Referee's report on PhD thesis
"Differential and geometric properties
of certain classes of homeomorphisms"
by Zofia Grochulska, 2024

Motivation.

This thesis is devoted to the construction of various homeomorphisms from a subset of $\mathbb{R}^n$ to a subset of $\mathbb{R}^n$ or even mappings from a manifold to a manifold. These results are important in Geometric Function Theory, Geometric Measure Theory and related fields. Homeomorphisms arise naturally for example when we study models of deformations in Elasticity, where domain $\Omega \subset \mathbb{R}^n$ corresponds to the deformed body and a mapping $f : \Omega \to \mathbb{R}^n$ describes the deformation of the body. The continuity of f is natural as the deformation cannot break the body and the invertibility of f follows from the interpenetration of the matter.

In those models one naturally minimizes some energy functional and hence we work with mapping in some Sobolev space $W^{1,p}$ and we know that such mappings are approximatively differentiable a.e. The big part of the thesis is devoted to the construction of wild homeomorphisms that are approximatively differentiable a.e. Such examples are crucial for developing an intuition as they tell us what kind of results we cannot expect.

Another part of the thesis is devoted to various extension theorems from some subset to the whole set where we keep the crucial property that our mapping is invertible, or bilipschitz or even a diffeomorphisms. Those are essential ingredients for developing a reasonable theory as they serve a basic building block to the construction of mappings and find those results incredibly useful.

Overview of the thesis.

In the first part of the thesis the author studies the possibility of extension of a given diffeomorphism on some "nice" subset of a manifold to a diffeomorphism of the whole manifold and similarly extension of a bilipschitz mapping to a bilipschitz mapping of the whole manifold and orientation preserving homeomorphism to an orientation preserving homeomorphism of the whole manifold.

These results are very useful and I have needed results of this kind several times in my life but unfortunately I did not know any useful reference. The proofs of these results depend on some deep results from topology like bi-Lipschitz stable homeomorphism conjecture or stable homeomorphism conjecture and the authors must have detailed understanding of the subject and they must master the techniques of both Algebraic Topology and Geometric Function Theory. You really need experts in both fields to prove results of this kind and I have no doubt

that these results will be useful for people working in Geometric Function Theory and related fields in the future.

The next part of the thesis is devoted to the construction of a.e. approximatively differentiable homeomorphism from $[0,1]^n$ to $[0,1]^n$ with prescribed derivative. As a useful tool the authors first need to develop some Lusin-type theorem for gradients which is of independent interest. This part of the thesis is the most difficult, deep, extremely technically complicated and involved and uses tools like Alberti's theorem, Dacorogna-Moser solution to the Jacobian equation and substantially improves some old constructions by Oxtoby and Ulam.

The main result of the second part answers also some open problem from the paper of Goldstein and Hajlasz from 2017. It shows that the derivative of a.e. approximatively differentiable homeomorphisms can be absolutely arbitrary and it carries no information about the quality of the mapping which is truly shocking and incredibly interesting.

Conclusion.

This theses contains two research papers written jointly with P. Goldstein and P. Hajlasz that were recently submitted for publication. I have checked that the results are correct, they are very interesting and let me point out that the second paper will be without any doubt accepted in one of the top journals. It is clear from the thesis that the contribution of the student to both papers was substantial.

Overall this is an interesting contribution to an exciting and important area of mathematics. It is well-written and pleasant to read. The author shows her overview of the field together with the ability to produce new interesting results. The proofs are involved and the authors clearly masters the techniques of Geometric Function Theory, Topology of Euclidean space and also many surrounding fields.

Let me compare this thesis to the average thesis at Charles University which is the very best school for mathematics in the Czech Republic. I could easily compare this thesis with the top one fourth there so for me it definitely deserves an honorary distinction.

Zofia Grochulska showed that she is an independent researcher with broad understanding of the subject. I strongly recommend the thesis for defense and I suggest to award it an honorary distinction.

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