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Report on the PhD thesis

The logical strength of Ramsey-theoretic principles over a weak base theory

by Katarzyna Wiesława Kowalik

The PhD thesis written by Katarzyna W. Kowalik is a major contribution to the study of combinatorial theorems in reverse mathematics.

Reverse mathematics is a foundational program started by Harvey Friedman in the 1970s, whose goal is to find optimal axioms to prove ordinary theorems. More precisely, it uses subsystems of second-order arithmetic to calibrate the logical strength of theorems in terms of set existence axioms. Historically, theorems are studied modulo a base theory, namely, RCA_0, standing for Recursive Comprehension Axiom. It is composed of some basic axioms, PA^-, stating that the natural numbers form a discretely ordered commutative semiring, together with the comprehension scheme for computable (Δ^0_1) predicates, and the induction scheme for computably enumerable (Σ^0_1) predicates. The theory RCA_0 arguably captures "computable mathematics", and its provably total functions are exactly the primitive recursive ones.

More recently, there has been a growing interest for reverse mathematics over a weaker base theory, RCA^*_0, where the induction scheme is further restricted to computable predicates. The motivation is two-fold: First, by the correspondence between the induction scheme and the bounded comprehension scheme, RCA_0 states the existence of finite c.e. sets, while RCA^*_0 only includes computable sets in the model, and thus arguably corresponds better to computable mathematics. Second, some recent work of David Belanger showed a formal connection between the study of theorems over RCA^*_0 and over RCA_0, the former increasing the understanding of the latter. Working over the base theory RCA^*_0 raises additional challenges, as there is no longer a robust notion of "infinity": some sets might be unbounded, but of finite cardinality.

In her PhD thesis, Katarzyna W. Kowalik studied Ramsey's theorem and some of its consequences that are of major importance in classical reverse mathematics. More precisely, she studied the Chain-Antichain principle (CAC), the Ascending Descending Sequence principle (ADS) and two variants of the cohesiveness principle (COH and CRT^2_2) which are equivalent when sufficient induction is available. The manuscript is divided into 4 chapters. In Chapter 1, she explains the preliminary concepts which will be used throughout the thesis. Chapters 2 and 3 are based on two articles published in renowned international journals:

- "How strong is Ramsey's theorem if infinity can be weak?", by Leszek Aleksander Kołodziejczyk, Katarzyna W. Kowalik and Keita Yokoyama, published in the Journal of Symbolic Logic
- "Weaker cousins of Ramsey's theorem over a weak base theory", by Marta Fiori-Carones, Leszek Aleksander Kołodziejczyk and Katarzyna W. Kowalik, published in Annals of Pure and Applied Logic

More precisely, as mentioned, there are two non-equivalent natural definitions of infinity over RCA^*_0: being unbounded, that is, $\forall x\ \exists y > x\ y \in S$ and being in bijection with $\mathbb{N}$. Many statements from

Ramsey theory state the existence of an infinite set with some structural properties, where infinity is naturally formulated in terms of unboundedness. These statements are referred to as *normal versions*, while a strengthening of these statements in terms of bijection with $\mathbb{N}$ are called *long versions*.

In Chapter 2, Kowalik successfully classifies the normal versions of the above statements over RCA_0^*, giving an almost complete picture. This classification relies on a very clever model-theoretic argument based on the existence of proper Σ^0_1-definable cuts in models where Σ^0_1-induction fails. Very surprisingly, in these models, Ramsey's theorem and its consequences behave like first-order statements, and in particular have "computable" solutions, contrary to well-known classical results showing that all these statements have computable instances with no computable solutions. She also manages to characterize the first-order consequences of Ramsey's theorem over RCA_0^*, while such a characterization for Ramsey's theorem for pairs over RCA_0 remains a major open problem. Kowalik does not only provide arguments for these specific statements, she defines a general machinery for a whole syntactical family of statements called pSO, standing for pseudo-Second-Order. In particular, she provides a very simple and general condition for such statements to be $\forall\Pi^0_3$-conservative over RCA_0^*, where $\forall\Pi^0_3$ is the class of formulas of the form $\forall X \forall x \exists y \forall z \varphi(X, x, y, z)$ with φ containing only bounded quantifiers.

In Chapter 3, she studied the long version of these statements. It is known that the statement of the equivalence between the two notions of infinity is itself equivalent to Σ^0_1-induction, so in some cases, Kowalik proves, as expected, that the long version implies Σ^0_1-induction. It follows that the classical reverse-mathematical analysis of these statements over RCA_0 applies. It is the case of the long versions of $\mathsf{RT}2_2$, CAC and ADS. However, in some other cases, the long versions of the statements exhibit behavior similar to their normal version, and are proven to be $\forall\Pi^0_3$-conservative over RCA_0^*, and in particular not to imply Σ^0_1-induction. This is the case of the long version of CRT^2_2. The cohesiveness principle (COH) plays a particular role, as it can be considered as a combination of a normal and a long Ramsey-type statement. It states, for every collection of sets $(R_n)_{n\in\mathbb{N}}$, the existence of an unbounded set C such that for every $n \in \mathbb{N}$, C is almost included in R_n, or in its complement. The family of sets is indexed by $\mathbb{N}$, and therefore suggests a long version, but the set C is only required to be unbounded. Kowalik proves that COH is closer to the long versions than to the normal versions, by constructing an instance of COH with no computable solution. It follows that the classical proof of $\mathsf{RT}^2_2 \to \mathsf{COH}$ over RCA_0 does not hold over RCA_0^*. Since then, Mengzhou Sun proved that COH actually implies Σ^0_1-induction over RCA_0^*.

The last chapter, of significant importance, is dedicated to the proof that $\mathsf{RCA}_0^* + \mathsf{CAC}$ is polynomially simulated by RCA_0^* with respect to $\forall\Pi^0_3$-formulas, meaning that every proof of a $\forall\Pi^0_3$-formula over $\mathsf{RCA}_0^* + \mathsf{CAC}$ can be translated into a proof over RCA_0^* using a polynomial-time procedure. As a consequence, the translated proof size is polynomial in the original proof size. This chapter merits further discussion. Contrary to Chapter 2 and Chapter 3, this chapter is some unpublished material, which constitutes half the length of the PhD thesis. The mathematical nature of this chapter is also very different: The previous chapters are more classical reverse mathematics, though in a weaker base theory with important new challenges. As such, they essentially use computability-theoretic and model-theoretic arguments. This chapter has a much more proof-theoretic flavor, and is a pioneer is the proof-length analysis of conservation results in reverse mathematics. To my knowledge, the only two previous results on this subject were by Jeremy Avigad and a paper by Kołodziejczyk, Wong and Yokoyama. Kowalik uses the techniques of *forcing interpretation* invented by Avigad, which consists of (1) formalizing model-theoretic forcing arguments to obtain a translation from a formula φ to a sentence "$s \Vdash \varphi$" in a theory T_2, meaning that the condition s forces φ to hold, (2) showing that T_2 proves the sentence "$s \Vdash A$" for every condition s and every axiom A of a theory T_1, and (3) proving over T_2 that is "$s \Vdash \varphi$" for every condition s, then φ actually holds. If each step is realized with a polynomial-time transformation, then the conservation theorem actually yields a polynomial simulation of T_1 by T_2, showing that the theory T_1 does not yield significantly shorter proofs than T_2. The whole construction is extremely subtle, with many intermediate challenges, such as showing that the classical theories are finitely axiomatizable thanks to arithmetization of the satisfaction predicate, or the use of intermediate theories obtained by case analysis. The argument is intrinsically complicated, and Kowalik does an excellent job in explaining the proof by first giving a general picture, and then considering each challenging part one by one, while not making the reader lose track of the goal.

A few points stand out to me, of equal importance. First, the maturity of the mathematical writing of Katarzyna W. Kowalik: she articulates all the definitions and theorems as a coherent whole, with a clear storyline, making it a real pleasure to read. I particularly appreciated the intuitions she gave for many technical parts of the constructions or new concepts, such as the correspondence between the cardinalities of unbounded sets and Σ^0_1-definable cuts, or how she clearly explained the proof strategy for the non-speedup theorem, starting with model-theoretic intuitions and relating constructions to existing literature. In many situations, she took time to explain where such argument would fail if applied in another situation, such as the failure of the generic cut construction for proving a non-speedup of RT^2_2 over RCA^*_0. Second, I was impressed by the breadth of the techniques Kowalik used to answer the questions in her thesis: The first part already exploits some very surprising phenomena due to the failure of Σ^0_1-induction to revisit combinatorial theorems which were extensively studied in classical reverse mathematics and for which one could have thought everything was already said. The second part, Chapter 4, employs a wide array of techniques, drawn from computability theory, model theory, proof theory, forcing, and more. Kowalik handles these concepts with brilliance, demonstrating a deep understanding of these fields. Last, I should point out the significance of her results. In particular, the polynomial simulation of $\mathsf{RCA}^*_0 + \mathsf{CAC}$ over RCA^*_0 should be put in contrast with the non-elementary speedup of $\mathsf{RCA}^*_0 + \mathsf{RT}^2_2$ (Ramsey's theorem for pairs) over RCA^*_0 proven by Kołodziejczyk, Wong and Yokoyama for the same class of formulas. Indeed, CAC is a natural fragment of RT^2_2, and I consider this as a result of great importance.

From the above, it is clear that Katarzyna W. Kowalik's PhD thesis is of exceptionally high quality, both in substance and in presentation. I therefore **deem the thesis sufficient to grant a PhD**. Moreover, given the depth of the results, the originality of the methods, and the clarity of exposition, I **recommend that the thesis be awarded an honorary distinction**.

Below, a few minor comments and typos:

- P8: "programme" $\rightarrow$ "program"
- P15, definition of ω-extension. It might be useful to add a warning, saying that the name ω-extension might be confusing, in that an ω-extension of a non-standard model is not an ω-model.
- P20, "it a substructure" $\rightarrow$ "it is a substructure"
- P29, proof of Theorem 1.21. It is a bit overkill to use Π^0_3 conservation of $\mathsf{WKL}_0 + \mathsf{RT}^2_2$ over RCA_0 to separate $\mathsf{WKL}_0 + \mathsf{RT}^2_2$ from RT^3_2. Indeed, simply use cone avoidance (Jockusch and Soare for WKL_0 and Seetapun for RT^2_2). This was known way earlier.
- P29, proof of Theorem 1.21. We can even separate $\mathsf{WKL}_0 + \mathsf{CRT}^2_2$ from ADS over ω-models. See Theorem 5.1 of Wang's paper "The definability strength of combinatorial principles". The idea is the following: By Hirschfeldt and Shore, given a computable instance of SADS with no computable solution, letting A be the ω-part, for every infinite set $H \subseteq A$, A is H-c.e, and of every infinite set $H \subseteq \overline{A}$, $\overline{A}$ is H-c.e. This shows that ADS does not preserve the non-c.e. definitions of A and $\overline{A}$ simultaneously. On the other hand, WKL_0 and COH preserve non-c.e. definitions of ω many sets simultaneously.
- P65. It would seem more natural to put (FI11) before (FI10) as the former is still about the axioms of equality, while (FI10) talks about the properties of functions.

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