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Referee report on “Linear Algebra in Orbit-finite Dimension” submitted for dissertation by Mr. Arka Ghosh

The thesis makes a significant contribution to the research programme exploring the power of computation with atoms – specifically, the study computations involving infinite sets that satisfy a finiteness condition, enabling their algorithmic representation. The condition is expressed using automorphisms, with the two most relevant groups being the full symmetric group on a countable set, S_ω, and the automorphism group of the ordered rationals, Aut(Q,<). These groups exhibit finitely many orbits in their actions on n-tuples for each n.

The thesis focuses on fundamental computational problems in theoretical computer science. The input for these problems consists of a (infinite, but orbit-finite) system of linear equations over some ring, often R, Q, or Z (Chapter 4), or a system of linear inequalities over Q (Chapter 5). The computational question is whether there exists an orbit-finite solution, or even a solution with finite support. The question whether there exists a solution (without imposing any assumptions on the solution) is also interesting, but not considered in the thesis (the decidability is then open, this is Question 8.4). The linear programming problem is examined as well, where the input additionally contains a linear objective function and the task is to compute the maximum of this function over all solutions to the given system; again the problem comes in different variants depending on the assumptions that we want to impose on the solutions.

All these computational problems are decidable, except for orbit-finite satisfiability of orbit-finite linear inequalities over Z, which is undecidable (even in atom dimension 3). The undecidability result is one of the major contributions of the thesis (decidability for atom dimension 2 remains open). When considering computational complexity, we have to discuss in more detail how the input is repesented; moreover, it again makes a big difference whether the input is restricted to a fixed atom dimension or not. In fixed atom dimension, many of the mentioned problems are even decidable in polynomial time, for instance orbit-finite satisfiability of orbit-finite linear inequalities over Q, or the corresponding linear programming problem (using the polynomial-time tractability for classical linear programming).

These results are strong and coherent. They are carefully elaborated and well presented and have led to two publications in the top-tier conference LICS (logic in computer science), which has a very competitive peer review system. The journal version of one of these papers has subsequently been published in JACM, the leading journal in theoretical computer science. The results of one chapter (about duality in linear programming in the orbit-finite setting) have not yet been published.

There are several minor linguistic issues that should be fixed in the final and published version of the thesis. Often, this concerns the proper usage of articles in English. To give an example: "A set x is finitely supported if it has some finite support; in this case, x has ***the*** least support, denoted support(x)": the use of the definite article `the' is clearly incorrect here, it should be `has ***a*** least support'. It would be even better to first clearly state the fact that for the particular underlying group (the full symmetric group!) there exists a least support with respect to inclusion, and only then to make the definition (in which case the definite article would be appropriate). In any case, it should be stressed more that this is something that needs to be proved. A reference [5] is given, but this could be interpreted as a reference for the definition, and the fact about least supports is hidden. It would be interesting and useful for readers with a background in model theory to link the notion of least supports to weak elimination of imaginaries.

Further suggestions for improvement.

- The definition of `finite solutions' conflicts with standard set theory (vectors are (special) sets, and sets can be finite). "finitary" (used in related publications) is preferable.
- I missed a discussion of the choice of R versus Q. It appears that for most of the considered computational problems, the difference doesn't matter, but I think this should be discussed.
- For the polynomial-time tractability of linear programming and the polynomial-time tractability of linear equations over Z, the thesis should provide references (each of these where open problems for quite some time).
- I recommend the usage of figures (actually, I didn't find any figure in the thesis). The publication [18] of the PhD candidate contains the useful Figure 1 about polynomial-time reductions, and an algorithm box – they would be valuable in the thesis as well.
- In Example 5.27: what is q in the final line?
- Please consider using \qedhere at appropriate places to avoid extra empty lines before QED symbols.
- Section 2.5.2: "In [13] the author has shown finite generation of equivalent polynomial ideals with equality atoms as variables": please provide a precise reference within [13] (and appropriate explanation, if necessary).
- On page 39, item 1: you use R and {\mathbb R}, I guess {\mathbb R} should be R?
- Section 3.2: "let S \subseteq_Fin A denote its support": do you mean "least support"?
- Section 5.1: "We represent linear inequalities similarly as linear equations": but in this section, we assume that the values are real numbers, so you cannot represent them in general on a classical computer (note that you later use notions like polynomial-time reductions, etc, which would require a definition once you move outside the setting of classical computability). This needs to be discussed more carefully.

- Remark 5.10: “as we can not simulate equalities...”: is this a formal claim? If not, rephrase to “we do not know” or something like this. If yes, what precisely is meant by `simulate'? And how do you prove that we cannot simulate?
- Page 94: “We leave it to the reader to verify the details”: IMO not appropriate for a thesis text. It is a standard fact, so one should give a reference.
- Please run a spell-checker (“simpliflying”). Also there should be software to spot capitalisations such as “only know that If” (page 105).
- Page 106: “we want THAT the solutions... differ” or “we want the solutions... TO differ”.
- Page 108: for some polynomials p_1',...,p_k',q: should it be q'?
- As mentioned above, there are many instances of missing definite and indefinite articles, but also many missing letters “s”. Automated tools (e.g., LLMs) can help detect these.
- Question 8.4 comes without a discussion. Have attempts been made? Can you give some more background, and/or describe the key challenges?

Overall, this is a thesis with strong results at the front of a successful research programme. The open problems are interesting, well-motivated, and natural and likely to inspire follow-up work. The work easily meets the criteria for a PhD, and I award it with the highest grade (1.0, very good). It is also a strong candidate for the *magna cum laude* distinction.

Sincerely,

Manuel Bodirsky

Manuel Bodirsky

Signature numérique de Manuel Bodirsky
Date : 2025.08.05 13:09:26 +02'00'