Review of the doctoral dissertation *Julia sets in random holomorphic dynamics* by Krzysztof Lech.

The dissertation is dedicated to random dynamics of two classes of maps: quadratic polynomials and exponential functions. It is mostly based on two successful papers ([20] and [21] in the dissertation bibliography) written by the author in collaboration with specialists in the respective areas and published in sufficiently prestigious journals. The results are quite complex and very interesting. They contain, in particular, some new surprising observations differentiating random dynamics of holomorphic maps from the regular one. The work is based on several complex theories, but also introduces some new methods and techniques. Finally, the dissertation text itself is well written and easy to follow. Based on the quality of the dissertation, I strongly recommend to award Krzysztof Lech the doctoral degree. Below, let me elaborate on the points listed above.

The main object of the dissertation is the Julia set of a random sequence of holomorphic maps from certain parameter spaces. The definition of a Julia set for non-atonomous systems is very similar to the regular one. However, as the author explains, there are many differences between the non-atonomous and classical dynamics of holomorphic maps already at the very basic level. They include

- absence of a reasonable notion of periodic points;
- existence of simple example of empty Julia sets;
- absence of invariance of the Fatou and Julia sets.

These differences make proofs of analogous statements in non-autonomous holomorphic dynamics, in general, more complicated than in the autonomous one.

Random polynomial dynamics was studied by many authors including R. Brück, M. Büger, M. Comerford, V. Mayer, S. Reitzwhich, M. Urbański. One of the main results of the dissertation answers a question by Brück, Büger and Reitzwhich which was open for more than 20 years. Denote $f_c(z) = z^2 + c$. Given a sequence $\omega = (c_0, c_1, c_2, \dots)$ of complex numbers let f_ω stand for the sequence of iterations $f_\omega^n = f_{c_{n-1}} \circ f_{c_{n-2}} \cdots \circ f_{c_0}$. Given $R > 1/4$ Brück, Büger and Reitzwhich asked whether the Julia set of f_ω is totally disconnected

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for almost all values of $\omega \in \mathbb{D}(0,R)^{\mathbb{N}}$ with respect to the product of Lebesgue measures. The author of the dissertation gives a positive answer and even proves a much more general result. Namely, he shows that the set V from which ω is chosen does not need to be a ball and it is sufficient that $V \supsetneq \mathbb{D}(0,1/4)$. Moreover, the measure defining randomness does not need to be Lebesgue, but can be a sufficiently general Borel probability measure.

The above described result shows a new and rather surprising difference between the random dynamics and the regular one. Namely, if B is the main cardioid of the Mandelbrot set, then for almost all sequences $\omega = (c_0, c_1, c_2, \ldots) \in B^{\mathbb{N}}$ the Julia set J_ω of the sequence of maps f_ω is totally disconnected. This result is in a striking contrast with the autonomous situation, in which *for every* $c \in B$ the Julia set J_c of $f_c(z)$ is connected. Another new phenomenon obtained in the dissertation is existence of simple sequences of exponential maps for which all singular orbits converge to infinity while the Fatou set of the non-autonomous system is non-empty. The latter is impossible for the dynamics of a single exponential map. Finally, the dissertation provides examples of sequences $\lambda_n > 1/e$ convergent to $1/e$ for which the Julia sets of the compositions of the sequence of maps $\lambda_n e^z$ are the whole plane. This is an interesting result since for the limiting value $\lambda = 1/e$ the Julia set of the map λe^z has empty interior.

In addition to classical Complex Dynamics and Probability Theory, to obtain some of the results the author uses a version of Potential Theory for non-atonomous systems (non-atonomous version of transfer operator studied by Brück in [7] and non-atonomous equilibrium measures). One of important tools the author uses is the generalization of Green function for non-autonomous dynamics from [17], which allows to describe the basin of infinity $\mathcal{A}_\omega$ of f_ω as a regular set of logarithmic potential theory. The doctorant describes the asymptotical behavior of the Green function g_ω of f_ω and uses it to study the escape rate of the critical point zero of f_ω. He then proves a series of neat technical estimates on the rate of escape of the critical point which allow him to obtain the main result of the dissertation concerning random quadratic dynamics. Another complex theory used in the dissertation is the formalism of bounded random systems of quadratic maps (Arnold [2] and Crauel [14]). Krzysztof Lech applies it to obtain results on dimensions of the maximal measure for the random quadratic maps f_ω.

In my opinion, the text of the dissertation itself is of a high quality. It contains all necessary details to check correctness of the main results. In particular, the dissertation contains exposition of necessary notions and statements from the theories involved, i.e. Potential Theory and bounded random systems of maps.

To summarize, in my opinion, the dissertation deserves a high praise. It certainly meets the requirements laid down for the degree of PhD. Moreover, I believe that the results of the thesis are sufficiently exceptional to award an honorary distinction.

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