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Review

for the doctoral thesis
Linear Algebra in Orbit-finite Dimension
by Arka Ghosh

This is a report of the doctoral thesis of Mr. Arka Ghosh, submitted in the University of Warsaw in April 2025.

The thesis makes several contributions to the computational theory of linear algebra in a setting where the set of dimensions is infinite but exhibits enough symmetry to be amenable to algorithmic manipulation. The main motivating question is solvability of infinite systems of linear equations, but the thesis takes the investigation further, to various forms of linear programming that involve inequalities as well as equations.

The underlying framework of this study is that of *sets with atoms*, actively developed over the last decade or so, with the Warsaw school making several key contributions. There, all structures under consideration (here: variables, equations, systems of equations, linear functions to be minimised etc.) are built in a usual set-theoretic manner starting from an infinite collection of atoms, and subject to a canonical action of the automorphism group of those atoms. An important class of structures are the *orbit-finite* ones, i.e. those where the action has finitely many orbits. These structures, although infinite, are finite up to permutation of the atoms, and so they can be represented by finite means and be seen as input to algorithms. In a connection to model theory, they are also equivalent to structures interpretable in the pure set.

Orbit-finite generalisations of finite computational problems often come in multiple variants, and solving systems of equations is not an exception: given such an orbit-finite system, one may look for (i) and arbitrary solution, i.e. a function from variables to values that satisfies every equation, (ii) a solution which is itself orbit-finite, or (iii) a solution that is finite, i.e. zero almost everywhere. This thesis does not concern itself with (i), and the common theme is that it attempts to reduce (ii) to (iii) and decide the latter. In several cases, both the reduction and the decision procedure (if they exist) are quite nontrivial.

After two introductory chapters, Chapter 3 investigates a fundamental and elementary question: the existence of bases of orbit-finite dimensional vector spaces. Obviously every vector space has a basis, but here the interest is in orbit-finite bases only. It is known that not every orbit-finitely spanned space has such a basis, so here the attention is restricted to a specific case: the space of all orbit-finite vectors over an orbit-finite set B of dimensions. This is not quite the free vector space over B (that one contains only almost-zero vectors, and trivially has a basis). The main result of the chapter is such spaces have orbit-finite bases. The proof is not very complicated, but the chapter serves a useful purpose: in introduces several concepts and proof ideas used later, and it illustrates very well the differences between three different kinds of spaces that play a role later: that of abirtrary vectors, that of orbit-finite ones, and of almost-zero ones.

Chapter 4 moves to the main subject of the thesis: solvability of orbit-finite systems of linear equations. First, the problem of finding orbit-finite solutions is reduced to finding finite (i.e. almost zero) ones; then the latter is proved decidable. Both steps are interesting and nontrivial. The second step is realised via an excursion to totally ordered atoms, in a way that is similar to a related result known from the literature about solving orbit-finite systems of *finite* equations. The first step is entirely new.

Chapter 5 generalises the situation to orbit-finite systems of inequalities. Here, over the field of the reals, the development is similar to Chapter 4, although additional complications arise. A new and interesting concept of a parametrically parametrised inequality, which is a linear inequality with coefficients parametrised, in a nonlinear way, by an integer unknown. This is a creative and interesting idea, and I enjoyed learning it. In another direction, over integers (rather than reals), solving systems of orbit-finite linear inequalities is proved undecidable. In effect, this proves the undecidability of orbit-finite integer linear programming, which is also a new result.

The technical development culminates in Chapters 6 and 7, where the full generality of orbit-finite linear programming is considered. The problem is proved decidable (working over the reals), and weak duality is proved. This is nontrivial, as weak duality is known to fail for infinite linear programs in general. However, strong duality fails for orbit-finite systems, and one has to restrict attention to systems of finite equations (or their transpose twins) to recover it.

All these results are interesting, and some of their proofs are quite involved. I particularly appreciated the technique of polynomially parametrised (in)equalities, which I think may be of independent interest, and the decidability results that rely on averaging out coefficients. I think that several results will attract a good degree of attention in the field; I myself plan to cite the results of Sec. 5.2 (undecidability of integer linear programming) soon. Most of the thesis has been published in very good venues, including a J.ACM article, so it is likely to find a permanent place in the literature.

The thesis is well explained and carefully presented. There is a small number of typos here and there, which is hard to avoid in a document of this size. I did not spot any mathematical mistakes to speak of.

I have some comments and questions to the author that do not detract from my general high opinion about the thesis, but are perhaps worth discussing during the defense. Here is a selection:

- Applying the terminology of vector spaces to structures over rings (rather than field) may be controversial. In particular, some readers may wonder what the point of Sec. 3.3 is, since it is well known that not all modules (which is the more standard terminology) have bases. The technical development makes sense, but it took me two readings to convince myself of that.

- I wonder if the technology used to prove Thm. 3.1 would work for totally ordered atoms. Is it easy to generalise the notion of tight orbit to that setting?

- I am curious to hear the author's thoughts about fully unrestricted solvability problems. To begin with: would the problem EQ(R) be decidable if one looked for arbitrary solutions of equations? This has been left out of scope of the thesis, but I wonder if the author has spent any significant time thinking about it.

- It seems to me that the reduction sketched in Sec. 5.5.2 would fail for totally ordered atoms, because the number of $\{T\}$-orbits grows exponentially with the size of T in that case. How bad is this problem? Is there anything that we can do in that case?

- Question 8.7 on p. 172 is not really open. The answer is negative, and a counterexample (the decidable problem of 3-colorability of orbit-finite graphs) is provided in the paper cited as [26].

To conclude, I think that this is an interesting and well presented work, and it easily satisfies the common standard for a doctoral thesis. I also think that the content and presentation of this thesis merits a distinction.

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