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Mathematics

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Report on the Ph.D thesis "Differential and geometric properties of certain classes of homeomorphisms" by Zofia Grochulska, submitted to University of Warsaw, Faculty of Mathematics, Informatics and Mechanics, 2024.

To whom it may concern:

I have carefully reviewed the doctoral thesis of Zofia Grochulska, and have the following assessment of its contents.

The strong interplay among the analysis and the geometric topology of mappings is the core of the thesis. The main theme is developing techniques to construct differentiable homeomorphisms. Such constructions have come into widespread use in variational problems, quasiconformal theory, and mathematical models of nonlinear elasticity, leading to solutions of numerous open problems.

In particular, the extension and mapping problems studied in this thesis are at the heart of Geometric Function Theory. Grochulska's thesis makes significant contributions to these problems. It is based on the two manuscripts "Gluing diffeomorphisms, bi-Lipschitz mappings and homeomorphisms" and "Constructing diffeomorphisms and homeomorphisms with prescribed derivative" both co-authored with P. Goldstein and P. Hajłasz.

The first paper addresses the following extension/gluing problem: given an orientation-preserving, differentiable/bi-Lipschitz homeomorphisms defined on several 'patches' of a manifold M, can it be extended to a global homeomorphism with the same regularity? The existence of such an extension is highly nontrivial. A caution already comes from the famous exotic 7-spheres example by Milnor. In the case of just one 'patch' Palais (1960) and a diffeomorphic extension gave a positive answer to this extension problem.

This thesis demonstrates that the Palais argument can be applied in a variety of other settings: the case of several patches, and in the lower regularity cases of bi-Lipschitz homeomorphisms and homeomorphisms. Since in geometric function theory the diffeomorphism assumption is too rigid, it is essential to have an explicit construction, as presented here, of gluing homeomorphisms without high regularity assumptions.

Yet, by far the strongest results of this thesis are the theorems proved in the second paper, "Constructing diffeomorphisms and homeomorphisms with prescribed derivative." Roughly, the authors prove a (highly nontrivial) homeomorphic version of Alberti's Lusin property for gradients. Recall, the classical result of Alberti shows that, for every Borel vector field u, there exists a function ϕ of class C^1 whose gradient $\nabla\phi$ agrees with u outside a set of arbitrary small measure. Applying

the result to components of a matrix-valued measurable mapping T one obtains a mapping $\Phi \in C^1$ whose differential $D\Phi$ agrees with T outside a set of arbitrary small measure.

The striking improvement to a homeomorphic version of Alberti's result involves a fascinating interplay between analysis, topology and geometry. Even though the proof relies on that result, ensuring that the maps are in fact diffeomorphisms involves deep mathematics. To ensure injectivity in the construction, the authors use a theorem of Dacorogna and Moser that guarantees smooth diffeomorphisms with prescribed Jacobians. The present paper is remarkable in that it is the first place where explicit constructions of homeomorphisms with prescribed differential properties are given. I expect that these results will find broad applications and the paper will be published in one of the very top mathematical journals.

In the strongest possible terms, I enthusiastically support this thesis. In it, Grohulska offers a wealth of insightful ideas and a clear vision for the future direction of geometric function theory. In both papers she demonstrates a broad and deep understanding of the subject. This thesis makes a substantial contribution to an exciting and rapidly developing area of mathematics. Applications are plentiful and quite significant. The proofs are intricate yet clearly written, and display a mastery of the techniques of Geometric Function Theory and related fields.

In my opinion, this thesis clearly exceeds expectations for a doctoral degree. If I can provide any other information to assist in your evaluation of this work, please contact me.

Sincerely,

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