Review of the Doctoral Thesis by Bogdan Petraszczuk

Armin Schikorra

September 11, 2025

Motivation

This thesis is devoted to the analysis of *nonlinear elliptic systems* under critical growth assumptions, focusing on *n-harmonic maps* and *H-systems*. Such problems arise naturally in Geometric Function Theory and the Calculus of Variations, where one studies deformations (e.g. in Elasticity) or energy-minimizing maps with boundary constraints. In these models, it is natural to assume certain Sobolev regularity on a mapping u (meaning that the map has finite "energy"), and try to prove or disprove better regularity properties such as continuity or differentiability. This is quite tricky in the situations the authors is considering: vectorial maps, i.e. PDE systems, and possibly with nonlinear leading order terms (in this case the so-called p-Laplacian).

Overview of the Thesis

The work divides naturally into two main parts:

- *Regularity Theory for a system involving the n-Laplace.* The regularity theory for n-harmonic maps into manifolds and n-systems of prescribed mean curvature H (H-systems) is a long-standing open problem for $n \geq 3$. It was formulated as such in the early 1990s, amongst others by the candidate's advisor, and it is apparent that some of the cleverest minds in the field tried and failed to obtain a complete understanding.

  Since it is so difficult, the community became interested instead to "simplify the problem", by finding mild, non-trivializing additional assumptions on the solution that would lead to regularity theory. "Simplify the problem" does not mean that the issues get easy at all, it just means that we are trying to get a better understanding between the original problem and what additional assumption we would have to show that solutions to the original problem satisfy in order to resolve the question at hand. So-to-speak we want to narrow the gap in our understanding as much as possible.

  This is what the candidate did in the first part of the thesis (in collaboration with co-authors). The "finite energy" assumption of the system is that the solution belongs to the Sobolev space $W^{1,n}$. The authors, in addition, assume that the solution belongs to $W^{\frac{n}{2},2}$ – we see that for the case $n = 2$ there is no change in assumptions. Then they argue by means of a sharp Hardy-BMO duality, commutator theorems, and Uhlenbeck-type gauge decompositions to obtain the result. These methods are a delicate combination of tools from Harmonic Analysis, and PDE. As the author mentions, by now their result has been improved to a "less strong" additional assumption in the work by Martino and myself, a work which was very much inspired by the candidate's work under consideration.

- *Prescribing Singularities.* The second part focuses on constructing *irregular* solutions in two-dimensional (2D) elliptic systems, building on Frehse's example of a single-point discontinuity to create solutions whose singular sets can be prescribed arbitrarily. This is highly technical yet provides a concrete demonstration that in the critical-growth regime one can have extremely pathological behavior. This portion of the thesis clarifies how local logarithmic patches and clever cut-off arguments let the author maintain $F \in L^1$ while orchestrating wild singularities on any compact set.

  By now the proofs are consistent and to the best of my understanding correct, and all details have been worked out reasonably.

Conclusion

Overall, this thesis makes a valuable contribution to an exciting and active area of modern mathematical analysis. The author clearly shows deep knowledge of *Geometric Function Theory*, *Sobolev spaces*, and *harmonic-map-type PDE*, as well as a talent for constructing ingenious examples. The proofs are involved, and the techniques employed combine ideas from functional analysis, geometric measure theory, and the theory of critical-growth equations.

I have carefully verified that the main statements appear correct and see no flaws in the overall reasoning. The results on regularity will be helpful for experts analyzing n-harmonic systems, and the constructions of singular solutions should serve as a key reference for those who investigate extremal or non-regular behavior. In short, it is a significant and original piece of work.

Given these observations, I find this thesis deserving of a doctoral degree.

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Armin Schikorra

Review of the Doctoral Thesis by Bogdan Petraszczuk

Armin Schikorra

January 31, 2025

Motivation

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The work divides naturally into two main parts:

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  Since it is so difficult, the community became interested instead to “simplify the problem”, by finding mild, non-trivializing additional assumptions on the solution that would lead to regularity theory. “Simplify the problem” does not mean that the issues get easy at all, it just means that we are trying to get a better understanding between the original problem and what additional assumption we would have to show that solutions to the original problem satisfy in order to resolve the question at hand. So-to-speak we want to narrow the gap in our understanding as much as possible.

  This is what the candidate did in the first part of the thesis (in collaboration with co-authors). The “finite energy” assumption of the system is that the solution belongs to the Sobolev space $W^{1,n}$. The authors, in addition, assume that the solution belongs to $W^{\frac{n}{2},2}$ – we see that for the case $n = 2$ there is no change in assumptions. Then they argue by means of a sharp Hardy-BMO duality, commutator theorems, and Uhlenbeck-type gauge decompositions to obtain the result. These methods are a delicate combination of tools from Harmonic Analysis, and PDE. As the author mentions, by now their result has been improved to a “less strong” additional assumption in the work by Martino and myself, a work which was very much inspired by the candidate’s work under consideration.

- *Prescribing Singularities.* The second part focuses on constructing *irregular* solutions in two-dimensional (2D) elliptic systems, building on Frehse's example of a single-point discontinuity to create solutions whose singular sets can be prescribed arbitrarily. This is highly technical yet provides a concrete demonstration that in the critical-growth regime one can have extremely pathological behavior. This portion of the thesis clarifies how local logarithmic patches and clever cut-off arguments let the author maintain $F \in L^1$ while orchestrating wild singularities on any compact set.

  At least that is the idea. I would like the candidate to be more clear on a few parts of his proof. Precisely, the following important points are not appropriately worked out as of now:

  - Why does ∇u_N converges to ∇u pointwise a.e.?
  - Why is u indeed discontinuous on the singular set?

  To be clear, I believe that both statements are true, and can be proven with the available technology – but this should be carried out carefully to make it useful for other's to use.

Conclusion

Assuming the changes above are taken care of, overall, this thesis makes a valuable contribution to an exciting and active area of modern mathematical analysis. The author clearly shows deep knowledge of *Geometric Function Theory*, *Sobolev spaces*, and *harmonic-map-type PDE*, as well as a talent for constructing ingenious examples. The proofs are involved, and the techniques employed combine ideas from functional analysis, geometric measure theory, and the theory of critical-growth equations.

I have carefully verified that the main statements appear correct (besides my complaints above) and see no flaws in the overall reasoning. The results on regularity will be helpful for experts analyzing n-harmonic systems, and the constructions of singular solutions should serve as a key reference for those who investigate extremal or non-regular behavior. In short, it is a significant and original piece of work.

Given these observations, I find this thesis *almost* deserving of a doctoral degree. I ask the author to make the specific corrections (bullet points) mentioned above. Once these points have been taken of I am very confident I will be able to recommend the acceptance of the dissertation.

DocuSigned by:
Armin Schikorra
8A0BF6030419490...
Armin Schikorra ARMIN@pitt.edu
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