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Report on the thesis of Patryk Jaśniewski, entitled :
Algebraic and homological properties of strict polynomial functors

General context. The thesis of Patryk Jaśniewski deals with the study of modules over Schur algebras, a classical subject which is related to many important topics such as algebraic geometry, representation theory, algebraic topology and K-theory.

The Schur algebras $S(n,d)$ were introduced over a field k of characteristic zero in the work of I. Schur at the begining of the 20th century. In modern formulation, his work shows that the category of $S(n,d)$-modules is equivalent to the category of d-homogeneous polynomial representations of $GL_n(k)$, and provides a complete description of this category : every $S(n,d)$-module is semi-simple, the simple modules are classified by partitions of d in at most n parts, and Schur gives a concrete construction of these simple modules.

In the 1980es there was a new start of the subject by considering the situation of a ground field k of prime characteristic, with major contributions of Green, Akin-Buchsbaum-Weyman, Donkin, Cline-Parshall-Scott (among others). In prime characteristic p, new phenomena arise :

1. Simple $S(n,d)$-modules are not completely understood. They are classified by partitions of d in at most n parts, they can be constructed as submodules of certain 'costandard modules' (which are very close to the simple modules of characteristic zero), but the complexity of these costandard modules is too important and a clear description of simple modules is out of reach. To illustrate this point, let me mention that a formula giving the dimension of simple modules is out of reach for the moment (though much progress has been made in the recent years with the work of S. Riche and G. Williamson, using methods of geometric representation theory).
2. $S(n,d)$-modules can be glued together in nontrivial ways to produce new modules. This phenomenon gives rise to (and is measured by) certain Ext-groups which are nontrivial to compute.

These two phenomenas are not completely independent – in particular, some work of Lusztig have shown some links between them. Understanding both phenomena are very challenging problems.

A new impulse in the study of the second phenomenon was given at the end of the 1990es by the celebrated work of Friedlander and Suslin on finite group schemes. They showed that modules over Schur algebras can be constructed in a functorial way in any characteristic (in characteristic zero this was already contained in the pioneering work of I. Schur). The functors appearing there in are called **strict polynomial functors**, and the category $\mathcal{P}_d$ of homogeneous strict polynomial functors of degree d is equivalent to $S(n,d)-\mathrm{Mod}$ for $n \geq d$. Working with $\mathcal{P}_d$ rather than with $S(n,d)$-modules has some advantages, a major one being the possibility of composing functors, an operation which is not available in $S(n,d)$-modules. Subsequent works of Franjou-Friedlander-Scorichenko-Suslin, Chałupnik, Touzé. . .have made full use of these advantages and greatly improved our understanding of Ext-groups in $\mathcal{P}_d$. However we are still very far from a complete understanding of Ext-groups in $\mathcal{P}_d$.

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Overview of the results of the thesis. The thesis of Patryck Jaśniewski undertakes a systematic study of the categories of $S(n,d)$-modules over a field of prime characteristic p for low values of d, namely $d \leq 2p$, in particular of its homological aspects (Ext-groups). Until now, Ext-groups were only computed between very specific functors (typically exterior and symmetric powers) in these low degrees, and no systematic computation had been obtained before. Chapter 2 and 3 of the thesis yield :

(a) A complete computation of $\mathrm{Ext}^*_{\mathcal{P}_d}(F,G)$ when $d \leq 2p$ and F and G are costandard. These results are given in Theorems 2.3.2, 2.4.1 and 3.4.2.

(b) A complete computation of $\mathrm{Ext}^*_{\mathcal{P}_d}(F,G)$ when $d < 2p$ and F and G are simple. These results are given in Theorems 2.4.1 and 2.5.2

It should be emphasized that when $d < 2p$, the results do not only provide formulas for the dimensions of Ext-groups, but also decribe the Yoneda products, and also give a formality result which has a important impact on the structure of the derived category.

As explained before computations of Ext in $\mathcal{P}_d$ yield computations in the equivalent categories $S(n,d)$-Mod, for $n \geq d$. The last chapter of the thesis deals with "unstable computations", i.e. when $n < d$. This thesis gives

(c) Complete Ext computations involving costandard and simple objects in $S(n,p)$-Mod for $n < p$. These computations are presented in Thm 4.2.1 and 4.2.2.

Here again, the results describe Yoneda products, as well as a formality result related to the derived category. There is also a computation of the derived functor of the Schur-Weyl duality functor (in corollary 4.2.6) which plays a prominent role in comparing homological computations for $S(n,p)$ and for the symmetric group $\mathfrak{S}_p$. A surprising side corollary 4.2.7 shows that the derived functor of Schur-Weyl duality functor induces an isomorphism on the level of K_0.

All these results are clearly new and interesting, and give a clear and rather complete portrayal of the homological properties of $S(n,d)$-modules in degrees $d \leq 2p$. The results form a very useful complement to the calculations previously obtained by Franjou-Friedlander-Scorichenko-Suslin, Chalupnik, and Touzé, which rather focused on understanding Ext-groups between specific families of functors (symmetric powers, exterior powers, twisted functors. . .) in all degrees d.

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Comments on some overcome difficulties and some techniques used. Let me stress out some remarkable aspects of the work accomplished in this thesis. Given two partitions of degree $d \leq 2p$, there is no doubt that the very experts of the domain are able to compute (probably by a long and intricated computation) the dimensions of the Ext-groups in $\mathcal{P}_d$ between say, the associated Schur functors. The thesis of Patryk Jaśniewski solves a significantly more difficult problem, namely the problem of performing *all* the computations simultaneously, and of expressing the results in an intelligible way.

As regards the formulation of the results, I can say that theorems 3.2.2, 3.2.5 and 4.2.2 give a beautifully compact and complete description of Yoneda Ext-algebras and of the derived category in degree $d = p$. As these results are exhaustive and have a clear and simple expression, there is no doubt that they can be used by others as a basis of further investigations in the future.

In order to obtain his results, Patryk Jaśniewski has to use and develop various techniques, including :

(i) *Combinatorics of partitions.* Simple functors and costandard functors are classified by partitions. It is well-known the blocks of the category $\mathcal{P}_d$ are known : two simple/costandard modules lie in the same block if and only if they have the same p-core. In particular if $d = p$ the only nontrivial block is the principal block, which corresponds to partition with a trivial p-core, which correspond only to hooks. But if $d > p$ some nontrivial calculation is needed to determine the nontrivial blocks and corresponding partitions (this is explained in section 3.1).

(ii) *Explicit resolutions.* Costandard functors associated to hooks (and some related functors) have explicit resolutions provided by certain Koszul and De Rham complexes. These resolutions date back from the works of Franjou-Lannes-Schwartz and Friedlander-Suslin at the end of the nineties, and they are fully exploited in this thesis. In particular they are used to describe the DG-algebra governing the complexes in the principal block (as far as I know, this is a new use of these complexes), and prove its formality.

Additional explicit complexes are used in the computations for $d = 2p$, namely the short exact sequences (3.5) and (3.6). These are constructed by Patryck Jaśniewski by analyzing the Littlewood Richardson filtration through the lens of the block decomposition of $\mathcal{P}_d$, and I think this is their first appearance in the literature on Ext-computations.

(iii) *Sum-diagonal adjunction and the decomposition formula.* This is a very general technique of computation for Ext in $\mathcal{P}_d$. It ultimately relies on the operation of composition of functors, hence it is not available in the categories of $S(n, d)$-modules. This technique is one of the classical techniques of computations in $\mathcal{P}_d$ and justifies to perform computations with functors rather than with modules over Schur algebras.

(iv) *Translation functors.* In theorem 2.4.1, Patryck Jaśniewski proves that there are some 'translation functors', which provide an equivalence of abelian categories between the principal block of $\mathcal{P}_p$ and any block corresponding to a partition of p-weight 1 in $\mathcal{P}_d$ for $d > p$. This is a nice and very general result, which shows that the computations performed in degree $d = p$ performed in this thesis actually also cover many computations in higher degrees.

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Overall appreciation of the thesis. As it should be clear from the previous paragraphs, the results of the thesis are very valuable. Although many results have a rather simple statement, the proofs are highly nontrivial and require a combination of a range of advanced techniques.

Let me also emphasize here that the thesis is well written. This point is crucial given the combinatorial complexity of the calculation performed.

As a consequence, there is no doubt that the thesis of Patryck Jaśniewski is more than sufficient to grant a PhD.

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