Report of v2 of PhDThesis of Ramazan Yozgyur

Pedro Vaz

The candidate has successfully addressed the comments raised on the previous version, and the exposition has been significantly improved. The text is clearly written, mathematically sound, and presents original and well-structured research results in link homology and its equivariant extensions. I have noticed only a few minor typographical errors (see below), which do not affect the overall quality or readability of the manuscript.

I therefore deem the thesis fully acceptable for the award of the PhD degree and strongly recommend its defense.

Minor remarks

- In the statement of Thm 5.29, there in an extra *a* in "in a Kom..."
- A period is missing at the end of the proof of Cor 8.12 (page 53).

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Report of PhDThesis of Ramazan Yozgyur

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Summary

Originating from M. Khovanov's influential 1999 paper[1] [Duke Math. J.], the categorification of topological invariants has become a highly competitive and rapidly evolving field of mathematical research. Following the introduction of *Khovanov homology*, M. Khovanov and L. Rozansky constructed, in 2004, a link homology theory categorifying the so-called $\mathfrak{sl}_N$-polynomial of links. Since then, a plethora of link homology theories have been developed. Notably, Khovanov homology coincides with Khovanov–Rozansky homology for $N = 2$.

Like any suitable homology theory, *Khovanov–Rozansky* homology defines a functor - specifically, from a category of links and cobordisms to a category of bigraded abelian groups. As a result, it encodes 4-dimensional information. For instance, in 2018, L. Piccirillo used Khovanov homology to prove a longstanding conjecture about the sliceness of the Conway knot, a statement in 4-dimensional topology [Ann. of Math.].

Like knot theory itself, link homology lies at the intersection of various fields beyond topology, including representation theory, symplectic geometry, algebraic geometry, physics, and combinatorics, for example. This multidisciplinary nature has fostered rich and fruitful interactions across disciplines.

This thesis investigates the structure of Khovanov–Rozansky $\mathfrak{sl}_N$-homology for a class of links known as *periodic links*. Diagrams of periodic links (with period $m \in \mathbb{N}$) naturally admit an action of the cyclic group $\mathbb{Z}_m$. This work explores an equivariant version of Khovanov–Rozansky homology tailored to this class of links. Equivariant Khovanov homology for periodic links was introduced by W. Politarczyk in 2015 [Michigan Math. J.] and later studied further by M. Borodzik and W. Politarczyk [Indiana Univ. Math. J.].

This thesis is based on a collaboration [Math. Res. Lett., to appear] between the candidate and M. Borodzik and W. Politarczyk, both of whom are supervisors. The introduction of the dissertation states that "the thesis is an expanded version of the paper". In the remainder of this report, I will refer to this article by the candidate and the supervisors as "the paper".

Main results and structure of the thesis

This thesis employs the theory of webs and foams to construct an equivariant version of Khovanov–Rozansky $\mathfrak{sl}_N$-homology for periodic links. It further investigates its structure and explores some implications for its decategorification, known as the Reshetikhin–Turaev $\mathfrak{sl}_N$ invariant.

Following a one-page introduction, the thesis includes a section summarizing knot theory, Khovanov homology (the $N = 2$ case of Khovanov–Rozansky homology), and homological algebra.

Section 3 summarizes periodic links and its basic properties.

Section 4 summarizes the theory of webs and foams, along with several associated categories. It also introduces the definition of the $\mathbb{S}_N$ -equivariant Khovanov–Rozansky complex, where $\mathbb{S}_N$ denotes the ring of symmetric polynomials in N variables with integer coefficients — an object in the bounded homotopy category of an additive category. The topological invariance of the homotopy class of the $\mathbb{S}_N$-equivariant Khovanov–Rozansky complex is explicitly proved. By applying a certain functor to an abelian category, the $\mathbb{S}_N$-equivariant Khovanov–Rozansky homology is then defined. Although this homology is a topological invariant, this fact is not explicitly stated in the thesis.

In Section 5, the specializations of the $\mathbb{S}_N$-equivariant Khovanov–Rozansky complex and, subsequently, of the $\mathbb{S}_N$-equivariant Khovanov–Rozansky homology are studied. This is achieved by passing to the field of complex numbers as the ground field of the theory and specializing the elements of $\mathbb{S}_N$ to N-tuples of elements of $\mathbb{C}$. It is proved that the specialization is functorial, and this property is used to recover the Khovanov–Rozansky homology, the Lee homology, as well as the well-known spectral sequence connecting them.

Then, the study of m-periodic links begins. From this point onward, all links are assumed to be m-periodic. An action of $\mathbb{Z}_m$ on the Khovanov–Rozansky complex is constructed, and it is proved that this action descends

[1] Here and below, dates refer to the preprint on the arXiv.

to an action of $\mathbb{C}[\mathbb{Z}_m]$ on the Khovanov–Rozansky homology. This gives rise to a $\mathbb{Z}_m$-equivariant version of the Khovanov–Rozansky $\mathfrak{sl}_N$-homology. A $\mathbb{Z}_m$-equivariant Lee homology is defined similarly.

It is further proved that the $\mathbb{Z}_m$-equivariant homologies (both Khovanov–Rozansky and Lee) decompose further, revealing a finer structure. This finer structure appears to be nonexistent (as far as I can tell) for knots that are not m-periodic.

Section 6 is devoted to the proof of topological invariance of the $\mathbb{Z}_m$-equivariant Khovanov–Rozansky complex.

Section 7 constructs several spectral sequences involving $\mathbb{Z}_m$-equivariant Khovanov–Rozansky homology.

Section 8 studies the invariant polynomials arising from the link homologies of periodic links, including the Poincaré polynomial (referred to as the Khovanov–Rozansky polynomial, or the Lee polynomial) and its specialization at $t = -1$, which corresponds to the Euler characteristic and coincides with the Reshetikhin–Turaev invariant. Formulas for the Khovanov–Rozansky polynomial of m–periodic links in terms of the Lee polynomial (along with a family of polynomials with nonnegative coefficients, whose explicit form requires other methods for computation) are provided, revealing the structure of these polynomial invariants

Fixing m as a power of a prime number p, a skein relation for the Euler characteristic of the $\mathbb{Z}_m$- equivariant theory is also constructed. Additionally, more is proven about the structure of the Khovanov–Rozansky polynomials: they can be expressed as a linear combination of other polynomials with non-negative coefficients, which satisfy a congruence relation (depending on p). Precise formulas are provided for $m = 3$ and $m = 4$.

Conclusion

Although the mathematical statements are correct, the thesis is not well written and falls short of the expected standard for a doctoral dissertation. The use of undefined or incorrectly defined notions invalidates the mathematical proofs, and such issues are present throughout the thesis. For example,

- Several notions are used without being properly defined or explained, e.g. what is the $s(K, \mathbb{F})$ appearing in the statement of Theorem 3.13.?
- It contains several mathematical errors, even in proofs that are correct in the paper, e.g. what is a rotation from $\mathbb{Z}_m$ to $\mathbb{Z}_m$, as mentioned in the proof of Proposition 5.1.?
- Some important parts of the text have been omitted in the transition from the paper to the thesis. For example, important explanations about group actions are missing in the beginning of section 5.6.1. and they are present in the paper (section 4.1 there). Example 6.13 ends up with "we only prove that"
- Some parts are copied from the paper and rephrased in a somewhat cryptic manner, e.g. in section 7.1 what is the content of the phrase starting with "This can be unique; otherwise g_i will be g_j"? This short section is written in a clear way in the paper, so why modify it?

I have the impression that the thesis was written hastily, with parts copied separately from the paper without ensuring the overall coherence and consistency of the text.

In light of the above I recommend that the thesis undergo several corrections before proceeding (see attached file).

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Khovanov-Rozansky $\mathfrak{sl}_N$-homology for periodic links

PhD Dissertation

submitted by

Ramazan Yozgyur

written under the supervision of

Maciej Borodzik and Wojciech Politarczyk

Warsaw
2024

Author's declaration:
I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

November 19, 2024 ....
Ramazan Yozgyur

Supervisors' declaration:
The dissertation is ready to be reviewed.

November 19, 2024 ....
dr hab. Maciej Borodzik

....
dr Wojciech Politarczyk

Abstract

The main goal of this dissertation is to construct equivariant $\mathfrak{sl}_N$ homology for periodic links. For this purpose, we use the approach to $\mathfrak{sl}_N$ homology via webs and foams. The action on webs and foams allows us to define equivariant Khovanov-Rozansky homology for periodic links.

Following this definition, we deal with Reshetikhin-Turaev polynomials for the newly constructed equivariant homology via the newly defined difference polynomials.

In the end, we provide a periodicity criterion originating from equivariant Khovanov-Rozansky $\mathfrak{sl}_N$ homology.

1. Introduction

Let $L \subset S^3$ be a link. For $m \geq 2$, we say that L is *m-periodic* if it is invariant under a semi-free $\mathbb{Z}_m$-action on S^3 and L is disjoint from the fixed point set. For a periodic link, we have a question: how is the symmetry of the link reflected in link invariants? As an example, we have the Murasugi formula [18] recalled in Theorem 3.5. Besides giving a useful periodicity criterion, it also establishes the relation between the Alexander polynomial of L and the Alexander polynomial of its quotient.

Equivariant Khovanov homology for periodic links was defined in [21]. The group action on S^3 induces a well-defined group action on the Khovanov homology modules $\mathrm{Kh}(L;R)$. The $\mathfrak{sl}_N$-homology for links was introduced in [10, 11] by Khovanov and Rozansky as a generalization of Khovanov homology. The first method to construct $\mathfrak{sl}_N$-homology was matrix via factorization. Over the years, other methods were constructed, see [5, 23, 25]. In this thesis, the combinatorial definition approach sketched in Section 4 turns out to be well-suited for studying periodic links. Basically, in this approach, for any link diagram D we define a cochain complex $[\![D]\!]$ living in a suitably defined foam category. To get $\mathfrak{sl}_N$ homology, we pass to the category of $\mathbb{S}_N$-modules. For this, we need the Evaluation functor $\mathcal{F}$ which takes webs and sends them to $\mathbb{S}_N$-modules. The goal of this thesis is to generalize the result of [15, 21] in the case of $\mathfrak{sl}_N$-homology. We show that the action of the symmetry group $\mathbb{Z}_m$ of the periodic link induces a $\mathbb{Z}_m$ action on its $\mathfrak{sl}_N$-homology. Precisely, we have the following theorem.

Theorem (see Proposition 5.21). Suppose D is a labelled periodic link diagram. Then, there is an action of $\mathbb{Z}_m$ on $[\![D]\!]$ induced by rotating resolution diagrams of D.

By using the evaluation functor $\mathcal{F}$, we obtain a chain complex of $\mathbb{S}_N$-modules $\mathcal{F}([\![D]\!])$. By Proposition 5.23, $\mathcal{F}$ commutes with the $\mathbb{Z}_m$ action. The $\mathbb{Z}_m$ action on $[\![D]\!]$ gives $\mathbb{S}_N[\mathbb{Z}_m]$-module structure on the chain complex $\mathcal{F}([\![D]\!])$. We prove the following result.

Theorem (see Theorem 5.24). Suppose L is a $\mathbb{Z}_m$-periodic link and D and D' are $\mathbb{Z}_m$-equivalent m−periodic link diagrams of L then we have an induced quasi isomorphism between $\mathcal{F}([\![D]\!])$ and $\mathcal{F}([\![D']\!])$ in the category of $\mathbb{S}_N[\mathbb{Z}_m]$ modules.

Theorem 5.24 is stated and proved only for links whose labels are equal to 1, that is, for usual links. Next, we establish a skein spectral sequence for a change of an orbit of crossings in $\mathfrak{sl}_N$-homology. An analogous skein spectral sequence was considered in [21] for the Khovanov homology of a periodic link. The skein spectral sequence gives a relation between the so-called difference $\mathfrak{sl}_N$-polynomials after a change of an orbit of crossings. Refer to Section 8 for details.

The graded Euler characteristic of the Khovanov homology is the Jones polynomial. In the presence of a $\mathbb{Z}_{p^\ell}$-action (with p prime) there is a refinement of the Jones polynomial, called the difference Jones polynomials. They essentially appear as the graded Euler characteristic associated with the eigenspaces of the action of $\mathbb{Z}_{p^\ell}$ on the Khovanov homology.

Similarly, the Euler characteristic of $\mathfrak{sl}_N$-homology gives a well-known polynomial, the Reshetikin-Turaev polynomial, also known as the $\mathfrak{sl}_N$-polynomial. For a periodic link, we define analogs of difference Jones polynomials in $\mathfrak{sl}_N$-homology. We call them *difference $\mathfrak{sl}_N$-polynomials*. We use the skein spectral sequence to study these polynomials for the link and its mirror. Moreover, we show that if a link where all labels are equal to 1, is p^l-periodic, then the Poincaré polynomial of its $\mathfrak{sl}_N$-homology admits a decomposition into a sum of polynomials with non-negative coefficients and satisfying specific congruence relations; see Theorem 8.17. The new periodicity criterion cannot distinguish 3 and 4 periodic links.

The thesis is an expanded version of the paper [4].

2. Khovanov homology

In this chapter, we define Khovanov homology. To define it, we first introduce some basic concepts from knot theory and some concepts from homological algebra.

2.1. **Short introduction to knot theory.**

2.1.1. *Introduction.* This subsection is based on [27] and [3]. To understand the definition of the Khovanov homology, we need some basic definitions and facts about knots and links.

Definition 2.1. A *knot* is an embedding of a circle S^1 in the 3-dimensional Euclidean space or in the 3-dimensional sphere S^3.

If we embed more than one circle, we call the image a *link*. Generally, we are interested in regular projections of knots (links) onto a 2-dimensional Euclidean subspace, meaning that the projection is injective everywhere except at finitely many points, called the crossing points, where the knot projection crosses itself once. We will call the projection diagram where we have an over-strand and under-strand a knot (link) diagram.

Example 2.2. We have some well-known knot diagrams below

Right-handed trefoil   Left-handed trefoil   Figure eight knot

A link can be given an orientation. For these intersections of over-strand and under-strand, we have a specific name. We call these intersections positive crossing and negative crossing. Changing the orientation of one component of a link, might affect positivity of the crossings; however if we change the orientation of every component of the link, the positivity of all crossings is preserved. We will denote n_+ for the total number of positive crossings and n_- for the total number of negative crossings in a diagram.

positive crossing   negative crossing

For these two crossing we have 0 and 1 resolution of crossings. For crossing ╳ we have 0 resolution ≍ and for 1 resolution we have)(. Furthermore, if we change under and over strand we swap the 0- and the 1-resolutions.

Definition 2.3. The writhe $\omega(D)$ of a diagram D of an oriented knot or link is the difference between the numbers of positive and negative crossings, i.e.,

$$\omega(D) = n_+ - n_-$$

Definition 2.4. The reverse rK of an oriented knot K is simply the same knot with the opposite orientation.

Definition 2.5. Change all crossing points from positive to negative and from negative to positive crossing. The final diagram will be called the mirror image $m(K)$ of a knot K. In other words, The mirror image of a knot diagram is obtained by reflecting the knot diagram with respect to a line $\mathbb{R}$ in the plane.

We consider the following equivalence relation between knots. It applies also for links.

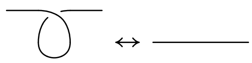
Definition 2.6. Two knots K_1 and K_2 are ambient isotopic if there is a smooth map $F : S^3 \times [0,1] \to S^3$ such that $F_x = F_{|S^3 \times \{x\}}$ is a diffeomorphism for each $x \in [0,1]$, $F_{|S^3 \times 0} = \text{id}$, and $F_{|S^3 \times 1}(K_1) = K_2$.

We want to understand if two knots are isotopic. The best way to understand this is by studying knot diagrams. We have an important theorem about equivalence in knot diagrams, but before this theorem, we need some definitions.

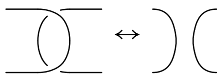
Definition 2.7. Isotopy of a knot projection is a continuous deformation of a plane in which the knot projection is drawn.

Definition 2.8. There are three local moves that are called Reidemeister moves for knot diagram equivalence.

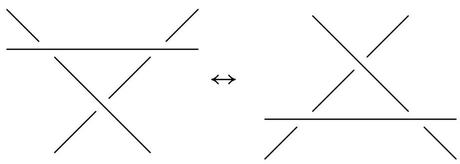
First Reidemeister move:



Second Reidemeister move:



Third Reidemeister move:



The following result was first proved by Reidemeister.

Theorem 2.9. *Two links are ambiently isotopic if and only if they are related by a finite number of Reidemeister moves and planar isotopies.*

A knot invariant is a property of a knot diagram that does not change under Reidemeister moves. For example, the writhe depends on the knot diagram, so it is not a knot invariant. A knot invariant only depends on the knot. Later, we will define the Jones polynomial and Khovanov homology. We will see that these are knot invariants.

2.2. **Jones Polynomial.** In this section, we will define the Jones polynomial. The Jones polynomial will be important for Khovanov homology. The definition of the Jones polynomial and its relation to Khovanov homology will be crucial to understanding concepts discussed in the following sections. We will start with the definition of the Kauffman bracket.

Definition 2.10. (see [3]) The Kauffman bracket is a function from the set of unoriented link diagrams in the plane to the ring of Laurent polynomials in variable q with integer coefficients. We denote by $\langle D \rangle \in \mathbb{Z}[q, q^{-1}]$ the Kauffman bracket of D. The Kauffman bracket is determined by the following three properties:

Need to assign an orientation to D first!

(1) $\langle \varnothing \rangle = 1$
(2) $\langle D \sqcup \bigcirc \rangle = (q^{-1} + q)\langle D \rangle$
(3) $\langle \times \rangle = \langle \asymp \rangle - q\langle)(\rangle$

where D is a diagram, $\varnothing$ is an empty diagram, and $\langle D \rangle$ is a Laurent polynomial.

The Kauffman bracket is invariant under RII and RIII moves. To make this definition invariant under the Reidemeister 1 move, we have to multiply $\langle D \rangle$ by $(-1)^{n_-} q^{n_+ - 2n_-}$ where n_+ is the number of positive crossings and n_- is the number of negative crossings. The resulting polynomial is a knot invariant.

Definition 2.11. (see [3]) The unnormalized Jones polynomial of a link L is defined as

↳ oriented!

$$\hat{J}(D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle,$$

where D is a diagram of L.

In addition, we define the normalized Jones polynomial

$$J(D) = \hat{J}(D)(q + q^{-1})^{-1}.$$

We generally use the unnormalized version in this paper. We assign numbers to each crossing by $1, \dots, n$. By applying 0 or 1 resolution to each crossing we get 2^n diagrams that we can index with the sequence which has 0 and 1. We call such a diagram a *smoothing*. With these 2^n smoothings D_α where $\alpha \in \{0,1\}^n$, we have an n-dimensional cube. When we resolve all crossings, we get a union of circles. To compute the unnormalized Jones polynomial, we replace each union of k-circles with a term $(-1)^{r_\alpha} q^{n_+ - 2n_- + r_\alpha}(q + q^{-1})^{k_\alpha}$.

$$J(D) = \sum_{\alpha \in \{0,1\}^n} (-1)^{r_\alpha} q^{n_+ - 2n_- + r_\alpha} (q + q^{-1})^{k_\alpha}$$

$$r_\alpha = \text{Number of 1s in } \alpha$$

$$k_\alpha = \text{Number of circles in the } D_\alpha$$

This thesis is about the sl_N-invariants. Why aren't they explained?

We will define Khovanov homology, but for that, we need some homological algebra.

2.3. Introduction to Homological Algebra.

In this section, we use [28] for the most definitions for some basic concepts of homological algebra that will be important for us

Definition 2.12. A chain complex $(C_\bullet, d_\bullet)$ is a sequence of modules $\cdots, C_{-2}, C_{-1}, C_0, C_1, C_2, \cdots$ connected by homomorphisms $d_n : C_n \to C_{n-1}$ where $d_{n-1} \circ d_n = 0$. We call $(C'_\bullet, d_\bullet)$ a subcomplex of $(C_\bullet, d_\bullet)$, if C'_i is a submodule of C_i and $d_n(C'_n) \subset C'_{n-1}$.

Definition 2.13. A cochain complex is a dual notion to a chain complex, it is a $(C_\bullet, d_\bullet)$ sequence of modules $\cdots, C_{-2}, C_{-1}, C_0, C_1, C_2, \cdots$ connected by homomorphism $d_n : C_n \to C_{n+1}$ where $d_{n+1} \circ d_n = 0$.

We define maps between chain complexes.

Definition 2.14. Assume we have $(C_\bullet, d_\bullet)$ and $(C'_\bullet, d'_\bullet)$ chain complexes. A chain map $F : C_\bullet \to C'_\bullet$ is a sequence of maps $\{F_n : C_n \to C'_n\}$ such that $F_{n-1} \circ d_n = d'_n \circ F_n$. In the diagram, we see that as below

$$\begin{array}{ccc} C_n & \xrightarrow{d_n} & C_{n-1} \\ {\scriptstyle F_n}\downarrow & & \downarrow{\scriptstyle F_{n-1}} \\ C'_n & \xrightarrow{d'_n} & C'_{n-1} \end{array}$$

Maps between cochain complexes can be defined similarly.

Definition 2.15. Assume we have a chain complex $(C_\bullet, d_\bullet)$, the homology of this sequence is $\ker(d_n)/\operatorname{im}(d_{n+1})$ and denoted by $H_n(C_\bullet)$.

Similarly, we define cohomology.

Definition 2.16. Assume we have a cochain complex

$$\cdots \to C^{n-1} \xrightarrow{d^{n-1}} C^n \to \cdots$$

The cohomology of this sequence is $\ker(d^n)/\operatorname{im}(d^{n-1})$ and denoted by $H^i(C^\bullet)$.

Proposition 2.17. *A chain map $F : C_\bullet \to C'_\bullet$ induces a homomorphism between the homology groups of these two complexes.*

Between two chain homotopy maps, we have equivalence also.

Definition 2.18. Suppose we have chain maps f and g between $(C_\bullet, d_\bullet)$ and $(C'_\bullet, d'_\bullet)$. A chain homotopy ϕ between f and g is a sequence of morphisms $\phi_n : C_n \to C'_{n+1}$ such that $f_n - g_n = d'_{n+1} \circ \phi_n + \phi_{n-1} \circ d_n$. We call f and g chain-homotopic chain maps and denote this relation $f \simeq g$.

We can define equivalence between two chain complexes.

Definition 2.19. We say chain complexes A and B are homotopy equivalent if and only if we have chain maps $f : (A_\bullet, d_\bullet) \to (B_\bullet, d'_\bullet)$ and $g : (B_\bullet, d'_\bullet) \to (A_\bullet, d_\bullet)$ such that $f \circ g \simeq id_{B_\bullet}$ and $g \circ f \simeq id_{A_\bullet}$.

Chain maps induce homomorphisms between the homology groups of chain complexes. Do we have any relation between the induced maps f_* and g_* where chain maps are chain-homotopic? The next proposition shows us this relation.

Proposition 2.20. *If we have f and g chain-homotopic chain maps, their induced maps f_* and g_* are the same on homology groups (i.e., $f_* = g_*$).*

Definition 2.21. Suppose $M_1, M_2, \cdots, M_n$ are modules over the fixed ring R, and $P_1, P_2, \cdots, P_n$ are module homomorphisms. We say that

$$M_1 \xrightarrow{P_1} M_2 \xrightarrow{P_2} M_3 \cdots \xrightarrow{P_{n-1}} M_n$$

is an exact sequence if $\operatorname{im}(P_{n-1}) = \ker(P_n)$.

Definition 2.22. Suppose A, B, C are modules over the fixed ring R. We say that

$$0 \to A \xrightarrow{\text{i}} B \xrightarrow{\text{p}} C \to 0$$

is a short exact sequence if i is a monomorphism, p is an epimorphism, and $\operatorname{im}(i) = \ker(p)$.

Furthermore, we define a short exact sequence in the category of chain complexes.

Definition 2.23. Suppose A, B, C are chain complexes, and i and p are chain maps. We say that the sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

is a short exact sequence if the induced sequence of maps

$$0 \to A_n \xrightarrow{i_n} B_n \xrightarrow{p_n} C_n \to 0$$

is a short exact sequence of modules.

Similarly, we define a long exact sequence for modules, and from the short exact sequence, we get a long exact sequence of homology groups.

Theorem 2.24. *Suppose A, B, C are chain complexes, and we have a short exact sequence of complexes given by:*

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

then we obtain a long homology sequence of homology groups

$$\cdots H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{j_*} H_{n-1}(C) \xrightarrow{\delta} \cdots$$

Proof. See [9, Theorem 2.16]. ☐

We have the same theory for cochain complexes

Theorem 2.25. *Suppose A,B and C are cochain complexes, and we have a short exact sequence of complexes given by*

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

then we obtain a long cohomology sequence of cohomology groups

$$\cdots H^n(A) \xrightarrow{i_*} H^n(B) \xrightarrow{j_*} H^n(C) \xrightarrow{\delta} H^{n+1}(A) \xrightarrow{i_*} H^{n+1}(B) \xrightarrow{j_*} H^{n+1}(C) \xrightarrow{\delta} \cdots$$

Definition 2.26. (see 1.5.1 [28]) Assume we have E and F be graded cochain complexes and $E \xrightarrow{f} F$ a chain map that preserves gradings. The mapping cone is a chain complex given in a degree k by

$$\mathrm{Cone}(f)_k = E_k \bigoplus F_{k-1}$$

with differential

$$\partial_{\mathrm{Cone}(f)} = \begin{pmatrix} -\partial_E & 0 \\ f & \partial_F \end{pmatrix} : \mathrm{Cone}(f)_k \to \mathrm{Cone}(f)_{k+1}.$$

We have the following lemma.

Lemma 2.27. *We have a short exact sequence which includes* $\mathrm{Cone}(f)$

$$0 \to F[1] \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} E \to 0$$

where $F[1]_n = F_{n-1}$, $i(a) = (0, a)$ *for* $a \in F$ *and* $p(e', a') = -e'$, *so we get a long exact sequence by Theorem 2.25*

$$\cdots \to H^d(E) \xrightarrow{H(f)} H^d(F) \xrightarrow{i_*} H^d(\mathrm{Cone}(f)) \xrightarrow{p_*} H^{d+1}(E) \to \cdots$$

Definition 2.28. Let C be an Abelian category. A homologically graded spectral sequence is a family of objects with differentials $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ which satisfy the rule $d^r \circ d^r = 0$ where $p, q, r \in \mathbb{Z}$. Moreover, for $E^{r+1}_{p,q}$ and $E^r_{p,q}$ for any r we have

$$E^{r+1}_{p,q} \equiv H(E^r_{p,q}) = \ker(d^r_{p,q})/\mathrm{Im}(d^r_{p-r,q+r-1})$$

For a fixed r, the family $E^r_{p,q}$ is called the page of the spectral sequence. Here we can think spectral sequences as a book. When we turn next page it means we increase r by 1 and take homology of the old page.

Definition 2.29. Let H_n be a collection of objects in category C.

- We say spectral sequence weakly converges to H_* if there is a filtration
$$\ldots \subseteq F_{p-1}H_n \subseteq F_pH_n \subseteq F_{p+1}H_n \subseteq \ldots \subseteq H_n$$
and isomorphism
$$\beta_{pq} : E^\infty_{pq} \cong F_pH_{p+q}/F_{p-1}H_{p+q}$$
- We say spectral sequence approaches to H_* if it weakly converges to H_* and
$$H_n = \bigcup F_pH_n \text{ and } \bigcap F_pH_n = 0$$
- We say sequence converges to H_* if it approaches to H_* and
$$H_n = \varprojlim(H_n/F_pH_n)$$
Convergence is denoted by $E^r_{pq} \implies H_{p+q}$

Definition 2.30. A first quadrant spectral sequence is a type of spectral sequence where all the information or data contained in its pages is confined or concentrated within the region of the (p, q)-plane where

$$p < 0 \text{ or } q < 0 \implies E^{p,q}_r = 0.$$

Proposition 2.31. *If the r-th page is confined to the first quadrant, then the (r+1)st page will also be so. Therefore, if the first one is, then all subsequent pages will be as well.*

Proposition 2.32. *For every first quadrant spectral sequence, convergence occurs at position (p, q) starting from the r-th term where r is greater than the maximum of p and $q + 1$.*

$$E^{p,q}_{max(p,q+1)+1} = E^{p,q}_{\infty}$$

Proposition 2.33. *If a first quadrant spectral sequence converges*

$$E^{p,q}_r \implies H^{p+q}$$

then each H^n has a filtration of length $n + 1$

$$0 = F^{n+1}H^n \subset F^nH^n \subset \dots F^1H^n \subset F^0H^n = H^n$$

We also have

- $F^nH^n \backsimeq E^{n,0}_{\infty}$
- $H^n/F^1H^n \backsimeq E^{0,n}_{\infty}$

2.4. **Introduction to the Khovanov homology.** In this paper, our main goal is to define $\mathfrak{sl}_N$ homology via web and foams. For $n = 2$, $\mathfrak{sl}_N$ homology is called Khovanov homology. In this subsection, we will define Khovanov homology in a basic way that will help us to understand $\mathfrak{sl}_N$ homology. We need the Khovanov bracket definition to define Khovanov homology. The definition is similar to the Kaufmann bracket definition. In this section we generally use papers [3] and [27].

Definition 2.34. We say that the vector space V is a graded vector space, if V can be decomposed into the direct sum of the form $V = \oplus_{n\in\mathbb{N}}V_n$ where V_n is a vector space for any n. Elements of V_n are called homogeneous element of degree n.

Definition 2.35 (see [3, Definition 3.1])**.** The q dimension for this new vector space is

$$qdim(V) \coloneqq \sum_m q^m dim(V_m)$$

Example 2.36. Suppose we have field F and we have graded vector space $F_{-1} \oplus F_1$ then $qdim(F_{-1} \oplus F_1) = q + q^{-1}$.

In this section we use vector space $V =< v_+, v_- >$ where $\deg v_+ = 1$ and $\deg v_+ = -1$. The $qdim(V) = q + q^{-1}$.

Definition 2.37 (see [3])**.** Khovanov bracket of a diagram D of a link L, denoted $[\![D]\!]$, is a cochain complex of graded $\mathbb{Z}$-vector spaces. It is characterized by the following properties:

(1) $[\![\varnothing]\!] = 0 \to \mathbb{Z} \to 0$
(2) $[\![\bigcirc \sqcup D]\!] = V \otimes [\![D]\!]$
(3) $[\![\times]\!] = \text{Cone}\left(0 \to [\![\asymp]\!] \xrightarrow{d} [\![)(]\!]\{1\} \to 0\right)$

Here, the $\{1\}$ operator is the degree shift operation $V\{l\}_m = V_{m-l}$.

The first axiom is about empty diagram, bracket sends empty diagram to cochain complex with 0 and $\mathbb{Z}$. The second axiom says that if we have diagram D which can be written as a disjoint sum of a circle and a diagram D', then to calculate $[\![D]\!]$ we need to calculate only $[\![D']\!]$. The third axiom gives a recipe how to find the Khovanov bracket of a general link diagram. If we have a link diagram D, the third axiom allows us to write

$$C^{i,*}(D) = C^{i,*}(D_0) + C^{i-1,*}(D_1)\{1\}$$

where D_0 and D_1 are the diagrams which we get them by resolving a fixed crossing by 0 and 1 respectively on the diagram D. In other words,the third axiom says that for a link diagram D, $C^{i,*}(D)$ is the mapping cone of $C^{i,*}(D_0)$ and $C^{i-1,*}(D_1)$ with the map d between $C^{i,*}(D_0)$ and $C^{i-1,*}(D_1)$, the map d will be defined in 2.42.

Now we define the modules that we use in the definition of Khovanov homology, see [27, Chapter 1.3]. We begin with the definition of the space V_α.

$$V_\alpha = V^{\otimes k_\alpha}\{r_\alpha + n_+ - 2n_-\},$$

where $\alpha \in \{0,1\}^n$, and:

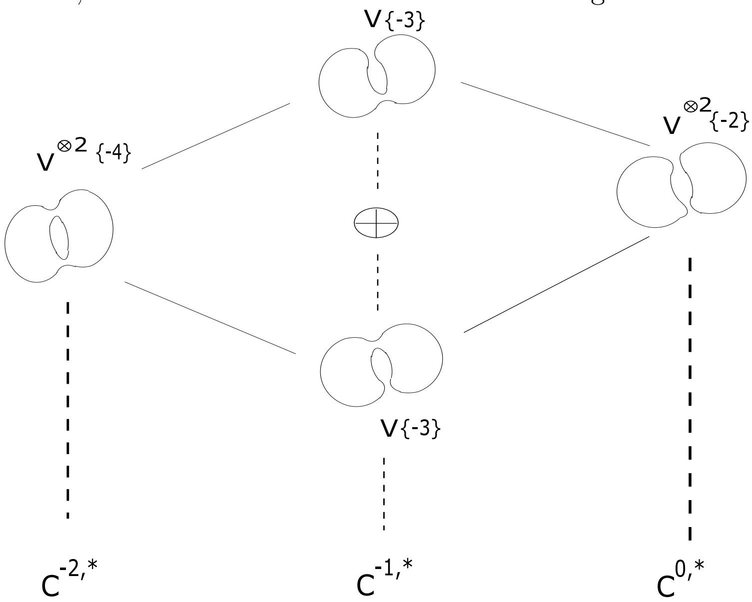
- k_α = the number of circles in the diagram D_α,
- r_α = the number of 1's in α,
- n_+ = number of positive crossings in L,
- n_- = number of negative crossings in L.

Why Khovanov bracket and not the Khovanov-Rozansky bracket?

We define our module now.

$$C^{i,*}(D) = \bigoplus_{\substack{\alpha\in\{0,1\}^n \\ i=r_\alpha - n_-}} V_\alpha$$

Example 2.38. (see figure 4 [27]) For the negative Hopf link. It is easy to see that $n_+ = 0$ and $n_- = 2$. In particular, the cube of resolutions has the following form:



2.5. **Definition of boundary map for Khovanov homology.** Have defined the modules underlying the Khovanov chain complex, we need to describe the boundary map. Consider a cube where nodes are diagrams which we get by different resolutions and we have edges between nodes. We define a map for the edge between two nodes which we get from different specific resolutions. We define the map d_ϵ where ϵ is the edge of our cube that lies between two resolutions that differ at one crossing. This edge can be labeled by sequences in $\{0, 1, *\}$ where the height of the ϵ is denoted by $|\epsilon|$ and is defined by the number of '1' in the domain of the d_ϵ. We turn edges into arrows by the rule $* = 0$ gives the tail and $* = 1$ gives the head. For instance, the edge between resolutions 001 and 011 is $0*1$ and the map between them is d_{0*1}. Prior to defining d_ϵ, we need to describe some elementary maps, from which d_ϵ is constructed. It might be helpful to remind here that V is vector space which is generated by v_+ and v_- where $\deg(v_+) = 1$ and $\deg(v_-) = -1$.

First, we define a map m that corresponds to merging two circles to one circle. Namely:

Definition 2.39. The *multiplication map* $m: V \otimes V \to V$ is defined as:

$$\begin{aligned} v_+ \otimes v_+ &\mapsto v_+ \\ v_+ \otimes v_- &\mapsto v_- \\ v_- \otimes v_+ &\mapsto v_- \\ v_- \otimes v_- &\mapsto 0. \end{aligned}$$

We extend it linearly to $V \otimes V$.

In addition to that, we define a map corresponding to splitting one circle into two circles:

Definition 2.40. The *comultiplication map* $\Delta : V \to V \otimes V$ is defined as:

$$\begin{aligned} v_+ &\mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- &\mapsto v_- \otimes v_- \end{aligned}$$

and it can be extended linearly on V.

We define the map d_ϵ.

Definition 2.41. We define d_ϵ as the identity on the tensor factors associated to circles which stay the same after smoothing. If two circles merge into one circle, d_ϵ is the map m on tensor factors associated to these two circles, see Definition 2.39. Another case is when we divide one circle into two circles, d_ϵ is the linear map Δ on this circle, see Definition 2.42.

We are ready to define the Khovanov differentials $d^i : C^{i,*}(D) \to C^{i+1,*}(D)$

Definition 2.42. For $v \in V_\alpha \subset C^{i,*}(D)$

$$d^i(v) = \sum_{\substack{\epsilon \\ \text{tail}(\epsilon)=\alpha}} \text{sign}(\epsilon) d_\epsilon(v)$$

where $\text{sign}(\epsilon) = (-1)^{\text{number of 1's to the left of the change place}}$
see Chapter 1 of [27].

For example, suppose we have ϵ the edge between 010 and 011, then $\text{sign}(\epsilon)$ is -1 because there is just one 1 before the change from 0 to 1 in the edge.

It can be shown that m and Δ preserve the quantum grading, and since d_ϵ is the sum of them, we say that d_ϵ preserve the q-grading.

With this definition, we have a lemma below:

Lemma 2.43 (see [27])**.** $d^r \circ d^{r-1} = 0$.

The above lemma shows that d^i is indeed a boundary map.

We defined the chain complex, so we define Khovanov homology on this chain complex.

Definition 2.44 (see [27])**.** $Kh^{*,*}(D) = H(C^{*,*}(D), d)$ where Kh stands for Khovanov homology.

The graded Euler characteristic of $C^{i,*}(L)$ for a link diagram L is

$$\sum_i (-1)^i q\text{dim}(C^{i,*}(D))$$

This is equal to the unnormalized Jones polynomial of the knot diagram $\langle D \rangle$ of a link L. See [27].

In order to say that this definition gives a well-defined link invariant, we need to show that if we have two different diagrams D_1 and D_2 of the same link L, we have $H(D_1) \simeq H(D_2)$. In particular, we need to check if homology will be the same after we apply Reidemeister move to link diagram. (See [3, Theorem 2])

Theorem 2.45. *Assume we have two diagrams D_1 and D_2 which are connected to each other with a single Reidemeister move, then $H(D_1) \simeq H(D_2)$.*

finite sequence of ...

Theorem 2.45 has in fact three parts, each corresponding to a different Reidemeister move.

There exist pairs of links where they have the same Jones polynomials but have different Khovanov homologies. This shows us that Khovanov homology is a stronger invariant.

Example 2.46 ([27, Example 3.2])**.** Two knots 5_1 and 10_{132} are the knots with the same Jones polynomial but different Khovanov homology. For the unnormalized Jones polynomial, we have $\hat{J}(10_{132}) = \hat{J}(5_1) = q^{-3} + q^{-5} + q^{-7} + q^{-15}$ whereas we have different Khovanov homology.

$$\mathrm{Kh}(5_1) = \mathbb{Q}_{(0,-3)} + \mathbb{Q}_{(0,-5)} + \mathbb{Q}_{(-2,-7)} + \mathbb{Q}_{(-3,-11)} + \mathbb{Q}_{(-4,-11)} + \mathbb{Q}_{(-5,-15)}$$

$$\mathrm{Kh}(10_{132}) = \mathbb{Q}_{(0,-1)} + \mathbb{Q}_{(0,-3)} + (\mathbb{Q} \oplus \mathbb{Q})_{(-2,-5)} + \mathbb{Q}_{(-3,-5)} + \mathbb{Q}_{(-3,-9)} + \mathbb{Q}_{(-4,-7)} + \mathbb{Q}_{(-4,-9)} + \mathbb{Q}_{(-5,-11)} + \mathbb{Q}_{(-6,-11)} + \mathbb{Q}_{(-7,-15)},$$

where $\mathbb{Q}_{i,j}$ means at the i and j th degree we have a copy of $\mathbb{Q}$.

3. Periodic links

Definition 3.1. Consider a link L in S^3 and semi-free $\mathbb{Z}_m$ action on S^3. We say L is m-periodic for the semi-free $\mathbb{Z}_m$ rotation action of order m on S^3, if the set of fixed points f of action is disjoint with L and L is invariant under the $\mathbb{Z}_m$ action.

Similarly, we define an action for a link diagram.

Definition 3.2. We say that the link diagram $D \subset \mathbb{R}^2$ of an m-periodic link L is m-periodic if it is invariant under the rotation action of $\mathbb{R}^2$ of order m, and D is disjoint from the set of fixed points of the action. In other words, an m-periodic link diagram is a diagram that is carried to itself by a rotation of $(360/m)^\circ$ about the origin.

Every m-periodic link admits an m-periodic link diagram.

Example 3.3. The trefoil knot is a 3-periodic knot.

Remark 3.4. Smith's conjecture states that a fixed point set of $\mathbb{Z}_m$ on S^3 cannot be a nontrivial knot.

To check whether a link is periodic, one may apply one of the following critieria.

Theorem 3.5 (Murasugi Conditions, see [18])**.** *Suppose we have $K \subset S^3$ a $q = p^r$-periodic knot with prime p, Δ the Alexander polynomial of K, and Δ' the Alexander polynomial of the quotient knot $K/\mathbb{Z}_p$. Furthermore, we have l the absolute value of the linking number of K with the symmetry axis. Then*

(1) $\Delta' | \Delta$
(2) $\Delta \equiv (\Delta')^q (1 + t + \ldots + t^{l-1})^{q-1} \pmod{p}$

Example 3.6. The left-handed trefoil knot has period 3; the quotient knot is the unknot, and the linking number l is 2.

- It is obvious that the first condition is satisfied, which means $1|\Delta$.
- The Alexander polynomial of the trefoil knot is $t^2 - t + 1$. So we have
$$(1)^3(1 + t^{2-1})^{3-1} = (1 + t)^2 \equiv t^2 - t + 1 \pmod{3}.$$
This means the second condition is satisfied.

Example 3.7. For the figure eight knot, the Alexander polynomial is $-t^{-1} + 3 - t$. Since $\Delta(t) = -t^{-1} + 3 - t$ is irreducible and since $\Delta'(1)|\Delta(1)$ we deduce $\Delta' = 1$.
We have
$$-t^{-1} + 3 - t = (1 + t + \ldots + t^{l-1})^{p-1} \pmod{3}.$$
We know that the Alexander polynomial is well-defined up to multiplication by powers of t. So we take Alexander polynomial here $\Delta(t) = -1 + 3t - t^2$. Hence the polynomial on the right-hand side should have the same degree with the polynomial on the left-hand side. Hence we should

have $(l-1)(p-1) = 2.$. We have two cases. Either $l = 3, p = 2$ or $l = 2, p = 3$. For $l = 2, p = 3$ on the right-hand side. We have $(1+t)^2 = 1 + 2t + t^2$ but

$$1 + 3t - t^2 \neq 1 + 2t + t^2 \pmod 3.$$

On the other hand, we have

$$1 + 3t - t^2 \neq 1 + t + t^2 \pmod 2.$$

This shows that figure eight knot is not p-periodic for $p \geq 3$.

Theorem 3.8 (Edmonds' Criterion, see [1])**.** *Assume we have K, a periodic knot of period q, and $\bar{K}$, the quotient knot of K. Then there are nonnegative integers $g_{\bar{K}}$ and σ such that*

$$g(K) = q g_{\bar{K}} + \frac{(q-1)(\sigma-1)}{2}.$$

where $g(K)$ and $g_{\bar{K}}$ is the Seifert genus.

Example 3.9. For a trefoil knot K, $\bar{K}$ is the unknot. The trefoil knot has genus 1, and the unknot has genus 0. If we take $\sigma = 2$, then we have $1 = 3 \cdot 0 + 2 \cdot \frac{1}{2}$.

Theorem 3.10 (Naik's Criterion, see [19])**.** *Suppose $K \subset S^3$ is a p-periodic knot with p a prime and let $k > 1$ whereas we denote $\bar{K}$ for quotient knot of K. For $\Sigma^m(K)$ the m-fold branched cover of K suppose that $H_1(\Sigma^m(K))$ has nontrivial q-torsion part, for some prime $q \neq p$, and let l_q to be the least positive integer such that $q^{l_q} \equiv \pm 1 \pmod p$. Then there exist non-negative integers $b_1, b_2, \ldots$ such that*

$$H_1(\Sigma^m(K); \mathbb{Z})_q / H_1(\Sigma^m(\bar{K}); \mathbb{Z})_q = \mathbb{Z}_q^{2b_1 l_q} \oplus \mathbb{Z}_{q^2}^{2b_2 l_q} \oplus \cdots.$$

Theorem 3.11 (HOMFLYPT Criterion, see [22])**.** *Assume we have the unital subring R in $\mathbb{Z}[a^{\pm}, z^{\pm}]$ where $R = \langle a, a^{-1}, \frac{a+a^{-1}}{z}, z \rangle$. If a knot is p-periodic and $P(a,z)$ is its HOMFLYPT polynomial, then*

$$P(a,z) \equiv P(a^{-1}, z) \pmod{< p, z^p >},$$

where $< p, z^p >$ is the ideal generated by p and z^p in R.

We have Borodzik-Politarczyk criterion for periodic knots. Before going to it, we need to give a remark.

Remark 3.12. The Khovanov polynomial is equal to the Jones polynomial where we have $t = -1$. In other words, We have the equation $\mathrm{KhP}(K, -1, q) = J(K)$.

Theorem 3.13 (Borodzik-Politarczyk Criterion, see [15, Theorem 1.1])**.** *Assume we have a p^n-periodic knot K, where p is an odd prime. Suppose that $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \mathbb{F}_r$ for a prime r where $r \neq p$ and r has the maximal order in $\mathbb{Z}_p^n$. Here since $gcd(r,p) = 1$ any prime $r \neq p$ will have maximal order p^n. Take $c = 1$ if $\mathbb{F} = \mathbb{F}_2$ and $c = 2$ otherwise. Then*

$$\mathrm{KhP}(K, t, q) = P_0 + \sum_{n=1}^{n} (p^j - p^{j-1}) P_j,$$

Where $P_0, P_1, \ldots, P_n \in \mathbb{Z}[q, q^{-1}, t, t^{-1}]$ are Laurent polynomials such that

(1) $P_0 = q^{s(K,\mathbb{F})}(q + q^{-1}) + \sum_{j=1}^{\infty} (1 + tq^{2cj}) S_{0j}(t,q)$, and the polynomials S_{0j} have non-negative coefficients;

(2) $P_k = \sum_{j=1}^{\infty} (1 + tq^{2cj}) S_{kj}(t,q)$ and the polynomials S_{kj} have non-negative coefficients for $1 \leq k \leq n$,

(3) $P_k(-1,q) - P_{k+1}(-1,q) \equiv P_k(-1,q^{-1}) - P_{k+1}(-1,q^{-1}) \pmod{q^{p^{n-k}} - q^{-p^{n-k}}}$;

The criterion is rather specific, easier to implement on a computer than to solve by hand. The following example is discussed in [15].

Example 3.14. Take the knot $15n1335221$. This knot satisfies all periodicity criteria for $p = 5$ we discussed in the thesis. In particular, it satisfies the HOMFLYPT criterion for $p = 5$. It has the Khovanov polynomial

$$q + q^{-1} + (1 + tq^4)(t^{-7}q^{-15} + 3t^{-6}q^{-13} + t^{-5}q^{-11} + 3t^{-4}q^{-9} + t^{-3}q^{-9} + 3t^{-2}q^{-7} + t^{-1}q^{-5} + 3t^{-1}q^{-3} + q^{-3} + q^{-1} + 3tq + t^2q^3 + 3t^3q^3 + t^4q^5 + 3t^5q^7 + t^6q^9 + 4(t^{-5}q^{-11} + t^{-4}q^{-9} + 2t^{-3}q^{-7} + 2t^{-2}q^{-5} + t^{-1}q^{-5} + t^{-1}q^{-3} + 2tq^{-1} + q^{-3} + q^{-1} + 2t'2q + t^3q^3 + t^4q^5)).$$

We write $\mathrm{KhP} = q + q^{-1} + (1 + tq^4)S'_{01} + 4(1 + tq^4)S'_{11}$ where

$$S'_{01} = t^{-7}q^{-15} + 3t^{-6}q^{-13} + t^{-5}q^{-11} + 3t^{-4}q^{-9} + t^{-3}q^{-9} + 3t^{-2}q^{-7} + t^{-1}q^{-5} + 3t^{-1}q^{-3} + q^{-3} + q^{-1} + 3tq + t^2q^3 + 3t^3q^3 + t^4q^5 + 3t^5q^7 + t^6q^9$$

and

$$S'_{11} = t^{-5}q^{-11} + t^{-4}q^{-9} + 2t^{-3}q^{-7} + 2t^{-2}q^{-5} + t^{-1}q^{-5} + t^{-1}q^{-3} + 2tq^{-1} + q^{-3} + q^{-1} + 2t'2q + t^3q^3 + t^4q^5$$

According to Theorem 3.13(3), $A(q) = q + q^{-1} + (1 + tq^4)S'_{01}(t, q) - (1 + tq^4)S'_{11}(t, q)$ and we have $A := (A(q) - A(q^{-1})) \mod q^5 - q^{-5}$. So we have $A = -10q + 5q^3 - 5q^7 + 10q^9$. Since $A \neq 0$, we need to change S'_{01} and S'_{11}. We need to satisfy Theorem 3.13(1) and (2). We must have $S'_{11} \to S'_{11} - \delta$ and $S'_{01} \to S'_{01} + 4\delta$. Here it is important that we must have non-negative coefficients for S'_{11} and S'_{01}. We have only finitely many possibilities for δ. In order to reduce the number of possibilities, we use the following argument. Take $\delta = at^iq^j$. Then, after changing S'_{01} and S'_{11}, we have $A \to A + aT_{ij}$, where $T_{ij} = (-1)^i 5(-q^{-j-4} + q^{-j} - q^j + q^{j+4}) \mod (q^5 - q^{-5})$. We deduce that $T_{ij} = (-1)R_{j'}$ with $j' = j \mod 10$ and

$$\begin{aligned} R_1 &= R_5 = 5(q - q^9), \\ R_3 &= 10(q^3 - q^7), \\ R_7 &= R_9 = 5(-q - q^3 + q^7 + q^9). \end{aligned}$$

For different δ, A will change by $-a_1R_1 - a_3R_3 - a_7R_7$. Note that coefficients change based on conditions that $S'_{11} - \delta$ must have non-negative coefficients. We must have coefficients

$$\begin{aligned} a_1 &\in \{-1, 0, 1, 2, 3, 4, 5, 6\}, \\ a_3 &\in \{-3, -2, -1, 0\}, \\ a_7 &\in \{-4, -3, -2, -1, 0, 1, 2\}. \end{aligned}$$

With these conditions, it is not possible to have $A = 0$. We deduce that a knot $15n1335221$ is not 5-periodic.

4. Webs, foams and categories

We have already studied Khovanov homology. Now, we want to define $\mathfrak{sl}_N$ homology. Actually, Khovanov homology is $\mathfrak{sl}_2$ homology, but for $\mathfrak{sl}_N$ homology, we have to use a more formalized approach. We will use webs and foams.

4.1. Webs and foams.

Definition 4.1. A trivalent graph Γ is a closed one-dimensional cell complex where three edges meet at each vertex.

Definition 4.2. In an oriented graph, the source vertex is a vertex that has zero indegree. In other words, it is a vertex where the number of incoming edges is 0. Similarly, a sink vertex is a vertex that has zero outdegree. In other words, it is a vertex where the number of outgoing edges is 0.

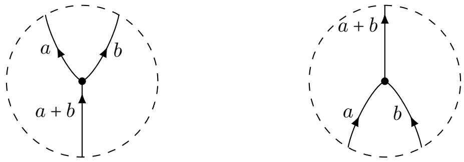


FIGURE 1. The flow condition of Definition 4.3

Definition 4.3 (web)**.** A *closed web* is a finite oriented trivalent graph V without sources and sinks properly embedded in $\mathbb{R}^2$. Each edge is labeled by numbers $0, \dots, N$. An edge with the 0 label can be deleted from the web, so in some papers, edge labeling starts from 1. The labelings of edges should satisfy an important condition called the *flow condition*; see Figure 1:

- If two edges with labels a and b enter a vertex, then the outgoing edge has label $a + b$. We call a vertex a *merge vertex* when the vertex has two incoming edges.
- If two edges with labels a and b exit from a vertex, then the incoming edge has label $a + b$. Similarly, we call a vertex a *split* vertex when the vertex has two outgoing edges.

In Figure 1, the web on the left has a split vertex, and the web on the right has a merge vertex.

Remark 4.4. An empty web is just a web with no vertices and no edges.

Assume that we have two webs W_0 and W_1 in $\mathbb{R}^2$. Think of W_0 in $\mathbb{R}^2 \times \{0\}$ and W_1 in $\mathbb{R}^2 \times \{1\}$.

Definition 4.5 (foam)**.** Assume we have two webs W_0 and W_1. An *N-undecorated foam* $F\colon W_0 \to W_1$ is a compact, finite 2-dimensional CW-complex properly embedded in $\mathbb{R}^2 \times [0,1]$ such that:

- If $x \in F \setminus (W_0 \cup W_1)$, then there exists a neighborhood U of x in F homeomorphic to one of the following three models:
  - a *smooth point*: U is homeomorphic to a disk in $\mathbb{R}^2$;
  - a Y-shaped point (codimension 1 singularity): U is homeomorphic to the union of three distinct rays stemming out of a common point, times $(0,1)$;
  - a cone over a 1-skeleton of a tetrahedron (codimension 2 singularity), when x is a triple point. Compare Figure 2.
- Every *facet* F_i of F, i.e., a connected component of the set of smooth points, carries an orientation and a label by an integer $0, \dots, N$;
- a binding: compact oriented 1 dimensional manifolds. Each binding has
  - an orientation that agrees with the orientation of facets with labels a and b whereas disagrees with the orientation of facet with label $a + b$.
  - cycling ordering of the three facets around binding: when foam embedded in $\mathbb{R}^3$ this ordering must be compatible with the left-hand rule with respect to its orientation.
- Every *seam* C_i, which is a connected component of the set of Y-shaped points of F, carries an orientation;
- The orientation of every seam agrees with the orientation of precisely two adjacent facets; if these two facets are labeled by a and b, then the third facet has the label $a + b$;
- The *bottom boundary* of each facet F_i, that is $\overline{F_i} \cap (\mathbb{R}^2 \times \{0\})$, is an edge of W_0 with the same label and the orientation opposite to the orientation induced by F_i;
- The *top boundary* of each facet F_i, that is $\overline{F_i} \cap (\mathbb{R}^2 \times \{1\})$, is an edge of W_1 with the same label and the orientation agreeing with the orientation induced by F_i;

A foam is a map between webs. We define the composition of foams.

Definition 4.6. Assume we have webs W_0, W_1 and W_2. Furthermore, we have foams F_{01} between W_0 and W_1, F_{12} between W_1 and W_2. We define composition F_{02} of F_{01} and F_{12} as the union of F_{01} and F_{12} along W_1 where we can think of F_{01} as a subset of $\mathbb{R}^2 \times [0, 1/2]$ and F_{12} as a subset of $\mathbb{R}^2 \times [1/2, 1]$.

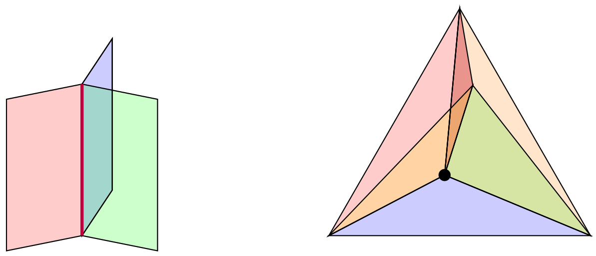


FIGURE 2. Codimension 1 and 2 singular points of a foam.

In our case, closed foams are crucial for us:

Example 4.7. A closed foam is the map from an empty web to an empty web.

4.2. **Coloring and decorations.** On webs and foams, we might have some extra structures, namely colorings and decorations. The coloring of a web is similar to the labeling.

Definition 4.8 (coloring of a web)**.** Let W be a web. A *coloring* is an assignment of a subset A_e of P=$\{1, 2, \dots, N\}$ to every edge e such that $|A_e|$ = labeling of the edge. In other words, for every edge, we assign a subset of $\{1, 2, \dots, N\}$. This assignment should satisfy two conditions:

- We have two edges with colorings A and B enter a vertex, then the outgoing edge should have coloring $A \cup B$ where in particular we have $A \cap B = \varnothing$.
- We have two edges with colorings A and B exiting from a vertex, then the incoming edge should have coloring $A \cup B$.

The colorings of foams are similar.

Definition 4.9 (coloring of a foam)**.** Assume we have a foam F. A *coloring* is an assignment of a subset $c(f)$ of $\{1, 2, \dots, N\}$ to a face f such that $|c(f)|$ = labeling of the foam. This assignment should be compatible with the composition of two foams. In other words, near each seam, the set of variables at one adjacent face is a disjoint sum of a set of variables assigned to the two other adjacent edges.

A *colored foam* is a foam with a coloring.

In addition to this structure on webs and foams, we have decorations of foams.

Definition 4.10.

- Assume we have a colored foam (F, c). We define surface $F_i(c)$ as a union of all the facets that contain $i \in P$. The restriction on orientations of facets ensures that $F_i(c)$ is also oriented.
- Assume we have a colored foam (F, c). We define surface $F_{ij}(c)$ as a union of all the facets which contain i or j but not both at the same time in their colors. The restriction on orientations of facets ensures that $F_{ij}(c)$ is also oriented.

Definition 4.11. Assume we have a colored foam (F, c) and we have $i < j$. A circle in $F_i(C) \cap F_j(C) \cap F_{ij}(C)$ is *positive* (respectively *negative*) with respect to (i, j) if it consists of positive (respectively negative) bindings. We denote the number of positive (respectively negative) by $\theta^+_{ij}(c)_F$ (respectively $\theta^-_{ij}(c)_F$). Furthermore, we have $\theta_{ij}(c) = \theta^+_{ij}(c) + \theta^-_{ij}(c)$.

4.3. **Decorations, degrees and evaluations.**

Definition 4.12 (Degree of an undecorated foam)**.** The degree $d^{un}(F)$ for a foam F is the sum of the following items.

- For a face f we have $d(f) = a(N - a)\chi(f)$ where a is the face label and χ is Euler characteristic;
- For seam i which is not a circle and is surrounded by faces with labels $a, b, a + b$ we have $d(i) = ab + (a + b)(N - a - b)$;

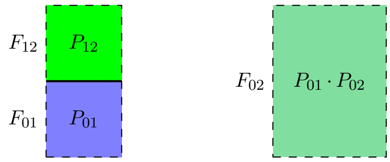


FIGURE 3. Rule for gluing decorated foams.

- For a singular point p surrounded by faces with label $a, b, c, a+b, b+c, a+b+c$ we have $d(p) = ab + bc + cm + ma + ac + bm$, where $m = N - a - b - c$.
- At the final stage we have the formula

$$d_N(F) = -\sum_{\text{f facet}} d(f) + \sum_{\text{i seam}} d(i) - \sum_{\text{p singular points}} d(p)$$

.

Another important definition for foams is the decoration.

Definition 4.13 (Decoration of a foam)**.** Assume we have a foam F and a face f with labeling a. A *decoration* is an assignement of a symmetric homogeneous polynomial p_f in a variables to the face f.

A decorated foam is a foam together with a decoration of each face. We have the composition of foams when decoration on foams respects composition rule also. Namely, assume we have two foams F_{01} and F_{12} with decoration P_{01} on face f_1 of F_{01} and P_{12} on face f_2 of F_{12}. Assume that composition happens on faces f_1 and f_2. Then the new face should have decoration $P_{01} \cdot P_{12}$.

Remark 4.14. We fix our variables for polynomials as $X_1, X_2, \dots, X_N$, and we declare that each variable X_i has degree 2.

Definition 4.15 (Degree of decorated foam)**.** The degree of a decorated foam F is equal to $d^{un}(F) + 2\sum_f \deg(P_f)$, where the sum runs over all faces f of the foam.

Definition 4.16 (Evaluation of a foam)**.** The evaluation of a foam involves assigning a polynomial to the foam. Assume we have a colored decorated foam (F, c). We have contributions:

$$s(F,c) = \sum_{i=1}^{N} i\left(\frac{\chi(F_i(c))}{2}\right) + \sum_{1\le i\le j\le N} \theta^+_{ij}(F,c)$$

$$P(F,c) = \prod_{f \text{ face of } F} P_f(c(f))$$

$$Q(F,c) = \prod_{1\le i\le j\le N} (X_i - X_j)^{\left(\frac{\chi(F_{ij}(c))}{2}\right)}$$

$$\langle F, c\rangle = (-1)^{s(F,c)} \frac{P(F,c)}{Q(F,c)}.$$

Assume we have a decorated foam F, we define the evaluation of a foam F:

$$\langle F\rangle = \sum_c \langle F, c\rangle,$$

where the sum runs over all colorings of F.

It is proved in [25] that $\langle F\rangle$ is a symmetric polynomial. The next observation is made in [25]; it follows promptly from the definition of $\langle F\rangle F$.

Lemma 4.17. *If we have two isotopic foams F_1 and F_2 in $\mathbb{R}^2 \times [0,1]$, then $\langle F_1\rangle = \langle F_2\rangle$.*

4.4. **Foam categories.** We want to define $\mathfrak{sl}_N$ homology. For this, it is convenient to package webs and foams into a category theory language.

Definition 4.18. $\mathbf{Foam}_N^*$ category is a category with N-webs as objects, and morphisms between two webs W_1 and W_2 are foams from W_1 to W_2.

The next category we want to have is the $\mathbb{S}\mathbf{Foam}_N^*$ category. We use the notation $\mathbb{S}_N := \mathrm{Sym}[X_1, X_2, \dots, X_N]$ as the graded ring of symmetric polynomials with integer coefficients. Recall that the variables X_i have degree 2.

Definition 4.19 ($\mathbb{S}\mathbf{Foam}_N^*$)**.** The category $\mathbb{S}\mathbf{Foam}_N^*$ is the $\mathbb{S}_N$ linear, $\mathbb{Z}$-graded category with

- Objects as formal shifts $q^k V$, where V is a web, q is a formal variable, and $k \in \mathbb{Z}$ is grading.
- Morphisms as $\mathbb{S}_N$ linear combinations of decorated foams. Foams are in the form of $F : q^s V \to q^m W$. Here, F has degree $m - w$. For $p \in \mathbb{S}_N$, pF has degree $\deg(p) + \deg(F)$.

We defined the $\mathbb{S}\mathbf{Foam}_N^*$ category. To understand it better, we define the evaluation functor from $\mathbb{S}\mathbf{Foam}_N^*$ to the category Sym_N^* of $\mathbb{S}_N$ projective modules.

Definition 4.20 (Naive evaluation functor)**.** We have functor $\widetilde{\mathcal{F}} : \mathbb{S}\mathbf{Foam}_N^* \to \mathrm{Sym}_N^*$

- For any shifted web $q^a V$, we have
$$\widetilde{\mathcal{F}}(q^a V) = \bigoplus_{G \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, q^a V)} \mathbb{S}_N\{d_N(G)\}$$
- For a morphism $F : q^a V \to q^b W$, we have the map
$$\mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, q^a V) \xrightarrow{\widetilde{\mathcal{F}}(F)(-) := F \circ (-)} \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, q^b W)$$

Recall that $\mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, q^a V)$ is an $\mathbb{S}_N$ module.

In our assignment for a web V, we took all foams, but it is logical to expect isotopic foams as defining the same objects, respectively the same morphisms. To overcome this problem, we need to take a suitable quotient using foam evaluation.

Suppose we have a web V and $F' \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(V, \varnothing)$, define
$$\phi_{F'} : \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, V) \to \mathbb{S}_N$$
$$\phi_{F'}(F) = \langle F' \circ F \rangle$$

Now we define
$$I(V) = \bigcap_{F' \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(V, \varnothing)} \ker \phi_{F'}$$

Actually, $I(V)$ consists of all $\mathbb{S}_N$ linear combinations of foams from $\varnothing$ to V that evaluate to zero when capped with any foam from V to $\varnothing$.

As closed isotopic foams evaluate to the same polynomial, we have the following observation, which we record for a future use.

Lemma 4.21. *For any two isotopic foams F and F' from $\varnothing$ to V, $F - F'$ is in $I(V)$.*

Definition 4.22 (Evaluation functor)**.** We define a new evaluation functor $\mathcal{F}$ which is similar to the naive evaluation functor $\widetilde{\mathcal{F}}$.

- For any web $q^a V$, we have $\mathcal{F}(q^a V) = \widetilde{\mathcal{F}}(q^a V)/I(q^a V) = \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, q^a V)/I(V)$.
- For morphism $G : V \to W$, we have the map
$$\mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, V)/I(V) \xrightarrow{\mathcal{F}(G)(-) := G \circ (-)} \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, W)/I(W)$$

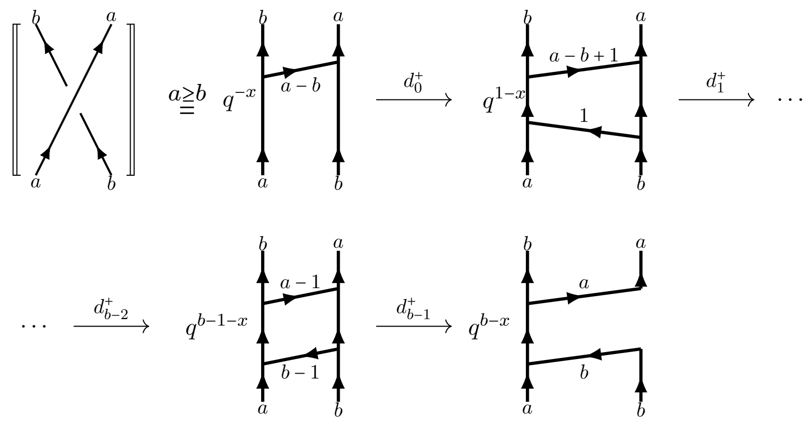


FIGURE 4. The resolution of a crossing. Here $x = b(N-b)$ and q denotes the quantum grading shift. The first term is at homological degree zero.

Now we define a new category where any two isotopic foams between two webs will be in the same class.

Definition 4.23. The category of $\mathbb{S}\mathbf{Foam}_N$ has the same objects as $\mathbb{S}\mathbf{Foam}_N^*$, but for morphisms, it is different: $\mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N}(V,W) := \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(V,W)/\ker\mathcal{F}$.

We need a bracket definition, and for this, we need to define a new category.

Definition 4.24 (Kom($\mathbb{S}\mathbf{Foam}_N$) category)**.** The category Kom($\mathbb{S}\mathbf{Foam}_N$) is defined as follows:

- objects are cochain complexes of finitely generated $\mathbb{S}_N$ modules generated by objects in the $\mathbb{S}\mathbf{Foam}_N$ category;
- morphisms are formal linear combinations of morphisms in the $\mathbb{S}\mathbf{Foam}_N$ category.

4.5. **$\mathbb{S}_N$-equivariant $\mathfrak{sl}_N$-homology.** We need to define the bracket $[\![D]\!] \in \mathrm{Kom}(\mathbb{S}\mathbf{Foam}_N)$ for any labeled link diagram D. For this, we just need to define the bracket for a straight strand, positive and negative crossing. For any diagram, we will take the tensor product of these three diagrams.

Definition 4.25 (Bracket definition, see [5, Defintion 3.3])**.**

- For a strand a bracket maps it to the corresponding web in homological degree zero.
- The bracket maps a positive crossing with a as an overstrand label and b as an understrand label, denoted as $a \geq b$, to the chain complex as in Figure 4. The differentials d_k^+ are given by the foams in Figure 6.
- The bracket maps a positive crossing when we have b as an overstrand and a as an understrand to the complex obtained by taking mirror images of webs and foams along the vertical axis and swapping the a and b.
- For a negative crossing, the bracket sends it to the complex where we can obtain it from positive crossing by inverting the q-degrees and homological degrees, and reflecting the differential foam in a horizontal plane; see Figure 5.

Strange way of spelling it out!

What is a labelled/unlabelled link diagram?

Remark 4.26. For an unlabelled link diagram of D, we declare all strands as labelled 1.

We create a cube of resolutions to better understand the bracket definition. Let D be a diagram with n crossings, enumerated from 1 to n. At each crossing, we have labels a_i and b_i. We define c_i as the minimum of these two labels and set $C_i = \{0,\dots,c_i\}$ for the i-th positive crossings and $C_i = \{-c_i,\dots,0\}$ for the i-th negative crossings. Similarly, we define $SC_I = [0,c_i]$ and $SC_I = [-c_i,0]$. Moreover, we consider SC_i to be a CW-complex where 0-cells are the

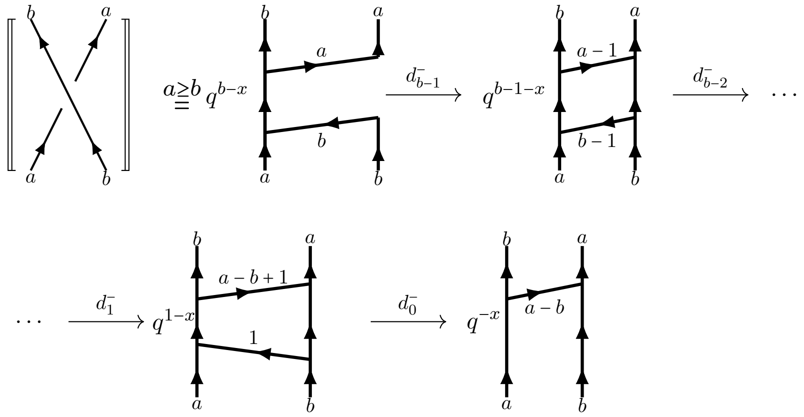


FIGURE 5. The complex associated with a negative crossing.

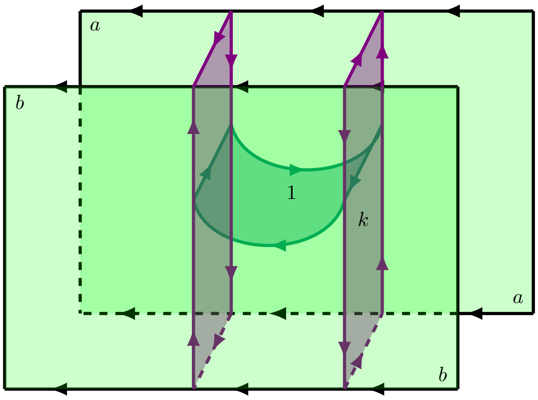


FIGURE 6. The foam that is the differential d_k^+ of the complex in Figure 4. It is decorated by constant polynomial equal 1.

integral points, and 1-cells are the intervals. Define $\mathrm{Cube}(D) = \prod_i C_I$ and $\mathrm{SCube}(D) = \prod_i SC_i$ where SCube carries a concrete CW-complex structure.

Definition 4.27 (Immediate successor)**.** For $I, I' \in \mathrm{Cube}(D)$, we say I' is an *immediate successor* of I if I and I' agree on all coordinates except one, and this one coordinate is one larger than that of I.

Definition 4.28 (Sign assignment)**.** A *sign assignment* $\mathfrak{s}$ is an assignment of $\mathfrak{s}(I, I') \in \mathbb{F}_2$ for any pair I and I' such that I' is an immediate successor of I.

We want $\mathfrak{s}$ to satisfy the following chain condition for I, I_1, I_2, and I_{12} where $I_1 \neq I_2$, I_1 and I_2 are the immediate successors of I, and I_{12} is the immediate successor of I_1 and I_2. We have

$$\mathfrak{s}(I, I_1) + \mathfrak{s}(I, I_2) + \mathfrak{s}(I_1, I_{12}) + \mathfrak{s}(I_2, I_{12}) = 1 \in \mathbb{F}_2$$

Remark 4.29. Algebraically, we think of $\mathfrak{s}$ as a cellular 1-cochain in the cellular cochain complex $C^1_{cell}(\mathrm{SCube}; \mathbb{F}_2)$ where $\delta\mathfrak{s}$ is a 2-cochain with a constant value of 1.

Lemma 4.30. *For any diagram D, there exists a sign assignment $\mathfrak{s}$. For any two assignments $\mathfrak{s}$ and $\mathfrak{s}'$, there is a coboundary such that $\mathfrak{s} - \mathfrak{s}' = \delta t$ where t is a cellular 0-cochain on* $\mathrm{SCube}(D)$. *Moreover, t is uniquely determined if it fixes its value on* $(0, \dots, 0)$.

Proof. For a $c \in C^2_{cell}(\mathrm{SCube}; \mathbb{F}_2)$ where c is a constant cochain with a value of 1, we have $\delta(c) = 0$ because the cube has an even number of rectangles. Since we take the sum of an even number of 1's, we get 0 in $\mathbb{F}_2$. Since the cube $\mathrm{SCube}(D)$ is contractible, we have a 1-cochain

FIGURE 7. Reidemeister one move. To the left: the source and the target of the map ϕ. To the right: the foam realizing this map (it is a product foam everywhere except near the crossing).

$e \in C^1_{cell}(\mathrm{SCube}; \mathbb{F}_2)$ such that $\delta(e) = c$. We have e as a sign assignment $\jmath$. Assume that we have two sign assignments $\jmath$ and $\jmath'$. We have $\delta(\jmath - \jmath') = \delta(\jmath) - \delta(\jmath') = 1 - 1 = 0$, so $\jmath - \jmath'$ is a 1-cocycle. Again, since the cube $\mathrm{SCube}(D)$ is contractible, we have t such that $\delta(t) = \jmath - \jmath'$. If we have another t' such that $\delta(t') = \jmath - \jmath'$, then we have $\delta(t - t') = 0$, which means $t - t'$ is a cellular 0-cocycle. This implies that $(t - t')(a) = (t - t')(b) = 0$ for any point a, b that belongs to any interval I. This means that for any points in the cube, $t - t'$ is equal to zero, so $t - t'$ is constant. □

Theorem 4.31. *For any diagrams D and D' of the link L, we have $[\![D]\!] \simeq [\![D']\!]$. In other words, the complexes for these two diagrams are homotopy equivalent in* $\mathrm{Kom}(\mathbb{S}\mathbf{Foam}_N)$.

Proof. The statement is well-known to the experts, with a few known proofs. To show how sign assignements work, we provide a proof in two special cases. Namely, if D' differs from D by a single Reidemeister move, and

- The case of a Reidemeister 1 move for general labelings.
- The case of a Reidemeister 2 move for diagrams with all labels equal to 1.

In this proof, the main issue will be clarifying signs. Namely, we will show how to relate sign assignments on D with sign assignments on D'. In the case of non-periodic links, we do not have a sign assignment problem, Koszul's sign rule being sufficient. We have proof of this theorem in [5, Theorem 3.5]. We will imitate [16, Section 7].

We denote the diagram obtained from D via a single Reidemeister 1 move with a positive crossing by $D\langle \circlearrowleft \rangle$. We assume that the strand at which the Reidemeister move is done is labeled by $a > 0$. We denote partial resolutions of $D\langle \circlearrowleft \rangle$ as $D\langle)\circ \rangle$, $D\langle)\supset_1 \rangle, \dots, D\langle)\supset_{a-1} \rangle$, and $D\langle)\supset \rangle$. Here, by putting i we mean we label the loop which is next to the diagram by i. We can write $[\![D]\!]$ as the following bicomplex

$$0 \to [\![D\langle)\circ \rangle]\!] \xrightarrow{d_0^+} [\![D\langle)\supset_1 \rangle]\!] \xrightarrow{d_1^+} \dots \xrightarrow{d_a^+} [\![D\langle)\supset \rangle]\!] \to 0, \tag{4.32}$$

Here d_i^+ is the identity except near the relevant crossing. The foam near the crossing is given by Figure 6. We have a chain map between $[\![D]\!]$ and $[\![D\langle \circlearrowleft \rangle]\!]$ given by

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & [\![D]\!] & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\phi} & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & [\![D\langle)\circ \rangle]\!] & \xrightarrow{d_0^+} & [\![D\langle)\supset_1 \rangle]\!] & \xrightarrow{d_1^+} & \dots & \xrightarrow{d_a^+} & [\![D\langle)\supset \rangle]\!] & \longrightarrow & 0, \end{array} \tag{4.33}$$

Here ϕ is the union of the identity foams and the cup foam. It is the identity foam away from the crossing, and the cup foam when we have a resolution for an extra crossing. In general, we can say that the map between $[\![D]\!]$ and $[\![D\langle \circlearrowleft \rangle]\!]$ is given by $(-1)^{d(I)}\phi_I$, where $d(I)$ is a choice of a sign. The main issue with choosing appropriate sign assignments is to show that the choice $d(I) \equiv 0$ is consistent. That is, for the rest of the proof, we will deal with sign assignments on $\mathrm{Cube}(D)$ and on $\mathrm{Cube}(D\langle \asymp \rangle)$ so that ϕ is the chain homotopy map. We know that $\mathrm{Cube}(D\langle \asymp \rangle) = \mathrm{Cube}(D) \times \{0, \epsilon, 2\epsilon, \dots, a\epsilon\}$. The following lemma will show us how to extend sign assignment from $\mathrm{Cube}(D)$ to $\mathrm{Cube}(D')$ where $\mathrm{Cube}(D') \cong \mathrm{Cube}(D) \times \{0, \epsilon, 2\epsilon, \dots, a\epsilon\}$ where ϵ is the sign of the new crossing.

Lemma 4.34. *Suppose we have a sign assignment $\mathscr{s}$ for the diagram D. Assume we have the diagram D' with one more crossing compared to D, so we have $\mathrm{Cube}(D') = \mathrm{Cube}(D) \times \{0, \epsilon, 2\epsilon, \dots, a\epsilon\}$ where ϵ is the sign of an additional crossing. There exists a unique sign assignment $\mathscr{s}'$ for D' satisfying the following conditions.*

- *The new sign assignment should be compatible with the old one, so to say for $I, I' \in \mathrm{Cube}(D)$ where I' is the immediate successor of I, we have*
$$\mathscr{s}'((I,0),(I',0)) = \mathscr{s}(I,I')$$
- *For $I \in \mathrm{Cube}(D)$*
$$\mathscr{s}'((I,j),(I,j+1)) = 0$$
for all j, i.e., for positive crossing $j = 0, \dots, a-1$ and for negative crossing $j = -a, \dots, -1$. Furthermore, suppose $\mathscr{s}_1, \mathscr{s}_2$ are two sign assignments on D and $\mathscr{s}_1 - \mathscr{s}_2 = \delta(t)$, denote $\mathscr{s}_1'$ and $\mathscr{s}_2'$ extensions of $\mathscr{s}_1$ and $\mathscr{s}_2$. Now define the cellular 1-cochain t' on $\mathrm{SCube}(D')$ by $t'(1,x) = t(I)$ for any $(I,x) \in \mathrm{Cube}(D)$. Then $\mathscr{s}_1' - \mathscr{s}_2' = \delta t'$.

Proof. We will prove it only for a positive added crossing; the proof for the negative crossing is similar. We need to address the case of elements of the cube with the same last coordinate. We set

$$\mathscr{s}'((I,j),(I',j)) = \mathscr{s}(I,I') + \begin{cases} 1 & j \text{ odd} \\ 0 & j \text{ even.} \end{cases} \tag{4.35}$$

To show that the choice gives actually a sign assignement, we need to check the cochain condition. We check each case separately:

- For $I', I_1', I_2', I_{12}' \in \mathrm{Cube}(D')$ where these all have 0 as their last coordinate, we have
$$\begin{aligned} &\mathscr{s}'(I', I_1') + \mathscr{s}'(I', I_2') + \mathscr{s}'(I_1', I_{12}') + \mathscr{s}'(I_2', I_{12}') \\ &= \mathscr{s}(I, I_1) + \mathscr{s}(I, I_2) + \mathscr{s}(I_1, I_{12}) + \mathscr{s}(I_2, I_{12}) = 1. \end{aligned}$$
- For $I', I_1', I_2', I_{12}' \in \mathrm{Cube}(D')$ where these all have j with the condition $j \neq 0$ as their last coordinate, either we have 1 or 0 in the definition 4.35. We have
$$\begin{aligned} &\mathscr{s}'(I', I_1') + \mathscr{s}'(I', I_2') + \mathscr{s}'(I_1', I_{12}') + \mathscr{s}'(I_2', I_{12}') \\ &= 1 + \mathscr{s}(I, I_1) + 1 + \mathscr{s}(I, I_2) + 1 + \mathscr{s}(I_1, I_{12}) + 1 + \mathscr{s}(I_2, I_{12}) = 1 \text{ in } \mathbb{F}_2. \end{aligned}$$
- For $I, I_1 \in \mathrm{Cube}(D)$ where I_1 is the immediate successor of I, we have $I' = (I,j), I_1' = (I_1, j), I_2' = (I, j+1), I_{12}' = (I_1, j+1)$. For these, we have
$$\begin{aligned} &\mathscr{s}'(I', I_1') + \mathscr{s}'(I', I_2') + \mathscr{s}'(I_1', I_{12}') + \mathscr{s}'(I_2', I_{12}') \\ &= \mathscr{s}(I, I_1) + 0/1 + 0 + 0 + \mathscr{s}(I, I_1) + 1/0 = 1 \text{ in } \mathbb{F}_2. \end{aligned}$$

Now, we prove the second part. Suppose we have sign assignments $\mathscr{s}_1$ and $\mathscr{s}_2$ for $\mathrm{Cube}(D)$. For I and I' where I' is the immediate successor of I, we have $\mathscr{s}_1(I,I') - \mathscr{s}_2(I,I') = t(I) - t(I')$. Now we consider $I_1', I_2' \in \mathrm{Cube}(D \times \{0, \epsilon, 2\epsilon, \dots, a\epsilon\})$ such that I_2' is the immediate successor of I_1'. We have three cases:

- $I_1' = (I_1, j)$ and $I_2' = (I_2, j)$. For j is even, we have
$$\mathscr{s}_1'(I_1', I_2') - \mathscr{s}_2'(I_1', I_2') = \mathscr{s}_1(I_1, I_2) - \mathscr{s}_2(I_1, I_2) = t(I_1) - t(I_2) = t'(I_1') - t'(I_2').$$
- $I_1' = (I_1, j)$ and $I_2' = (I_2, j)$. For j is odd, we have
$$\mathscr{s}_1'(I_1', I_2') - \mathscr{s}_2'(I_1', I_2') = 1 + \mathscr{s}_1(I_1, I_2) - 1 - \mathscr{s}_2(I_1, I_2) = t(I_1) - t(I_2) = t'(I_1') - t'(I_2').$$

- In this case, we have $I'_1 = (I, j)$ and $I'_2 = (I, j+1)$, then we have
$$\mathfrak{s}'_1(I'_1, I'_2) - \mathfrak{s}'_2(I'_1, I'_2) = 0 = t(I) - t(I) = t'(I'_1) - t'(I'_2).$$

□

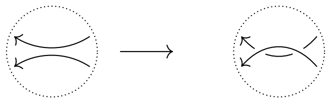
With the sign assignment from Lemma 4.34, we have the following corollary:

Corollary 4.36. *For any $I \in \mathrm{Cube}(D)$, the I-th component of the map $\phi : [\![D, \mathfrak{s}]\!] \to [\![D\langle()\circ\rangle, \mathfrak{s}']\!]$ is given by $\phi_I : D_I \to D_{(I,0)}$ without any sign correction.*

Proof. Let $I_1 \in \mathrm{Cube}(D)$ and I_2 be an immediate successor of I_1. We chose $I'_1 = (I_1, 0)$ and $I'_2 = (I_2, 0)$. The map $d' \circ \phi_{I_1}$ is the composition of the foams ϕ_{I_1} and $\delta'(I'_1, I'_2)$ with the sign $(-1)^{d(I_1)+\mathfrak{s}'(I'_1,I'_2)}$. On the other hand, we have another map which is a composition of the foams $\delta(I_1, I_2)$ and ϕ_{I_2} with the sign $(-1)^{d(I_2)+\mathfrak{s}(I_1,I_2)}$. By Lemma 4.34, $\mathfrak{s}(I_1, I_2) = \mathfrak{s}'(I'_1, I'_2)$ and we took $d(I) = 0$ for any $I \in \mathrm{Cube}(D)$, this implies ϕ is commutative with differential for any $I \in \mathrm{Cube}(D)$. □

In [16], it is proved that the map ϕ is indeed a chain homotopy equivalence. In fact, there exist explicit foams giving the inverse map. As the sign choice for D and for $D\langle()\circ\rangle$ is the same, there is no extra sign correction needed for the inverse maps either. □

We will now sketch the proof of the Reidemeister 2a move which means the move

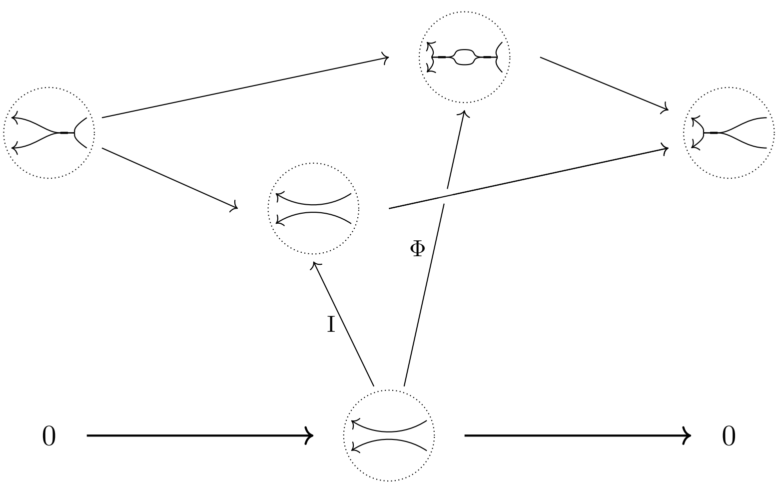


does not change homotopy type of $[\![D]\!]$. Recall that we have 1 for all labels here. Furthermore we assume that the left crossing of D' is first new crossing and right crossing is the second one. We have $\mathrm{Cube}(D') = \mathrm{Cube}(D) \times \{0,1\} \times \{-1,0\}$. We have sign assignment $\mathfrak{s}$ of D. We can extend this sign assignment on D' by firstly extending it on $\mathrm{Cube}(D) \times \{0,1\}$ by 4.34 and later on $\mathrm{Cube}(D') = \mathrm{Cube}(D) \times \{0,1\} \times \{-1,0\}$ by 4.34. We have the following observation.

Lemma 4.37. *The sign assignment $\mathfrak{s}'$ on $\mathrm{Cube}(D) \times \{1\} \times \{-1\}$ agrees with $\mathfrak{s}$.*

Proof. Let $I, I' \in \mathrm{Cube}(D)$ with I' an immediate successor of I. By (4.35), we have $\mathfrak{s}_1((I,1),(I',1)) = 1 + \mathfrak{s}(I, I')$. Again by (4.35) , we obtain $\mathfrak{s}'((I,1,-1),(I',1,-1)) = 1 + \mathfrak{s}_1((I,1),(I',1)) = \mathfrak{s}(I, I')$. □

We define the following cochain map



In the figure, we have the local cochain complex of $[\![D]\!]$ at the bottom and at the top we have the local cochain complex of $[\![D']\!]$. Here I is the identity foam with the sign $+1$ and by

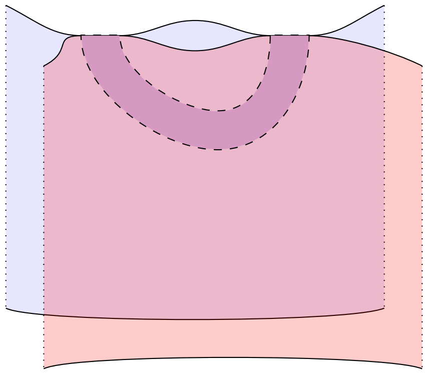


FIGURE 8. The map Φ for the Reidemeister $2a$ move in the proof of Theorem 4.31. The dashed part is the seam singularity on the foam.

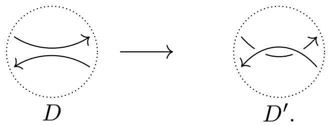


FIGURE 9. Reidemeister 2b move.

Lemma 4.37, Φ is a cochain map between the complex $[\![D]\!]$ and the subcomplex of D' obtained by a $(-1,1)$-resolution of the crossing created in the Reidemeister 2a move. In [16] it can be seen that $I \oplus \Phi$ is a chain map, the description of the inverse map and that $I \oplus \Phi \colon [\![D]\!] \to [\![D']\!]$ is a cochain homotopy equivalence.

Proof. The description of sign assignments for the Reidemeister 2b move, drawn in Figure 9, is the same. For the Reidemeister 3 move, we do not encounter problems with sign assignment, because the move preserves the crossings. The sign assignment on D induces a natural sign assignment for D'.

This implies that $[\![D]\!] \simeq [\![D']\!]$.

□

By Theorem 4.31, we know that we define the bracket independently of the diagram of a link.

Definition 4.38. (Khovanov-Rozansky homology) For any diagram D of a link L, we define the $\mathbb{S}_N$-valued Khovanov-Rozansky homology as the homology of the chain complex $\mathcal{F}([\![D]\!])$.

5. SPECIALIZATION

Recall that we have $\mathbb{S}_N$, the ring of symmetric polynomials in fixed $N \geq 0$ variables with complex coefficients. We know that $\mathbb{S}_N$ is naturally isomorphic to the ring of polynomials in N variables.

Theorem 5.1 (Quillen–Suslin, see [2, 24])**.** *Every finitely generated projective module over a polynomial ring is a free module.*

5.1. **Algebraic Specialization of Modules.** Recall that Sym_N is the category of graded projective $\mathbb{S}_N$-modules. An object of Sym_N is a direct sum of finite copies of $\mathbb{S}_N\{q^a\}$. A morphism in this category is a matrix of symmetric polynomials.

Assume we have Σ, an (unordered) N-tuple of points in $\mathbb{C}$, not necessarily distinct. We denote $P(\Sigma)$ as the evaluation of $P \in \mathbb{S}_N$ at Σ. Then, Σ specifies a left $\mathbb{S}_N$-module structure on $\mathbb{C}$, via $Pz = P(\Sigma)z$, for $P \in \mathbb{S}_N$ and $z \in \mathbb{C}$.

Definition 5.2. (Specialization functor) We have a functor

$$\mathrm{ev}^{\Sigma} : \mathrm{Sym}_N \to \mathbf{Vect}(\mathbb{C})$$

given by $\mathrm{ev}^{\Sigma}(M) \to M \otimes_{\mathbb{S}_N} \mathbb{C}$ and for a morphism

$$F : M \to N$$

$$\mathrm{ev}^{\Sigma}(F) : M \otimes_{\mathbb{S}_N} \mathbb{C} \to N \otimes_{\mathbb{S}_N} \mathbb{C}$$

$$\mathrm{ev}^{\Sigma}(F)(m \otimes c) = F(m) \otimes c$$

We call this a *specialization functor.* If Σ consists of pairwise distinct complex numbers, then the functor is called a generic specialization functor. On the other hand, if $\Sigma = (0, \dots, 0)$, then the functor is called a singular specialization functor.

5.2. **Algebraic Specialization of Cochain Complexes.** Let M be a symmetric finitely graded free $\mathbb{S}_N$ module.

$$M = \bigoplus_{j=1}^{m} \mathbb{S}_N\{q^{a_m}\}$$

where q^{a_m} denotes the grading shift with a_m as an integer. Between two modules $\mathbb{S}_N\{q^{a_m}\}$ and $\mathbb{S}_N\{q^{a_n}\}$, we have a morphism ϕ. We say that ϕ is a degree k morphism, when it is a map $\mathbb{S}_N \to \mathbb{S}_N$ with degree $k + b - a$. Therefore ϕ is a multiplication by a homogenous polynomial of degree $(k + a_n - a_m)/2$.
Note that we have degree 2 for variables $X_1, \dots, X_N$. Recall that the graded cochain complex is the complex that has differentials with degree zero. Assume we have the graded cochain complex C_* with graded, free $\mathbb{S}_N$ modules. Now we form two cochain complexes.

Definition 5.3 (Generic and Singular Specialization of Complexes)**.**

- For $\Sigma = (0, \dots, 0)$, we have the singular specialization C_*^0, which is obtained by applying ev^{Σ} to C_*.
- For Σ with the set of pairwise distinct complex numbers, we have a generic specialization C_*^{gen}, which is obtained by applying ev^{Σ} to C_*.

If we have $C_i = \bigoplus_{j=1}^{n_i} \mathbb{S}_N\{q^{a_{ij}}\}$, then $C_i^0 = C_i^{gen} = \bigoplus_{j=1}^{n_i} \mathbb{C}\{q^{a_{ij}}\}$ because $\mathbb{S}_N \otimes \mathbb{C} = \mathbb{C}$. The boundary maps d_i^0 and d_i^{gen} are equal to $\mathrm{ev}^{\Sigma}(d)$, where d is the boundary map in $C_i = \bigoplus_{j=1}^{n_i} \mathbb{S}_N\{q^{a_{ij}}\}$.

Assume we have a chain complex $C_i = \bigoplus_{j=1}^{n_i} \mathbb{S}_N\{q^{a_{ij}}\}$, the differential map $d^i : C_i \to C_{i+1}$ is the sum of the maps $d_{i,kl} : \mathbb{S}_N\{q^{a_{ik}}\} \to \mathbb{S}_N\{q^{a_{i+1,l}}\}$. The map having the degree 0 is the multiplication of a homogeneous polynomial of degree $(a_{i+1,l} - a_{i,k})/2$. The singular evaluation of any homogeneous polynomial of degree at $(a_{i+1,l} - a_{i,k})/2$ can be non-zero only if $a_{i+1,l} - a_{i,k} = 0$ when we apply ev^{Σ} for $\Sigma = (0, \dots, 0)$. We can deduce that with the ev^{Σ} functor, the differential d_0^i of the complex C_0^i keeps grading.

On the other hand, for the cochain complex $(C_{\Sigma}^i, d_{\Sigma}^i)$, the situation is different. Homogeneous polynomials can be nonzero when evaluated at Σ when $a_{i+1,l} - a_{i,k} \geq 0$. This means $(C_{\Sigma}^i, d_{\Sigma}^i)$ is filtered.

Proposition 5.4. *There exists a spectral sequence, whose first page is $H^*(C_*^0)$ and whose homology is $H^*(C_*^{gen})$.*

Proof. The differentials $d^i : C^i \to C^{i+1}$ can be decomposed as a sum $d^{i0} + d^{i1} + \dots$, where d^{is} is given by a matrix of homogeneous polynomials of degree s. After performing a generic specialization, d^{is} becomes the map d_{gen}^{is} increasing the grading by $2s$. That is, $d_{gen}^i = d_{gen}^{i0} + d_{gen}^{i1} + \dots$. The graded part of d_{gen}^i is equal to d_{gen}^{i0}.

Specialization of d^{is} with all variables zero gives the zero map, unless $s = 0$. That is, $d_0^i = d_0^{i0}$. The non-zero map d_0^{i0} is equal to d_{gen}^{i0} because a degree-zero polynomial is necessarily constant. Therefore, the graded part of d_{gen}^i is equal to the differential d_0^i.

Summarizing, (C^*_{gen}, d^i_{gen}) is a filtered cochain complex, whose graded part is d^i_0. A classical argument shows the existence of the spectral sequence. □

5.3. **Geometric Specialization.** For $\Sigma \in \mathbb{C}^n$, we have the evaluation for any foam F, denoted by $\langle F\rangle_\Sigma$, which is obtained by evaluating the polynomial $\langle F\rangle$ at Σ. For any $G \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}^*_N}(V, \varnothing)$, we have

$$\Phi_{G,\sigma} : \widetilde{\mathcal{F}}(V) \to \mathbb{C}, \Phi_{G,\sigma}(F) = \langle G \circ F\rangle_\Sigma.$$

→ ΣFoam$_N$ is used to define itself?

Based on this construction, we define the $\Sigma\mathbf{Foam}_N$ category.

Definition 5.5. For the category $\Sigma\mathbf{Foam}_N$, the objects are the same as the objects of $\mathbb{S}\mathbf{Foam}^*_N$. In other words, objects are webs with a formally assigned quantum grading. The morphisms are given by $\mathrm{Hom}_{\Sigma\mathbf{Foam}_N}(V, W) := \mathrm{Hom}_{\Sigma\mathbf{Foam}_N}(V, W)/I_\Sigma(V)$, where $I_\Sigma(V) = \cap \ker \Phi_{G,\Sigma}$.

We have a functor $\mathcal{F}_\Sigma$ from the category $\Sigma\mathbf{Foam}_N$ to the category of vector spaces.

Definition 5.6. $\mathcal{F}_\Sigma(V) = \mathrm{Hom}_{\Sigma\mathbf{Foam}_N}(\varnothing, V)$ and for a map $f : V \to W$, we have $\mathcal{F}_\Sigma(f) : \mathrm{Hom}_{\Sigma\mathbf{Foam}_N}(\varnothing, V) \to \mathrm{Hom}_{\Sigma\mathbf{Foam}_N}(\varnothing, W)$.

For a specific Σ, we have special cases. For example, for $\Sigma = (0, \dots, 0)$, we have the $0\mathbf{Foam}_N$ category. For other $\Sigma = \Sigma'$, we call it $\Sigma'\mathbf{Foam}_N$ category.

5.4. **Geometric versus Algebraic Specialization.** We know that both $\Sigma\mathbf{Foam}_N$ and $\mathbb{S}\mathbf{Foam}_N$ are quotient categories of $\mathbb{S}\mathbf{Foam}^*_N$, but the kernel is larger in $\Sigma\mathbf{Foam}_N$ compared to the kernel in $\mathbb{S}\mathbf{Foam}_N$. This is because for $\mathbb{S}\mathbf{Foam}^*_N$, in the kernel, we have foams F such that $\langle F\rangle$ is zero, but on the other hand, for $\Sigma\mathbf{Foam}_N$, we have $\langle F\rangle$ which is zero when evaluated on Σ. We have the following diagram of functors.

$$\begin{array}{ccc} \mathbb{S}\mathbf{Foam}_N & \xrightarrow{\mathcal{F}} & \mathrm{Sym}_N \\ \downarrow & & \downarrow \mathrm{ev}^\Sigma \\ \Sigma\mathbf{Foam}_N & \xrightarrow{\mathcal{F}_\Sigma} & \mathbf{Vect}_\mathbb{C} \end{array}$$

Here, $\mathbf{Vect}_\mathbb{C}$ is a category of graded vector spaces over $\mathbb{C}$.

Proposition 5.7. *The diagram above is commutative.*

Proof. This is the statement of [25, Proposition 4.1]. □

Based on these definitions, we define Khovanov-Rozansky $\mathfrak{sl}_N$-homology and Lee $\mathfrak{sl}_N$-homology.

Definition 5.8. For $\Sigma = (0, \dots, 0)$ we have a chain complex $\mathcal{F}_0[\![D]\!]$ for a link diagram D of L. We define Khovanov-Rozansky $\mathfrak{sl}_N$-homology as the cohomology space $H^k(\mathrm{ev}^\Sigma \circ \mathcal{F}([\![D]\!])) = H^k(\mathcal{F}_0[\![D]\!]) = \mathrm{KR}^{k,r}_N(L)$ of L where k is the homological grading and r is the quantum grading. Furthermore, by 5.7 we know that algebraic and geometric specialization give the same result so we can define Khovanov-Rozansky $\mathfrak{sl}_N$ homology on the cochain complex $(C^i_{(0,\dots,0)}, d^i_{(0,\dots,0)})$.

N ?

Definition 5.9. For Σ being a set of pairwise distinct complex numbers, we have a chain complex $\mathcal{F}_\Sigma[\![D]\!]$ for a link diagram D of L. We define Lee $\mathfrak{sl}_N$-homology as the cohomology space $H^k(\mathrm{ev}^\Sigma \circ \mathcal{F}([\![D]\!])) = H^k(\mathcal{F}_\Sigma[\![D]\!]) = \mathrm{Lee}^k_N(L)$ of L where k is the homological grading. Similarly, we can define Lee $\mathfrak{sl}_N$ homology on the cochain (C^i_Σ, d^i_Σ) for generic Σ.

Theorem 5.10 (Lee-Gornik spectral sequence)**.** *Let D be a link diagram. There is a spectral sequence whose first page is* $\mathrm{KR}^{k,r}_N(L)$ *abutting to* $\mathrm{Lee}^k_N(L)$.

Proof. Here take $C_* = \mathcal{F}([\![D]\!])$ over $\mathbb{S}_N$. The cochain $C^0_* = \mathcal{F}_0([\![D]\!])$ and $C^{gen}_* = \mathcal{F}_0([\![D]\!])$ are the specialization of C_*. The statement follows from 5.4. □

Now assume we have a link L and its mirror L'. For Khovanov homology we have $\mathrm{Kh}^{i,j}(L) \cong \mathrm{Kh}^{-i,-j}(L')$. For Khovanov-Rozansky homology we have a similar relation.

Proposition 5.11. *For Khovanov-Rozansky $\mathfrak{sl}_N$ homology we have an isomorphism* $\mathrm{KR}_N^{k,r}(L) \cong \mathrm{KR}_N^{-k,-r}(L')$.

Proof. Assume we have a diagram D of link L with n crossings. Enumerate $\mathrm{Cr}(D) = \{1,\dots,n\}$. For each vertex I, we associate $\mathcal{F}_0(D_I)$. Now assume we have a mirror diagram D'. For $I \in \mathrm{Cube}(D)$, $I = (i_1,\dots,i_n)$, denote by I' the dual resolution $(-i_1,\dots,-i_n) \in \mathrm{Cube}(D')$. The webs $D'_{I'}$ and D_I are isomorphic because D and D' are mirrors to each other. We have a map $i : \mathcal{F}_0([\![D]\!]) \to \mathcal{F}_0([\![D']\!])$. In other words, we have $i : \mathcal{F}_0([\![D_I]\!]) \to \mathcal{F}_0([\![D'_{I'}]\!])$.

The differentials in the mirror complex are dualized. For example, if we have a differential from $\mathcal{F}_0([\![D_{I_1}]\!])$ to $\mathcal{F}_0([\![D_{I_2}]\!])$ then for the mirror complex we have a differential from $\mathcal{F}_0([\![D'_{I'_2}]\!])$ to $\mathcal{F}_0([\![D'_{I'_1}]\!])$; and if the first differential is given by matrix A, then the second differential in the mirror complex is given by A^T. Now fix the basis of $\mathcal{F}_0([\![D]\!])$. We have just showed that $\mathcal{F}_0([\![D]\!])$ and $\mathcal{F}_0([\![D']\!])$ have the same basis. If we send the basis of $\mathcal{F}_0([\![D']\!])$ to its dual basis, that is, the basis of $\mathrm{Hom}_{\mathbb{C}}(\mathcal{F}_0([\![D]\!]),\mathbb{C})$ we get an isomorphism between $\mathrm{Hom}_{\mathbb{C}}(\mathcal{F}_0([\![D]\!]),\mathbb{C})$ and $\mathcal{F}_0([\![D]\!])$. In other words, we have

$$\mathcal{F}_0([\![D']\!]) \cong \mathrm{Hom}_{\mathbb{C}}(\mathcal{F}_0([\![D]\!]),\mathbb{C})$$

with underlying gradings reversed. By the universal coefficient theorem, since we work over the field $\mathbb{C}$, we obtain

$$H^{-k,-r}(\mathrm{Hom}_{\mathbb{C}}(\mathcal{F}_0([\![D]\!]),\mathbb{C})) = H^{k,r}(\mathcal{F}_0([\![D]\!]))$$

so we get

$$H^{k,r}([\![D]\!]) = H^{-k,-r}([\![D']\!])$$

□

5.5. **Computation of Lee-Gornik homology.** Recall that the decoration of a foam F is an assignment of a symmetric polynomial to every face of a foam F according to specific rules.

Definition 5.12 (Algebra of decorations)**.** Let F be a foam and f be its face. The algebra of decorations A_f is an algebra that is generated by all possible decorations on the face f modulo all decorations that make F a zero map in $\Sigma'\mathbf{Foam}_N$.

Theorem 5.13. *Let f be the foam facet with label a. The algebra of decorations is the direct sum of one-dimensional algebras indexed by the subsets of Σ with cardinality a. In each summand, we have a generator 1_A, which is an idempotent in A_f.*

Proof. [26, Lemma 4.2], [5, Lemma 2.28]

□

We define an algebra associated to a foam F. Assume we have a web W and a foam F from W to W. Then the algebra A_F is generated by all possible decorations on the foam F modulo the decorations that evaluate to zero under F_Σ.

Theorem 5.14. *The algebra A_F can be written as a direct sum of one-dimensional summands. The summands are in bijection with colorings of all facets by a subset of Σ as in Theorem 5.13, satisfying the admissibility condition.* Where are they defined?

Definition 5.15 (Karoubi envelope)**.** Assume we have a category C. The Karoubi envelope of C is the category obtained by formally splitting all idempotents of C. More precisely, the category $\mathbf{Kar}(C)$ has objects as pairs (O,e) where O is an object in C and $e : O \to O$ is an idempotent. A morphism between (O,e) and (O',e') is a map $f \in \mathrm{Mor}(O,O')$ such that $f \circ e = e' \circ f$.

Consider the category $\Sigma'\mathbf{Foam}_N$ and consider the identity foam F on W. As mentioned in Theorem 5.14, a decoration on F induces a coloring on W.

Definition 5.16. The category $\mathbf{Kar}^0(\Sigma'\mathbf{Foam}_N)$ is the full subcategory of the Karoubi envelope of $\Sigma'\mathbf{Foam}_N$ whose objects are webs decorated by Σ'.

meaning? Ob = (O, e)!

- What is Σ'? Is it an idempotent?
- Why passing to Kar here?

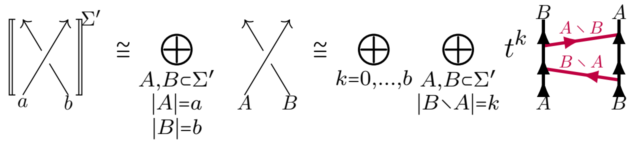
Example 5.17. We depict any web W in $\mathbf{Kar}^0(\Sigma'\mathbf{Foam}_N)$ as a direct sum of its decorations:

$$W \equiv \sum_D (W, D)$$

where D runs through all admissible decorations.

Theorem 5.18. *Let D be a diagram of a link with chain complex $[\![D]\!]_{\Sigma'}$ in $\mathbf{Kar}^0(\Sigma'\mathbf{Foam}_N)$. In the category $\mathbf{Kar}^0(\Sigma'\mathbf{Foam}_N)$, the complex is isomorphic to the complex with trivial differentials. Locally, we write:*

(5.19)



Proof. [5, Lemma 3.13], and [25, Lemma 5.9] □

As we know, when we apply $F_{\Sigma'}$ to $[\![D]\!]_{\Sigma'}$, we get the $\mathfrak{sl}_N$ Lee homology. Therefore, we can compute the Lee homology of labeled links with this formula.

Theorem 5.20. *Let L be a link with labels equal to 1. The Lee homology of L is isomorphic to $\mathbb{C}^{N^{\#L}}$. Furthermore, for each map $\Phi : \{$components of $L\} \to \{1, \dots, N\}$, we can assign a class $\ell_\phi \in Lee_N(L)$ of homological degree*

$$\deg(\ell_\phi) = \sum_{a \neq b, a, b \in \{1,\dots,n\}} \mathrm{lk}(\phi^{-1}(a), \phi^{-1}(b))$$

These classes generate $\mathrm{Lee}_N(L)$.

Proof. [8, Theorem 2] □

5.6. **$\mathfrak{sl}_N$-homology for periodic links.** In this section, we study group action on homology. For that, we take $G = \mathbb{Z}_m$. We have a group action on $\mathbb{R}^2 \times \mathbb{R}$ by rotating about the axis $(0,0) \times \mathbb{R}$. Assume L is a link in $\mathbb{R}^3$ preserved by G and disjoint from the rotation axis, and D is a periodic link diagram of L.

5.6.1. *Group actions on* $[\![D]\!]$. We want to construct the $\mathbb{Z}_m$-equivariant $\mathfrak{sl}_N$-homology of a periodic link. For this, we need to prove:

- Existence of an action on $[\![D]\!]$
- Equivariance of the evaluation functor $\mathcal{F}$, implying the existence of a $\mathbb{Z}_m$-action on $F([\![D]\!])$
- Independence of the action on the diagram.

Proposition 5.21. *Assume we have a link diagram D. We have an action of $\mathbb{Z}_m$ on $[\![D]\!]$ by rotating resolution diagrams.*

Proof. Fix a generator $g \in \mathbb{Z}_m$. For this generator g, we define $\rho_g : \mathbb{Z}_m \to \mathbb{Z}_m$ as a rotation cobordism. We have the action of $\mathbb{Z}_m$ on D, and this action induces an action on $\mathrm{Cube}(D)$ denoted $(g, I) \mapsto gI$. Furthermore, we can define an action for D_I. We have $gD_I = D_{gI}$, where g acts on D_I by rotation. Let $\rho_{g,I} : D_I \to D_{gI}$ denote the foam realizing the rotation. We need a sign assignment to construct an action on $[\![D]\!]$. The sign assignment needs to satisfy some invariance property. We define the action of g on sign assignments via $\mathfrak{s} \mapsto g\mathfrak{s}$, where $g\mathfrak{s}(gI, gI') = \mathfrak{s}(I, I')$. The sign assignment $g\mathfrak{s}$ does not need to be equal to $\mathfrak{s}$. But we must have $g\mathfrak{s} - \mathfrak{s} = \delta t$ for some 0-cochain t. Define $\mathcal{G}_{g,I} = (-1)^{t(I)}\rho_{g,I}$ for any $I \in \mathrm{Cube}(D)$. The map $\mathcal{G}$ depends on t, but by Lemma 4.30, we know that for two different t_1 and t_2, we have $t_1 = t_2 + a$ where a is constant. There are two options. Either we fix t by requiring that $t(0, \dots, 0) = 0$, or we emphasize the dependence of t by writing $\mathcal{G}_g = [\![\rho_g, t]\!]$. Unless specified explicitly otherwise, we adopt the first convention.

We need to prove that if d is a differential on $[\![D]\!]$, then $d\mathcal{G}_g = \mathcal{G}_g d$. Take resolutions $I, I' \in \mathrm{Cube}(D)$ such that I' is an immediate successor of I. Let us have ϕ as the foam which gives the component of the differential from D_I to D'_I. We have the following diagram:

$$\begin{array}{ccc} D_I & \xrightarrow{(-1)^{t(I)}\rho_{g,I}} & D_{gI} \\ {\scriptstyle (-1)^{s(I,I')}\phi}\downarrow & & \downarrow{\scriptstyle (-1)^{s(gI,gI')}g\phi} \\ D_{I'} & \xrightarrow{(-1)^{t(I')}\rho_{g,I'}} & D_{gI'}. \end{array}$$

Here the vertical maps are differentials, and the horizontal maps are given by $\mathcal{G}_{g,I}$ and $\mathcal{G}_{g,I'}$. The foams $\rho_{g,I'} \circ \phi$ and $g\phi\rho_{g,I}$ are isotopic. By the property $\delta t = gs - s$, the diagram commutes. This means $\mathcal{G}_g$ is a chain map in $\mathrm{Kom}(\mathbb{S}\mathbf{Foam}_N)$. Lastly, we need to prove that $\mathcal{G}_g$ generates an action of G. In other words, we need to show that $(\mathcal{G}_g)^m = \mathrm{Id}$. For $m = 2$, we have $(\mathcal{G}_g)^2(D_I) = (-1)^{t(I)+t(gI)}\rho_{g,gI} \circ \rho_{g,I}$. Now for general m, we have

$$\mathcal{G}_g^m(D_I) = (-1)^{t(I)+\cdots+t(g^{m-1}I)}\rho_{g,g^{m-1}I} \circ \rho_{g,gI} \circ \rho_{g,I}$$

$\rho_{g,I}$ is basically rotation so when we apply this m times we will get the identity map.

$$\rho_{g,g^{m-1}I} \circ \ldots \circ \rho_{g,gI} \circ \rho_{g,I} = \mathrm{Id}$$

Define $t'(I) = t(I) + \cdots + t(g^{m-1}I)$. We have $\delta(t') = \delta(t) + \delta(tg) + \cdots \delta(t(g^{m-1})) = gs - s + g^2 s - gs + \cdots + g^m s - g^{m-1}s = 0$ by telescope sum. Since $\delta(t') = 0$, we deduce that t' is a constant function. For $I = (0, \ldots, 0)$, we have $t'(I) = t(I) + \cdots + t(g^{m-1}I)$, but since $(0, \ldots, 0)$ is fixed in any action, we have $t'(I) = 0 + \cdots + 0 = 0$. As a result, we have $\mathcal{G}_g^m(D_I) = \mathrm{Id}$. □

Remark 5.22. The proof that $\mathcal{G}_{g,I}^m$ is the identity uses the fact that $t(0, \ldots, 0) = 0$. Another choice, if m is odd, leads to an action such that $\mathcal{G}_{g,I}^m$ is minus the identity.

Proposition 5.23. *Suppose D is a periodic diagram; then, the functor $\mathcal{F}$ extends to a $\mathbb{Z}_m$-equivariant functor with values in the category of graded $\mathbb{S}_N[\mathbb{Z}_m]$-modules that are free as $\mathbb{S}_N$ modules.*

Proof. Assume we have a web V and $g \in \mathbb{Z}_m$. We want to show that $g\mathcal{F}[\![V]\!] = \mathcal{F}[\![gV]\!]$ for that firstly we show $g\widetilde{\mathcal{F}}[\![V]\!] = \widetilde{\mathcal{F}}[\![gV]\!]$. The web gW is obtained by rotating the web V. We have

$$g\widetilde{\mathcal{F}}[\![V]\!] = g \bigoplus_{G \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, V)} \mathbb{S}_N\{d_N(G)\} = \bigoplus_{G \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, V)} \mathbb{S}_N\{d_N(gG)\}$$

For the functor $\widetilde{\mathcal{F}}$, since the degree is preserved by the group action ($d_N(F) = d_N(gF)$):

$$g\widetilde{\mathcal{F}}[\![V]\!] = \bigoplus_{G \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, V)} \mathbb{S}_N\{d_N(gG)\} = \bigoplus_{G \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, V)} \mathbb{S}_N\{d_N(G)\}$$

$$\bigoplus_{G \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, V)} \mathbb{S}_N\{d_N(G)\} = \bigoplus_{G \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, V)} \mathbb{S}_N\{d_N(g^{-1}gG)\}$$

Set $A = gG$, we have

$$\bigoplus_{G \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, V)} \mathbb{S}_N\{d_N(g^{-1}gG)\} = \bigoplus_{A \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, gV)} \mathbb{S}_N\{d_N(g^{-1}A)\}$$

$$\bigoplus_{A \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, gV)} \mathbb{S}_N\{d_N(g^{-1}A)\} = \bigoplus_{A \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, gV)} \mathbb{S}_N\{d_N(A)\} = \widetilde{\mathcal{F}}[\![gV]\!]$$

By this equation, we conclude that $\widetilde{\mathcal{F}}$ is $\mathbb{Z}_m$-equivariant. For functor $\mathcal{F}$, we need to show that $gI(V) = I(gV)$ for a web V. For F' and F, we have

$$\phi_{gF'}(gF) = \langle gF' \circ gF \rangle = \langle F' \circ F \rangle = \phi_{F'}(F)$$

This means that $g \ker \phi_{F'} = \ker \phi_{gF'}$ which implies $gI(V) = I(gV)$. As we know, $\mathcal{F}(V) = \widetilde{\mathcal{F}}(V)/I(V)$, since $\widetilde{\mathcal{F}}$ is G-equivariant and $gI(V) = I(gV)$ we deduce that $\mathcal{F}$ is G-equivariant also. ☐

Suppose D_1, D_2 are two $\mathbb{Z}_m$-equivariant diagrams of the same periodic link. By [21], they can be connected by a combination of equivariant Reidemeister moves, that is by moves that affect a $\mathbb{Z}_m$-orbit of places. For example, an equivariant Reidemeister 1 move creates or destroys an orbit of loops.

Theorem 5.24. *Suppose we have two different $\mathbb{Z}_m$-equivalent $m-$periodic link diagrams D and D' of an $m-$periodic link L, then there is a chain homotopy equivalence between $[\![D]\!]$ and $[\![D']\!]$ in the category $\mathrm{Kom}(\mathbb{S}\mathbf{Foam}_N)$ and induced quasi-isomorphism between $\mathcal{F}([\![D]\!])$ and $\mathcal{F}([\![D']\!])$ in the category of $Kom(\mathbb{S}_N)[\mathbb{Z}_m]$ modules.*

Proof. We know that D and D' are connected by equivariant Reidemeister moves. We need to use the theorem below to prove Theorem 5.24. ☐

Theorem 5.25. *Suppose we can obtain D' from D by a single equivariant Reidemeister move. Then there exists a map $\phi : [\![D]\!] \to [\![D']\!]$ and a map $\mathcal{F}(\phi) : \mathcal{F}([\![D]\!]) \to \mathcal{F}([\![D']\!])$ where ϕ is $\mathbb{S}_N[\mathbb{Z}_m]$-equivariant chain homotopy map and $\mathcal{F}(\phi)$ is a quasi-isomorphism in the category of complexes of $\mathbb{S}_N[\mathbb{Z}_m]$ modules.*

Proof. We will prove this theorem later in Section 6 ☐

5.7. **Equivariant $\mathfrak{sl}_N$-homology.** As shown in Theorem 5.24, we have the same chain complex for $\mathcal{F}([\![D]\!])$ and $\mathcal{F}([\![D']\!])$. We define the cohomology of $\mathcal{F}([\![D]\!])$ as a $\mathbb{S}_N[\mathbb{Z}_m]$-module and denote this cohomology by $\mathrm{KR}^*_{\mathbb{S}_N[\mathbb{Z}_m]}(L)$.

Proposition 5.26. *The $\mathbb{S}_N[\mathbb{Z}_m]$-module structure on $\mathrm{KR}^*_{\mathbb{S}_N[\mathbb{Z}_m]}(L)$ induces a $\mathbb{C}_N[\mathbb{Z}_m]$-module structure on $\mathrm{KR}^*_N(L)$ and $\mathrm{Lee}^*_N(L)$. The Lee-Gornik spectral sequence exists in the category of finitely generated $\mathbb{C}[\mathbb{Z}_m]$-modules.*

Proof. Suppose D and D' are m-periodic link diagrams of L. We know that D and D' are related to each other with a sequence of equivariant Reidemeister moves. Hence, we have a chain homotopy equivalence $h : [\![D]\!] \to [\![D']\!]$ in the category of complexes $\mathbb{S}_N$-modules. By Theorem 5.24, we have a quasi-isomorphism $\mathcal{F}(h)$ between $\mathcal{F}([\![D]\!])$ and $\mathcal{F}([\![D']\!])$, and this quasi-isomorphism is $\mathbb{S}_N[\mathbb{Z}_m]$-equivariant.

Now choose Σ a set of N complex numbers. We apply the ev^Σ evaluation functor. The map $h_\Sigma = \mathrm{ev}^\Sigma(h)$ is a chain homotopy equivalence between $\mathrm{ev}^\Sigma(\mathcal{F}([\![D]\!]))$ and $\mathrm{ev}^\Sigma(\mathcal{F}([\![D']\!]))$. Specifically, h_Σ induces an isomorphism between cohomology spaces of $\mathrm{ev}^\Sigma(\mathcal{F}([\![D]\!]))$ and $\mathrm{ev}^\Sigma(\mathcal{F}([\![D']\!]))$.

Since ev^Σ commutes with the $\mathbb{Z}_m$ action, h_Σ is $\mathbb{Z}_m$-equivariant. Hence $[h_\Sigma]$ is $\mathbb{Z}_m$ equivariant. We deduce that the $\mathbb{Z}_m$ equivariant isomorphism of vector spaces is an isomorphism of $\mathbb{C}[\mathbb{Z}_m]$ modules. In other words, h_Σ is a quasi-isomorphism in the category of $\mathbb{C}[\mathbb{Z}_m]$ modules.

Since we have a $\mathbb{Z}_m$ action on $\mathcal{F}[\![D]\!]$, we have $\mathbb{Z}_m$-action on $\mathrm{ev}^\Sigma(\mathcal{F}([\![D']\!]))$. For any $g \in \mathbb{Z}_m$, $gev^\Sigma(\mathcal{F}([\![D']\!] = ev^\Sigma(g\mathcal{F}([\![D']\!]))$. This shows that we have a $\mathbb{C}[\mathbb{Z}_m]$ structure on $\mathrm{KR}^*_N(L)$ and on $\mathrm{Lee}^*_N(L)$. ☐

Definition 5.27. Assume we have an m-periodic link. The equivariant Khovanov-Rozansky $\mathfrak{sl}_N$-homology $\mathrm{EKR}^{k;r}_N$ is the group which inherits its $\mathbb{C}[\mathbb{Z}_m]$ module structure from the action of $\mathbb{Z}_m$ on $\mathrm{KR}^*_N(L)$. Similarly, the equivariant Lee $\mathfrak{sl}_N$- homology ELee^k_N is the group Lee^k with its $\mathbb{C}[\mathbb{Z}_m]$ module structure which comes from $\mathbb{Z}_m$ action on $\mathrm{Lee}^*_N(L)$.

We have a mirror property in link diagrams at this equivariant homologies also.

Proposition 5.28. *Suppose L is the m-periodic link with its periodic link diagram D, and suppose L' is the mirror image of L. Then for any k, r, there is a map of $\mathbb{C}[\mathbb{Z}_m]$-modules*

$$\mathrm{EKR}_N^{k,r}(L) \cong \mathrm{EKR}_N^{-k,-r}(L')$$

Proof. We already set an isomorphism in 5.11 at the level of vector spaces over $\mathbb{C}$. Now we need to show this isomorphism is $\mathbb{Z}_m$-equivariant. We have a $\mathbb{Z}_m$ action on Cube(D) and on Cube(D'). For $g \in \mathbb{Z}_m$, we have $(gI)' = gI'$. Furthermore, taking a mirror of resolution commutes with the action. We have $gD_I = D_{gI}$ and $gD'_{I'} = D'_{gI'}$. We define $i : \mathcal{F}_0([\![D]\!]) \to \mathcal{F}_0([\![D']\!])$, $i(\mathcal{F}_0(D_I)) = \mathcal{F}_0(D'_{I'})$. Now we show that i map commutes with the group action. For $g \in \mathbb{Z}_m$

$$gi(\mathcal{F}_0(D_I)) = g\mathcal{F}_0(D'_{I'}) = \mathcal{F}_0(gD'_{I'}) = \mathcal{F}_0(D'_{(gI)'})$$

$$\mathcal{F}_0(D'_{(gI)'}) = \mathcal{F}_0(D'_{(gI)'}) = i(\mathcal{F}_0(D_{gI})) = i(\mathcal{F}_0 g(D_I)) = i(g\mathcal{F}_0(D_I))$$

We have $\mathcal{F}_0([\![D]\!])$ as $\mathbb{C}[\mathbb{Z}_m]$- module. Then obviously $\mathcal{F}_0([\![D']\!])$ has the same basis. On this basis, we can write

$$\Phi : \mathcal{F}_0([\![D']\!]) \to \mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_m]}(\mathcal{F}_0([\![D]\!]), \mathbb{C}[\mathbb{Z}_m])$$

where Φ sends the basis of $\mathcal{F}_0([\![D']\!])$ to the dual basis of $\mathcal{F}_0([\![D]\!])$. This is an isomorphism. With the choice of basis, the differential in the chain complex $\mathcal{F}_0([\![D']\!])$ is the transpose of the differential on $\mathcal{F}_0([\![D]\!])$. Actually, $\mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_m]}(\mathcal{F}_0([\![D]\!]), \mathbb{C}[\mathbb{Z}_m])$ has the same differential as $\mathcal{F}_0([\![D']\!])$ so Φ is actually an isomorphism of chain complexes. □

5.8. **Decomposition of $\mathfrak{sl}_N$-homology.** We note that $\mathrm{EKR}_N^{k,r}$ and ELee_N^k are $\mathbb{C}[\mathbb{Z}_m]$-modules, and we aim to decompose these modules.

Since the group algebra $\mathbb{C}[\mathbb{Z}_m]$ is semisimple, we have a decomposition:

$$\mathbb{C}[\mathbb{Z}_m] = \bigoplus_{i=0}^{m-1} \mathbb{C}_{\xi_m^i},$$

where $\mathbb{C}_{\xi_m^j}$ denotes the ξ_m^j-eigenspace of $\mathbb{C}[\mathbb{Z}_m]$, and $\xi_m = \exp(\frac{2\pi\sqrt{-1}}{m})$ for $j = 0, 1, \ldots, m-1$. We express this decomposition using pairwise orthogonal idempotents, denoted as $e_0, e_1, \ldots, e_{m-1}$, where $e_j e_k = \delta_{jk} e_j$. Moreover, we have $g.e_j = \xi_m^j e_j$, and

$$1 = \sum_{j=0}^{m-1} e_j.$$

Similarly, we can decompose any $\mathbb{C}[\mathbb{Z}_m]$ module M:

$$M = \bigoplus_{i=0}^{m-1} M_{\xi_m^i},$$

where $M_{\xi_m^i} := e_i M$ is the ξ_m^i-eigenspace of M for $i = 0, 1, \ldots, m-1$.

Theorem 5.29. *For any finitely generated $C[\mathbb{Z}_m]$-module M, we have*

$$\mathrm{Hom}_{C[\mathbb{Z}_m]}(M_{\xi_m^j}, M_{\xi_m^k}) = 0$$

unless $j = k$.

Proof. Assume we have a homomorphism $\Phi : M_{\xi_m^j} \to M_{\xi_m^k}$.Then, for any morphism $A : M \to M$, we have

$$\begin{array}{ccc} M_{\xi_m^j} & \xrightarrow{A} & M_{\xi_m^j} \\ \downarrow \Phi & & \downarrow \Phi \\ M_{\xi_m^k} & \xrightarrow{A} & M_{\xi_m^k}. \end{array}$$

$A\Phi = \Phi A$, $e_k A\Phi = e_k \Phi A$, $\sigma_k e_k \Phi = e_k \Phi A$, $\sigma_k \Phi = e_k \Phi A$, $\sigma_k \Phi = \Phi A$, $\sigma_k \Phi = \Phi \sigma_j$, from here we deduce $\sigma_k = \sigma_j$, which means $k = j$. We write the third equality above because we have

$Ae_k = \sigma_k e_k$ and for the fourth equation e_k behaves like Id because the projection of $M_{\xi_m^k}$ to itself is an identity. Similarly, for the fifth equation, e_k is again an identity. For the sixth equation, we know A behaves like multiplication on the eigenspace $M_{\xi_m^j}$, so we can write σ_j instead of A. □

Similarly, we can apply this decomposition to $\mathbb{S}_N[\mathbb{Z}_m]$, as $\mathbb{S}_N$ is a ring of complex polynomials. For a chosen generator g, we have:

$$\mathbb{S}_N[\mathbb{Z}_m] = \bigoplus \mathbb{S}_{N,\xi_m^i}, \ \mathbb{S}_{N,\xi_m^i} := e_j \mathbb{S}_N[\mathbb{Z}_m],$$

where $\mathbb{S}_{N,\xi_m^i}$ is the ideal consisting of ξ_m^i. Consequently, for any finitely generated $\mathbb{S}_N[\mathbb{Z}_m]$ module M, we have the decomposition:

$$M = \bigoplus_{i=0}^{m-1} M_{\xi_m^i},$$

where $M_{\xi_m^i} := e_i M$. Moreover, for any finitely generated $\mathbb{S}_N[\mathbb{Z}_m]$ module M, we have:

$$\mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_m]}(M_{\xi_m^j}, M_{\xi_m^k}) = 0$$

unless $j = k$.

Now assume we have an m-periodic link diagram D. The $\mathbb{S}_N$-equivariant Khovanov-Rozansky homology of D admits a decomposition into the eigenspaces of the action $\mathbb{Z}_m$:

$$H^{k,r}(\mathcal{F}([\![D]\!])) = \bigoplus_{i=0}^{m-1} H^{k,r}_{\xi_m^i}(\mathcal{F}([\![D]\!])).$$

In particular, we have a decomposition at the level of the cochain complex:

$$\mathcal{F}([\![D]\!]) = \bigoplus_{i=0}^{m-1} (\mathcal{F}_{\xi_m^i}([\![D]\!])).$$

We can continue the decomposition by grouping $i < m$ such that we will have another composition. Namely, for any d dividing m, we define

$$M_d = \bigoplus_{\substack{0 \le i < m \\ \gcd(i,m)=m/d}} M_{\xi_m^i} = \bigoplus_{\substack{0 \le i < m \\ \gcd(i,d)=1}} M_{\xi_d^i}.$$

According to this decomposition, we can write

$$\mathrm{EKR}_N^{k,r}(L) = \bigoplus_{d|m} \mathrm{EKR}_N^{k,r,}(L,d). \tag{5.30}$$

Here also, we have a definition for every d dividing m:

$$\mathrm{EKR}_N^{*,*}(L,d) := \mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_m]}(\mathbb{C}[\mathbb{Z}_m]_d, \mathrm{EKR}_N^{*,*}(L)) \cong H^{*,*}\left(\mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_m]}(\mathbb{C}[\mathbb{Z}_m]_d, C_0^{*,*}(L))\right).$$

This should be come earlier, it is part of 5.30 Df.

We have this isomorphism because $\mathbb{C}[\mathbb{Z}_m]$ is semisimple, and so $\mathrm{Ext}^i_{\mathbb{C}[\mathbb{Z}_m]}(M,N) = 0$ for $i > 0$ for any $\mathbb{C}[\mathbb{Z}_m]$-modules M, N. We write a similar decomposition for Lee homology. We know that $\mathrm{Lee}_N^k(L)$ depends only on the linking numbers of components of L. Since $\mathrm{ELee}_N^k(L)$ depends on the action on $\mathrm{Lee}_N^k(L)$, we need to understand the action on components of L.

Recall that $\mathrm{Lee}_N^k(L)$ was generated by classes l_ψ where $\psi : \{\text{components of L}\} \to \{1, \dots, N\}$ is any coloring. $\mathbb{Z}_m$ acts on S^3 preserving L and acts on the components of L. Specifically, there exists an action $g \in \mathbb{Z}_m$ on the set of all colorings of components of L. We denote this action $(g,\psi) \to g\psi$. We call an order of coloring the minimum number i such that $g^i\psi = \psi$ for all g. We denote the order of coloring as $\theta(\psi)$. We can see l_ψ as a vector, and we can see $\mathrm{ELee}_N^k(L, \theta(\psi))$ as an eigenspace which is generated by the coloring with the order $\theta(\psi)$. As a result, we have the decomposition:

$$\mathrm{ELee}_N^k(L) = \bigoplus_{d|m} \mathrm{ELee}_N^k(L,d).$$

Lemma 5.31. *Suppose the group $\mathbb{Z}_m$ acts trivially on the components of an unlabeled L. Then,* $\mathrm{ELee}_N^k(L,d)$ *is trivial unless $d = 1$.*

Proof. Since $d = 1$, there are no other components in the decomposition. □

6. Proof of Lemma 5.25

Proof. In this proof, we have $G = \mathbb{Z}_m$, which acts on $\mathbb{R}^2$ by rotating the angle $e^{2\pi i/m}$. Without loss of generality, we assume that D' has no fewer crossings than D. We construct ϕ as a family of foams $\phi_{I,J}$ for $(I,J) \subset \mathrm{Cube}(D) \times \mathrm{Cube}(D')$ and signs $\mathscr{d}(I,J)$ so that the component of ϕ from ϕ_I to ϕ_J is $(-1)^{\mathscr{d}(I,J)}\phi_{I,J}$.

We need to deal with two problems: a geometric one and an algebraic one.

- Geometrical problem: The group G acts on $\mathrm{Cube}(D)$ and on $\mathrm{Cube}(D')$. The action is the permutation of crossings. We need to form foams $\phi_{I,J}$ such that $\phi_{gI,gJ}$ is isotopic to foam $g\phi_{I,J}$ between gD_I and gD_J.
- Algebraical problem: We need to show that the sign assignment on D reduces to a good sign assignment on D'. Specifically, we need to show that the following diagram commutes.

$$\begin{array}{ccc} D_I & \xrightarrow{(-1)^{\mathscr{t}(I)}\rho_{g,I}} & D_{gI} \\ {\scriptstyle (-1)^{\mathscr{d}(I,J)}\Phi_{I,J}}\big\downarrow & & \big\downarrow{\scriptstyle (-1)^{\mathscr{d}(gI,gJ)}\Phi_{gI,gJ}} \\ D'_J & \xrightarrow{(-1)^{\mathscr{t}'(J)}\rho'_{g,J}} & D'_{gJ}, \end{array} \tag{6.1}$$

where t is the cochain on $\mathrm{SCube}(D)$ defined by the property $\delta t = \mathscr{s} - g\mathscr{s}$. We give a proof in three steps:

- We prove all details for a positive Reidemeister 1 move for $G = \mathbb{Z}_2$;
- We prove the algebra part of the Reidemeister 1 move for $G = \mathbb{Z}_m$;
- We discuss the algebra part of a Reidemeister 2a move and $G = \mathbb{Z}_2$;

The cases Reidemeister 2b and negative Reidemeister 1 move are direct consequences of what we prove. Regarding the Reidemeister 3 move, it is the easiest one because the number of crossings does not change, and the sign assignment on D induces the same assignment on D'.

6.1. Positive Reidemeister move, $\mathbb{Z}_m$ action for general m. We have a diagram D', which is the diagram obtained by applying two times Reidemeister 1 moves to diagram D, denoted by $D' = D\langle \rho\rho \rangle$. Furthermore, we have $[\![D]\!] = D\langle \uparrow\uparrow \rangle$ and the diagram D with one Reidemeister move applied to one crossing is denoted by $D\langle \uparrow\rho \rangle$ and for the other crossing, it is denoted by $D\langle \rho\uparrow \rangle$. We want to prove that $[\![D']\!] \simeq [\![D]\!]$.

For the cochain complex $[\![D\langle \rho\uparrow \rangle]\!]$ we have

$$[\![D\langle \rho\uparrow \rangle]\!] = \left\{0 \to [\![D\langle \uparrow\circ\uparrow \rangle]\!] \xrightarrow{d} [\![D\langle \tilde\rho\uparrow \rangle]\!] \to 0\right\},$$

and for $[\![D\langle \uparrow\rho \rangle]\!]$ we have

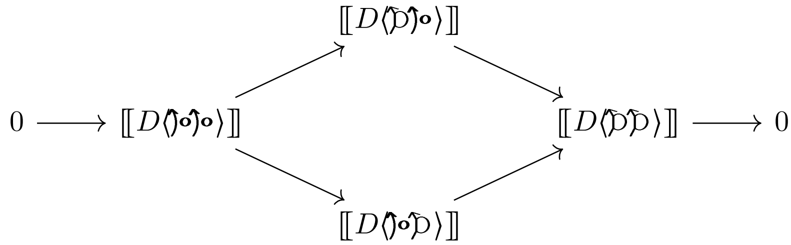
$$[\![D\langle \uparrow\rho \rangle]\!] = \left\{0 \to [\![D\langle \uparrow\uparrow\circ \rangle]\!] \xrightarrow{d} [\![D\langle \uparrow\tilde\rho \rangle]\!] \to 0\right\},$$

In terms of Cube notation, we have relations

$$\mathrm{Cube}(D\langle \rho\rho \rangle) \cong \mathrm{Cube}(D\langle \uparrow\uparrow \rangle) \times \{0,1\}^2 \tag{6.2}$$

$$\mathrm{Cube}(D\langle\cdots\rangle) = \bigcup_{x=0,1} \mathrm{Cube}(D\langle\cdots\rangle) \times \{(0,x)\}$$
$$\mathrm{Cube}(D\langle\cdots\rangle) = \bigcup_{x=0,1} \mathrm{Cube}(D\langle\cdots\rangle) \times \{(x,0)\}.$$

Here on the right side of the equation, we label extra crossing points by $\{0,1\}$ and $\{0,x\},\{x,0\}$. We split the cochain complex $[\![D\langle\cdots\rangle]\!]$. Namely, we have

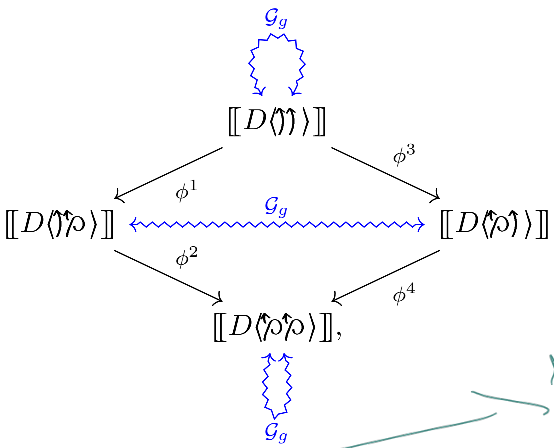


We have the following maps corresponding to the non-equivariant Reidemeister move:

$$\phi^1 \colon [\![D\langle\cdots\rangle]\!] \to [\![D\langle\cdots\rangle]\!] \qquad \phi^2 \colon [\![D\langle\cdots\rangle]\!] \to [\![D\langle\cdots\rangle]\!]$$
$$\phi^3 \colon [\![D\langle\cdots\rangle]\!] \to [\![D\langle\cdots\rangle]\!] \qquad \phi^4 \colon [\![D\langle\cdots\rangle]\!] \to [\![D\langle\cdots\rangle]\!]$$

With these maps, we have the following diagram in $\mathrm{Kom}(\mathbb{S}\mathbf{Foam}_N)$

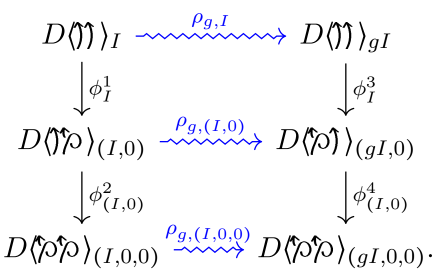
(6.3)



Here the blue arrows mean g action on the cochain complex. For example, the blue arrow in the middle means a 180-degree rotation of diagram $D\langle\cdots\rangle$. In order to understand this diagram better, we specify to a single resolution I of D.

Lemma 6.4. *For any $I \in \mathrm{Cube}(D)$, the diagram below is commutative in $\mathbb{S}\mathbf{Foam}_N$.*

(6.5)



In this diagram, the map $\rho_{g,I}$ is the foam from the diagram with resolution in I to the diagram with resolution in gI. We have a similar definition for $\rho_{g,(I,0)}$ and $\rho_{g,(I,0,0)}$.

No! Look at the orientations!

Proof. For the above square, we have $\rho_{g,(I,0)} \circ \phi^1_I$. This foam arises from $\rho_{g,(I,0)}$ when we rotate the upper side of it. On the other hand, for $\phi^3_I \circ \rho_{g,I}$ we have the foam where firstly we rotate the lower part of ϕ^1_I and apply ϕ^3_I. These foams are isotopic rel boundary, so they are equivalent in the $\mathbb{S}\mathbf{Foam}_N$ category. The second square is similar, which means that the big square is commutative. This proves the geometric part of our proof. □

For the algebraic proof, we fix the sign assignment s on D and take $g \in \mathbb{Z}_2$ as a generator from $\mathbb{Z}_2$. By Lemma 4.30, we have $gs = s + \delta t$ where t is the 0-cochain on $\mathrm{SCube}(D)$ with the property $t(0,\dots,0) = 0$. We can extend s to the diagram $D\langle \wp\wp \rangle$ in two ways. The first one is extending s on the diagram $D\langle)\wp \rangle$ and then to $D\langle \wp\wp \rangle$. The second one is extending s on the diagram $D\langle \wp) \rangle$ and then to $D\langle \wp\wp \rangle$. We have a relation between these sign assignments. We have s on D and s_1 on $D\langle \wp) \rangle$. Write $s_2 = gs_1$, s_2 is the sign assignment on $D\langle)\wp \rangle$. We extend s_1 on $D\langle \wp\wp \rangle$ by Lemma 4.34 we denote the new sign assignment s_3. Similarly, we can extend s_2 on $D\langle \wp\wp \rangle$, and denote this new sign assignment s_4. We have two sign assignments on $D\langle \wp\wp \rangle$. For these two assignments, there exists t' such that we have $s_3(I, I') - s_4(I, I') = t'(I) - t'(I')$ for any $I, I' \in \mathrm{Cube}(D\langle \wp\wp \rangle)$ where I' is an immediate successor of I.

Lemma 6.6. *(a) We have $gs_3 = s_4$.*

(b) If t is a 0-cochain on $\mathrm{SCube}(D)$ such that $s - gs = \partial t$, then the 0-cochain t' on $\mathrm{SCube}(D\langle \wp\wp \rangle)$ defined by

$$t'((I,x,y)) = xy + t(I) \in \mathbb{F}_2,$$

satisfies $s_3 - s_4 = \partial t'$.

Proof. Let $I'_1, I'_2 \in \mathrm{Cube}(D\langle \wp\wp \rangle)$, where I'_2 is an immediate successor of I'_1. The action of g switches the last two crossings. We write $I'_k = (I_k, x_k, y_k)$ for $k = 1, 2$ with $I_k \in \mathrm{Cube}(D)$, $x_k, y_k \in \{0,1\}$.

For the action g, we have $gI'_k = (gI_k, y_k, x_k)$. By Lemma 4.34, we extend s_3 and have

$$s_3((I_1,x_1,y_1),(I_2,x_2,y_2)) = \begin{cases} 0 & \text{if } y_2 = y_1 + 1 \\ s_1((I_1,x_1),(I_2,x_2)) + y_1 & \text{if } y_1 = y_2. \end{cases}$$

Similarly,

$$s_4((I_1,x_1,y_1),(I_2,x_2,y_2)) = \begin{cases} 0 & \text{if } x_2 = x_1 + 1 \\ s_2((I_1,y_1),(I_2,y_2)) + x_1 & \text{if } x_1 = x_2. \end{cases}$$

More precisely,

$$s_3((I_1,x_1,y_1),(I_2,x_2,y_2)) = \begin{cases} 0 & y_2 = y_1 + 1 \\ y_1 & y_1 = y_2 \text{ and } x_2 = x_1 + 1 \\ s(I_1,I_2) + x_1 + y_1 & y_1 = y_2 \text{ and } x_1 = x_2. \end{cases} \tag{6.7}$$

and

$$s_4((I_1,x_1,y_1),(I_2,x_2,y_2)) = \begin{cases} 0 & x_2 = x_1 + 1 \\ x_1 & x_1 = x_2 \text{ and } y_2 = y_1 + 1 \\ gs(I_1,I_2) + x_1 + y_1 & y_1 = y_2 \text{ and } x_1 = x_2. \end{cases} \tag{6.8}$$

By equations 6.7 and 6.8, we have $gs_3 = s_4$. □

For the second part of the proof, we observe

$$s_3((I_1,x_1,y_1),(I_2,x_2,y_2)) - s_4((I_1,x_1,y_1),(I_2,x_2,y_2)) = \begin{cases} x_1 & x_1 = x_2 \text{ and } y_2 = y_1 + 1 \\ y_1 & y_1 = y_2 \text{ and } x_2 = x_1 + 1 \\ s(I_1,I_2) - gs(I_1,I_2) & x_1 = x_2,\ y_1 = y_2. \end{cases} \tag{6.9}$$

In this equation, we think of possible variation. Namely, we cannot have the case $x_2 = x_1 + 1$ and $y_2 = y_1 + 1$. In addition to this equation with the definition $\mathscr{t}'(I,x,y) = \mathscr{t}(I) + xy$, we have:

$$\delta \mathscr{t}' = \mathscr{t}'(I_1, x_1, y_1) - \mathscr{t}'(I_2, x_2, y_2) = x_1y_1 + x_2y_2 + \mathscr{t}(I_1) - \mathscr{t}(I_2). \tag{6.10}$$

We want to show $\mathscr{s}_3 - \mathscr{s}_4 = x_1y_1 + x_2y_2 + \mathscr{t}(I_1) - \mathscr{t}(I_2)$. We have two cases

- First case $I_1 = I_2$: Note that I_2' is an immediate successor of I_1'. Since we have $I_1' \neq I_2'$, we cannot have $x_1 = x_2$ and $y_1 = y_2$ so we can have $(x_1,y_1) = (0,0), (x_2,y_2) = (0,1)$, $(x_1,y_1) = (0,0), (x_2,y_2) = (1,0)$, $(x_1,y_1) = (0,1), (x_2,y_2) = (1,1)$ or $(x_1,y_1) = (1,0), (x_2,y_2) = (1,1)$. For the cases where we have $(x_2,y_2) = (1,1)$, $\mathscr{s}_3 - \mathscr{s}_4 = 1$ and $x_1y_1 + x_2y_2 + \mathscr{t}(I_1) - \mathscr{t}(I_2) = 1$. For the cases where we have $(x_1,y_1) = (0,0)$, $\mathscr{s}_3 - \mathscr{s}_4 = 0$ and $x_1y_1 + x_2y_2 + \mathscr{t}(I_1) - \mathscr{t}(I_2) = 0$.
- $I_1 \neq I_2$: Note that we study in $\mathbb{Z}_2$. In this case, we have $x_1 = x_2$ and $y_1 = y_2$ because I_2' is an immediate successor of I_1'. We have $\mathscr{s}_3 - \mathscr{s}_4 = \mathscr{s} - g\mathscr{s} = \delta t = t(I_1) - t(I_2) = x_1y_1 + x_2y_2 + \mathscr{t}(I_1) - \mathscr{t}(I_2)$.

Continuing the proof of the algebraic part, we claim that the following diagram is commutative in $\mathrm{Kom}(\mathbb{S}\mathbf{Foam}_N)$.

$$\begin{array}{ccc}
[\![D\langle\upuparrows\rangle, \mathscr{s}]\!] & \xrightarrow{[\![\rho_g, \mathscr{t}]\!]} & [\![D\langle\upuparrows\rangle, \mathscr{s}]\!] \\
\downarrow \phi^1 & & \downarrow \phi^3 \\
[\![D\langle\uparrow\rho\rangle, \mathscr{s}_1]\!] & & [\![D\langle\rho\uparrow\rangle, \mathscr{s}_2]\!] \\
\downarrow \phi^2 & & \downarrow \phi^4 \\
[\![D\langle\rho\rho\rangle, \mathscr{s}_3]\!] & \xrightarrow{[\![\rho_g, \mathscr{t}']\!]} & [\![D\langle\rho\rho\rangle, \mathscr{s}_4]\!].
\end{array} \tag{6.11}$$

Note that $\rho_g, \mathscr{t}$ are the same as defined in Lemma 6.4. For any $I \in \mathrm{Cr}(D)$, we show in Lemma 6.4 that the diagram is commutative in $\mathbb{S}\mathbf{Foam}$. We can generalize it in $\mathrm{Kom}(\mathbb{S}\mathbf{Foam})$ without a sign. We just need to show the sign that makes no problem for commutativity. By the definition of $G_{\rho,I}$, the sign we get from D_I starting with $\phi^2 \circ \phi^1$ and then through $(\rho_g, \mathscr{t}')$ is $(-1)^{\mathscr{t}'((I,0,0))}$. Similarly, when we start with $[\![\rho_g, \mathscr{t}]\!]$ and then by $\phi^4 \circ \phi^3$ gives the sign of $(-1)^{\mathscr{t}(I)}$. By the definition of $\mathscr{t}'$, $(-1)^{\mathscr{t}'((I,0,0))} = (-1)^{\mathscr{t}(I)}$. □

Lemma 6.12. *The compositions $\phi^4 \circ \phi^3$ and $\phi^2 \circ \phi^1$ are equal as maps in* $\mathrm{Kom}(\mathbb{S}\mathbf{Foam}_N)$.

Proof. For any $I \in \mathrm{Cr}(D)$, the map $\phi_I^4 \circ \phi_I^3$ is given by the foams that start with a Reidemeister move for the first crossing and then for the second crossing, i.e., $\upuparrows \to \rho\uparrow \to \rho\rho$. Similarly, the other foam $\phi_I^2 \circ \phi_I^1$ is given by the foams that start with a Reidemeister move for the second crossing and then for the first crossing, i.e., $\upuparrows \to \uparrow\rho \to \rho\rho$. All the foams $\phi_I^1, \ldots, \phi_I^4$ are product foams of the identity except for the relevant crossings. □

Denote ϕ as the composition $\phi_I^4 \circ \phi_I^3 = \phi_I^2 \circ \phi_I^1$. It is induced by a composition of individual, non-equivariant Reidemeister moves. Specifically, ϕ is a (nonequivariant) chain homotopy equivalence. The horizontal maps in 6.11 are group actions on $[\![D]\!]$ and $[\![D\langle\rho\rho\rangle]\!]$. The commutativity of 6.11 implies that ϕ commutes with the group action. This proves the first part of 5.25 for the specific case of Reidemeister move 1 and $\mathbb{Z}_2$.

For the proof of the second part, we apply the evaluation functor $\mathcal{F}$ from the category $\mathrm{Kom}(\mathbb{S}\mathbf{Foam}_N)$ to the category $\mathrm{Kom}(\mathrm{Sym}_N)$. The map $\mathcal{F}(\phi) : \mathcal{F}([\![D]\!]) \to \mathcal{F}([\![D']\!])$ is a chain homotopy equivalence. More specifically, it is a quasi-isomorphism in $\mathrm{Kom}(\mathrm{Sym}_N)$. By Proposition 5.23, $\mathbb{Z}_m$ acts on $\mathcal{F}([\![D]\!])$ and on $\mathcal{F}([\![D']\!])$. By 6.11 and 6.12, $\mathcal{F}(\phi)$ commutes with the $\mathbb{Z}_m$ action. A $\mathbb{Z}_m$-equivariant quasi-isomorphism is a quasi-isomorphism in $\mathrm{Kom}(\mathrm{Sym}_N[\mathbb{Z}_m])$.

Remark 6.13. If ϕ is a chain homotopy equivalence, then $\phi - \mathrm{Id} = dh + hd$ for some map h. However, even if ϕ is equivariant, we cannot claim that h is equivariant. We only prove that ...

Text missing?

6.2. **Positive Reidemeister one move, $\mathbb{Z}_m$ action for general m.** This step is similar to the previous one. Let D be a periodic link diagram, and D' be the link diagram obtained by applying the Reidemesiter one move. We again identify $\text{Cube}(D') \equiv \text{Cube}(D) \times \{0,1\}^m$. For any $I \in \text{Cr}(D)$ and a generator $g \in Z_m$ with $x_1, x_2, \dots, x_m \in \{0,1\}$, we have

$$g(I, x_1, x_2, \dots, x_m) = (gI, x_2, x_3, \dots, x_m, x_1)$$

We define two maps ϕ_I^A and ϕ_I^B, $\phi_I^A = \phi_I^m \circ \phi_I^{m-1} \circ \dots \circ \phi_I^1$ where ϕ_I^i is the foam that realizes the i-th Reidemeister move as in Figure 7. For any $I \in \text{Cr}(D)$, we have the following diagram.

$$\begin{array}{ccc} D\langle \circlearrowleft \dots \circlearrowright \rangle_I & \xrightarrow{\rho_{g,I}} & D\langle \circlearrowleft \dots \circlearrowright \rangle_{gI} \\ \downarrow{\scriptstyle \phi_I^A} & & \downarrow{\scriptstyle \phi_I^B} \\ D\langle \circlearrowleft\!\!\rho \dots \circlearrowright\!\!\rho \rangle_{(I,0,\dots,0)} & \xrightarrow{\rho_{g,(I,0,\dots,0)}} & D\langle \circlearrowleft\!\!\rho \dots \circlearrowright\!\!\rho \rangle_{(gI,0,\dots,0)}, \end{array} \tag{6.14}$$

This diagram is a generalization of the diagram (6.5). We prove the geometric part of this step as in the proof of step 1. We omit details. We pass to the algebraic part directly. Take $\mathscr{s}$ sign assignment for the diagram D, and let $\mathscr{t}$ be such that $\mathscr{s} - g\mathscr{s} = \delta\mathscr{t}$. We get the sign assignment $\mathscr{s}'$ on $\text{Cube}(D')$ by Lemma 4.34.

Lemma 6.15. *Assume $I_1', I_2' \in \text{Cube}(D')$ and I_2' is an immediate successor of I_1'. Write $I_1' = (I_1, x_1, \dots, x_m)$, $I_1' = (I_1, y_1, \dots, y_m)$ where $I_1, I_2 \in \text{Cube}(D)$. If $x_k \neq y_k$ for some k, then*

$$\mathscr{s}'(I_1', I_2') = x_{k+1} + \dots + x_m.$$

If $x_k = y_k$ for all k, then

$$\mathscr{s}'(I_1', I_2') = x_1 + \dots + x_m + \mathscr{s}(I_1, I_2).$$

Proof. Define the sign assignment $\mathscr{s}_l'$ on diagram D_l', which is the sign assignment obtained after the first l Reidemeister moves. Assume $x_k \neq y_k$ for some k. By 4.34, we have:

$$\mathscr{s}_k'((I_1, x_1, \dots, x_k), (I_2, y_1, \dots, y_k))) = 0.$$

We continue to apply inductively for $j = k+1, \dots, m$, and by either 4.34 (if $x_j = 0$) or (4.35) (if $x_j = 1$), we get

$$\mathscr{s}_k'((I_1, x_1, \dots, x_k), (I_2, y_1, \dots, y_k))) = x_{k+1} + \dots + x_j.$$

This leads to the result $\mathscr{s}'(I_1', I_2') = x_{k+1} + \dots + x_m$. Similarly, if we take $x_1 = y_1, \dots, x_m = y_m$, then we can again apply induction. If $x_i = y_i = 0$ by 4.34, we have the result; if $x_i = y_i = 1$, then by (4.35), we have the result. □

We have a generalization of Lemma 6.6.

Lemma 6.16. *Assume $\mathscr{s} - g\mathscr{s} = \delta\mathscr{t}$. Define the 0-cochain on $\mathbb{S}\,\text{Cube}(D')$ defined by $\mathscr{t}'(I, x_1, \dots, x_m) = \mathscr{t}(I) + x_1(x_2 + \dots + x_m)$. Then for any $I_1', I_2' \in \text{Cube}(D')$ where I_2' is an immediate successor of I_1', we have*

$$\mathscr{s}'(I_1', I_2') - \mathscr{s}'(gI_1', gI_2') = \mathscr{t}'(I_1') - \mathscr{t}(I_2'). \tag{6.17}$$

Proof. We have two cases:

- Suppose $I_1 = I_2$ and $x_i = y_i$ except for k, $x_k = 0$, and $y_k = 1$. By Lemma 6.15, we have

  $$\mathscr{s}'(I_1', I_2') = x_{k+1} + \dots + x_m.$$

  In addition to that, we have

  $$\mathscr{s}'(gI_1', gI_2') = x_{k+1} + \dots + x_m.$$

  Thus,

  $$\mathscr{s}'(I_1', I_2') - \mathscr{s}'(gI_1', gI_2') = \begin{cases} x_1 & k > 1 \\ x_2 + \dots + x_m & k = 1. \end{cases}$$

On the other hand, for $k > 1$, we have

$$\mathfrak{t}'(I_1') - \mathfrak{t}'(I_2') = \mathfrak{t}(I_1') + x_1(x_2 + \cdots + x_m) - (\mathfrak{t}(I_2') + x_1(x_2 + \cdots + x_m + 1)) = x_1.$$

For $k = 1$, we have

$$\mathfrak{t}'(I_1') - \mathfrak{t}'(I_2') = \mathfrak{t}(I_1') + x_1(x_2 + \cdots + x_m) - (\mathfrak{t}(I_2') + (x_1 + 1)(x_2 + \cdots + x_m)) = x_2 + \cdots + x_m.$$

- Suppose $I_1 \neq I_2$, then $x_k = y_k$ for all k. Thus, we have

$$\mathfrak{s}'(I_1', I_2') - \mathfrak{s}'(gI_1', gI_2') = \mathfrak{s}(I_1, I_2) - \mathfrak{s}(gI_1, gI_2).$$

$$\mathfrak{s}(I_1, I_2) - \mathfrak{s}(gI_1, gI_2) = \mathfrak{t}(I_1) - \mathfrak{t}(I_2)$$

$$\mathfrak{t}'(I_1') - x_1(x_2 + \cdots + x_m) - (\mathfrak{t}'(I_2') - x_1(x_2 + \cdots + x_m)) = \mathfrak{t}'(I_1') - \mathfrak{t}'(I_2').$$

The remaining part of the step is similar to part $m = 2$. In short, we repeat the proof of Lemma 6.12 to show that ϕ_I^A and ϕ_I^B induce the same map

$$\Phi\colon \llbracket D\langle \upuparrows, \dots, \upuparrows \rangle \rrbracket \xrightarrow{\simeq} \llbracket D\langle \looparrowleft, \dots, \looparrowleft \rangle \rrbracket.$$

The corresponding diagram of 6.11 is

$$\begin{array}{ccc}
\llbracket D\langle \upuparrows, \dots, \upuparrows \rangle, \mathfrak{s} \rrbracket & \xrightarrow{\llbracket \rho_g, t \rrbracket} & \llbracket D\langle \upuparrows, \dots, \upuparrows \rangle, \mathfrak{s} \rrbracket \\
\downarrow \Phi & & \downarrow \Phi \\
\llbracket D\langle \looparrowleft, \dots, \looparrowleft, \mathfrak{s}' \rangle \rrbracket & \xrightarrow{\llbracket \rho_g, t' \rrbracket} & \llbracket D\langle \looparrowleft, \dots, \looparrowleft \rangle, g\mathfrak{s}' \rrbracket.
\end{array}$$

The same argument as in the previous step implies that this diagram is commutative. Specifically, $\mathcal{F}(\phi)$ induces a $\mathbb{Z}_m$-equivariant chain homotopy equivalence, which means $\mathcal{F}(\phi)$ is a quasi-isomorphism in the category $\mathrm{Kom}(\mathrm{Sym}_N[\mathbb{Z}_m])$. □

6.3. **Step 3: Reidemeister 2a move, $\mathbb{Z}_2$ action.** Let D be a periodic link diagram, and D' be the link diagram obtained by applying the equivariant Reidemesiter 2a move. We have the identification

$$\mathrm{Cube}(D') \cong \mathrm{Cube}(D) \times \{0,1\} \times \{-1,0\} \times \{0,1\} \times \{-1,0\}.$$

For $I \in \mathrm{Cr}(D)$, denote

$$I_1' = (I,0,0,0,0), \quad I_2' = (I,1,-1,0,0), \quad I_3' = (I,0,0,1,-1), \quad I_4' = (I,1,-1,1,-1).$$

There are four different foams for I_1', I_2', I_3', I_4'. These foams are part of the I-th component of the map $\phi\colon \llbracket D \rrbracket \to \llbracket D' \rrbracket$, where we define $\phi_I := \phi_{I,1} + \phi_{I,2} + \phi_{I,3} + \phi_{I,4}$.

These four foams are as follows:

- $\phi_{I,1}$ is the identity foam;
- $\phi_{I,2}$ is the foam from Figure 8 at the first place where the Reidemeister move is applied, followed by the identity foam;
- $\phi_{I,3}$ is the identity foam followed by the foam from Figure 8 for the second Reidemeister move;
- $\phi_{I,4}$ is the foam from Figure 8 for the first Reidemeister 2a move, followed by the foam from Figure 8 for the second move.

We have $g \in \mathbb{Z}_2$, where the action involves switching pairs of points. For example, $gI_2 = I_3$ because g sends $(1,-1)$ to $(0,0)$ and $(0,0)$ to $(1,-1)$. Therefore, we have

$$g\phi_{I,1} = \phi_{gI,1}, \;\; g\phi_{I,2} = \phi_{gI,3}, \;\; g\phi_{I,3} = \phi_{gI,2}, \;\; g\phi_{I,4} = \phi_{gI,4}. \tag{6.18}$$

This implies $g\phi_I = \phi_{gI}$. Thus, g commutes with $\Phi\colon \llbracket D \rrbracket \to \llbracket D' \rrbracket$ up to sign. This proves the geometric part of step 3.

Let $\mathfrak{s}$ be a sign assignment on diagram D. We extend $\mathfrak{s}$ to a sign assignment $\mathfrak{s}'$ on D' by adding crossings and applying Lemma 4.34. We add x_1, then x_2 and x_3, x_4. The analogy of Lemma 6.6 is as follows:

Lemma 6.19. *Assume $\mathfrak{s} - g\mathfrak{s} = \partial \mathfrak{t}$. Define the 0-cochain on* $\mathrm{SCube}(D')$ *as* $\mathfrak{t}'(I, x_1, \dots, x_4) = \mathfrak{t}(I) + (x_1 + x_2)(x_3 + x_4)$. *Then,* $\mathfrak{s}' - g\mathfrak{s}' = \partial \mathfrak{t}'$.

Proof. Take $I'_1, I'_2 \in \mathrm{Cube}(D')$ such that I'_2 is an immediate successor of I'_1. Write $I'_s = (I_s, x_{1s}, x_{2s}, x_{3s}, x_{4s})$. By Lemma 6.15, we have

$$\mathfrak{s}'(I'_1, I'_2) = \begin{cases} x_{j+1,1} + \dots + x_{41} & x_{j1} \neq x_{j2} \\ \mathfrak{s}(I_1, I_2) & x_{j1} = x_{j2} \text{ for all } j. \end{cases}$$

We know that if $I' = (I, x_1, \dots, x_4)) \in \mathrm{Cube}(D')$, then $gI' = (gI, x_3, x_4, x_1, x_2)$. Thus, we have

$$\mathfrak{s}'(I'_1, I'_2) - \mathfrak{s}'(gI'_1, gI'_2) = \begin{cases} x_{31} + x_{41} & x_{11} \neq x_{12} \text{ or } x_{21} \neq x_{22} \\ x_{11} + x_{21} & x_{31} \neq x_{32} \text{ or } x_{41} \neq x_{42} \\ \mathfrak{t}(I_1) - \mathfrak{t}(I_2) & x_{j1} = x_{j2} \text{ for all } j. \end{cases}$$

The proof is the same as in Lemma 6.16. In order to finish the proof of 5.25 at step 3, consider the diagram:

$$\begin{array}{ccc} [\![D, \mathfrak{s}]\!] & \xrightarrow{[\![\rho_g, \mathfrak{t}]\!]} & [\![D, \mathfrak{s}]\!] \\ \downarrow \Phi & & \downarrow \Phi \\ [\![D, \mathfrak{s}']\!] & \xrightarrow{[\![\rho_g, \mathfrak{t}']\!]} & [\![D, g\mathfrak{s}']\!]. \end{array}$$

We have already showed that the diagram above is commutative up to sign, and now by Lemma 6.19, we conclude that this diagram is commutative. This shows that Φ is $\mathbb{Z}_m$-equivariant. By Theorem 4.31, we know that Φ is a chain homotopy equivalence. Similarly to Steps 1 and 2, we conclude that $\mathcal{F}(\Phi)$ is a quasi-isomorphism in the $\mathrm{Kom}(\mathrm{Sym}_N[\mathbb{Z}_m])$ category. □

The proofs of the Reidemeister move 2a for any m and the Reidemeister move 2b are analogous, so we do not provide them again. For the case of the Reidemeister 3 move, we have a natural bijection between crossings of D and D', so we do not need to extend our sign assignment. Only a geometric part is needed, but it is similar to Step 1; we omit the details.

7. The skein spectral sequence

7.1. **Review of the Ind and Res Functors.** We review the Ind and Res functors before constructing the spectral sequence. For a finite group G, we denote BG as the category with a single object $*$ and $\mathrm{Hom}_{BG}(*, *) = G$. If $\mathcal{B}$ is an additive category, we denote by $\mathcal{B}[G] = \mathrm{Fun}(BG, \mathcal{B})$ the category of G-objects in $\mathcal{B}$. For a subgroup H of G, we have a canonical inclusion of categories $BH \subset BG$, leading to the restriction functor $\mathrm{Res}^G_H : \mathcal{B}[G] \to \mathcal{B}[H]$. We also have the functor $\mathrm{Ind}^G_H : \mathcal{B}[H] \to \mathcal{B}[G]$, the biadjoint functor of Res^G_H. For $C \in B[H]$ and $D \in B[G]$, we have

$$\mathrm{Hom}_{\mathcal{B}[G]}(\mathrm{Ind}^G_H(C), D) \cong \mathrm{Hom}_{\mathcal{B}[H]}(C, \mathrm{Res}^G_H(D)), \qquad (7.1)$$
$$\mathrm{Hom}_{\mathcal{B}[G]}(C, \mathrm{Ind}^G_H(D)) \cong \mathrm{Hom}_{\mathcal{B}[H]}(\mathrm{Res}^G_H(C), D).$$

Assuming $G/H = \{g_1H, g_2H, \dots, g_kH\}$, then $\mathrm{Ind}^G_H(C)$ can be written as the direct sum

$$\mathrm{Ind}^G_H(C) = \bigoplus_{i=1}^{k} g_i C, \qquad (7.2)$$

For $g \in G$, we can write $g = g_i h$. This can be unique; otherwise, g_i will be g_j. We have

$$g \cdot (-) : g_j C \to g_k C, \quad x \mapsto (h' g_j^{-1} h g_j) \cdot x,$$

where $g_k = g_i \cdot g_j \cdot h'$, with $h' \in H$ and g_k representing the coset of $g_i \cdot g_j$.

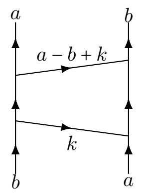


FIGURE 10. The k-smoothing of a positive crossing, for $0 \le k \le b \le a$. In order to obtain the k-smoothing of a negative crossing, reflect the above picture about the vertical line and switch labels.

7.2. **Construction of the Spectral Sequence.** The initial construction will be done for the general link diagram, and later, we will focus on periodic link diagrams. Let D be a labeled link diagram, where each link component is labeled by $c \in \{1, 2, \dots N\}$. Recall that $\mathrm{Cr}(D)$ is the set of crossings.

We define the extended cube of resolutions $\mathrm{Cube}^+(D)$. For a crossing $i \in \mathrm{Cr}(D)$, we have $C_i = \{0, \dots, c_i\}$ where $c_i = \min(a_i, b_i)$ if the crossing is positive, and $C_i = \{-c_i, \dots 0\}$ if the crossing is negative. We extend C_i by the definition $\hat{C}_i = C_i \cup \{*\}$. $\mathrm{Cube}^+(D)$ is the product of the $\hat{C}_i$.

For $\hat{I} \in \mathrm{Cube}^+(D)$, we define the resolution diagram $D_{\hat{I}}$. If the i-th crossing in $\hat{I}$ is equal to $*$, we do not resolve the crossing. Otherwise, we resolve the crossing as in the standard case. The resolutions are depicted in Figure 10, see also the skein relation in Figure 4. For $\hat{I} \in \mathrm{Cube}^+$, we define $\mathrm{supp}\,\hat{I}$ to be the set of crossings $i \in \mathrm{Cr}(D)$ where $\hat{I}(i) \neq *$. If $\hat{I}, \hat{J} \in \mathrm{Cube}^+$ and $\mathrm{supp}\,\hat{I} \cap \mathrm{supp}\,\hat{J} = \varnothing$, we define $\hat{I} \cup \hat{J}$ to be the resolution such that

$$(\widehat{I} \vee \widehat{J})(i) = \begin{cases} \widehat{I}(i) & i \in \mathrm{supp}\,\widehat{I} \\ \widehat{J}(i) & i \in \mathrm{supp}\,\widehat{J} \\ * & \text{otherwise.} \end{cases}$$

For $\hat{I}$ with support X, we define $[\![D_{\widehat{I}}]\!]$ as a cochain complex generated by those D_I for which I and $\hat{I}$ coincide on X. Furthermore, the differential is given by foams of Figure 6 with the sign assignment $\mathfrak{s}_I$ inherited from the sign assignment $\mathfrak{s}$ on D. We also define the degree for $\hat{I}$ as

$$\deg \widehat{I} = \sum_{i \in \mathrm{supp}\,\widehat{I}} I(i).$$

For a subset $X \subset \mathrm{Cr}(D)$, we let

$$A(X) = \{\widehat{I} \in \mathrm{Cube}^+(D) \colon \mathrm{supp}\,\widehat{I} = X\}, A_k(X) = \{\widehat{I} \in A(X) \colon \deg \widehat{I} = k\}$$

Let $X \subset \mathrm{Cr}(D)$ be a subset of positive crossings (for negative subset discussion will be similar). Set $Y = \mathrm{Cr}(D) - X$. We let $\mathrm{Cube}(X)$, $\mathrm{Cube}(Y)$ be the cubes of resolution for X and Y. In other words, we have $\mathrm{Cube}(X) = \prod_{i \in X} C_i$, $\mathrm{Cube}(Y) = \prod_{i \in Y} C_i$. For $\mathrm{Cube}(D)$ we have $\mathrm{Cube}(D) = \mathrm{Cube}(X) \times \mathrm{Cube}(Y)$. For $I \in \mathrm{Cube}(D)$, we denote I_X, I_Y its projections on $\mathrm{Cube}(X)$ and on $\mathrm{Cube}(Y)$ respectively.

We introduce one more piece of notation. Assume we have $I \in \mathrm{Cube}(D)$. Let $\hat{I} \in \mathrm{Cube}^+(D)$ be gained by taking I_X and extending it by putting $*$ for all crossings in Y. This means that crossings in X are already resolved so $D_{\hat{I}}$ has a set of crossings Y. This means $(D_{\hat{I}})_{I_Y} = D_I$. We can write $[\![D]\!]$ as the following bicomplex.

$$0 \to \bigoplus_{\widehat{I} \in A_0(X)} [\![D_{\widehat{I}}]\!]\{q^{-b(N-b)|X|}\} \xrightarrow{\pm d_0} \bigoplus_{\widehat{I} \in A_1(X)} [\![D_{\widehat{I}}]\!]\{q^{1-b(N-b)|X|}\} \xrightarrow{\pm d_1} \dots \tag{7.3}$$

Here q is the grading shift. The differentials d_i are defined as follows. For I, $J \in \mathrm{Cube}(D)$ where J is an immediate successor of I. We have two cases

- Assume $I_X = J_X$, the part of the differential on $[\![D]\!]$ from I to J contributes to the differential on $[\![D_{\hat{I}}]\!]$. It goes from $(D_{\hat{I}})_{I_Y}$ to $(D_{\hat{I}})_{J_Y}$ with the sign $(-1)^{\jmath(I,J)}$. We call this differential part the internal differential or horizontal differential.
- Assume $I_Y = J_Y$, we set $s = \deg \widehat{I}$ the part of the differential on $[\![D]\!]$ that contributes to the differential d_s going from $[\![D_{\widehat{I}}]\!]$ to $[\![D_{\widehat{J}}]\!]$. In particular, it goes from $(D_{\hat{I}})_{I_Y}$ to $(D_{\hat{J}})_{J_Y}$ with sign $\jmath(I,J)$. We call this differential part the external differential or vertical differential.

The sum of these two differentials is equal to the differential on $[\![D]\!]$. Therefore, we have the following result.

Lemma 7.4. *The total complex* (7.3) *is equal to* $[\![D]\!]$.

In general, a bicomplex leads to a spectral sequence. To apply this principle, we apply functor F to (7.3) to work in an Abelian category. To be more precise, we define the triply graded bicomplex

$$M(D,X)^{k,\ell,h} = \bigoplus_{\widehat{I}\in A_k(X)} \mathcal{F}([\![D_{\widehat{I}}]\!])\{q^{k-|X|b(N-b)}\}_{\ell,h}.$$

Here ℓ is the homology grading and q is the quantum grading. If X is a subset of negative crossings, we define

$$M(D,X)^{k,\ell,h} = \bigoplus_{\widehat{I}\in A_k(X)} \mathcal{F}([\![D_{\widehat{I}}]\!])\{q^{-k+|X|b(N-b)}\}_{\ell,h}.$$

In the bicomplex $M(D,X)^{\bullet,\ell,h}$, we have an internal (horizontal) differential and the external (vertical) differential going from $M(D,X)^{\bullet,\ell,h}$ to $M(D,X)^{\bullet+1,\ell,h}$.

Lemma 7.5. *The cohomology of the total complex* $Tot^{r,h}M(D,X) = \bigoplus_{k+\ell=r} M(D,X)^{k,\ell,h}$ *is the* $\mathbb{S}_N$*-valued Khovanov-Rozansky homology of the link.*

Proof. Firstly, focus on the horizontal differentials; if we do not consider grading shifts, the cohomology of the complex $M(D,X)^{k,\ell,h}$ is equal to the sum of Khovanov-Rozansky homologies of webs $D_{\hat{I}}$. We can say that by 7.4 we have a spectral sequence, whose E^1 page is the cohomology of the complex $M(D,X)^{k,\ell,h}$. This spectral sequence approaches the Khovanov-Rozansky homology of D. □

Assume that D is a $\mathbb{Z}_m$ periodic link diagram. Our primary focus will be on the case when X is an orbit of crossings. In this case, $\mathbb{Z}_m$ acts on $\mathrm{Cr}(D)$ and it preserves X. For any k, this action can be induced on $A_k(X)$. For $\hat{I} \in A_k(X)$, define the isotropy group of $\hat{I}$ $\mathrm{Iso}(\widehat{I}) = \{g \in \mathbb{Z}_m : \widehat{I} \circ g = \widehat{I}\}$. For any $d|m$ define

$$A_k^d(X) = \{\widehat{I} \in A_k(X) : \mathrm{Iso}(\widehat{I}) = \mathbb{Z}_d\} \tag{7.6}$$

and denote by $\overline{A}_k^d(X)$ the quotient of $A_k^d(X)$ by the action of $\mathbb{Z}_m$. Notice that for $\hat{I} \in A_k^d(X)$, the diagram $D_{\widehat{I}}$ is d-periodic. Furthermore, for any $g \in G$ with the group action on $[\![D]\!]$, we have a map $\mathcal{G}_{g,\widehat{I}} : [\![D_{\widehat{I}}, \jmath_{\widehat{I}}]\!] \to [\![D_{g\widehat{I}}, \jmath_{g\widehat{I}}]\!]$, where $\jmath_{\widehat{I}}$ and $\jmath_{g\widehat{I}}$ denote restrictions of the sign assignment $\jmath$ on $\mathrm{Cube}(D)$ to $\mathrm{Cube}(D_{\widehat{I}})$ and $\mathrm{Cube}(D_{g\widehat{I}})$, respectively.

Lemma 7.4 can be generalized for an equivariant setting. Assume X is a set of crossings in which either all crossings in X are positive or all crossings in X are negative and where X is $\mathbb{Z}_m$ invariant. Note that $D_{\hat{I}}$ is d–periodic diagram for any $\hat{I} \in A_k^d(X)$. We have the natural $\mathbb{Z}_d$-action on $[\![D_{\widehat{I}}]\!]$ and $\mathcal{F}([\![D_{\widehat{I}}]\!])$ as defined in Proposition 5.21. We define the equivariant version of the bicomplex $M(D,X)^{k,\ell,h}$ by

$$\mathrm{EM}(D,X)^{k,l,\bullet} = \begin{cases} \bigoplus_{d|k} \bigoplus_{\widehat{I}\in\overline{A}_k^d(X)} \mathrm{Ind}_{\mathbb{Z}_d}^{\mathbb{Z}_m}\left(\mathcal{F}([\![D_{\widehat{I}}]\!])\{q^{-|X|b(N-b)+k}\} \otimes \mathbb{C}_{s(m,d,\widehat{I})}\right) & \text{X is positive,} \\ \bigoplus_{d|k} \bigoplus_{\widehat{I}\in\overline{A}_k^d(X)} \mathrm{Ind}_{\mathbb{Z}_d}^{\mathbb{Z}_m}\left(\mathcal{F}([\![D_{\widehat{I}}]\!])\{q^{|X|b(N-b)-k}\} \otimes \mathbb{C}_{s(m,d,\widehat{I})}\right) & \text{X is negative.} \end{cases} \tag{7.7}$$

where $s(m,d,\widehat{I}) \in \mathbb{F}_2$:

$$s(m,d,\widehat{I}) = t(I_0) + t(gI_0) + \cdots + t(g^{m/d-1}I_0) \tag{7.8}$$

Here, we take the tensor product over the ring $\mathbb{C}[\mathbb{Z}_d]$ and we think $\mathrm{Sym}_N[\mathbb{Z}_d]$-module $\mathcal{F}([\![D]\!])$ as a right $\mathbb{C}[\mathbb{Z}_d]$-module with the standard action of $\mathbb{C}$ on Sym_N. On the one-dimensional complex vector space $\mathbb{C}_j$, $\mathbb{Z}_d$ acts either trivially if $j = 0$ or it acts as the sign action, i.e., the generator of $\mathbb{Z}_d$ acts on $\mathbb{C}$ by multiplication by -1, if $j = 1$. Lately here, $I_0 = \widehat{I} \vee J_0$ for $J_0 = (0,\dots,0) \in \mathrm{Cube}(D_{\widehat{I}})$ and t a 0-cochain on $\mathrm{SCube}(D)$ satisfying $gs - s = \partial t$, $t((0,\dots,0)) = 0$.

Lemma 7.9. *We have an isomorphism* $\mathrm{EM}(D,X) \cong M(D,X)$ *as complexes of* $\mathbb{S}_N$*-modules.*

Proof. Since we need to show they are isomorphic as $\mathbb{S}_N$-modules, we do not care about the action of $\mathbb{C}_{s(m,d,\widehat{I})}$. Enough to show that both sides have the same $\mathcal{F}([\![D_{\widehat{I}}]\!])$. For any $\widehat{I} \in A_k(X)$, this $\widehat{I}$ must be in one of $A^d_k(X)$ for $d|k$. Furthermore, we can get this $\widehat{I}$ from $\widehat{J} \in \overline{A}^d_k(X)$ such that $g\widehat{J} = \widehat{I}$ where $g \in \mathbb{Z}_m/\mathbb{Z}_d$. For any $\widehat{I}$ for $\widehat{I} \in A_k(X)$, we have $\mathcal{F}([\![D_{\widehat{I}}]\!] = \mathcal{F}([\![D_{g\widehat{J}}]\!])$ for $J \in \overline{A}^d_k(X)$ and for $g \in \mathbb{Z}_m/\mathbb{Z}_d$. ☐

Lemma 7.10. *We have an isomorphism between the total complex of* $\mathrm{EM}(D,X) \cong M(D,X)$ *as complexes of* $\mathbb{S}_N[\mathbb{Z}_m]$*-modules.*

Proof. By Lemma 7.9, we need to show that the isomorphism between $\mathrm{EM}(D,X)$ and $\mathcal{F}([\![D]\!])$ as $\mathbb{S}_N$-modules is $\mathbb{Z}_m$-equivariant. Recall that we have g as a generator of $\mathbb{Z}_m$ acting on the plane by rotation by the angle $\frac{2\pi}{m}$. Fix a sign assignment s on D, and let t be the 0-cochain satisfying $\partial t = gs - s$, $t((0,\dots,0)) = 0$. For a divisor d of m set $h = g^{m/d}$ to be a generator of $\mathbb{Z}_d \subset \mathbb{Z}_m$. Take $\widehat{I} \in \overline{A}^d_k(X)$ and consider the partial resolution $D_{\widehat{I}}$. Define $s_{\widehat{I}}$ to be the sign assignment on $\mathrm{Cube}(D_{\widehat{I}})$, defined as $s_{\widehat{I}}(J,J') = s(\widehat{I} \vee J, \widehat{I} \vee J')$. Since $D_{\widehat{I}}$ is a d-periodic diagram, by Proposition 5.21 we can define an action of $\mathbb{Z}_d$ on $[\![D_{\widehat{I}}]\!]$. Specifically, we let $t_{\widehat{I}}$ be the 0-cochain on $\mathrm{Cube}(D_{\widehat{I}})$ such that $hs_{\widehat{I}} - s_{\widehat{I}} = t_{\widehat{I}}$ and $t_{\widehat{I}}(0,\dots,0) = 0$. Corresponding to the action of h (i.e. rotation by the angle $\frac{2\pi}{d}$) we have the map $\mathcal{H}_h \colon [\![D_{\widehat{I}}]\!] \to [\![D_{\widehat{I}}]\!]$.

There are two maps that are induced by the action of h on $[\![D_{\widehat{I}}]\!]$. The external one is $(\mathcal{G}_g)^{m/d}$, where $\mathcal{G}_g$ is the action constructed in Proposition 5.21 for $[\![D]\!]$. The other map is $\mathcal{H}_h$. Since these two maps are obtained from the same sets of foams, these two maps are actually equal up to a sign choice. To complete the proof of Lemma 7.10, we need to compare $t_{\widehat{I}}(J)$ and $t(\widehat{I} \vee \widehat{J})$ for $J \in \mathrm{Cube}(D_{\widehat{I}})$. We know for any two 0-cochains t_1, t_2 such that $hs_{\widehat{I}} - s_{\widehat{I}} = \partial t_1 = \partial t_2$. We can say $J \mapsto t_{\widehat{I}}(J)$ and $J \mapsto t(\widehat{I} \vee J)$ are either equal or differ by an overall sign. To understand this sign issue, let $J_0 = (0,\dots,0) \in \mathrm{Cube}(D_{\widehat{I}})$, set $I_0 = J_0 \vee \widehat{I}$. Suppose that $\rho_g \colon D_{I_0} \to D_{gI_0}$ is the foam realizing the rotation of D_{I_0} by $g \in \mathbb{Z}$, i.e., the I_0-th component of $\mathcal{G}_g$ is equal to $(-1)^{t(I_0)}\rho_g$. Let $\rho_h = \rho_{g^{m/d-1}I_0} \circ \cdots \circ \rho_{gI_0} \circ \rho_{I_0}$ and $t_h(I_0) = t(I_0) + t(gI_0) + \cdots + t(g^{m/d-1}I_0)$, then the I_0-th component of $\mathcal{H}_h$ is equal to $(-1)^{t_h(I_0)}\rho_h$. By the proof of Proposition 5.21, $t_{\widehat{I}}(0,\dots,0) = 0$. In other words, the J_0-th component of $\mathcal{H}_h$ is equal to ρ_h. Therefore $s(m,d,\widehat{I}) = t_h$. We conclude by (7.8). ☐

Proposition 7.11 (Skein spectral sequence)**.** *Let D be an $m = p^\ell$-periodic labeled link diagram, with p an odd prime and $\ell \geq 1$. Let $X \subset \mathrm{Cr}(D)$ be an orbit of crossings between an a-labeled overstrand and a b-labeled understrand, where $a \geq b$. If $0 \leq u \leq \ell$ and X is a set of positive crossings, we obtain, for any $1 \leq s \leq |X|b$, a spectral sequence with*

$$E_1^{k,l,\bullet}(D,X,p^{\ell-u}) = \bigoplus_{p^s|k} \bigoplus_{\widehat{I} \in \overline{A}^s_k(D,X)} \mathrm{EKR}_N^{\bullet,\bullet}(D_{\widehat{I}}, \kappa(u,s))^{\oplus\lambda(u,s)} t^k q^{-|X|b(N-b)-k}, \tag{7.12}$$

with $0 \leq k \leq p^\ell b$ and

$$\kappa(u,s) = \begin{cases} 1, & u \geq s, \\ p^{s-u}, & \textit{otherwise}, \end{cases} \qquad \lambda(u,s) = \begin{cases} \phi(p^{\ell-u}), & u \geq s, \\ p^{\ell-s}, & u < s, \end{cases}$$

converging to $\mathrm{EKR}_N^{\bullet;\bullet}(D, p^{\ell-u})$. *On the other hand, if* X *is the set of negative crossings, we obtain a spectral sequence with*

$$E_1^{k,l,\bullet}(D, X, p^{\ell-u}) = \bigoplus_{p^s|k} \bigoplus_{\widehat{I}\in\overline{A}^s_{p^\ell+k}(D,X)} \mathrm{EKR}_N^{\bullet;\bullet}(D_{\widehat{I}}, \kappa(u,s))^{\oplus\lambda(u,s)} t^k q^{|X|b(N-b)-k},$$

where $-p^\ell b \le k \le 0$.

Proof. We prove this proposition only in the positive case. Note that the total complex of $\mathrm{EM}(D, X)$ is the complex of $\mathcal{F}([\![D]\!])$ by Lemma 7.10. We will denote the singular specialization of $\mathrm{EM}(D, X)$ by $\mathrm{EM}_0(D, X)$. We fix $0 \le u \le \ell$ and consider the bicomplex derived from $\mathrm{EM}(D, X)$:

$$\mathrm{EM}^{k,l,*}(D, X, p^{\ell-u}) := \mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_{p^\ell}]}(\mathbb{C}[\mathbb{Z}_{p^\ell}]_{p^{\ell-u}}, \mathrm{EM}_0^{k,l,*}(D, X)).$$

On considering separately the internal (vertical) and the external (horizontal) differentials in $\mathrm{EM}(D, X, p^{\ell-u})$, we obtain a spectral sequence of $\mathbb{C}[\mathbb{Z}_m]$-modules converging to $\mathrm{EKR}_N^{*,*}(D, p^{\ell-u})$, whose E_1-page is given by

$$\begin{aligned} E_1^{k,l,*}(D, X, p^{\ell-u}) &= H^{k,*}(\mathrm{EM}_0^{*,l,*}(D, X, p^{\ell-u}), d_{\mathrm{vert}}) \\ &\cong \mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_{p^\ell}]}(\mathbb{C}[\mathbb{Z}_{p^\ell}]_{p^{\ell-u}}, H^{k,*}(\mathrm{EM}_0^{*,l,*}(D, X), d_{\mathrm{vert}})). \end{aligned}$$

i.e., we take the vertical homology of $\mathrm{EM}(D, X, p^{\ell-u})$. The aim of the proof is to show that this page is isomorphic to (7.12). Consider the decomposition of the group algebra $\mathbb{C}[\mathbb{Z}_{p^\ell}]$. Recall from Section 5.8 that

$$\mathbb{C}[\mathbb{Z}_{p^\ell}]_{p^{\ell-u}} = \bigoplus_{\substack{0\le i<p^\ell \\ \gcd(i,p^\ell)=p^u}} \mathbb{C}_{\xi^i_{p^\ell}}.$$

Observe that for any $0 \le s \le \ell$ we have

$$\mathrm{Res}^{\mathbb{Z}_{p^\ell}}_{\mathbb{Z}_{p^s}}(\mathbb{C}_{\xi^j_{p^{\ell-u}}}) = \mathbb{C}_{(\xi^j_{p^{\ell-u}})^{p^{\ell-s}}} = \begin{cases} \mathbb{C}_1, & s \le u, \\ \mathbb{C}_{\xi^j_{p^{s-u}}}, & s > u. \end{cases}$$

Therefore,

$$\mathrm{Res}^{\mathbb{Z}_{p^\ell}}_{\mathbb{Z}_{p^s}}(\mathbb{C}[\mathbb{Z}_{p^\ell}]_{p^{\ell-u}}) = \begin{cases} \mathbb{C}_1^{\phi(p^{\ell-u})}, & s \le u, \\ \mathbb{C}[\mathbb{Z}_{p^s}]^{p^{\ell-s}}_{p^{s-u}}, & s > u. \end{cases} \tag{7.13}$$

By the definition of $\mathrm{EM}_0(D, X)$, we obtain

$$\begin{aligned} E_1^{k,l,\bullet}(D, X, p^{\ell-u}) &= \mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_{p^\ell}]}(\mathbb{C}[\mathbb{Z}_{p^\ell}]_{p^{\ell-u}}, H^{k,*}(\mathrm{EM}_0^{*,l,*}(D, X), d_{\mathrm{vert}})) \\ &\cong \bigoplus_{p^s|k} \bigoplus_{\widehat{I}\in\overline{A}^s_k(D,X)} \mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_{p^\ell}]}\left(\mathbb{C}[\mathbb{Z}_{p^\ell}]_{p^{\ell-u}}, \mathrm{Ind}^{\mathbb{Z}_{p^\ell}}_{\mathbb{Z}_{p^s}} \mathrm{EKR}(D_{\widehat{I}}) t^k q^{-|X|b(N-b)+k}\right). \end{aligned}$$

Consider the right-hand side of the above equation:

$$\begin{aligned} &\mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_{p^\ell}]}\left(\mathbb{C}[\mathbb{Z}_{p^\ell}]_{p^{\ell-u}}, \mathrm{Ind}^{\mathbb{Z}_{p^\ell}}_{\mathbb{Z}_{p^s}} \mathrm{EKR}(D_{\widehat{I}}) t^k q^{-|X|b(N-b)+k}\right) \\ &\overset{(7.1)}{\cong} \mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_{p^s}]}\left(\mathrm{Res}^{\mathbb{Z}_{p^\ell}}_{\mathbb{Z}_{p^s}}(\mathbb{C}[\mathbb{Z}_{p^\ell}]_{p^{\ell-u}}), \mathrm{EKR}(D_{\widehat{I}}) t^k q^{-|X|b(N-b)+k}\right) \\ &\overset{(7.13)}{=} \begin{cases} \mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_{p^\ell}]}(\mathbb{C}_1^{\phi(p^{\ell-u})}, \mathrm{EKR}(D_{\widehat{I}}) t^k q^{-|X|b(N-b)+k}), & s \le u, \\ \mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_{p^\ell}]}(\mathbb{C}[\mathbb{Z}_{p^s}]^{p^{\ell-s}}_{p^{s-u}}, \mathrm{EKR}(D_{\widehat{I}}) t^k q^{-|X|b(N-b)+k}) & s > u \end{cases} \\ &= \begin{cases} \mathrm{EKR}^{*,*}(D_{\widehat{I}}, 1)^{\oplus\phi(p^{\ell-u})} t^k q^{-|X|b(N-b)+k}, & s \le u \\ \mathrm{EKR}^{*,*}(D_{\widehat{I}}, p^{s-u})^{\oplus p^{\ell-s}} t^k q^{-|X|b(N-b)+k}, & s > u \end{cases} \\ &= \mathrm{EKR}^{*,*}(D_{\widehat{I}}, \kappa(u,s))^{\lambda(u,s)} t^k q^{-|X|b(N-b)+k}. \end{aligned}$$

The proposition follows. □

8. Polynomial Invariants

8.1. **Poincare polynomials of $\mathfrak{sl}_N$ and Lee homology.** First, we remind a common construction.

Definition 8.1. Let L be a link. The LeeP_N polynomial is

$$\mathrm{LeeP}_N(L) = \sum_{k,r} \dim_{\mathbb{C}} \mathrm{Gr}^r \mathrm{Lee}_N^k(L) t^k q^r,$$

where Gr^r is the r-th graded part of the filtered Lee_N homology, and the Khovanov–Rozansky polynomial $\mathrm{KRP}_N(K)$ is the Poincaré polynomial of $\mathfrak{sl}_N$-homology:

$$\mathrm{KRP}_N(L) = \sum_{k,r} t^k q^r \dim_{\mathbb{C}} \mathrm{KR}_N^{k,r}(L).$$

For an m-periodic link, we modify the definition above and generalize the approach of [20].

Definition 8.2. Assume we have an m-periodic link L and let $d|m$. The *equivariant Khovanov–Rozansky polynomial*, for $\mathfrak{sl}_N$-homology, is

$$\mathrm{KRP}_{N,d}(L) = \sum_{k,r} t^k q^r \dim_{\mathbb{C}_d} \mathrm{EKR}_N^{k,r,d}(L). \tag{8.3}$$

The *equivariant Lee polynomial* is:

$$\mathrm{LeeP}_{N,d}(L) = \sum_{k,r} \dim_{\mathbb{C}_d} \mathrm{Gr}^r \mathrm{ELee}_N^{k;d}(L) t^k q^r,$$

We have the following relation between the Khovanov–Rozansky polynomial and the equivariant Khovanov–Rozansky polynomial.

$$\mathrm{KRP}_N(L) = \sum_{d|m} \phi(d) \mathrm{KRP}_{N,d}(L), \tag{8.4}$$

where $\phi(d) = \#\{1 \le i \le d \colon \gcd(i,d) = 1\}$ is Euler's totient function.

We can compute Lee homology from Proposition 5.20. For the precise formula for the knot, we refer to [12, Proposition 2.6]. Other references include [8, 13, 14, 26, 29].

Lemma 8.5. *For any knot K, we have $\mathrm{LeeP}_N(K) = q^{s_N(K)}(q^{-N+1} + q^{-N+3} + \cdots + q^{N-1})$, where $s_N(K)$ is the Lewark's s_N-invariant; see* [12].

We have the following statement as a consequence of Lemma 5.31.

Lemma 8.6. *If the action of $\mathbb{Z}_m$ on the components of L is trivial, then $\mathrm{LeeP}_{N,d}$ is equal to LeeP_N if $d = 1$, and $\mathrm{LeeP}_{N,d}$ is equal to 0 otherwise.*

The following proposition shows the relation between polynomials KRP and LeeP. Its proof is the same as in the Khovanov case, see [15, Proposition 2.17]. See also [6, Theorem 5.1] and [12, Proposition 5.2].

Proposition 8.7. *For a link L, there are polynomials $R_1, R_2, \dots$ with non-negative coefficients such that*

$$\mathrm{KRP}_N(L) = \mathrm{LeeP}_N(L) + (1 + tq^{2N})R_1 + (1 + tq^{4N})R_2 + \dots.$$

Furthermore, for an m-periodic link L where $d|m$, we have

$$\mathrm{KRP}_{N,d}(L) = \mathrm{LeeP}_{N,d}(L) + (1 + tq^{2N})R_1^d + (1 + tq^{4N})R_2^d + \dots$$

for polynomials $R_1^d, R_2^d, \dots$ with non-negative coefficients.

8.2. **The Reshetikhin-Turaev RT_N polynomials.** We recall that for a link L, the HOMFLYPT polynomial $X(a,b)$ is defined by its value on the unknot and skein relation.

$$aX_{L_+}(a,b) - a^{-1}X_{L_-}(a,b) = bX_{L_0}(a,b), \tag{8.8}$$

where L_0 is the 0 resolution, L_+ is the positive crossing, and L_- is the negative crossing.

Reshetikhin-Turaev is a specific case of the HOMFLYPT polynomial. For $N \geq 0$ Reshetikhin-Turaev is

$$\mathrm{RT}_N(q) = X(q^N, q - q^{-1}) \tag{8.9}$$

The normalization of this polynomial is

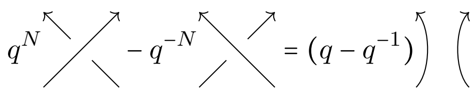
$$\mathrm{RT}_N(\text{unknot}) = \frac{q^N - q^{-N}}{q - q^{-1}}.$$

For $N = 0$, RT_0 is the Alexander polynomial, and for $N = 1$, $RT_1 \equiv 1$, and for $N = 2$, we have the Jones polynomial which categorifies Khovanov homology. For $N > 2$, we call these polynomials as $\mathfrak{sl}_N$ polynomials of L. In [10, 11] it was proved that $\mathfrak{sl}_N$ homology categorifies the $\mathfrak{sl}_N$ polynomial.

Lemma 8.10. *For a link L and for $\mathrm{KR}_N^{k,r}(L)$ its $\mathfrak{sl}_N$-homology, we have*

$$\mathrm{RT}_N(L) = \sum_{k,r} (-1)^k q^r \dim \mathrm{KR}_N^{k,r}(L) = \mathrm{KRP}_N|_{t=-1}.$$

The skein relation for RT_N polynomial is a particular version of the skein relation of the HOMFLYPT polynomial.

(8.11) 

8.3. **Difference polynomials.** Fix $m = p^l$ for a prime p, and let D be an m-periodic diagram of an m-periodic link L. The $\mathfrak{sl}_N$ homology of L decomposes as in (5.30). We have

$$\mathrm{RT}_{N,j} = \mathrm{KRP}_{N,p^j}|_{t=-1},$$

where $\mathrm{KRP}_{N,d}$ is as in (8.3).

We have the corollary that will be used in the future.

Corollary 8.12. *Assume L is a p^n periodic link for the prime p, and assume L' is its mirror; then $\mathrm{RT}_{N,j}(L)(q) = \mathrm{RT}_{N,j}(L')(q^{-1})$.*

Proof. We know from Proposition 5.28 we have an isomorphism of $\mathbb{C}$-vector spaces $\mathrm{EKR}^{k,r,m}(L) = \mathrm{EKR}^{-k,-r,m}(L')$. Thus, by (8.3), we have the statement. It seems there is no need for $m = p^n$ here! □

For Reshetikhin-Turaev, we have difference $\mathfrak{sl}_N$ polynomials.

Definition 8.13. $\mathrm{DRT}_{N,j}(D) = \begin{cases} \mathrm{RT}_{N,p^j}(D) - \mathrm{RT}_{N,p^{j+1}}(D) & 0 \leq j < \ell \\ \mathrm{RT}_{N,p^\ell}(D) & j = \ell. \end{cases}$

Proposition 8.14. $\mathrm{DRT}_{N,j}(D)$ *polynomials have the following relations between each other.*

(1) *For $j = 0$ we have*

$$q^{mN}\,\mathrm{DRT}_{N,0}(L_+) - q^{-mN}\,\mathrm{DRT}_{N,0}(L_-) = (q^{-m} - q^m)\,\mathrm{DJ}_{N,0}(L_0).$$

(2) *For any $0 \leq j < \ell$, we have*

$$q^{mN}\,\mathrm{DRT}_{N,\ell-j}(L_+) - q^{-mN}\,\mathrm{DRT}_{N,\ell-j}(L_-) \equiv (q^{-m} - q^m)\,\mathrm{DJ}_{N,\ell-j}(L_0) \pmod{q^{p^j} - q^{-p^j}}.$$

Proof. We use [20, Theorem 3.6]. Assume that $\{E_r^{*,*}, d_r\}_{r\geq 1}$ is a spectral sequence of graded finite-dimensional $\mathbb{C}$-vector spaces which converge to a double-graded $\mathbb{C}$-vector space $H^{*,*}$. Moreover, assume the spectral sequence collapses at a finite stage. Consider the Poincaré polynomials of the page $E_r^{*,*}$:

$$P(E_r^{*,*}) = \sum_{i,j} t^{i+j}\, \mathrm{qdim}_{\mathbb{C}}\, E_r^{i,j}.$$

For a graded $\mathbb{C}$-vector space V^*, we have

$$\mathrm{qdim}_{\mathbb{C}}\, V^* = \sum_i q^i \dim_{\mathbb{C}} V^i.$$

By [17, Exercise 1.7], we conclude that for any $r \geq 1$,

$$P(E_r^{*,*})(-1,q) = P(E_\infty^{*,*})(-1,q) = \sum_{i,j} (-1)^i\, \mathrm{qdim}_{\mathbb{C}}\, H^{i,*}. \tag{8.15}$$

For a fixed p^l-periodic diagram D we apply (8.15) to spectral sequences constructed in Proposition 7.11. We get

$$P(E_1^{*,*}(p^{\ell-u}))(-1,q) = P(E_\infty^{*,*})(-1,q) = \mathrm{RT}_{N,\ell-u}(D).$$

Recall that $E_1^{*,*}(p^{\ell-u})$ is the first page of the homology of a diagram which is invariant under the action of a subgroup of order p^t for $t \leq \ell - u$ smaller order. The description of $E_1^{*,*}(p^{\ell-u})$ implies that $P(E_1^{*,*})(-1,q)$ is a linear combination of polynomials $\mathrm{RT}_{N,j}(D_{\widehat{I}})$, where $\widehat{I} \in A_k(X)$ and appropriate j. Consequently,

$$\begin{aligned} DP_{N,\ell-u}(D) &= \mathrm{RT}_{N,\ell-u}(D) - \mathrm{RT}_{N,\ell-u+1}(D) = \\ &= P(E_1^{*,*}(p^{\ell-u}))(-1,q) - P(E_1^{*,*}(p^{\ell-u+1}))(-1,q). \end{aligned} \tag{8.16}$$

we apply formula (8.16) to $DP_{N,\ell-u}(L_+)$ and $DP_{N,\ell-u}(L_+)$. We get

$$DP_{N,\ell-u}(L_+) = \sum_{k=0}^{p^\ell} \sum_{s=u}^{\ell} \sum_{\widehat{I}\in A_k^s(L_+,X)} (-1)^k q^{-p^\ell(N-1)-k} DP_{N,s-u}(D_{\widehat{I}}),$$

$$DP_{N,\ell-u}(L_-) = \sum_{k=-p^\ell}^{0} \sum_{s=u}^{\ell} \sum_{\widehat{I}\in A^s_{p^\ell+k}(L_-,X)} (-1)^k q^{p^\ell(N-1)+p^\ell-k} DP_{N,s-u}(D_{\widehat{I}}).$$

By $A_k(L_+,X) = A_{p^\ell-k}(L_-,X)$, we get

$$q^{p^\ell N} DP_{N,\ell-u}(L_+) - q^{-p^\ell N} DP_{N,\ell-u}(L_-) = \sum_{k=0}^{p^\ell} \sum_{s=u}^{\ell} \sum_{\widehat{I}\in A_k^s(L_+,X)} (-1)^k (q^{p^\ell-k} - q^{-p^\ell+k}) DP_{N,s-u}(D_{\widehat{I}}).$$

We know $A_k^s(L_+,X)$ is empty unless p^s divides k. In the above equation observe that for $k=0$ we have $q^{p^\ell} - q^{-p^\ell})\,\mathrm{DRT}_{N,\ell-u}(L_0)$ and for $k = p^\ell$ the sum is zero Hence we have

$$\begin{aligned} &q^{p^\ell N}\, \mathrm{DRT}_{N,\ell-u}(L_+) - q^{-p^\ell N}\, \mathrm{DRT}_{N,\ell-u}(L_-) - (q^{p^\ell} - q^{-p^\ell})\, \mathrm{DRT}_{N,\ell-u}(L_0) = \\ &\sum_{k=1}^{p^\ell-1} \sum_{s=u}^{\ell} \sum_{\widehat{I}\in A_k^s(L_+,X)} (-1)^k (q^{p^\ell-k} - q^{-p^\ell+k})\, \mathrm{DRT}_{N,s-u}(D_{\widehat{I}}). \end{aligned}$$

For $u=\ell$, since $s=u$ and $u=\ell$ we have $\ell = s$ which implies $k \leq p^s - 1$ so p^s can not divide k. Hence the right-hand side is zero. We have

$$q^{p^\ell N} DP_{N,0}(L_+) - q^{-p^\ell N} DP_{N,0}(L_-) = (q^{p^\ell} - q^{-p^\ell}) DP_{N,0}(L_0),$$

as we want.

For $0 \leq u < \ell$ and for $u \leq s \leq \ell$ and k divisible by p^s, we write $k = k'p^s$.

$$p^\ell - k = p^\ell - k'p^s = p^s(p^{\ell-s} - k')$$

Set $p^{\ell-s} - k' = A$. We have

$$q^{p^\ell - k} - q^{-p^\ell + k} = q^{p^s A} - q^{-p^s A} = q^{-p^s A}(q^{2p^s A} - 1)$$

Since $q^{p^u} - q^{-p^u} \equiv 0 \pmod{q^{p^u} - q^{-p^u}}$, we have $q^{2p^u} \equiv 1 \pmod{q^{p^u} - q^{-p^u}}$ Hence

$$q^{p^\ell - k} - q^{-p^\ell + k} = q^{p^s A} - q^{-p^s A} = q^{-p^s A}(q^{2p^s A} - 1) \equiv 0 (mod q^{p^u} - q^{-p^u})$$

We deduce by the above equations

$$(q^{p^\ell - k} - q^{-p^\ell + k}) \, \mathrm{DRT}_{N,s-u}(D_{\widehat{T}}) \equiv 0 \pmod{q^{p^u} - q^{-p^u}}.$$

Consequently,

$$q^{p^\ell N} \mathrm{DRT}_{N,0}(L_+) - q^{-p^\ell N} \mathrm{DRT}_{N,0}(L_-) \equiv (q^{p^\ell} - q^{-p^\ell}) \mathrm{DRT}_{N,0}(L_0) \pmod{q^{p^u} - q^{-p^u}}.$$

□

8.4. **Periodicity criterion.** The result in this section ports the periodicity criterion of [15] to the case of $\mathfrak{sl}_N$-homology.

In which sense is Thm 8.17 a criterion? It looks more like a property ...

Theorem 8.17. *Assume L is an $m = p^\ell$ periodic knot with p a prime. Then, there exist polynomials $\mathcal{P}_0, \mathcal{P}_1, \ldots$ such that*

$$\mathrm{KRP}_N = \mathcal{P}_0 + \sum_{j=1}^{\ell} (p^j - p^{j-1}) \mathcal{P}_j.$$

In this equation $\mathcal{P}_0, \ldots$ are Laurent polynomials in t, q such that

(P-1) The Laurent polynomial $\mathcal{P}_0$ can be presented as

$$\mathcal{P}_0 = q^{s_N(L)}(q^{1-N} + q^{3-N} + \cdots + q^{N-1}) + \sum_{j=1}^{\infty} (1 + tq^{Nj}) \mathcal{S}_{0j}(t,q),$$

while the Laurent polynomials $\mathcal{P}_k$, $k > 0$, can be presented as

$$\mathcal{P}_k = \sum_{j=1}^{\infty} (1 + tq^{Nj}) \mathcal{S}_{kj}(t,q).$$

(P-2) The Laurent polynomials $\mathcal{S}_{kj}$, $k \geq 0$, from item (P-1) have non-negative coefficients.

(P-3) The polynomials $\mathcal{P}_k$, $k \geq 0$, satisfy the following congruence relation:

$$\mathcal{P}_k(-1,q) - \mathcal{P}_{k+1}(-1,q) \equiv \mathcal{P}_k(-1,q^{-1}) - \mathcal{P}_{k+1}(-1,q^{-1}) \pmod{q^{p^{\ell-k}} - q^{-p^{\ell-k}}}.$$

Proof. For integral k, r, we have $\mathrm{KR}_N^{k,r}(L) = \mathrm{EKR}_N^{k,r}(L)$ as vector spaces. The latter have decomposition as in (5.30):

$$\mathrm{EKR}_N^{k,r}(L) = \bigoplus_{d|m} \mathrm{EKR}_N^{k,r,d}(L).$$

We have $m = p^\ell$, and we have $\mathcal{P}_j = P_{N,p^j}$ as the Poincaré polynomial of $\mathrm{EKR}_N^{\bullet,\bullet,p^j}(L)$. By the (8.4), we have

$$\mathrm{KRP}_N(L) = \sum_{j=0}^{\infty} (p^j - p^{j-1}) \mathcal{P}_j,$$

where $p^j - p^{j-1}$ is the Euler's totient function for p^j. In this equation, $\mathcal{P}_j$ is equal to the $\mathrm{KRP}_{N,d}$ in Proposition 8.7. The sum above is finite because E_1 page has modules of finite dimension over $\mathbb{C}$. Since E_1 is a finite spectral sequence that degenerates in a finite page, so the Poincaré polynomial of the page gets zero. Hence, write $\mathcal{S}_{jk} = R_k^{p^j}$ we have

$$\mathcal{P}_j = \mathrm{LeeP}_{N,p^j}(L) + \sum_{k=1}^{\infty} (1 + tq^{2Nk}) \mathcal{S}_{jk}.$$

By Proposition 8.7, we know $\mathcal{S}_{jk}$ is non-negative. The computation of ELee in Lemma 8.6, together with Lemma 8.5, gives

$$\mathrm{LeeP}_{N,p^0}(L) = q^{s_N(L)}(q^{-N+1} + q^{-N+3} + \cdots + q^{N-1}),$$

while $\mathrm{LeeP}_{N,p^j}(L) = 0$ for $j > 0$. This proves (P-1) and (P-2).

For (P-3), we use Proposition 8.14. Specifically, we have

$$(\mathcal{P}_j - \mathcal{P}_{j+1})|_{t=-1} = \mathrm{DRT}_{N,j}$$

where $\mathrm{DRT}_{N,j}$ is a difference polynomial. Proposition 8.14 implies that changing an orbit of crossings on a diagram does not affect $\mathrm{DRT}_{N,j}$ modulo the ideal generated by $q^{p^{n-j}} - q^{-p^{n-j}}$. We get a mirror of the link by changing all orbits of crossings. Since changing the orbit of crossing does not affect $\mathrm{DRT}_{N,j}$ modulo the ideal generated by $q^{p^{n-j}} - q^{-p^{n-j}}$, we stay in the same relation after the first change, i.e., changing the orbit of the first crossing. By Corollary 8.12, we get the result. □

8.5. **Periodicity 3 and 4.** Now we will show that the periodicity criteria cannot hinder a knot from being 3 or 4 periodic. We begin with the following result.

Theorem 8.18 ([7]). *If K is a knot and X is its HOMFLY-PT polynomial, then $X(a,b) = T(a,b)q(a,b) + 1$, where $q(a,b)$ is a Laurent polynomial with integer coefficients and $T(a,b) = a^4 - 2a^2 + 1 - a^2b^2$ is the HOMFLY-PT polynomial for the trefoil.*

The following result is followed by Theorem 8.18 and (8.9).

Corollary 8.19. *For a knot K, the RT_N polynomial has the form*

$$\mathrm{RT}_N(q) = A(q)(q^{4N} - 2q^{2N} + 1 - q^{2N}(q - q^{-1})^2) + 1,$$

where $A(q)$ is a Laurent polynomial with integer coefficients, and define

$$T_N = q^{4N} - 2q^{2N} + 1 - q^{2N}(q - q^{-1})^2.$$

Lemma 8.20.

- *If ζ_6 is a root of unity of order 6, then $T_N(\zeta_6) = 0$ unless $3|N$;*
- *If ζ_8 is a root of unity of order 8 and N is odd, then $T_N(\zeta_8) = 0$.*

Proof. Firstly, we prove the first part for $N \equiv 1 \pmod 3$ and $N \equiv 2 \pmod 3$.
For $N = 1$, we have

$$\begin{aligned} T_N &= q^4 - 2q^2 + 1 - q^2(q - q^{-1})^2 \\ &= (q^2 - 1)^2 - (q^3 - q)(q - q^{-1}) \\ &= (q^2 - 1)(q^2 - 1 - q^2 + 1) = 0 \end{aligned}$$

For $N \equiv 1 \pmod 3$, we write $N = 3k + 1$. We have

$$\begin{aligned} T_N &= q^{4(3k+1)} - 2q^{2(3k+1)} + 1 - q^{2(3k+1)}(q - q^{-1})^2 \\ &= q^{12k}q^4 - 2q^{6k}q^2 + 1 - q^{6k}q^2(q - q^{-1})^2 \end{aligned}$$

Since $(\zeta_6)^6 = 0$, we have

$$T_N(\zeta_6) = (\zeta_6)^4 - 2(\zeta_6)^2 + 1 - (\zeta_6)^2((\zeta_6) - (\zeta_6)^{-1})^2 = 0$$

For $N = 2$, we have

$$T_N = q^8 - 2q^4 + 1 - q^4(q - q^{-1})^2$$

Since $(\zeta_6)^6 = 0$, we have

$$\begin{aligned} T_N(\zeta_6) &= (\zeta_6)^8 - 2(\zeta_6)^4 + 1 - (\zeta_6)^4((\zeta_6) - (\zeta_6)^{-1})^2 \\ &= (\zeta_6)^2 - 2(\zeta_6)^4 + 1 - (\zeta_6)^4((\zeta_6)^2 - 2 + (\zeta_6)^{-2}) \\ &= (\zeta_6)^2 - 2(\zeta_6)^4 + 1 - 1 + 2(\zeta_6)^4 - (\zeta_6)^2 = 0 \end{aligned}$$

For $N \equiv 2 \pmod 3$, we write $N = 3k+2$. We have

$$T_N(\zeta_6) = (\zeta_6)^{4(3k+2)} - 2(\zeta_6)^{2(3k+2)} + 1 - (\zeta_6)^{2(3k+2)}((\zeta_6) - (\zeta_6)^{-1})^2$$
$$= (\zeta_6)^{12k}(\zeta_6)^8 - 2(\zeta_6)^{6k}(\zeta_6)^4 + 1 - (\zeta_6)^{6k}(\zeta_6)^4((\zeta_6) - (\zeta_6)^{-1})^2$$

Since $(\zeta_6)^6 = 0$, we have

$$= (\zeta_6)^8 - 2(\zeta_6)^4 + 1 - (\zeta_6)^4((\zeta_6) - (\zeta_6)^{-1})^2 = 0 .$$

For the second part, we do the same. Firstly, we show for $N = 1$ and later for $N = 2k+1$. □

Corollary 8.21. *For any knot K, we have the congruences* $\mathrm{RT}_N(q) - \mathrm{RT}_N(q^{-1}) \equiv 0 \bmod q^3 - q^{-3}$, $\mathrm{RT}_N(q) - \mathrm{RT}_N(q^{-1}) \equiv 0 \bmod q^4 - q^{-4}$.

Proof. We start with the first part. This congruence is the same as saying that for any root of unity ζ_6 of order 6, it satisfies $\mathrm{RT}_N(\zeta_6) - \mathrm{RT}_N(\zeta_6^{-1}) = 0$. We have two cases here. The first case, suppose that N is not a multiple of 3. By Corollary 8.19 and Lemma 8.20, we have $\mathrm{RT}_N(\zeta_6) = 1$, so $\mathrm{RT}_N(\zeta_6) - \mathrm{RT}_N(\zeta_6^{-1}) = 0$. The second case, suppose $3|N$. We write the Khovanov-Rozansky polynomial as follows, see Proposition 8.7.

$$\mathrm{KRP}_N(t,q) = q^s(q^{1-N} + q^{3-N} + \cdots + q^{N-1}) + \sum_j (1 + tq^{2Nj}) R_j(t,q).$$

and we have $\mathrm{RT}_N(q) = \mathrm{KRP}_N(-1,q)$. For the term $(1 + tq^{2Nj})$ for $t = -1$ and $q = \zeta_6$ is equal to zero because $(\zeta_6)^6 = 0$. At the same time, we have

$$q^{1-N} + q^{3-N} + \cdots + q^{N-1} = \frac{q^N - q^{-N}}{q - q^{-1}}.$$

The latter expression is zero when evaluated at a root of unity of order dividing $2N$. That is to say

$$\mathrm{RT}_N(\zeta_6) = \mathrm{KRP}_N(-1,\zeta_6) = 0.$$

For the second part, first assume that N is odd. Then, $\mathrm{RT}_N(\zeta_8) = 1$ by the same argument combining. Again, we have $\mathrm{RT}_N(\zeta_6) = 1$ and

$$\mathrm{RT}_N(\zeta_6) = \mathrm{KRP}_N(-1,\zeta_6) = 0.$$

Now assume N is even. Assume that $4|N$ then as the same argument above we have

$$\mathrm{RT}_N(\zeta_8) = \mathrm{KRP}_N(-1,\zeta_8) = 0.$$

We have only one case, namely when $N = 4k+2$. Assume we split this case into two cases. For some k we can write $N = 4k+2 = 8m+2$, and for some k we can write $N = 4k+2 = 8m-2$. For $N = 8k+2$, take ζ_8 such that $\zeta_8^4 = 1$. From the formula of HOMFLYPT polynomial $X(a,b)$ we have $\mathrm{RT}_N(q) = X(q^N, q - q^{-1})$. Since $\zeta_8^4 = 1$, $\mathrm{RT}_N(\zeta_8) = X(\zeta_8^{4k+2}, \zeta_8 - \zeta_8^{-1}) = X(\zeta_8^2, \zeta_8 - \zeta_8^{-1}) = \mathrm{RT}_2(\zeta_8)$.

Now, RT_2 is the Jones polynomial. It was proved in [15, Section 4.6] that $\mathrm{RT}_2(\zeta_8) - \mathrm{RT}_2(\zeta_8^{-1}) = 0$. The same proof is valid for when $\zeta_8^4 = -1$. The remaining case is when $N = 8k-2$ and $\zeta_8^4 = -1$. Write $X(a,b) = \sum \alpha_{ij} a^i b^j$. Since $\mathrm{RT}_N(q) = X(q^N, q - q^{-1})$, we have

$$\mathrm{RT}_N(\zeta_8) - \mathrm{RT}_N(\zeta_8^{-1}) = \sum \alpha_{ij} \zeta_8^{Ni}(\zeta_8 - \zeta_8^{-1})^j - \zeta_8^{-Ni}(-\zeta_8 + \zeta_8^{-1})^j =$$
$$\sum \alpha_{ij}(\zeta_8^{-2i} - \zeta_8^{2i})(\zeta_8 - \zeta_8^{-1})^j = -\mathrm{RT}_2(\zeta_8) + \mathrm{RT}_2(\zeta_8^{-1}).$$

After all, RT_2 is the Jones polynomial. It was proved in [15, Section 4.6] that $\mathrm{RT}_2(\zeta_8) - \mathrm{RT}_2(\zeta_8^{-1}) = 0$. The same proof is valid for when $\zeta_8^4 = 1$. □

Corollary 8.22. *Assume K is a knot. Set $\mathcal{P}_0 = \mathrm{KRP}_N$, $\mathcal{S}_{0j} = R_j$, where R_j is as in Proposition 8.7. Then $\mathcal{S}_{0j}, \mathcal{P}_0$ satisfy the statement of Theorem 8.17 regardless of whether or not K is 3 or 4-periodic.*

Proof. We prove this corollary just for 3-periodic knots. The proof for 4-periodic knots is similar. Item (P-1) is satisfied by definition. By Proposition 8.7, $\mathcal{S}_{0j}$ has non-negative coefficients. The congruence (P-3) is a direct consequence of Corollary 8.21. $\square$

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alphabetic order!

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H. Wu : good

Charles A. Weibel : bad