REPORT ON THE PHD DISSERTATION 'MOTIVIC CHERN CLASSES AND STABLE ENVELOPES' BY JAKUB KONCKI

This highly interesting, important and very well written PhD dissertation 'Motivic Chern classes and stable envelopes' of Jakub Koncki from the University of Warsaw (and with Andrzej Weber as his Supervisor) explains in an equivariant K-theoretical context a new relation between 'characteristic classes of singular spaces' in complex algebraic geometry and the recently introduced and studied 'stable envelopes' of Okounkov and his coworkers for symplectic varieties in the context of Nakajima quiver varieties for a geometric approach to suitable quantum groups via their R-matrices. Both theories can be worked out on three different levels using either (equivariant) cohomology, K-theory or elliptic cohomology.

The corresponding cohomological context for the equivariant Chern-Schwartz-MacPherson classes has been studied by Rimanyi-Varchenko and Aluffi-Mihalcea-Schürmann-Su, the K-theoretic context for equivariant motivic Chern classes a la Brasselet-Schürmann-Yokura has been recently started by the two groups Feher-Rimanyi-Weber and Aluffi-Mihalcea-Schürmann-Su, and finally suitable elliptic classes are the latest studied by Kumar-Rimanyi-Weber.

In his dissertation Jakub Koncki compares in the K-theoretical context the equivariant (twisted) motivic Chern classes of Bialynicki-Birula cells (in short BB-cells), coming from the Bialynicki-Birula decomposition (in short BB-decomposition) of an algebraic torus action on the cotangent bundle T^*M of a complex projective manifold M, with the corresponding K-theoretical stable envelopes of Okounkov and his coworkers for the corresponding cotangent bundle as underlying symplectic manifold. Note that such a comparison is only possible for cotangent bundles.

Here one assumes that the algebraic torus A is acting on M with a finite fixed point set M^A, with the corresponding Bialynicki-Birula decomposition into cells $M = \bigsqcup_{e \in M^A} M_e^+$. These Bialynicki-Birula cells $M_e^+ \subset M$ are locally closed submanifolds of M, which are isomorphic to some $\mathbb{C}^{n_e}$. But their closures $\overline{M_e^+} \subset M$ are usually singular subvarieties of M. An important class of examples comes from homogenous varieties $M = G/P$ for a complex reductive group G and a parabolic subgroup P, with A the maximal torus of a Borel subgroup $B \subset P \subset G$, with M_e^+ and $\overline{M_e^+}$ the corresponding Schubert cells and varieties. For the theory of 'stable envelopes' one uses the additional $\mathbb{C}^*$-action by multiplication on the fibers of T^*M preserving its standard symplectic form, with

$$K^{A\times\mathbb{C}^*}(T^*M) \simeq K^{A\times\mathbb{C}^*}(M) \simeq K^A(M)[y, y^{\pm 1}]\,,$$

since $\mathbb{C}^*$ is acting trivially on M. Then both players of the story of this dissertation are living in these K-groups of equivariant algebraic vector bundles:

(1) The pushforwards under the closed inclusion $i : \overline{M_e^+} \hookrightarrow M$ of (A-equivariant) motivic Chern classes $mC_y^A(M_e^* \subset \overline{M_e^+}) \in G^A(\overline{M_e^+})[y]$, with y a parameter and G^A the corresponding K-group of equivariant coherent sheaves.
(2) The K-theoretic stable envelopes $Stab^s(e) \in K^{A\times\mathbb{C}^*}(T^*M)$ (for the 'polarization' of T^*M given by TM), indexed by the fixed points $e \in M^A$. These depend in addition on the choice of a (good) one-parameter subgroup $\sigma : \mathbb{C}^* \to A$, with $M^A = M^\sigma$, and a 'slope' given by a fractional line bundle $s \in Pic(M) \otimes_{\mathbb{Z}} \mathbb{Q}$.

While the motivic Chern classes can be constructed explicitly in terms of a resolution of singularities, the stable envelopes are defined axiomatically. The axioms used by Jakub Koncki for his definition of stable envelopes are the (see Section 1.4): 1. Support axiom, 2. Normalization axiom, 3. Newton inlusion property and 4. Distinguished point (property).

If the slope s is general enough, then the Axiom 4 is redundant and the other ones are equivalent to the Axioms used by Okounkov (as shown here in Proposition 7.11 of the Appendix). By the Example 7.11, the Axiom 4 is needed to get the uniqueness result of Proposition 7.12 for the stable envelopes and all slopes. Axiom 2 is a simple normalization condition for $Stab^s(e)|_e$, whereas Property 3 is an important 'smallness condition' introduced by Okounkov in terms of Newton polytopes $\mathcal{N}^A(Stab^s(e)|_{e'})$ attached to elements in the Laurent polynomial ring $K^A(e') = \mathbb{Q}[Hom(A, \mathbb{C}^*)]$ of a fixed point $e' \in M^A$, with $Hom(A, \mathbb{C}^*)$ the corresponding character lattice. The Support Axiom 1 bounding the support $supp(Stab^s(e)) \subset T^*M$ of a stable envelope is of different nature using also that $\sigma : \mathbb{C}^* \to A$ is 'admissible' in the sense that the union of conormal bundles to the BB-cells is closed in

T^*M (Definition 1.64), or in other words that the BB-decomposition of M is a Whitney A-stratification (Remark 1.67).

In Chapter 1 all needed definitions and properties are recalled about equivariant K-theory (including an important Lefschetz-Riemann-Roch Theorem 1.24), BB-decompositions, motivic Chern classes and stable envelopes, as well as a very interesting 'limit technique' in Sections 1.2.2-3, needed later on for the study of Newton Polytopes as in the Axioms 3 and 4 above.

Chapter 2 presents the proof of the first comparison **Theorem 2.2**: Let $e \in M^A$ be a fixed point. Then the (rescaled) motivic Chern class

$$\mathfrak{h}^{-dim(M_e^+)}\pi^*\rho\left(mC_y^A(M_e^+ \to M)\right) \in K^{A\times\mathbb{C}^*}(T^*M) \tag{1}$$

satisfies the Axioms 2,3 and 4 above. Here $\pi : T^*M \to M$ and $\mathfrak{h} \in Hom(A\times\mathbb{C}^*, \mathbb{C}^*)$ are the projections, with $\rho : K^A(M)[y] \to K^{A\times\mathbb{C}^*}(M)$ given by $y \mapsto -\mathfrak{h}$.

But these motivic Chern classes do not depend on the choice of an additional 'slope' given by a fractional line bundle $s \in Pic(M) \otimes_{\mathbb{Z}} \mathbb{Q}$. For this reason 'twisted motivic Chern classes' $mC_y^A(W, \partial W; \triangle) \in G^A(W)[y]$ are introduced and studied in Charpter 3. Here W is a possible singular quasi-projective A-variety, with a closed invariant subvariety ∂W (called a boundary) such that $W\backslash\partial W$ is smooth. Moreover $\triangle$ is a $\mathbb{Q}$-Cartier divisor on W with support in the boundary ∂W. Then the **'twisted motivic Chern classes'** $mC_y^A(W, \partial W; \triangle) \in G^A(W)[y]$ are defined in Definition 3.4 with the help of a suitable A-equivariant resolution of singularities $f : (Y, \partial Y) \to (W, \partial W)$. This construction is a combination of the definition of motivic Chern class a la Brasselet-Schürmann-Yokura and ideas coming from the theory of 'multiplier ideals' (compare with Remark 3.8). These 'twisted motivic Chern classes' can also be interpreted as a limit of the elliptic classes constructed by Libgober-Borisov only for Kawamata log-terminal pairs, but working in this more general context. The deep 'weak factorization theorem' is used to show that the resulting classes are well defined, i.e. do not depend on the choice of a resolution (see Corollary 3.13).

In Chapter 4 these twisted Chern classes are then further compared with (the Axioms of) stable envelopes. The key result is the second comparison **Theorem 4.2**:

Let $e \in M^A$ be a fixed point and consider a slope $s \in Pic(M) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then the (rescaled) twisted motivic Chern class

$$(2) \qquad \mathfrak{h}^{-dim(M_e^+)} \pi^* \rho \left(i_{e*} mC_y^A(\overline{M_e^+}, \partial \overline{M_e^+}; \triangle_{e,s}) \right) \in K^{A \times \mathbb{C}^*}(T^*M)$$

satisfies the Axioms 2,3 and 4 above. Here $i_e : \overline{M_e^+} \hookrightarrow M$ is the closed inlusion of the closure of a BB-cell, with $\triangle_{e,s}$ an associated $\mathbb{Q}$-Cartier divisor on $\overline{M_e^+}$ with support in the boundary $\partial \overline{M_e^+}$ (constructed in Section 4.2).

So for a complete indentification of twisted motivic Chern classes and stable envelopes only the 'Support Axiom 1' is missing (by the uniqueness result of Proposition 7.12). This 'Support Axiom' doesn't depend on the choice of a slope and is studied in Chapter 5. A sufficient 'local product property' for the pair (M, σ) with a good one parameter subgroup $\sigma : \mathbb{C}^* \to A$ (with M^A finite) is introduced in Definition 5.7. It automatically implies that σ is admissible (Proposition 5.8), as well as that the (twisted) motivic Chern classes

$$\mathfrak{h}^{-dim(M_e^+)} \pi^* \rho \left(mC_y^A(M_e^+ \to M) \right)$$

and

$$\mathfrak{h}^{-dim(M_e^+)} \pi^* \rho \left(i_{e*} mC_y^A(\overline{M_e^+}, \partial \overline{M_e^+}; \triangle_{e,s}) \right)$$

satisfy the 'Support Axiom 1' (see Proposition 5.10). Putting everything together one gets the following main comparison **Theorem 5.12** (and its **Corollary 5.13**):
Let $e \in M^A$ be a fixed point, with $i_e : \overline{M_e^+} \hookrightarrow M$ ithe closed inlusion and consider a slope $s \in Pic(M) \otimes_{\mathbb{Z}} \mathbb{Q}$. Assume the pair (M, σ) satisfies the 'local product property' (with M^A finite). Then the stable envelopes of T^*M are equal to the (rescaled) twisted motivic Chern class of the BB-cells of M:

$$(3) \qquad \mathfrak{h}^{-dim(M_e^+)} \pi^* \rho \left(i_{e*} mC_y^A(\overline{M_e^+}, \partial \overline{M_e^+}; \triangle_{e,s}) \right) = Stab^s(e) \,.$$

If the slope s is trivial or small and antiample, then

$$(4) \qquad \mathfrak{h}^{-dim(M_e^+)} \pi^* \rho \left(mC_y^A(M_e^+ \to M) \right) = Stab^s(e) \,.$$

As shown in **Theorem 5.15**, the 'local product property' holds in the case of a homogenous variety $M = G/P$ as before and a general $\sigma : \mathbb{C}^* \to A$, with A the maximal torus of a Borel subgroup $B \subset P \subset G$. Therefore the main comparison Theorem 5.12 (and its Corollary 5.13) apply to the Schubert cells M_e^+ and varieties $\overline{M_e^+}$ in G/P (Corollary 5.16). For $M = G/B$ and small antiample slope one gets a new proof of the identifcation (4) of motivic Chern classes of Schubert cells with suitable stable envelopes due to Aluffi-Mihalcea-Schürmann-Su (based on a corresponding Hecke-algebra action).

The dissertation of Jakub Koncki finishes in Chapter 6 with the Example and explicit calculations for the case of the Lagrangian Grassmanian $LG(2,4)$, as well as the Appendix A in the final Chapter 7 for the uniqueness and comparison of different Axioms for stable envelopes (as used by Jakub Koncki and Okounkov).

This very important dissertation of Jakub Koncki proves new and interesting results in equivariant intersection theory, relating K-theoretical characteristic classes of BB-cells to stable envelope invariants for a geometric approch to suitable quantum groups. It is almost selfcontained, very well written and the proofs are correct, so that

I deem this thesis of Jakub Koncki as sufficient to grant a PhD.

In addition I suggest to award **an honorary distinction** to this excellent thesis.