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University of Warsaw
Faculty of Mathematics, Informatics and Mechanics

Exam for 2nd cycle studies of MACHINE LEARNING

June 28, 2024

Solving time: 150 minutes

*In each of the 30 problems there are three variants: (a), (b), and (c). For each variant you should answer if it is true, writing **YES** or **NO** in the box close to it. In case of an error you should cross out the box and write the correct word on its left side.*

Example of a correctly solved problem

4. Every integer of the form $10^n - 1$, where n is integer and positive,

   YES (a) is divisible by 9;

   NO (b) is prime;

   YES (c) is odd.

*You can write only in the indicated places and only the words **YES** and **NO**. Use pen.*

Scoring

You get "big" points (0 – 30) and "small" points (0 – 90):

- *one "big" point for each problem, in which you correctly solved all three variants;*
- *one "small" point for each correctly solved variant. So 3 "small" points in a single problem give one "big" point.*

The final result of the exam is the number

$$W = \min(30, D + m/100),$$

where D is the number of "big" points, and m is the number of "small" points, e.g. score 5.50 means that a candidate correctly solved 50 variants in the whole test, but gave correct answers to all three variants in a set for some five problems.

"Big" points are more important. "Small" points are just to increase resolution in the case when many candidates get the same number of "big" points.

Good luck!

1. There exists an infinite bounded sequence of reals such that

   - (a) the sequence is divergent;
   - (b) the sequence has a convergent subsequence and a divergent subsequence;
   - (c) every subsequence of the sequence is divergent.

2. Series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

   - (a) is convergent;
   - (b) has an increasing sequence of partial sums;
   - (c) has a bounded sequence of partial sums.

3. The number $u \in \mathbb{C} \setminus \mathbb{R}$ is a 3rd root of 1. It follows that

   - (a) u^2 is a root of the polynomial $z^2 + z + 1$;
   - (b) $\frac{1}{u}$ is a root of the polynomial $z^2 + z + 1$;
   - (c) u is a root of the polynomial $z^9 - 1$.

4. Let X be a vector space over $\mathbb{R}$. Let the sequence of vectors a, b, c, d from X be linearly independent. It follows that

   - (a) $\dim X = 4$;
   - (b) the sequence a, $a + b$, $a + c$, $a + d$ is linearly independent;
   - (c) $\operatorname{span}(a, b, c, d) = \operatorname{span}(a, a + b, a + c, a + d)$.

5. Let X be a vector space over $\mathbb{C}$ such that the dimension of X is an odd natural number. It follows that

   - (a) there exist isomorphic linear subspaces U and V of X such that $X = U \oplus V$;
   - (b) there exist nonzero linear subspaces U and V of X such that $X = U \oplus V$ and the dimension of U or V is an even natural number;
   - (c) there exists a natural number k such that X and $\mathbb{C}^{2k+1}$ are isomorphic.

6. Let $A \in \mathbb{C}^{2024,2024}$ be a matrix such that A^{2024} is the identity matrix. It follows that

   - (a) 1 is an eigenvalue of A;
   - (b) the set of eigenvalues of A has 2024 elements;
   - (c) the sum of eigenvalues of A is nonzero.

7. There exists an infinite set of infinite subsets of $\mathbb{N}$ the intersection of which is

   - (a) empty;
   - (b) finite;
   - (c) infinite.

8. In every nonempty partial order $\langle A, \leq \rangle$

☐ (a) there exists a subset with the least upper bound and the greatest lower bound;
☐ (b) every subset that has the greatest lower bound has the least upper bound;
☐ (c) for every element $a \in A$ the greatest lower bound of $\{b \in A \mid a \leq b \wedge a \neq b\}$ is a.

9. There exist $a_0, a_1 \in \mathbb{Z}$ such that the recurrence $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$ defines

☐ (a) a sequence with limit $-\infty$;
☐ (b) a constant sequence;
☐ (c) a sequence with limit ∞.

10. We want to distribute 12 indistinguishable apples to three children so that each child gets at least one apple. The number of ways to do it is

☐ (a) 55;
☐ (b) 10 if each child gets at most 5 apples;
☐ (c) 7 if each child gets at most 2 apples more than any other.

11. Let $A(x)$ be the generating function of the constant sequence $a_n = 1$ for $n \geq 0$. Let $B(x)$ be the generating function of the sequence $\langle b_n \rangle_{n \geq 0}$ defined by the formula $b_n = 3^n$ for $n \geq 0$. Suppose that the generating function of the sequence $\langle c_n \rangle_{n \geq 0}$ is $C(x) = A(x)B(x)$. It follows that

☐ (a) $c_3 = 40$;
☐ (b) $c_4 = 81$;
☐ (c) $c_{2024} = \frac{1}{2}(3^{2025} - 1)$.

12. Graph G was created by removing four edges from the complete graph K_6. It follows that

☐ (a) G is connected;
☐ (b) the chromatic number of G is at most 4;
☐ (c) G has a Hamilton cycle.

13. Wilhelm Tell hits the target with probability $\frac{9}{10}$ in each attempt independently. It follows that the probability that

☐ (a) he hits the target for the first time in the third attempt is greater than $\frac{1}{125}$;
☐ (b) he hits the target for the first time in the third attempt is less than $\frac{9}{100}$;
☐ (c) he misses the target in three consecutive attempts is less than $\frac{1}{900}$.

14. Let X and Y be random variables ranging over the set $\{0, 1\}$. Variables X and Y are independent when

☐ (a) $\mathrm{P}(Y = 1) = 0$;
☐ (b) $\mathrm{E}(XY) = \mathrm{E}(X)\mathrm{E}(Y)$;
☐ (c) X and $1 - Y$ are independent.

15. Let X be a random variable such that $\mathrm{P}(X = k) = \frac{k}{10}$ for $k = 1, 2, 3, 4$. It follows that

- [ ] (a) $\mathrm{E}(X) = 2.5$;
- [ ] (b) $\mathrm{E}(|X - \mathrm{E}(X)|) = 0$;
- [ ] (c) $\mathrm{Var}(X) = 1$.

16. A Markov chain of n states, where $n \in \mathbb{N}$, with two transient classes

- [ ] (a) exists for $n = 2$;
- [ ] (b) exists for some odd n where $n \leq 5$;
- [ ] (c) does not exist for any n.

17. Assume that X is a continuous random variable with the density function $f_X \colon \mathbb{R} \to \mathbb{R}$, a strictly increasing cumulative distribution function $F_X \colon \mathbb{R} \to [0, 1]$, a finite expected value and a finite variance. Consider a random variable $Y = -X$ with its density function f_Y and cumulative distribution function F_Y. It follows that

- [ ] (a) $f_Y(t) = f_X(-t)$;
- [ ] (b) $F_Y(t) = 1 - F_X(-t)$;
- [ ] (c) $\mathrm{Var}(Y) = -\mathrm{Var}(X)$.

18. We approximate $I(f) = \int_{-1}^{1} f(x)\,dx$ using the interpolation quadrature

$$Q(f) = A \cdot (f(-c) + f(c)).$$

It follows that

- [ ] (a) for some A and c, the order of $Q(f)$ is 4;
- [ ] (b) for $A = c = 1$ and $f \in C^3[-1, 1]$ such that $\max_{-1 \leq x \leq 1} |f^{(k)}(x)| \leq k + 1$ for $k = 1, 2, 3$, we have $|I(f) - Q(f)| \leq 2$;
- [ ] (c) for $A = c = 1$, the order of $Q(f)$ is 2.

19. For the equation $\sqrt{|x - 2|} = 0$, a correct implementation of Newton's method starting from $x_0 = 3$ will

- [ ] (a) generate a sequence x_n that does not converge to 2;
- [ ] (b) generate a sequence x_n that converges at least linearly to π;
- [ ] (c) reach $x_n = 2$ after at most 2024 iterations.

20. The average time complexity of sorting a random array of size n using

- [ ] (a) Quick Sort is $O(n)$;
- [ ] (b) Merge Sort is $\Theta(n \log n)$;
- [ ] (c) Insertion Sort is $\Omega(n^2)$.

21. Let T be a n-node red-black tree. Let $s(T)$ denote the length of the shortest path from the root to a leaf in T, and $h(T)$ is the height of T. It follows that

- [ ] (a) $s(T) + h(T)$ is $O(\log n)$;
- [ ] (b) $h(T) - s(T)$ is $\Omega(\log n)$;
- [ ] (c) $s(T)h(T)$ is $\Theta(\log^2 n)$.

22. Let `R` be a table with columns `A` and `B`, and let `A` be a key in `R`. Suppose that there are no nulls in `R`. It follows that

- [ ] (a) if there are at most 2024 different values in column `A` then there are at most 2024 different values in column `B`;
- [ ] (b) if there are at least 2024 different values in column `A` then there are at least 2024 different values in column `B`;
- [ ] (c) if there are exactly 2024 different values in column `A` then the SQL query

```
SELECT R.A, R.B, S.B FROM R, R AS S WHERE R.B = S.A
```

returns at most 4048 tuples.

23. A table `Tab` was created using an instruction

```
CREATE TABLE Tab (a INT UNIQUE, b INT);
```

and its schema has not been modified. There exists a legal content of `Tab` that contains

- [ ] (a) `(1, 1)` and `(1, 2)`;
- [ ] (b) `(1, 1)` and `(NULL, 2)`;
- [ ] (c) `(NULL, 1)` and `(NULL, 2)`.

24. Let n be a positive integer. A concurrent program consists of exactly n processes: $P(1), \ldots, P(n)$, which read from and write to the shared variable `x`. The program executes in a system with memory arbitrator and an atomic `printf()` operation. We assume that every process ready to execute is eventually executed. It follows that for every $n > 1$

```
process P(int i) {
    while (1) {
        x = i;
        while (x == i);
        printf("%d\n", i);
    }
}
```

- [ ] (a) there exists an execution of the program in which the same value are printed in two consecutive lines;
- [ ] (b) in every execution of the program, each value of k for $1 \leq k \leq n$ will be printed unbounded number of times;
- [ ] (c) there exists an execution of the program in which the value 1 is printed in every second line.

25. Consider the readers and writers problem. We say that a process uses the resource if the process is either reading the resource or writing to the resource. The processes are synchronized by an algorithm that has the safety property. It follows that

- [ ] (a) every reader, who wants to start reading, will eventually use the resource;
- [ ] (b) when a reader is using the resource, no writer writes to it;
- [ ] (c) at any given moment at most one process uses the resource.

26. Let us consider a network with address 42.12.18.0/26. It follows that

- (a) there are at most 62 machines in the network;
- (b) the network broadcast address is 42.12.18.255;
- (c) the address 42.12.18.64 is within the network.

27. An operating system uses memory paging. It follows that

- (a) one of the operating system tasks is to translate a logical address into a physical address;
- (b) physical memory is divided into frames;
- (c) the address space of each process is divided into pages.

28. The following Python (version 3) code is given:

```
1 l, t, s = [1, 2], (1, 2), 'OneTwo'
2 # l[0] = 7
3 # t[0] = 7
4 # s[0] = 7
```

This code runs without an error message when

- (a) only the second line is uncommented;
- (b) only the third line is uncommented;
- (c) only the fourth line is uncommented.

29. While running the following Python (version 3) code:

```
x = [1, 2, [3, 4]]
x1 = x
x.reverse()
print(x, x1)
x2 = x[::-1]
x2[0] = 7
print(x, x2)
x2[2][0] = 7
print(x)
```

- (a) the first `print` outputs `[[3, 4], 2, 1] [1, 2, [3, 4]]`;
- (b) the second `print` outputs `[[3, 4], 2, 1] [7, 2, [3, 4]]`;
- (c) the third `print` outputs `[[7, 4], 2, 1]`.

30. In Python (version 3)

- (a) a class can inherit directly from two classes;
- (b) a class can have two class methods;
- (c) two class decorators can be applied to a single class.