Report on the doctoral thesis
Corson-like compacta and related function spaces
by Krzysztof Zakrzewski, University of Warsaw

The main part of the thesis is focused on Corson-like compact spaces, more precisely to κ-Corson compact spaces for uncountable regular κ, ω-Corson compact spaces and NY compact spaces. The main results are contained in Chapters 2 and 3 and are taken from two recent papers – [Za] (authored by the applicant and published in Results in Mathematics) and [MPZ] (co-authored by the applicant, his supervisor and G. Plebanek and accepted in Israel Journal of Mathematics). The thesis additionally contains an introductory chapter (Chapter 1) and Chapter 4 with some abstract results on topological vector spaces, locally convex spaces and operators between such spaces.

Chapter 1 contains an overview of the topic and some terminology. The terminology is more or less standard. I have only one comment on 'generating topologies':

- In Section 1.2 the weak* topology is defined as the topology generated by a certain family of sets. This family is a subbase.
- In Section 1.3 the topology of pointwise convergence is defined as the topology generated by a certain family of sets. This family is in this case a base.
- The compact-open topology is defined explicitly using a subbase.

All these definitions are correct, but the presentation is not very consistent. Why to use once a subbase, once a base (without explicitly mentioning it) and once a subbase (with explicitly pointing it out)?

In **Chapter 2** definitions of κ-Corson compact spaces and NY compact spaces are given, together with some characterizations and basic results. The key sections (2.3–2.5) are rather nice and well presented (except for two details mentioned below). Some parts of Sections 2.1 and 2.2 are hard to read or not completely correct. I continue with comments on Chapter 2:

- The proof of assertion (b) of Remark 2.1.1 is probably correct, but hard to read and overcomplicated. An easier approach giving a bit stronger statement reads as follows:

  Assume that K is κ-Corson. If $w(K) < \kappa$, then K may be embedded to $[0,1]^{w(K)} = \Sigma_\kappa([0,1]^{w(K)})$ and hence the assertion holds. Next assume that $w(K) \geq \kappa$. Consider $K \subset \Sigma_\kappa([0,1]^\Gamma)$ (necessarily $|\Gamma| \geq w(K) \geq \kappa$). Let $(x_\alpha)_{\alpha<w(K)}$ be a dense subset of K. Let $\Gamma' = \bigcup_{\alpha<w(K)} \operatorname{supp} x_\alpha$. Each support has cardinality $< \kappa \leq w(K)$. Thus $|\Gamma'| \leq w(K)$. Further, K may be clearly embedded to $\Sigma_\kappa([0,1]^{\Gamma'})$.

  Note that for $w(K) \geq \kappa$ we may use density (more precisely $\max\{\kappa, \operatorname{dens} K\}$) instead of weight.
- Remark 2.2.4 is partially false. Note that in assertions (i) – (iv) only the space K appears and not the Boolean algebra $\mathfrak{B}$. Therefore the assumption $K = \operatorname{ult}(\mathfrak{B})$ for a Boolean algebra $\mathfrak{B}$ means just that K is zero-dimensional. Moreover, the Cantor space $\{0,1\}^\omega$ is zero-dimensional, ω-Corson but not scattered. Hence (i) is satisfied, but (iii) is not satisfied. This remark is copied from [MPZ, Remark 3.5] which is equally false.

  Perhaps assertion (i) should rather say that the Boolean algebra $\mathfrak{B}$ is ω-Corson. At least Lemma 2.1.3 and the following remark suggests it. Note that in [MPZ] equivalence $(i) \Leftrightarrow (ii)$ is 'proved' by referring to [MPZ, Lemma 2.3], where the assumption that κ is uncountable is missing (in the original source [BKT, Corollary 3.1] it is present, as well as in Lemma 2.1.3 of the thesis).

  Assume that (i) is changed to say that the Boolean algebra $\mathfrak{B}$ is ω-Corson. Then the claim that $(i) \Leftrightarrow (ii)$ follows from the very definition is almost true. More precisely, (i) then means that there is a point-finite family of clopen sets generating whole algebra of clopen sets. Hence $(i) \Rightarrow (ii)$ is really obvious, but the converse requires a small argument. Further, implications

$(ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (iii)$ are explained in the text, $(iii) \Rightarrow (iv)$ is known to hold always. But an argument for $(iii) \Rightarrow (ii)$ is missing.

- In the proof of Proposition 2.2.8, in the formula for ψ some cases are missing ($x \in K_t$, $t \in T$, $\gamma \in T \setminus \{t\}$ or $\gamma \in \Gamma_s$ for some $s \neq t$).
- In Theorem 2.5.5(iii), the assumption that K is compact is probably missing.
- In the proof of $(iii) \Rightarrow (iv)$ of Theorem 2.5.5 the argument is a bit strange. More precisely, $\mathfrak{B}$ is κ-Corson by the very definition. Lemma 2.1.3 then yields that ult($\mathfrak{B}$) is κ-Corson.

Chapter 3 is devoted to function spaces on κ-Corson or NY compact spaces. The results are taken mainly from [Za]. This chapter is well worked out, the results are nontrivial, proofs are correct and understandable. I have only two minor comments:

- The title of the chapter should rather read 'Function spaces on Corson-like compacta' ('on' is missing).
- The references used to prove Corollary 3.5.4 are not exact. Instead of Corollary 2.5.6 one should invoke [MPZ, Theorem 12.4(c)] which is a known fact. Alternatively, one may complete Corollary 2.5.6 by Theorem 2.5.5 (implication $(i) \Rightarrow (iv)$). But it would be rather strange because Corollary 2.5.6 follows from Theorem 2.5.5 using [MPZ, Theorem 12.4(c)].

Chapter 4 is named 'Miscellaneous results' and contains a collection of rather unrelated abstract results of varying difficulty, importance and quality of proofs. Most of the questions addressed in this chapter are somehow natural, but I see no relation to Corson-like compacta and related function spaces. Therefore I see no reason why to include this chapter to the thesis. Anyway I include comments to this chapter:

- When considering spaces with several topologies, one should be careful with the terminology. In particular, I assume that 'weakly continuous' means 'weak-to-weak continuous', weakly open means 'weak-to-weak open' and similarly for the weak* topologies.
- Proof of Theorem 4.1.4 is a bit strange and unnecessarily complicated. In particular, it is not clear why just weakly open subbasic sets are considered. It would be easier to do the proof in two steps:

  Step 1: T is additionally one-to-one. Then the assertion follows from Lemma 4.1.1 applied to T^{-1} restricted to the range of T (using the paragraph after Lemma 4.1.2).

  Step 2: T not one-to-one. Consider the factorization T' as in the text. Then T' is one-to-one and satisfies the assumptions of Theorem 4.1.4. Indeed, if $U \subset X/\ker(T)$ is open, then $T'(U) = T'(\pi(\pi^{-1}(U))) = T(\pi^{-1}(U))$, so it is open in the range of T (which is the same as the range of T'). Thus T' is weakly open by Step 1, hence T is weakly open by the argument in the text.
- At the end of the proof of Lemma 4.1.6, in the formula $\psi \in \dots$, V_1 should be replaced by $V_1 \cap U_1$, similarly for V_{n-1}.
- Theorem 4.1.7 is probably true, but its proof is not correct. Firstly, Y should be 'locally convex', not 'convex'. (The same applies to Theorem 4.1.8.) Further, in the statement there are assertions $(i), (ii)$, but in the proof they are referred as $a), b)$. But there are also mathematical problems:

  In the proof of $(i) \Rightarrow (ii)$ it is correctly proved that $T^*(V)$ is weak*-open in the range for any subbasic weak* open set V. But this is not enough, it should be proved for each basic set. (Note that for the proof of continuity it is enough to take subbasic sets, but for openness we need to consider basic sets.) I think it may be done by a modification of the current proof, using Lemma 4.1.6.

  In the proof of $(ii) \Rightarrow (i)$ the set V is not subbasic (in the standard sense) but basic. Moreover, Lemma 4.1.6 says that $x_1, \dots, x_n$ may be linearly independent, not necessarily $Tx_1, \dots, Tx_n$. This gap may be repaired if Y is Hausdorff, as in this case the linear span of $Tx_1, \dots, Tx_n$ is closed and does not contain y. Thus ϕ may be found due to non-emptyness of V.

- In the paragraph before Proposition 4.1.9 one should rather write 'we may drop' than 'we need to drop'.
- The proof of Proposition 4.1.9 is a bit misleading. Firstly, [Sc, Theorem IV.7.3] (note that the reference is not exact, this applies also to other places where [Sc] is referred to) should be applied to T, not to $\widetilde{T}$. Then one should use that $\widetilde{T}^* = T^* \circ i^*$ and that i^* is a surjective isometry, thus ranges of T^* and $\widetilde{T}^*$ coincide.
- In the proof of Lemma 4.2.2 (by the way, not marked as a proof) one should say that x_0 is a fixed point from X.
- In the proof of Lemma 4.2.3 the described sets are not **all** neighborhoods of a, but basic ones.

Overall evaluation: The thesis presents nontrivial new results on the border of general topology and functional analysis, based mainly on two recent papers authored or co-authored by the applicant. The core of the thesis is contained in Chapters 2 and 3 which are essentially well written, with some exceptions mentioned above. These exceptions apply mainly to Chapter 2. The mistakes were copied from the paper [MPZ] co-authored by the applicant, which serves as the source for Chapter 2. But these mistakes are easy to correct. The quality of the thesis is a bit decreased by Chapter 4, which seems to be rather an artificially added appendix and some proofs are not completely correct (see the above comments). Anyway, in my opinion the thesis fulfills conditions of a doctoral dissertation (Chapters 2 and 3 provide sufficient value) and should be accepted.

Prague, August 27, 2025

Ondřej Kalenda