Report on the PhD Dissertation of Maciej Markiewicz

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This is a very substantial thesis making advances on a rather classical subject in the theory of links in the 3-sphere. The overall subject is the link signature, thought of as a function of many variables (as many as the number of link components). The main results relate the limit of the link signature as one variable approaches $1 \in S^1 \subset \mathbb{C}$ to the signature function on the link with the the corresponding component removed. This is a very natural thing to do, because simply plugging in 1 for that variable usually just gives 0 as an answer.

I was surprised to realize that this limiting behavior hadn't been fully considered in the (substantial) literature in this area. Indeed, the work of Torres from more than 70 years ago considers the relationship between the multivariable Alexander polynomial of a link and that of its sublinks. The signature function (and its related nullity) pose new and interesting challenges, which have been well considered in the thesis.

In many applications of knot and link signatures, a key role is played by interpreting the link signature (as defined, say, via a Seifert matrix) in 4-dimensional terms. The traditional approach to this makes use of a bounding 4-manifold over which the twisting character extends. This is very effective, but has the defect that the (twisted) signature of the 4-manifold typically used coincides with the link signature only when the twisting character is non-trivial for all components. The candidate resolves this by finding a new construction for a bounding manifold whose twisted signature coincides with the link signature for all characters. This is a potentially very useful contribution of the thesis.

The first two chapters of the thesis are introductory in nature, with the first being an outline of results, and the second being a discussion of underlying ideas and constructions. The main contributions of the thesis are in the next two chapters. Chapter 3 treats the signature and nullity from the 3-dimensional point of view, and chapter 4 gives the 4-dimensional interpretation and applications to concordance.

The material in chapter 3 makes use of the calculation of the multivariable Alexander polynomial and signature in terms of a C-complex. This is done with great care, and leads to the understanding of the limiting behavior as some (taken to be the first) twisting variable tends to 1. The answer is conceptually satisfying, in that the limit is a link homotopy invariant, in fact determined by the linking numbers between the first component and the other components. The proof of this fact is fairly straightforward, but in addition a rather complicated explicit formula is given, and some examples are computed.

Chapter 4 presents a 4-dimensional interpretation of the multivariable signature function, greatly generalizing the classic one variable case originating in work of Viro and Kauffman-Taylor in the 1970s. The main construction is that of a certain bounding manifold coming from a singular surface (with embedded components) spanning the link. The signature and nullity of the twisted intersection pairing on this bounding manifold are computed, and shown to coincide with the corresponding notions for the link away from the locus where some twisting variable is equal to 1. The advantage of the 4-dimensional viewpoint is that the signature function extends naturally over that locus, and can be non-zero. This is potentially very useful in applications.

The last portion of chapter 4 contains a natural application of the theory to link concordance. It shows that away from potential roots of the Alexander polynomial, the signature gives a link concordance invariant.

The thesis is itself very well written, with clear and detailed proofs. I did not find any mathematical errors, although I note a few small points of writing below the signature line. I do not need to see these corrections in a final version. The work is original and substantial, and would be an excellent PhD thesis at any University. I deem the thesis as sufficient to grant a PhD.

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Notes for the author:

Page 3, introduction Should specify that all links are to be considered as oriented links, right at the beginning. This is implied, but should be said explicitly.

Page 3, line 3 useful tool of knot theory → useful tool *in* knot theory

Page 3, line 5 discussion of properties of it → discussion of *its* properties

Page 3, Theorem 1.1 The theorem, or perhaps the following text, should state that the correction terms depend on linking numbers of L_1 with the other components.

Page 10, Definition Should indicate here, or perhaps earlier, that Z^μ is considered to have a fixed basis.

Page 20, line 16 is coherent → is coherently

Page 24, line 19 such that $am_1 + bl_1$ any element of the kernel. → such that $am_1 + bl_1$ *is an* element of the kernel.