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Report on the Doctoral Thesis of Maja Szlenk

The mathematical analysis of fluids has always been very challenging, with many fundamental problems still wide open. A main issue is the possibility of turbulent behaviour, which requires the study of weak solutions. For the incompressible Navier-Stokes equations, which model viscous incompressble fluids, it has long been known through the work of Jean Leray that weak solutions exist in the three-dimensional space for any initial datum with finite energy. The compressible case is much harder and was solved only decades later by P.-L. Lions and Feireisl. Another more recent landmark result is the one by Vasseur and Yu on the case of degenerate viscosity, based on prior techniques of Bresch-Desjardins and Mellet-Vasseur which are also used in Maja Szlenk's thesis.

Ms Szlenk considers in her thesis three problems of existence of weak solutions in compressible fluid models: The compressible Stokes system with Newtonian stress and very general pressure (Chapter 2); a non-Newtonian compressible Stokes system (Chapter 3); and a pressureless degenerate compressible Navier-Stokes system with a nonlocal interaction (Chapter 4). In an introductory chapter, Ms Szlenk gives a very readable overview of the literature and of her own achievements. She clearly and concisely summarises her original contributions and the methods to achieve them.

Chapter 2 on the compressible Stokes system on the torus has been published by Ms Szlenk in sole authorship in the highly reputable journal *J. Differential Equations*. The chapter gives a nice result on global existence and uniqueness for the compressible Stokes system, which is derived from the standard compressible Navier-Stokes equations by neglecting intertia. One arrives at the problem given by

$$\begin{aligned}\partial_t\rho + \operatorname{div}(\rho u) &= 0,\\ \operatorname{div} S(\nabla u) + \nabla p(\rho) &= 0,\end{aligned}$$

where the stress tensor is chosen linear. The system has a semi-stationary character, because the velocity can be recovered from the density, whose evolution therefore determines the full dynamics. This is seen in the active scalar reformulation (2.2) in the thesis. Ms Szlenk choo-

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ses a Lagrangian approach, which allows her to achieve stronger results as previously known in the literature. More precisely, she shows existence and uniqueness of solutions (ρ, u) which are bounded, and whose velocity is 'almost Lipschitz' (that is, $\nabla u \in BMO$) uniformly in time. The pressure law, remarkably, can be very general and only needs to satisfy condition (2.3), which includes a vast class of relevant (also non-monotone) pressures. Apparently the uniqueness part is completely new. The proof relies on the transformation between Lagrangian and Eulerian coordinates. However, due to the usual breakdown of Calder'on-Zygmund theory in L^∞, one cannot work in the classical Lipschitz framework, but has to use more sophisticated techniques from renormalisation theory (DiPerna-Lions, Crippa-De Lellis), and some harmonic analysis ingredients such as the Mucha-Rusin inequality. The method is quite appealing and nicely employs several tools from recent literature.

In Chapter 3, again a Stokes system is considered, but now for certain non-Newtonian fluids. In joint work with Milan Pokorný (published in *Math. Methods Appl. Sci.*), Ms Szlenk has achieved an existence result for the Stokes system with velocity-dependent, monotonic shear and bulk viscosities; specifically, the shear viscosity takes the form $\mu = \mu_0(|D_{sym}u|) + 2\mu_1$ and the bulk viscosity $\lambda = \lambda(|\operatorname{div} u|)$, and a crucial assumption is $0 \leq \mu_0, \lambda \leq C/\cdot$. I found it a bit confusing that, after the statement of these assumptions on p. 30 of the thesis, a discussion is made about power-law and Herschel-Bulkley fluids, although except in the special case $r = 1$ or $n = 1$, respectively, such fluids do not satisfy the assumptions made. Anyway, the chapter follows previous work of Feireisl-Liao-Málek, but in the thesis the assumptions on the viscosity functions are weaker and do, in particular, not ensure an L^∞ a priori bound on the divergence of the velocity field. Instead, a BMO bound is achieved for the effective viscous flux (Lemma 3.3), which can later be handled by another use of the Mucha-Rusin inequality and Osgood's Lemma.

Chapter 4 is the most extensive and difficult part of the thesis, and has meanwhile been uploaded to ArXiv jointly with Piotr Mucha and Ewelina Zatorska. It contains a detailed proof of existence of weak solutions to a degenerate compressible Navier-Stokes system with a nonlocal attraction-repulsion term of the form $\rho\nabla(K * \rho)$, where

$$K(x) = \frac{c_1}{|x|^\alpha} + \frac{c_2}{2}|x|^2.$$

The construction proceeds by regularization as in (4.11) with more than ten parameters (the nice table 4.1 doesn't even contain all of them!) that are successively, and in the appropriate order, sent to zero. Along the way, the observations of Bresch-Desjardins and Mellet-Vasseur on additional entropies are employed, where some of the regularization terms are needed to guarantee the appropriate renormalization. This construction, as a positive side-effect, leads to weak solutions which additionally satisfy the Bresch-Desjardins and Mellet-Vasseur estimates. While the mere mastery of these recent techniques is already an intellectual achievement, Ms Szlenk and her coauthors have to go even further and find a way to include the nonlocality in the Mellet-Vasseur estimate (this is then the second term in (4.10)). This is carried out in Section 4.4. The technical effort taken in this chapter is considerable and adds up to almost 50 pages. Therefore I understand that some details (such as why it suffices to consider continuity only at $t = 0$ in the proof of Proposition 4.16) are omitted, although they might have been helpful for the reader.

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Maja Szlenk has strongly impressed me with her thesis. As a result of her doctoral studies, she has produced three results of very high scientific quality: All the problems considered are relevant, difficult, and require a thorough mastery of some of the deepest state-of-art techniques, as well as clever novel ingredients. Let me emphasise that all three results are of different flavour and require different approaches, so that Ms Szlenk has already reached substantial breadth in her knowledge. She can now be considered an expert in a broad range of compactness and transport methods for nonlinear PDE.

In summary, this is a well-written thesis which contains substantial scientific progress via sophisticated methods and novel ideas. The level of technical difficulty, particularly in the last chapter, is extremely high. I therefore recommend to **accept this thesis and award Maja Szlenk a distinction**.

Kind regards,

Emil Wiedemann

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