UNIVERSITY OF
OREGON

June 25, 2024

PhD committee
Academic Council of Mathematics and CIS
University of Warsaw
Krakowskie Przedmieście 26/28
00-927 Warsaw, Poland

To the committee:

This is an independent evaluation of Jakub Paliga's doctoral dissertation, "Equivariant Khovanov homotopy types." I corresponded with Paliga once roughly six months ago but, as far as I remember, that has been our only contact. In particular, I have not collaborated with him or his thesis advisor. His thesis has been submitted to a journal where I am an editor, but because I am a referee for the thesis itself, it is being handled by a different editor at the journal. So, I am confident that I can provide an unbiased evaluation of the work.

I feel the thesis is sufficient to grant a PhD. I have a number of minor suggestions that I recommend be incorporated before the final version of the thesis. None of the changes I suggest is mathematically significant, and I am convinced the thesis is correct. My recommendation is that Paliga's thesis advisor approve the revisions to the thesis once they have been made, but I would of course be willing to review the revised version if the committee feels that is necessary.

Paliga's thesis is on a subject of substantial current interest: equivariant Khovanov homology and its space-level refinement. Khovanov homology is an invariant of knots in $\mathbb{R}^3$, introduced by Khovanov about 25 years ago. Its definition starts from a knot diagram: one defines a chain complex in terms of certain labelings of resolutions of the knot diagram, and then takes the homology of that chain complex. In 2011, Sarkar and I gave a refinement of the Khovanov complex, replacing a chain complex of abelian groups with a chain complex of spaces (that is, replacing direct sums of $\mathbb{Z}$ with wedge sums of spheres). The refinement has more information about knots than Khovanov

Department of Mathematics
1222 University of Oregon, Eugene OR 97403-1222
541-346-4705 | FAX 541-346-0987 http://math.uoregon.edu

homology does. More importantly, though, maps of spaces are much more interesting than maps of abelian groups. One way this manifests is that the Khovanov space can be used to study knots with symmetries, giving results that seem not to be accessible without it.

There are several different constructions of an equivariant space in the presence of symmetries: one due to Borodzik, Politarczyk, and Silvero; another due to Stoffregen and Zhang; and a third due to a former PhD student of mine, Musyt. (Musyt's construction is quite close to Stoffregen-Zhang's, and he has not published his thesis.) It has been expected that these constructions agree—that is, the spaces they produce should be equivariantly stably homotopy equivalent. Paliga's thesis proves this conjecture.

So, Paliga's thesis fills a gap in the literature that has been open since 2018. This is a valuable contribution, and seems to me substantial enough for a dissertation. I found the arguments in Paliga's thesis clear and convincing. Somewhat to my surprise, I also learned interesting background and context for the constructions while reading the thesis; for example, the notion of the free topological category generated by a category was new to me, as was the paper of Steimle he cites about homotopy coherent diagrams. The work in his dissertation involves comparing the details of two machines, and the clear way he explains those machines is likely to be useful to researchers entering the area.

The fraction of the thesis spent on reviewing background from the literature seemed to me a bit larger than typical, though not so large as to be a concern. In particular, it would be hard to write a self-contained proof of his result without including a lot of background. On the other hand, the new argument itself was quite short—most of it is in Section 7.2, though the rest of Section 7 and much of Sections 6 and 8 have also not appeared in print (but seemed more straightforward). As such, I think it would be nice to expand the proof of Theorem 3—the main new work. While I agree that the identification in that theorem follows fairly directly from the proof of [LLS20, Theorem 8] and [BPS21, Appendix B], since writing down this proof is the main new contribution, I recommend giving more details.

I give a number of more specific comments on the attached pages. I want to reiterate that I enjoyed reading this thesis and learned from it, and that I recommend accepting it, after some minor revisions.

Sincerely,

Robert Lipshitz
Professor of Mathematics
University of Oregon

Here are some specific comments and questions. I think they are all minor. Some questions might simply be because I overlooked something, and I can always be mistaken in the comments.

1. p. 1. Where you say "composition defined by the group law of G", it might be worth specifying if $g \circ h$ is gh or hg.
2. p. 6, displayed equation *before* equation (2.3). Should the instances of $g\cdot$ on the left and right be ψ_g, for consistency with equation (2.3)?
3. p. 9, (FC-2). This is true for $x \neq y$. The way you have defined things, I guess $Hom(x,x)$ is a point, which does not satisfy this condition.
4. p. 9, (FC-3). I think the formula with a big coproduct is not quite right: the different spaces in that coproduct are not disjoint, but rather overlap on their codimension-1 boundary (the codimension-2 boundary of $Hom(x,y)$).
5. p. 10, (EFC-7), second line. Where you write $G_{x,y} \subset \mathcal{M}_{\mathcal{C}}(x,y)$, you could replace the $\subset$ by an equality, by (EFC-3), right? Next line: has the notion of a G-manifold been introduced? If not, maybe give a citation to a place the reader can find the definition.
6. p. 11, after Proposition 3.2.1. The notation $\Pi_{P_1,\dots,P_i\cup\cdots\cup P_i,\dots,P_{k+1}}$ is a bit misleading, right?
7. p. 11, just before Proposition 3.2.3. "some other $\{0,1\}^{n'}$": reading this, I wondered what n' was. Maybe refer to the formula in [BPS21], if you do not want to repeat it.
8. p. 12. Is Figure 3.1 referenced anywhere? (Usually, it is recommended that every figure be referenced somewhere in the text.)
9. p. 13, before Proposition 3.3.2. Was $\mathbb{R}^{u\Delta 0}$ defined as a $(\mathbb{Z}_m)_u$-representation before?
10. p. 13, Proposition 3.3.2. Should $C(n)$ be $C(nm)$?
11. p. 14, displayed equation before Definition 3.4.1 ($g \cdot (-) :\dots$). I did not understand this formula. For example, what is f? What does the line after the displayed formula mean?
12. p. 16, Section 3.5, 4th line. Was $\underline{2}^n_+$ introduced somewhere earlier?
13. p. 16, towards the end. "cells of increasing gradings $|x| = |f(x)|$". Specifically, is this the cell $EX(x)$?

14. p. 19, Lemma 4.1.5. [SZ18] cites Sarkar-Scaduto-Stoffregen for the proof. Sarkar-Scaduto-Stoffregen cite Lawson-Lipshitz-Sarkar for the proof. It might be more helpful for the reader to cite LLS directly.

15. p. 19, last line. "action of G by ψ on $\mathcal{C}$": the "by ψ" should be deleted, right?

16. p. 20, near top. Point 2 refers go "2-isomorphisms". Point 3 refers to "2-morphism". Aren't all 2-morphisms in this category isomorphisms? If so, it is less confusing to be consistent about referring to them all as 2-morphisms or all as 2-isomorphisms.

17. p. 20, Example 4.2.3. I think this could be unpacked further, if you chose. In particular, an "invertible correspondence" is just (the graph of) a bijection, and point 2 just says that $\psi_{gh,v} = \psi_{g,v} \circ \psi_{h,v}$ as bijections, right?

18. p. 22. There's a reference to Definition 6.1.1, which comes later. It might be worth reordering somewhat so that the definition comes before this section. Otherwise, I suggest at least adding a few words acknowledging that the definition comes later (e.g., "This section assumes familiarity with Definition 6.1.1 and the fact that Musyt data is equivalent to SZ data, but the rest of Section 6 is not needed.")

19. p. 25, Proposition 5.2.5, point 2. Has "weakly equivalent" been defined? If not, maybe give a citation to somewhere the reader can find a definition.

20. p. 27, Construction 6.2.1 point 3. It might be worth connecting the notation here with the notation in Definition 4.2.1 by reminding the reader that the cube category has a unique morphism from u to v when $u \geq v$, so what is denoted $\psi_{g,u,v}$ here would have been denoted $\psi_{g,A}$ in Definition 4.2.1. (I was initially confused by the difference. In particular, $\psi_{g,u,v}$ from Construction 6.2.1 looks a lot like $\psi_{g,h,v}$ from Definition 4.2.1, but the meaning is totally different.)

21. p. 33, just before Proposition 7.1.1. I was a little confused what the goal of this section was initially. I think the point is that since $\tilde{F}$ is a G-coherent refinement of F and Stoffregen-Zhang's construction is independent of the choice of G-coherent refinement, you can use this specific refinement when taking the homotopy colimit of $\tilde{F}_+$. Maybe it would be worth saying some version of that.

22. p. 34, Proof of Theorem 3. It seems to me that this is where the main new work is. So, it felt a little odd reading many pages of background and then having half the argument be "The non-equivariant identification is proven in [LLS20, Proposition 6.1]. Taking into account the external action, the cell $C'(x)$ becomes a $G_{f(x)}$-space with action split over…". (It felt especially odd since the result is one people probably expected to be true.) I think there should be more details here: a step-by-step explanation of why the homeomorphisms in [LLS20, Proposition 6.1] do respect the action. It does not have to be long (maybe a paragraph or half a page is enough) and does not necessarily have to be self-contained: you can refer to specific formulas in [LLS20] rather than re-writing them. Similarly, I think the phrase later on the page, "the point is that the boxes B_x are identifal in both $C'(x)$ and $C(x)$" should be expanded—why is this true and why is it the point? Again, the explanation might only be a paragraph.
    Similarly, I think the explanation at the end, "the cells $C(x)^H$ and $C'(x)^H$… respectively" should expand a bit.
    Also on the same page, isn't the big formula starting $C(x) = EX(x) = \cdots$ on page 16? If not, maybe briefly indicate why this is different. If it is on page 16, maybe it does not need to appear in both places.
23. p. 35. I did not understand what exactly this paragraph meant or what its role is. Maybe it is used in the proof of Theorem 4? Can the paragraph be said more precisely?
24. p. 38, before Theorem 4. "It is clear that…". Since this is again one of the main new parts of the thesis, I think there should be a few more details.

I also noticed a small number of typos, which should be corrected in the final version of the thesis:

- T1. p. v. "as form communications" should be "as from communications".
- T2. p. 1, third paragraph. "The fixed points of the action of $\mathbb{Z}_m$ on…" should, I think, be "For the action of $\mathbb{Z}_m$…"
- T3. p. 5, line 6. "wih" should be "with".
- T4. p. 5, line 11 (displayed equation). I think an instance of g_n should be f_n.

T5. p. 6, line 2. "i.e." should be "e.g.".

T6. p. 9, Definition 3.1.5. I would italicize "G-equivariant flow category".

T7. p. 9, (EFC-6). "by a map" should be "by the map".

T8. p. 10, Definition 3.1.6, first bullet point. "commutes with group" should be "commutes with the group".

T9. p. 14, Definition 3.4.1. "relative representation" should be "relative a representation". "relative sequence" should be "relative a sequence".

T10. p. 17, second line. "relative orthogonal" should be "relative an orthogonal"

T11. p. 17, last line. Should the last instance of V be deleted?

T12. p. 21, Lemma 4.2.4(2). In the displayed formula, $\psi_{g,h,v}$ should be $\psi_{gh,v}$, right?

T13. p. 27, Construction 6.2.1 point 3. "From bijection" should be "From the bijection".

T14. p. 27, Lemma 6.2.2. "ψ_g, u" should be $\psi_{g,u}$.

T15. p. 28, Construction 6.2.3, point 1. "From 1-isomorphism" should be "From the 1-isomorphism" and "produce bijection" should be "produce the bijection".

T16. p. 30, Construction 6.3.3. "from data" should be "from the data".

T17. p. 33, second line. "by identity" should be "by the identity"

T18. p. 34, middle. "show that" should be "shows that". Later: "is carries" should just be "carries".