivalence of $\mathcal{T}$ with $\mathcal{C}_\sigma^{\mathrm{tr}}$ for some DG category $\mathcal{C}_\sigma$. The category $\mathcal{C}_\sigma$ is a subcategory of the DG enhancement of $\mathcal{T}$ and thus $H^i(\mathrm{Hom}^\bullet_{\mathcal{C}_\sigma}(T_j, T_l)) = \mathrm{Hom}_{\mathcal{T}}(T_j, T_l[i])$ for all $j, l \in \{1, \ldots, n\}$.

Remark 1.2.4. If an enhanced triangulated category $\mathcal{T}$ is Karoubian then a full exceptional collection σ in $\mathcal{T}$ leads to an equivalence of $\mathcal{T}$ and $\mathcal{D}^b(\mathcal{C}_\sigma)$.

For a smooth projective variety X the category $\mathcal{D}^b(X)$ is an enhanced triangulated category. Its DG enhancement is the full DG subcategory of $\mathrm{Kom}(\mathrm{Inj}_{\mathrm{QCoh}(X)})$ consisting of complexes having only finitely many non-zero and coherent cohomology sheaves. Lunts and Orlov in [48, Theorem 9.9] show that in this case the DG enhancement of $\mathcal{D}^b(X)$ is strongly unique, i.e. for any two DG categories $\mathcal{B}$ and $\mathcal{B}'$ and equivalences $\varepsilon\colon H^0(\mathcal{B}) \to \mathcal{D}^b(X)$, $\varepsilon'\colon H^0(\mathcal{B}') \to \mathcal{D}^b(X)$ there exists a quasi-functor $F\colon \mathcal{B} \to \mathcal{M}od - \mathcal{B}'$ such that $\varepsilon' \circ H^0(F)$ and ε are isomorphic.

Let $\sigma = \langle \mathcal{E}_1, \ldots, \mathcal{E}_n \rangle$ be a full exceptional collection of coherent sheaves on X. By Theorem 1.2.3 there exists a DG category $\mathcal{C}_\sigma$ with objects $\mathcal{E}_1, \ldots, \mathcal{E}_n$ such that $\mathcal{D}^b(X)$ is equivalent to $\mathcal{C}_\sigma^{\mathrm{tr}}$. The category $\mathcal{D}^b(X)$ is Karoubian and so $\mathcal{D}^b(X)$ and $\mathcal{D}^b(\mathcal{C}_\sigma)$ are equivalent. Also, as σ is an exceptional collection the category $H(\mathcal{C}_\sigma)$ is ordered and finite.

Remark 1.2.5. If a DG category $\mathcal{C}$ has only zeroth cohomology then it is quasi-equivalent to $H(\mathcal{C}) = H^0(\mathcal{C})$. Indeed, for $C_1, C_2 \in \mathrm{ob}\,\mathcal{C}$ let $\mathrm{Hom}^\bullet_{\mathcal{C}}(C_1, C_2) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^n_{\mathcal{C}}(C_1, C_2)$ with differential

$$\partial^n_{C_1,C_2} : \mathrm{Hom}^n_{\mathcal{C}}(C_1, C_2) \to \mathrm{Hom}^{n+1}_{\mathcal{C}}(C_1, C_2)$$

and let $\mathcal{C}_I$ be a DG category with the same objects as $\mathcal{C}$ and morphisms defined by

$$\mathrm{Hom}^\bullet_{\mathcal{C}_I}(C_1, C_2) = \bigoplus_{n<0} \mathrm{Hom}^n_{\mathcal{C}}(C_1, C_2) \oplus \mathrm{Ker}\partial^0_{C_1,C_2}.$$

Then the natural inclusion functor $\mathcal{C}_I \to \mathcal{C}$ is a quasi-equivalence. Let us also set

$$J_{C_1,C_2} = \mathrm{Im}(\partial^{-1}_{C_1,C_2}) \oplus (\bigoplus_{n<0} \mathcal{C}^n(C_1, C_2))$$

for any objects C_1 and C_2 of $\mathcal{C}$ and consider the category $\mathcal{C}_{I/J}$ with $\mathrm{ob}\,\mathcal{C}_{I/J} = \mathrm{ob}\,\mathcal{C}$ and

$$\mathrm{Hom}^\bullet_{\mathcal{C}_{I/J}}(C_1, C_2) = \mathrm{Hom}^\bullet_{\mathcal{C}_I}(C_1, C_2)/J_{C_1,C_2}.$$

Then $\mathcal{C}_{I/J}$ is isomorphic to $H(\mathcal{C})$ and the natural functor $\mathcal{C}_I \to \mathcal{C}_{I/J}$ is a quasi-equivalence.

In particular, if σ is a strong exceptional collection then the DG category $\mathcal{C}_\sigma$ is quasi-equivalent to an ordinary category.

1.3 A_∞-categories

The category $\mathcal{D}^b(X)$ has a strongly unique DG enhancement given by the full DG subcategory of $\mathrm{Kom}(\mathrm{Inj}_{\mathrm{QCoh}(X)})$ and hence, a priori, calculating the category $\mathcal{C}_\sigma$ from Theorem 1.2.3 requires taking injective resolutions. This suggest that the category $\mathcal{C}_\sigma$ can have infinitely dimensional morphisms spaces. Using the techniques of A_∞-categories we will show that $\mathcal{C}_\sigma$ is quasi-isomorphic to a finite and ordered DG category. First, we recall definitions and properties of A_∞-categories following Keller's survey papers [37], [38] and Lefèvre-Hasegawa unpublished thesis [45].

Definition 1.3.1. *([37, Definition 7.2]) An A_∞-*category *$\mathcal{G}$ over k consists of*

- *a set of objects $ob(\mathcal{G})$,*
- *for any two $G_1, G_2 \in ob(\mathcal{G})$ a $\mathbb{Z}$ - graded k-vector space* $\mathrm{Hom}^{\bullet}_{\mathcal{G}}(G_1, G_2)$*,*
- *for any $n \geq 1$ and a sequence $G_0, G_1, \ldots, G_n \in ob(\mathcal{G})$ a graded k-linear map:*

$$m_n\colon \mathrm{Hom}^{\bullet}_{\mathcal{G}}(G_{n-1}, G_n) \otimes_k \ldots \otimes_k \mathrm{Hom}^{\bullet}_{\mathcal{G}}(G_0, G_1) \to \mathrm{Hom}^{\bullet}_{\mathcal{G}}(G_0, G_n)$$

  of degree $2-n$ such that for any $n \geq 0$ and all $(n+1)$-tuples of objects $G_0, \ldots, G_n$ we have the identity

$$\sum_{\{r+s+t=n,\, s\geq 1,\, r,t\geq 0\}} (-1)^{r+st} m_{r+1+t}(id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = 0$$

  of maps

$$\mathrm{Hom}^{\bullet}_{\mathcal{G}}(G_{n-1}, G_n) \otimes_k \ldots \otimes_k \mathrm{Hom}^{\bullet}_{\mathcal{G}}(G_0, G_1) \to \mathrm{Hom}^{\bullet}_{\mathcal{G}}(G_0, G_n)$$

When these formulae are applied to elements additional signs appear because of the Koszul sign rule:

$$(f \otimes g)(x \otimes y) = (-1)^{|x||g|} f(x) \otimes g(y).$$

An A_∞-*algebra* is an A_∞-category with one object.

An A_∞-category $\mathcal{G}$ is *ordered* if there exists a partial order $\preceq$ on the set $\mathrm{ob}(\mathcal{G})$ such that $\mathrm{Hom}_{\mathcal{G}}(G, G') = 0$ for $G' \preceq G$. It is *finite* if $\mathrm{ob}(\mathcal{G})$ is a finite set and $\mathrm{Hom}^{\bullet}_{\mathcal{G}}(G, G')$ is finite-dimensional for any G and G'.

An A_∞-category $\mathcal{G}$ is *strictly unital* if for any object G of $\mathcal{G}$ there exists a morphism $1_G \in \mathrm{Hom}^\bullet_{\mathcal{G}}(G,G)$ of degree 0 such that for any object G' of $\mathcal{G}$ and any morphisms $\phi \in \mathrm{Hom}^\bullet_{\mathcal{G}}(G,G')$, $\psi \in \mathrm{Hom}^\bullet_{\mathcal{G}}(G',G)$ we have $m_2(\phi, 1_G) = \phi$ and $m_2(1_G, \psi) = \psi$. Moreover, for $n \neq 2$ the operation m_n equals 0 if any of its arguments is equal to 1_G.

An A_∞-category $\mathcal{G}$ is *homologically unital* if there exist units for the homotopy category $H(\mathcal{G})$.

The operation m_1 gives for any pair of objects G, G' of $\mathcal{G}$ a structure of a complex on $\mathrm{Hom}^\bullet_{\mathcal{G}}(G,G')$. Using m_1, analogously as for DG categories, we can associate to a homologically unital A_∞-category a graded homology category $H(\mathcal{G})$.

An A_∞-category is called *minimal* if the operation m_1 is trivial.

Remark 1.3.2. Any DG category $\mathcal{C}$ is an A_∞-category. Indeed, if ∂ is the differential on the spaces of morphisms of $\mathcal{C}$ and μ is the composition of morphisms then putting $m_1 = \partial$, $m_2 = \mu$ and $m_i = 0$ for $i \geq 3$ we obtain an A_∞-category.

Remark 1.3.3. Lefèvre-Hasegawa in [45, Section 1.1] considers a k-linear abelian semi-simple, cocomplete category C with filtered exact limits. Recall that an abelian category C is semi-simple if every subobject is a direct summand and C is cocomplete if it has all colimits.

Then, Lefèvre-Hasegawa defines in [45, Section 1.2] an A_∞-algebra A as a graded module over C together with a family of graded maps $m_i \colon A^{\otimes i} \to A$ satisfying

$$\sum_{\{r+s+t=n,\, s\geq 1,\, r,t\geq 0\}} (-1)^{r+st} m_{r+1+t}(\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) = 0.$$

In order to define A_∞-categories he introduces in [45, Section 5.1] a a k-linear abelian, semi-simple, cocomplete category $\mathrm{C}(\mathfrak{G}, \mathfrak{G})$ with filtered exact limits for any set $\mathfrak{G}$. Then, he defines an A_∞-category $\mathcal{G}$ with the set of objects $\mathfrak{G}$ as an A_∞-algebra over the monoidal category $\mathrm{C}(\mathfrak{G}, \mathfrak{G})$. In [45, Remarque 5.1.2.2] he proves that this definition agrees with Definition 1.3.1 given above.

Therefore, all the claims proved in [45] for A_∞-algebras remain valid for A_∞-categories.

For any set S there exists an A_∞-category $k\,S$. The objects of $k\,S$ are elements of S and

$$\mathrm{Hom}^\bullet_{k\,S}(s_1, s_2) = \begin{cases} k & \text{if } s_1 = s_2, \\ 0 & \text{otherwise.} \end{cases}$$

All operations m_n in $k\,S$ are trivial.

In particular, for any set S the category $k\,S$ is strictly unital.

Definition 1.3.4. *A functor of A_∞-categories $F \colon \mathcal{G} \to \mathcal{G}'$ is a map $F_0 \colon ob(\mathcal{G}) \to ob(\mathcal{G}')$ and a family of graded maps*

$$F_n \colon \mathrm{Hom}^\bullet_{\mathcal{G}}(G_{n-1}, G_n) \otimes_k \ldots \otimes_k \mathrm{Hom}^\bullet_{\mathcal{G}}(G_0, G_1) \to \mathrm{Hom}^\bullet_{\mathcal{G}'}(F_0(G_0), F_0(G_n))$$

of degree $1-n$ *such that*

$$\sum_{r+s+t=n}(-1)^{r+st}F_{r+1+t}(id^{\otimes r}\otimes m_s\otimes id^{\otimes t})=\sum_{i_1+\ldots+i_r=n}(-1)^p m_r(F_{i_1}\otimes\ldots\otimes F_{i_r}),$$

where $p=(r-1)(i_1-1)+(r-2)(i_2-1)+\ldots+2(i_{r-2}-1)+(i_{r-1}-1)$.

Composition of functors is given by

$$(F\circ G)_n=\sum_{i_1+\ldots+i_s=n}F_s\circ(G_{i_1}\otimes\ldots\otimes G_{i_s}).$$

A functor $F:\mathcal{G}\to\mathcal{G}'$ is called an A_∞-*quasi-equivalence* if F_1 induces an equivalence between homotopy categories $H(\mathcal{G})$ and $H(\mathcal{G}')$.

Theorem 1.3.5. *([45, Theorem 3.2.1.1]) Minimal homologically unital* A_∞*-category is* A_∞*-quasi-equivalent to a minimal strictly unital* A_∞*-category.*

Remark 1.3.6. A minimal A_∞-category is equal to its homology category. Hence, an A_∞-quasi-isomorphism F given by the above proposition satisfies $F_1=\mathrm{id}$.

An A_∞-category $\mathcal{G}$ is *augmented* if there exists a strict unit preserving functor $\epsilon\colon k\,\mathrm{ob}(\mathcal{G})\to\mathcal{G}$. Then $\mathcal{G}$ decomposes as $\mathcal{G}=k\,\mathrm{ob}(\mathcal{G})\oplus\overline{\mathcal{G}}$.

Any A_∞-category $\mathcal{G}$ is A_∞-quasi-equivalent to its homotopy category $H(\mathcal{G})$.

Theorem 1.3.7. *([31, Theorem 1], see also [44, Theorem 4.3]) If* $\mathcal{G}$ *is an* A_∞*-category, then* $H(\mathcal{G})$ *admits an* A_∞*-category structure such that*

1. $m_1=0$ *and* m_2 *is induced from* $m_2^{\mathcal{G}}$ *and*

2. *there is an* A_∞*-quasi-equivalence* $\mathcal{G}\to H(\mathcal{G})$.

Moreover, this structure is unique up to a non unique A_∞*-equivalence.*

The A_∞-category $H(\mathcal{G})$ is called the *minimal model* of $\mathcal{G}$.

Remark 1.3.8. Let $\mathcal{C}$ be a DG category. Its minimal model $H(\mathcal{C})$ is A_∞-quasi-equivalent to a strictly unital A_∞-category. Indeed, the category $\mathcal{C}$ is strictly unital and hence homologically unital. Its homotopy category is also homologically unital and Theorem 1.3.5 guarantees that there exists a strictly unital minimal category A_∞-quasi-equivalent to $H(\mathcal{C})$.

1.3.1 The universal DG category of an A_∞-category

For any augmented A_∞-category $\mathcal{G}$ there exists a DG category $U(\mathcal{G})$ and an A_∞-quasi-isomorphism $\mathcal{G} \to U(\mathcal{G})$. To define the category $U(\mathcal{G})$ we need the following definitions taken from [45, Section 1.2], see also [49, Section 2].

Definition 1.3.9. *A* DG cocategory $\mathcal{B}$ *consists of*

- *the set of objects $ob(\mathcal{B})$,*
- *for any pair of objects $B_i, B_j \in ob(\mathcal{B})$ a complex of k-vector spaces* $\mathrm{Hom}^\bullet_{\mathcal{B}}(B_i, B_j)$ *with a differential d^{ij} of degree one, and*
- *a coassociative cocomposition, that is a family of linear maps*

$$\Delta\colon \mathrm{Hom}^\bullet_{\mathcal{B}}(B_i, B_j) \to \sum_{B_k \in ob(\mathcal{B})} \mathrm{Hom}^\bullet_{\mathcal{B}}(B_k, B_j) \otimes_k \mathrm{Hom}^\bullet_{\mathcal{B}}(B_i, B_k).$$

These data have to satisfy the condition

$$\Delta \circ d = (d \otimes id + id \otimes d) \circ \Delta.$$

For any set S the A_∞-category $k\,S$ is also a DG cocategory.

A functor Φ between DG cocategories $\mathcal{B}$ and $\mathcal{B}'$ preserves the grading and differentials on morphisms and satisfies the condition

$$\Delta \circ \Phi = (\Phi \otimes \Phi) \circ \Delta.$$

A DG cocategory $\mathcal{B}$ is *counital* if it admits a counit, a functor $\eta\colon \mathcal{B} \to k\,\mathrm{ob}(\mathcal{B})$. The category $\mathcal{B}$ is *coaugmented* if it is counital and admits a coaugmentation functor $\varepsilon\colon k\,\mathrm{ob}(\mathcal{B}) \to \mathcal{B}$ such that the composition $\eta\varepsilon$ is the identity on $k\,\mathrm{ob}(\mathcal{B})$.

Let $\mathcal{B}$ be a coaugmented DG cocategory. Denote by $\overline{\mathcal{B}}$ a cocategory with the same objects as $\mathcal{B}$ and morphisms $\mathrm{Hom}^\bullet_{\overline{\mathcal{B}}}(B_i, B_j) = \ker \varepsilon$.

For an augmented A_∞-category $\mathcal{G}$ one can define its *bar* DG cocategory $B_\infty(\mathcal{G})$. Recall that as an augmented category $\mathcal{G}$ can be written as $\overline{\mathcal{G}} \oplus k\,\mathrm{ob}(\mathcal{G})$. Then $B_\infty(\mathcal{G}) = T^c(S\bar{\mathcal{G}})$ is a tensor cocategory of the suspension of $\overline{\mathcal{G}}$. Here $S\overline{\mathcal{G}}$ denotes the category $\overline{\mathcal{G}}$ with a shift in a morphisms spaces $\mathrm{Hom}^n_{S\overline{\mathcal{G}}}(G, G') = \mathrm{Hom}^{n+1}_{\overline{\mathcal{G}}}(G, G')$. Note that $S\overline{\mathcal{G}}$ is not an A_∞-category. However, the operations m_n in $\overline{\mathcal{G}}$ define the following graded maps of degree 1 in $S\overline{\mathcal{G}}$:

$$b_n\colon \mathrm{Hom}^\bullet_{S\overline{\mathcal{G}}}(G_{n-1}, G_n) \otimes_k \ldots \otimes_k \mathrm{Hom}^\bullet_{S\overline{\mathcal{G}}}(G_0, G_1) \to \mathrm{Hom}^\bullet_{S\overline{\mathcal{G}}}(G_0, G_n),$$
$$b_n = -s \circ m_n \circ \omega^{\otimes n},$$

where $s\colon V \to SV$ is a suspension of a graded vector space V and $\omega = s^{-1}$. That is, $(SV)^i \simeq V^{i+1}$ and $s\colon V \to SV$ is the map of degree one induced by the identity morphism of V.

The cocategory $B_\infty(\mathcal{G})$ has the same objects as $\mathcal{G}$ and the morphisms in this category are defined by

$$\begin{aligned}\mathrm{Hom}^\bullet_{B_\infty(\mathcal{G})}(G,G') := \mathrm{Hom}^\bullet_{S\overline{\mathcal{G}}}(G,G') \oplus \bigoplus_{G_1\in \mathrm{ob}(\mathcal{G})} \mathrm{Hom}^\bullet_{S\overline{\mathcal{G}}}(G_1,G') \otimes_k \mathrm{Hom}^\bullet_{S\overline{\mathcal{G}}}(G,G_1) \\ \oplus \bigoplus_{G_1,G_2\in \mathrm{ob}(\mathcal{G})} \mathrm{Hom}^\bullet_{S\overline{\mathcal{G}}}(G_2,G') \otimes_k \mathrm{Hom}^\bullet_{S\overline{\mathcal{G}}}(G_1,G_2) \otimes_k \mathrm{Hom}^\bullet_{S\overline{\mathcal{G}}}(G,G_1) \oplus \ldots\end{aligned}$$

for $G \neq G'$. In the case $G = G'$ we have to add to the above sum one copy of the base filed k in degree 0 corresponding to the identity morphism 1_G.

To simplify the notation we shall write $(\alpha_n, \ldots, \alpha_1)$ for $\alpha_n \otimes \ldots \otimes \alpha_1$. The differential in $B_\infty(\mathcal{G})$ is given by

$$\begin{aligned}&d(\alpha_n,\ldots,\alpha_1) = \\ &\sum_{s=1}^{n}\sum_{l=1}^{n-s+1} (-1)^{|\alpha_{l-1}|+\ldots+|\alpha_1|}(\alpha_n,\ldots\alpha_{l+s}, b_s(\alpha_{l+s-1},\ldots,\alpha_l),\alpha_{l-1},\ldots\alpha_1),\end{aligned}$$

for $(\alpha_n,\ldots,\alpha_1) \in \mathrm{Hom}_{B_\infty(\mathcal{G})}(G,G')$. The cocomposition is given by

$$\begin{aligned}&\Delta(\alpha_n,\ldots,\alpha_1) = \\ &1_{G'} \otimes (\alpha_n,\ldots,\alpha_1) + (\alpha_n,\ldots,\alpha_1) \otimes 1_G + \sum_{l=1}^{n-1}(\alpha_n,\ldots,\alpha_{l+1}) \otimes (\alpha_l,\ldots,\alpha_1).\end{aligned}$$

With these definitions $B_\infty(\mathcal{G})$ is an augmented DG cocategory.

Remark 1.3.10. Let $\mathcal{G}$ be an ordered and finite A_∞-category such that $\mathrm{Hom}^\bullet_{\overline{\mathcal{G}}}(G,G) = 0$ for any object $G \in \mathrm{ob}\mathcal{G}$. Then the DG cocategory $B_\infty(\mathcal{G})$ also satisfies these conditions; i.e. it is ordered, finite and $\mathrm{Hom}^\bullet_{\overline{B_\infty(\mathcal{G})}}(G,G) = 0$ for any object G.

Analogously, via a *cobar* construction one can assign to an augmented DG cocategory $\mathcal{B}$ a DG category $\Omega(\mathcal{B})$. Let $\mathcal{B}$ be a DG cocategory with a differential d and cocomposition Δ. Its cobar DG category is equal to $T(S^{-1}\overline{\mathcal{B}})$. Here $S^{-1}\overline{\mathcal{B}}$ denotes the shift of the cocategory $\mathcal{B}$ and $T(S^{-1}\overline{\mathcal{B}})$ is the tensor DG category of it. As before the morphisms spaces in $T(S^{-1}\overline{\mathcal{B}})$ are given by

$$\begin{aligned}\mathrm{Hom}^\bullet_{\Omega(\mathcal{B})}(B,B') := \mathrm{Hom}^\bullet_{S^{-1}\overline{\mathcal{B}}}(B,B') \oplus \bigoplus_{B_1\in \mathrm{ob}(\mathcal{B})} \mathrm{Hom}^\bullet_{S^{-1}\overline{\mathcal{B}}}(B_1,B') \otimes_k \mathrm{Hom}^\bullet_{S^{-1}\overline{\mathcal{B}}}(B,B_1) \\ \oplus \bigoplus_{B_1,B_2\in \mathrm{ob}(\mathcal{B})} \mathrm{Hom}^\bullet_{S^{-1}\overline{\mathcal{B}}}(B_2,B') \otimes_k \mathrm{Hom}^\bullet_{S^{-1}\overline{\mathcal{B}}}(B_1,B_2) \otimes_k \mathrm{Hom}^\bullet_{S^{-1}\overline{\mathcal{B}}}(B,B_1) \oplus \ldots\end{aligned}$$

for $B \neq B'$. Again, for $B = B'$ one has to add one copy of the base filed k in degree zero, corresponding to the identity morphism, 1_B.

The composition in $\Omega(\mathcal{B})$ is defined by concatenation and the differential ∂ on the morphisms spaces is given by

$$\partial = \sum 1 \otimes \ldots \otimes 1 \otimes (d + \Delta) \otimes 1 \otimes \ldots \otimes 1.$$

Remark 1.3.11. If an ordered and finite DG cocategory $\mathcal{B}$ satisfies the condition $\mathrm{Hom}^{\bullet}_{\overline{\mathcal{B}}}(B, B) = 0$, then the category $\Omega(\mathcal{B})$ is also finite, ordered and $\mathrm{Hom}^{\bullet}_{\overline{\Omega(B)}}(B, B) = 0$.

For an augmented A_∞-category $\mathcal{G}$ its universal DG category $U(\mathcal{G})$ is defined as $\Omega(B_\infty(\mathcal{G}))$. There is a natural map $\mathcal{G} \to U(\mathcal{G})$. [45, Lemma 1.3.2.3] proves that this map extends to a functor and it is an A_∞-quasi-equivalence. Moreover, for an A_∞-quasi-equivalence ϕ the functor $U(\phi)$ is a quasi-equivalence of DG categories.

1.3.2 A_∞-modules

Definition 1.3.12. *An A_∞-module over an A_∞-category $\mathcal{G}$ is an A_∞-functor $M\colon \mathcal{G} \to DGVect_k$. A morphism of modules $F\colon M \to N$ is given by a family $\{F_G\colon M_0(G) \to N_0(G)\}_{G \in ob\,\mathcal{G}}$ of morphisms in $DGVect_k$ such that the diagrams*

$$\begin{tikzcd} M_0(G_0) \arrow[r, "M_n(\alpha_{n-1}\otimes\ldots\otimes\alpha_0)"] \arrow[d, "F_{G_0}"] & M_0(G_n) \arrow[d, "F_{G_n}"] \\ N_0(G_0) \arrow[r, "N_n(\alpha_{n-1}\otimes\ldots\otimes\alpha_0)"] & N_0(G_n) \end{tikzcd}$$

commute for any $n \in \mathbb{N}$, $G_i \in ob\mathcal{G}$ and $\alpha_i \in \mathrm{Hom}^{\bullet}_{\mathcal{G}}(G_i, G_{i+1})$.

The category of A_∞-modules over an A_∞-category $\mathcal{G}$ will be denoted as $\mathcal{M}od_\infty$-$\mathcal{G}$.

The morphism $F\colon M \to N$ of modules is an A_∞*-quasi-isomorphism* if F_G is a quasi-isomorphism of complexes for any $G \in \mathrm{ob}\,\mathcal{G}$.

The *derived category* $\mathcal{D}_\infty(\mathcal{G})$ of an A_∞-category $\mathcal{G}$ is defined as a localization of the category $\mathcal{M}od_\infty$-$\mathcal{G}$ with respect to the class of A_∞-quasi-isomorphisms. For A_∞-quasi-equivalent A_∞-categories their derived categories are equivalent.

Remark 1.3.13. ([45, Lemme 2.4.3.2]) For a DG category $\mathcal{C}$ the derived categories $\mathcal{D}^b(\mathcal{C})$ and $\mathcal{D}_\infty(\mathcal{C})$ are equivalent. Hence, for any augmented A_∞-category $\mathcal{G}$ the category $\mathcal{D}_\infty(\mathcal{G})$ is equivalent to $\mathcal{D}(U(\mathcal{G}))$.

1.4 Existence of DG quivers

Theorem 1.4.1. *Let $\mathcal{C}$ be a DG category with finitely many objects. Assume that $H(\mathcal{C})$ is an ordered and finite graded category such that $\mathrm{Hom}^{\bullet}_{H(\mathcal{C})}(C, C) = k$ for any object C of*

$\mathcal{C}$. *Then there exists an ordered and finite DG category* $\widetilde{\mathcal{C}}$ *such that* $\mathcal{D}(\mathcal{C})$ *is equivalent to* $\mathcal{D}(\widetilde{\mathcal{C}})$.

Proof. Theorem 1.3.7 guarantees existence of the minimal model of $\mathcal{C}$ given by the homology A_∞-category $H(\mathcal{C})$. By Remark 1.3.8 we can assume that $H(\mathcal{C})$ is strictly unital. As $\mathrm{Hom}^\bullet_{H(\mathcal{C})}(C, C) = k$ for any C the category $H(C)$ is an augmented ordered A_∞-category. We have

$$\mathcal{D} = \mathcal{D}(\mathcal{C}_\sigma) = \mathcal{D}_\infty(H(\mathcal{C}_\sigma)) = \mathcal{D}(\mathcal{U}(H(\mathcal{C}_\sigma)).$$

Remarks 1.3.10 and 1.3.11 show that the category $\mathcal{U}(H(\mathcal{C}_\sigma))$ is ordered and finite. $\square$

As a corollary we get

Theorem 1.4.2. *Let* X *be a smooth projective variety and let* $\sigma = \langle \mathcal{E}_1, \ldots, \mathcal{E}_n \rangle$ *be a full exceptional collection on* X. *Then there exists an ordered, finite DG category* $\widetilde{\mathcal{C}_\sigma}$ *such that* $\mathcal{D}^b(X)$ *is equivalent to* $\mathcal{D}^b(\widetilde{\mathcal{C}_\sigma})$.

Proof. By Theorem 1.2.3 there exists a DG category $\mathcal{C}_\sigma$ such that $\mathcal{D}^b(X) = \mathcal{D}^b(\mathcal{C}_\sigma)$. Since the sheaves $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are exceptional and the category $\mathcal{D}^b(X)$ is Ext-finite, by construction the category $\mathcal{C}_\sigma$ satisfies conditions of Theorem 1.4.1. Hence there exists a DG category $\widetilde{\mathcal{C}_\sigma}$ such that the categories $\mathcal{D}(\mathcal{C}_\sigma)$ and $\mathcal{D}(\widetilde{\mathcal{C}_\sigma})$ are equivalent. It follows that $\mathcal{D}^b(\widetilde{\mathcal{C}_\sigma}) \simeq \mathcal{D}^b(X)$. $\square$

The category $\widetilde{\mathcal{C}_\sigma}$ is A_∞-quasi-equivalent to the category $\mathcal{C}_\sigma$ and hence $\mathrm{Hom}^l_{H(\widetilde{\mathcal{C}_\sigma})}(\mathcal{E}_i, \mathcal{E}_j) = \mathrm{Ext}^l_X(\mathcal{E}_i, \mathcal{E}_j)$.

To graphically present the DG category $\widetilde{\mathcal{C}_\sigma}$ we introduce the following definition.

A *DG quiver* is a quiver Q endowed with a $\mathbb{Z}$-grading on the set of arrows Q_1 and a structure of a DG algebra on the path algebra $k[Q]$ such that the following conditions are satisfied:

- The $\mathbb{Z}$-grading of $k[Q]$ is compatible with the grading of Q_1.
- For any path $p \in k[Q]$ the differential $\partial(p)$ is a combination of paths with the same head and tail as p.
- For any vertex i of Q the trivial path $\varepsilon_i \in k[Q]$ is closed of degree 0.

The DG category $\widetilde{\mathcal{C}_\sigma}$ has finitely dimensional space of morphisms and so one can associate to it a DG quiver Q_σ. If the collection σ is strong it follows from Remark 1.2.5 that the DG quiver coincides with the quiver introduced in [9].

Remark 1.4.3. The DG quiver, or the DG category $\widetilde{\mathcal{C}_\sigma}$, is determined only up to a quasi-isomorphism. Therefore, we can always modify it by finding a quasi-isomorphic DG subcategory or a quasi-isomorphic quotient.

For example, let S be a subspace of the space of morphisms in $\widetilde{\mathcal{C}_\sigma}$ which has a basis consisting of pairs α, $\partial(\alpha)$. Clearly, if S is an ideal then the quotient DG category $\widetilde{\mathcal{C}_\sigma}'$ is quasi-isomorphic to $\widetilde{\mathcal{C}_\sigma}$. On the other hand, if there is no morphism in S which is a non-trivial composition of two morphisms not in S then the subcategory $\widetilde{\mathcal{C}_\sigma}''$ of $\widetilde{\mathcal{C}_\sigma}$ obtained by removing all the morphisms in S is quasi-isomorphic to $\widetilde{\mathcal{C}_\sigma}$.

Chapter 2

Calculating DG quivers

In this chapter we describe how to calculate DG quivers given by Theorem 1.4.2. We give three algorithms depending on available data.

In Section 2.1 we define after Bondal and Kapranov, [10] mutations of DG categories. They allow us to calculate a DG quiver of an exceptional collection, provided we know the DG quiver of one of its mutations. In particular, they allow us to calculate a DG quiver of a collection that can be mutated to a strong one. In Section 2.1.1 we show how to use this method to find a DG quiver of a full exceptional collection on a surface X obtained from $\mathbb{P}^2$ by blowing up a point and then blowing up a point on the exceptional divisor.

In Section 2.2 we show how to calculate the DG quiver from the A_∞-quiver of an exceptional collection. The A_∞-quiver can be in some cases computed using some homological operations called Massey products are. Results of Sections 2.2.2 - 2.2.6 lead to Theorem 2.2.12 which proves that a non-empty n-tuple Massey product in a uniquely enhanced triangulated category $\mathcal{T}$ provides important information about the structure of A_∞-category.

In Section 2.2.2 we recall definition of Massey products in a triangulated category. This definition uses Postnikov systems and convolutions of complexes. In Sections 2.2.3 and 2.2.4 we provide explicit calculations of triple and quadruple Massey products. Observations made in these sections are generalized to n-tuple Massey products in Section 2.2.5. Proposition 2.2.8 of this section describes how Massey products are connected with Postnikov systems and existence of convolutions. Then, we introduce after [41] defining systems which are Massey products in homotopy category of a DG category. Lemma 2.2.10 proves that in the case when $\mathcal{T}$ is an enhanced triangulated category both definitions agree up to a sign.

In Section 2.2.6 we use these results, together with Merkulov's description of a minimal A_∞-model recalled in 2.2.1, Theorem 2.2.12 relating Massey products with the minimal A_∞-structure.

These techniques are then used in Section 2.2.7 to calculate the A_∞-structure of a DG quiver of an exceptional collection on the surface X considered before, that is the surface obtained from $\mathbb{P}^2$ by blowing up a point and then blowing up a point on the exceptional divisor. In this section we also explain why Massey products are not sufficient to find the A_∞-structure on the full collection. We also briefly recall that Polishchuk used the same methods in [57] in order to calculate A_∞-structure on the category of line bundles on elliptic curves.

In Section 2.3 we describe how to find DG quivers of exceptional collections $\sigma = \langle E_1, \ldots, E_n \rangle$ such that $\mathrm{Ext}^i(E_j, E_l) = 0$ for $i \neq 0, 1$ and all pairs j, l. The method relies on universal extensions and coextensions defined after Hille and Perling, [25] in Section 2.3.1. Both constructions systematically make the first Ext groups between objects vanish and allow us to construct a tilting object in the category $\mathcal{D}^b(X)$ from an exceptional collection with vanishing higher Ext-groups. Theorem 2.3.1 states that the endomorphism algebra of this tilting object determines the DG quiver of σ.

Universal extensions and coextensions first appeared in representation theory of quasi-hereditary algebras (see [17]). In Section 2.3.2 we recall definitions of quasi-hereditary algebras. We also prove in that endomorphism algebras of tilting generators obtained from σ by universal extensions and coextensions are so called Ringel dual quasi-hereditary algebras (see Proposition 2.3.3).

In Section 2.3.3 we come back to the example of an exceptional collection on the surface X obtained from $\mathbb{P}^2$ as a two-step blow-up. We calculate the DG quiver using universal extensions.

In Section 2.2.7 we showed that this DG quiver could not be calculated by Massey products only. Therefore, in Section 2.4 we investigate what additional information is provided by the endomorphism algebra of the tilting object given by universal extensions. We describe a cohomological operation, a relative triple Massey product. We also conclude that triple Massey products and relative triple Massey products are sufficient to calculate the DG quiver of the exceptional collection on X.

In Section 2.4.1 we show that there are many equivalent ways to calculate n-tuple Massey products in enhanced triangulated categories. This observation explains the name "relative triple Massey product".

2.1 Action of the braid group

First, let us recall after [9, Section 2] action of the braid group on the set of exceptional collections in a triangulated category $\mathcal{T}$.

Let $\langle E, F \rangle$ be an exceptional pair in $\mathcal{T}$. Then $\langle L_E F, E \rangle$ and $\langle F, R_F E \rangle$ are also exceptional pairs for $L_E F$ and $R_F E$ defined by means of distinguished triangles in $\mathcal{T}$.

$$L_E F \to \mathrm{Hom}^\bullet_{\mathcal{T}}(E,F) \otimes E \to F \to L_E F[1],$$
$$E \to \mathrm{Hom}^\bullet_{\mathcal{T}}(E,F)^* \otimes F \to R_F E \to E[1].$$

Here, $\mathrm{Hom}^\bullet_{\mathcal{T}}(E,F) = \bigoplus_l \mathrm{Hom}_{\mathcal{T}}(E, F[l])$ denotes a complex of k-vector spaces with trivial differential. For an element $E \in \mathcal{T}$ and a complex $V^\bullet$ the tensor product is defined by $E \otimes V^\bullet = \bigoplus_{l\in\mathbb{Z}} \bigoplus_{i=1}^{\dim V^l} E[-l]$.

For an exceptional collection $\sigma = \langle E_1, \dots, E_n\rangle$ the *i-th left mutation* $L_i\sigma$ and the *i-th right mutation* $R_i\sigma$ are defined by

$$L_i\sigma = \langle E_1, \dots, E_{i-1}, L_{E_i}E_{i+1}, E_i, E_{i+2}, \dots, E_n\rangle,$$
$$R_i\sigma = \langle E_1, \dots, E_{i-1}, E_{i+1}, R_{E_{i+1}}E_i, E_{i+2}, \dots, E_n\rangle.$$

Clearly, if the collection σ is full then the same is true about $L_i\sigma$ and $R_i\sigma$.

The action of the braid group on the set of full exceptional collections can be lifted to associated DG categories (see also [10, Section 5]).

Twisted complexes provide description of the categories $\widetilde{\mathcal{C}_{L_i\sigma}}$ and $\widetilde{\mathcal{C}_{R_i\sigma}}$ by means of $\widetilde{\mathcal{C}_\sigma}$. To see it we need to define tensor product of a twisted complex with a complex of vector spaces.

Let $\mathcal{C}$ be a finite DG category, let $C \in \mathrm{ob}\,\mathcal{C}^{\text{pre-tr}}$ be a twisted complex and let $V^\bullet$ be a finite dimensional complex of vector spaces with differential $\partial^i \colon V^i \to V^{i+1}$. $C \otimes V$ is defined as $(\bigoplus_{\{i|V^i\neq 0\}} C[-i]^{\oplus \dim V^i}, q_{i,j}) \in (\mathcal{C}^{\text{pre-tr}})^{\text{pre-tr}}$. The morphisms $q_{i,i+1}$ are induced by the differential ∂^i tensored with the identity on C and $q_{i,j} = 0$ for $j \neq i+1$.

Now let $C, D \in \mathcal{C}^{\text{pre-tr}}$ be twisted complexes. There exist closed morphisms of degree 0

$$\phi \colon C \otimes \mathrm{Hom}^\bullet_{\mathcal{C}^{\text{pre-tr}}}(C,D) \to D,$$

where $\phi_{i,0} \colon C[-i]^{\oplus \dim \mathrm{Hom}^i(C,D)} \to D$ is given by morphisms of degree i between C and D. The morphism

$$\psi \colon C \to \mathrm{Hom}^\bullet_{\mathcal{C}^{\text{pre-tr}}}(C,D)^* \otimes D.$$

is defined analogously.

Now we define new twisted complexes over $\mathcal{C}$

$$L_C D = \mathrm{Tot}(\mathrm{Cone}(\phi)[-1]),$$
$$R_D C = \mathrm{Tot}(\mathrm{Cone}(\psi)).$$

Let $\sigma = \langle \mathcal{E}_1, \dots, \mathcal{E}_n \rangle$ be a full exceptional collection on a smooth projective variety X and let $\widetilde{\mathcal{C}_\sigma}$ be the category described in Theorem 1.4.1. Let $E_1, \dots, E_n$ denote the objects of $\widetilde{\mathcal{C}_\sigma}$. We define two full subcategories of $\widetilde{\mathcal{C}_\sigma}^{\text{pre-tr}}$; category $\widetilde{\mathcal{C}_\sigma^{L_i}}$ with objects $E_1, \dots, E_{i-1}, L_{E_i}E_{i+1}, E_i, E_{i+2}, \dots E_n$ and category $\widetilde{\mathcal{C}_\sigma^{R_i}}$ with objects $E_1, \dots, E_{i-1}, E_{i+1}, R_{E_{i+1}}E_i, E_{i+2}, \dots, E_n$.

Proposition 2.1.1. *Let $\sigma = \langle \mathcal{E}_1, \dots, \mathcal{E}_n \rangle$ be a full exceptional collection on X and let $\widetilde{\mathcal{C}_\sigma}$ be a finite DG category with objects $E_1, \dots, E_n$ with $H^l\mathcal{C}_\sigma(E_i, E_j) = Ext^l_X(\mathcal{E}_i, \mathcal{E}_j)$ and such that $\mathcal{D}^b(X)$ is equivalent to $\mathcal{D}^b(\widetilde{\mathcal{C}_\sigma})$. Then the categories $\widetilde{\mathcal{C}_\sigma^{L_i}}$ and $\widetilde{\mathcal{C}_\sigma^{R_i}}$ satisfy analogous conditions for the collections $L_i\sigma$ and $R_i\sigma$, respectively.*

Proof. As the category $\widetilde{\mathcal{C}_\sigma}$ is finite, mutations of twisted complexes over $\widetilde{\mathcal{C}_\sigma}$ are well defined. Furthermore, the category $\mathcal{D}^b(X)$ is equivalent to $\widetilde{\mathcal{C}_\sigma}^{\text{tr}}$. Under this equivalence $L_{\mathcal{E}_i}\mathcal{E}_{i+1}$ corresponds to $L_{E_i}E_{i+1}$. Hence the category $\widetilde{\mathcal{C}_\sigma^{L_i}}$ is the DG category described by Theorem 1.2.3. By construction it is also finite.

Let $\mathrm{Hom}^\bullet_{\widetilde{\mathcal{C}_\sigma^{L_i}}}(L_{E_i}E_{i+1}, L_{E_i}E_{i+1}) = k \cdot \mathrm{id} \oplus S'$ be any splitting of the space of morphisms and let $S = S' \oplus \mathrm{Hom}^\bullet_{\widetilde{\mathcal{C}_\sigma^{L_i}}}(E_i, L_{E_i}E_{i+1})$. Then Remark 1.4.3 guarantees that the DG subcategory $\widetilde{\mathcal{C}_\sigma^{L_i}}''$, obtained from $\widetilde{\mathcal{C}_\sigma^{L_i}}$ by removing morphisms in S, is quasi-isomorphic to $\widetilde{\mathcal{C}_\sigma^{L_i}}$. Then the category $\widetilde{\mathcal{C}_\sigma^{L_i}}''$ is both finite and ordered. □

2.1.1 Example

Let X be a smooth surface obtained from $\mathbb{P}^2$ by blowing up a point and then blowing up the point on the exceptional divisor. Let E_2 denote the exceptional divisor of the second blow up, let E_1 be a strict transform of the exceptional divisor of the first blow up and let $\mathcal{O}_X(H)$ be a pullback to X of $\mathcal{O}_{\mathbb{P}^2}(1)$. By Orlov's theorem, [54, Theorem 4.3]

$$\langle \mathcal{O}_{E_2}(E_2)[-1], \mathcal{O}_{E_1+E_2}(E_1+E_2)[-1], \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_X(H), \mathcal{O}_X(2H) \rangle$$

is a full exceptional collection. This collection is not strong and it has the following Ext-quiver

$$\begin{tikzcd} \mathcal{O}_{E_2}(E_2)[-1] \arrow[r, shift left, "\bar{\gamma}"] \arrow[r, shift right, "\gamma"'] & \mathcal{O}_{E_1+E_2}(E_1+E_2)[-1] \arrow[r, "\bar{\delta}"] & \mathcal{O}_X \arrow[r, shift left=2, "\alpha_1"] \arrow[r, "\alpha_2"] \arrow[r, shift right=2, "\alpha_3"'] & \mathcal{O}_X(H) \arrow[r, shift left=2, "\beta_1"] \arrow[r, "\beta_2"] \arrow[r, shift right=2, "\beta_3"'] & \mathcal{O}_X(2H) \end{tikzcd}$$

with morphisms

$$\gamma \in \mathrm{Hom}(\mathcal{O}_{E_2}(E_2)[-1], \mathcal{O}_{E_1+E_2}(E_1+E_2)[-1]),$$
$$\bar{\gamma} \in \mathrm{Ext}^1(\mathcal{O}_{E_2}(E_2)[-1], \mathcal{O}_{E_1+E_2}(E_1+E_2)[-1]),$$
$$\bar{\delta} \in \mathrm{Hom}(\mathcal{O}_{E_1+E_2}(E_1+E_2)[-1], \mathcal{O}_X) = \mathrm{Ext}^1(\mathcal{O}_{E_1+E_2}(E_1+E_2), \mathcal{O}_X)$$

and relations given by

$$\begin{aligned}
&\beta_1\alpha_2 = \beta_2\alpha_1, && \beta_1\alpha_3 = \beta_3\alpha_1, && \beta_2\alpha_3 = \beta_3\alpha_2,\\
&\bar{\delta}\circ\bar{\gamma} = 0, && \alpha_1\circ\bar{\delta} = 0, && \alpha_2\circ\bar{\delta} = 0.
\end{aligned}$$

If we mutate $\mathcal{O}_{E_1+E_2}(E_1+E_2)[-1]$ to the right over $\mathcal{O}_X$ then the resulting collection $\langle \mathcal{O}_{E_2}(E_2)[-1], \mathcal{O}_X, \mathcal{O}_X(E_1+E_2), \mathcal{O}_X(H), \mathcal{O}_X(2H)\rangle$ is strong and has quiver

$$\begin{tikzcd}
\mathcal{O}_{E_2}(E_2)[-1] \arrow[r, "a"] \arrow[rr, bend left, "b"] & \mathcal{O}_X \arrow[r, "c"] \arrow[rr, bend right, "d_3"'] & \mathcal{O}_X(E_1+E_2) \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_2"'] & \mathcal{O}_X(H) \arrow[r, shift left=2, "e_1"] \arrow[r, "e_2"] \arrow[r, shift right=2, "e_3"'] & \mathcal{O}_X(2H)
\end{tikzcd}$$

with relations

$$\begin{aligned}
&e_1\circ d_2 = e_2\circ d_1, && e_1\circ d_3 = e_3\circ d_1\circ c, && e_2\circ d_3 = e_3\circ d_2\circ c,\\
&c\circ a = 0, && d_1\circ b = 0, && d_3\circ a = d_2\circ b.
\end{aligned}$$

We can thus present $\mathcal{O}_{E_1+E_2}(E_1+E_2)$ as a twisted complex $\{\mathcal{O}_X \xrightarrow{c} \mathcal{O}_X(E_1+E_2)\}$. This shows that the sought after DG quiver is

$$\begin{tikzcd}
\mathcal{O}_{E_2}(E_2)[-1] \arrow[r, shift left, "\bar{\gamma}"] \arrow[r, shift right, "\gamma"'] & \mathcal{O}_{E_1+E_2}(E_1+E_2)[-1] \arrow[r, "\bar{\delta}"] \arrow[rr, bend right, "\epsilon_1"'] \arrow[rr, bend right=40, "\epsilon_2"'] & \mathcal{O}_X \arrow[r, shift left=2, "\alpha_1"] \arrow[r, "\alpha_2"] \arrow[r, shift right=2, "\alpha_3"'] & \mathcal{O}_X(H) \arrow[r, shift left=2, "\beta_1"] \arrow[r, "\beta_2"] \arrow[r, shift right=2, "\beta_3"'] & \mathcal{O}_X(2H)
\end{tikzcd}$$

with ϵ_1 and ϵ_2 in degree minus one, $\bar{\gamma}$ in degree one and the remaining arrows in degree zero. The differentials in this quiver are given by

$$\partial(\epsilon_1) = \alpha_1\circ\bar{\delta}, \qquad \partial(\epsilon_2) = \alpha_2\circ\bar{\delta}$$

and relations

$$\begin{aligned}
&\beta_1\circ\alpha_2 = \beta_2\circ\alpha_1, && \beta_1\circ\alpha_3 = \beta_3\circ\alpha_1, && \beta_2\circ\alpha_3 = \beta_3\circ\alpha_2,\\
&\beta_2\circ\epsilon_1 = \beta_1\circ\epsilon_2, && \bar{\delta}\circ\bar{\gamma} = 0, && \epsilon_1\circ\bar{\gamma} = 0, && \epsilon_2\circ\bar{\gamma} = \alpha_3\circ\bar{\delta}\circ\gamma.
\end{aligned}$$

2.2 Massey products

Instead of finding the DG quiver of an exceptional collection σ one can try to calculate the minimal A_∞-structure on the Ext-quiver of σ. Since every DG category has a unique minimal model and every A_∞-category is A_∞-quasi-isomorphic to a DG category, both approaches are equivalent.

The structure of an A_∞-category on the Ext-quiver of σ can be sometimes calculated by means of Massey products.

2.2.1 Minimal model by Merkulov's construction

In [51, Section 3] Merkulov gives an algorithm for calculating the A_∞-category $H(\mathcal{C})$ for any DG-category $\mathcal{C}$. As in Remark 1.3.2 we denote by ∂ the differential on the space of morphisms of $\mathcal{C}$ and by μ the composition of morphisms. To find higher multiplications on $H(\mathcal{C})$ we need maps $i\colon H(\mathcal{C}) \to \mathcal{C}$, $\pi\colon \mathcal{C} \to H(\mathcal{C})$ of degree zero and $h\colon \mathcal{C} \to \mathcal{C}$ of degree minus one such that

$$\pi \circ i = \mathrm{id}, \qquad i \circ \pi = \mathrm{id} - \partial(h), \qquad h^2 = 0.$$

With chosen π, i and h we define operations

$$\lambda_n\colon \mathrm{Hom}^\bullet_{\mathcal{C}}(X^{n-1}, X^n) \otimes_k \ldots \otimes_k \mathrm{Hom}^\bullet_{\mathcal{C}}(X^0, X^1) \to \mathrm{Hom}^\bullet_{\mathcal{C}}(X^0, X^n[2-n]).$$

We put $\lambda_1 = -h^{-1}$, $\lambda_2 = \mu$ and

$$\lambda_n = \sum_{l=1}^{n-1} (-1)^{n-l+1} \mu(h(\lambda_{n-l}), h(\lambda_l)).$$

Then, the operations $m_n\colon \mathrm{Hom}^\bullet_{H(\mathcal{C})}(X^{n-1}, X^n) \otimes_k \ldots \otimes_k \mathrm{Hom}^\bullet_{H(\mathcal{C})}(X^0, X^1) \to \mathrm{Hom}^\bullet_{H(\mathcal{C})}(X^0, X^n[2-n])$ defined by

$$m_n = \pi \circ \lambda_n \circ i^{\otimes n}$$

give the A_∞-structure on $H(\mathcal{C})$.

2.2.2 Massey products in triangulated and DG setting

We will later see that Merkulov's construction can be sometimes identified with Massey products in enhanced triangulated categories. First, we recall some definitions after [20, Chapter IV, Section 2.10] and [41].

Let $\mathcal{T}$ be a triangulated category and let $T^\bullet = \{T^0 \xrightarrow{\partial^0} T^1 \xrightarrow{\partial^1} \ldots \to T^n\}$ be a finite complex in $\mathcal{T}$.

Definition 2.2.1. *A* right Postnikov system *of $T^\bullet$ is a diagram*

$$\begin{tikzcd}
& T^0[n] \arrow[rr, "\widetilde{\partial^0}"] \arrow[dr, "i_0"] & & T^1[n-1] \arrow[r] & \ldots \arrow[r] & T^{n-1}[1] \arrow[rr, "\widetilde{\partial^{n-1}}"] \arrow[dr, "i_{n-1}"'] & & T^n \\
Y^0 \arrow[ur, "{[1]}", "j_{-1}"'] & \ast & Y^1 \arrow[ll] \arrow[ur, "{[1]}", "j_0"'] \arrow[u, phantom, "\circlearrowleft"] & {} \arrow[l] & \ldots & {} & Y^n = T^n[1] \arrow[ll] \arrow[ur, "{[1]}", "j_0 = id[1]"'] \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

in which all triangles

$$T^l[n-l] \xrightarrow{i_l} Y^{l+1} \to Y^l \xrightarrow{j_{l-1}} T^l[n-l+1]$$

are distinguished and all diagrams of the form

$$
\begin{tikzcd}
T^l[n-l] \arrow[rr, "\widetilde{\partial}^l"] \arrow[dr, "i_l"'] & & T^{l+1}[n-l-1] \\
& Y^{l+1} \arrow[ur, "j_l"'] &
\end{tikzcd}
$$

commute. Here, $\widetilde{\partial}^i \colon T^i[i] \to T^{i+1}[i-1]$ *is a map of degree 1 corresponding to the map* ∂^i *under the equality* $\mathrm{Hom}(T^i, T^{i+1}) \simeq \mathrm{Ext}^1(T^i[1], T^{i+1})$*.*

Definition 2.2.2. *A* left Postnikov system *of* $T^\bullet$ *is a diagram*

$$
\begin{tikzcd}
T^0 \arrow[rr, "\partial^0"] \arrow[dr, "i_0=id"'] & \arrow[d, phantom, "\circlearrowleft"] & T^1 \arrow[rr, "\partial^1"] \arrow[dr, "i_1"] \arrow[d, phantom, "*"] & \arrow[d, phantom, "\circlearrowleft"] & T^2 \arrow[r] & \dots \arrow[r] & T^n \arrow[dr, "i_n"] \arrow[d, phantom, "*"] & \\
& Z^0 = T^0 \arrow[ur, "j_0"] & & Z^1 \arrow[ll, "{[1]}"] \arrow[ur, "j_1"] & \arrow[l] & \dots & & Z^n \arrow[ll, "{[1]}"]
\end{tikzcd}
$$

in which all triangles of the form

$$
Z^{l-1} \xrightarrow{j_{l-1}} T^l \xrightarrow{i_l} Z^l \to Z^{l-1}[1]
$$

are distinguished and all diagrams

$$
\begin{tikzcd}
T^l \arrow[rr, "\partial^l"] \arrow[dr, "i_l"'] & & T^{l+1} \\
& Z^l \arrow[ur, "j_l"'] &
\end{tikzcd}
$$

commute.

Definition 2.2.3. *An object S of $\mathcal{D}$ is called a* right (left) convolution *of* $T^\bullet$ *if there exists a right (left) Postnikov system of* $T^\bullet$ *with* $S = Y^0[1]$ *(*$S = Z^n$*, respectively).*

Lemma 2.2.4. *([20, Chapter III.4, Exercise 1]) The set of left convolutions of* $T^\bullet$ *coincides with the set of right convolutions of* $T^\bullet$*.*

Proof. Assume first that the complex has length 3. Then $S = Y^0[-1]$ is a convolution of $T^0 \xrightarrow{\partial^0} T^1 \xrightarrow{\partial^1} T^2$ if there is a diagram

$$
\begin{tikzcd}
& T^0[2] \arrow[rr, "\partial^0"] \arrow[dr, "i_0"] \arrow[d, phantom, "\star"] & & T^1[1] \arrow[dr, "\partial^1"] \arrow[d, phantom, "\star"] & \\
Y^0 \arrow[ur, "j_{-1}", "{[1]}"'] & & Y^1 \arrow[ll] \arrow[ur, "j_0", "{[1]}"'] & & T^2[1] \arrow[ll]
\end{tikzcd}
$$

Then the octahedron axiom

$$
\begin{tikzcd}
T^0[2] \arrow[dd, "{[1]}"', "\partial^0"] \arrow[dr, "i_0", "{[1]}"'] & \arrow[d, phantom, "\star"] & Y^0[-1] = S \arrow[ll, "j_{-1}"'] \\
\arrow[r, phantom, "\circlearrowleft"] & Y^1[-1] \arrow[ur] \arrow[dl, "j_0"'] \arrow[d, phantom, "\star"] \arrow[r, phantom, "\circlearrowleft"] & \\
T^1[1] \arrow[rr, "{[1]}", "\partial^1"'] & & T^2 \arrow[ul] \arrow[uu, "\epsilon"']
\end{tikzcd}
\qquad
\begin{tikzcd}
T^0[2] \arrow[dd, "{[1]}"', "\partial^0"] & \arrow[d, phantom, "\circlearrowleft"] & Y^0[-1] = S \arrow[ll, "j_{-1}"'] \arrow[dl, "\alpha"] \\
\arrow[r, phantom, "\star"] & V \arrow[ul, "\delta"] \arrow[dr, "{[1]}", "\beta"'] \arrow[d, phantom, "\circlearrowleft"] \arrow[r, phantom, "\star"] & \\
T^1[1] \arrow[ur, "\gamma"] \arrow[rr, "{[1]}", "\partial^1"'] & & T^2 \arrow[uu, "\epsilon"']
\end{tikzcd}
$$

tell us that S is a convolution of a right Postnikov system

$$\begin{tikzcd}
 & T^1 \arrow[rr, "\partial^1"] \arrow[dr, "\gamma"] \arrow[d, phantom, "\star"] & & T^2 \arrow[dr, "\epsilon"] \arrow[d, phantom, "\star"] & \\
T^0 \arrow[ur, "\partial^0"] & & V \arrow[ll, "\delta", "{[1]}"'] \arrow[ur, "\beta"] & & S \arrow[ll, "\alpha"]
\end{tikzcd}$$

For a complex of length $n+1$ we proceed by induction.

Assume that $S = Y^0[-1]$ is a right convolution of $T^0 \to \ldots \to T^n$. Then we have a diagram

$$\begin{tikzcd}
 & & & & & & Z \arrow[dl, "{[1]}", "v"'] & & \\
 & T^0[n] \arrow[r] \arrow[dr] & \ldots & \ldots & \ldots \arrow[r] & T^{n-2}[2] \arrow[rr, "\partial^{n-2}"] \arrow[dr, "i_{n-2}"] & & T^{n-1}[1] \arrow[dr, "\partial^{n-1}"] \arrow[ul, "u"'] & \\
Y^0 \arrow[ur, "{[1]}", "j_{-1}"'] & & Y^1 \arrow[ll] & \ldots \arrow[l] & Y^{n-2} \arrow[l] \arrow[ur, "{[1]}", "j_{n-3}"'] & & Y^{n-1} \arrow[ll] \arrow[ur, "{[1]}", "j_{n-2}"'] & & T^n \arrow[ll]
\end{tikzcd}$$

where Z is the cone of $T^{n-2}[2] \to T^{n-1}[1]$. Again, the octahedron

$$\begin{tikzcd}
T^{n-2}[2] \arrow[dd, "\partial^{n-2}"', "{[1]}"] \arrow[dr, "{[1]}", "i_{n-2}"'] & & Y^{n-2}[-1] \arrow[ll, "j_{n-3}", "\star"'] \\
 & Y^{n-1}[-1] \arrow[ur] \arrow[dl, "j_{n-2}"'] & \\
T^{n-1}[1] \arrow[rr, "{[1]}", "\partial^{n-1}"'] & & T^n \arrow[ul] \arrow[uu]
\end{tikzcd}
\qquad
\begin{tikzcd}
T^{n-2}[2] \arrow[dd, "\partial^{n-2}"', "{[1]}"] & & Y^{n-2}[-1] \arrow[ll, "j_{n-3}"'] \arrow[dl, "r"] \\
 & Z \arrow[ul, "v"'] \arrow[dr, "q", "{[1]}"'] & \\
T^{n-1}[1] \arrow[ur, "u"] \arrow[rr, "{[1]}", "\partial^{n-1}"'] & & T^n \arrow[uu]
\end{tikzcd}$$

gives maps $r\colon Y^{n-2}[-1] \to Z$ and $q\colon Z \to T^n[1]$ such that $v \circ r = j_{n-3}[1]$ and $q \circ u = \partial^{n-1}$.

Now, we consider a map $p\colon T^{n-3}[1] \xrightarrow{i_{n-3}[-2]} Y^{n-2}[-2] \xrightarrow{r[-1]} Z[-1]$ and a complex

$$T^0[1] \xrightarrow{\partial^0} T^1[1] \to \ldots \to T^{n-3}[1] \xrightarrow{p} Z[-1] \xrightarrow{q[-1]} T^n.$$

We will show that $Y^0[-1]$ is a right convolution of this complex. Indeed, we have a diagram

$$\begin{tikzcd}
 & T^0[n] \arrow[r, "\partial^0"] \arrow[dr, "i_0"] & \ldots & \ldots \arrow[r] & T^{n-3}[3] \arrow[rr, "{p[2]}"] \arrow[dr, "i_{n-3}"] & & Z[1] \arrow[dr, "q"] & \\
Y^0 \arrow[ur, "{[1]}", "j_{-1}"'] & & Y^1 \arrow[ll] & \ldots \arrow[l] & \ldots & Y^{n-2} \arrow[l] \arrow[ur, "{[1]}", "r"'] & & T^n \arrow[ll]
\end{tikzcd}$$

which is a Postnikov system because $r \circ i_{n-3} = p[2]$.

The new complex has length n and hence each right convolution is also a left one, i.e. we have a diagram

$$\begin{tikzcd}
 & T^1[1] \arrow[r] & \ldots \arrow[r] & T^{n-3}[1] \arrow[rr, "p"] \arrow[dr, "\alpha_{n-2}"] & & Z \arrow[rr, "{q[-1]}"] \arrow[dr, "\alpha_{n-1}"] & & T^n \arrow[dr, "\alpha_n"] & \\
T^0[1] \arrow[ur, "\partial^0"] & \ldots \arrow[l, "{[1]}"] & \ldots & \ldots & P^{n-1} \arrow[l, "{[1]}"] \arrow[ur, "\beta_{n-2}"] & & P^n \arrow[ll, "{[1]}"] \arrow[ur, "\beta_{n-1}"] & & Y^0[-1] \arrow[ll, "{[1]}"]
\end{tikzcd}$$

It follows that P^n is a left convolution of a complex $T^0[1] \to \ldots \to T^{n-3}[1] \to Z$. As this complex has length $n-1$, P^n is also its right convolution. Then, from the diagram

$$\begin{tikzcd}
 & T^0[n-1] \arrow[rr, "\partial^0"] \arrow[dr] & & T^1[n-2] & \cdots & T^{n-3}[2] \arrow[rr, "\partial^{n-3}"] \arrow[dr, "{p[1]}"] & & T^{n-2}[1] \arrow[dr, "\partial^{n-2}"] & \\
P^n[1] \arrow[ur, "{[1]}"] & & R^{n-1} \arrow[ll] \arrow[ur, "{[1]}"] & \ldots \arrow[l] & & \ldots & Z \arrow[l] \arrow[ur, "{[1]}", "{v[1]}"'] & & T^{n-1} \arrow[ll]
\end{tikzcd}$$

it follows that P^n is a convolution of $T^0 \to \ldots \to T^{n-1}$ (because $v[1] \circ p[1] = v[1] \circ r \circ i_{n-3}[-2] = \partial^{n-3}$). Finally, from the fact that Y^0 is a cone of a map P^n to T^n it follows that Y^0 is also a left convolution of $T^0 \to \ldots \to T^n$.

The proof that every left convolution is a right convolution is analogous. $\square$

Definition 2.2.5. *Let* $T^\bullet = \{T^0 \xrightarrow{\partial^0} T^1 \xrightarrow{\partial^1} T^2 \to \cdots \to T^{n-1} \xrightarrow{\partial^{n-1}} T^n\}$ *be a complex in* $\mathcal{T}$ *and let*

$$\begin{tikzcd}
T^1 \arrow[r] & \ldots \arrow[r] & T^{n-1} \arrow[dl, "i"] \\
 & S \arrow[ul, "\rho", "{[n-2]}"'] &
\end{tikzcd}$$

be a convolution.

The n-fold Massey product $\mu_n(\partial^{n-1}, \ldots, \partial^0)$ *is defined as the subset of* $\mathrm{Hom}(T^0, T^n[2-n])$ *consisting of compositions*

$$q \circ p \colon T^0 \to T^n[2-n]$$

where

$$p \colon T^0 \to S[2-n], \qquad q \colon S \to T^n$$

are such that

$$\rho \circ p = \partial^0, \qquad q \circ i = \partial^{n-1},$$

$$\begin{tikzcd}
T^0 \arrow[r, "\partial^0"] \arrow[drr, "p", "{[2-n]}"'] & T^1 \arrow[r] & \ldots \arrow[r] & T^{n-1} \arrow[r, "\partial^{n-1}"] \arrow[dl, "i"] & T^n \\
 & & S. \arrow[ul, "\rho", "{[n-2]}"'] \arrow[urr, "q"] & &
\end{tikzcd}$$

For any pair of morphisms $T^0 \xrightarrow{f} T^1 \xrightarrow{g} T^2$ we can also put

$$\mu_2(g, f) = g \circ f.$$

2.2.3 Triple Massey product

The triple Massey product is always defined for a complex $T^0 \xrightarrow{\partial^0} T^1 \xrightarrow{\partial^1} T^2 \xrightarrow{\partial^2} T^3$ in $\mathcal{T}$.

Let

$$\begin{tikzcd} T^0 \arrow[r, "\partial^0"] & T^1 \arrow[rr, "\partial^1"] & & T^2 \arrow[r, "\partial^2"] \arrow[dl, "\alpha"] & T^3 \\ & & S \arrow[ul, "\beta", "{[1]}"'] \arrow[u, phantom, "\star"] & & \end{tikzcd}$$

be a distinguished triangle.

Applying the functor $\mathrm{Hom}_{\mathcal{T}}(T^0,-)$ to this distinguished triangle we get the following long exact sequence

$$\mathrm{Hom}_{\mathcal{T}}(T^0,T^2[-1]) \xrightarrow{\alpha[-1]^*} \mathrm{Hom}_{\mathcal{T}}(T^0,S[-1]) \xrightarrow{\beta[-1]^*} \mathrm{Hom}_{\mathcal{T}}(T^0,T^1) \xrightarrow{(\partial^1)^*} \mathrm{Hom}_{\mathcal{T}}(T^0,T^2)$$

Since, $(\partial^1)^*(\partial^0) = \partial^1 \circ \partial^0 = 0$ there exists $q \in \mathrm{Hom}_{\mathcal{T}}(T^0,S[-1])$ such that $\partial^0 = \beta[-1]^*(q)$. Moreover, an element $q \in \mathrm{Hom}_{\mathcal{T}}(T^0,S[-1])$ is determined up to elements of the form $\alpha[-1] \circ \widetilde{q}$ for $\widetilde{q} \in \mathrm{Hom}_{\mathcal{T}}(T^0,T^2[-1])$.

Analogously, applying $\mathrm{Hom}_{\mathcal{T}}(-\,,T^3)$ we get a long exact sequence

$$\mathrm{Hom}_{\mathcal{T}}(T^1[1],T^3) \xrightarrow{(-)\circ\beta} \mathrm{Hom}_{\mathcal{T}}(S,T^3) \xrightarrow{(-)\circ\alpha} \mathrm{Hom}_{\mathcal{T}}(T^2,T^3) \xrightarrow{(-)\circ\partial^1} \mathrm{Hom}_{\mathcal{T}}(T^1,T^3).$$

Then there exists $p \in \mathrm{Hom}_{\mathcal{T}}(S,T^3)$ such that $\partial_2 = p \circ \alpha$ and an element p is determined up to elements of the form $\widetilde{p} \circ \beta[-1]$ for $\widetilde{p} \in \mathrm{Hom}_{\mathcal{T}}(T^1[1],T^3)$.

The triple Massey product $\mu_3(\partial^2,\partial^1,\partial^0)$ is then the composition $p \circ q \in \mathrm{Hom}_{\mathcal{T}}(T^0,T^3[-1])$.

As we can choose $q + \alpha \circ \widetilde{q}$ instead of q and $p + \widetilde{p} \circ \beta[-1]$ instead of p, the composition $p \circ q$ is determined only up to

$$(p + \widetilde{p} \circ \beta[-1]) \circ (q + \alpha \circ \widetilde{q}) - p \circ q = \widetilde{p} \circ \partial^0 + \partial^2[-1] \circ \widetilde{q}.$$

Hence the triple Massey product takes value in $\mathrm{Hom}_{\mathcal{T}}(T^0,T^3[-1])/A$ for

$$A = \mathrm{Hom}_{\mathcal{T}}(T^1[1],T^3) \circ \partial^0 + \partial^2[-1] \circ \mathrm{Hom}_{\mathcal{T}}(T^0,T^2[-1]).$$

The following lemma can be found as [20, Chapter IV 2, Exercise 3].

Lemma 2.2.6. *The complex $T^\bullet = \{T^0 \xrightarrow{\partial^0} T^1 \xrightarrow{\partial^1} T^2 \xrightarrow{\partial^2} T^3\}$ has a convolution if and only if the image of $\mu_3(\partial^2,\partial^1,\partial^0)$ in $\mathrm{Hom}_{\mathcal{T}}(T^0,T^3[-1])/A$ vanishes.*

Proof. We have a diagram

$$\begin{tikzcd} & T^1 \arrow[rr, "\partial^1"] \arrow[dr, "i_1"] \arrow[d, phantom, "*"] & & T^2 \arrow[rr, "\partial^2"] \arrow[dr, "i_2"] \arrow[d, phantom, "*"] & & T^3 \\ T^0 \arrow[ur, "\partial_0"] & {} \arrow[r, phantom, "\circlearrowleft"] & Z^1 \arrow[ll, "{[1]}"] \arrow[ur, "j_1"] & {} & Z^2. \arrow[ll, "{[1]}"] \arrow[ur, dashed, "j_2"] & \end{tikzcd}$$

Thus, $T^\bullet$ has a convolution if and only if there exists a map $j_2\colon Z^2 \to T^3$ such that $j_2 \circ i_2 = \partial^2$.

Z^2 is a convolution of the complex $\{T^0 \xrightarrow{\partial^0} T^1 \xrightarrow{\partial^1} T^2\}$. By Lemma 2.2.4 it can be also calculated by means of a right Postnikov system, i.e. we have a diagram

$$\begin{tikzcd}
& T^0[1] \arrow[rr, "\partial^0[1]"] \arrow[dr, "p"] \arrow[d, phantom, "*"] & \arrow[d, phantom, "\circlearrowleft"] & T^1 \arrow[dr, "\partial^1"] \arrow[d, phantom, "*"] & \\
Z^2 \arrow[ur] & & Y^1 \arrow[ll, "\alpha_2"] \arrow[ur, "\beta"] & & X^2 \arrow[ll, "\alpha"]
\end{tikzcd}$$

Moreover, as $\partial^2 \circ \partial^1 = 0$ there exists $q\colon Y^1 \to T^3$ such that $q \circ \alpha = \partial^2$.

The octahedron axiom tells us that the map $i_2\colon T^2 \to Z^2$ equals $\alpha_2 \circ \alpha$:

$$\begin{tikzcd}
T^1 \arrow[rr, "\partial^1"] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & T^2 \arrow[dd, "i_2"] \\
\arrow[r, phantom, "*"] & Z^1 \arrow[ur] \arrow[dl] \arrow[d, phantom, "\circlearrowleft"] \arrow[r, phantom, "*"] & \\
T^0 \arrow[uu, "\partial^0"] & & Z^2 \arrow[ll] \arrow[ul]
\end{tikzcd}
\qquad
\begin{tikzcd}
T^1 \arrow[rr, "\partial^1"] & \arrow[d, phantom, "*"] & T^2 \arrow[dd, "i_2"] \arrow[dl, "\alpha"] \\
\arrow[r, phantom, "\circlearrowleft"] & Y^1 \arrow[ul] \arrow[dr, "\alpha_2"] \arrow[d, phantom, "*"] \arrow[r, phantom, "\circlearrowleft"] & \\
T^0 \arrow[uu, "\partial^0"] \arrow[ur] & & Z^2 \arrow[ll]
\end{tikzcd}.$$

Now, we can apply the functor $\mathrm{Hom}_{\mathcal{T}}(-, T^3)$ to the distinguished triangle

$$T^0[1] \xrightarrow{p} Y^1 \xrightarrow{\alpha_2} Z^2 \to T^0[2].$$

We obtain

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{T}}(Z^2, T^3) \arrow[r] & \mathrm{Hom}_{\mathcal{T}}(Y^1, T^3) \arrow[r] & \mathrm{Hom}_{\mathcal{T}}(T^0[1], T^3) \\
& q \arrow[r, mapsto] & q \circ p
\end{tikzcd}$$

If the triple Massey product $q \circ p = \mu_3(d^2, d^1, d^0)$ can be chosen to be zero then there exists $j_2\colon Z^2 \to T^3$ such that $j_2 \circ \alpha_2 = q$. It follows that $j_2 \circ i_2 = j_2 \circ \alpha_2 \circ \alpha = q \circ \alpha = \partial^3$.

Conversely, if there exists $j_2\colon Z^2 \to T^3$ such that $j_2 \circ i_2 = j_2 \circ \alpha_2 \circ \alpha = \partial^2$ then taking $q = j_2 \circ \alpha_2$ we get that $q \circ p = j_2 \circ \alpha_2 \circ p = 0$ (because α_2 and p are consecutive maps in a distinguished triangle). $\square$

Thus the triple Massey product is defined for any complex of length four and such a complex has a convolution if the triple Massey product vanishes. The picture gets slightly more complicated when the length of the complex increases.

2.2.4 Quadruple Massey product

Let $T^\bullet = \{T^0 \xrightarrow{\partial^0} T^1 \xrightarrow{\partial^1} T^2 \xrightarrow{\partial^2} T^3 \xrightarrow{\partial^3} T^4\}$ be a complex of length five and let

$$\begin{tikzcd}
T^1 \arrow[r, "\partial^1"] & T^2 \arrow[r, "\partial^2"] \arrow[d, phantom, "{[2]}"] & T^3 \arrow[dl, "i"] \\
& S \arrow[ul, "\rho"] &
\end{tikzcd}$$

be a convolution.

We show that the quadruple Massey product $\mu_4(\partial^3, \partial^2, \partial^1, \partial^0)$ is non-empty if the triple Massey products $\mu_3(\partial^3, \partial^2, \partial^1)$ and $\mu_3(\partial^2, \partial^1, \partial^0)$ vanish simultaneously. Note that [20, Chapter IV, Exercise 2.3.b)] omits this condition. However, already in 1978 O'Neill in [53] gave an example of a topological space X and elements in $H^*(X, \mathbb{Z})$ such that $0 \in \mu_3(u_1, u_2, u_3)$, $0 \in \mu_3(u_2, u_3, u_4)$ but $\mu_4(u_1, u_2, u_3, u_4) = \emptyset$.

The quadruple Massey product is defined as the set

$$\mu_4(\partial^3, \partial^2, \partial^1, \partial^0) = \{q \circ p \colon T^0 \to T^4[-2] \mid p \colon T^0 \to S[-2],\ q \colon S \to T^4,\ \rho \circ p = \partial^0,\ q \circ i = \partial^3\};$$

$$\begin{tikzcd}
T^0 \arrow[r, "\partial^0"] \arrow[drr, "p"', "{[-2]}"] & T^1 \arrow[r, "\partial^1"] & T^2 \arrow[r, "\partial^2"] & T^3 \arrow[r, "\partial^3"] \arrow[dl, "i"'] & T^4 \\
 & & S. \arrow[ul, "\rho", "{[2]}"'] \arrow[urr, "q"'] & &
\end{tikzcd}$$

The quadruple Massey product $\mu_4(\partial^3, \partial^2, \partial^1, \partial^0)$ is non-empty if the triple Massey products $\mu_3(\partial^2, \partial^1, \partial^0)$ and $\mu_3(\partial^3, \partial^2, \partial^1)$ vanish simultaneously. To see it consider the diagram

$$\begin{tikzcd}
T^1 \arrow[rr, "\partial^1"] \arrow[dr, "\mathrm{id}"'] & \arrow[d, phantom, "\circlearrowleft"] & T^2 \arrow[rr, "\partial^2"] \arrow[dr, "i_2"] \arrow[d, phantom, "*"] & \arrow[d, phantom, "\circlearrowleft"] & T^3 \arrow[rr, "\partial^3"] \arrow[dr, "i_3"] \arrow[d, phantom, "*"] & & T^4 \\
 & T^1 \arrow[ur, "\partial^1"] & \arrow[d, phantom, "\circlearrowleft"] & Z^2 \arrow[ll, "{[1]}", "\rho_2"'] \arrow[ur, "j_2"] & & S^1 \arrow[ll, "{[1]}", "\rho_3"'] & \\
 & & T^0 \arrow[ul, "\partial^0"] \arrow[ur, "{[-1]}", "p_1"'] \arrow[urrr, dashed, "{[-2]}", "p"'] & & & &
\end{tikzcd}$$

Here, Z^2 is the cone of the map $\partial^1 \colon T^1 \to T^2$. As $\partial^1 \circ \partial^0 = 0$ and $\partial^2 \circ \partial^1 = 0$ there exist $p_1 \colon T^0 \to Z^2[-1]$ and $j_2 \circ Z^2 \to T^3$ such that $\rho_2 \circ p_1 = \partial^0$ and $j_2 \circ i_2 = \partial^2$. Then S^1 is defined by means of a distinguished triangle

$$Z^2 \xrightarrow{j_2} T^3 \xrightarrow{i_3} S^1 \xrightarrow{\rho_3} Z^2[1].$$

After applying the functor $\mathrm{Hom}_{\mathcal{T}}(T^0, -)$ to it we get

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{T}}(T^0, S^1[-2]) \arrow[r] & \mathrm{Hom}_{\mathcal{T}}(T^0, Z^2[-1]) \arrow[r] & \mathrm{Hom} -\mathcal{T}(T^0, T^3[-1]) \\
 & p_1 \arrow[r, mapsto] & j_2 \circ p_1 \in \mu_3(\partial^2, \partial^1, \partial^0)
\end{tikzcd}$$

Hence, there exists $p \colon T^0 \to S^1[-2]$ such that $\rho_3 \circ p = p_1$ (and also $\rho_2 \circ \rho_3 \circ p = \partial^0$) if and only if one can choose p_1 and j_2 in such a way that $j_2 \circ p_1 = 0$, i.e. if $0 \in \mu_3(\partial^2, \partial^1, \partial^0)$. Note, that S^1 is defined as a cone of j_2 and hence depends on the choice of j_2.

To give a necessary condition for the existence of $q \colon S^1 \to T^4$ such that $q \circ i_3 = \partial^3$ we use Lemma 2.2.4 again.

Consider the diagram

$$
\begin{tikzcd}
 & & & & T^4 \arrow[d, phantom, "\circlearrowleft"] & & \\
 & S^2 \arrow[urrr, dashed, "q"] \arrow[dr, "\beta_2", "{[2]}"'] & & Y^2 \arrow[ll, "\alpha_2"] \arrow[ur, "q_1"] \arrow[dr, "\beta_1", "{[1]}"'] \arrow[d, phantom, "\circlearrowleft"] & & T^3 \arrow[ll, "\alpha_1"] \arrow[ul, "\partial^3"'] \arrow[dr, "\mathrm{id}"] \arrow[d, phantom, "\circlearrowleft"] & \\
T^0 \arrow[rr, "\partial^0"'] & & T^1 \arrow[rr, "\partial^1"'] \arrow[ur, "\gamma_1", "{[-1]}"'] \arrow[u, phantom, "*"] & & T^2 \arrow[rr, "\partial^2"'] \arrow[ur, "\partial^2"] \arrow[u, phantom, "*"] & & T^3
\end{tikzcd}
$$

Here, Y^2 is the cone of the map $d^2\colon T^2 \to T^3$. As $\partial^3 \circ \partial^2 = 0$ and $\partial^2 \circ \partial^1 = 0$ there exist $q_1\colon Y^2 \to T^4$ and $\gamma_1\colon T^1 \to Y^2[-1]$ such that $q_1 \circ \alpha_1 = \partial^3$ and $\beta_1 \circ \gamma_1 = \partial^1$.

S^2 is defined by means of a distinguished triangle

$$T^1 \xrightarrow{\gamma_1} Y^2[-1] \xrightarrow{\alpha_2} S^2[-1] \xrightarrow{\beta_2} T^1[1].$$

Applying $\mathrm{Hom}_{\mathcal{T}}(-, T^4)$ to this triangle we get

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{T}}(S^2, T^4) \arrow[r] & \mathrm{Hom}_{\mathcal{T}}(Y^2, T^4) \arrow[r] & \mathrm{Hom}_{\mathcal{T}}(T^1[1], T^4) \\
 & q_1 \arrow[r, mapsto] & q_1 \circ \gamma_1 \in \mu_3(\partial^3, \partial^2, \partial^1)
\end{tikzcd}
$$

Hence, there exists $q\colon S^2 \to T^4$ such that $q \circ \alpha_2 = q_1$ (and hence $q \circ \alpha_2 \circ \alpha_1 = \partial^3$) if and only if one can choose $q_1\colon Y^2 \to T^4$ and $\gamma_1\colon T^1 \to Y^2[-1]$ in such a way that $\gamma_1 \circ q_1 = 0$, i. e. if $0 \in \mu_3(\partial^3, \partial^2, \partial^1)$. Again, note that S^2 is defined as a cone of γ_1 and hence depends on the choice made.

The quadruple Massey product $\mu_4(\partial^3, \partial^2, \partial^1, \partial^0)$ is non-empty if one can choose maps j_2, p_1, γ_1 and q_1 in such a way that S^1 - the cone of j_2 is equal to S^2 - the cone of γ_1 that is when $\mu_3(\partial^2, \partial^1, \partial^0)$ and $\mu_3(\partial^3, \partial^2, \partial^1)$ *vanish simultaneously.*

2.2.5 n-tuple Massey product

Definition 2.2.7. *Consider a complex $T^0 \xrightarrow{\partial^0} T^1 \to \ldots \to T^n \xrightarrow{\partial^n} T^{n+1}$. The n-tuple Massey products $\mu_n(\partial^{n-1}, \ldots, \partial^0)$ and $\mu_n(\partial^n, \ldots, \partial^1)$* vanish simultaneously *if there exists a convolution S of $T^1 \to \ldots \to T^{n-1}$, a convolution $\widetilde{S}$ of $T^2 \to \ldots \to T^n$ and maps $p\colon T^0 \to S[2-n]$, $q\colon S \to T^n$, $\widetilde{p}\colon T^1 \to \widetilde{S}[2-n]$ and $\widetilde{q}\colon \widetilde{S} \to T^{n+1}$ defining Massey products such that*

$$q \circ p = 0, \qquad \widetilde{q} \circ \widetilde{p} = 0, \qquad \mathit{Cone}\, q = \mathit{Cone}\, \widetilde{p}.$$

$$\begin{tikzcd}
 & & & S \arrow[dll] \arrow[drrr, "q"] & & & \\
T^0 \arrow[r, "\partial^0"] \arrow[urrr, "p"] & T^1 \arrow[r, "\partial^1"] \arrow[drr, "\widetilde{p}"'] & T^2 \arrow[r] & \dots \arrow[r] & T^{n-1} \arrow[r, "\partial^{n-1}"] \arrow[ul] & T^n \arrow[r, "\partial^n"] \arrow[dll] & T^{n+1} \\
 & & & \widetilde{S} \arrow[ul] \arrow[urrr, "\widetilde{q}"'] & & &
\end{tikzcd}$$

Proposition 2.2.8. *The n-tuple Massey product $\mu_n(\partial^{n-1}, \dots, \partial^0)$ is non-empty if and only if $\mu_{n-1}(\partial^{n-1}, \dots, \partial^1)$ and $\mu_{n-1}(\partial^{n-2}, \dots, \partial^0)$ vanish simultaneously. The complex $T^\bullet = \{T^0 \xrightarrow{\partial^0} T^2 \to \cdots \to T^{n-1} \xrightarrow{\partial^{n-1}} T^n\}$ has a Postnikov system if and only if $0 \in \mu_n(\partial^{n-1}, \dots, \partial^0)$.*

Proof. We have already seen the proof for small n.

Assume by induction that for any complex $\{A^0 \xrightarrow{f^0} A^1 \to \cdots \to A^{n-2} \xrightarrow{f^{n-2}} A^{n-2}\}$ of length $n-1$ there exists a Postnikov system if and only if $0 \in \mu_{n-1}(f^{n-2}, \dots, f^0)$ and that for any complex $\{B^0 \xrightarrow{g^0} B^1 \to \cdots \to B^{n-1} \xrightarrow{g^{n-1}} B^n\}$ of length n the n-fold Massey product $\mu_n(g^{n-1}, \dots, g^0)$ is non-empty if and only if $\mu_{n-1}(g^{n-2}, \dots, g^0)$ and $\mu_{n-1}(g^{n-1}, \dots, g^1)$ vanish simultaneously.

Now, take $T^\bullet = \{T^0 \xrightarrow{\partial^0} T^1 \to \cdots \to T^n \xrightarrow{\partial^n} T^{n+1}\}$ and consider the diagram

$$\begin{tikzcd}
T^0 \arrow[r, "\partial^0"] \arrow[drr, "p_{n-1}", "{[2-n]}"'] & T^1 \arrow[r, "\partial^1"] & T^2 \arrow[r] & \dots \arrow[r] & T^{n-1} \arrow[r, "\partial^{n-1}"] \arrow[dll, "i_{n-1}"'] & T^n \arrow[r, "\partial^n"] \arrow[d, "i_n"] & T^{n+1} \\
 & & S^{n-1} \arrow[ull, "\rho_{n-1}", "{[n-2]}"'] \arrow[urrr, "j_{n-1}"] & & & S^n \arrow[lll, "{[1]}"', "k_n"] &
\end{tikzcd}$$

The morphisms $j_{n-1} : S^{n-1} \to T^n$ such that $j_{n-1} \circ i_{n-1} = \partial^{n-1}$ exists if and only if $\{T^1 \to \cdots \to T^n\}$ has a Postnikov system, if and only if $0 \in \mu_{n-1}(\partial^{n-1}, \dots, \partial^1)$.

The map $p_{n-1} \colon T^0 \to S^{n-1}[2-n]$ such that $\rho_{n-1} \circ p_{n-1} = \partial^0$ exists if and only if $\{T^0 \to \cdots \to T^{n-1}\}$ has a Postnikov system, if and only if $0 \in \mu_{n-1}(\partial^{n-2}, \dots, \partial^0)$.

We have a distinguished triangle

$$S^{n-1} \xrightarrow{j_{n-1}} T^n \xrightarrow{i_n} S^n \xrightarrow{l_n} S^{n-1}[1]$$

which leads to

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{T}}(T^0, S^n[1-n]) \arrow[r] & \mathrm{Hom}_{\mathcal{T}}(T^0, S^{n-1}[2-n]) \arrow[r] & \mathrm{Hom}_{\mathcal{T}}(T^0, S^n[2-n]) \\
 & p_{n-1} \arrow[r, mapsto] & j_{n-1} \circ p_{n-1} \in \mu_n(\partial^{n-1}, \dots, \partial^0).
\end{tikzcd}$$

It follows that there exists $p_n \colon T^0 \to S^n[1-n]$ such that $\rho_{n-1} \circ l_n \circ p_n = \partial^0$ if and only if there exists a Postnikov system for $\{T^0 \xrightarrow{\partial^0} T^1 \to \cdots \to T^{n-1} \xrightarrow{\partial^{n-1}} T^n\}$, if and only if $0 \in \mu_n(\partial^{n-1}, \dots, \partial^0)$.

It finishes the proof that a complex of length $n+1$ has a Postnikov system if and only if the n-tuple Massey product contains zero.

Now consider a diagram

$$
\begin{tikzcd}
 & \widetilde{S}^1 \arrow[d, "{[n-1]}" ', "\beta_1"] & \widetilde{S}^2 \arrow[l, "\alpha_1" '] \arrow[d, "\beta_2" ', "{[n-2]}"] \arrow[rrrr, "q_2"] & & & & \\
T^0 \arrow[r, "\partial^0" '] & T^1 \arrow[r, "\partial^1" '] \arrow[ur, "{[2-n]}", "\gamma_1" '] & T^2 \arrow[r] & \dots \arrow[r] & T^{n-1} \arrow[r, "\partial^{n-1}" '] & T^n \arrow[r, "\partial^n" '] \arrow[ulll, "\alpha_2" '] & T^{n+1}
\end{tikzcd}
$$

The map $\gamma_1 \colon T^1 \to \widetilde{S}^2[2-n]$ such that $\beta_2 \circ \gamma_1 = \partial^1$ exists if and only if $0 \in \mu_{n-1}(\partial^{n-1}, \dots, \partial^1)$ and $q_2 \colon \widetilde{S}^2 \to T^{n+1}$ such that $q_2 \circ \alpha_2 = \partial^n$ exists if and only if $0 \in \mu_{n-1}(\partial^n, \dots, \partial^2)$.

The distinguished triangle

$$T^1 \xrightarrow{\gamma_1} \widetilde{S}^2[2-n] \xrightarrow{\alpha_1} \widetilde{S}^1[2-n] \xrightarrow{\beta_1} T^1[1]$$

gives

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{T}}(\widetilde{S}^1, T^{n+1}) \arrow[r] & \mathrm{Hom}_{\mathcal{T}}(\widetilde{S}^2, T^{n+1}) \arrow[r] & \mathrm{Hom}_{\mathcal{T}}(T^1[2-n], T^{n+1}) \\
 & q_2 \arrow[r, mapsto] & q_2 \circ \gamma_1 \in \mu_n(\partial^n, \dots, \partial^1).
\end{tikzcd}
$$

Hence, $q_1 \colon \widetilde{S}^1 \to T^{n+1}$ such that $q_1 \circ \alpha_1 \circ \alpha_2 = \partial^n$ exists if and only if $0 \in \mu_n(\partial^n, \dots, \partial^1)$.

As in the case of the fourfold Massey product, the choice of maps $j_{n-1} \colon S^{n-1} \to T^n$ and $\gamma_1 \colon T^1 \to \widetilde{S}^2$ determines objects S^n and $\widetilde{S}^1$. Therefore, the $(n+1)$-tuple Massey product $\mu_{n+1}(\partial^n, \dots, \partial^0)$ is not empty if and only if there exist objects S^{n-1} - a convolution of $\{T^1 \to \dots \to T^{n-1}\}$ and $\widetilde{S}^2$ - a convolution of $\{T^2 \to \dots \to T^n\}$ and maps $j_{n-1} \colon S^{n-1} \to T^n$, $p_{n-1} \colon T^0 \to S^{n-1}[2-n]$, $\gamma_1 \colon T^1 \to \widetilde{S}^2[2-n]$ and $q_2 \colon \widetilde{S}^2 \to T^{n+1}$ such that

$$
\begin{array}{lll}
j_{n-1} \circ p_{n-1} = 0, & q_2 \circ \gamma_1 = 0, & q_2 \circ \alpha_2 = \partial^n, \\
\beta_2 \circ \gamma_1 = \partial^1, & j_{n-1} \circ i_{n-1} = \partial^{n-1}, & \rho_{n-1} \circ p_{n-1} = \partial^0
\end{array}
$$

and

$$\mathrm{Cone}(\gamma_1) = \mathrm{Cone}(j_{n-1}),$$

that is when $\mu_n(\partial_n, \dots, \partial_1)$ and $\mu_n(\partial_{n-1}, \dots, \partial_0)$ vanish simultaneously. □

If $H(\mathcal{C})$ is a homotopy category of a DG category $\mathcal{C}$ then Massey products on $H(\mathcal{C})$ can be defined using the DG structure of $\mathcal{C}$. These two definitions coincide for enhanced triangulated categories.

Definition 2.2.9. *Let* $T^0 \xrightarrow{\partial^0} T^1 \to \ldots \to T^{n-1} \xrightarrow{\partial^{n-1}} T^n$ *be a complex in* $H(\mathcal{C})$. *A* defining system *for morphisms* $\partial^{n-1}, \ldots, \partial^0$ *is a set of morphisms* $\alpha_{i,j} \in \mathrm{Hom}^{\bullet}_{\mathcal{C}}(T^i, T^j)$, $0 \le i < j \le n$, $(i,j) \neq (0,n)$ *such that the class of* $\alpha_{i,i+1}$ *in* $H(\mathcal{C})$ *is* ∂^i *and*

$$\partial(\alpha_{i,j}) = \sum_{l=1}^{j-i-1} (-1)^{j+i+l+1+(l+1)(|\partial^{i+l}|+\ldots+|\partial^{j-1}|)} \alpha_{i+l,j} \circ \alpha_{i,i+l}.$$

Then, the n-tuple Massey product $\overline{\mu_n}(\partial^{n-1}, \ldots, \partial^0) =$ *is defined as the set of*

$$\left[\sum_{l=1}^{n-1} (-1)^{l+n+1+(l+1)(|\partial^{i+l}|+\ldots+|\partial^{n-1}|)} \alpha_{l,n} \circ \alpha_{0,l} \right]$$

for all defining systems $(\alpha_{i,j})$ *for* $\partial^{n-1}, \ldots, \partial^0$. *Here, the square brackets denote the class in* $H(\mathcal{C})$.

A set of maps $(\alpha_{i,j})$ *for any pair* $0 \le i < j \le n$ *satisfying the above conditions is called* extended defining system.

Lemma 2.2.10. *Let* $T^0 \xrightarrow{\partial^0} T^1 \to \ldots \to T^n$ *be a complex in* $H^0(\mathcal{C})$ *for a pretriangulated DG category* $\mathcal{C}$. *Then*

$$\mu_n(\partial^{n-1}, \ldots, \partial^0) = -\overline{\mu_n}(\partial^{n-1}, \ldots, \partial^0).$$

Proof. First, we will show that for a complex $T^1 \xrightarrow{\partial^1} T^2 \to \ldots \to T^{n-2} \xrightarrow{\partial^{n-2}} T^{n-1}$ a convolution S in $H^0(\mathcal{C})$ is an object representing a twisted complex $(C_i, q_{i,j})[1-n]$ with $C_i = T^i$ and $q_{i,j} = (-1)^j \alpha_{i,j}$ for $(\alpha_{i,j})$ - an extended defining system for $\partial^{n-2}, \ldots, \partial^1$. Indeed, $(C_i, q_{i,j})$ is a twisted complex as

$$(-1)^j q_{i,j} + \sum_l q_{l,j} q_{i,l} = \sum_l (-1)^{l+j+1} \alpha_{l,j} \alpha_{i,l} + \sum_l (-1)^{l+j} \alpha_{l,j} \alpha_{i,l} = 0.$$

Moreover, a map $f = (f_{0,n})$ of degree $n-2$ from T^0 to $(C_i, q_{i,j})[1-n]$ which is the same as a map of degree 0 from $T^0[-1]$ to $(C_i, q_{i,j})$ is given by $f_{0,l} = (-1)^l \beta_{0,l}$. Here $\beta_{i,j}$ is an extended defining system for $\partial^{n-2}, \ldots, \partial^0$ such that $\beta_{i,j} = \alpha_{i,j}$ if $i \neq 0$. Indeed, it is a closed map as

$$\begin{aligned}(-1)^l \partial(\varphi_{0,l}) + \sum_s q_{s,l} \circ f_{0,s} &= \partial(\beta_{0,l}) + \sum_s (-1)^l \alpha_{s,l} (-1)^s f_{0,s} \\ &= \sum_s (-1)^{s+l+1} \beta_{s,l} \beta_{0,s} + \sum_s (-1)^{s+l} \beta_{s,l} \beta_{0,s} = 0.\end{aligned}$$

The cone of this map, which is the convolution of a complex $T^0 \to \ldots \to T^{n-1}$ is a twisted complex with differentials given by $(-1)^j \beta_{i,j}$, where $(\beta_{i,j})$ is an extended defining system for $\partial^{n-2}, \ldots, \partial^0$. Inductive argument proves that the convolution of a complex

$T^1 \to \ldots \to T^n$ in $H^0(\mathcal{C})$ is given by a twisted complex $(C_i, q_{i,j})$ with $C_i = T^{n+i}$ and $q_{i-n,j-n} = (-1)^{j+n}\alpha_{i,j}$ for an extended defining system $(\alpha_{i,j})$.

Now, let $g = (g_{l,n})$ be a map of degree zero from $(C_i, q_{i,j})[1-n]$ to $T^n[1-n]$. Then $g_{l,n} = (-1)^n\gamma_{l,n}$ where $\gamma_{i,j}$ is an extended defining system for $\partial^{n-1}, \ldots, \partial^1$ such that $\gamma_{i,j} = \alpha_{i,j}$ if $j \neq n$. Indeed, it is a closed map of degree zero, as its differential is

$$(-1)^{n-1}\partial(g_{l,n}) - \sum_s g_{s,n}q_{l,s} = -\partial(\gamma_{l,n}) - \sum_s (-1)^n\gamma_{s,n}(-1)^k\gamma_{l,s} =$$
$$-\sum_s (-1)^{s+n+1}\gamma_{s,n}\gamma_{l,s} - \sum_s (-1)^{s+n}\gamma_{s,n}\gamma_{l,s} = 0.$$

Thus, a choice of a convolution S of a complex $T^1 \to \ldots \to T^{n-1}$ in $H^0(\mathcal{C})$ is the same as the choice of an extended defining system in $\mathcal{C}$, a choice of a map from T^0 to S is the same as a choice of an extended defining system for $T^0 \to \ldots \to T^{n-1}$ and a choice of a map from S to T^n is the same as a choice of an extended defining system for $T^1 \to \ldots \to T^n$. Moreover, these defining systems have to agree and together they form a defining system $(\delta_{i,j})$ for $T^0 \to \ldots \to T^n$.

The n-tuple Massey product $\mu_n(\partial^{n-1}, \ldots, \partial^0)$ is a composition of g and f, i.e. a class of a map

$$\mu_n(\partial^{n-1}, \ldots, \partial^0) = [\sum_{s=1}^{n-1} g_{s,n}f_{0,s}] = [\sum_{s=1}^{n-1} (-1)^n\gamma_{s,n}(-1)^s\beta_{0,s}]$$
$$= -[\sum_{s=1}^{n-1} (-1)^{s+n+1}\delta_{s,n} \circ \delta_{0,s}] = -\overline{\mu_n}(\partial^{n-1}, \ldots, \partial^0).$$

$\square$

Remark 2.2.11. From the proof of Lemma 2.2.10 it follows that if $\mathcal{C}$ is a pretriangulated category then a complex in $H^0(\mathcal{C})$ can be lifted to a twisted complex over $\mathcal{C}$ if and only if it has a convolution.

2.2.6 Massey products and A_∞-structure.

Lemma 2.2.10 allows us to prove the following Theorem, see also [46, Theorem 3.1].

Theorem 2.2.12. *Let $\mathcal{C}$ be a pretriangulated DG category and let*

$$T^0 \xrightarrow{\partial^0} T^1 \to \ldots \to T^{n-1} \xrightarrow{\partial^{n-1}} T^n$$

be a complex in $H^0(\mathcal{C})$. Assume that $\mu_n(\partial^{n-1}, \ldots, \partial^0) \neq \emptyset$ and choose $f \in \mu_n(\partial^{n-1}, \ldots, \partial^0)$. Then, there exists a minimal A_∞-structure on $H(\mathcal{C})$ such that $m_n(\partial^{n-1}, \ldots, \partial^0) = -f$ and $m_l(\partial^{i+l-1}, \ldots, \partial^i) = 0$ for $i \in \{0, \ldots, n-1\}$ and l such that $i + l \leq n - 1$.

Proof. Recall from 2.2.1 that Merkulov's construction of a minimal model of a DG category $\mathcal{C}$ is based on a choice of maps $i\colon H(\mathcal{C}) \to \mathcal{C}$, $\pi\colon \mathcal{C} \to H(\mathcal{C})$ and $h\colon \mathcal{C} \to \mathcal{C}$ which define operations λ_n.

Note also that the defining system $(\alpha_{i,j})$ for a complex $T^0 \xrightarrow{\partial^0} T^1 \to \ldots \to T^{n-1} \xrightarrow{\partial^{n-1}} T^n$ is such that $\alpha_{i,i+1}$ is some lift of ∂^i to $\mathcal{C}$ and

$$\partial(\alpha_{i,j}) = \lambda_{j-i}(\alpha_{j-1,j}, \ldots, \alpha_{i,i+1}).$$

By Lemma 2.2.10 we know that $\mu_n(\partial^{n-1}, \ldots, \partial^0) = -\overline{\mu_n}(\partial^{n-1}, \ldots, \partial^0)$. As $\partial(\alpha_{i,j}) = \lambda_{j-i}(\alpha_{j-1,j}, \ldots, \alpha_{i,i+1})$ for any pair i, j such that $(i,j) \neq (0,n)$ the element $\lambda_{j-i}(\alpha_{j-1,j}, \ldots, \alpha_{i,i+1})$ has a trivial cohomology class and hence by Merkulov's construction $m_{j-i}(\partial^{j-1}, \ldots, \partial^i) = 0$.

Also, for a suitable choice of h

$$\begin{aligned}\lambda_n(\alpha_{n-1,n}, \ldots, \alpha_{0,1}) &= \sum(-1)^{n-l+1} h\lambda_{n-l}(\partial^{n-1}, \ldots, \partial^l) \circ h\lambda_l(\partial^{l-1}, \ldots, \partial^0) \\ &= \sum(-1)^{n-l+1}\alpha_{l,n} \circ \alpha_{0,n}\end{aligned}$$

which shows that one can choose $(\alpha_{i,j})$ in such a way that

$$m_n(\partial^{n-1}, \ldots, \partial^0) = \pi(\lambda_n(\alpha_{n-1,n}, \ldots, \alpha_{0,1})) = -f.$$

□

2.2.7 Examples

Recall that in Section 2.1.1 we have been considering a smooth surface X obtained from $\mathbb{P}^2$ by blowing up a point x_0 and then a point x_1 on an exceptional divisor. In the previous notation, we know that the collection $\langle \mathcal{O}_{E_2}(E_2)[-1], \mathcal{O}_{E_1+E_2}(E_1 + E_2)[-1], \mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H)\rangle$ is full. If we mutate $\mathcal{O}_{E_1+E_2}(E_1 + E_2)[-1]$ and $\mathcal{O}_{E_2}(E_2)$ to the right over $\mathcal{O}_X$ we obtain a full exceptional collection of line bundles on X, namely

$$\langle \mathcal{O}_X, \mathcal{O}_X(E_2), \mathcal{O}_X(E_1 + E_2), \mathcal{O}_X(H), \mathcal{O}_X(2H)\rangle.$$

We show in Section 2.2.7 that Massey products are not enough to calculate the A_∞-structure on the Ext-quiver of this collection. However, they provide sufficient information to calculate the A_∞-structure on the Ext-quiver of $\langle \mathcal{O}_X, \mathcal{O}_X(E_2), \mathcal{O}_X(E_1 + E_2), \mathcal{O}_X(H)\rangle$. We do all the necessary calculations in Section 2.2.7 and present the sought for A_∞-structure.

Example of A_∞-structure that can be calculated via Massey products

The Ext-quiver of the collection

$$\langle \mathcal{O}_X, \mathcal{O}_X(E_2), \mathcal{O}_X(E_1 + E_2), \mathcal{O}_X(H), \mathcal{O}_X(2H)\rangle.$$

is

$$
\begin{tikzcd}
\mathcal{O}_X \arrow[r, "\gamma"] \arrow[rrr, bend right, "\eta"'] & \mathcal{O}_X(E_2) \arrow[r, "\delta"] \arrow[r, bend left, "\theta"] & \mathcal{O}_X(E_1+E_2) \arrow[r, shift left, "\alpha_1"] \arrow[r, shift right, "\alpha_2"'] & \mathcal{O}_X(H) \arrow[r, shift left=2, "\beta_1"] \arrow[r, "\beta_2"] \arrow[r, shift right=2, "\beta_3"'] & \mathcal{O}_X(2H)
\end{tikzcd}
$$

with θ in degree 1 and relations

$$
\begin{aligned}
&\beta_1\alpha_2 = \beta_2\alpha_1, && \beta_1\eta = \beta_3\alpha_1\delta\gamma,\\
&\beta_2\eta = \beta_3\alpha_2\delta\gamma, && \theta\gamma = 0,\\
&\alpha_1\theta = 0, && \alpha_2\theta = 0.
\end{aligned}
$$

Note that the pushforward of $\mathcal{O}_X(-E_1 - E_2)$ to $\mathbb{P}^2$ is an ideal of a 0-dimensional scheme Z of length 2. This scheme determines the tangent direction at the point x_0 that we blow up. In particular, there exists a unique line containing this subscheme Z. Let D_2 denote the strict transform of a line on $\mathbb{P}^2$. Then $D_2 \cap E_2 \neq \emptyset$. The morphism α_2 in the above quiver is zero along $D_2 + E_2$. γ has zeros along E_2 and δ has zeroes along E_1. These three morphisms are determined uniquely up to a constant. Let D_1 be a line in $\mathbb{P}^2$ passing through x_0 and not containing Z. Then, $D_1 \cap E_1 \neq \emptyset$. The map α_1 is zero along D_1. The divisor D_1 is not determined uniquely and one can change α_1 in the above quiver by adding some multiplicity of α_2.

Note that morphisms form $\mathcal{O}_X$ to $\mathcal{O}_X(H)$ are pull backs of sections of $\mathcal{O}_{\mathbb{P}^2}(H)$. As the preimage of x_0 in X is $E_1 \cup E_2$, a section of $\mathcal{O}_X(H)$ on X is either zero on both E_1 and E_2 (possibly with some multiplicities) or it is non-zero on every point of E_1 and E_2. Sections having zeros along E_1 and E_2 are linear combinations of $\alpha_1\delta\gamma$ and $\alpha_2\delta\gamma$.

The morphisms η in the quiver is any section of $\mathcal{O}_X(H)$ which is non-zero on E_1 and E_2. Hence, it can be changed by adding any linear combination of $\alpha_1\delta\gamma$ and $\alpha_2\delta\gamma$.

To obtain the above relations we note that $\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X(H)) \simeq \mathrm{Hom}(\mathcal{O}_X(H), \mathcal{O}_X(2H))$. Then we put β_1 as the morphism corresponding to $\alpha_1\delta\gamma$, β_2 as the morphism corresponding to $\alpha_2\delta\gamma$ and β_3 as the morphism corresponding to η.

To sum up, γ, δ and α_2 are determined uniquely. Other morphisms can be changed by

$$
\begin{aligned}
\alpha_1 &\rightsquigarrow \alpha_1 + a\,\alpha_2\\
\eta &\rightsquigarrow \eta + b\,\alpha_1\delta\gamma + c\,\alpha_2\delta\gamma
\end{aligned}
$$

for $a, b, c \in k$. These morphism determine β_i's in such a way that the above relations are satisfied.

Due to degree reasons only m_3 operations on the Ext-quiver of this collection can be non-trivial. The A_∞-structure is thus determined by the value of

$$
\begin{array}{lll}
m_3(\alpha_1,\theta,\gamma), & m_3(\alpha_2,\theta,\gamma), & \\
m_3(\beta_1,\alpha_1,\theta), & m_3(\beta_2,\alpha_1,\theta), & m_3(\beta_3,\alpha_1,\theta), \\
m_3(\beta_1,\alpha_2,\theta), & m_3(\beta_2,\alpha_2,\theta), & m_3(\beta_3,\alpha_2,\theta), \\
m_3(\beta_1\alpha_1,\theta,\gamma), & m_3(\beta_2\alpha_1,\theta,\gamma), & m_3(\beta_3\alpha_1,\theta,\gamma), \\
m_3(\beta_2\alpha_2,\theta,\gamma), & m_3(\beta_3\alpha_2,\theta,\gamma). &
\end{array}
$$

Let us denote by $(\mathcal{G}, m_n)$ the A_∞-category given by this quiver. The A_∞-structure is unique only up to an A_∞-quasi-isomorphism. Thus, we can choose a quasi-isomorphism $F\colon (\mathcal{G}, m_n) \to (\mathcal{G}, \widetilde{m_n})$. Such a functor F is a quasi-isomorphism if $F_1 = \mathrm{id}$. Putting

$$
\begin{aligned}
F_2(\beta_1\alpha_1,\theta) &= -m_3(\beta_1,\alpha_1,\theta),\\
F_2(\beta_1\alpha_2,\theta) &= -m_3(\beta_1,\alpha_2,\theta),\\
F_2(\beta_2\alpha_2,\theta) &= -m_3(\beta_2,\alpha_2,\theta),\\
F_2(\beta_3\alpha_1,\theta) &= -m_3(\beta_3,\alpha_1,\theta),\\
F_2(\beta_3\alpha_2,\theta) &= -m_3(\beta_3,\alpha_2,\theta),
\end{aligned}
$$

and all other values of F_n to be equal to zero we obtain a new A_∞-structure on $\mathcal{G}$ with

$$
\begin{aligned}
&\widetilde{m_2} = \widetilde{m_2}(F_1 \otimes F_1) = F_1 m_2 = m_2,\\
&\widetilde{m_3}(F_1\otimes F_1\otimes F_1) - \widetilde{m_2}(F_2\otimes F_1) + \widetilde{m_2}(F_1\otimes F_2) = F_1 m_3 + F_2(m_2\otimes \mathrm{id}) - F_2(\mathrm{id}\otimes m_2).
\end{aligned}
$$

In particular

$$
\begin{array}{lll}
\widetilde{m_2} = m_2, & & \\
\widetilde{m_3}(\beta_1,\alpha_1,\theta) = 0, & \widetilde{m_3}(\beta_1,\alpha_2,\theta) = 0, & \\
\widetilde{m_3}(\beta_2,\alpha_2,\theta) = 0, & \widetilde{m_3}(\beta_3,\alpha_1,\theta) = 0, & \widetilde{m_3}(\beta_3,\alpha_2,\theta) = 0
\end{array}
$$

and

$$\widetilde{m_3}(\beta_2,\alpha_1,\theta) = m_3(\beta_2,\alpha_1,\theta) - m_3(\beta_1,\alpha_2,\theta).$$

Moreover, the relation

$$m_3(m_2\otimes\mathrm{id}\otimes\mathrm{id}) - m_3(\mathrm{id}\otimes m_2\otimes\mathrm{id}) + m_3(\mathrm{id}\otimes\mathrm{id}\otimes m_2) - m_2(m_3\otimes\mathrm{id}) - m_2(\mathrm{id}\otimes m_3) = 0$$

gives that always

$$
\begin{aligned}
m_3(\beta_1\alpha_1,\theta,\gamma) &= m_3(\beta_1,\alpha_1,\theta)\gamma - \beta_1 m_3(\alpha_1,\theta,\gamma),\\
m_3(\beta_1\alpha_2,\theta,\gamma) &= m_3(\beta_1,\alpha_2,\theta)\gamma - \beta_1 m_3(\alpha_2,\theta,\gamma),\\
m_3(\beta_2\alpha_2,\theta,\gamma) &= m_3(\beta_2,\alpha_2,\theta)\gamma - \beta_2 m_3(\alpha_2,\theta,\gamma),\\
m_3(\beta_3\alpha_1,\theta,\gamma) &= m_3(\beta_3,\alpha_1,\theta)\gamma - \beta_3 m_3(\alpha_1,\theta,\gamma),\\
m_3(\beta_3\alpha_2,\theta,\gamma) &= m_3(\beta_3,\alpha_2,\theta)\gamma - \beta_3 m_3(\alpha_2,\theta,\gamma).
\end{aligned}
$$

Unfortunately, Theorem 2.2.12 does not allow to calculate the A_∞-structure of the collection $\langle \mathcal{O}_X, \mathcal{O}_X(E_2), \mathcal{O}_X(E_1+E_2), \mathcal{O}_X(H), \mathcal{O}_X(2H)\rangle$ as it does not provide a way to calculate $m_3(\beta_1, \alpha_2, \theta) - m_3(\beta_2, \alpha_1, \theta)$. However, Massey products allow us to determine the A_∞-structure on the (not full) exceptional collection $\langle \mathcal{O}_X, \mathcal{O}_X(E_2), \mathcal{O}_X(E_1+E_2), \mathcal{O}_X(H)\rangle$.

Calculations of triple Massey products

In order to calculate the A_∞-structure of this collection we have to determine $m_3(\alpha_1, \theta, \gamma)$ and $m_3(\alpha_2, \theta, \gamma)$. These values can be determined by Massey products $\mu_3(\alpha_1, \theta, \gamma)$ and $\mu_3(\alpha_2, \theta, \gamma)$.

The cone of θ is a vector bundle V which is a non-trivial extension of $\mathcal{O}_X(E_2)$ by $\mathcal{O}_X(E_1+E_2)$.

$$0 \longrightarrow \mathcal{O}_X(E_1+E_2) \xrightarrow{\phi_1} V \xrightarrow{\phi_2} \mathcal{O}_X(E_2) \longrightarrow 0. \tag{*}$$

Because $\theta \circ \gamma = 0$ there exists $\zeta\colon \mathcal{O}_X \to V$ such that $\phi_2\zeta = \gamma$. Analogously, from the fact that compositions $\alpha_1 \circ \theta$ and $\alpha_2 \circ \theta$ are zero follows the existence of maps $\iota_1, \iota_2\colon V \to \mathcal{O}_X(H)$ such that $\iota_i\phi_1 = \alpha_i$. Then,

$$\iota_1 \circ \zeta \in \mu_3(\alpha_1, \theta, \gamma), \qquad \iota_2 \circ \zeta \in \mu_3(\alpha_2, \theta, \gamma).$$

Massey products $\mu_3(\alpha_1, \theta, \gamma)$ and $\mu_3(\alpha_1, \theta, \gamma)$ are not defined uniquely. On the other hand, from Merkulov's construction it follows that the A_∞-structure depends on the choice of the lifting homotopy on $\overline{\alpha_1} \circ \overline{\theta}$, $\overline{\alpha_2} \circ \overline{\theta}$ and $\overline{\theta} \circ \overline{\gamma}$ where by $\overline{f}$ we denote some lift of a morphism f in $H^0(\mathcal{C})$ to a morphism in $\mathcal{C}$. Then one can treat maps ι_1, ι_2 and ζ as the choice of values of the lifting homotopy on these three compositions. Thus for any ι_1, ι_2 and ζ putting

$$m_3(\alpha_1, \theta, \gamma) = \iota_1 \circ \zeta, \qquad m_3(\alpha_2, \theta, \gamma) = \iota_2 \circ \zeta$$

leads to a correct A_∞-structure.

The maps ζ, ι_1 and ι_2 are not determined uniquely, they can change by

$$\begin{aligned} &\zeta \rightsquigarrow \zeta + d\,\phi_1\delta\gamma, \\ &\iota_1 \rightsquigarrow \iota_1 + e\,\alpha_1\delta\phi_2 + f\,\alpha_2\delta\phi_2, \\ &\iota_2 \rightsquigarrow \iota_2 + g\,\alpha_1\delta\phi_2 + h\,\alpha_2\delta\phi_2 \end{aligned}$$

for $d, \ldots, h \in k$.

Short exact sequence (*) gives $c_1(V) = E_1 + 2E_2$ and $c_2(V) = 0$. Let us consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\zeta} V \xrightarrow{\psi} \mathcal{F} \longrightarrow 0.$$

Let $T \hookrightarrow \mathcal{F}$ be the torsion part of $\mathcal{F}$. Then we have the commutative diagram

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & 0 \arrow[r] & \mathcal{F}/T \arrow[u] \arrow[r, "="] & \mathcal{F}/T \arrow[u] & \\
0 \arrow[r] & \mathcal{O}_X \arrow[u] \arrow[r, "\zeta"] & V \arrow[u] \arrow[r, "\psi"] & \mathcal{F} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[u, "="] \arrow[r] & \mathcal{G} \arrow[u, "\chi"] \arrow[r] & T \arrow[u] \arrow[r] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

in which $\mathcal{G}$ is a sheaf of rank 1. $\mathcal{G}$ fits into a short exact sequence

$$0 \to \mathcal{G} \to V \to \mathcal{F}/T \to 0$$

with V locally free and $\mathcal{F}/T$ torsion-free. Hence, $\mathcal{G}$ is torsion-free and it injects into its double dual $\mathcal{G}^{**}$ with cokernel T',

$$0 \to \mathcal{G} \to \mathcal{G}^{**} \to T' \to 0.$$

T' is a torsion sheaf and a subsheaf of $\mathcal{F}/T$, hence $T' = 0$ and $\mathcal{G}$ is reflexive. Hartshorne in [24, Corollary 1.4] proves that every reflexive sheaf on a smooth surface is locally free and therefore we obtain that $\mathcal{G} = \mathcal{O}_X(D)$ for some effective divisor D. The composition of morphisms

$$\mathcal{O}_X \longrightarrow \mathcal{O}_X(D) \xrightarrow{\chi} V \xrightarrow{\psi_2} \mathcal{O}_X(E_2)$$

is equal to δ, so the morphism $\mathcal{O}_X(D) \to \mathcal{O}_X(E_2)$ is non-zero. It follows that D is equal either to 0 or to E_2. If $D = E_2$ then we would have a splitting

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(E_1+E_2) \arrow[r, "\phi_1"] & V \arrow[r, "\phi_2"] & \mathcal{O}_X(E_2) \arrow[l, bend right, "\chi"'] \arrow[r] & 0.
\end{tikzcd}$$

As V is a non-trivial extension of $\mathcal{O}_X(E_2)$ by $\mathcal{O}_X(E_1+E_2)$ we get a contradiction. Hence, $D = 0$ and $\mathcal{F}$ is a torsion-free sheaf of rank 1. As $c_2(\mathcal{F}) = 0$, $\mathcal{F}$ is a line bundle. $c_1(\mathcal{F}) = E_1 + 2E_2$ shows that $\mathcal{F} = \mathcal{O}_X(E_1 + 2E_2)$. Hence, ζ fits into the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\zeta} V \xrightarrow{\psi} \mathcal{O}_X(E_1 + 2E_2) \longrightarrow 0.$$

Let $\mathcal{L}_i = \ker(\iota_i)$, $\mathcal{K}_i = \mathrm{coker}(\iota_i)$ and $\mathcal{M}_i = \mathrm{Im}(\iota_i)$. There are three short exact sequences:

$$0 \longrightarrow \mathcal{L}_i \longrightarrow V \longrightarrow \mathcal{M}_i \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{M}_i \longrightarrow \mathcal{O}_X(H) \longrightarrow \mathcal{K}_i \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_X(E_1+E_2) \longrightarrow \mathcal{M}_i \longrightarrow \mathcal{N}_i \longrightarrow 0.$$

The diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathcal{O}_X(E_1+E_2) \arrow[u] \arrow[r] & \mathcal{M}_i \arrow[u] \arrow[r] & \mathcal{N}_i \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_X(E_1+E_2) \arrow[u, "="'] \arrow[r] & V \arrow[u] \arrow[r] & \mathcal{O}_X(E_2) \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] \arrow[r] & \mathcal{L}_i \arrow[u] \arrow[r, "="] & \mathcal{L}_i \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

gives relations between the Chern classes:

$$\begin{aligned}
&c_1(V) = E_1 + 2E_2 = c_1(\mathcal{L}_i) + c_1(\mathcal{M}_i),\\
&c_2(V) = 0 = c_1(\mathcal{L}_i)c_1(\mathcal{M}_i) + c_2(\mathcal{M}_i),\\
&H = c_1(\mathcal{M}_i) + c_1(\mathcal{K}_i),\\
&c_2(\mathcal{O}_X(H)) = 0 = c_1(\mathcal{M}_i)c_1(\mathcal{K}_i) + c_2(\mathcal{M}_i) + c_2(\mathcal{K}_i),\\
&c_1(\mathcal{M}_i) = E_1 + E_2 + c_1(\mathcal{N}_i),\\
&c_2(\mathcal{M}_i) = (E_1+E_2)c_1(\mathcal{N}_i) + c_2(\mathcal{N}_i),\\
&E_2 = c_1(\mathcal{L}_i) + c_1(\mathcal{N}_i),\\
&0 = c_1(\mathcal{L}_i)c_1(\mathcal{N}_i) + c_2(\mathcal{N}_i).
\end{aligned}$$

Diagrams

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & 0 \arrow[r] & \mathcal{K}_1 \arrow[u] \arrow[r, "="] & \mathcal{K}_1 \arrow[u] & \\
0 \arrow[r] & \mathcal{O}_X(E_1+E_2) \arrow[u] \arrow[r] & \mathcal{O}_X(H) \arrow[u] \arrow[r] & \mathcal{O}_{D_1}(H) \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_X(E_1+E_2) \arrow[u, "="'] \arrow[r] & \mathcal{M}_1 \arrow[u] \arrow[r] & \mathcal{N}_1 \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & 0 \arrow[r] & \mathcal{K}_2 \arrow[u] \arrow[r, "="] & \mathcal{K}_2 \arrow[u] & \\
0 \arrow[r] & \mathcal{O}_X(E_1+E_2) \arrow[u] \arrow[r] & \mathcal{O}_X(H) \arrow[u] \arrow[r] & \mathcal{O}_{D_2+E_2}(H) \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_X(E_1+E_2) \arrow[u, "="'] \arrow[r] & \mathcal{M}_2 \arrow[u] \arrow[r] & \mathcal{N}_2 \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

show that the support of N_1 is D_1, and the support of N_2 is contained in $D_2 \cup E_2$.

$\mathcal{K}_1$ can be supported on a finite number of points or on the whole D_1. In the second case we would get that $\mathcal{M}_1$ - the image of ι_1 is equal to $\mathcal{O}_X(E_1+E_2)$ and hence the sequence

$$0 \longrightarrow \mathcal{O}_X(E_1+E_2) \xrightarrow{\psi_1} V \xrightarrow{\psi_2} \mathcal{O}_X(E_2) \longrightarrow 0$$

splits. But V is a non-trivial extension of $\mathcal{O}_X(E_2)$ by $\mathcal{O}_X(E_1+E_2)$ so we know that $\mathcal{K}_1$ must be supported on a finite number of points. Hence, $c_1(\mathcal{K}_1) = 0$ and $c_1(\mathcal{N}_1) = D_1$.

An analogous argument shows that the support of $\mathcal{K}_2$ can not be equal to $D_2 + E_2$. Hence, $c_1(\mathcal{K}_2)$ is either E_2, D_2 or 0 and $c_1(N_2) = D_2$, E_2 or $D_2 + E_2$.

The above diagrams give additional relations

$$H - E_1 - E_2 = c_1(\mathcal{N}_i) + c_1(\mathcal{K}_i),$$
$$0 = c_1(\mathcal{N}_i)c_1(\mathcal{K}_i) + c_2(\mathcal{N}_i) + c_2(\mathcal{K}_i).$$

For ι_1 we have:

$$\begin{aligned}
c_1(\mathcal{N}_1) &= D_1,\\
c_1(\mathcal{K}_1) &= c_1(\mathcal{O}_{D_1}(H)) - c_1(\mathcal{N}_1) = H - E_1 - E_2 - D_1,\\
c_1(\mathcal{M}_1) &= H - c_1(\mathcal{K}_1) = E_1 + E_2 + D_1,\\
c_1(\mathcal{L}_1) &= E_2 - c_1(\mathcal{N}_1) = E_2 - D_1,\\
c_2(\mathcal{M}_1) &= -c_1(\mathcal{L}_1)c_1(\mathcal{M}_1) = 1,\\
c_2(\mathcal{K}_i) &= -c_1(\mathcal{M}_1)c_1(\mathcal{K}_1) - c_2(\mathcal{M}_1) = 0,\\
c_2(\mathcal{N}_1) &= c_2(\mathcal{M}_1) - (E_1 + E_2)c_1(\mathcal{N}_1) = 0.
\end{aligned}$$

- $\mathcal{M}_1 = m_x \otimes \mathcal{O}_X(E_1+E_2+D_1)$, where m_x is the maximal ideal of functions vanishing at a point $x \in D_1$,
- $\mathcal{L}_1 = \mathcal{O}_X(E_2 - D_1)$,
- $\mathcal{N}_1 = \mathcal{O}_{D_1}$,
- $\mathcal{K}_1 = \mathcal{O}_x(H) \simeq \mathcal{O}_x$, (because of the exact sequences $\mathcal{N}_1 \to \mathcal{O}_{D_1}(H) \to \mathcal{K}_1$ and $m_x \otimes \mathcal{O}_X(E_1 + E_2 + D_1) \to \mathcal{O}_X(H) \to \mathcal{O}_x(H)$).

We want to know whether the composition

$$\mathcal{O}_X \xrightarrow{\zeta} V \xrightarrow{\iota_1} \mathcal{O}_X(H)$$

is equal to zero or η.

ζ and ι_1 fit into a diagram

$$\begin{tikzcd}
 & & 0 & & \\
 & \mathcal{O}_X \arrow[r] & m_x \otimes \mathcal{O}_X(E_1 + E_2 + D_1) \arrow[r] \arrow[u] & \mathcal{O}_{\widetilde{D}} & \\
0 \arrow[r] & \mathcal{O}_X \arrow[r, "\zeta"] \arrow[u, "="'] & V \arrow[r] \arrow[u, "\iota_1"] & \mathcal{O}_X(E_1 + 2E_2) \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[r] \arrow[u] & \mathcal{O}_X(E_2 - D_1) \arrow[r, "="] \arrow[u] & \mathcal{O}_X(E_2 - D_1) \arrow[u] & \\
 & & 0 \arrow[u] & &
\end{tikzcd}$$

We tensor this diagram with $\mathcal{O}_{E_2}$.

$\mathcal{O}_X(E_1 + E_2 + D_1) \otimes \mathcal{O}_{E_2} = \mathcal{O}_{E_2}$, $m_x \otimes \mathcal{O}_{E_2} = \mathcal{O}_{E_2}$ as $x \notin E_2$ so there exists an epimorphism $V|_{E_2} \to \mathcal{O}_{E_2}$ with kernel $\mathcal{O}_{E_2}(-1)$.

$$\begin{tikzcd}
 & & 0 & & \\
 & & \mathcal{O}_{E_2} \arrow[u] & & \\
0 \arrow[r] & \mathcal{O}_{E_2} \arrow[r, "\zeta|_{E_2}"'] & V|_{E_2} \arrow[r, "\psi|_{E_2}"'] \arrow[u, "\iota_1|_{E_2}"'] & \mathcal{O}_{E_2}(-1) \arrow[r] & 0 \\
 & & \mathcal{O}_{E_2}(-1) \arrow[u] & & \\
 & & 0 \arrow[u] & &
\end{tikzcd}$$

It follows that $V|_{E_2} = \mathcal{O}_{E_2} \oplus \mathcal{O}_{E_2}(-1)$ and the composition

$$\mathcal{O}_X \xrightarrow{\zeta} V \xrightarrow{\iota_1} \mathcal{M}_1 \hookrightarrow \mathcal{O}_X(H)$$

restricted to E_2 is an isomorphism. Hence $\iota_1\zeta$ does not have zeros along E_2. Changing ι_1 if necessary we obtain that

$$\iota_1\zeta = \eta.$$

For ι_2 there are three possibilities

$$c_1(\mathcal{N}_2) = \begin{cases} D_2, & \text{case } (A) \\ E_2, & \text{case } (B) \\ E_2 + D_2. & \text{case } (C) \end{cases}$$

Hence,

$$c_1(\mathcal{M}_2) = \begin{cases} E_1 + E_2 + D_2, & \text{case } (A) \\ E_1 + 2E_2, & \text{case } (B) \\ E_1 + 2E_2 + D_2. & \text{case } (C) \end{cases}$$

$\mathcal{M}_2$ is a subsheaf of $\mathcal{O}_X(H)$ and hence it is of the form $\mathcal{O}_X(L) \otimes \mathcal{I}_Z$, where $\mathcal{I}_Z$ is an ideal sheaf of a set of points $Z \in D_2 \cup E_2$. Then $c_1(\mathcal{M}_2) = L$ and $c_2(\mathcal{M}_2) = \deg(Z) \geq 0$.

In case (A) we have:

$$\mathcal{L}_2 = \mathcal{O}_X(E_2 - D_2), \quad \mathcal{N}_2 = \mathcal{O}_{D_2}, \quad c_1(\mathcal{M}_2) = E_1 + E_2 + D_2, \quad c_2(\mathcal{M}_2) = -1.$$

The second Chern class of $\mathcal{M}_2$ is negative and it follows that this case cannot happen.

In case (B) we have:

$$\mathcal{L}_2 = \mathcal{O}_X, \quad \mathcal{N}_2 = \mathcal{O}_{E_2}, \quad c_1(\mathcal{M}_2) = E_1 + 2E_2, \quad c_2(\mathcal{M}_2) = 0.$$

On X we have an inclusion $\mathcal{O}_X(E_1+E_2) \to \mathcal{M}_2 \to \mathcal{O}_X(H)$. Tensoring with $\mathcal{O}_{D_2}$ we get $\mathcal{O}_{D_2}(E_1+E_2) = \mathcal{O}_{D_2}(1)$ and $\mathcal{O}_{D_2}(H) = \mathcal{O}_{D_2}(1)$. It follows that $\mathcal{M}_2 \otimes \mathcal{O}_{D_2}$ modulo torsion is equal to $\mathcal{O}_{D_2}(1)$. Whereas, in this case it equals to $\mathcal{O}_{D_2}(2)$.

Thus, we are left we case (C):

$$\mathcal{L}_2 = \mathcal{O}_X(-D_2), \quad \mathcal{N}_2 = \mathcal{O}_{D_2+E_2}, \quad c_1(\mathcal{M}_2) = E_1 + 2E_2 + D_2, \quad c_2(\mathcal{M}_2) = 1.$$

We obtain that $\mathcal{M}_2 = m_y \otimes \mathcal{O}_X(E_1 + 2E_2 + D_2)$ for some $y \in E_2 \cup D_2$. If $y \in D_2$ then $\mathcal{M}_2 \otimes \mathcal{O}_{D_2} = \mathcal{O}_{D_2}$ and there is no epimorphism from $V|_{D_2} = \mathcal{O}_{D_2}(1) \oplus \mathcal{O}_{D_2}(1)$ onto $\mathcal{O}_{D_2}$. So $y \in E_2 \setminus D_2$.

Thus, we know that $\iota_2\zeta$ is zero on a point $y \in E_2$. Hence it has zeros along E_2 and one can choose ι_2 in such a way that

$$\iota_2\zeta = 0.$$

Thus the A_∞-structure sought for is

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "\gamma"] \arrow[rrr, bend right, "\eta"'] & \mathcal{O}_X(E_2) \arrow[r, "\delta"] \arrow[r, bend left, "\theta"] & \mathcal{O}_X(E_1+E_2) \arrow[r, shift left, "\alpha_1"] \arrow[r, shift right, "\alpha_2"'] & \mathcal{O}_X(H) \end{tikzcd}$$

with

$$\begin{aligned} &m_2(\theta,\gamma) = 0, && m_2(\alpha_1,\theta) = 0, && m_2(\alpha_2,\theta) = 0, \\ &m_3(\alpha_1,\theta,\gamma) = \eta, && m_3(\alpha_2,\theta,\gamma) = 0. \end{aligned}$$

Remark 2.2.13. In [57] Polishchuk considered A_∞-structure on the category of line bundles on an elliptic curve E. In [57, Theorem 2.2] he showed that the A_∞-structure is uniquely determined by operations

$$m_3\colon \operatorname{Hom}(\mathcal{O}_E, L_1) \otimes \operatorname{Ext}^1(L_1, L_1 \otimes M) \otimes \operatorname{Hom}(L_1 \otimes M, L_1 \otimes M \otimes L_2) \to \operatorname{Hom}(\mathcal{O}_E, L_1 \otimes M \otimes L_2)$$

with $\deg L_1 = \deg L_2 = 1$ and $\deg M = -1$. Moreover, these operations can be calculated by means of triple Massey products on E.

2.3 Calculating DG quivers via universal extensions

Another way of calculating DG quivers of exceptional collections was presented in [7]. It is based on a construction of universal extensions and coextensions defined by Hille and Perling in [25].

2.3.1 Universal extensions and coextensions

Let E, F be objects of a k-linear abelian category. Note that

$$\mathrm{End}(\mathrm{Ext}^1(E,F)) = \mathrm{Ext}^1(E, F \otimes \mathrm{Ext}^1(E,F)^*).$$

As $\mathrm{id} \in \mathrm{End}(\mathrm{Ext}^1(E,F))$ there exists a distinguished element $\widetilde{\mathrm{id}}$ of $\mathrm{Ext}^1(E, F \otimes \mathrm{Ext}^1(E,F)^*)$.

Following [25] we say that an object $\overline{E}$ is the *universal extension of E by F* if $\overline{E}$ is the extension of E by $F \otimes \mathrm{Ext}^1(E,F)^*$ corresponding to $\widetilde{\mathrm{id}} \in \mathrm{Ext}^1(E, F \otimes \mathrm{Ext}^1(E,F)^*)$. $\overline{E}$ fits into the short exact sequence:

$$0 \longrightarrow F \otimes \mathrm{Ext}^1(E,F)^* \longrightarrow \overline{E} \longrightarrow E \longrightarrow 0.$$

This short exact sequence gives the long exact sequence

$$0 \longrightarrow \mathrm{Hom}(E,F) \longrightarrow \mathrm{Hom}(\overline{E},F) \longrightarrow \mathrm{Hom}(F,F) \otimes \mathrm{Ext}^1(E,F) \longrightarrow \mathrm{Ext}^1(E,F) \longrightarrow \dots.$$

Assume that $\mathrm{Ext}^n(F,F) = 0$ for $n \geq 0$. Then the above sequence shows that $\mathrm{Ext}^1(\overline{E},F) = 0$ and the groups $\mathrm{Ext}^n(\overline{E},F)$ and $\mathrm{Ext}^n(E,F)$ are isomorphic for $n > 1$.

If we assume that F is simple, i.e. $\mathrm{Hom}(F,F) = k$ and that $\mathrm{Hom}(F,E) = 0$ then the long exact sequence

$$0 \longrightarrow \mathrm{Hom}(F,F) \otimes \mathrm{Ext}^1(E,F)^* \longrightarrow \mathrm{Hom}(F,\overline{E}) \longrightarrow \mathrm{Hom}(F,E) \longrightarrow \dots$$

shows that $\mathrm{Hom}(F,\overline{E}) = \mathrm{Ext}^1(E,F)^*$.

Thus, if F is exceptional and $\mathrm{Ext}^n(F,E) = 0$ for all n then the objects $\overline{E}$, F and morphisms between them determine E as a cone of the canonical morphism

$$F \otimes \mathrm{Hom}(F,\overline{E}) \xrightarrow{\mathrm{can}} \overline{E} \longrightarrow E.$$

Assume that (E,F) is an exceptional pair in an enhanced triangulated category with the enhancement $\widetilde{\mathcal{C}}$ and denote by $\mathcal{C}$ the DG subcategory of $\widetilde{\mathcal{C}}$ with objects $\overline{E}$ and F. Then the cone of the canonical morphism $F \otimes \mathrm{Hom}_{\mathcal{C}}(F,\overline{E}) \to \overline{E}$ in $\mathcal{C}^{\text{pre-tr}}$ corresponds to E in $\widetilde{\mathcal{C}}$. Hence, as in the case of mutations, if we know $\mathcal{C}$ we can calculate the DG subcategory of $\widetilde{\mathcal{C}}$ with objects E and F.

Analogously, we define an object $\overline{F}$ to be the *universal coextension of E by F* if $\overline{F}$ is the extension of $E \otimes \mathrm{Ext}^1(E,F)$ by F corresponding to the above defined $\widetilde{\mathrm{id}} \in \mathrm{Ext}^1(E \otimes \mathrm{Ext}^1(E,F), F) \simeq \mathrm{Ext}^1(E, F \otimes \mathrm{Ext}^1(E,F)^*)$. The object $\overline{F}$ fits into a short exact sequence

$$0 \longrightarrow F \longrightarrow \overline{F} \longrightarrow E \otimes \mathrm{Ext}^1(E,F) \longrightarrow 0.$$

The same arguments as in the case of universal extensions show that if E is exceptional and $\mathrm{Hom}(F,E) = 0$ then $\mathrm{Ext}^1(E,\overline{F}) = 0$, $\mathrm{Ext}^n(E,\overline{F}) \simeq \mathrm{Ext}^n(E,F)$ for $n > 1$ and

$\mathrm{Ext}^1(E,F) \simeq \mathrm{Hom}(\overline{F},E)^*$. From the last equality it follows that F is the shift by -1 of the cone of the canonical map

$$\overline{F} \xrightarrow{\mathrm{can}} E \otimes \mathrm{Hom}^\bullet(\overline{F},E).$$

Let $\langle E_1,\dots,E_n\rangle$ be an exceptional collection on a smooth projective variety X such that $\mathrm{Ext}^i(\mathcal{E}_j,\mathcal{E}_l)=0$ for $i\geq 2$ and all j,l. Following [25] we define $\mathcal{E}_i(j)$ as E_i if $j=1,\dots,i$. For $j>i$ we define $\mathcal{E}_i(j)$ as the universal extension of $\mathcal{E}_i(j-1)$ by $\mathcal{E}_j$, so that $\mathcal{E}_i(j)$ fits into the short exact sequence

$$0 \longrightarrow E_j \otimes \mathrm{Ext}^1(\mathcal{E}_i(j-1),E_j)^* \longrightarrow \mathcal{E}_i(j) \longrightarrow \mathcal{E}_i(j-1) \longrightarrow 0.$$

Then, by [25, Theorem 4.1], $\mathscr{E}=\bigoplus_{i=1}^n \mathcal{E}_i(n)$ is a tilting object. Recall that a tilting object is an object $\mathscr{E}$ such that $\mathrm{Ext}^i(\mathscr{E},\mathscr{E})=0$ for $i>0$ and $\mathcal{D}^b(X)$ is the smallest strictly full subcategory of $\mathcal{D}^b(X)$ containing $\mathscr{E}$ and closed under taking direct summands.

Theorem 2.3.1. *Let $\sigma=\langle E_1,\dots,E_n\rangle$ be a full exceptional collection on a smooth projective variety X. Let us assume that $Ext^i(E_j,E_l)=0$ for $i\neq 0,1$ and any l, j. Then the DG category $\mathcal{A}_\sigma$ is determined up to a quasi-isomorphism by the endomorphism algebra of the tilting generator $Hom_X(\bigoplus_i E_i^n,\bigoplus E_i^n)$.*

Proof. The E_i's are exceptional and there are no morphisms from E_j to $\mathcal{E}_i(j-1)$. Thus the objects $\mathcal{E}_1(n),\dots,\mathcal{E}_n(n)$ determine $\mathcal{E}_i$'s. Thus, as described above, the endomorphisms algebra of $\bigoplus_{i=1}^n \mathcal{E}_i(n)$ determines the DG structure of the collection $\langle E_1,\dots,E_n\rangle$. $\square$

Remark 2.3.2. In the above theorem instead of universal extensions we can use universal coextensions and define object $\mathcal{F}_i(j)$ for $i,j=1,\dots,n$. We put $\mathcal{F}_i(i)=\ldots=\mathcal{F}_i(n)=E_i$ and for $j\leq i$ define $\mathcal{F}_i(j-1)$ as the universal coextension of $\mathcal{F}_i(j)$ by E_{j-1},

$$0\to\mathcal{F}_i(j)\to\mathcal{F}_i(j-1)\to E_{j-1}\otimes\mathrm{Ext}^1(E_{j-1},\mathcal{F}_i(j))\to 0.$$

Then, $\mathscr{F}=\bigoplus_i \mathcal{F}_i(1)$ is a tilting object in $\mathcal{D}^b(X)$ and from its endomorphism algebra one can read off the DG quiver of collection $\langle E_1,\dots,E_n\rangle$.

2.3.2 Relations to quasi-hereditary algebras

The construction of universal extensions and coextensions originates in the theory of quasi-hereditary algebras. We recall some definitions after Klucznik and König, [40].

Let A be a finite dimensional k-algebra. Let Λ be an indexing set for the isomorphism classes of simple A-modules. For $\lambda\in\Lambda$ the corresponding isomorphism class of a simple A-module is denoted by $L(\lambda)$. Let $\preceq$ be a partial order on Λ.

The algebra (A,Λ) is *quasi-hereditary* if for every $\lambda\in\Lambda$ there exists a left A-module $\Delta(\lambda)$ such that

- there is a surjection $\phi_\lambda \colon \Delta(\lambda) \to L(\lambda)$ for which all the composition factors $L(\mu)$ of the kernel satisfy $\mu \prec \lambda$. The module $\Delta(\lambda)$ is called a *standard module* (or a *Verma module*).

- The indecomposable projective cover $P(\lambda)$ of $L(\lambda)$ maps onto $\Delta(\lambda)$ via a map $\psi_\lambda \colon P(\lambda) \to \Delta(\lambda)$ whose kernel is filtered by modules $\Delta(\mu)$ with $\mu \succ \lambda$.

Analogously, one can consider *costandard modules* $\nabla(\lambda)$ and injective covers of simple modules $I(\lambda)$. An algebra A is quasi-hereditary if there is an injective morphism $L(\lambda) \to \nabla(\lambda)$ whose cokernel has a filtration with composition factors $L(\mu)$ for $\mu \prec \lambda$ and if there is an injective morphism $\nabla(\lambda) \to I(\lambda)$ whose cokernel has a filtration with composition factors $\nabla(\mu)$ for $\lambda \prec \mu$.

From the above conditions it follows that $\mathrm{Hom}(\Delta(\lambda), \Delta(\lambda)) = k$, (see [18, Lemma 1.6]).

Universal extensions and coextensions defined in Section 2.3.1 play an important role in the theory of quasi-hereditary algebras.

The algorithm presented in the proof of Theorem 2.3.1 and Remark 2.3.2 allow us to remove all Ext^1-groups between any sequence $E_1, \dots, E_n$ of objects in an abelian category $\mathcal{A}$. Universal extensions lead to an object $\mathscr{E} = \bigoplus \overline{E_i}$, while universal coextesions to $\mathscr{F} = \bigoplus \overline{F_i}$.

If the starting set is the partially ordered set of standard modules over a quasi-hereditary algebra then the objects obtained by universal extensions are $\overline{E_\lambda} = P(\lambda)$, the projective cover of $\Delta(\lambda)$ and $\overline{F_\lambda} = T(\lambda)$, a direct summand of the characteristic tilting module.

The endomorphism algebra of $T = \bigoplus T(\lambda)$ is again quasi-hereditary and it is called the *Ringel dual* of the initial algebra A. The associated poset is dual to the poset of A (see [58, Theorem 6]).

A quasi-hereditary algebra can be also constructed from a standardisable set of objects in an abelian category. We recall some relevant definitions and constructions from [18]. Let $\mathcal{A}$ be an abelian category. For a set $\Psi \subset \mathrm{ob}\,\mathcal{A}$ we denote by $\mathcal{F}(\Psi)$ a full subcategory of $\mathcal{A}$ consisting of objects that admit a filtration with quotients in Ψ.

Let $\Theta = \{\Theta(\lambda) \mid \lambda \in \Lambda\}$ be a finite set of objects of an abelian k-linear category $\mathcal{A}$. We define a quiver $Q(\Lambda)$ with vertex set Λ. There is an arrow $\lambda \to \mu$ in $Q(\Lambda)$ provided the set of non-invertible maps $\Theta(\lambda) \to \Theta(\mu)$ or the set $\mathrm{Ext}^1(\Theta(\lambda), \Theta(\mu))$ is non-empty.

We say that a finite set $\Theta = \{\Theta(\lambda) \mid \lambda \in \Lambda\}$ is *standardisable* if for all $\lambda, \mu \in \Lambda$ the spaces $\mathrm{Hom}(\Theta(\Lambda), \Theta(\mu))$ and $\mathrm{Ext}^1(\Theta(\Lambda), \Theta(\mu))$ are finite-dimensional over k and if the quiver $Q(\Lambda)$ has no oriented cycles. In particular, any object $\Theta(\lambda)$ has one-dimensional endomorphism space and $\mathrm{Ext}^1(\Theta(\lambda), \Theta(\lambda)) = 0$. Moreover, the quiver $Q(\Lambda)$ defines a partial order on the set Λ with $\mu \prec \lambda$ if there is an arrow $\mu \to \lambda$.

The universal extensions algorithm applied to a standardisable set Θ gives a quasi-hereditary algebra A such that the subcategory $\mathcal{F}(\Theta)$ and the category $\mathcal{F}(\Delta_A)$ are equivalent. Similarly, universal extensions allow us to construct a quasi-hereditary algebra B such that the subcategory $\mathcal{F}(\Theta)$ of $\mathcal{A}$ and the category $\mathcal{F}(\nabla_B)$ are equivalent.

Let $\langle E_1, \ldots, E_n \rangle = \mathcal{T}$ be an full exceptional collection in a triangulated category $\mathcal{T}$. Then $E_1, \ldots, E_n$ is a standardisable set in $\mathcal{T}$ and one can construct two associated quasi-hereditary algebras out of it. We shall also assume that only Hom and the first Ext-groups between E_i, E_j are non-zero.

First, using universal extensions, we define the objects $\overline{E_i}$ and put

$$\mathscr{E} = \oplus_{i=1}^n \overline{E_i}.$$

Note that $\mathscr{E}$ is a tilting generator of $\mathcal{T}$. Moreover, $\mathrm{Ext}^1(\mathscr{E}, \overline{E_j}) = 0$ for all j.

Analogously, using universal coextensions, we define the objects $\overline{F_i}$ and put

$$\mathscr{F} = \oplus_{i=1}^n \overline{F_i}$$

to be another tilting generator of $\mathcal{T}$. Then for all j we have $\mathrm{Ext}^1(E_j, \mathscr{F}) = 0$.

The endomorphism algebra of $\mathscr{E}$, $\mathcal{A}_{\mathscr{E}} = \mathrm{End}(\mathscr{E})$ is quasi-hereditary. The order on the set $\{1, \ldots, n\}$ is the standard one. The projective modules are $P_{\mathscr{E}}(i) = \mathrm{Hom}(\mathscr{E}, \overline{E_i})$. Because $\mathrm{Ext}^1(\mathscr{E}, E_j)$ vanish we have short exact sequences

$$0 \to \mathrm{Hom}(\mathscr{E}, E_j) \otimes \mathrm{Ext}^1(E_i^{j-1}, E_j)^* \to \mathrm{Hom}(\mathscr{E}, E_i^j) \to \mathrm{Hom}(\mathscr{E}, E_i^{j-1}) \to 0.$$

If we compose the surjections $\mathrm{Hom}(\mathscr{E}, \overline{E_i}) \twoheadrightarrow \mathrm{Hom}(\mathscr{E}, E_i^{n-1}) \twoheadrightarrow \ldots \twoheadrightarrow \mathrm{Hom}(\mathscr{E}, E_i)$ we get the required map from $P\mathscr{E}(i)$ to

$$\Delta_{\mathscr{E}}(i) = \mathrm{Hom}(\mathscr{E}, E_i).$$

Similarly, the endomorphism algebra $\mathcal{A}_{\mathscr{F}} = \mathrm{End}(\mathscr{F})$ is quasi-hereditary. The order on the set $\{1, \ldots, n\}$ is the opposite one, $i \preceq j$ if and only if $j \leq i$. The projective modules are $P_{\mathscr{F}}(i) = \mathrm{Hom}(\overline{F_i}, \mathscr{F})$. Again, the sequence

$$0 \to \mathrm{Hom}(E_j, \mathscr{F}) \otimes \mathrm{Ext}^1(E_j, F_i^j) \to \mathrm{Hom}(F_i^{j-1}, \mathscr{F}) \to \mathrm{Hom}(F_i^j, \mathscr{F}) \to 0$$

is exact. This shows that the composition $\mathrm{Hom}(\overline{F_i}, \mathscr{F}) \twoheadrightarrow \mathrm{Hom}(F_i^2, \mathscr{F}) \twoheadrightarrow \ldots \twoheadrightarrow \mathrm{Hom}(E_i, \mathscr{F})$ is injective and we get the required map from $P\mathscr{F}(i)$ to

$$\Delta_{\mathscr{F}}(i) = \mathrm{Hom}(E_i, \mathscr{F}).$$

Functors

$$\Phi = \mathrm{Hom}(\mathscr{E}, -) \colon \mathcal{F}(\{E_i \,|\, i = 1, \ldots, n\}) \to \mathcal{F}(\Delta_{\mathcal{A}_{\mathscr{E}}}),$$
$$\Psi = \mathrm{Hom}(-, \mathscr{F})^{\vee} \colon \mathcal{F}(\{E_i \,|\, i = 1, \ldots, n\}) \to \mathcal{F}(\nabla_{\mathcal{A}_{\mathscr{F}}})$$

define equivalences of categories.

In order to find the Ringel dual of the algebra $A_{\mathscr{E}}$ we have to compute the characteristic modules. We know that $T(1) = \Delta(1) = \Phi(E_1)$ and that the remaining $T(i)$'s are obtained as universal coextensions from $\Delta_{A_{\mathscr{E}}}(i)$. Under the equivalence of $\mathcal{F}(\{E_i \,|\, i = 1, \ldots, n\})$ and $\mathcal{F}(\Delta_{A_{\mathscr{E}}})$ we see that $T(i)$ corresponds exactly to the module $F_i(1)$ and hence $R(A_{\mathscr{E}}) = \mathrm{End}(\bigoplus T(i)) = \mathrm{End}(\bigoplus F_i(1)) = A_{\mathscr{F}}$ which proves

Proposition 2.3.3. *Algebras $A_{\mathscr{E}}$ and $A_{\mathscr{F}}$ are Ringel dual to each other.*

2.3.3 Example

Let us go back to the example of Section 2.2.7. Recall that X is obtained from $\mathbb{P}^2$ by blowing up a point and then blowing up a point of the exceptional divisor. The collection

$$\langle \mathcal{O}_X, \mathcal{O}_X(E_2), \mathcal{O}_X(E_1 + E_2), \mathcal{O}_X(H), \mathcal{O}_X(2H) \rangle$$

is a full exceptional collection on X. There is one non-zero Ext^1 group between objects in this collection, namely $\mathrm{Ext}^1(\mathcal{O}_X(E_2), \mathcal{O}_X(E_1 + E_2)) = k$.

In order to calculate the DG quiver of this collection using universal extensions one has to understand endomorphism algebra of

$$\mathcal{O}_X \oplus V \oplus \mathcal{O}_X(E_1 + E_2) \oplus \mathcal{O}_X(H) \oplus \mathcal{O}_X(2H)$$

where V is a non-trivial extension of $\mathcal{O}_X(E_1 + E_2)$ by $\mathcal{O}_X(E_2)$ defined by short exact sequence

$$0 \longrightarrow \mathcal{O}_X(E_1 + E_2) \xrightarrow{\phi_1} V \xrightarrow{\phi_2} \mathcal{O}_X(E_2) \longrightarrow 0. \qquad (\star)$$

Recall, that we have considered maps $\zeta\colon \mathcal{O}_X \to V$, $\iota_1\colon V \to \mathcal{O}_X(H)$ and $\iota_2\colon V \to \mathcal{O}_X(H)$ such that

$$\phi_2 \circ \zeta = \gamma, \qquad \iota_1 \circ \phi_1 = \alpha_1, \qquad \iota_2 \circ \phi_1 = \alpha_2.$$

Calculations of the previous section show that

$$\iota_1 \circ \zeta = \eta, \qquad \iota_2 \circ \zeta = 0.$$

Then, the collection (not exceptional!) $\langle \mathcal{O}_X, V, \mathcal{O}_X(E_1 + E_2), \mathcal{O}_X(H), \mathcal{O}_X(2H) \rangle$ has a quiver

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, "\zeta"] & V \arrow[r, shift left, "\delta\phi_2"] \arrow[rr, bend right, "\iota_1"'] \arrow[rr, bend right=50, "\iota_2"'] & \mathcal{O}_X(E_1 + E_2) \arrow[l, shift left, "\phi_1"] & \mathcal{O}_X(H) \arrow[r, shift left=2, "\beta_1"] \arrow[r, "\beta_2"] \arrow[r, shift right=2, "\beta_3"'] & \mathcal{O}_X(2H)
\end{tikzcd}$$

with relations:

$$\iota_1 \circ \zeta = \eta, \quad \delta\phi_2 \circ \phi_1 = 0, \quad \iota_2 \circ \zeta = 0,$$
$$\beta_1 \circ \iota_1 \circ \zeta = \beta_3 \circ \iota_1 \circ \phi_1 \circ \delta\phi_2 \circ \zeta, \quad \beta_2 \circ \iota_1 \circ \zeta = \beta_3 \circ \iota_2 \circ \phi_1 \circ \delta\phi_2 \circ \zeta,$$

$$\beta_1\iota_2 - \beta_2\iota_1 \in \operatorname{span}\{\beta_1 \circ \iota_1 \circ \phi_1 \circ \delta\phi_2,\ \beta_1 \circ \iota_2 \circ \phi_1 \circ \delta\phi_2,\ \beta_2 \circ \iota_2 \circ \phi_1 \circ \delta\phi_2,$$
$$\beta_3 \circ \iota_1 \circ \phi_1 \circ \delta\phi_2,\ \beta_3 \circ \iota_2 \circ \phi_1 \circ \delta\phi_2\}.$$

The last inclusion follows from applying functor $\operatorname{Hom}(-, \mathcal{O}_X(2H))$ to the short exact sequence (*).

Recall also that some of the morphisms can be changed:

$$\begin{aligned}
\alpha_1 &\rightsquigarrow \alpha_1 + a\,\alpha_2\\
\eta &\rightsquigarrow \eta + b\,\alpha_1\delta\gamma + c\,\alpha_2\delta\gamma,\\
\zeta &\rightsquigarrow \zeta + d\,\phi_1\delta\gamma,\\
\iota_1 &\rightsquigarrow \iota_1 + e\,\alpha_1\delta\phi_2 + f\,\alpha_2\delta\phi_2,\\
\iota_2 &\rightsquigarrow \iota_2 + g\,\alpha_1\delta\phi_2 + h\,\alpha_2\delta\phi_2.
\end{aligned}$$

A change of a morphism from $\mathcal{O}_X$ to $\mathcal{O}_X(H)$ results in a change of a morphism from $\mathcal{O}_X(H)$ to $\mathcal{O}_X(2H)$ according to the following correspondence

$$\beta_1 \leftrightarrow \alpha_1\delta\gamma, \qquad \beta_2 \leftrightarrow \alpha_2\delta\gamma, \qquad \beta_3 \leftrightarrow \eta.$$

A change $\alpha_1 \rightsquigarrow \alpha_1 + a\,\alpha_2$ changes ι_1 to $\iota_1 + a\,\iota_2$ (because $\iota_1\phi_1 = \alpha_1$). As also β_1 depends on α_1 we get

$$\beta_1\iota_2 - \beta_2\iota_1 \rightsquigarrow (\beta_1 + a\,\beta_2)\iota_2 - \beta_2(\iota_1 + a\,\iota_2) = \beta_1\iota_2 + a\,\beta_2\iota_2 - \beta_2\iota_1 - a\,\beta_2\iota_2 = \beta_1\iota_2 - \beta_2\iota_1$$

and hence the parameter a has no influence on the relation between $\beta_1\iota_2$ and $\beta_2\iota_1$.

As we want $\iota_2\zeta = 0$ the calculation

$$(\iota_2 + g\,\alpha_1\delta\phi_2 + h\,\alpha_2\delta\phi_2)(\zeta + d\,\phi_1\delta\gamma) = \iota_2\zeta + (d+h)\,\alpha_2\delta\gamma + g\,\alpha_1\delta\gamma$$

show that g must be 0 and h can be arbitrary, as putting $d = -h$ will not lead to any changes in this relation.

The morphism $\eta = \iota_1\zeta$ can be changed by any combination of $\alpha_1\delta\gamma$ and $\alpha_2\delta\gamma$ so there is no condition on e and f.

Calculating

$$\beta_1(\iota_2 + h\,\alpha_2\delta\phi_2) - \beta_2(\iota_1 + e\,\alpha_1\delta\phi_2 + f\,\alpha_2\delta\phi_2) = \beta_1\iota_2 - \beta_2\iota_1 + (h-e)\,\beta_1\alpha_2\delta\phi_2 - f\,\beta_2\alpha_2\delta\phi_2$$

we see that e and f can be chosen in such a way that

$$\beta_1\iota_2 - \beta_2\iota_1 = A\,\beta_1\iota_1\phi_1\delta\phi_2 + B\,\beta_3\iota_1\phi_1\delta\phi_2 + C\,\beta_3\iota_2\phi_1\delta\phi_2.$$

Furthermore, the relation $\beta_3\iota_2\phi_1\delta\phi_2\zeta = \beta_2\iota_1\zeta$ gives

$$\begin{aligned} 0 = \beta_3\iota_2\phi_1\delta\phi_2\zeta - \beta_2\iota_1\zeta &= \beta_3\iota_2\phi_1\delta\phi_2\zeta - \beta_1\iota_2\zeta + A\,\beta_1\iota_1\phi_1\delta\phi_2\zeta + B\,\beta_3\iota_1\phi_1\delta\phi_2\zeta \\ +C\,\beta_3\iota_2\phi_1\delta\phi_2\zeta &= A\,\beta_1\iota_1\phi_1\delta\phi_2\zeta + B\,\beta_3\iota_1\phi_1\delta\phi_2\zeta + (C+1)\,\beta_3\iota_2\phi_1\delta\phi_2\zeta \end{aligned}$$

and leads to $A = 0 = B$ and $C = -1$ due to linear independence of $\beta_1\iota_1\phi_1\delta\phi_2\zeta$, $\beta_3\iota_1\phi_1\delta\phi_2\zeta$ and $\beta_3\iota_2\phi_1\delta\phi_2\zeta$.

Note that the relation $\beta_1\iota_2 + \beta_3\iota_2\phi_1\delta\phi_2 = \beta_2\iota_1$ composed with ζ gives $\beta_3\iota_2\phi_1\delta\phi_2\zeta = \beta_2\iota_1\zeta$ which is exactly the relation $\beta_3\alpha_2\delta\gamma = \beta_2\eta$ in the Ext quiver of the collection.

Hence relations in the endomorphism algebra of $\mathcal{O}_X \oplus V \oplus \mathcal{O}_X(E_1+E_2) \oplus \mathcal{O}_X(H) \oplus \mathcal{O}_X(2H)$ in the notation introduced before are:

$$\begin{aligned} \iota_2\zeta &= 0, & \delta\phi_2\phi_1 &= 0, \\ \beta_1\iota_1\zeta &= \beta_3\iota_1\phi_1\delta\phi_2\zeta, & \beta_1\iota_2 + \beta_3\iota_2\phi_1\delta\phi_2 &= \beta_2\iota_1. \end{aligned}$$

The collection $\langle \mathcal{O}_X, \mathcal{O}_X(E_2), \mathcal{O}_X(E_1+E_2), \mathcal{O}_X(H), \mathcal{O}_X(2H)\rangle$ has the DG quiver of the collection

$$\begin{tikzcd} (\mathcal{O}_X, & \mathcal{O}_X(E_1+E_2) \arrow[d, "\phi_1"] & \mathcal{O}_X(E_1+E_2), \quad \mathcal{O}_X(H), \quad \mathcal{O}_X(2H)) \\ & V & \end{tikzcd}$$

Morphisms from $\mathcal{O}_X$ to $\mathcal{O}_X(E_2)$ have basis

$$a_1 = \begin{tikzcd} & \mathcal{O}_X(E_1+E_2) \arrow[d, "\phi_1"] \\ \mathcal{O}_X \arrow[ur, "\delta\phi_2\zeta"] & V \end{tikzcd} \qquad a_2 = \begin{tikzcd} & \mathcal{O}_X(E_1+E_2) \arrow[d, "\phi_1"] \\ \mathcal{O}_X \arrow[r, "\delta\phi_2\zeta\phi_1"] & V \end{tikzcd}$$

$$a_3 = \begin{tikzcd} & \mathcal{O}_X(E_1+E_2) \arrow[d, "\phi_1"] \\ \mathcal{O}_X \arrow[r, "\zeta"] & V \end{tikzcd}$$

with $\partial(a_1) = a_2$.

Morphisms from $\mathcal{O}_X(E_2)$ to $\mathcal{O}_X(E_1+E_2)$ have basis:

$$b_1 = \begin{tikzcd} \mathcal{O}_X(E_1+E_2) \arrow[d, "\phi_1"] \arrow[dr, "\mathrm{id}"] & \\ V & \mathcal{O}_X(E_1+E_2) \end{tikzcd} \qquad b_2 = \begin{tikzcd} \mathcal{O}_X(E_1+E_2) \arrow[d, "\phi_1"] & \\ V \arrow[r, "\delta\phi_2"] & \mathcal{O}_X(E_1+E_2) \end{tikzcd}$$

with b_1 in degree 1.

Morphisms from $\mathcal{O}_X(E_2)$ to $\mathcal{O}_X(H)$ are

$$c_1 = \begin{tikzcd} \mathcal{O}_X(E_1+E_2) \arrow[d, "\phi_1"] & \\ V \arrow[r, "\iota_1"] & \mathcal{O}_X(H) \end{tikzcd} \qquad c_2 = \begin{tikzcd} \mathcal{O}_X(E_1+E_2) \arrow[d, "\phi_1"] \arrow[dr, "\iota_1\phi_1"] & \\ V & \mathcal{O}_X(H) \end{tikzcd}$$

$$c_3 = \begin{tikzcd} \mathcal{O}_X(E_1+E_2) \arrow[d, "\phi_1"] & \\ V \arrow[r, "\iota_1\phi_1\delta\phi_2"] & \mathcal{O}_X(H) \end{tikzcd} \qquad c_4 = \begin{tikzcd} \mathcal{O}_X(E_1+E_2) \arrow[d, "\phi_1"] & \\ V \arrow[r, "\iota_2"] & \mathcal{O}_X(H) \end{tikzcd}$$

$$c_5 = \begin{tikzcd} \mathcal{O}_X(E_1+E_2) \arrow[d, "\phi_1"] \arrow[dr, "\iota_2\phi_1"] & \\ V & \mathcal{O}_X(H) \end{tikzcd} \qquad c_6 = \begin{tikzcd} \mathcal{O}_X(E_1+E_2) \arrow[d, "\phi_1"] & \\ V \arrow[r, "\iota_2\phi_1\delta\phi_2"] & \mathcal{O}_X(H) \end{tikzcd}$$

with $\partial(c_1) = c_2$ and $\partial(c_4) = c_5$.

Morphisms from $\mathcal{O}_X(E_2)$ to $\mathcal{O}_X(2H)$ factor through $\mathcal{O}_X(H)$ so can be written as $\beta_i c_j$.

Relations between these morphisms are

$$\begin{aligned} b_1a_3 &= 0, & b_2a_3 &= \delta\phi_2\zeta, & \iota_1\phi_1b_1 &= c_2, \\ \iota_1\phi_1b_2 &= c_3, & \iota_2\phi_1b_1 &= c_5, & \iota_2\phi_1b_2 &= c_6, \\ \partial(c_1) &= \iota_1\phi_1b_1, & \partial(c_4) &= \iota_2\phi_1b_1, & c_1a_3 &= \iota_1\zeta, \\ c_4a_3 &= 0, & \beta_2c_1a_3 &= \beta_3\iota_2\phi_1b_2a_3, & \beta_1c_4 &= \beta_2c_1, \\ \beta_1c_1a_3 &= \beta_3\iota_1\phi_1b_2a_3, & \beta_1c_4 + \beta_3c_6 &= \beta_2c_1. \end{aligned}$$

Putting $\gamma = a_3$, $\delta = b_2$, $\theta = b_1$, $\alpha_1 = \iota_1\phi_1$, $\alpha_2 = \iota_2\phi_2$, $\epsilon_1 = c_1$ and $\epsilon_2 = c_4$ we obtain the following DG quiver

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, "\gamma"] & \mathcal{O}_X(E_2) \arrow[r, bend left, "\theta"] \arrow[r, "\delta"] \arrow[rr, bend right, "\epsilon_1"'] \arrow[rr, bend right=40, "\epsilon_2"'] & \mathcal{O}_X(E_1+E_2) \arrow[r, shift left, "\alpha_1"] \arrow[r, shift right, "\alpha_2"'] & \mathcal{O}_X(H) \arrow[r, shift left=2, "\beta_1"] \arrow[r, "\beta_2"] \arrow[r, shift right=2, "\beta_3"'] & \mathcal{O}_X(2H)
\end{tikzcd}$$

with relations and differentials given by

$$\begin{aligned}
&\partial(\epsilon_1) = \alpha_1\theta, \qquad &&\partial(\epsilon_2) = \alpha_2\overline{\delta}, \qquad &&\epsilon_2\gamma = 0, \\
&\beta_1\alpha_2 = \beta_2\alpha_1, \qquad &&\beta_2\epsilon_1\gamma = \beta_3\alpha_2\delta\gamma, \qquad &&\beta_1\epsilon_2 + \beta_3\alpha_2\delta = \beta_2\epsilon_1.
\end{aligned}$$

Calculating A_∞-structure on the cohomology of this category gives

$$m_3(\alpha_1, \theta, \gamma) = \eta, \qquad m_3(\alpha_2, \theta, \gamma) = 0, \qquad m_3(\beta_2, \alpha_1, \theta) - m_3(\beta_1, \alpha_2, \theta) = \beta_3\alpha_2\delta.$$

where η denotes the cohomology class of $\epsilon_1\gamma$.

2.4 Comparison of the methods

We have seen in Section 2.2.7 that Massey products are not enough to determine the A_∞-structure on the Ext-quiver of the exceptional collection

$$\langle \mathcal{O}_X, \mathcal{O}_X(E_2), \mathcal{O}_X(E_1+E_2), \mathcal{O}_X(H), \mathcal{O}_X(2H)\rangle.$$

More precisely, Massey products were not sufficient to calculate the value of the difference $m_3(\beta_1, \alpha_2, \theta) - m_3(\beta_2, \alpha_1, \theta)$ which is an invariant of the A_∞-quasi-isomorphism class of the corresponding A_∞-category.

However, methods of Section 2.3.3 allowed us to calculate the A_∞-category in this case. The value $m_3(\beta_1, \alpha_2, \theta) - m_3(\beta_2, \alpha_1, \theta)$ was in this case determined by the cohomological operation $\widetilde{\mu_3}(\beta_1, \alpha_2; \beta_2, \alpha_1, \theta)$. We call it a *relative triple Massey product.* We will motivate this name in Section 2.4.1.

Such a product is defined for maps in a triangulated category $\mathcal{T}$

$$\begin{tikzcd}
 & & C \arrow[dr, "c"] & \\
A \arrow[r, "a"] & B \arrow[ur, "b"] \arrow[dr, "d"'] & & D \\
 & & E \arrow[ur, "e"'] &
\end{tikzcd}$$

such that

$$b \circ a = 0, \qquad d \circ a = 0, \qquad e \circ d = c \circ b.$$

If we denote by F the cone of a

$$A \xrightarrow{a} B \xrightarrow{\iota} F \xrightarrow{\pi} A[1]$$

the conditions $b \circ a = 0$ and $d \circ a = 0$ guarantee that there exist maps $\beta\colon F \to C$ and $\delta\colon F \to E$ such that

$$\beta \circ \iota = b, \qquad\qquad \delta \circ \iota = d.$$

Then, from the fact that $e \circ d = c \circ b$ it follows that there exists a map $\gamma\colon A[1] \to D$ such that $\gamma \circ \pi = c \circ \beta - e \circ \delta$. Let us denote the set of all maps γ like this by

$$\widetilde{\mu_3}(c, b; e, d; a).$$

Then

$$\begin{aligned} m_3(\alpha_1, \theta, \gamma) &\in \mu_3(\alpha_1, \theta, \gamma), \\ m_3(\alpha_2, \theta, \gamma) &\in \mu_3(\alpha_2, \theta, \gamma), \\ m_3(\beta_1, \alpha_2, \theta) - m_3(\beta_2, \alpha_1, \theta) &\in \widetilde{\mu_3}(\beta_1, \alpha_2; \beta_2, \alpha_1; \theta). \end{aligned}$$

2.4.1 Further properties of Massey products in enhanced triangulated categories

In Section 2.2.5 we have seen that in order to calculate an n-tuple Massey product of a complex in an enhanced triangulated category $\mathcal{T} = H^0(\mathcal{C})$ one has to find a defining system in $\mathcal{C}$. In this section we will use this observation to give an equivalent definition of Massey products in triangulated language.

Lemma 2.4.1. *Let $S \in TotT^{\bullet}$ be a convolution of a complex $T^{\bullet}$ in an enhanced triangulated category $\mathcal{T} = H^0(\mathcal{C})$ and let $i \in \{0, \dots, n-1\}$. Finally, let S^i be a cone of ∂^i;*

$$T^i \xrightarrow{\partial^i} T^{i+1} \xrightarrow{\iota_i} S^i \xrightarrow{\pi_i} T^i[1].$$

Then, there exist $\widetilde{\partial}^{i-1}\colon T^{i-1} \to S^i[1]$, $\widetilde{\partial}^{i+1}\colon S^i \to T^{i+2}$ such that $\pi_i \circ \widetilde{\partial}^{i-1} = \partial^{i-1}$, $\widetilde{\partial}^{i+1} \circ \iota_i = \partial^{i+1}$ and S is a convolution of a complex

$$T_i^{\bullet}\colon T^0 \xrightarrow{\partial^0} T^1 \to \dots \xrightarrow{\partial^{i-2}} T^{i-1} \xrightarrow{\widetilde{\partial}^{i-1}} S^i[-1] \xrightarrow{\widetilde{\partial}^{i+1}} T^{i+1}[-1] \xrightarrow{\partial^{i+2}} \dots \xrightarrow{\partial^{n-1}} T^n[-1].$$

Proof. From Section 2.2.5 it follows that there exists an extended defining system $\alpha_{i,j}$ for the complex $T^{\bullet}$ such that $S \in \mathcal{T}$ represents a twisted complex $(T^i, \alpha_{i,j}) \in \mathcal{C}^{\text{pre-tr}}$.

Object S_i can be presented as a twisted complex $\{T^i \xrightarrow{\alpha_{i,i+1}} T^{i+1}\}$ in $\mathcal{C}^{\text{pre-tr}}$, where T^i is in gradation minus one.

$\widetilde{\partial}^{i-1}$ and $\widetilde{\partial}^{i+1}$ are maps in $\mathcal{T}$ corresponding respectively to maps

$$\begin{tikzcd} T^{i-1} \arrow[r, "\alpha_{i-1,i}"] \arrow[dr, "-\alpha_{i-1,i+1}"'] & T^{i} \arrow[d, "-\alpha_{i,i+1}"] \\ & T^{i+1} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T^{i} \arrow[d, "\alpha_{i,i+1}"'] \arrow[dr, "\alpha_{i,i+2}"] & \\ T^{i+1} \arrow[r, "\alpha_{i+1,i+2}"'] & T^{i+2} \end{tikzcd}$$

in $\mathcal{C}^{\text{pre-tr}}$. □

Remark 2.4.2. The same argument as in the proof of the above Lemma shows that any defining system for the complex $T^{\bullet}$ is also a defining system for $T_i^{\bullet}$.

Proposition 2.4.3. *Let $T^{\bullet} = T^0 \to \ldots \to T^n$ be a complex in an enhanced triangulated category $\mathcal{T}$, φ an element of $\mu_n(\partial^{n-1}, \ldots, \partial^0)$ and let $i \in \{0, \ldots, n-1\}$.*

If $i \neq 0, n-1$ then φ is an element of $\mu_{n-1}(\partial^{n-1}, \ldots, \partial^{i+2}, \widetilde{\partial}^{i+1}, \widetilde{\partial}^{i-1}, \partial^{i-2}, \ldots, \partial^0)$ for the complex $T_i^{\bullet}$ defined in Lemma 2.4.1.

If $i = 0$ then $\varphi \circ \iota_0$ is an element of $\mu_{n-1}(\partial^{n-1}, \ldots, \partial^2, \widetilde{\partial}^0)$ for the complex $T_0^{\bullet}$. Similarly, for $i = n-1$ morphism $\pi_{n-1} \circ \varphi$ belongs to $\mu_{n-1}(\widetilde{\partial}^{i-1}, \partial^{i-2}, \ldots, \partial^0)$.

Moreover, $\mu_{n-1}(\partial^{n-1}, \ldots, \partial^{i+2}, \widetilde{\partial}^{i+1}, \widetilde{\partial}^{i-1}, \partial^{i-2}, \ldots, \partial^0)$ is the subset of $\mu_{n-1}(\partial^{n-1}, \ldots, \partial^0)$ for $i \neq 0, n-1$.

If $i = 0$ then any element of $\mu_{n-1}(\partial^{n-1}, \ldots, \partial^2, \widetilde{\partial}^0)$ is of the form $\psi \circ \iota_0$ for some $\psi \in \mu_n(\partial^{n-1}, \ldots, \partial^0)$. Analogously, if $i = n-1$ then any element of $\mu_{n-1}(\widetilde{\partial}^{i-1}, \partial^{i-2}, \ldots, \partial^0)$ is of the form $\pi_{n-1} \circ \psi$ for some $\psi \in \mu_n(\partial^{n-1}, \ldots, \partial^0)$.

Proof. By Lemma 2.2.10 Massey products in $\mathcal{T}$ coincide up to a sign with Massey products defined using the DG structure.

Then, the claim follows from Remark 2.4.2.

To see that any $(n-1)$-tuple Massey product defined for $T_0^{\bullet}$ factors through ι_0 we note that S^0 can be considered as a twisted complex $\{T^0 \xrightarrow{\alpha_{0,1}} T^1\}$. Similar argument works for $T_{n-1}^{\bullet}$. □

Let

$$T^0 \xrightarrow{\partial^0} T^1 \xrightarrow{\partial^1} T^2 \xrightarrow{\partial^2} T^3$$

be a complex in an enhanced triangulated category and let

$$T^0 \xrightarrow{\partial^0} T^1 \xrightarrow{\iota} S \xrightarrow{\pi} T^0[1]$$

be a distinguished triangle.

Since $\partial^1 \circ \partial^0 = 0$ there exists $\widetilde{\partial}^1 \colon S \to T^2$ such that $\widetilde{\partial}^1 \circ \iota = \partial^1$.

Note that $(\partial^2 \circ \widetilde{\partial}^1) \circ \iota = \partial^2 \circ \partial^1 = 0$ and hence there exists $\varphi \colon T^0[1] \to T^3$ such that $\varphi \circ \pi = \partial^2 \circ \widetilde{\partial}^1$.

By Proposition 2.4.3

$$\varphi \in \mu_3(\partial^2, \partial^1, \partial^0).$$

This motivates the name "relative Massey product" used in Section 2.4.

Remark 2.4.4. As there are many ways to calculate Massey products, the relative Massey product described in Section 2.4 is in fact a triple Massey product of the following complex

$$A \xrightarrow{a} B \xrightarrow{\begin{pmatrix} b \\ -d \end{pmatrix}} C \oplus E \xrightarrow{(c,e)} D.$$

Chapter 3

DG quivers of smooth rational surfaces

Smooth rational surfaces have full exceptional collections with vanishing second and higher Ext-groups. Therefore, results of Section 2.3 allow us to calculate corresponding DG quivers. It turns out that a large class of full exceptional collections can be mutated to collections for which the tilting object and its endomorphism algebra can be written down explicitly in terms of generators and relations. Such a presentation is given in Proposition 3.2.16 and Lemma 3.2.18.

In Section 3.1 we recall basic facts about rational surfaces and prove some useful lemmas.

In Section 3.2 we describe exceptional collections on smooth rational surfaces obtained from exceptional collections on $\mathbb{P}^2$ and Hirzebruch surfaces $\mathbb{F}_a$. In Section 3.2.1 for a rational surface X we calculate Ext-quiver of the part of the collection corresponding to the map $f\colon X \to X_0$ to its minimal model. In Section 3.2.2 we choose basis of Hom- and Ext-groups appearing in this collection and in Section 3.2.3 we show how composition of Hom and Ext-groups can be written in this basis. We also describe in Section 3.2.4 how to calculate which compositions are zero purely in terms of the intersection matrix of irreducible components of the exceptional divisor of f.

In Section 3.2.5 we describe the full Ext-quiver of the exceptional collection on X, provided the exceptional collection on X_0 we start with is strong.

In Section 3.2.6 we describe the DG quiver of such a collection. In Section 3.2.7 we explicitly describe the tilting object on X obtained by universal coextensions. With Proposition 3.2.16 and Lemma 3.2.18 we describe how the endomorphism algebra of this tilting object looks like in a large family of examples.

We then show that above construction allows to calculate DG quiver of full exceptional collections of line bundles naturally arising on smooth toric surfaces. We recall after [19] basis definitions in Section 3.3.1 and give examples of full exceptional collections on smooth toric surfaces in Section 3.3.2. We show that all of these exceptional collections give an exceptional collection on the total space of the canonical bundle, therefore we introduce a canonical DG algebra of a toric surface in Section 3.3.3. We describe how methods of Section 3.2.6 allow us to calculate this DG algebra. We conclude with some examples of these DG algebras in Section 3.3.4.

3.1 Geometry of rational surfaces

The first examples of smooth rational surfaces are the projective plane $\mathbb{P}^2$ and Hirzebruch surfaces $\mathbb{F}_a$ for $a \geq 0$.

Recall that the Hirzebruch surface $\mathbb{F}_a$ is a projectivisation of a bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a)$. Its Picard group is generated by the class of fiber F of $\mathbb{F}_a \to \mathbb{P}^1$ and the class C of the zero section. The intersection form on $\mathbb{F}_a$ is given by

$$F^2 = 0, \qquad F.C = 1, \qquad C^2 = -a.$$

The canonical divisor of $\mathbb{P}^2$ is $K_X = -3H$ and the canonical divisor of $\mathbb{F}_a$ is $K_{\mathbb{F}_a} = -2C - (a+2)F$.

The collection $\langle \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(H), \mathcal{O}_{\mathbb{P}^2}(2H)\rangle$ is a full strong exceptional collection on $\mathbb{P}^2$ while $\langle \mathcal{O}_{\mathbb{F}_a}, \mathcal{O}_{\mathbb{F}_a}(F), \mathcal{O}_{\mathbb{F}_a}(aF+C), \mathcal{O}_{\mathbb{F}_a}((a+1)F+C)\rangle$ is a full strong exceptional collection on $\mathbb{F}_a$ (see for example [26]).

Any smooth rational surface X is obtained by a sequence of blow-ups from $\mathbb{P}^2$ or $\mathbb{F}_a$. We have a sequence of maps

$$X = X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \dots \longrightarrow X_1 \xrightarrow{\pi_1} X_0,$$

where $X_0 = \mathbb{P}^2$ or $\mathbb{F}_a$. We can also assume that every π_i is a blow up of X_{i-1} at one point x_{i-1}.

Let us denote by $\pi\colon X \to X_0$ the composition of all π_i's.

Let $E_i \subset X_i$ be the exceptional divisor of π_i. By an abuse of notation the strict transform of E_i in X is also denoted by E_i. By $R_i \subset X$ we denote the pullback of E_i under $\pi_{i+1} \dots \pi_n$. By definition we can write R_i as a sum $\sum a_i^j E_j$ for some $a_i^j \in \mathbb{Z}^{\geq 0}$.

The intersection form is preserved by π^*, i.e. for $\pi\colon X \to Y$ and divisors D_1, D_2 on Y we have $D_1.D_2 = \pi^*(D_1).\pi^*(D_2)$. Hence, R_i's form an orthogonal basis and $R_i^2 = -1$ for every i.

We have a natural partial order on the set of R_i's, namely

$$R_i \geq R_j \quad \Leftrightarrow \quad R_i - R_j \geq 0.$$

Lemma 3.1.1. *We have*

$$R_i.E_j = \begin{cases} 0 & j > i, \\ -1 & j = i \\ \geq 0 & j < i. \end{cases}$$

Proof. We proceed by induction on the number n of blow ups. Assume that the statement is true for $R_i^{n-1} = \pi_{n-1}^* \circ \ldots \circ \pi_{i+1}^*(E_i) \subset X_{n-1}$. Then obviously the lemma is true for $R_n = E_n$. Also, since $R_i = \pi_n^*(R_i^{n-1})$ we know that $R_i.E_n = 0$ for $i < n$. Finally, for $i, j < n$

$$\pi_n^*(R_i^{n-1}).\pi_n^*(E_j) = R_i^{n-1}.E_j$$

□

and the assertion follows from the induction assumption.

Remark 3.1.2. In fact, one can say more about the intersection form. Assume that the order on the set $\{R_j\}_{j=1,\ldots,n}$ is linear and that in the exceptional divisor $E(\pi)$ of the map π the irreducible component E_i intersects E_l for some $l > i$. Then

$$E_i.R_j = \begin{cases} -1 & \text{for } j = i, \\ 1 & \text{for } j = i+1, \ldots, l \\ 0 & \text{for } j > l. \end{cases}$$

In order to see it, we look at X_l, where the irreducible components of $E(\pi)$ intersect as $E_i \text{ —— } E_l \text{ —— } E_{l-1} \text{ —— } \ldots \text{ —— } E_{i+1}$. Then, for $j = i+1, \ldots, l$ the divisor $R_j = E_j + E_{j+1} + \ldots + E_l$. When we pass to X these coefficients do not change. Also, on X the divisors R_j with $j > l$ are sums of E_s with $s \geq j$.

It follows that

$$E_i^2 = \sharp\{j \mid E_i R_j = 1\} - 1.$$

Lemma 3.1.3. *Let $E_i E_{i+l} = 1$ for some $l \geq 1$. Then $R_i = E_i + \sum_{j=i+1}^{i+l} R_j$.*

Proof. Without loss of generality we can assume that the order on the set $\{R_l\}$ is linear, i.e. the map $\pi_l \colon X_l \to X_{l-1}$ blows up a point on $E_{l-1} \subset X_{l-1}$.

We look at the irreducible component E_i in the exceptional fiber of π. Assume that E_i intersects E_{i+l} for some $l > 0$. We have seen before that in this case in X_{i+l} the irreducible curves intersect as

$$E_i \text{ —— } E_{i+l} \text{ —— } E_{i+l-1} \text{ —— } \ldots \text{ —— } E_{i+1}.$$

Then, the pull back of E_i to X_l is equal to

$$E_i + E_{i+1} + 2E_{i+2} + \ldots + lE_{i+l} \; = \; E_i + R_{i+1} + R_{i+2} + \ldots + R_l.$$

The map from X to X_{i+l} blows up points on $E_{i+1}, \ldots, E_{i+l}$ and hence for $j > i+l$ the multiplicity of E_j in LHS coincides with multiplicity in RHS. □

Finally, the canonical divisor of X is $K_X = \pi^* K_{X_0} + \sum_i R_i$. In particular $R_i K_X = -1$ for any i.

3.2 Full exceptional collections on rational surfaces and their DG quivers

Let $\pi\colon X \to X_0$ be a map from a smooth rational surface to $\mathbb{P}^2$ or $\mathbb{F}_a$ and let $\langle \mathcal{O}_{X_0}, \mathcal{N}_1, \dots, \mathcal{N}_t\rangle$ be a full exceptional collection on X_0. By [54, Theorem 4.3] in this situation $\langle \mathcal{O}_{R_n}(R_n), \dots, \mathcal{O}_{R_1}(R_1), \mathcal{O}_X, \pi^*(\mathcal{N}_1), \dots, \pi^*(\mathcal{N}_t)\rangle$ is a full exceptional collection on X.

Let us understand the Ext-quiver of this collection.

3.2.1 Ext-quiver of $\langle \mathcal{O}_{R_n}(R_n), \dots, \mathcal{O}_{R_1}(R_1)\rangle$

Lemma 3.2.1. *For $\mathcal{O}_{R_i}(R_i)$ and $\mathcal{O}_{R_j}(R_j)$ we have*

$$\mathrm{Hom}(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_j}(R_j)) = \begin{cases} k & \textit{if } R_i \le R_j \\ 0 & \textit{otherwise;} \end{cases} \qquad \mathrm{Ext}^1(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_j}(R_j)) = \begin{cases} k & \textit{if } R_i < R_j \\ 0 & \textit{otherwise} \end{cases}$$

and the higher Ext-groups are always 0.

Proof. From the fact that sheaves $\mathcal{O}_{R_i}(R_i)$ are in the right orthogonal to $\mathcal{O}_X$ and short exact sequences

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(R_i) \to \mathcal{O}_{R_i}(R_i) \to 0$$

it follows that

$$\mathrm{Ext}^l(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_j}(R_j)) \simeq \mathrm{Ext}^l(\mathcal{O}_X(R_i), \mathcal{O}_X(R_j)) \simeq H^l(X, \mathcal{O}_X(R_j - R_i)).$$

From the above sequence we also conclude that $H^0(\mathcal{O}_X(R_i)) = k$ and $H^j(\mathcal{O}_X(R_i)) = 0$ for $j \neq 0$.

From Riemann-Roch theorem it follows that $\chi(R_j - R_i) = 0$.

Then, from the short exact sequence

$$0 \to \mathcal{O}_X(R_j - R_i) \to \mathcal{O}_X(R_j) \to \mathcal{O}_{R_i}(R_j) \to 0$$

we deduce that $H^0(\mathcal{O}_X(R_j - R_i)) = k = H^1(\mathcal{O}_X(R_j - R_i))$ if only the divisor $R_j - R_i$ is effective and all cohomology groups of $\mathcal{O}_X(R_j - R_i)$ are zero otherwise. $\square$

Now, let us calculate the composition of Hom and Ext^1 groups between $\mathcal{O}_{R_i}(R_i)$.

Clearly, the composition

$$\mathrm{Ext}^1(\mathcal{O}_{R_j}(R_j), \mathcal{O}_{R_i}(R_i)) \otimes \mathrm{Ext}^1(\mathcal{O}_{R_l}(R_l), \mathcal{O}_{R_j}(R_j)) \to \mathrm{Ext}^2(\mathcal{O}_{R_l}(R_l), \mathcal{O}_{R_i}(R_i))$$

is zero as groups $\mathrm{Ext}^2(\mathcal{O}_{R_l}(R_l), \mathcal{O}_{R_i}(R_i))$ are zero.

Remark 3.2.2. In order to understand the composition in the Ext-algebra of the exceptional collection $\langle \mathcal{O}_{R_n}(R_n), \ldots, \mathcal{O}_{R_1}(R_1)\rangle$ it is enough to understand it in the case when the partial order on the set of R_i's is in fact linear. To see it we consider the partially ordered set I of R_i's and a subset $J \subset I$ satisfying

- for $i, j \in J$ and $l \in I$ such that $i \leq l \leq j$ the element l also belongs to J,

- for $j \in J$ and $l \in I$ such that $j \leq l$ the element l also belongs to J.

Then, the map $\pi = \pi_I$ which leads to the poset I factors through a map π_J which gives a partial order J on divisors $\{R_j \,|\, j \in J\}$.

It is slightly easier to answer questions about compositions of Hom and Ext^1 groups in a different category. More precisely, let $\mathcal{C}_\pi$ denote the category generated by $\mathcal{O}_{R_n}(R_n), \ldots, \mathcal{O}_{R_1}(R_1)$. If we mutate this category to the right over $\mathcal{O}_X$ we obtain an equivalent category $\widetilde{\mathcal{C}_\pi}$ with a full exceptional collection $\langle \mathcal{O}_X(R_n), \ldots, \mathcal{O}_X(R_1)\rangle$. We will do the calculations in the category $\widetilde{\mathcal{C}_\pi}$.

Till the end of this section we will assume that the divisors $R_1, \ldots, R_n$ are linearly ordered. It means that the map $\pi_{j+1} \colon X_{j+1} \to X_j$ blows up a point on the exceptional divisor $E_j \subset X_j$.

Lemma 3.2.3. *Let $R_l < R_j < R_i$. The composition*

$$\mathrm{Hom}(\mathcal{O}_X(R_j), \mathcal{O}_X(R_i)) \otimes \mathrm{Hom}(\mathcal{O}_X(R_l), \mathcal{O}_X(R_j)) \to \mathrm{Hom}(\mathcal{O}_X(R_l), \mathcal{O}_X(R_i))$$

is always non-zero.

Proof. Let $\widetilde{\alpha_{lj}} \in \mathrm{Hom}(\mathcal{O}_X(R_l), \mathcal{O}_X(R_j))$ and $\widetilde{\alpha_{ji}} \in \mathrm{Hom}(\mathcal{O}_X(R_j), \mathcal{O}_X(R_i))$ be non-zero. The map $\widetilde{\alpha_{ji}}$ is injective and hence

$$\widetilde{\alpha_{ji}} \circ \widetilde{\alpha_{lj}} \neq 0.$$

□

Lemma 3.2.4. *Let $R_l < R_j < R_i$. The composition*

$$\mathrm{Ext}^1(\mathcal{O}_X(R_j), \mathcal{O}_X(R_i)) \otimes \mathrm{Hom}(\mathcal{O}_X(R_l), \mathcal{O}_X(R_j)) \to \mathrm{Ext}^1(\mathcal{O}_X(R_l), \mathcal{O}_X(R_i))$$

is always non-zero.

Proof. Consider non-zero elements $\widetilde{\alpha_{lj}} \in \mathrm{Hom}(\mathcal{O}_X(R_l), \mathcal{O}_X(R_j))$ and $\widetilde{\beta_{ji}} \in \mathrm{Ext}^1(\mathcal{O}_X(R_j), \mathcal{O}_X(R_i))$. The first one fits into a short exact sequence

$$0 \to \mathcal{O}_X(R_l) \to \mathcal{O}_X(R_j) \to \mathcal{O}_{R_j - R_l}(R_j) \to 0.$$

Applying $\mathrm{Hom}(-, \mathcal{O}_X(R_i))$ to it we get a long exact sequence from which it follows that

$$\widetilde{\beta_{ji}} \circ \widetilde{\alpha_{lj}} = 0 \Leftrightarrow \mathrm{Ext}^1(\mathcal{O}_{R_j - R_l}(R_j), \mathcal{O}_X(R_i)) = k.$$

We have short exact sequences

$$\begin{aligned}
&0 \to \mathcal{O}_X(R_j) \to \mathcal{O}_X(R_j + R_l) \to \mathcal{O}_{R_l}(R_j + R_l) \simeq \mathcal{O}_{R_l}(R_l) \to 0\\
&0 \to \mathcal{O}_X(R_l) \to \mathcal{O}_X(R_l + R_j) \to \mathcal{O}_{R_j}(R_j + R_l) \to 0,\\
&0 \to \mathcal{O}_{R_j - R_l}(R_j) \to \mathcal{O}_{R_j}(R_j + R_l) \to \mathcal{O}_{R_l}(R_l + R_j) \simeq \mathcal{O}_{R_l}(R_l) \to 0.
\end{aligned}$$

From the first sequence and the fact that $\mathrm{Hom}(\mathcal{O}_X(R_j + R_l), \mathcal{O}_X(R_i)) = 0$ it follows that $\mathrm{Ext}^1(\mathcal{O}_X(R_j + R_l), \mathcal{O}_X(R_i)) = k$. Then, the second sequence gives $\mathrm{Ext}^1(\mathcal{O}_{R_j}(R_l + R_j), \mathcal{O}_X(R_i)) = k$. Finally, from the last sequence we learn that $\mathrm{Ext}^1(\mathcal{O}_{R_j - R_l}(R_j), \mathcal{O}_X(R_i)) = 0$. Hence,

$$\widetilde{\beta_{ji}} \circ \widetilde{\alpha_{lj}} \neq 0.$$

$\square$

Lemma 3.2.5. *Let $R_l < R_j < R_i$. The composition*

$$\mathrm{Hom}(\mathcal{O}_X(R_j), \mathcal{O}_X(R_i)) \otimes \mathrm{Ext}^1(\mathcal{O}_X(R_l), \mathcal{O}_X(R_j)) \to \mathrm{Ext}^1(\mathcal{O}_X(R_l), \mathcal{O}_X(R_i))$$

is non-zero if and only if $R_i - R_j - E_l$ is not effective.

Proof. Let $\widetilde{\beta_{lj}} \in \mathrm{Ext}^1(\mathcal{O}_X(R_l), \mathcal{O}_X(R_j))$, $\widetilde{\alpha_{ji}} \in \mathrm{Hom}(\mathcal{O}_X(R_j), \mathcal{O}_X(R_i))$ be non-zero. $\widetilde{\alpha_{ji}}$ fits into an exact sequence

$$0 \to \mathcal{O}_X(R_j) \to \mathcal{O}_X(R_i) \to \mathcal{O}_{R_i - R_j}(R_i) \to 0.$$

Applying to it the functor $\mathrm{Hom}(\mathcal{O}_X(R_l), -)$ we see that

$$\widetilde{\alpha_{ji}} \circ \widetilde{\beta_{lj}} \neq 0 \Leftrightarrow \mathrm{Hom}(\mathcal{O}_X(R_l), \mathcal{O}_{R_i - R_j}(R_i)) = H^0(\mathcal{O}_{R_i - R_j}(R_i - R_l)) = 0.$$

Let $\psi_l \colon X \to X_l$ be the composition $\psi_l = \pi_{l+1} \circ \ldots \circ \pi_n$. The functor $L^\bullet \psi_l^* \colon D^b(X_l) \to D^b(X)$ is fully faithful and hence the composition $\widetilde{\alpha_{ji}} \circ \widetilde{\beta_{lj}}$ is a pullback of the composition of the corresponding morphisms on X_l. Thus, without loss of generality, we can assume that $l = n$, i.e. $R_l = E_l$.

First, we consider the case when the divisor $R_i - R_j - R_l = R_i - R_j - E_l$ is effective. Then,

$$0 \to \mathcal{O}_{E_l}(R_j) \to \mathcal{O}_{R_i - R_j}(R_j + R_i - R_j - E_l) \simeq \mathcal{O}_{R_i - R_j}(R_i - E_l) \to \mathcal{O}_{R_i - R_j - E_l}(R_i - E_l) \to 0$$

is exact.

From the fact that $E_l = R_l$ and $R_l.R_j = 0$ it follows that $\mathcal{O}_{E_l}(R_j) \simeq \mathcal{O}_{E_l}$ which shows that $k = H^0(\mathcal{O}_{E_l}(R_j)) \hookrightarrow H^0(\mathcal{O}_{R_i-R_j}(R_i - R_l))$.

Now, we consider the case when $R_i - R_j - R_l$ is not an effective divisor.

The divisor $R_i - R_j$ on X is a pullback of a divisor from X_j and recall that we assume that the order on the set of $\{R_s\}_{\{s=1,\dots,l\}}$ is linear. It follows that for $\pi_s \colon X_{s+1} \to X_s$ we have ${\pi_s}^*(E_s) = E_s + E_{s+1}$ (where $E_s \subset X_{s+1}$ denotes the strict transform of $E_s \subset X_s$). Hence, if the coefficient a_s in front of E_s in the presentation $R_i - R_j = \sum_{s=i}^{l} a_s E_s$ is non-zero for some $s \geq j$, so is the coefficient a_l. And if the coefficient a_l is non-zero then the divisor $R_i - R_j - E_l = R_i - R_j - R_l$ is effective.

Let us assume first that $j = i + 1$. By the above observation, the condition that $R_i - R_j - E_l$ is not effective is equivalent to the condition that $R_i - R_{i+1} = E_i$ (because $R_i = E_i + E_{i+1} + \sum_{s=i+2}^{l} b_s E_s$ and hence the coefficient in front of E_i in the difference $R_i - R_{i+1}$ is equal to 1 and the coefficient in front of E_{i+1} is zero). In this case

$$\mathcal{O}_{R_i-R_j}(R_i - E_l) = \mathcal{O}_{E_i}(R_i - E_l).$$

From Lemma 3.1.1 we know that $E_i.R_i = -1$. It remains to determine whether E_i intersects E_l.

Assume that it does. We assume that the order on the set of $\{R_s\}_{\{s=1,\dots,l\}}$ is linear from which it follows that some part of the exceptional divisor of π has irreducible components intersecting in the following order

$$E_i \text{ --- } E_l \text{ --- } E_{l-1} \text{ --- } \dots \text{ --- } E_{i+1}.$$

It follows that $R_i = E_i + E_{i+1} + \ldots + E_{l-1} + 2E_l$ and $R_j = E_j + \ldots + E_l$ which contradicts the condition $R_k - R_j - E_l$ not being effective. Hence, $E_l E_i = 0$.

We have thus proved that for $j = i + 1$

$$\mathcal{O}_{R_i-R_j}(R_i - E_l) = \mathcal{O}_{E_i}(R_i - E_l) = \mathcal{O}_{E_i}(-1)$$

which has no sections.

Now, we proceed to the general case when $j > i$. The short exact sequence

$$0 \to \mathcal{O}_{R_i-R_j-E_i}(R_i - E_l - E_i) \to \mathcal{O}_{R_i-R_j}(R_i - E_l) \to \mathcal{O}_{E_i}(R_i - E_l) \to 0$$

together with the observation that $\mathcal{O}_{E_i}(R_i - E_l) \simeq \mathcal{O}_{E_i}(-1)$ gives $H^s(\mathcal{O}_{R_i-R_j}(R_k - E_l)) = H^s(\mathcal{O}_{R_i-R_j-E_i}(R_i - E_l - E_i))$. We will argue that, under the assumption that $R_i - R_j - E_l$ is not effective, $R_i - E_i = R_{i+1}$ and then proceed by induction.

If $R_i - E_i \neq R_{i+1}$ the intersection point of E_i and E_{i+1} on X_{i+1} must be blown up by one of the maps π_s for some $s > i + 1$. We assume that the order on the set of R_s's is linear and hence this point must be blown up by $\pi_{i+1} \colon X_{i+2} \to X_{i+1}$. In this case

$R_i = E_i + E_{i+1} + 2E_{i+2} + \sum_{s=i+3}^{l} b_s E_s$ and $b_s \geq 2$ for $s \geq i+3$. In particular $b_j \geq 2$. We know also that $R_j = E_j + \sum_{s=i+1}^{l} c_s E_s$. It follows that $R_i - R_j$ considered as a divisor on X_j is equal to $(b_j - 1)E_j \neq 0$ and hence the coefficient in front of E_l of the pullback of this divisor to X_j is also non-zero. This contradicts our assumption. $\square$

Remark 3.2.6. From the condition that the coefficient in front of E_j in the difference $R_i - R_j$ is zero it does not follow that the coefficient in front of E_l is. In the example with irreducible components of the exceptional divisor of π intersecting as

$$E_1 \longrightarrow E_4 \longrightarrow E_3 \longrightarrow E_2$$

we have $R_1 = E_1 + E_2 + 2E_3 + 3E_4$, $R_2 = E_2 + E_3 + E_4$ and $R_1 - R_2 = E_1 + E_3 + 2E_4$.

On the other hand, under the assumption that the order of R_i's in linear the condition that $R_i - R_j - E_l$ is effective is equivalent to the condition that $R_i - R_j - E_{j+1}$ is. In the proof of Proposition 3.2.5 we have seen one implication. In the other direction, assume that $R_i - R_j - E_l$ is effective and $R_i - R_j - E_{j+1}$ is not. From the second assumption it follows that in X_{j+1} the divisor E_{j+1} intersects E_j and possibly some E_s for $s < i$. The order on R_p's is linear and hence $f_{j+1}\colon X_{j+2} \to X_{j+1}$ is a blow up of a point on $E_{j+1} \subset X_{j+1}$ and so on. In the first case, when E_{j+1} intersects only E_j, it follows that the coefficients in front of E_s's both in R_i and R_j will agree for $s \geq j$. Then, of course, $R_i - R_j - E_s$ will not be effective for $s \geq j$. In the second case, when $E_{j+1} \subset X_j$ intersects some E_s, we know that on X_j some part of the exceptional divisor of the map $X_j \to X_0$ has irreducible components intersecting as

$$E_s \longrightarrow E_{j+1} \longrightarrow E_j \longrightarrow \ldots \longrightarrow E_i \longrightarrow \ldots \longrightarrow E_{s+1}.$$

Hence, $R_i = E_i + E_{i+1} + \ldots + E_{j+1} + \sum_{s=j+2}^{l} b_s E_s$ and again we see that coefficients b_s agree with the coefficients c_s of $R_j = E_j + E_{j+1} + \sum_{s=l+2}^{l} c_s E_s$.

Finally, we will show that it is possible to choose basis of $\mathrm{Hom}(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_j}(R_j))$ and $\mathrm{Ext}^1(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_j}(R_j))$ in such a way that no parameters appear in the composition laws.

3.2.2 The basis of $\mathrm{Hom}(\mathcal{O}_{R_l}(R_l), \mathcal{O}_{R_j}(R_j))$ and $\mathrm{Ext}^1(\mathcal{O}_{R_l}(R_l), \mathcal{O}_{R_j}(R_j))$

For every $i \in \{1, \ldots, n\}$ let us fix all maps in the distinguished triangle

$$\mathcal{O}_X \xrightarrow{s_i} \mathcal{O}_X(R_i) \xrightarrow{t_i} \mathcal{O}_{R_i}(R_i) \xrightarrow{r_i} \mathcal{O}_X[1].$$

Definition 3.2.7. *For any $l > j$ let $\alpha_{lj}\colon \mathcal{O}_{R_l}(R_l) \to \mathcal{O}_{R_j}(R_j)$ be such that the diagram*

$$\begin{array}{ccc} \mathcal{O}_X[1] & \xrightarrow{id} & \mathcal{O}_X[1] \\ {\scriptstyle r_l}\uparrow & & \uparrow{\scriptstyle r_j} \\ \mathcal{O}_{R_l}(R_l) & \xrightarrow{\alpha_{lj}} & \mathcal{O}_{R_j}(R_j) \end{array}$$

commutes.

The maps α_{lj} give also maps $\widetilde{\alpha_{lj}}\colon \mathcal{O}_X(R_l) \to \mathcal{O}_X(R_j)$ defined by the commutative diagrams

$$\begin{tikzcd}
\mathcal{O}_{R_l}(R_l) \arrow[r, "\alpha_{lj}"] & \mathcal{O}_{R_j}(R_j) \\
\mathcal{O}_X(R_l) \arrow[u, "t_l"] \arrow[r, "\widetilde{\alpha_{lj}}"] & \mathcal{O}_X(R_j). \arrow[u, "t_j"']
\end{tikzcd}$$

Finally, for any $l > j$ we have $\mathcal{O}_{R_l}(R_j) \simeq \mathcal{O}_{R_l}$. We fix explicit isomorphisms

$$\psi_{lj}\colon \mathcal{O}_{R_l}(R_l) \to \mathcal{O}_{R_l}(R_l) \otimes \mathcal{O}_X(R_j)$$

such that for all $l > j > i$ diagrams

$$\begin{tikzcd}
\mathcal{O}_{R_l}(R_l) \otimes \mathcal{O}_{R_i}(R_i) \arrow[r, "\alpha_{lj}\otimes\mathrm{id}"] & \mathcal{O}_{R_j}(R_j) \otimes \mathcal{O}_X(R_i) \\
\mathcal{O}_{R_l}(R_l) \arrow[u, "\psi_{li}"] \arrow[r, "\alpha_{lj}"] & \mathcal{O}_{R_j}(R_j) \arrow[u, "\psi_{ji}"']
\end{tikzcd} \tag{3.1}$$

commute. Moreover, for triples $l > j > i$ such that some part of the exceptional divisor $E(\pi)$ of π has the form E_l —— $\ldots$ —— E_j —— $\ldots$ —— E_i let also the diagrams

$$\begin{tikzcd}
\mathcal{O}_{R_l}(R_l) \otimes \mathcal{O}_X(R_j) \arrow[r, "\mathrm{id}\otimes\widetilde{\alpha_{ji}}"] & \mathcal{O}_{R_l}(R_l) \otimes \mathcal{O}_X(R_i) \\
\mathcal{O}_{R_l}(R_l) \arrow[u, "\psi_{lj}"] \arrow[r, "\mathrm{id}"] & \mathcal{O}_{R_l}(R_l) \arrow[u, "\psi_{li}"']
\end{tikzcd} \tag{3.2}$$

commute. Notice, that under this condition $Z(\widetilde{\alpha_{ji}}) = E_j + \ldots + E_{i-1}$ and hence all maps in diagram (3.2) are invertible.

All the above conditions on the maps ψ_{lj} can be fulfilled. Indeed, the first set of diagrams tell us that fixing ψ_{li} we fix also $\psi_{(l-1)i}, \ldots, \psi_{(k+1)i}$. The second set of diagrams leads to relations between ψ_{lj} and ψ_{li}. However, these relations are not independent. To see it consider the diagram

$$\begin{tikzcd}
\mathcal{O}_{R_j}(R_j) \arrow[rrr, leftrightarrow, "\mathrm{id}"] \arrow[dr, leftrightarrow, "\psi_{ji}"] & & & \mathcal{O}_{R_j}(R_j) \\
& \mathcal{O}_{R_j}(R_j) \otimes \mathcal{O}_X(R_i) \arrow[r, leftrightarrow, "\mathrm{id}\otimes\widetilde{\alpha_{im}}"] & \mathcal{O}_{R_j}(R_j) \otimes \mathcal{O}_X(R_m) \arrow[ur, leftrightarrow, "\psi_{jm}"'] & \\
& \mathcal{O}_{R_l}(R_l) \otimes \mathcal{O}_X(R_i) \arrow[u, "\alpha_{lj}\otimes\mathrm{id}"] \arrow[r, leftrightarrow, "\mathrm{id}\otimes\widetilde{\alpha_{im}}"] & \mathcal{O}_{R_l}(R_l) \otimes \mathcal{O}_X(R_m) \arrow[u, "\alpha_{lj}\otimes\mathrm{id}"] \arrow[dr, leftrightarrow, "\psi_{lm}"] & \\
\mathcal{O}_{R_l}(R_l) \arrow[uuu, "\alpha_{lj}"] \arrow[rrr, leftrightarrow, "\mathrm{id}"] \arrow[ur, leftrightarrow, "\psi_{lk}"] & & & \mathcal{O}_{R_l}(R_l) \arrow[uuu, "\alpha_{lj}"']
\end{tikzcd}$$

Clearly, the inner and the outer squares commute. Moreover, if upper, lower and right squares commute the same is true about the left one.

Definition 3.2.8. *With the above morphisms chosen we define* $\beta_{ij}\colon \mathcal{O}_{R_i}(R_i) \to \mathcal{O}_{R_j}(R_j)[1]$ *as* $\beta_{ij} = t_j[1] \circ (r_i \otimes id) \circ \psi_{ij}$*,*

$$\mathcal{O}_{R_i}(R_i) \xrightarrow{\psi_{ij}} \mathcal{O}_{R_i}(R_i) \otimes \mathcal{O}_X(R_j) \xrightarrow{r_i \otimes id} \mathcal{O}_X[1] \otimes \mathcal{O}_X(R_j) \xrightarrow{t_j[1]} \mathcal{O}_{R_j}(R_j)[1].$$

Remark 3.2.9. Short exact sequences

$$0 \to \mathcal{O}_X(R_j - R_i) \to \mathcal{O}_X(R_j) \to \mathcal{O}_{R_i} \to 0$$

lead to identifications

$$\begin{aligned}
&\mathrm{Hom}(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_j}(R_j)) \simeq H^0(\mathcal{O}_X(R_j)) \simeq H^0(\mathcal{O}_X),\\
&\mathrm{Ext}^1(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_j}(R_j)) \simeq H^0(\mathcal{O}_{R_i}) \simeq H^0(\mathcal{O}_X).
\end{aligned}$$

Under these identifications maps α_{ij} and β_{ij} correspond to identity morphism in $\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X)$.

3.2.3 Parameters entering the composition in the basis

Proposition 3.2.10. *Let* $R_l < R_j < R_i$*. No parameters enter the composition*

$$\mathrm{Hom}(\mathcal{O}_{R_j}(R_j), \mathcal{O}_{R_i}(R_i)) \otimes \mathrm{Hom}(\mathcal{O}_{R_l}(R_l), \mathcal{O}_{R_j}(R_j)) \to \mathrm{Hom}(\mathcal{O}_{R_l}(R_l), \mathcal{O}_{R_i}(R_i)).$$

Proof. This is clear from the definition of the maps α_{lj}. □

Proposition 3.2.11. *Let* $R_l < R_j < R_i$*. With the above choice of basis no parameters appear in the composition*

$$\mathrm{Ext}^1(\mathcal{O}_{R_j}(R_j), \mathcal{O}_{R_i}(R_i)) \otimes \mathrm{Hom}(\mathcal{O}_{R_l}(R_l), \mathcal{O}_{R_j}(R_j)) \to \mathrm{Ext}^1(\mathcal{O}_{R_l}(R_l), \mathcal{O}_{R_i}(R_i)).$$

Proof. It follows the equality $r_j \otimes \alpha_{ij} = r_i$ and the fact that diagram (3.1) commutes;

$$\begin{tikzcd}
\mathcal{O}_{R_j}(R_j) \arrow[r, "\psi_{ji}"] \arrow[rrr, bend left, "\beta_{ji}"] & \mathcal{O}_{R_j}(R_j) \otimes \mathcal{O}_X(R_i) \arrow[r, "r_j\otimes \mathrm{id}"] & \mathcal{O}_X[1] \otimes \mathcal{O}_X(R_i) \arrow[r, "t_i[1]"] & \mathcal{O}_{R_i}(R_i)[1] \\
\mathcal{O}_{R_l}(R_l) \arrow[r, "\psi_{li}"] \arrow[u, "\alpha_{lj}"] \arrow[rrru, bend right, "\beta_{li}"] & \mathcal{O}_{R_l}(R_l) \otimes \mathcal{O}_X(R_i) \arrow[u, "\alpha_{lj}\otimes \mathrm{id}"] \arrow[ru, "r_l\otimes \mathrm{id}"] & &
\end{tikzcd}$$

□

Proposition 3.2.12. *Let* $R_l < R_j < R_i$ *be such that the composition*

$$\mathrm{Hom}(\mathcal{O}_{R_j}(R_j), \mathcal{O}_{R_i}(R_i)) \otimes \mathrm{Ext}^1(\mathcal{O}_{R_l}(R_l), \mathcal{O}_{R_j}(R_j)) \to \mathrm{Ext}^1(\mathcal{O}_{R_l}(R_l), \mathcal{O}_{R_i}(R_i)).$$

is non-zero. With the above choice of basis of these spaces no parameters appear in this composition.

Proof. It follows from the fact that diagram (3.2) commutes;

$$\begin{tikzcd}
\mathcal{O}_{R_l}(R_l) \arrow[r, "\psi_{li}"] \arrow[rrr, bend left, "\beta_{li}"] & \mathcal{O}_{R_l}(R_l)\otimes\mathcal{O}_X(R_i) \arrow[r, "r_l\otimes\mathrm{id}"] & \mathcal{O}_X[1]\otimes\mathcal{O}_X(R_i) \arrow[r, "t_i"] & \mathcal{O}_{R_i}(R_i)[1] \\
\mathcal{O}_{R_l}(R_l) \arrow[u, "\mathrm{id}"] \arrow[r, "\psi_{lj}"] \arrow[rrr, bend right, "\beta_{lj}"'] & \mathcal{O}_{R_l}(R_l)\otimes\mathcal{O}_X(R_j) \arrow[u, "\mathrm{id}\otimes\widetilde{\alpha_{ji}}"] \arrow[r, "r_l\otimes\mathrm{id}"] & \mathcal{O}_X[1]\otimes\mathcal{O}_X(R_j) \arrow[u, "\mathrm{id}\otimes\widetilde{\alpha_{ji}}"] \arrow[r, "t_j[1]"] & \mathcal{O}_{R_j}(R_j)[1] \arrow[u, "\alpha_{ji}[1]"]
\end{tikzcd}$$

□

3.2.4 Combinatorics of the composition

We describe a combinatorial way of determining the Ext-quiver of the collection $\langle\mathcal{O}_{R_n}(R_n),\ldots,\mathcal{O}_{R_1}(R_1)\rangle$.

To the map π we can associate two graphs $\mathcal{G}$ and $\mathcal{H}$. Both of them have vertices $\{1,\ldots,n\}$. Graph $\mathcal{G}$ is the dual graph to the graph of $E(\pi)$. It is not oriented; vertices i, j are connected by an edge if irreducible components E_i, E_j of the exceptional divisor $E(\pi)$ of π intersect.

Notice that an oriented graph with vertices $\{1,\ldots,n\}$ and no cycles allows us to define an order on the set $\{R_i\}_{i=1,\ldots,n}$ by putting $R_i < R_j$ if there is an arrow $i \to j$ and taking the transitive closure of this relation. Let $\mathcal{H}$ be a minimal graph such that the order on the set $\{R_i\}$ defined by it coincides with the partial order $R_i \geq R_j$ which was considered before.

The Ext-quiver of collection $\langle\mathcal{O}_{R_n}(R_n),\ldots,\mathcal{O}_{R_1}(R_1)\rangle$ is obtained from the graph $\mathcal{H}$ by changing a vertex i to $\mathcal{O}_{R_i}(R_i)$ and every arrow $i \to j$ to arrows $\alpha_{ij}\colon \mathcal{O}_{R_i}(R_i)\to\mathcal{O}_{R_j}(R_j)$ and $\beta_{ij}\colon \mathcal{O}_{R_i}(R_i)\to\mathcal{O}_{R_j}(R_j)[1]$. From the previous section we know that $\beta_{js}\circ\beta_{lj}=0$ for any $l\to j\to s$.

The path algebra of this quiver has generators in degree zero, α_{ls} defined as compositions of maps α_{ij} and generators in degree one, β_{lk} defined as $\beta_{ls}=\beta_{is}\circ\alpha_{li}$ where i is the vertex of $\mathcal{H}$ such that there exists an arrow $i\to s$ in $\mathcal{H}$.

We already know that

$$\alpha_{js}\circ\alpha_{ij}=\alpha_{is},\quad \beta_{js}\circ\alpha_{ij}=\beta_{is}.$$

It remains to determine when $\alpha_{js}\circ\beta_{ij}=\beta_{is}$ and when this composition is zero. In order to do it we consider a graph $\mathcal{H}^{\mathrm{op}}$ obtained from $\mathcal{H}$ by reverting all the arrows. Notice, that if $E(\pi)$ is connected then $\mathcal{H}^{\mathrm{op}}$ is a tree with the root 1.

Now we look at a vertex $\mathcal{O}_{R_s}(R_s)$ of $\mathcal{H}$. In the graph $\mathcal{H}^{\mathrm{op}}$ we can find all leaves l for which there exists a path from s to l in $\mathcal{H}^{\mathrm{op}}$. For any such leaf l let P_l be the set of all vertices of the path connecting s with l. Then the order on the set $\{R_i\}_{i\in P_l}$ is linear and

R_s is its maximal element. In the set $\{R_i\}_{i \in P_l}$ there exists an element $\mathcal{O}_{R_j}(R_j)$ such that for $t < j$ and $\forall\, i > t$ we have $\alpha_{ts} \circ \beta_{it} \neq 0$ and for $t \geq j$ and $\forall\, i > t$ $\alpha_{ts} \circ \beta_{it} = 0$. The index $j \in P_l$ is the minimal one such that on a path from s to j in the graph $\mathcal{G}$ there exists a vertex $t \in P_l$ such that $t > j$. If no such index exists then we put $j = l$.

Repeating this procedure for every leaf l will give all relations of the form $\alpha_{js} \circ \beta_{ij} = 0$. If we do the same algorithm for all vertices of the Ext-quiver we get all the relations.

The algorithm for any s finds j such that $R_s = E_s + \ldots + E_{j-1} + 2E_j + \ldots$. Then, by Remark 3.2.6, we know that for $t < j - 1$ and any $i > t$ the coefficient in front of E_i in $R_s - R_t$ is zero and that for $t \geq j - 1$ and any $i > t$ the coefficient in front of E_i in $R_s - R_t$ is not zero. By Proposition 3.2.5 we can then conclude about compositions of the corresponding morphisms and elements of the first Ext-groups.

Let us consider the following example. Let the graph $\mathcal{G}$ be

$$\begin{tikzcd}
 & & 6 \arrow[d, dash] & & & \\
7 \arrow[r, dash] & 2 \arrow[r, dash] & 3 \arrow[r, dash] & 1 \arrow[r, dash] & 4 \arrow[r, dash] & 5
\end{tikzcd}$$

Then the graph $\mathcal{H}^{\mathrm{op}}$ is

$$\begin{tikzcd}
 & 7 & & \\
6 & 3 \arrow[l] & 2 \arrow[l] \arrow[ul] & 1 \arrow[l] \arrow[dl] \\
 & 5 & 4 \arrow[l] &
\end{tikzcd}$$

and the Ext-quiver of the collection $\langle \mathcal{O}_{R_7}(R_7), \ldots, \mathcal{O}_{R_1}(R_1) \rangle$ is

$$\begin{tikzcd}
 & \mathcal{O}_{R_7}(R_7) \arrow[dr, shift left, "\alpha_{72}"] \arrow[dr, shift right, "\beta_{72}"'] & & \\
\mathcal{O}_{R_6}(R_6) \arrow[r, shift left, "\alpha_{63}"] \arrow[r, shift right, "\beta_{63}"'] & \mathcal{O}_{R_3}(R_3) \arrow[r, shift left, "\alpha_{32}"] \arrow[r, shift right, "\beta_{32}"'] & \mathcal{O}_{R_2}(R_2) \arrow[r, shift left, "\alpha_{21}"] \arrow[r, shift right, "\beta_{21}"'] & \mathcal{O}_{R_1}(R_1) \\
 & \mathcal{O}_{R_5}(R_5) \arrow[r, shift left, "\alpha_{54}"] \arrow[r, shift right, "\beta_{54}"'] & \mathcal{O}_{R_4}(R_4) \arrow[ur, shift left, "\alpha_{41}"] \arrow[ur, shift right, "\beta_{41}"'] &
\end{tikzcd}$$

- For $k = 1$ and $l = 5$ we have $P_5 = \{1, 4, 5\}$, $j = 5$ and hence

$$\beta_{54} \circ \alpha_{41} = \beta_{51}.$$

- For $k = 1$ and $l = 6$ we have $P_6 = \{1, 2, 3, 6\}$, $j = 2$ and hence

$$\beta_{32} \circ \alpha_{21} = 0, \quad \beta_{62} \circ \alpha_{21} = 0, \quad \beta_{63} \circ \alpha_{31} = 0.$$

- If $k = 1$ and $l = 7$ then $P_7 = \{1, 2, 7\}$, $j = 7$ and hence

$$\beta_{72} \circ \alpha_{21} = \beta_{71}.$$

- If $k = 2$ and $l = 6$ then $P_6 = \{2, 3, 6\}$, $j = 6$ and hence

$$\beta_{63} \circ \alpha_{32} = \beta_{62}.$$

In the remaining cases (for example $k = 2$, $l = 7$) we do not learn anything new about compositions as there is not enough arrows to compose.

3.2.5 Maps from $\mathcal{O}_{R_k}(R_k)$ to $L^{\bullet}\pi^*\mathcal{D}^b(X_0)$

It remains to understand what are the maps from $\mathcal{O}_{R_s}(R_s)$ to $\pi^*\mathcal{N}_i$ and the compositions between them.

As $\pi^*\mathcal{N}_i$ are torsion-free we know that $\mathrm{Hom}(\mathcal{O}_{R_s}(R_s), \pi^*\mathcal{N}_i) = 0$. From the short exact sequence

$$0 \to \pi^*\mathcal{N}_i \to \pi^*\mathcal{N}_i \otimes \mathcal{O}_X(R_s) \to \mathcal{O}_{R_s}(R_s) \to 0. \tag{3.3}$$

we deduce that $\mathrm{Ext}^1(\mathcal{O}_{R_s}(R_s), \pi^*\mathcal{N}_i) \simeq \mathrm{Hom}(\mathcal{O}_{R_s}(R_s), \mathcal{O}_{R_s}(R_s)) = k$. Let ζ_s^i denote the non-zero element of this group.

The diagram

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & & \mathcal{O}_{R_s-R_j}(R_s) \arrow[u] \arrow[r, "="] & \mathcal{O}_{R_s-R_j}(R_s) \arrow[u] & \\
0 \arrow[r] & \pi^*\mathcal{N}_i \arrow[r] & \pi^*\mathcal{N}_i \otimes \mathcal{O}_X(R_s) \arrow[u] \arrow[r] & \mathcal{O}_{R_s}(R_s) \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \pi^*\mathcal{N}_i \arrow[u, "="] \arrow[r] & \pi^*\mathcal{N}_i \otimes \mathcal{O}_X(R_j) \arrow[u] \arrow[r] & \mathcal{O}_{R_j}(R_j) \arrow[u] \arrow[r] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

shows that the composition

$$\mathrm{Hom}(\mathcal{O}_{R_j}(R_j), \mathcal{O}_{R_s}(R_s)) \otimes \mathrm{Ext}^1(\mathcal{O}_{R_s}(R_s), \pi^*\mathcal{N}_i) \to \mathrm{Ext}^1(\mathcal{O}_{R_j}(R_j), \pi^*\mathcal{N}_i)$$

is an isomorphism.

To understand the composition

$$\mathrm{Ext}^1(\mathcal{O}_{R_s}(R_s), \pi^*\mathcal{N}_i) \otimes \mathrm{Hom}(\pi^*\mathcal{N}_i, \pi^*\mathcal{N}_l) \to \mathrm{Ext}^1(\mathcal{O}_{R_s}(R_s), \pi^*\mathcal{N}_l)$$

we apply the functor $\mathrm{Hom}(-, \pi^*\mathcal{N}_l)$ to the short exact sequence (3.3). It follows that for $\phi \in \mathrm{Hom}(\pi^*\mathcal{N}_i, \pi^*\mathcal{N}_l)$ the composition $\phi \circ \zeta_s^i$ is zero if and only if ϕ factors through $\pi^*\mathcal{N}_l(-R_s)$.

3.2.6 DG quiver of $\langle \mathcal{O}_{R_n}(R_n)[-1], \ldots, \mathcal{O}_{R_1}(R_1)[-1], \mathcal{O}_X, \pi^*\mathcal{N}_1, \ldots, \pi^*\mathcal{N}_t \rangle$

Now, we will present calculations allowing to determine the DG quiver of the full exceptional collection on X. We will assume that the collection $\langle \mathcal{O}_{X_0}, \mathcal{N}_1, \ldots, \mathcal{N}_t \rangle$ on X_0 is strong.

To calculate the DG category of the collection $\langle \mathcal{O}_{R_n}(R_n)[-1], \ldots, \mathcal{O}_{R_1}(R_1)[-1], \mathcal{O}_X, \mathcal{N}_1, \ldots, \mathcal{N}_t \rangle$ we substitute some objects with universal coextensions.

3.2.7 Tilting object

Note that if $R_1 > R_2$ we have a unique non-trivial extension

$$0 \to \mathcal{O}_{R_1}(R_1) \to \mathcal{O}_{R_1+R_2}(R_1+R_2) \to \mathcal{O}_{R_2}(R_2) \to 0.$$

Hence $\mathcal{O}_{R_1+R_2}(R_1+R_2)$ is the universal coextension of $\mathcal{O}_{R_1}(R_1)$ by $\mathcal{O}_{R_2}(R_2)$.

We will show that for $R_{i_k} < \ldots < R_{i_1} < R_s$ the universal coextension of $\mathcal{O}_{R_{i_1}+\ldots+R_{i_l}}(R_{i_1}+\ldots+R_{i_l})$ by $\mathcal{O}_{R_s}(R_s)$ is $\mathcal{O}_{R_s+R_{i_1}+\ldots+R_{i_l}}(R_s+R_{i_1}+\ldots+R_{i_l})$.

Theorem 3.2.13. *Let $\langle \mathcal{O}_{R_n}(R_n)[-1], \ldots, \mathcal{O}_{R_1}(R_1)[-1], \mathcal{O}_X, \pi^*\mathcal{N}_1, \ldots, \pi^*\mathcal{N}_t \rangle$ be an exceptional collection on X such that $\langle \mathcal{O}_{X_0}, \mathcal{N}_1, \ldots, \mathcal{N}_t \rangle$ is a strong exceptional collection on X_0. Then*

$$\mathcal{O}_{S_n}(S_n)[-1] \oplus \mathcal{O}_{S_{n-1}}(S_{n-1})[-1] \oplus \ldots \oplus \mathcal{O}_{S_1}(S_1)[-1] \oplus \mathcal{O}_X \oplus \pi^*\mathcal{N}_1 \oplus \ldots \oplus \pi^*\mathcal{N}_t$$

is tilting on X, where S_k are defined as

$$S_l = \sum_{R_j \leq R_l} R_j.$$

To prove Theorem 3.2.13 we shall need the following Lemma.

Lemma 3.2.14. *For $R_i < R_j < R_l$ we have*

$$\begin{aligned}
&Hom(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_l+\ldots+R_j}(R_l+\ldots+R_j)) \simeq \\
&Hom(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_{j+1}}(R_{j+1})) \otimes Hom(\mathcal{O}_{R_{j+1}}(R_{j+1}), \mathcal{O}_{R_l+\ldots+R_j}(R_l+\ldots+R_j)) = k, \\
&Ext^1(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_l+R_{l+1}+\ldots+R_j}(R_l+R_{l+1}+\ldots+R_j)) \simeq \\
&Hom^1(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_{j+1}}(R_{j+1})) \otimes Ext^1(\mathcal{O}_{R_{j+1}}(R_{j+1}), \mathcal{O}_{R_l+\ldots+R_j}(R_l+\ldots+R_j)) = k,
\end{aligned}$$

where the sum $R_l+\ldots+R_j$ is taken over all divisors R_s such that $j \succeq s \succeq l$.

Proof. We proceed by induction. The basis case, for $j = l$ follows from Lemma 3.2.4. The induction step follows from applying the functor $\mathrm{Hom}(\mathcal{O}_{R_i}(R_i), -)$ to the short exact sequence

$$0 \to \mathcal{O}_{R_l+\ldots+R_{j-1}}(R_l+\ldots+R_{j-1}) \to \mathcal{O}_{R_l+\ldots+R_j}(R_l+\ldots+R_j) \to \mathcal{O}_{R_j}(R_j) \to 0. \quad (3.4)$$

□

Proof of Theorem 3.2.13. From the above lemma and the short exact sequence (3.4) it follows that if $R_i < R_j < R_l$ the sheaf $\mathcal{O}_{R_l+\ldots+R_j}(R_l+\ldots+R_j)$ is the universal coextension of $\mathcal{O}_{R_l+\ldots+R_{j-1}}(R_l+\ldots+R_{j-1})$ by $\mathcal{O}_{R_i}(R_i)$. Hence, by the construction of universal coextensions the object

$$\mathcal{O}_{S_n}(S_n)[-1] \oplus \mathcal{O}_{S_{n-1}}(S_{n-1})[-1] \oplus \ldots \oplus \mathcal{O}_{S_1}(S_1)[-1] \oplus \mathcal{O}_X \oplus \pi^*\mathcal{N}_1 \oplus \ldots \oplus \pi^*\mathcal{N}_t$$

is tilting on X. $\square$

We describe how to calculate the endomorphism algebra of $\bigoplus \mathcal{O}_{S_j}(S_j)$ from the intersection form on irreducible components E_i's.

Lemma 3.2.15. *Assume that $R_n < R_{n-1} < \ldots < R_1$. Then*

$$\dim \operatorname{Hom}(\mathcal{O}_{R_i+\ldots+R_n}(R_i+\ldots+R_n), \mathcal{O}_{R_j+\ldots+R_n}(R_j+\ldots+R_n)) = n - \max\{i,j\} + 1.$$

Proof. First, consider the case when $i \leq j$. Because $H^j(\mathcal{O}_{R_j+\ldots+R_n}(R_j+\ldots+R_n)) = 0$ for all j

$$\hom(\mathcal{O}_{R_i+\ldots+R_n}(R_i+\ldots+R_n), \mathcal{O}_{R_j+\ldots+R_n}(R_j+\ldots+R_n)) = h^0(\mathcal{O}_{R_j+\ldots+R_n}(-R_i-\ldots-R_{j-1})).$$

Because $i < j-1 < j$ we have an isomorphism $\mathcal{O}_{R_j+\ldots+R_n}(-R_i-\ldots-R_{j-1}) \simeq \mathcal{O}_{R_j+\ldots+R_n}$. Short exact sequence

$$0 \to \mathcal{O}_{R_{j+1}+\ldots+R_n}(-R_j) \to \mathcal{O}_{R_j+\ldots+R_n} \to \mathcal{O}_{R_j} \to 0$$

together with an isomorphism $\mathcal{O}_{R_{j+1}+\ldots+R_n}(-R_j) \simeq \mathcal{O}_{R_{j+1}+\ldots+R_n}$ show that indeed $h^0(\mathcal{O}_{R_j+\ldots+R_n}) = n - j + 1$.

Now, assume that $i > j$. Then as before

$$\hom(\mathcal{O}_{R_i+\ldots+R_n}(R_i+\ldots+R_n), \mathcal{O}_{R_j+\ldots+R_n}(R_j+\ldots+R_n)) = h^0(\mathcal{O}_{R_j+\ldots+R_n}(R_j+\ldots+R_{i-1})).$$

From the short exact sequence

$$0 \to \mathcal{O}_{R_i+\ldots+R_n} \to \mathcal{O}_{R_j+\ldots+R_n}(R_j+\ldots+R_{i-1}) \to \mathcal{O}_{R_j+\ldots+R_{i-1}}(R_j+\ldots+R_{i-1}) \to 0$$

and the fact that $R^\bullet\pi_*(\mathcal{O}_{R_j+\ldots+R_{i-1}}(R_j+\ldots+R_{i-1})) = 0$ it follows that $h^0(\mathcal{O}_{R_j+\ldots+R_n}(R_j+\ldots+R_{i-1})) = n - i + 1$. $\square$

It is easy to give the basis of morphisms between $\mathcal{O}_{S_i}(S_i)$. If i is smaller or equal than j we have a map

$$\gamma^l_{i,j} \colon \mathcal{O}_{R_i+\ldots+R_n}(R_i+\ldots+R_n) \twoheadrightarrow \mathcal{O}_{R_l+\ldots+R_n}(R_l+\ldots+R_n) \hookrightarrow \mathcal{O}_{R_j+\ldots+R_n}(R_j+\ldots+R_n)$$

for $l = j, j+1, \ldots, n$.

If i is greater than j then the same map is defined for $l = i, i+1, \ldots, n$.

Still assuming that the partial order on R_i's is linear we denote by

$$\alpha_i\colon \mathcal{O}_{R_{i+1}+\ldots+R_n}(R_{i+1}+\ldots+R_n) \hookrightarrow \mathcal{O}_{R_i+\ldots+R_n}(R_i+\ldots+R_n),$$
$$\beta_i\colon \mathcal{O}_{R_i+\ldots+R_n}(R_i+\ldots+R_n) \twoheadrightarrow \mathcal{O}_{R_{i+1}+\ldots+R_n}(R_{i+1}+\ldots+R_n)$$

Proposition 3.2.16. *If $R_n < R_{n-1} < \ldots < R_1$ the endomorphism algebra of $\bigoplus \mathcal{O}_{S_i}(S_i)$ is a path algebra of the quiver*

$$\begin{tikzcd} \mathcal{O}_{S_n}(S_n) \arrow[r, shift left, "\alpha_{n-1}"] & \mathcal{O}_{S_{n-1}}(S_{n-1}) \arrow[l, shift left, "\beta_{n-1}"] \arrow[r, shift left] & \ldots \arrow[l, shift left] \arrow[r, shift left] & \mathcal{O}_{S_2}(S_2) \arrow[l, shift left] \arrow[r, shift left, "\alpha_1"] & \mathcal{O}_{S_1}(S_1) \arrow[l, shift left, "\beta_1"] \end{tikzcd}$$

with

$$\beta_{n-1}\circ\alpha_{n-1} = 0, \quad \beta_i \circ \alpha_i = \begin{cases} \alpha_{i+1}\circ\ldots\circ\alpha_{i+j}\circ\beta_{i+j}\circ\ldots\circ\beta_{i+1} & i+j<n \\ 0 & i+j=n \end{cases} \tag{3.5}$$

where j is such that $E_iE_{i+j} = 1$.

Proof. First, notice that $\beta_i\alpha_i$ is a map

$$\mathcal{O}_{R_{i+1}+\ldots+R_n}(R_{i+1}+\ldots+R_n) \hookrightarrow \mathcal{O}_{R_i+\ldots+R_n}(R_i+\ldots+R_n) \twoheadrightarrow \mathcal{O}_{R_{i+1}+\ldots+R_n}(R_{i+1}+\ldots+R_n).$$

The map β_i has a kernel $\mathcal{O}_{R_i}(R_i)$. Therefore the kernel of $\beta_i\alpha_i$ is $\mathcal{O}_D(D)$ where D is such that the positive part of $R_{i+1}+\ldots+R_n-R_i$ is equal to $R_{i+1}+\ldots+R_n-D$. The positive part of $R_{i+1}+\ldots+R_n-R_i$ is $R_{i+1}+\ldots+R_n-R_i+E_i$. By Lemma 3.1.3 $R_i-E_i = R_{i+1}+\ldots+R_{i+j}$ for $j>0$ such that $E_iE_{i+j} = 1$. Hence, $D = R_{i+1}+\ldots+R_{i+j}$ and the map $\beta_i\alpha_i$ factors through the surjection $\mathcal{O}_{R_i+\ldots+R_n}(R_i+\ldots+R_n) \twoheadrightarrow \mathcal{O}_{R_{i+j+1}+\ldots+R_n}(R_{i+j+1}+\ldots+R_n)$.

Because there are $n-2$ relations of this form and $2n-2$ arrows the basis elements α_i's and β_j's can be chosen in such a way that no parameters enter the picture.

The maps $\gamma^l_{i,j}$ given before are

$$\gamma^l_{i,j} = \alpha_j\circ\ldots\circ\alpha_{l-1}\circ\beta_{l-1}\circ\ldots\circ\beta_i.$$

Given relations allow to present every path in a form in which all α's are to the left. Thus, the relations allow us to reduce any path in the quiver to a basis morphism and hence all other relations between paths follow from relations 3.5. $\square$

Remark 3.2.17. When the order on the set $R_1, \ldots, R_n$ is not linear parameters can occur in the endomorphism algebra of $\bigoplus \mathcal{O}_{S_i}(S_i)$. For example, let us consider a surface X_4 obtained from $\mathbb{P}^2$ in the following five blow-ups of points.

$\pi_1\colon X_1 \to \mathbb{P}^2$ is a blow up of a point. $\pi_2\colon X_2 \to X_1$ is a blow up of a point on the exceptional divisor of π_1. $\pi_3\colon X_3 \to X_2$ is a blow up of the intersection of the exceptional divisor of π_2 and the strict transform of the exceptional divisor of π_1. Finally, $\pi_4\colon X_4 \to X_3$ is a blow up of two points on the exceptional divisor of π_3.

If we denote by E_i strict transform of the exceptional divisor of π_i for $i = 1, 2, 3$ and by E_4, E_5 irreducible components of the exceptional divisor of π_4 then the intersection form is

$$
\begin{array}{lllll}
E_1^2 = -3, & E_1E_2 = 0, & E_1E_3 = 1, & E_1E_4 = 0, & E_1E_5 = 0, \\
E_2^2 = -2, & E_2E_3 = 1, & E_2E_4 = 0, & E_2E_5 = 0, & E_3^2 = -3, \\
E3E_4 = 1, & E_3E_5 = 1, & E_4^2 = -1, & E_4E_5 = 0, & E_5^2 = -1.
\end{array}
$$

The divisors R_i are

$$
\begin{aligned}
R_1 &= E_1 + E_2 + 2E_3 + 2E_4 + 2E_5, \\
R_2 &= E_2 + E_3 + E_4 + E_5, \\
R_3 &= E_3 + E_4 + E_5, \\
R_4 &= E_4, \\
R_5 &= E_5
\end{aligned}
$$

and the endomorphism algebra of $\mathcal{O}_{S_i}(S_i)$ is given by the following quiver

$$
\begin{tikzcd}
\mathcal{O}_{S_4}(S_4) \arrow[dr, shift left, "\alpha_{43}"] & & & \\
& \mathcal{O}_{S_3}(S_3) \arrow[ul, shift left, "\alpha_{34}"] \arrow[dl, shift left, "\alpha_{35}"] \arrow[r, shift left, "\alpha_{32}"] & \mathcal{O}_{S_2}(S_2) \arrow[l, shift left, "\alpha_{23}"] \arrow[r, shift left, "\alpha_{21}"] & \mathcal{O}_{S_1}(S_1) \arrow[l, shift left, "\alpha_{12}"] \\
\mathcal{O}_{S_5}(S_5) \arrow[ur, shift left, "\alpha_{53}"] & & &
\end{tikzcd}
$$

with relations

$$
\begin{array}{ll}
\alpha_{34}\alpha_{43} = 0, & \alpha_{23}\alpha_{32} = a\,\alpha_{53}\alpha_{35} + b\,\alpha_{43}\alpha_{34}, \\
\alpha_{15}\alpha_{53} = 0, & \alpha_{12}\alpha_{21} = c\,\alpha_{32}\alpha_{53}\alpha_{35}\alpha_{23} + d\,\alpha_{32}\alpha_{43}\alpha_{34}\alpha_{23}
\end{array}
$$

for some $a, b, c, d \in \mathbb{C}^*$. Changing basis by

$$
\widetilde{\gamma_{53}} = a\gamma_{53}, \qquad \widetilde{\gamma_{43}} = b\gamma_{43}, \qquad \widetilde{\gamma_{12}} = \frac{a}{c}\gamma_{12}
$$

we obtain relations

$$
\begin{array}{ll}
\alpha_{34}\alpha_{43} = 0, & \alpha_{23}\alpha_{32} = \alpha_{53}\alpha_{35} + \alpha_{43}\alpha_{34}, \\
\alpha_{15}\alpha_{53} = 0, & \alpha_{12}\alpha_{21} = \alpha_{32}\alpha_{53}\alpha_{35}\alpha_{23} + \lambda\,\alpha_{32}\alpha_{43}\alpha_{34}\alpha_{23}
\end{array}
$$

for

$$
\lambda = \frac{ad}{bc}.
$$

3.2.8 Ext^1 groups between $\mathcal{O}_{S_l}(S_l)$ and $\pi^*\mathcal{N}_i$

Lemma 3.2.18. *Let $R_{i_l} < R_{i_{l-1}} < \ldots < R_{i_1}$. Then*

$$Ext^1(\mathcal{O}_{R_{i_1}+\ldots+R_{i_l}}(R_{i_1}+\ldots+R_{i_l}), \pi^*\mathcal{N}_i) = k^l$$

and the remaining Ext groups are zero.

Proof. We proceed by induction. The short exact sequence

$$0 \to \mathcal{O}_{R_{i_1}+\ldots+R_{i_{l-1}}}(R_{i_1}+\ldots+R_{i_{l-1}}) \to \mathcal{O}_{R_{i_1}+\ldots+R_{il}}(R_{i_1}+\ldots+R_{i_l}) \to \mathcal{O}_{R_{i_l}}(R_{i_l}) \to 0$$

together with an equality

$$\mathrm{Ext}^i(\mathcal{O}_{R_{i_l}}(R_{i_l}), \pi^*\mathcal{N}_i) = \mathrm{Ext}^i(\mathcal{O}_{E_{i_l}}(E_{i_l}), \pi^*\mathcal{N}_i)$$

completes the proof. □

If we apply the functor $\mathrm{Hom}(\mathcal{O}_{S_j}(S_j), -)$ to the short exact sequence

$$0 \to \pi^*\mathcal{N}_i \to \pi^*\mathcal{N}_i \otimes \mathcal{O}_X(S_l) \to \mathcal{O}_{S_l}(S_l) \to 0$$

we get an isomorphism

$$\mathrm{Ext}^1(\mathcal{O}_{S_l}(S_l), \pi^*\mathcal{N}_i) \simeq \mathrm{Hom}(\mathcal{O}_{S_l}(S_l), \mathcal{O}_{S_l}(S_l)). \tag{3.6}$$

The identity morphism in the latter space corresponds to an element $\zeta_k^i \in \mathrm{Ext}^1(\mathcal{O}_{S_l}(S_l), \pi^*\mathcal{N}_i)$.

The diagram

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & & \mathcal{O}_{S_j-S_l}(S_j) \arrow[r, "="] \arrow[u] & \mathcal{O}_{S_j-S_l}(S_j) \arrow[u] & \\
0 \arrow[r] & \mathcal{N}_i \arrow[r] & \mathcal{N}_i \otimes \mathcal{O}_X(S_j) \arrow[r] \arrow[u] & \mathcal{O}_{S_j}(S_j) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{N}_i \arrow[r] \arrow[u, "="] & \mathcal{N}_i \otimes \mathcal{O}_X(S_l) \arrow[r] \arrow[u] & \mathcal{O}_{S_l}(S_l) \arrow[r] \arrow[u, "\iota"] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

shows that for an inclusion $\iota\colon \mathcal{O}_{S_l}(S_l) \to \mathcal{O}_{S_j}(S_j)$ we have $\zeta_j^i \circ \iota = \zeta_l^i$.

The isomorphism (3.6) allows also to calculate the Yoneda composition

$$\mathrm{Hom}(\pi^*\mathcal{N}_i, \pi^*\mathcal{N}_k) \otimes \mathrm{Ext}^1(\mathcal{O}_{S_l}(S_l), \pi^*\mathcal{N}_i) \to \mathrm{Ext}^1(\mathcal{O}_{S_l}(S_l), \pi^*\mathcal{N}_k).$$

Thus, if the collection $\langle \mathcal{O}_{X_0}, \mathcal{N}_1, \ldots, \mathcal{N}_t \rangle$ on X_0 is strong we know the endomorphism algebra of the tilting object

$$\mathcal{O}_{S_n}(S_n)[-1] \oplus \mathcal{O}_{S_{n-1}}(S_{n-1})[-1] \oplus \ldots \oplus \mathcal{O}_{S_1}(S_1)[-1] \oplus \mathcal{O}_X \oplus \pi^*\mathcal{N}_1 \oplus \ldots \oplus \pi^*\mathcal{N}_t.$$

Using twisted complexes one can then calculate the DG quiver of the collection $\langle \mathcal{O}_{R_n}(R_n)[-1], \ldots, \mathcal{O}_{R_1}(R_1)[-1], \ \mathcal{O}_X, \pi^*\mathcal{N}_1, \ldots, \pi^*\mathcal{N}_t \rangle$ and of any of its mutations.

3.3 Canonical DG algebras of toric surfaces

We apply results of the previous section to full exceptional collections on smooth toric surfaces.

3.3.1 Toric surfaces

We recall some information about toric surfaces. More details can be found for example in [19, Chapter 1].

A smooth projective toric surface Y is determined by its fan, spanned by a collection of elements $\rho_1, \ldots, \rho_n$ in a lattice $N = \mathrm{Hom}(k^*, T) \cong \mathbb{Z}^2$, where $T = (k^*)^2$ is a two-dimensional torus. We enumerate ρ_i's clockwise and consider their indexes, i's, to be elements of $\mathbb{Z}/n\mathbb{Z}$. Then, for every $i \in \mathbb{Z}/n\mathbb{Z}$, vectors ρ_i and ρ_{i+1} form an oriented basis of N. Moreover, for every such pair there is no other ρ_k lying in the rational polyhedral cone generated by ρ_i and ρ_{i+1} in $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$.

There is a one-to-one correspondence between one-dimensional orbits of the T-action on Y and the rays in the fan generated by ρ_i's. For every i we denote by D_i the closure of this orbit. Then D_i's are T-invariant divisors on X. Every D_i is isomorphic to $\mathbb{P}^1$ and the intersection form is given by

$$D_i D_j = \begin{cases} a_i & \text{if } i = j, \\ 1 & \text{if } j \in \{i-1, i+1\} \\ 0 & \text{otherwise,} \end{cases}$$

where $a_i \in \mathbb{Z}$ are such that $\rho_{i-1} + a_i \rho_i + \rho_{i+1} = 0$. Conversely, the numbers $(a_1, \ldots, a_n)$ determine the toric surface Y.

Divisors D_i and D_{i+1} intersect transversely in a T-fixed point p_i corresponding to the cone spanned by vectors ρ_i and ρ_{i+1}.

A surface Y_1 obtained from Y by a blow-up of a torus-fixed point p_i is again a toric surface. The fan of Y_1 is determined by vectors $\rho_1, \ldots, \rho_i, \rho_i + \rho_{i+1}, \rho_{i+1}, \ldots, \rho_n$. Moreover, every toric surface different from $\mathbb{P}^2$ can be obtained from some Hirzebruch surface $\mathbb{F}_a$ by a finite sequence of blow-ups of T-fixed points.

A canonical divisor of a toric surface is given by $K_Y = -\sum_{i=1}^n D_i$. The Picard group of Y is $\mathrm{Pic}(Y) = \mathbb{Z}^{n-2}$.

3.3.2 Exceptional collections on toric surfaces

The a-th Hirzebruch surface $\mathbb{F}_a$ has a fan with four vectors and we can assume that $w_1 = (1,0)$, $w_2 = (0,-1)$, $w_3 = (-1,a)$ and $w_4 = (0,1)$. The collection $\langle \mathcal{O}_{\mathbb{F}_a}, \mathcal{O}_{\mathbb{F}_a}(D_1), \mathcal{O}_{\mathbb{F}_a}(D_1+D_2), \mathcal{O}_{\mathbb{F}_a}(D_1+D_2+D_3)\rangle$ is a full strong exceptional collection on $\mathbb{F}_a$.

If Y is obtained from $\mathbb{F}_a$ by a sequence of T-equivariant blow-ups then we can assume that the vectors $\rho_1,\ldots,\rho_n$ determining Y are numbered in such a way that $\rho_n = w_4 = (0,1)$. Then the collection $\langle \mathcal{O}_Y, \mathcal{O}_Y(D_1), \mathcal{O}_Y(D_1+D_2), \ldots, \mathcal{O}_Y(D_1+\ldots+D_{n-1})\rangle$ on Y is also full (see [26, Proposition 5.5]). The following lemma tells us that in fact the numeration of T-invariant divisors is not important.

Lemma 3.3.1. *(cf. [9, Theorem 4.1]) Let $\langle \mathcal{E}_1,\ldots,\mathcal{E}_n\rangle$ be a full exceptional collection on a smooth projective variety Z of dimension m. Then the n-fold mutation of $\mathcal{E}_n$ to the left is $L^n\mathcal{E}_n = \mathcal{E}_n \otimes \omega_Z[m-n]$, where ω_Z is the canonical line bundle on Z.*

Let $\sigma_1 = \langle \mathcal{O}_Y, \mathcal{O}_Y(D_1), \mathcal{O}_Y(D_1+D_2), \ldots, \mathcal{O}_Y(D_1+\ldots+D_{n-1})\rangle$ be a full exceptional collection on Y. Then, by the above lemma

$$L^n\, \mathcal{O}_Y(D_1+\ldots+D_{n-1}) = \mathcal{O}_Y(-D_n)[2-n].$$

Hence, σ_1 can be mutated to a collection

$$\langle \mathcal{O}_Y(-D_n)[2-n], \mathcal{O}_Y, \mathcal{O}_Y(D_1), \mathcal{O}_Y(D_1+D_2), \ldots, \mathcal{O}_Y(D_1+\ldots+D_{n-2})\rangle$$

which, in turn, after a shift and a twist by $\mathcal{O}_Y(D_n)$ is equivalent to the collection

$$\sigma_n = \langle \mathcal{O}_Y, \mathcal{O}_Y(D_n), \mathcal{O}_Y(D_n+D_1), \ldots, \mathcal{O}_Y(D_n+D_1+\ldots+D_{n-2})\rangle\,.$$

One can repeat this operation and obtain full exceptional collections

$$\sigma_i = \langle \mathcal{O}_Y, \mathcal{O}_Y(D_i), \ldots, \mathcal{O}_Y(D_i+\ldots+D_{i+n-2})\rangle$$

for any $i \in \mathbb{Z}/n$.

3.3.3 Canonical DG algebra of a toric surface

Let $Z = \operatorname{Tot}\omega_Y$ be the total space of the canonical bundle on Y and let $p\colon Z \to Y$ denote the canonical projection. As the vector bundle

$$\mathcal{E} = \mathcal{O}_Y \oplus \mathcal{O}_Y(D_1) \oplus \ldots \oplus \mathcal{O}_Y(D_1+\ldots+D_{n-1})$$

is a generator of $D^b(Y)$, we know that $p^*(\mathcal{E})$ is a generator of $D^b(Z)$. Moreover,

$$\begin{aligned}\operatorname{Hom}_Z(p^*(\mathcal{E}), p^*(\mathcal{E})) &= \operatorname{Hom}_Y(\mathcal{E}, p_*p^*(\mathcal{E})) = \\ &= \operatorname{Hom}_Y(\mathcal{E}, \mathcal{E}\otimes p_*(\mathcal{O}_Z)) = \bigoplus_{n\geq 0} \operatorname{Hom}_Y(\mathcal{E}, \mathcal{E}\otimes \mathcal{O}_Y(-nK_Y)).\end{aligned}$$

On Y we can consider an infinite sequence $(A_l)_{l=0}^{\infty}$ of line bundles

$$A_{sn+r} = \mathcal{O}(sK_Y + D_1 + \ldots + D_r), \quad \text{for } 0 \leq r < n.$$

Denote by $\mathcal{A}_Y = \bigoplus A_k$ the sum of all elements in this sequence. It is proved in [59, Lemma 5.1] that the DG enhancement of $\mathrm{Hom}^\bullet(\mathcal{A}_Y, \mathcal{A}_Y)$ can be calculated via the Čech enhancement. It follows that the DG enhancement of $\mathrm{Hom}_Z(p^*(\mathcal{E}), p^*(\mathcal{E}))$ is the same as the DG enhancement of $\mathrm{Hom}_Y(\mathcal{A}_Y, \mathcal{A}_Y)$.

Exceptional collection $\langle \mathcal{O}_Y, \mathcal{O}_Y(D_1), \ldots, \mathcal{O}_Y(D_1 + \ldots + D_{n-1})\rangle$ can be mutated to a collection of the form described in Section 3.2 and hence the DG algebra of endomorphisms of $\bigoplus_{l=0}^{n-1} A_l$ can be calculated by universal coextensions. Lemma 3.3.1 guarantees that up to shifts the remaining elements of the sequence (A_k) are obtained by mutations from $A_0, \ldots, A_{n-1}$. Therefore, twisted complexes allow to calculate the DG endomorphism algebra of $\mathrm{Hom}(\mathcal{A}_Y, \mathcal{A}_Y)$, *the canonical DG algebra of* Y.

The composition provides a natural map

$$\mathrm{Hom}(A_{i_{l-1}}, A_{i_l}) \otimes_k \ldots \otimes_k \mathrm{Hom}(A_{i_1}, A_{i_2}) \xrightarrow{\Psi_{i_1,\ldots,i_k}} \mathrm{Hom}(A_{i_1}, A_{i_l})$$

and an analogous one for elements of $\mathrm{Ext}^1(A_{i_1}, A_{i_l})$. If there exists $K \in \mathbb{N}$ such that for any i, j any element of $\mathrm{Hom}(A_i, A_j)$ or $\mathrm{Ext}^1(A_i, A_j)$ is in the image of some $\Psi_{i_1,\ldots,i_l}$ such that $i_{s+1} - i_s < K$ for all $s \in \{1, \ldots, k-1\}$ then the canonical DG algebra of Y can be presented as a path algebra of a cyclic DG quiver with K vertices.

If one can choose K to be the number n of T-invariant divisors of Y then the DG quivers Q_i's of exceptional collections σ_i can be read from the canonical DG quiver Q of Y

$$\begin{aligned}
(Q_i)_0 &= (Q_Y)_0, \\
(Q_i)_1 &= (Q_Y)_1 \setminus \{a \in (Q_Y)_1 \,|\, t(a) > i-1 > h(a)\}
\end{aligned}$$

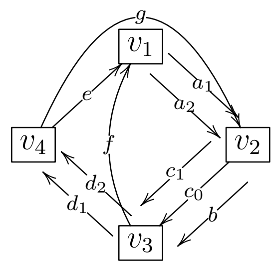
and the canonical DG quiver Q is obtained by glueing of the DG quivers Q_i.

Remark 3.3.2. The canonical DG algebra of $\mathbb{F}_3$ cannot be presented as a path algebra of such a quiver, i.e. in this case $K > 4$. If, as before, we consider the fan of $\mathbb{F}_3$ with $w_1 = (1, 0)$, $w_2 = (0, -1)$, $w_3 = (-1, 3)$ and $w_4 = (0, 1)$ then the map $\phi : \mathcal{O}_{\mathbb{F}_3}(D_1 + D_2) \to \mathcal{O}_{\mathbb{F}_3}(2D_1 + 2D_2 + 2D_3 + D_4)$ with zeroes along $2D_2$ is not a non-trivial composition of any maps between line bundles.

3.3.4 Examples

We conclude with some examples of canonical DG quivers of toric surfaces.

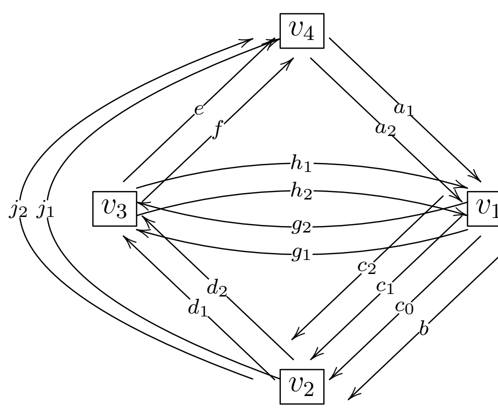
The canonical DG algebra of $\mathbb{F}_1$ is a path algebra of the quiver



with relations

$$
\begin{array}{lll}
c_0\, a_1 = c_1\, a_2, & d_1\, c_0 = d_2\, c_1, & d_1\, b\, a_2 = d_2\, b\, a_1, \\
a_1\, e\, d_2 = a_2\, e\, d_1, & a_1\, f = g\, d_1, & a_2\, f = g\, d_2, \\
b\, a_1\, e = c_1\, g, & b\, a_2\, e = c_0\, g, & f\, c_0 = e\, d_2\, b, \\
f\, c_1 = e\, d_1\, b. & &
\end{array}
$$

The canonical DG algebra of $\mathbb{F}_2$, with intersection numbers $(0, 2, 0, -2)$, is a path algebra of the following DG quiver



with

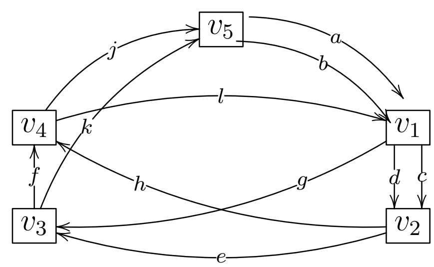
$$
\begin{array}{llll}
\deg(a_1) = 0, & \deg(a_2) = 0, & \deg(b) = 0, & \deg(c_0) = 0, \\
\deg(c_1) = 0, & \deg(c_2) = 0, & \deg(d_1) = 0, & \deg(d_2) = 0, \\
\deg(e) = 0, & \deg(f) = 1, & \deg(g_1) = -1, & \deg(g_2) = -1, \\
\deg(h_1) = 0, & \deg(h_2) = 0, & \deg(j_1) = 0, & \deg(j_2) = 0,
\end{array}
$$

$$
\begin{array}{lll}
\partial(g_1) = d_2\, c_1 \,-\, d_1\, c_0, & \partial(g_2) = d_2\, c_2 \,-\, d_1\, c_1, & \partial(h_1) = a_1\, f, \\
\partial(h_2) = a_2\, f, & \partial(j_1) = f\, d_1, & \partial(j_2) = f\, d_2
\end{array}
$$

and relations

$$\begin{array}{llll}
c_0\,a_1 = c_1\,a_2, & c_1\,a_1 = c_2\,a_2, & d_1\,b\,a_2 = d_2\,b\,a_1, & c_1\,h_2 = c_0\,h_1 + b\,a_2\,e, \\
c_2\,h_2 = c_1\,h_1 + b\,a_1\,e, & a_1\,j_2 = a_2\,j_1, & h_1\,d_2 = h_2\,d_1, & a_1\,e\,d_2 = a_2\,e\,d_1, \\
a_1\,f\,d_2 = a_2\,f\,d_1, & f\,d_1\,c_0 = f\,d_2\,c_1, & f\,d_1\,c_1 = f\,d_2\,c_2, & f\,g_1 = e\,d_2\,b, \\
f\,g_2 = e\,d_1\,b, & j_1\,c_0 = j_2\,c_1, & j_1\,c_1 = j_2\,c_1, & a_1\,j_1 = 0, \\
a_2\,j_2 = 0, & h_1\,d_1 = 0, & h_2\,d_2 = 0. &
\end{array}$$

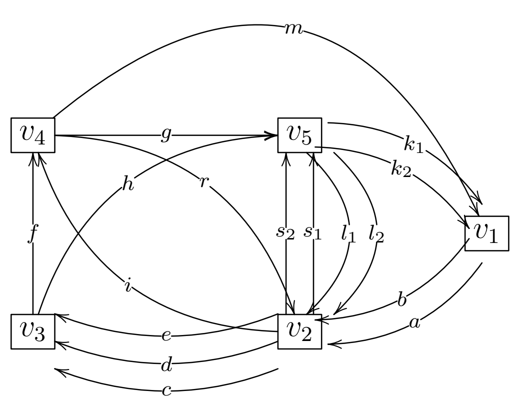
If we blow up $\mathbb{F}_1$ in such a way that the obtained toric surface Y_1 has intersection numbers $(-1,-1,0,0,-1)$ then the canonical algebra of Y_1 is a path algebra of the following quiver



with relations

$$\begin{array}{llll}
g\,b = e\,d\,a, & h\,d = f\,g, & h\,c\,b = f\,e\,c\,a, & k\,g = j\,h\,c, \\
k\,e\,d = j\,f\,e\,c, & b\,k = l\,f, & b\,j\,h = e\,j\,f\,e, & l\,h = a\,k\,e, \\
l\,f\,e = b\,k\,e, & d\,l = c\,b\,j, & b\,k = l\,f, & d\,a\,k = c\,a\,j\,f, \\
g\,l = e\,c\,a\,j. & & &
\end{array}$$

If we blow up $\mathbb{F}_1$ in another point, to obtain Y_2 with intersection numbers $(0,1,-1,-1,-2)$, then the canonical DG algebra is a path algebra of the following DG quiver:



with

$$
\begin{array}{llll}
\deg(a) = 0, & \deg(b) = 0, & \deg(c) = 0, & \deg(d) = 0, \\
\deg(e) = 0, & \deg(f) = 0, & \deg(g) = 0, & \deg(h) = 0, \\
\deg(i) = 0, & \deg(k_1) = 1, & \deg(k_2) = 0, & \deg(l_1) = 0, \\
\deg(l_2) = 0, & \deg(m) = 0, & \deg(r) = 0, & \deg(s_1) = -1, \\
\deg(s_2) = -1, & & &
\end{array}
$$

$$
\begin{array}{lll}
\partial(l_1) = b\,k_1, & \partial(l_2) = b\,k_2, & \partial(m) = k_1\,g, \\
\partial(r) = k_1\,h, & \partial(s_1) = h\,e\,-\,g\,i, & \partial(s_2) = h\,d\,-\,g\,f\,e
\end{array}
$$

and relations

$$
\begin{array}{llll}
e\,b = d\,a, & i\,b = f\,e\,a, & g\,f\,c\,a = h\,c\,b, & e\,l_1 = c\,b\,k_2 + d\,l_2, \\
i\,l_1 = f\,e\,l_2 + f\,c\,a\,k_2, & l_1\,g = b\,m, & l_2\,g = a\,m, & a\,r = l_2\,h, \\
a\,m\,f = b\,r, & b\,k_2\,h = a\,k_2\,g\,f, & b\,k_1\,h = a\,k_1\,g\,f, & k_1\,s_1 = k_2\,h\,c, \\
k_1\,s_2 = k_2\,g\,f\,c. & & &
\end{array}
$$

Chapter 4

Birational morphisms of surfaces and derived categories

In the first part of Chapter 3 we focused on the category

$$\mathcal{C}_f = \{E \in \mathcal{D}^b(X) \,|\, Rf_*(\mathcal{E}) = 0\} \subset \mathcal{D}^b(X)$$

defined for a morphism $f\colon X \to X_0$ from a smooth rational surface to its minimal model. In this chapter we define two t-structures on the category $\mathcal{C}_f$ and describe the relation between the category $\mathcal{C}_f$ and discrepancy of f.

First, we notice that $\mathcal{C}_f$ is well defined for any birational morphism of smooth projective surfaces. Thus we no longer assume that X is rational and we consider any birational morphism $f\colon X \to Y$ between smooth projective surfaces.

In Section 3.2 we described how the category $\mathcal{C}_f$ depends on the exceptional divisor of f. Theorem 4.4.4 shows that all objects of $\mathcal{C}_f$ are scheme-theoretically supported on the discrepancy divisor of f. Moreover, $\mathrm{Coh}(X) \cap \mathcal{C}_f$ has a structure of a highest weight category with duality (see Proposition 4.5.4).

To prove Theorem 4.4.4 we use t-structures on $\mathcal{C}_f$ given by tilting generators of $\mathcal{C}_f$ constructed by universal extensions and coextensions. Therefore, in Section 4.1 we recall definitions of t-structures and tilting after [4]. We also give after [56] definitions of highest weight categories.

In Section 4.2 we define $\mathcal{C}_f$ and give its basic properties. Then in Section 4.3 we describe three natural t-structures on $\mathcal{C}_f$. In Section 4.4 we use one of these t-structures to prove Theorem 4.4.4. In Section 4.5 we show that two of the three t-structures on $\mathcal{C}_f$ coincide. We describe simple objects in the heart of the non-standard t-structure in Section 4.5.1.

4.1 t-structures, tilting and highest weight categories

In this section we recall some definitions after [4].

Definition 4.1.1. *A* semi-orthogonal decomposition *of a triangulated category $\mathcal{T}$ is a family of full triangulated subcategories $\mathcal{T}_1, \ldots, \mathcal{T}_n$ of $\mathcal{T}$ such that*

- *if $i > j$ then $\mathrm{Hom}_{\mathcal{T}}(T_i, T_j) = 0$ for any $T_i \in \mathcal{T}_i$ and $T_j \in \mathcal{T}_j$.*
- *Any object $T \in \mathcal{T}$ has a filtration $0 = T_n \to T_{n-1} \to \ldots \to T_0 = T$ such that a cone of the map $T_i \to T_{i-1}$ is an element of $\mathcal{T}_i$.*

Semi-orthogonal decompositions generalise the notion of full exceptional collections. Any full exceptional collection $\langle E_1, \ldots, E_n\rangle$ in $\mathcal{T}$ gives a semi-orthogonal decomposition with $\mathcal{T}_i = \langle E_i\rangle \simeq \mathcal{D}^b(k)$.

Definition 4.1.2. *A* t-structure *on a triangulated category $\mathcal{T}$ consists of two strictly full triangulated subcategories $\mathcal{T}^{\leq 0}$, $\mathcal{T}^{\geq 1}$ such that*

- *$\mathcal{T}^{\leq 0}[1] \subset \mathcal{T}^{\leq 0}$, $\mathcal{T}^{\geq 1}[-1] \subset \mathcal{T}^{\geq 1}$. We denote those categories by $\mathcal{T}^{\leq -1} = \mathcal{T}^{\leq 0}[1]$, $\mathcal{T}^{\geq 2} = \mathcal{T}^{\geq 1}[-1]$.*
- $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}^{\geq 1}, \mathcal{T}^{\leq 0}) = 0$.
- *any object $T \in \mathcal{T}$ fits into a distinguished triangle*

$$T^- \to T \to T^+ \to T^-[1]$$

with $T^+ \in \mathcal{T}^{\geq 1}$ and $T^- \in \mathcal{T}^{\leq 0}$.

The category $\mathcal{A} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ is an abelian category and is called a *heart* a the t-structure. Short exact sequences in $\mathcal{A}$ are distinguished triangles in $\mathcal{T}$ with all objects contained in $\mathcal{A}$.

For any $T \in \mathcal{T}$ objects T^+ and T^- are determined uniquely. In fact, there exist functors $\tau_{\leq 0} \colon \mathcal{T} \to \mathcal{T}^{\leq 0}$ and $\tau_{\geq 1} \colon \mathcal{T} \to \mathcal{T}^{\geq 1}$ such that $\tau_{\leq 0}(T) = T^-$ and $\tau_{\geq 1}(T) = T^+$. More generally, we can define functors $\tau_{\leq n} \colon \mathcal{T} \to \mathcal{T}^{\leq n}$ and $\tau_{\geq n} \colon \mathcal{T} \to \mathcal{T}^{\geq n}$. The functor $\tau_{\leq n}\tau_{\geq n} \colon \mathcal{T} \to \mathcal{A}[-n]$ is a cohomology functor on $\mathcal{T}$ with respect to the given t-structure. We will denote in by $H^n_{\mathcal{A}}$.

A t-structure is *bounded* if every object $T \in \mathcal{T}$ lies in $\mathcal{T}^{\leq n} \cap \mathcal{T}^{\geq m}$ for some m, n. Recall the following [15, Lemma 3.2].

Lemma 4.1.3. *An abelian subcategory $\mathcal{A} \subset \mathcal{T}$ of a triangulated category is a heart of a bounded t-structure if and only if*

- $\mathrm{Hom}_{\mathcal{T}}(A_1[k_1], A_2[k_2]) = 0$ *for any $A_1, A_2 \in \mathcal{A}$ and $k_1 > k_2$.*

- *For any non-zero $T \in \mathcal{T}$ there exist $k_1 > k_2 > \ldots > k_n$ and maps $0 = T_0 \to T_1 \to \ldots \to T_{n-1} \to T_n = T$ such that a cone of $T_i \to T_{i-1}$ is an element of $\mathcal{A}[k_i]$.*

The first example of a t-structure on a triangulated category $\mathcal{T}$ occurs in the case when $\mathcal{T}$ is the derived category of an abelian category $\mathcal{A}$. Then $\mathcal{D}(A)^{\leq 0}$ is the full subcategory of objects whose cohomology is concentrated in degrees less or equal to zero. Similarly, $\mathcal{D}(A)^{\geq 1}$ is the full subcategory of objects whose cohomology is concentrated in degrees greater than zero. This defines the so called, *standard t-structure* on $\mathcal{D}(\mathcal{A})$. Its heart is equivalent to $\mathcal{A}$. If $\mathcal{T}$ is a bounded derived category of some abelian category $\mathcal{A}$ then the standard t-structure on $\mathcal{T}$ is also bounded.

This example allows us to construct a t-structure from an equivalence $\mathcal{T} \simeq \mathcal{D}(\mathcal{A})$ for some abelian category $\mathcal{A}$. In particular, if $\mathscr{E}$ is a tilting object on a smooth projective variety X then the equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(\text{Mod-}\operatorname{End}(\mathscr{E}))$ defines a new t-structure on $\mathcal{D}^b(X)$.

The following lemma proves to be very useful when comparing two t-structures on a triangulated category.

Lemma 4.1.4. *Let $\mathcal{A}$, $\mathcal{B}$ be abelian subcategories of a triangulated category $\mathcal{C}$ such that both are hearts of bounded t-structures and $\mathcal{A} \subset \mathcal{B}$. Then $\mathcal{A} = \mathcal{B}$.*

Proof. By Lemma 4.1.3 if an abelian subcategory $\mathcal{A} \subset \mathcal{C}$ is a heart of a bounded t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ the every object $C \in \mathcal{C}$ admits a sequence of maps

$$0 = C_0 \to C_1 \to \ldots \to C_{n-1} \to C_n = C$$

such that the cone A_i of the map $C_{i-1} \to C_i$ lies in $\mathcal{A}[l_i]$ and $l_1 > l_2 > \ldots > l_n$. Moreover, such a sequence is unique. Then also $\mathcal{C}^{\leq 0} = \{C \mid l_1 \leq 0\}$ and $\mathcal{C}^{\geq 0} = \{C \mid l_n \geq 0\}$.

Let $C \in \mathcal{C}$ be an object in $\mathcal{C}^{\leq_{\mathcal{A}} 0}$. Then the objects $A_i \in \mathcal{A}[l_i] \subset \mathcal{B}[l_i]$ and $l_1 \leq 0$. It follows that C is also an object of $\mathcal{C}^{\leq_{\mathcal{B}} 0}$. A similar argument shows that $\mathcal{C}^{\geq_{\mathcal{A}} -1} \subset \mathcal{C}^{\geq_{\mathcal{B}} -1}$. Then ${}^{\perp}(\mathcal{C}^{\geq_{\mathcal{B}} -1}) = \mathcal{C}^{\leq_{\mathcal{B}} 0} \subset \mathcal{C}^{\leq_{\mathcal{A}} 0} = {}^{\perp}(\mathcal{C}^{\geq_{\mathcal{A}} -1})$. It follows that $\mathcal{C}^{\leq_{\mathcal{A}} 0} = \mathcal{C}^{\leq_{\mathcal{B}} 0}$ and hence the two t-structures in question are equal. □

From a t-structure on $\mathcal{T}$ and a torsion pair on its heart $\mathcal{A}$ we can construct a new t-structure on $\mathcal{T}$.

Definition 4.1.5. *A* torsion pair *on an abelian category $\mathcal{A}$ is a pair of additive subcategories $\mathfrak{F}, \mathfrak{T} \subset \mathcal{A}$ such that*

- $\operatorname{Hom}(\mathfrak{T}, \mathfrak{F}) = 0$.

- *For any $A \in \mathcal{A}$ there exists a short exact sequence*

$$0 \to T \to A \to F \to 0$$

in $\mathcal{A}$ with $T \in \mathfrak{T}$ and $F \in \mathfrak{F}$.

A torsion pair $(\mathfrak{T}, \mathfrak{F})$ on a heart $\mathcal{A}$ of a bounded t-structure on $\mathcal{T}$ allows to define a new t-structure on $\mathcal{T}$ with a heart

$$\mathcal{A}^\sharp = \{E \in \mathcal{T} \mid H^0_{\mathcal{A}}(E) \in \mathfrak{T},\ H^{-1}_{\mathcal{A}}(E) \in \mathfrak{F}\}.$$

Then a t-structure with heart $\mathcal{A}^\sharp$ is called a *tilt* of t-structure with heart $\mathcal{A}$.

If a triangulated category $\mathcal{T}$ has a semi-orthogonal decomposition $\mathcal{T} = \langle \mathcal{S}, \mathcal{R} \rangle$ then sometimes t-structures on $\mathcal{S}$ and $\mathcal{R}$ can be glued to give a t-structure on $\mathcal{T}$.

Definition 4.1.6. *Let $\mathcal{T}$, $\mathcal{S}$, $\mathcal{R}$ be triangulated categories. The category $\mathcal{T}$* admits recollement relative to $\mathcal{S}$ and $\mathcal{R}$ *if there exist exact functors $i_* = i_! \colon \mathcal{S} \to \mathcal{T}$, $j^* = j^! \colon \mathcal{T} \to \mathcal{R}$, $i^*, i^! \colon \mathcal{T} \to \mathcal{S}$ and $j_!, j_* \colon \mathcal{R} \to \mathcal{T}$ such that*

- *$(i^* \dashv i_* = i_! \dashv i^!)$ and $(j_! \dashv j^! = j^* \dashv j_*)$, where $i^* \dashv i_*$ means that i^* is left adjoint to i_*.*
- *The composition $i^! j_* = 0$.*
- *Functors i_*, $j_!$, j_* are full embeddings and thus $i^* i_* \simeq id_{\mathcal{S}} \simeq i^! i_*$ and $j^* j_! \simeq id_{\mathcal{R}} \simeq j^* j_!$.*
- *for any object $T \in \mathcal{T}$ triangles*

$$\begin{aligned} &i_! i^* T \to T \to j_* j^* T \to i_! i^* T[1], \\ &j_! j^! T \to T \to i_* i^* T \to j_! j^! T[1] \end{aligned}$$

are exact.

We denote such a recollement by

$$\mathcal{S} \rightleftarrows \mathcal{T} \rightleftarrows \mathcal{R}$$

Proposition 4.1.7. *([4, Theorem 1.4.10]) Suppose that $\mathcal{T}$ admits a recollement $\mathcal{S} \rightleftarrows \mathcal{T} \rightleftarrows \mathcal{R}$. Then t-structures $(\mathcal{S}^{\le 0}, \mathcal{S}^{\ge 1})$, $(\mathcal{R}^{\le 0}, \mathcal{R}^{\ge 1})$ induce a t-structure on $\mathcal{T}$ with*

$$\begin{aligned} \mathcal{T}^{\le 0} &= \{T \in \mathcal{T} \mid j^*(T) \in \mathcal{R}^{\le 0},\ i^*(T) \in \mathcal{S}^{\le 0}\}, \\ \mathcal{T}^{\ge 1} &= \{T \in \mathcal{T} \mid j^!(T) \in \mathcal{R}^{\ge 1},\ i^!(T) \in \mathcal{S}^{\ge 1}\}. \end{aligned}$$

We will also need the following definitions (see [56]).

A k-linear abelian category $\mathcal{A}$ is *locally artinian* if $\mathcal{A}$ admits arbitrary unions of subobjects and if every object in $\mathcal{A}$ is an union of its subobjects of finite length. Moreover, we require that the category $\mathcal{A}$ has enough injectives and $B \cap (\bigcup_\alpha A_\alpha) = \bigcup_\alpha (B \cap A_\alpha)$ for all subobjects B and all family of objects $\{A_\alpha\}$ of each object A.

A poset Λ is *locally finite* if the set $[\lambda, \mu] = \{\tau \mid \lambda \le \tau \le \mu\}$ are finite for any pair $\lambda, \mu \in \Lambda$.

Definition 4.1.8. *A* highest weight category *is a locally artinian abelian category $\mathcal{A}$ such that there exists a locally finite poset Λ (the poset of* weights*) satisfying the following conditions*

- *There is a complete collection $\{S(\lambda)\}_{\lambda\in\Lambda}$ of isomorphism classes of simple object in $\mathcal{A}$.*

- *There is a collection $\{\Delta(\lambda)\}$ of* standard objects *of $\mathcal{A}$ and a surjection $\Delta(\lambda)\to S(\lambda)$ such that all composition factors $S(\mu)$ of the kernel satisfy $\mu<\lambda$.*

- *The projective cover $P(\lambda)$ of $S(\lambda)$ admits a surjection to $\Delta(\lambda)$. The kernel of this surjection has a filtration with quotients $\Delta(\mu)$ for $\mu>\lambda$ and such that every $\Delta(\mu)$ appears in this filtration only finitely many times.*

The motivating example for considering highest weight categories is the category $\mathcal{O}$ introduced in [5] (see also [28]). Therefore, standard modules in the above definition are sometimes called *Verma modules*. Another class of examples is given by categories of modules over quasi-hereditary algebras as defined in Section 2.3.2.

Following [29, Section 2] we say that a highest weight category $\mathcal{A}$ has a *duality* if there exists an antiequivalence $\mathcal{D}$ of $\mathcal{A}$ such that $\mathcal{D}^2$ is equivalent to $\mathrm{id}_{\mathcal{A}}$ and $\mathcal{D}(S(\lambda))\simeq S(\lambda)$ for any $\lambda\in\Lambda$.

4.2 Category $\mathcal{C}_f$

Section 3 focuses on smooth rational surfaces. However, Sections 3.2.1 - 3.2.4 deal only an arbitrary birational morphism between smooth surfaces.

Let $f\colon X\to Y$ be such a morphism. f can be decomposed into a sequence of blow-ups of single points

$$X=X_n\xrightarrow{f_n}X_{n-1}\to\ldots\to X_1\xrightarrow{f_1}X_0=Y.$$

For simplicity, we denote by π_j the composition $f_{j+1}\circ\ldots\circ f_n\colon X\to X_j$.

In this case $R^{\bullet}f_*(\mathcal{O}_X)=\mathcal{O}_Y$ and by the projection formula the functor $L^{\bullet}f^*\colon\mathcal{D}^b(Y)\to\mathcal{D}^b(X)$ is fully faithful,

$$\begin{aligned}&\mathrm{Hom}_X(L^{\bullet}f^*\mathcal{E},L^{\bullet}f^*\mathcal{F})=\mathrm{Hom}_Y(\mathcal{E},R^{\bullet}f_*(L^{\bullet}f^*(\mathcal{F})))=\\&=\mathrm{Hom}_Y(\mathcal{E},R^{\bullet}f_*(\mathcal{O}_X)\otimes^L\mathcal{F})=\mathrm{Hom}_Y(\mathcal{E},\mathcal{F}).\end{aligned}$$

Therefore we have semi-orthogonal decompositions

$$\mathcal{D}^b(X)=\langle\mathcal{C}_f,\,\mathcal{D}^b(Y)\rangle,\quad\mathcal{D}^b(X)=\langle\mathcal{D}^b(Y),\mathcal{D}_f\rangle,$$

where

$$\mathcal{C}_f = \{\mathcal{E} \in \mathcal{D}^b(X) \mid \mathrm{Hom}(L^{\bullet} f^* \mathcal{F}, \mathcal{E}) = 0 \ \forall \mathcal{F} \in \mathcal{D}^b(Y)\} = \{\mathcal{E} \in \mathcal{D}^b(X) \mid R^{\bullet} f_*(\mathcal{E}) = 0\}$$

and

$$\mathcal{D}_f = \{\mathcal{E} \in \mathcal{D}^b(X) \mid \mathrm{Hom}(\mathcal{E}, L^{\bullet} f^* \mathcal{F}) = 0 \ \forall \mathcal{F} \in \mathcal{D}^b(Y)\}.$$

With the notation from Section 3, $\langle \mathcal{O}_{R_n}(R_n), \ldots, \mathcal{O}_{R_1}(R_1) \rangle$ is a full exceptional collection in $\mathcal{C}_f$. Its Ext-quiver is described in Section 3.2.1.

Remark 4.2.1. A_∞-structure on this Ext-quiver does not have to be trivial. The first example when a non-zero higher multiplication appears is in the case when the exceptional divisor of f has four irreducible components intersecting as

$$E_1 \text{———} E_3 \text{———} E_4 \text{———} E_2.$$

In this case

$$\begin{aligned} R_1 &= E_1 + E_2 + 2E_3 + 3E_4, & R_3 &= E_3 + E_4, \\ R_2 &= E_2 + E_3 + 2E_4, & R_4 &= E_4, \end{aligned}$$

and the operation

$$m_3 \colon \mathrm{Hom}(\mathcal{O}_{R_2}(R_2), \mathcal{O}_{R_1}(R_1)) \otimes \mathrm{Ext}^1(\mathcal{O}_{R_3}(R_3), \mathcal{O}_{R_2}(R_2)) \otimes \mathrm{Ext}^1(\mathcal{O}_{R_4}(R_4), \mathcal{O}_{R_3}(R_3)) \to$$
$$\to \mathrm{Ext}^1(\mathcal{O}_{R_4}(R_4), \mathcal{O}_{R_1}(R_1))$$

is non-zero in any A_∞-category A_∞-quasi-isomorphic to the Ext-algebra of this collection.

The category $\mathcal{D}_f$ is obtained from $\mathcal{C}_f$ by mutation over $L^{\bullet} f^* \mathcal{D}^b(Y)$. Direct calculations show that the equivalence functor $\varphi \colon \mathcal{C}_f \to \mathcal{D}_f$ is given by $\varphi(\mathcal{E}) = \mathcal{E} \otimes \omega_{X/Y}$, where $\omega_{X/Y} = \mathcal{O}(\sum R_i)$ is the relative canonical class.

4.3 t-structures on $\mathcal{C}_f$

Results of Section 2.3.1 allow us to find two tilting objects in the category $\mathcal{C}_f$. Thus we have two hearts. By $\mathcal{A}_e$ we denote the heart corresponding to a tilting generator obtained from $\langle \mathcal{O}_{R_n}(R_n), \ldots, \mathcal{O}_{R_1}(R_1) \rangle$ by universal extensions. By $\mathcal{A}_{\mathrm{coe}}$ we denote the heart given by the tilting object constructed from the above collection by universal coextensions.

By construction the tilting object obtained by universal extensions decomposes as a direct sum of n objects. Let us denote it by $\bigoplus_{i=1}^n \mathcal{P}_i$. Here, $\mathcal{P}_i$ is obtained from $\mathcal{O}_{R_i}(R_i)$ by universal extension by $\mathcal{O}_{R_{i-1}}(R_{i-1}), \mathcal{O}_{R_{i-2}}(R_{i-2}), \ldots, \mathcal{O}_{R_1}(R_1)$. In particular,

$\mathcal{P}_1 = \mathcal{O}_{R_1}(R_1)$. Thus $\mathcal{P}_i$ are indecomposable projective objects in the abelian category $\mathcal{A}_e$.

In Section 3.2.7 we have seen that projective objects in $\mathcal{A}_{\mathrm{coe}}$ are sheaves $\mathscr{D}_l = \mathcal{O}_{S_l}(S_l)$ for $S_l = \sum_{R_i \leq R_l} R_i$. Let us note that the decomposition of f into $X \xrightarrow{\pi_l} X_l \xrightarrow{h} Y$ shows that S_k is the relative canonical divisor for the map π_l.

Proposition 4.3.1. *The category $\mathcal{C}_f$ has two natural t-structures. The first t-structure is obtained from the exceptional collection by means of universal extensions and $\bigoplus \mathcal{P}_i$ is the projective object in its heart, $\mathcal{A}_e$. The second t-structure is obtained from this collection by universal coextensions and $\bigoplus \mathscr{D}_i$ is the projective object in its heart, $\mathcal{A}_{coe}$. The endomorphism algebras of $\bigoplus \mathcal{P}_i$ and $\bigoplus \mathscr{D}_i$ are Ringel dual quasi-hereditary algebras. Moreover, both t-structures are bounded.*

Proof. We only need to check that the described t-structures are bounded. As the category $\mathcal{C}_f$ is a subcategory of $\mathcal{D}^b(X)$, it is Ext-finite. It follows that for any $\mathcal{F} \in \mathcal{C}_f$ only finitely many of $\mathrm{Ext}^i(\bigoplus \mathcal{P}_i, \mathcal{F})$ (or $\mathrm{Ext}^i(\bigoplus \mathscr{D}_i, \mathcal{F})$) are non-zero and hence the induced t-structure is bounded. □

Also by [14, Lemma 3.1] the standard t-structure on $\mathcal{D}^b(X)$ induces a t-structure on $\mathcal{C}_f$. We call it a standard t-structure and denote its heart by $\mathcal{A}_{\mathrm{st}} = \mathrm{Coh}(X) \cap \mathcal{C}_f$.

4.4 Support of objects in the category $\mathcal{C}_f$

We show by induction that every object of $\mathrm{Coh}(X) \cap \mathcal{C}_f$ is supported on a subscheme of the relative canonical divisor $K_{X/Y} = R_1 + \ldots + R_n$.

Lemma 4.4.1. *Let $f\colon X \to Y$ be a birational morphism of smooth surfaces and let $X \xrightarrow{h} Z \xrightarrow{g} Y$ be its factorization with $g\colon Z \to Y$ a blow up of a single point. Then there is a recollement of the form*

$$\mathcal{C}_h \to \mathcal{C}_f \xrightarrow{Rh_*} \mathcal{C}_g$$

and the t-structure with heart $\mathcal{A}_e$ is glued from the corresponding t-structures on $\mathcal{C}_h$ and $\mathcal{C}_g$.

Proof. The t-structure on $\mathcal{C}_f$ glued from $\mathrm{Coh}(X) \cap \mathcal{C}_h$ and $\mathrm{Coh}(Z) \cap \mathcal{C}_g$ is bounded and given by

$$\mathcal{C}_f^{\leq 0} = \{E \in \mathcal{C}_f \mid Rh_*(E) \in \mathcal{C}_g^{\leq 0},\ \mathrm{Hom}(E, \mathcal{C}_h^{\geq 1}) = 0\},$$
$$\mathcal{C}_f^{\geq 0} = \{E \in \mathcal{C}_f \mid Rh_*(E) \in \mathcal{C}_g^{\geq 0},\ \mathrm{Hom}(\mathcal{C}_h^{\geq -1}, E) = 0\}.$$

Let us denote, temporarily, the heart of this t-structure by $\mathcal{C}_f^0$. Then $\mathrm{Coh}(X) \cap \mathcal{C}_h \subset \mathcal{C}_f^0$. Also we have $\mathcal{O}_{R_1}(R_1) = Lh^*(\mathcal{O}_{E_1}(E_1)) \in \mathcal{C}_f^0$. The first condition, $Rh_*(\mathcal{O}_{R_1}(R_1)) \in$

$\mathrm{Coh}(Z)$, is satisfied because of the projection formula. The necessary Hom's vanish because $\mathcal{O}_{R_1}(R_1)$ is a sheaf on X. Hence, $\mathcal{O}_{R_1}(R_1) \in \mathcal{C}_f^0$.

The simple objects $\mathcal{O}_{E_2}(R_2), \ldots, \mathcal{O}_{E_n}(R_n) \in \mathrm{Coh}(X) \cap \mathcal{C}_h \subset \mathcal{C}_f^0$ by the induction assumption. By Lemma 3.1.3 we have a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_{R_1}(R_1) \to \mathcal{O}_{E_1}(R_1) \to 0$$

with $\mathcal{K} \in \mathcal{C}_h$ and hence $Rh_*(\mathcal{O}_{E_1}(R_1)) = \mathcal{O}_{E_1}(E_1)$. Also, $\mathcal{O}_{E_1}(E_1)$ is a sheaf on X which makes all the Hom's spaces in the definition of the heart $\mathcal{C}_f^0$ vanish. It proves that $\mathcal{O}_{E_1}(R_1) \in \mathcal{C}_f^0$.

Thus objects $\mathcal{O}_{E_1}(R_1), \ldots, \mathcal{O}_{E_n}(R_n) \in \mathcal{C}_f^0$ and it follows that $\mathrm{Coh}(X) \cap \mathcal{C}_f \subset \mathcal{C}_f^0$. As both are hearts of bounded t-structures we conclude by Lemma 4.1.4. $\square$

Recall that a *scheme theoretic support* of a coherent sheaf F on X is the smallest closed subscheme $i\colon Z \to X$ such that there exists a coherent sheaf G on Z with $i_*(G) = F$. If X is affine then Z is the zero locus of $\mathrm{Ann}(M)$ where $F = M^\sim$ (see [60, Lemma 28.5.4]).

Proposition 4.4.2. *Let $E \in Coh(X) \cap \mathcal{C}_f$. Then the scheme theoretic support of E is a subscheme of $K_{X/Y} = R_1 + \ldots + R_n$.*

Proof. First, we notice that if E is a sheaf in $\mathcal{C}_f$ then the support of E is of pure dimension one (because a direct image of a non-zero sheaf supported on a point is a non-zero sheaf supported on a point). Hence, the annihilator of E is of the form $\mathcal{O}(-D)$.

Now we proceed by induction. If f is a blow up of a single point then $\mathrm{Coh}(X) \cap \mathcal{C}_f$ consists of direct sums of $\mathcal{O}_{E_1}(E_1)$ and the support of every object is equal to $E_1 = K_{X/Y}$.

For $f\colon X \to Y$ let $f\colon X \xrightarrow{h} Z \xrightarrow{g} Y$ be a factorization with $g\colon Z \to Y$ a blow up at a single point. Let us take $\mathcal{E} \in \mathrm{Coh}(X) \cap \mathcal{C}_f$. By the previous lemma there exists a distinguished triangle in $\mathcal{C}_f$

$$Lh^* Rh_*(\mathcal{E}) \to \mathcal{E} \to \mathcal{F} \to Lh_* Rh^*(\mathcal{E})[1],$$

where $\mathcal{F} \in \mathcal{C}_h$ and $Rh_*(\mathcal{E})$ is a direct sum of copies of $\mathcal{O}_{E_1}(E_1)$. The last observation follows from the fact that $\mathcal{E}$ is an object of the glued heart. Hence $Lh^*(\mathcal{O}_{E_1}(E_1)) = h^*(\mathcal{O}_{E_1}(E_1)) = \mathcal{O}_{R_1}(R_1)$ and $Lh^* Rh_*(\mathcal{E}) = h^* h_*(\mathcal{E})$. Taking cohomology sheaves of the above triangle we get an exact sequence of sheaves on X:

$$0 \to \mathcal{H}^{-1}(\mathcal{F}) \to h^* h_*(\mathcal{E}) \to \mathcal{E} \to \mathcal{H}^0(\mathcal{F}) \to 0$$

Both $\mathcal{H}^{-1}(\mathcal{F})$, $\mathcal{H}^0(\mathcal{F})$ are elements of $\mathrm{Coh}(X) \cap \mathcal{C}_h$ (because $\mathrm{Coh}(X) \cap \mathcal{C}_h \subset \mathrm{Coh}(X) \cap \mathcal{C}_f$ and the cohomology functors with respect to these two hearts coincide for objects in $\mathcal{C}_h$). Hence, by the induction assumption, the scheme theoretic support of $\mathcal{H}^{-1}(\mathcal{F})$ and $\mathcal{H}^0(\mathcal{F})$ is contained in $K_{X/Z} = R_2 + \ldots + R_n$ and the scheme theoretic support of $h^* h_*(\mathcal{E})$ is equal to R_1 (if $h_*(\mathcal{E}) = 0$ then $\mathcal{E} \in \mathcal{C}_h$ and there is nothing to prove).

Clearly, if

$$M_1 \to M_2 \to M_3 \to 0$$

is an exact sequence of A-modules then $\operatorname{Ann}(M_1) \cdot \operatorname{Ann}(M_3) \subset \operatorname{Ann}(M_2)$.

In our case $\operatorname{Ann}(h^* h_*(\mathcal{E})) = \mathcal{O}(-R_1)$, $\operatorname{Ann}(\mathcal{H}^0(\mathcal{F})) = \mathcal{O}(-R_2 - \ldots - R_n)$ and $\operatorname{Ann}(\mathcal{E}) = \mathcal{O}(-D)$ for some divisor D. For effective divisors D_1 and D_2 we have an equality $\mathcal{O}(-D_1) \cdot \mathcal{O}(-D_2) = \mathcal{O}(-D_1) \otimes \mathcal{O}(-D_2)$ so from the fact that $\mathcal{O}(-R_1) \otimes \mathcal{O}(-R_2 - \ldots - R_n) \subset \mathcal{O}(-D)$ we conclude that $D \leq R_1 + \ldots + R_n$. □

Let $\iota\colon E_f \to X$ denote the inclusion of the non-reduced discrepancy divisor of f. By Proposition 4.4.2 for every object $F \in \operatorname{Coh}(X) \cap \mathcal{C}_f$ there exists $\mathcal{F} \in \operatorname{Coh}(E_f)$ such that $F = \iota_* \mathcal{F}$. Thus the functor $\iota^*\colon \operatorname{Coh}(X) \cap \mathcal{C}_f \to \operatorname{Coh}(E_f)$ gives a commutative diagram

$$\begin{tikzcd} \operatorname{Coh}(X) \cap \mathcal{C}_f \arrow[r, "\iota^*"] \arrow[dr, hook] & \operatorname{Coh}(E_f) \arrow[d, "\iota_*"] \\ & \operatorname{Coh}(X). \end{tikzcd}$$

Indeed, if $F \in \operatorname{Coh}(X) \cap \mathcal{C}_f$ then $\iota_* \iota^*(F) = \iota_* \iota^* \iota_*(\mathcal{F}) = \iota_*(\mathcal{F}) = F$ as $\iota^* \iota_* = \mathrm{id}$.

Proposition 4.4.3. *Let $\mathcal{E}$ be any element of $\mathcal{C}_f$. Then there exists $\widetilde{\mathcal{E}} \in \mathcal{D}^b(E_f)$ such that $\mathcal{E} = \iota_*(\widetilde{\mathcal{E}})$.*

Proof. We know that $\mathcal{C}_f$ is equivalent to a bounded derived category of $\mathcal{A}_{\mathrm{coe}}$. This category has enough projectives and thus every object in $\mathcal{C}_f$ can be represented by a complex of projective objects in $\mathcal{A}_{\mathrm{coe}}$,

$$\mathcal{E} \simeq \{0 \to P_{-n} \to \ldots \to P_m \to 0\}.$$

Moreover, by Proposition 4.3.1 all projective objects in $\mathcal{A}_{\mathrm{coe}}$ are supported on E_f, i.e. $P_l = \iota_* \widetilde{P}_l$ for some sheaf $\widetilde{P}_l \in \operatorname{Coh}(E_f)$. Therefore

$$\operatorname{Hom}_X(\iota_* \widetilde{P}_l, \iota_* \widetilde{P}_j) = \operatorname{Hom}_Z(\widetilde{P}_l, \iota^* \iota_* \widetilde{P}_j) = \operatorname{Hom}_Z(\widetilde{P}_l, \widetilde{P}_j).$$

Hence the homomorphisms between $\iota_* \widetilde{P}_l$'s have a unique lift to homomorphisms between $\widetilde{P}_l$'s. Thus a complex on X lifts to a complex on E_f.

If we define $\widetilde{\mathcal{E}}$ by a complex

$$\{0 \to \widetilde{P}_{-n} \to \ldots \to \widetilde{P}_m \to 0\}$$

then $\mathcal{E} = \iota_*(\widetilde{\mathcal{E}})$. □

Theorem 4.4.4. *Let $f\colon X \to Y$ be a birational morphism of smooth projective surfaces and let $E = \sum E_i$ be the exceptional divisor of f. It has a non-reduced scheme structure $\widetilde{E} = \sum a_i E_i$ given by the discrepancy of f, $K_X = f^* K_Y + \widetilde{E}$.*

Let $\mathcal{C}_f = \{\mathcal{E} \in \mathcal{D}^b(X) \,|\, R^\bullet f_*(\mathcal{E}) = 0\}$ be a full subcategory of $\mathcal{D}^b(X)$. Then $\mathcal{D}^b(X)$ has a semi-orthogonal decomposition, $\mathcal{D}^b(X) = \langle \mathcal{C}_f, L^\bullet f^* \mathcal{D}^b(Y)\rangle$ and any object $\mathcal{E} \in \mathcal{C}_f$ is of the form $\iota_* \widetilde{\mathcal{E}}$ for some $\widetilde{\mathcal{E}} \in \mathcal{D}^b(\widetilde{E})$, where $\iota\colon \widetilde{E} \to X$ is the closed embedding.

Proof. In Section 4.2 we have proved that the functor $L^\bullet f^*\colon \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$ is fully faithful and gives the requested semi-orthogonal decomposition of $\mathcal{D}^b(X)$.

In Section 4.3 we constructed a titling object $\bigoplus \mathscr{D}_i$ in $\mathcal{C}_f$. By Proposition 4.3.1 $\mathcal{C}_f$ is equivalent to the bounded derived category of $\mathcal{A}_{\mathrm{coe}}$. Moreover, $\mathcal{A}_{\mathrm{coe}}$ is equivalent to the category of finite dimensional modules over the endomorphism algebra of $\bigoplus \mathscr{D}_i$.

Finally, Proposition 4.4.3 proves that every object in $\mathcal{C}_f$ is supported on the discrepancy divisor of f. ◻

4.5 Comparing the t-structures

Proposition 4.5.1. *The t-structures with hearts $\mathcal{A}_{st}$ and $\mathcal{A}_{coe}$ are different.*

Proof. Let us consider the case when $f\colon X \to Y$ is a composition of two blow ups. In Section 4.5.1 we shall see that in this case simple objects in $\mathcal{A}_{\mathrm{coe}}$ are $\mathcal{O}_{E_1}(E_1 + E_2)[1]$ and $\mathcal{O}_{E_1+E_2}(E_1 + 2E_2)$. ◻

In order to show that $\mathcal{A}_e = \mathcal{A}_{\mathrm{st}}$ we use properties of perverse coherent sheaves introduced by Bridgeland in [14]. Let us recall after [14, Section 3] that for a morphism $f\colon X \to Y$ with fibers of dimensions at most one and such that $R^\bullet f_*(\mathcal{O}_X) = \mathcal{O}_Y$ one can define a t-structure on $\mathcal{D}^b(X)$ which is glued from the standard t-structure on $\mathcal{D}^b(Y)$ and a t-structure on $\mathcal{C}_f$ with heart $\mathcal{A}_{\mathrm{st}}[p]$ for any $p \in \mathbb{Z}$. The heart of the new t-structure on $\mathcal{D}^b(X)$ is denoted by ${}^p\mathrm{Per}(X/Y)$.

Let $Y = \mathrm{Spec}(R)$ with R a noetherian complete local ring with maximal ideal $\mathfrak{m}$ such that $k = R/\mathfrak{m}$ is algebraically closed and $k \subset R$. By [62, Proposition 3.5.7] in this case simple objects in ${}^{-1}\mathrm{Per}(X/Y)$ are $\mathcal{O}_{R_1}$ and $\mathcal{O}_{E_i}(-1)[-1]$. It follows that simple objects in $\mathcal{A}_{\mathrm{st}}$ are of the form $\mathcal{O}_{E_i}(-1) \simeq \mathcal{O}_{E_i}(R_i)$.

Lemma 4.5.2. *For any i the sheaf $\mathcal{O}_{E_i}(R_i)$ is an element of $\mathcal{A}_e$.*

Proof. According to Lemma 3.1.1 we have $E_i R_i = -1$ so $\mathcal{O}_{E_i}(R_i) \simeq \mathcal{O}_{E_i}(-1)$.

Let us recall that for any effective divisors D_1, D_2 on a surface we have a short exact sequence

$$0 \to \mathcal{O}_{D_2}(D_2) \to \mathcal{O}_D(D) \to \mathcal{O}_{D_1}(D) \to 0,$$

where $D = D_1 + D_2$. We apply this to the equality $R_i - E_i = \sum_{j=1}^{s} R_{i+j}$ for some $s > 0$ (which follows from Lemma 3.1.3). Hence we have a short exact sequence

$$0 \to \mathcal{O}_{R_{i+1}+\ldots+R_{i+s}}(R_{i+1} + \ldots + R_{i+s}) \to \mathcal{O}_{R_i}(R_i) \to \mathcal{O}_{E_i}(-1) \to 0. \tag{4.1}$$

Short exact sequences of the form

$$0 \to \mathcal{O}_{R_{i+1}+\ldots+R_{i+l}}(R_{i+1}+\ldots+R_{i+l}) \to \mathcal{O}_{R_{i+1}+\ldots+R_{i+l+1}}(R_{i+1}+\ldots+R_{i+l+1}) \\ \to \mathcal{O}_{R_{i+l+1}}(R_{i+l+1}) \to 0$$

show that the kernel of the canonical map $\mathcal{O}_{R_i}(R_i) \to \mathcal{O}_{E_i}(-1)$ has filtration with quotients $\mathcal{O}_{R_j}(R_j)$ for $j > i$.

By [18, Theorem 2] we have $\mathrm{Ext}^k(\mathcal{P}_i, \mathcal{O}_{R_j}(R_j)) = 0$ for $k \neq 0$ and any i, j, where $\mathcal{P}_i$ denotes the tilting generator constructed from $\mathcal{O}_{R_i}(R_i)$ by universal extensions Hence, the sheaves $\mathcal{O}_{R_j}(R_j)$ lie in the heart of t-structure given by $\bigoplus \mathcal{P}_i$. It also follows that $\mathcal{O}_{R_{i+1}+\ldots+R_{i+l}}(R_{i+1}+\ldots+R_{i+l})$ lie in $\mathcal{A}_e$.

Finally, (4.1) implies that $\mathcal{O}_{E_i}(R_i) \in \mathcal{A}_e$. □

Proposition 4.5.3. *t-structures on $\mathcal{C}_f$ with heart $\mathcal{A}_e$ and $\mathcal{A}_{st}$ are equal.*

Proof. By Lemma 4.5.2 we know that all simple objects in $\mathcal{A}_{\mathrm{st}}$ are elements of $\mathcal{A}_e$. It follows that $\mathcal{A}_{\mathrm{st}} \subset \mathcal{A}_e$. As both t-structures are bounded we can apply Lemma 4.1.4 to finish the proof. □

Thus the category $\mathcal{A}_{\mathrm{st}} = \mathrm{Coh}(X) \cap \mathcal{C}_f$ is a highest weight category. The order on the set of standard modules $\mathcal{O}_{R_i}(R_i)$ is induced from the order on R_i that we considered before.

Proposition 4.5.4. *The category Coh$(X) \cap \mathcal{C}_f$ is a highest weight category with a duality.*

Proof. By Proposition 4.5.3 the heart $\mathcal{A}_{\mathrm{st}} = \mathrm{Coh}(X) \cap \mathcal{C}_f$ of the standard t-structure on $\mathcal{C}_f$ coincides with $\mathcal{A}_e$. The endomorphism algebra of the projective generator $\bigoplus \mathcal{P}_i$ of $\mathcal{A}_e$ is quasi-hereditary and hence the category of modules over it, equivalent to the category $\mathrm{Coh}(X) \cap \mathcal{C}_f$, is a highest weight category.

The standard modules are $\mathcal{O}_{R_i}(R_i)$ and the simple modules are $\mathcal{O}_{E_i}(R_i) = \mathcal{O}_{E_i}(-1)$. Let us denote by $\widetilde{P}_i$ the indecomposable projective objects in $\mathcal{C}_f \cap \mathrm{Coh}(X)$.

There is an exact functor $D_f \colon \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ given by

$$D_f(\mathcal{F}) = R\mathcal{H}om_X(\mathcal{F},\, f^!(\mathcal{O}_Y)),$$

which maps $\mathcal{C}_f$ into itself.

Knowing that $f^!(\mathcal{O}_Y) = \mathcal{O}_X(R_1 + \ldots + R_n)$ we can calculate

$$\begin{aligned}
&D_f(\mathcal{O}_{E_i}(R_i)) = R\mathcal{H}om(\mathcal{O}_{E_i}(R_i), f^!\mathcal{O}_Y) \simeq \\
&\simeq R\mathcal{H}om([\mathcal{O}(R_i - E_i)) \to \mathcal{O}(R_i)], \mathcal{O}(R_1 + \ldots + R_n)) = \\
&= [R\mathcal{H}om(\mathcal{O}(R_i), \mathcal{O}(R_1 + \ldots + R_n)) \to R\mathcal{H}om(\mathcal{O}(R_i - E_i), \mathcal{O}(R_1 + \ldots + R_n))] = \\
&= [\mathcal{O}(R_1 + \ldots + R_{i-1} + R_{i+1} + \ldots + R_n) \to \mathcal{O}(E_i + R_1 + \ldots + R_{i-1} + R_{i+1} + \ldots + R_n)] = \\
&= \mathcal{O}_{E_i}(E_i + R_1 + \ldots + R_{i-1} + R_{i+1} + \ldots + R_n)[1] \simeq \mathcal{O}_{E_i}(-1)[1].
\end{aligned}$$

Hence, up to a shift, the functor D_f preserves simple objects in $\mathcal{A}_{\text{st}}$.

As the functor $D_f\colon \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ is exact, if $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is a short exact sequence of sheaves such that $D_f(\mathcal{A})$ and $D_f(\mathcal{C})$ are complexes concentrated in only one degree, the same is true about $D_f(\mathcal{B})$. As every object of $\mathrm{Coh}(X) \cap \mathcal{C}_f$ is an iterated extension of sheaves of the form $\mathcal{O}_{E_i}(-1)$ we know that D_f is a contravariant exact functor which maps $\mathrm{Coh}(X) \cap \mathcal{C}_f$ to $\mathrm{Coh}(X) \cap \mathcal{C}_f$ and preserves simple sheaves in this category.

□

Remark 4.5.5. The functor $D_f = R\mathcal{H}om_X(-, f^!(\mathcal{O}_Y))$ is not a duality on $\mathcal{A}_{\text{coe}}$. Already when f is a composition of two blowups, the projective objects in $\mathcal{A}_{\text{coe}}$ are $\mathscr{D}_1 = \mathcal{O}_{E_1+2E_2}(E_1+2E_2)$, $\mathscr{D}_2 = \mathcal{O}_{E_2}(E_2)$ and the simple objects are $S_1 = \mathcal{O}_{E_1+E_2}(E_1+2E_2)[1]$ and $S_2 = \mathcal{O}_{E_1+2E_2}(E_1+2E_2)$. The functor D_f maps S_2 to S_2 but

$$D_f(\mathcal{O}_{E_1+E_2}(E_1+2E_2)) = \mathcal{O}_{E_1+E_2}(E_1+E_2) \notin \mathcal{A}_{\text{coe}}$$

as $\mathrm{Hom}(\mathcal{O}_{E_2}(E_2), \mathcal{O}_{E_1+E_2}(E_1+E_2)) = k = \mathrm{Ext}^1(\mathcal{O}_{E_2}(E_2), \mathcal{O}_{E_1+E_2}(E_1+E_2))$.

4.5.1 Simple objects in $\mathcal{A}_{\text{coe}}$

In [61] Toda constructed a bounded t-structure on the category $\mathcal{D}_f$ defined in Section 4.2 by specifying simple objects in its heart. We show that the t-structure he considers on $\mathcal{D}_f$ corresponds to the t-structure with heart $\mathcal{A}_{\text{coe}}$ on $\mathcal{C}_f$ under the equivalence $\varphi\colon \mathcal{C}_f \to \mathcal{D}_f$ defined in Section 4.2.

Toda's t-structure is constructed by induction. If a map $f\colon X \to Y$ factors as n blow-ups of single points then the heart $\mathcal{D}_f^0 = |S_1, \dots, S_n|$, where $|S_1, \dots, S_n|$ denotes the smallest abelian subcategory of $\mathrm{Coh}(X)$ closed under extensions and containing objects $S_1, \dots, S_n$. These objects are constructed in the following way.

Let $X \xrightarrow{h} X' \xrightarrow{g} Y$ be a decomposition of f with $g\colon X' \to Y$ a blow up of a single point. Then $\mathcal{D}_h^0 = |S'_2, \dots, S'_n|$. We first consider a t-structure on $\mathcal{D}_f$ glued from the t-structure on $\mathcal{D}_h$ and a standard t-structure on $\mathcal{D}_g$ (there, the heart is $\langle \mathcal{O}_{E_1} \rangle$). Then

$$\widetilde{D_f^0} = |\mathcal{D}_h^0, \mathcal{O}_{R_1}|$$

and $(\mathcal{D}_h^0, |\mathcal{O}_{R_1}|)$ is a torsion pair in $\widetilde{D_0^f}$. Also, $(\langle \mathcal{O}_{R_1} \rangle, \mathcal{O}_{R_1}^{\perp})$ is a torsion pair in $\widetilde{D_f^0}$ and $\mathcal{D}_f$ is defined as a tilt with respect to this torsion pair, i.e.

$$\mathcal{D}_f = |\mathcal{O}_{R_1}^{\perp}, \mathcal{O}_{R_1}[-1]|.$$

Then

$$\mathcal{D}_f = |S_1, \dots, S_n|,$$

where $S_1 = \mathcal{O}_{R_1}[-1]$ and S_i are determined by the following universal extensions

$$0 \to S'_i \to S_i \to \mathcal{O}_{R_1} \otimes \mathrm{Ext}^1(\mathcal{O}_{R_1}, S'_i) \to 0.$$

The simple objects S_i can be also constructed explicitly. Recall that f has a decomposition $X_n \xrightarrow{f_n} X_{n-1} \to \ldots \to X_1 \xrightarrow{f_1} X_0$. We put $\pi_i = f_{i+1} \circ \ldots \circ f_n \colon X_n \to X_i$ and $g_{ij} = f_{j+1} \circ \ldots \circ f_i \colon X_i \to X_j$.

Simple objects S_i correspond to irreducible components E_i of the exceptional divisor of f. If E_i is such a curve that the point $f_i(E_i)$ does not lie on any E_j for $j < i$ then the corresponding simple object is $S_i = \mathcal{O}_{R_i}[-1]$. If this is not the case, let us set

$$\kappa(i) = \max\{j \mid f_i(E_i) \in E_j\}.$$

Then on X_i we have a short exact sequence

$$0 \to \bar{S}_i \to g^*_{i,\kappa(i)}(\mathcal{O}_{E_{\kappa(i)}}) \to \mathcal{O}_{E_i} \to 0$$

and

$$S_i = \pi_i^*(\bar{S}_i).$$

The t-structure with heart $\mathcal{A}_{\mathrm{coe}}$ on $\mathcal{C}_f$ has projective objects

$$\mathscr{D}_i = \mathcal{O}_{\sum_{R_j \le R_i} R_j}(\sum_{R_j \le R_i} R_j).$$

Therefore the corresponding t-structure on $\mathcal{D}_f$ has projective objects $\mathcal{R}_i = \mathcal{O}_{\sum_{R_j \le R_i} R_j}$. This follows from the fact that the equivalence $\phi \colon \mathcal{C}_f \to \mathcal{D}_f$ of Section 4.2 is given by tensor product with $\mathcal{O}_X(-R_1 - \ldots - R_n)$ and from the isomorphism $\mathcal{O}_{R_i}(R_k) \simeq \mathcal{O}_{R_i}$ for $R_i < R_k$. We will show that the objects $S_i[1]$ lie in the heart of this t-structure. More precisely, we have the following lemma

Lemma 4.5.6. *Let S_i, $\mathcal{R}_i$ be as above. Then*

$$\hom(\mathcal{R}_i, S_j[1]) = \delta_i^j.$$

Proof. The t-structure with simple objects S_j is defined inductively, so we proceed by induction on the rank of $\mathrm{Pic}(X/Y)$. We also assume that $f(E(f))$ is a single point. If it is not the case the morphism f decomposes as $f = h_1 \circ \ldots \circ h_j$ such that $h_1, \ldots, h_j$ have connected and mutually disjoint exceptional divisors. Then the category $\mathcal{D}_f$ equals $\mathcal{D}_{h_1} \oplus \ldots \oplus \mathcal{D}_{h_l}$ and for each $\mathcal{D}_{h_l}$ this assumption is satisfied.

If the rank of $\mathrm{Pic}(X/Y)$ is one then $S_1 = \mathcal{O}_{E_1}[-1]$ and $\mathcal{R}_1 = \mathcal{O}_{E_1}$. If the rank equals n then f factors as $X \xrightarrow{h} X' \xrightarrow{g} Y$, where $\mathrm{Pic}(X'/Y) = \mathbb{Z}$, $\mathcal{D}_h = \langle S'_2, \ldots, S'_n \rangle$ and $\hom(\mathcal{R}_i, S'_j[1]) = \delta_i^j$ for $i, j \ge 2$. Then the projective object $\mathcal{R}_1$ equals $\mathcal{O}_{R_1 + \ldots + R_n}$ and the simple object S_1 is equal to $\mathcal{O}_{R_1}[-1]$.

Then for $i \geq 2$,

$$\begin{aligned}
&\mathrm{Hom}^{\bullet}(\mathcal{R}_i, S_1[1]) = \mathrm{Hom}^{\bullet}(\mathcal{O}_{R_1}, \mathcal{O}_{\sum_{R_j \leq R_i} R_j}(\sum_{R_j \leq R_i} R_j))^{\vee} = \\
&= \mathrm{Hom}^{\bullet}(\mathcal{O}_{R_1}(R_1), \mathcal{O}_{\sum_{R_j \leq R_i} R_j}(\sum_{R_j \leq R_i} R_j))^{\vee} = 0
\end{aligned}$$

because

$$\mathcal{O}_{\sum_{R_j \leq R_i} R_j}(\sum_{R_j \leq R_i} R_j) \in \langle \mathcal{O}_{R_n}(R_n), \ldots, \mathcal{O}_{R_2}(R_2) \rangle$$

and

$$\langle \langle \mathcal{O}_{R_n}(R_n), \ldots, \mathcal{O}_{R_2}(R_2) \rangle, \mathcal{O}_{R_1}(R_1) \rangle$$

is a semi-orthogonal decomposition of $\mathcal{C}_f$.

From the definition of the objects S_i as universal extensions of $S_1[1]$ by S_i' is also follows that $\mathrm{hom}(\mathcal{R}_i, S_j[1]) = \delta_i^j$ for $i \geq 2$ and $j \geq 1$.

In order to calculate $\mathrm{hom}(\mathcal{R}_1, S_1[1])$ we use the short exact sequence

$$0 \to \mathcal{O}(-R_1) \to \mathcal{O} \to \mathcal{O}_{R_1} \to 0.$$

Applying the functor $\mathrm{Ext}^{\bullet}(-, \mathcal{O}_{R_1+\ldots+R_n}(R_1 + \ldots + R_n))$ we see that

$$\begin{aligned}
&\mathrm{Ext}^{i+1}(\mathcal{O}_{R_1}, \mathcal{O}_{R_1+\ldots+R_n}(R_1 + \ldots + R_n)) \simeq \\
&\simeq \mathrm{Ext}^{i}(\mathcal{O}, \mathcal{O}_{R_1+\ldots+R_n}(2R_1 + R_2 + \ldots + R_n) = \begin{cases} k & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

The last equality follows from the sequence

$$0 \to \mathcal{O}_{R_1}(2R_1) \to \mathcal{O}_{R_1+\ldots+R_n}(2R_1 + R_2 + \ldots + R_n) \to \mathcal{O}_{R_2+\ldots+R_n}(R_2 + \ldots + R_n) \to 0.$$

Thus we know that the only non-zero Ext group between $\mathcal{O}_{R_1}$ and $\mathcal{O}_{R_1+\ldots+R_n}(R_1+\ldots+R_n)$ is Ext^2 which by Serre duality proves that $\mathrm{Hom}(\mathcal{R}_1, S_1[1]) = k$ and other Ext-groups vanish.

It remains to show that $\mathrm{Hom}^{\bullet}(\mathcal{R}_i, S_1) = 0$ for $i \neq 1$. In order to prove this we first check when the group $\mathrm{Ext}^1(\mathcal{O}_{R_1}, S_i')$ vanishes.

Let us choose some i. First, let us assume that $\kappa(i) = 1$. Then $S_i' = \mathcal{O}_{R_i}[-1]$ and applying $\mathrm{Hom}(-, S_i'[1])$ to the exact sequence $0 \to \mathcal{O}(-R_1) \to \mathcal{O} \to \mathcal{O}_{R_1} \to 0$ we get

$$\mathrm{Hom}(\mathcal{O}_{R_1}, S_i'[1]) = k, \quad \mathrm{Ext}^1(\mathcal{O}_{R_1}, S_i'[1]) = k,$$

i.e. Ext^1 and Ext^2 groups between $\mathcal{O}_{R_1}$ and S_i' do not vanish. From the definition of S_i as the universal extension we immediately obtain that

$$\mathrm{Ext}^2(\mathcal{O}_{R_1}, S_i) = k, \quad \mathrm{Ext}^j(\mathcal{O}_{R_1}, S_i) = 0 \text{ for } j \neq 2.$$

On the other hand, if $\kappa(i) > 1$ then

$$0 \to \bar{S}_i \to g^*_{i,\kappa i}(\mathcal{O}_{E_{\kappa(i)}}) \to \mathcal{O}_{E_i} \to 0,$$

from which it follows that $g_{i,\kappa(i)_*}(\bar{S}_i) = \mathcal{O}_{E_{\kappa(i)}}(-1)$ and hence

$$f_* S_i = g_{i,1_*}(\bar{S}_i) = g_{\kappa(i),1_*}(\mathcal{O}_{E_{\kappa(i)}}(-1)) = 0.$$

Then, by adjunction $\operatorname{Hom}(\mathcal{O}_{R_1}, S_i) = \operatorname{Hom}(\mathcal{O}_{E_1}, f_*(S_i)) = 0$.

Finally, we notice that $\mathcal{R}_1$ fits into a short exact sequence

$$0 \to \bigoplus_{\{j \mid \kappa(j)=1\}} \mathcal{R}_j \to \mathcal{R}_1 \to \mathcal{O}_{R_1} \to 0.$$

Applying $\operatorname{Hom}(-, S_i)$ to this sequence we get that $\operatorname{Hom}^\bullet(\mathcal{R}_1, S_i) = 0$, which finishes the proof. $\square$

Proposition 4.5.7. *Let S_i be as above. Then $S_i \otimes \mathcal{O}(\sum R_j)$ are, up to isomorphsism, all simple objects in $\mathcal{A}_{coe}$.*

Proof. The category $\mathcal{A}_{\mathrm{coe}}$ is equivalent to the category of modules over the endomorphism algebra of $\bigoplus_{i=1}^n \mathscr{D}_i$. Hence it has n simple objects and every object in $\mathcal{A}_{\mathrm{coe}}$ admits a filtration by those simple modules. In particular, a simple object S_i appears in a filtration of M exactly $\dim_k \operatorname{Hom}(\mathscr{D}_i, M)$ times. The proposition follows from Lemma 4.5.6. $\square$

Chapter 5

Birational morphisms of surfaces and representation theory of simple Lie algebras

In this chapter we show that for some smooth morphisms $f\colon X \to Y$ the highest weight category $\mathcal{C}_f \cap \mathrm{Coh}(X)$ described in Chapter 4 appears also in representation theory of semisimple Lie algebras. More precisely, this happens if f is a composition of n blow-ups of points such that the $(i+1)$'st map blows up a point on the exceptional divisor of the i'th map. Equivalently, the exceptional divisor of f is a chain of $n-1$ rational curves of self-intersection -2 and one curve of self-intersection -1. In Theorem 5.3.12 we prove that for such an f the category $\mathcal{C}_f \cap \mathrm{Coh}(X)$ is equivalent to a certain block $\mathcal{O}_{\omega_1}$ of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for $\mathrm{sl}(n,k)$.

The proof of Theorem 5.3.12 is based on comparing endomorphism algebras of projective generators of $\mathcal{C}_f \cap \mathrm{Coh}(X)$ and $\mathcal{O}_{\omega_1}$. In fact, those endomorphism algebras are uniquely determined by Ext-algebras of standard objects $\mathcal{O}_{R_i}(R_i)$ in $\mathcal{C}_f \cap \mathrm{Coh}(X)$ and Verma modules $M(\lambda_i)$ in $\mathcal{O}_{\omega_1}$. In Chapter 3 we calculated the first Ext-algebra. In this chapter we calculate the Ext-algebra of Verma modules in $\mathcal{O}_{\omega_1}$ and prove that it is isomorphic to the Ext-algebra of $\mathcal{O}_{R_i}(R_i)$.

We begin with Section 5.1 is which we recall after [27] and [28] all definitions and facts about Lie algebras that we will use. We also compute the linkage class of weight $\lambda_1 = \omega_1 - \rho$ that determines the block $\mathcal{O}_{\omega_1}$ (see Lemma 5.1.2).

Then in Sections 5.2.1 and 5.2.2 we define the category $\mathcal{O}$ and give, following [28], all of its properties that we use in our calculations in Section 5.3. We particularly focus on constructing projective covers $P(\lambda)$ of Verma modules $M(\lambda)$. The detailed description of these modules is the most important part of Section 5.3.

5.1 Review of simple Lie algebras

Let $\mathfrak{g}$ be a simple Lie algebra. Let $\mathfrak{h}$ be its *Cartan subalgebra*. For a character α of $\mathfrak{h}$ let

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \,|\, [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$$

denote a *root space*. The algebra $\mathfrak{g}$ has a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$$

where $\Phi \subset \mathfrak{h}^*$ is a root system, i.e. the set of all characters such that $\mathfrak{g}_\alpha$ in non-zero. For any $\alpha \in \Phi$ the dimension on $\mathfrak{g}_\alpha$ is one. A hyperplane in $\mathfrak{h}^*$ which does not contain any root gives a decomposition of Φ into a set of positive, and negative roots denoted by Φ^+ and Φ^- respectively. *Simple roots* $\Delta \subset \Phi^+$ are such roots that any positive root is a $\mathbb{Z}$-linear combination of simple roots with non-negative coefficients.

We are mostly interested in the case $\mathfrak{g} = \mathrm{sl}(n, k)$. The Cartan subalgebra $\mathfrak{h} \subset \mathrm{sl}(n, k)$ consists of diagonal matrices. Let $e_{i,j}$ denote the basis matrix with 1 in the i-th row and j-th column. Let us denote by $e^{i,j}$ the dual basis. If $i < j$ the algebra of diagonal matrices acts on $e_{i,j}$ via the character $e^{i,i} - e^{j,j}$. Thus, the root system of $\mathrm{sl}(n, k)$ consists of elements $e^{i,i} - e^{j,j}$. If we denote $\alpha_i = e^{i,i} - e^{i+1,i+1}$ then $\alpha_1, \ldots, \alpha_{n-1}$ are simple roots and

$$\Phi^+ = \{\alpha_i + \alpha_{i+1} + \ldots + \alpha_l\}_{\{1\leq i\leq l<n\}}.$$

With such a choice the subalgebra $\mathfrak{n}^+ \subset \mathfrak{g}$ generated by $x \in \mathfrak{g}_\alpha$ for $\alpha \in \Phi^+$, is the algebra of upper triangular matrices and the algebra $\mathfrak{n}^- \subset \mathfrak{g}$ generated by $y \in \mathfrak{g}_{-\alpha}$, $\alpha \in \Phi^+$ is the algebra of lower triangular matrices.

The Killing form

$$\kappa(x, y) = \mathrm{tr}(\mathrm{ad}x \circ \mathrm{ad}y)$$

is a non-degenerate bilinear form on $\mathfrak{g}$. If $\mathfrak{g} = \mathrm{sl}(n, k)$ then $\kappa(x, y) = 2n \,\mathrm{tr}(xy)$.

The root system spans a $\mathbb{Q}$-subspace E of $\mathfrak{h}^*$ on which the Killing form in non-degenerate. Thus E is an euclidean space with inner product (λ, μ).

For any $\alpha \in \Phi$ we define a *coroot* $\alpha^\vee \in E^*$. The element $\alpha^\vee$ is uniquely determined by the equality

$$\langle\beta, \alpha^\vee\rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)},$$

where $\langle\, ,\, \rangle \colon E^* \times E \to k$ is the canonical pairing of a vector space and its dual.

We have already seen that for $\mathfrak{g} = \mathrm{sl}(n, k)$ there are $n - 1$ simple roots. We can order them in such a way that

$$\langle\alpha_i, \alpha_i^\vee\rangle = 2, \qquad \langle\alpha_i, \alpha_{i+1}^\vee\rangle = -1, \qquad \langle\alpha_i, \alpha_j^\vee\rangle = 0 \text{ for } |i - j| > 1.$$

The *Weyl group* W is the group of all automorphisms of $\mathfrak{h}^*$ which preserve the root system Φ. For any $\alpha \in \Phi$ the group W contains a reflection in a hyperplane orthogonal to α. This reflection is given by

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha.$$

Moreover, the Weyl group is generated by s_α for $\alpha \in \Delta$.

For $\mathrm{sl}(n,k)$ every simple root has the form $\alpha_i = e^{i,i} - e^{i+1,i+1}$ and so s_{α_i} acts on $\mathfrak{h}^* \simeq k^n$ as a transposition of i-th and $(i+1)$-st coordinates. It shows that the Weyl group for $\mathrm{sl}(n,k)$ is a symmetric group Σ_n. A root $\alpha = \alpha_i + \ldots + \alpha_l \in \Phi^+$ acts by transposition of i-th and l-th element. Hence,

$$s_\alpha = s_{\alpha_i} \circ s_{\alpha_{i+1}} \circ \ldots \circ s_{\alpha_{l-1}} \circ s_{\alpha_l} \circ s_{\alpha_{l-1}} \circ \ldots \circ s_{\alpha_{i+1}} \circ s_{\alpha_i}.$$

Next we consider the *universal enveloping algebra* $U(\mathfrak{g})$, i.e. an associative unital k-algebra with a morphism $\varphi \colon \mathfrak{g} \to U(\mathfrak{g})$ such that any morphism from $\mathfrak{g}$ to an associative unital k-algebra factors through $U(\mathfrak{g})$. The Poincaré-Bikhoff-Witt theorem gives a basis of $U(\mathfrak{g})$ (see [39, Theorem 5.11]).

Theorem 5.1.1 (Poincaré-Birkhoff-Witt)**.** *Let $x_1, \ldots, x_r$ be an ordered basis of $\mathfrak{g}$. Then the monomials $x_1^{t_1} \ldots x_r^{t_t}$ form a basis of $U(\mathfrak{g})$.*

For $\mathrm{sl}(n,k)$ we denote by $x_i = e_{i,i+1}$, $y_i = e_{i+1,i}$ elements corresponding to a simple root α_i and by $x_{i,k} \in \mathfrak{n}^+$, $y_{i,k} \in \mathfrak{n}^-$ elements corresponding to a root $\alpha_i + \ldots + \alpha_k$, for $i < k$. Representing x and y as matrices gives the commutation laws in $U(\mathrm{sl}(n,k))$.

We fix a lexicographic order on Φ^+. This establishes an order on the bases of $\mathfrak{n}^-$, $\mathfrak{h}$ and $\mathfrak{n}^+$. Finally, we order the basis of $\mathrm{sl}(n,k)$ by saying that vectors of $\mathfrak{n}^-$ are smaller than vectors of $\mathfrak{h}$ and $\mathfrak{n}^+$ while vectors of $\mathfrak{h}$ are smaller than vectors of $\mathfrak{n}^+$. According to Theorem 5.1.1 the order on the basis of $\mathrm{sl}(n,k)$ establishes a basis of $U(\mathrm{sl}(n,k))$.

Inside E we have the root lattice Λ_r spanned by Φ. We also consider a lattice

$$\Lambda = \{\lambda \in E \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \ \forall \alpha \in \Delta\}$$

spanned by *fundamental weights* ω_i, i.e. elements of E determined by the condition

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}.$$

Finally, we denote by $\Gamma \subset \Lambda$ the set of linear combinations of simple roots with coefficients in $\mathbb{Z}^+$ which gives us a *partial order* on Λ

$$\mu \leq \lambda \Leftrightarrow \lambda - \mu \in \Gamma.$$

The Weyl group W can also act on $\mathfrak{h}^*$ by the *dot action*,

$$w \cdot \lambda = w(\lambda + \rho) - \rho,$$

where

$$\rho = \omega_1 + \ldots + \omega_l$$

is a sum of fundamental weights. Equivalently, we can define ρ as

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

The orbit $W \cdot \lambda = \{w \cdot \lambda \,|\, w \in W\}$ is called the *linkage class* of λ.

We say that a weight λ is *regular* if $|W \cdot \lambda| = |W|$. Otherwise, λ is called *singular.* One can see that λ is regular if $\langle \lambda + \rho, \alpha^\vee \rangle \neq 0$ for all $\alpha \in \Phi$.

We say that a weight $\lambda \in \Lambda$ is *dominant* if $s_\alpha \cdot \lambda \leq \lambda$ for all $\alpha \in \Phi^+$. Analogously, $\lambda \in \Lambda$ is *antidominant* if $s_\alpha \cdot \lambda \geq \lambda$ for all $\alpha \in \Phi^+$. More generally, a weight $\lambda \in \mathfrak{h}^*$ is *dominant* if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^{<0}$ for all $\alpha \in \Phi^+$ and it is *antidominant* if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}^{>0}$ for all $\alpha \in \Phi^+$.

For weights $\lambda, \mu \in \mathfrak{h}^*$ we write $\mu \uparrow \lambda$ if $\mu = \lambda$ or there is a root $\alpha \in \Phi^+$ such that $\mu = s_\alpha \cdot \lambda < \lambda$; in other words $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}^{>0}$. More generally, we say that μ is *strongly linked* to λ if $\mu = \lambda$ or there exist $\alpha_1, \ldots, \alpha_r \in \Phi^+$ such that $\mu = (s_{\alpha_1} \ldots s_{\alpha_r}) \cdot \lambda \uparrow (s_{\alpha_2} \ldots s_{\alpha_r}) \cdot \lambda \uparrow \ldots \uparrow s_{\alpha_r} \cdot \lambda \uparrow \lambda$.

For $\lambda \in \Lambda$ the set of positive roots α such that $s_\alpha \cdot \lambda < \lambda$ is

$$\Phi^+(\lambda) = \{\alpha \in \Phi^+ | \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{N} \setminus 0\}.$$

Thus, $\mu \uparrow \lambda$ if and only if $\mu = s_\alpha \cdot \lambda$ and $\alpha \in \Phi^+(\lambda)$.

Lemma 5.1.2. *Let $\mathfrak{g} = \mathrm{sl}(n,k)$ with simple roots $\alpha_1, \ldots, \alpha_{n-1}$. Then the linkage class of $\lambda_1 = \omega_1 - \rho$ is equal to*

$$W \cdot \lambda_1 = \{\lambda_1, \ldots, \lambda_n\}; \quad \lambda_i = \lambda_1 - \sum_{l=1}^{i-1} \alpha_l.$$

Moreover, for $i > j$ the weight λ_i is strongly linked to λ_j and $|\Phi^+(\lambda_i)| = n - i$.

Proof. First, we notice that for $n-1 > i > 1$ we have

$$\langle \alpha_{i-1} + \alpha_i + \alpha_{i+1}, \alpha_i^\vee \rangle = -1 + 2 - 1 = 0.$$

For $i = 1$ we have

$$\langle \omega_1 - \alpha_1 + \alpha_2, \alpha_1^\vee \rangle = 1 - 2 + 1 = 0.$$

Thus,

$$\langle \omega_1 - \sum_{k=1}^{i-1} \alpha_k, \alpha_j \rangle = 0$$

if $j \neq i-1, i$. Also

$$\langle \omega_1 - \sum_{k=1}^{i-1} \alpha_k, \alpha_{i-1} \rangle = -1, \quad \langle \omega_1 - \sum_{k=1}^{i-1} \alpha_k, \alpha_i \rangle = 1.$$

Hence

$$s_{\alpha_i} \cdot \lambda_j = \begin{cases} \lambda_j & \text{if } i \neq j-1, j, \\ \lambda_{j-1} & \text{if } i = j-1, \\ \lambda_{j+1} & \text{if } i = j. \end{cases}$$

As s_{α_i} are generators of the Weyl group W, the linkage class of λ_1 consists of $\lambda_1, \ldots, \lambda_n$. We have also seen that for a positive root $\alpha = \alpha_i + \ldots + \alpha_l$ we have $s_\alpha = s_{\alpha_i} \circ \ldots s_{\alpha_l} \circ \ldots \circ s_{\alpha_i}$. Hence,

$$s_{\alpha_i + \ldots + \alpha_l} \cdot \lambda_i = \lambda_{l+1} \quad s_{\alpha_i + \ldots + \alpha_l} \cdot \lambda_{l+1} = \lambda_i$$

and the remaining positive roots fix λ_i. It proves that for $l \geq i$ we have $\lambda_{l+1} \uparrow \lambda_i$ and

$$|\Phi^+(\lambda_i)| = |\{\alpha_i,\, \alpha_i + \alpha_{i+1}, \ldots, \alpha_i + \ldots + \alpha_{n-1}\}| = n - i.$$

□

5.2 The Bernstein Gelfand Gelfand category $\mathcal{O}$

5.2.1 Definition

For a module M over $\mathfrak{g}$ we define

$$M_\lambda = \{v \in M \mid h(v) = \lambda(h)v \,\forall h \in \mathfrak{h}\} \quad \Pi(M) = \{\lambda \in E \mid M_\lambda \neq \{0\}\}.$$

Following [28, Definition 1.1] we define the category $\mathcal{O}$ as a full subcategory of Mod$-U(\mathfrak{g})$ whose objects are $U(\mathfrak{g})$-modules M such that

($\mathcal{O}1$) M is a finitely generated $U(\mathfrak{g})$-module.

($\mathcal{O}2$) M is $\mathfrak{h}$-semisimple, i.e. M is a weight module, $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$.

($\mathcal{O}3$) M is locally $\mathfrak{n}$-finite; for each $v \in M$ the subspace $U(\mathfrak{n})v$ of M is finite dimensional.

($\mathcal{O}4$) All weight spaces of M are finite dimensional.

($\mathcal{O}5$) The set $\Pi(M)$ of all weights is contained in the union of finitely many sets of the form $\lambda - \Gamma$, where $\lambda \in \mathfrak{h}^*$ and Γ is the semigroup in Λ_r generated by Φ^+.

In fact, the last two conditions follow from the first three (see [28, Proposition 1.1]).

Let M be a module in the category $\mathcal{O}$. A non-zero vector $v^+ \in M$ is a *maximal vector* of weight λ if $v^+ \in M_\lambda$ and $\mathfrak{n} \cdot v^+ = 0$. Every non-zero module in $\mathcal{O}$ has at least one maximal vector.

A module M is a *highest weight module* of weight λ if there exists a maximal vector $v^+ \in M_\lambda$ such that $M = U(\mathfrak{g})\, v^+$. By the PBW decomposition such an M has the form $M = U(\mathfrak{n}^-)\, v^+$.

Theorem 5.2.1. *([28, Theorem 1.2]) Let M be the highest weight module of weight $\lambda \in \mathfrak{h}^*$ generated by a maximal vector v^+. Fix an ordering of positive roots $\alpha_1, \ldots, \alpha_m$ and choose corresponding root vectors y_i in $\mathfrak{g}_{-\alpha_i}$. Then*

- *M is spanned by vectors $y_1^{i_1} \ldots y_m^{i_m} v^+$ with $i_j \in \mathbb{Z}^+$ having respective weights $\lambda - \sum i_j \alpha_j$. Thus M is a semisimple $\mathfrak{h}$-module.*

- *All weights μ of M satisfy $\mu \leq \lambda$, i.e. $\mu = \lambda -$ (sum of positive roots).*

- *For all weights μ of M we have $\dim M_\mu < \infty$ while $\dim M_\lambda = 1$. Thus M is a weight module, locally finite as an $\mathfrak{n}$-module and $M \in \mathcal{O}$.*

- *Each non-zero quotient of M is again a highest weight module of weight λ.*

- *Each submodule of M is a weight module. A submodule generated by a maximal vector of weight $\mu < \lambda$ is proper; in particular, if M is simple its maximal vectors are all multiples of v^+.*

- *M has a unique maximal submodule and a unique simple quotient. In particular, M is indecomposable.*

- *All simple highest weight modules of weight λ are isomorphic.*

Verma modules are the first examples of highest weight modules. In order to construct them we choose a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ which has an abelian quotient algebra $\mathfrak{b}/\mathfrak{n} \simeq \mathfrak{h}$. For any $\lambda \in \mathfrak{h}^*$ we consider a one-dimensional module k_λ on which $\mathfrak{h}$ acts via the character. k_λ can be also considered as a $\mathfrak{b}$ module with a trivial $\mathfrak{n}$ action. We set

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\lambda, \quad M(\lambda) \simeq U(\mathfrak{n}^-) \otimes k_\lambda \text{ as left } U(\mathfrak{n}^-)\text{-module.}$$

As $M(\lambda)$ is generated by one vector v^+ on which $\mathfrak{n}$ acts trivially and $\mathfrak{h}$ acts via λ we get another description, namely $M(\lambda) = U(\mathfrak{g})/I$ where I is an ideal generated by $\mathfrak{n}$ and $(h - \lambda(h)1)$ for $h \in \mathfrak{h}$.

We will denote by $L(\lambda)$ the unique simple quotient of $M(\lambda)$.

Let $Z(\mathfrak{g})$ be the centre of $U(\mathfrak{g})$ and let $M \in \mathcal{O}$ be a highest weight module generated by a maximal vector v^+ of weight λ. Then, for $z \in Z(\mathfrak{g})$ and $h \in \mathfrak{h}$ we have

$$h \cdot (z \cdot v^+) = z \cdot (h \cdot v^+) = z \cdot (\lambda(h)v^+) = \lambda(h)\, z \cdot v^+.$$

Because $\dim M_\lambda = 1$ above equality forces $z \cdot v^+ = \chi_\lambda(z)v^+$ for some scalar $\chi_\lambda(z) \in k$. The element z acts by this scalar on all vectors of M because M is spanned $U(\mathfrak{n}) \cdot v^+$ and z lies in the centre of $U(\mathfrak{g})$.

Thus, a fixed λ gives a character $\chi_\lambda \colon Z(\mathfrak{g}) \to k$. It turns out that $\chi_\lambda = \chi_\mu$ if μ is linked to λ (see [28, Proposition 1.8]). Moreover, every character $\chi\colon Z(\mathfrak{g}) \to k$ is of the form χ_λ for some $\lambda \in E$ (see [28, Theorem 1.10]).

For any $M \in \mathcal{O}$ and a character χ of $Z(\mathfrak{g})$ we define

$$M^\chi = \{v \in M \,|\, \forall z \in Z(\mathfrak{g})\, (z - \chi(z))^n v = 0 \,\text{for some}\, n > 0 \,\text{depending on}\, z\}.$$

We denote by $\mathcal{O}_\chi$ the full subcategory of $\mathcal{O}$ containing modules M such that $M = M^\chi$. Then the category $\mathcal{O}$ is a direct sum of categories $\mathcal{O}_\chi$ as χ ranges over the characters of $Z(\mathfrak{g})$ (see [28, Proposition 1.12]). Moreover, we have the following proposition

Proposition 5.2.2. *([28, Proposition 1.13]) If $\lambda \in \Lambda$ the subcategory $\mathcal{O}_{\chi_\lambda}$ is a block of $\mathcal{O}$.*

Let us recall that a full subcategory $\mathcal{A}$ of an abelian category $\overline{\mathcal{A}}$ is called a *block* if $\mathcal{A}$ is closed under extensions and it is minimal with this property.

It turns out that for $\lambda \notin \Lambda$ the category $\mathcal{O}_{\chi_\lambda}$ can be further decomposed into blocks.

5.2.2 Properties of the category $\mathcal{O}$

By [28, Theorem 1.11] each module $M \in \mathcal{O}$ is both artinian and noetherian. Moreover, each M possesses a composition series with simple quotients $L(\lambda)$ (recall that $L(\lambda)$ is the unique simple quotient of $M(\lambda)$). The multiplicity of $L(\lambda)$ is independent of the composition series and is denoted by $[M : L(\lambda)]$. It is called Jordan-Höelder multiplicity. Thus the Grothendieck group of $\mathcal{O}$ is generated by classes of simple modules $[L(\lambda)]$.

Furthermore, thanks to [28, Proposition 1.15 and 6.14] we have

Proposition 5.2.3. *For any $N \in \mathcal{O}$ we have*

$$[N] = \sum_\mu \mathcal{X}(\mathrm{Ext}^\bullet_{\mathcal{O}}(M(\mu), N))[M(\mu)],$$

where $\mathcal{X}(\mathrm{Ext}^\bullet_{\mathcal{O}}(M(\mu), N)) = \sum (-1)^i \dim \mathrm{Ext}^i_{\mathcal{O}}(M(\mu), N)$ is the Euler characteristic.

It turns out that the weights $\lambda \in \mathfrak{h}$ give a lot of information about the corresponding Verma modules $M(\lambda)$ and relations between them.

Theorem 5.2.4. *([28, Theorem 4.8]) Let $\lambda \in \mathfrak{h}^*$. Then $M(\lambda) = L(\lambda)$ if and only if λ is antidominant.*

Proposition 5.2.5. *([28, Proposition 3.8]) If $\lambda \in \mathfrak{h}^*$ is dominant then $M(\lambda)$ is projective in $\mathcal{O}$. Moreover, if P is projective and L is finite-dimensional then $P \otimes L$ is also projective.*

Following [28, Section 3.6] we describe a module $T = M(\lambda) \otimes_k L$ for a finite-dimensional module L. If $\dim L = l$ then the module T admits a filtration $0 = T_0 \subset T_1 \subset \ldots \subset T_l = T$ with quotients $M(\lambda + \mu)$, where μ ranges over the weights of L (see [28, Theorem 3.6]). To construct the filtration we choose a basis $v_1, \ldots, v_l$ of L. We assume that v_i has weight μ_i and that from the fact that $\mu_i < \mu_j$ it follows that $i > j$. If we denote by v^+ the maximal vector of $M(\lambda)$ of weight λ then vectors $t_1 = v^+ \otimes v_1, \ldots t_l = v^+ \otimes v_l$ generate T as a $U(\mathfrak{g})$-module. The filtration $T_\bullet$ is obtained by taking T_i to be the submodule of T generated by $t_1, \ldots, t_i$. Direct calculations show that the image of t_i under the quotient map $T_i \to T_i/T_{i-1}$ is the maximal vector of weight $\lambda+\mu_i$ of the module $M(\lambda+\mu_i) \simeq T_i/T_{i-1}$ (see [28, Section 3.6]).

In general, we say that a module $M \in \mathcal{O}$ has a *standard filtration* if it admits a filtration with quotients $M(\lambda)$ for $\lambda \in \mathfrak{h}^*$. The multiplicity of $M(\lambda)$ in this filtration is denoted by $(M : M(\lambda))$. In the above example $T = M(\lambda) \otimes_k L$ has a standard filtration and $(T : M(\lambda + \mu)) = \dim L_\mu$.

The above description allows us to find for any simple module $L(\lambda)$ a projective module P and an epimorphism $P \twoheadrightarrow L(\lambda)$. In fact, instead of dealing with $L(\lambda)$, itself we shall construct a map $P \twoheadrightarrow M(\lambda)$ and then compose it with the projection $M(\lambda) \twoheadrightarrow L(\lambda)$. For any λ there exists $l \in \mathbb{N}$ such that $\lambda + l\rho$ is dominant. Then, thanks to Proposition 5.2.5, the module $M(\lambda+l\rho)$ is projective and so are all the modules obtained by tensoring $M(\lambda+l\rho)$ by a finitely-dimensional module. Finally, module $L(l\rho)$ is finitely-dimensional (see [28, Theorem 1.6]) and its minimal weight is $-l\rho$. Thus, the filtration of $M(\lambda+l\rho)\otimes_k L(l\rho)$ described above gives an epimorphism $M(\lambda + l\rho) \otimes_k L(l\rho) \twoheadrightarrow M(\lambda)$.

We denote the projective cover of $L(\lambda)$ by $P(\lambda)$. From the above discussion it follows that $P(\lambda)$ is a direct summand of $M(\lambda + k\rho) \otimes_k L(k\rho)$ for any $k \in \mathbb{N}$ such that $\lambda + k\rho$ is dominant.

Theorem 5.2.6. *([28, Theorem 3.9]) The modules $P(\lambda)$ with $\lambda \in \mathfrak{h}^*$ satisfy the following property:*

- *Every indecomposable projective module in category $\mathcal{O}$ is isomorphic to some projective cover $P(\lambda)$.*
- *When a projective module $P \in \mathcal{O}$ is written as a direct sum of indecomposables modules then the number of direct summands isomorphic to $P(\lambda)$ is equal to* $\dim \operatorname{Hom}_{\mathcal{O}}(P, L(\lambda))$.

- *For all* $M \in \mathcal{O}$
$$\dim \operatorname{Hom}_{\mathcal{O}}(P(\lambda), M) = [M : L(\lambda)].$$
In particular, $\dim End_{\mathcal{O}} P(\lambda) = [P(\lambda) : L(\lambda)]$.

We obtained the module $P(\lambda)$ as a submodule of $M(\lambda + l\rho) \otimes_k L(l\rho)$. The module $P(\lambda)$ also has a standard filtration. Moreover, there are restrictions on which $M(\mu)$ can appear as quotients of this filtration.

Theorem 5.2.7. *([28, Theorem 3.10]) Each projective module in* $\mathcal{O}$ *has a standard filtration. In a standard filtration of* $P(\lambda)$*,* $(P(\lambda) : M(\lambda)) = 1$ *and the multiplicity* $(P(\lambda) : M(\mu))$ *is non-zero only if* $\mu \geq \lambda$*.*

It turns out that there exists a relation between multiplicities of $L(\lambda)$ in a composition series and $M(\mu)$ in a standard filtration.

Theorem 5.2.8. *(BGG reciprocity, [28, Theorem 3.11]) Let* $\lambda, \mu \in \mathfrak{h}^*$*. Then*
$$(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)].$$

Moreover, there cannot be many maps between Verma modules, i.e. we have the following:

Theorem 5.2.9. *([28, Theorem 4.2]) Let* $\lambda, \mu \in \mathfrak{h}^*$*.*

- *Any non-zero homomorphism* $\phi \colon M(\mu) \to M(\lambda)$ *is injective.*

- *In all cases,* $\dim \operatorname{Hom}_{\mathcal{O}}(M(\mu), M(\lambda)) \leq 1$*.*

Furthermore, we have

Theorem 5.2.10. *([28, Theorem 5.1]) Let* $\lambda, \mu \in \mathfrak{h}^*$*. If* μ *is strongly linked to* λ *then* $M(\mu) \hookrightarrow M(\lambda)$*. In particular* $[M(\lambda) : L(\mu)] \neq 0$*. Moreover, if* $[M(\lambda) : L(\mu)] \neq 0$ *then* μ *is strongly linked to* λ*.*

5.3 Singular block of the category $\mathcal{O}$ and the category $\operatorname{Coh}(X) \cap \mathcal{C}_f$

Let $\mathfrak{g} = \mathrm{sl}(n, k)$. The aim of this section is to show that the singular block of the category $\mathcal{O}$ associated with the linkage class of $\lambda_1 = \omega_1 - \rho$ is equivalent to the category $\operatorname{Coh}(X) \cap \mathcal{C}_f$ for a birational morphism $f \colon X \to Y$ of smooth surfaces such that the exceptional divisor of f is a chain of $n-1$ curves of self-intersection (-2) and one curve of self-intersection (-1).

We have seen in Section 4.5 that in this case $\mathrm{Coh}(X) \cap \mathcal{C}_f$ is a highest weight category with n standard objects $\mathcal{O}_{R_n}(R_n), \ldots, \mathcal{O}_{R_1}(R_1)$ such that

$$\mathrm{Hom}(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_j}(R_j)) = \begin{cases} k & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover,

$$\mathrm{Ext}^1(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_j}(R_j)) = \begin{cases} k & \text{if } i > j, \\ 0 & \text{otherwise} \end{cases}$$

and the compositions

$$\mathrm{Hom}(\mathcal{O}_{R_j}(R_j), \mathcal{O}_{R_k}(R_k)) \otimes \mathrm{Ext}^1(\mathcal{O}_{R_l}(R_l), \mathcal{O}_{R_j}(R_j)) \to \mathrm{Ext}^1(\mathcal{O}_{R_l}(R_l), \mathcal{O}_{R_k}(R_k)),$$
$$\mathrm{Ext}^1(\mathcal{O}_{R_j}(R_j), \mathcal{O}_{R_k}(R_k)) \otimes \mathrm{Hom}(\mathcal{O}_{R_l}(R_l), \mathcal{O}_{R_j}(R_j)) \to \mathrm{Ext}^1(\mathcal{O}_{R_l}(R_l), \mathcal{O}_{R_k}(R_k)),$$

are non-zero whenever the domain is non-zero.

Remark 5.3.1. For a map f as above calculating the projective generator of $\mathrm{Coh}(X) \cap \mathcal{C}_f$ via universal extensions can be done explicitly; we have $\mathcal{P}_i = \mathcal{O}_{R_i + \ldots + R_n}(R_i + \ldots + R_n)$ and the endomorphism algebra of $\bigoplus \mathcal{P}_i$ is a path algebra of the following quiver

$$\begin{tikzcd} \mathcal{P}_n \arrow[r, shift left, "\alpha_n"] & \mathcal{P}_{n-1} \arrow[l, shift left, "\beta_{n-1}"] \arrow[r, shift left, "\alpha_{n-1}"] & \cdots \arrow[l, shift left, "\beta_{n-2}"] \arrow[r, shift left, "\alpha_3"] & \mathcal{P}_2 \arrow[l, shift left, "\beta_2"] \arrow[r, shift left, "\alpha_2"] & \mathcal{P}_1 \arrow[l, shift left, "\beta_1"] \end{tikzcd}$$

with relations

$$\alpha_2\beta_1 = 0, \qquad \alpha_{i+1}\beta_i = \beta_{i-1}\alpha_i, \qquad i = 2, \ldots, n-1.$$

Thus, in this case the endomorphism algebra of the projective generator of $\mathrm{Coh}(X) \cap \mathcal{C}_f$ is uniquely determined by the Ext-algebra of standard objects.

Jordan-Hölder multiplicities in the singular block

Thanks to Theorem 5.2.10 we know that $[M(\lambda_i) : L(\lambda_j)] = 0$ for $i > j$. Now, we will prove

Proposition 5.3.2. *For $i \leq j$ we have*

$$[M(\lambda_i) : L(\lambda_j)] = 1.$$

In order to do this we use the following

Theorem 5.3.3. *([30, Theorem 5.20]) Let $\lambda, \mu \in \mathfrak{h}^*$ with $\mu \uparrow \lambda$. Then, the following are equivalent*

- $[M(\lambda) : L(\mu)] = 1,$

- *For all $\mu' \in \mathfrak{h}^*$ such that $\mu' \neq \lambda$ and $\mu \uparrow \mu' \uparrow \lambda$ we have $[M(\mu') : L(\mu)] = 1$ and*

$$|\Phi^+(\lambda)| - |\Phi^+(\mu)| = |\{\alpha \in \Phi^+(\lambda) \mid \mu \uparrow s_\alpha \cdot \lambda\}|.$$

Proof of Proposition 5.3.2. .

From Theorem 5.2.4 it follows that $M(\lambda_n) = L(\lambda_n)$. For λ_i with $i > n$ we proceed by induction.

Assume that $[M(\lambda_i) : L(\lambda_j)] = 1$ for all $i < l$ and $j \leq i$. From the BGG reciprocity (5.2.8) and Theorem 5.2.7 it follows that $[M(\lambda_l) : L(\lambda_l)] = 1$ and $[M(\lambda_l) : L(\lambda_j)] = 0$ for $j < l$. If $j > l$ then by Lemma 5.1.2 if $\mu \neq \lambda_l$ and $\lambda_j \uparrow \mu \uparrow \lambda_l$ then $\mu \in \{\lambda_j, \lambda_{j-1}, \ldots, \lambda_{l+1}\}$. By the induction hypothesis for all such μ we have $[M(\mu) : L(\lambda_j)] = 1$. Moreover, thanks to Lemma 5.1.2 $|\Phi^+(\lambda_l)| = n - l$, $|\Phi^+(\lambda_j)| = n - j$ and

$$\{\alpha \in \Phi^+(\lambda_l) \mid \lambda_j \uparrow s_\alpha \cdot \lambda_l\}, = \{\alpha_l, \alpha_l + \alpha_{l+1}, \ldots, \alpha_l + \ldots + \alpha_{j-1}\}.$$

Hence $[M(\lambda_l) : L(\lambda_j)] = 1$ by Theorem 5.3.3. □

By the BGG reciprocity we immediately get

$$(P(\lambda_j) : M(\lambda_i)) = \begin{cases} 1 & i \geq j, \\ 0 & \text{otherwise} \end{cases} \tag{5.1}$$

By Theorem 5.2.9 every map of Verma modules is injective and hence, by looking at the class in the Grothendieck group, we get short exact sequences

$$0 \to M(\lambda_{i+1}) \to M(\lambda_i) \to L(\lambda_i) \to 0.$$

Projective resolutions of Verma modules

By Proposition 5.2.5

$$P(\lambda_1) = M(\lambda_1)$$

because λ_1 is dominant. In order to find projective covers of the remaining simple modules we use the algorithm described in Section 5.2.2.

First, we notice that for any $i \in 2, \ldots, n$ the weight $\lambda_i + \rho$ is dominant and hence $M(\lambda_i+\rho) \otimes_k L(\rho)$ is a projective module with a map to $M(\lambda_i)$. Thus, $P(\lambda_i)$ is a submodule of $M(\lambda_i + \rho) \otimes_k L(\rho)$.

In order to better understand the structure of $P(\lambda_i)$ we shall use some properties of the module $L(\rho)$. [28, Theorem 1.6] guarantees that $\dim L(\rho)_{-\rho} = 1$. We fix a non-zero vector $v \in L(\rho)_{-\rho}$. As $-\rho$ is the minimal weight of $L(\rho)$, for any $y \in U(\mathfrak{n}^-)$ we have $y \cdot v = 0$.

On the other hand, for any $i \in \{1, \ldots, n-1\}$ we have $x_i\, v \neq 0$ as

$$y_i x_i\, v = x_i y_i\, v - h_i\, v = -\rho(h_i)\, v = v.$$

Analogous argument shows that $x_k x_{k+1} \ldots x_i\, v \neq 0$ for any $1 \leq k \leq i < n$.

Lemma 5.3.4. *Let $\lambda \in \Lambda$ and $\alpha \in \Delta$ be such that $\langle \lambda, \alpha^\vee \rangle = 0$. Consider a short exact sequence*

$$0 \to M(\lambda) \xrightarrow{\iota} P \xrightarrow{\pi} M(\lambda - \alpha) \to 0$$

with P generated by a vector w such that $\pi(w)$ is the generating vector of $M(\lambda - \alpha)$ and $x_\alpha w = \iota(v)$, where v is the generating vector of $M(\lambda)$. Then the sequence does not split.

Proof. The subspace of weight $\lambda - \alpha$ of P, $P_{\lambda-\alpha}$ is spanned by w and $y_\alpha x_\alpha w$. Because $\langle \lambda, \alpha^\vee \rangle = 0$, the vector $y_\alpha x_\alpha w$ is a maximal vector of weight $\lambda - \alpha$ in P, i.e. $U(\mathfrak{n}^+) \cdot y_\alpha x_\alpha \cdot w = 0$.

Let $\overline{w} = \pi(w)$ be the generating vector of $M(\lambda - \alpha)$. A morphism $s \colon M(\lambda - \alpha) \to P$ lifting π must satisfy $s(\overline{w}) = w + c\, y_\alpha x_\alpha w$ for some $c \in k$. On the other hand $s(x_\alpha \overline{w}) = x_\alpha s(\overline{w}) = 0$ while $x_\alpha(w + c\, y_\alpha x_\alpha w) = x_\alpha w \neq 0$. So s is not a map of $U(\mathfrak{g})$-modules. $\square$

If $\mathfrak{g} = \mathrm{sl}(n, k)$ we can generalise the above construction of the module P for a sequence of simple roots $\alpha_i, \alpha_{i+1}, \dots \alpha_l \in \Delta$. For $\lambda \in \mathfrak{h}^*$ we define $\lambda_i = \lambda$ and $\lambda_{j+1} = s_{\alpha_j} \cdot \lambda_j$ for $j = i, \dots, l$. We assume that $\lambda \in \mathfrak{h}^*$ was chosen in such a way that $\langle \lambda_j, \alpha_j^\vee \rangle = 0$ for $j = i, \dots, l$. Then, $\lambda_{j+1} = \lambda - \sum_{l=i}^{j} \alpha_l$.

We consider a module $P_{i,l+1} \in M$ generated by a vector w_{l+1} of weight λ_{l+1} with the following $U(\mathfrak{g})$-action. We define $w_j = x_j w_{j+1}$ and assume that $w_l \neq 0$, $w_{l-2} \neq 0, \dots, w_i \neq 0$ and $w_{i-1} = x_{i-1} w_i = 0$. Furthermore, $x_s w_j = 0$ for $s \neq j-1$ and $U(\mathfrak{n}^-)$ acts freely on each w_j. It follows that w_i is a maximal vector in $P_{i,l+1}$ of weight $\lambda_i = \lambda$.

If the simple root α considered in Lemma 5.3.4 is the i-th simple root then the module P of this lemma becomes in the new notation $P_{i,i+1}$.

Lemma 5.3.5. *Let $\mathfrak{g} = \mathrm{sl}(n, k)$, $\alpha_i, \dots \alpha_l \in \Delta$, $\lambda \in \mathfrak{h}^*$ and a module $P_{i,l}$ be as above. The natural inclusion $P_{i,l} \subset P_{i,l+1}$ leads to a short exact sequence*

$$0 \to P_{i,l} \to P_{i,l+1} \to M(\lambda_l) \to 0.$$

which does not split.

Proof. The argument is the same as in the proof of Lemma 5.3.4. Namely, if $\overline{w}$ is the generating vector of $M(\lambda_l)$ then the splitting homomorphism $s \colon M(\lambda_l) \to P_{i,l+1}$ would map $\overline{w}$ to a vector of weight λ_l of the form $\widetilde{v} = w_{l+1} + \sum c_i v_i$, for some $v_i \in (P_{i,l+1})_{\lambda_l}$, $v_i \neq w_{l+1}$. Moreover, as s is a homomorphism of $U(\mathfrak{g})$ modules, $U(\mathfrak{n}^+)\widetilde{v} = 0$. We will show that such a vector v does not exist.

As $U(\mathfrak{n}^+)\widetilde{v} = 0$ we have $x_l \widetilde{v} = 0$. On the other hand, by assumption $x_l w_{l+1} = w_l \neq 0$. The only vector of weight λ_l in the $U(\mathfrak{n}^-)$-submodule spanned by w_l is $y_l w_l$. As $\langle \lambda_{l-1}, \alpha_l^\vee \rangle = 0$, we have $x_l y_l w_l = 0$. Relations between w_j guarantee that vectors of weight λ_l in $U(\mathfrak{n}^-)$-submodules of $P_{i,l+1}$ generated by $w_{l-1}, \dots, w_i$ would not give w_l while multiplied by x_l. Finally, any vector of $P_{i,l+1}$ lies in a $U(\mathfrak{n}^-)$-submodule generated by w_j for some j. Hence, a sought for vector $\widetilde{v}$ does not exist. $\square$

Now, we are ready to describe $P(\lambda_i)$.

Proposition 5.3.6. *For λ_i, $i \neq 1$ the projective cover of $M(\lambda_i)$ is isomorphic to the module $P_{1,i}$.*

Proof. From Section 5.2.2 we know that $P(\lambda_i)$ is a direct summand of

$$T^i = M(\lambda_i + \rho) \otimes_k L(\rho).$$

The module T^i admits a filtration with quotients $M(\lambda_i + \rho + \mu)$ for μ such that $L(\rho)_\mu \neq 0$. We order the basis of $L(\rho)$, $v_1, \dots, v_s$ in such a way that $v = v_s$ is the only vector of weight $-\rho$. We assume that the weight of v_i is ν_i. On the other hand, vectors $v_{i_\mu}, v_{i_\mu+1}, \dots, v_{j_\mu}$ are all the basis elements with weight μ.

The only weights of the form $\lambda_i + \rho + \mu$, for μ as above, linked to λ_i are $\lambda_1, \dots, \lambda_i$. Putting $\mu_l = -\rho + \sum_{j=l}^{i-1} \alpha_j$ we have $\lambda_i + \rho + \mu_l = \lambda_l$. Then $L(\rho)_{\mu_l} \neq 0$ because $x_l \dots x_{i-1} v \neq 0$.

We can also assume that the basis of $L(\rho)$ is ordered in such a way that $v_{i_{\mu_l}} = x_l \dots x_{i-1} v$.

The standard filtration of T^i is obtained by considering submodules $T^i_l \subset T^i$ generated by $t_1, \dots, t_l$, where $t_l = v^+ \otimes v_l$ and $v^+ \in M(\lambda_i + \rho)$ is the maximal vector of weight $\lambda_i + \rho$.

For $l < i_{i_{\mu_1}}$ the weight λ_1 is not linked to any $\lambda_i + \rho + \nu_l$ and hence

$$T^i_{i_{\mu_2}-1} = T^i_{i_{\mu_1}-1} \oplus M(\lambda_1)^{\oplus \dim L(\rho)_{\mu_1}}.$$

From Lemma 5.3.4 we deduce that $T^i_{i_{\mu_2}}$ has as a direct summand a module P containing $P_{1,2}$.

Analogously, using Lemma 5.3.5 we see that the submodule $T^i_{i_{\mu_l}}$ has as a direct summand a module $\widetilde{P}$ containing $P_{i,l}$.

Finally, the submodule of T^i containing t_s contains also $P_{1,i}$.

This finishes the proof. We know that both $P_{1,i}$ and $P(\lambda_i)$ have standard filtrations. By (5.1) we know that $(P(\lambda_i) : M(\lambda_l)) = 1$ for $l \leq i$. On the other hand, by construction $(P_{1,i} : M(\lambda_l)) = 1$ for $l \leq i$ and $P_{1,i} \subset P(\lambda_i)$. $\square$

Using Lemma 5.3.5 and the above proposition we obtain

Corollary 5.3.7. *Let $i \in \{2, \dots, n\}$. Then*

$$0 \to P(\lambda_{i-1}) \to P(\lambda_i) \to M(\lambda_i) \to 0$$

is a projective resolution of the Verma module $M(\lambda_i)$.

Ext-algebra of Verma modules

Theorem 5.2.10 implies that

$$\mathrm{Hom}(M(\lambda_i), M(\lambda_j)) = \begin{cases} k & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, using Proposition 5.2.3 we know that

$$\chi(\mathrm{Ext}^\bullet(M(\lambda_i), M(\lambda_j)) = 0 \quad \text{for } i \neq j.$$

Finally, Corollary 5.3.7 implies that $\mathrm{Ext}^{>1}(M(\lambda_i), M) = 0$ for any module $M \in \mathcal{O}$. Thus

$$\mathrm{Ext}^1(M(\lambda_i), M(\lambda_j)) = \begin{cases} k & \text{if } i > j, \\ 0 & \text{otherwise.} \end{cases}$$

and the remaining Ext-groups vanish.

From Theorem 5.2.9 we know that a non-zero map $M(\lambda_i) \to M(\lambda_j)$ is injective and hence the composition

$$\mathrm{Hom}(M(\lambda_i), M(\lambda_j)) \otimes \mathrm{Hom}(M(\lambda_l), M(\lambda_i)) \to \mathrm{Hom}(M(\lambda_l), M(\lambda_j))$$

is non-zero for any triple $l > i > j$.

Lemma 5.3.8. *For $j < i$ the composition*

$$\mathrm{Hom}(M(\lambda_i), M(\lambda_j)) \otimes \mathrm{Ext}^1(M(\lambda_{i+1}), M(\lambda_i)) \to \mathrm{Ext}^1(M(\lambda_{i+1}), M(\lambda_j))$$

is non-zero.

Proof. Let $\beta \in \mathrm{Ext}^1(M(\lambda_{i+1}), M(\lambda_i))$, $\gamma \in \mathrm{Hom}(M(\lambda_i), M(\lambda_j))$ be non-zero. Assume that $\gamma \circ \beta = 0$. Then, applying the functor $\mathrm{Hom}(-, M(\lambda_j))$ to the short exact sequence

$$0 \to M(\lambda_i) \to P_{i,i+1} \to M(\lambda_{i+1}) \to 0$$

we get that

$$\mathrm{Hom}(P_{i,i+1}, M(\lambda_j)) = k^2.$$

As $P(\lambda_{i+1})$ can be constructed as an iterated universal extension of Verma modules $M(\lambda_{i+1}), M(\lambda_i), \dots M(\lambda_1)$ we have an epimorphism $P(\lambda_{i+1}) \to P_{i,i+1}$. Hence, $\mathrm{Hom}(P_{i,i+1}, M(\lambda_j)) \subset \mathrm{Hom}(P(\lambda_{i+1}), M(\lambda_j))$. Thus, $\dim \mathrm{Hom}(P(\lambda_{i+1}), M(\lambda_j)) \geq 2$. This contradicts the fact that $\dim \mathrm{Hom}(P(\lambda_{i+1}), M(\lambda_j)) = [M(\lambda_j) : L(\lambda_{i+1})] = 1$ (see Theorem 5.2.6 and 5.2.10). $\square$

Proposition 5.3.9. *For any triple i, k, l such that $l > i > j$ the composition*

$$\mathrm{Hom}(M(\lambda_i), M(\lambda_j)) \otimes \mathrm{Ext}^1(M(\lambda_l), M(\lambda_i)) \to \mathrm{Ext}^1(M(\lambda_l), M(\lambda_j))$$

is non-zero.

Proof. Let $\beta \in \mathrm{Ext}^1(M(\lambda_l), M(\lambda_i))$, $\gamma \in \mathrm{Hom}(M(\lambda_i), M(\lambda_j))$ be non-zero elements. We know that $\mathrm{Ext}^1(M(\lambda_l), M(\lambda_i)) = k$ and hence by Lemma 5.3.8 $\beta = \gamma_1 \circ \beta_1$ for $\beta_1 \in \mathrm{Ext}^1(M(\lambda_l), M(\lambda_{l-1}))$, $\gamma_1 \in \mathrm{Hom}(M(\lambda_{l-1}), M(\lambda_i))$. Then

$$\gamma \circ \beta = (\gamma \circ \gamma_1) \circ \beta_1 \neq 0$$

again by Lemma 5.3.8 and remark before Lemma 5.3.8. $\square$

Lemma 5.3.10. *The composition*

$$\mathrm{Ext}^1(M(\lambda_i), M(\lambda_{i-1})) \otimes \mathrm{Hom}(M(\lambda_{i+1}), M(\lambda_i)) \to \mathrm{Ext}^1(M(\lambda_{i+1}), M(\lambda_{i-1}))$$

is non-zero.

Proof. Let $\gamma \in \mathrm{Hom}(M(\lambda_{i+1}), M(\lambda_i))$, $\beta \in \mathrm{Ext}^1(M(\lambda_i), M(\lambda_{i-1}))$ be non-zero elements. We already know that β corresponds to a short exact sequence

$$0 \to M(\lambda_{i-1}) \to P_{i-1,i} \to M(\lambda_i) \to 0.$$

Applying to this sequence $\mathrm{Hom}(M(\lambda_{i+1}), -)$ we get that

$$\beta \circ \gamma \neq 0 \quad \Leftrightarrow \quad \dim \mathrm{Hom}(M(\lambda_{i+1}), P_{i-1,i}) = 1.$$

As the Verma module $M(\lambda_{i+1})$ is generated by a maximal vector of weight λ_{i+1}, the dimension of the space of homomorphisms from $M(\lambda_{i+1})$ to $P_{i-1,i}$ equals to the dimension of the subspace of $(P_{i-1,i})_{\lambda_{i+1}}$ spanned by the maximal vectors, that is by vectors on which $U(\mathfrak{n}^+)$ acts by zero.

From the construction of $P_{i-1,i}$ it follows that the space of vectors of weight λ_{i+1} is three dimensional with basis

$$y_i\, w_i,\ y_{i-1} y_i\, w_{i-1},\ y_{i-1,i}\, w_{i-1},$$

where $w_{i-1} = x_{i-1} w_i$.

Direct calculations show that

$$\begin{array}{lll} x_{i-1} y_i\, w_i = y_i w_{i-1}, & x_{i-1} y_{i-1} y_i\, w_{i-1} = y_i\, w_{i-1}, & x_{i-1} y_{i-1,i}\, w_{i-1} = -y_i\, w_{i-1}, \\ x_i y_i\, w_i = 0, & x_i y_{i-1} y_i\, w_{i-1} = -y_{i-1}\, w_{i-1}, & x_i y_{i-1,i}\, w_{i-1} = y_{i-1}\, w_{i-1}, \\ x_{i-1,i} y_i\, w_i = w_{i-1}, & x_{i-1,i} y_{i-1} y_i\, w_{i-1} = w_{i-1}, & x_{i-1,i} y_{i-1,i}\, w_{i-1} = -w_{i-1} \end{array}$$

and all other elements of $U(\mathfrak{n}^-)$ act on the above elements by zero. Hence, up to scalar there is only one maximal vector of weight λ_{i+1} in $P_{i-1,i}$ which proves that $\dim \mathrm{Hom}(M(\lambda_{i+1}), P_{i-1,i}) = 1$. $\square$

Proposition 5.3.11. *For any triple $l > i > j$ the composition*

$$\mathrm{Ext}^1(M(\lambda_i), M(\lambda_j)) \otimes \mathrm{Hom}(M(\lambda_l), M(\lambda_i)) \to \mathrm{Ext}^1(M(\lambda_l), M(\lambda_j))$$

is non-zero.

Proof. Let $\beta_{i,j} \in \mathrm{Ext}^1(M(\lambda_i), M(\lambda_j))$ and $\gamma_{l,i} \in \mathrm{Hom}(M(\lambda_l), M(\lambda_i))$ be non-zero. Then by Lemma 5.3.8 $\beta_{i,j} = \gamma_{i-1,j} \circ \beta_{i,i-1}$ for some $\beta_{i,i-1} \in \mathrm{Ext}^1(M(\lambda_i), M(\lambda_{i-1}))$ and $\gamma_{i-1,j} \in \mathrm{Hom}(M(\lambda_{i-1}), M(\lambda_j))$. Moreover, $\gamma_{l,i} = \gamma_{i+1,i} \circ \ldots \gamma_{l,l-1}$ for some $\gamma_{s,s-1} \in \mathrm{Hom}(M(\lambda_s), M(\lambda_{s-1}))$.

By Lemmas 5.3.10, 5.3.8 and the fact that $\mathrm{Ext}^1(M(\lambda_{i+1}), M(\lambda_{i-1})) = k$ we know that for any non-zero element $\gamma_{i+1,i} \in \mathrm{Hom}(M(\lambda_{i+1}), M(\lambda_i))$ and $\beta_{i,i-1} \in \mathrm{Ext}^1(M(\lambda_i), M(\lambda_{i-1}))$ there exist non-zero $\beta_{i+1,i} \in \mathrm{Ext}^1(M(\lambda_{i+1}), M(\lambda_i))$ and $\gamma_{i,i-1} \in \mathrm{Hom}(M(\lambda_{i+1}), M(\lambda_i))$ such that

$$\beta_{i,i-1} \circ \gamma_{i+1,i} = \gamma_{i,i-1} \circ \beta_{i+1,i}.$$

Hence

$$\beta_{i,k} \circ \gamma_{l,i} = \gamma_{i-1,k} \circ \beta_{i,i-1} \circ \gamma_{i+1,i} \circ \ldots \gamma_{l,l-1} = \gamma_{i-1,k} \circ \gamma_{i,i-1} \circ \beta_{i+1,i} \circ \gamma_{i+2,i+1} \circ \ldots \circ \gamma_{l,l-1} =$$
$$= \ldots = \gamma_{i-1,k} \circ \gamma_{i,i-1} \circ \ldots \circ \gamma_{l-1,l-2} \circ \beta_{l,l-1} \neq 0$$

by Proposition 5.3.9. □

Theorem 5.3.12. *Let $f\colon X \to Y$ be a birational morphism of smooth surfaces such that the exceptional divisor of f is a chain of $(n-1)$ curves of self-intersection (-2) and one curve of self-intersection (-1). Let $\mathcal{C}_f$ be the full subcategory of $\mathcal{D}^b(X)$ with objects $\mathcal{F} \in \mathcal{D}^b(X)$ such that $R^\bullet f_*(\mathcal{F}) = 0$.*

Then, the category $\mathcal{C}_f \cap \mathrm{Coh}(X)$ is equivalent to a block $\mathcal{O}_{\omega_1}$ of category $\mathcal{O}$ for the Lie algebra $\mathrm{sl}(n,k)$ determined by the (dominant) weight $\lambda = \omega_1 - \rho$.

Proof. We use notation of Chapter 3.

From Proposition 4.5.4 we know that $\mathrm{Coh}(X) \cap \mathcal{C}_f$ is a highest weight category with standard objects $\mathcal{O}_{R_n}(R_n), \ldots, \mathcal{O}_{R_1}(R_1)$. Moreover, we calculated in Section 3.2.1 the Ext-algebra of these standard objects.

With Lemma 5.1.2 we proved that the block of the category $\mathcal{O}$ determined by weight λ has n Verma modules, $M(\lambda_n), \ldots, M(\lambda_1)$.

Proposition 5.3.9 and 5.3.11 show that the Ext-algebras of $\bigoplus \mathcal{O}_{R_i}(R_i)$ and $\bigoplus M(\lambda_i)$ are isomorphic.

Finally, from Remark 5.3.1 we conclude that endomorphism algebras of projective generators of both $\mathrm{Coh}(X) \cap \mathcal{C}_f$ and $\mathcal{O}_{\omega_1}$ are uniquely determined by Ext-algebras of standard objects and Verma modules respectively. □

Remark 5.3.13. We could also start with a dominant weight $\mu = \omega_2 - \rho$. Then, for $\mathrm{sl}(n,k)$ with $n \geq 4$ the linkage class of μ is

$$\{\mu,\, \mu - \alpha_2,\, \mu - \alpha_1 - \alpha_2,\, \mu - \alpha_2 - \alpha_3,\, \mu - \alpha_1 - \alpha_2 - \alpha_3,\, \mu - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4, \ldots\}$$

If we put

$$\begin{aligned}
&\mu_2 = s_{\alpha_2} \cdot \mu = \mu - \alpha_2,\\
&\mu_1 = s_{\alpha_1} \cdot \mu_2 = \mu - \alpha_1 - \alpha_2,\\
&\mu_3 = s_{\alpha_3} \cdot \mu_2 = \mu - \alpha_2 - \alpha_3,\\
&\mu_{13} = s_{\alpha_3} \cdot \alpha_1 = s_{\alpha_1} \cdot \alpha_3 = \mu - \alpha_1 - \alpha_2 - \alpha_3\\
&\mu_4 = s_{\alpha_4} \cdot \mu_{13} = \mu - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4,\\
&\mu_5 = \ldots
\end{aligned}$$

then homomorphisms between Verma modules are

$$\begin{tikzcd}
 & & & M(\mu_1) \arrow[dr] & & \\
\ldots \arrow[r] & M(\mu_4) \arrow[r] & M(\mu_{13}) \arrow[ur] \arrow[dr] & & M(\mu_2) \arrow[r] & M(\mu). \\
 & & & M(\mu_3) \arrow[ur] & &
\end{tikzcd}$$

A category with the above morphisms between standard objects cannot be of the form $\mathrm{Coh}(X) \cap \mathcal{C}_f$ for a birational morphism between smooth projective surfaces because of the non-zero maps from $M(\mu_{13})$ to two incomparable objects $M(\mu_1)$ and $M(\mu_3)$.

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