ON THE SPECTRAL NORM OF RADEMACHER MATRICES

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Abstract. We discuss two-sided non-asymptotic bounds for the mean spectral norm of non-homogenous weighted Rademacher matrices. We show that the recently formulated conjecture holds up to $\log\log\log n$ factor for arbitrary $n\times n$ Rademacher matrices and the triple logarithm may be eliminated for matrices with $\{0,1\}$-coefficients.

1. Introduction and main results

One of the basic issues of the random matrix theory are bounds on the spectral norm (largest singular value) of various families of random matrices. This question is very well understood for classical ensembles of random matrices [2], when one may use methods based on the large degree of symetry. Recently, a substantial progress was attained in the understanding of unhomogenous models [13], especially in the Gaussian case [9, 3]. However, there are still many open questions in this area, the one concerning Rademacher matrices is discussed here.

In this paper we investigate the mean operator (spectral) norm of weighted Rademacher matrices, i.e., quantities of the form

$$\mathbb{E}\|(a_{i,j}\varepsilon_{i,j})\| := \mathbb{E}\sup_{\|s\|_2,\|t\|_2\le 1}\sum_{i,j} a_{i,j}\varepsilon_{i,j}s_it_j,$$

where $(a_{i,j})$ is a deterministic matrix and $(\varepsilon_{i,j})_{i,j\ge 1}$ is the double indexed sequence of i.i.d. symmetric ± 1 r.v's.

Since operator norm is bigger than length of every column and row we get

$$\mathbb{E}\|(a_{i,j}\varepsilon_{i,j})\| \sim (\mathbb{E}\|(a_{i,j}\varepsilon_{i,j})\|^2)^{1/2} \ge \max\Big\{\max_i \|(a_{i,j})_j\|_2, \max_j \|(a_{i,j})_i\|_2\Big\}.$$

For two nonnegative functions f and g we write $f \gtrsim g$ (or $g \lesssim f$) if there exists an absolute constant C such that $Cf \ge g$; the notation $f \sim g$ means that $f \gtrsim g$ and $g \gtrsim f$. Seginer [11] proved that for $n \ge 2$,

$$\mathbb{E}\|(a_{i,j}\varepsilon_{i,j})_{i,j\le n}\| \lesssim \log^{1/4} n\Big(\max_i \|(a_{i,j})_j\|_2 + \max_j \|(a_{i,j})_i\|_2\Big)$$

and constructed an example showing that in general the constant $\log^{1/4} n$ cannot be improved.

In [8, Theorem 1.1] it was shown that for any matrix (a_{ij}),

$$\mathbb{E}\|(a_{i,j}\varepsilon_{i,j})_{i,j\le n}\| \gtrsim \max_{1\le i\le n}\|(a_{i,j})_j\|_2 + \max_{1\le j\le n}\|(a_{i,j})_i\|_2 + \max_{1\le k\le n}\min_{I\subset[n],|I|\le k}\sup_{\|s\|_2,\|t\|_2\le 1}\Big\|\sum_{i,j\notin I} a_{i,j}\varepsilon_{i,j}s_it_j\Big\|_{\mathrm{Log}\,k}. \qquad (1)$$

Here and in the sequel $\mathrm{Log}\,x = \log(x\vee e)$ and $\|S\|_p = (\mathbb{E}|S|^p)^{1/p}$ denotes L_p-norm of a r.v. S.

It was also conjectured that bound (1) may be reversed, i.e., for any scalar matrix $(a_{i,j})_{i,j\leq n}$,

$$\mathbb{E}\|(a_{i,j}\varepsilon_{i,j})_{i,j\leq n}\| \lesssim \max_{1\leq i\leq n}\|(a_{i,j})_j\|_2 + \max_{1\leq j\leq n}\|(a_{i,j})_i\|_2 + \max_{1\leq k\leq n}\min_{I\subset[n],|I|\leq k}\sup_{\|s\|_2,\|t\|_2\leq 1}\Big\|\sum_{i,j\notin I} a_{i,j}\varepsilon_{i,j}s_it_j\Big\|_{\mathrm{Log}\,k}. \qquad (2)$$

The proof of [8, Remark 4.5], based on the permutation method from [9], shows that in order to establish (2) it is enough to show that for any submatrix $(b_{i,j})_{i,j\leq m}$ of $(a_{i,j})_{i,j\leq n}$ one has

$$\mathbb{E}\|(b_{i,j}\varepsilon_{i,j})_{i,j\leq m}\| \lesssim \max_{1\leq i\leq m}\|(b_{i,j})_j\|_2 + \max_{1\leq j\leq m}\|(b_{i,j})_i\|_2 + R_B(\mathrm{Log}\,m), \qquad (3)$$

where for a matrix $A=(a_{i,j})$ and $p\geq 1$ we put

$$R_A(p) := \sup_{\|s\|_2\leq 1,\|t\|_2\leq 1}\Big\|\sum_{i,j} a_{i,j}\varepsilon_{i,j}s_it_j\Big\|_p.$$

Our first result states that this conjectured bounds holds for $\{0,1\}$-matrices.

Theorem 1. *Inequality* (3) *holds if* $b_{i,j}\in\{0,1\}$ *for any* i,j. *As a consequence, for any* $E\subset[n]\times[n]$,

$$\mathbb{E}\|(\mathbb{1}_E(i,j)\varepsilon_{i,j})_{i,j\leq n}\| \sim \max_{1\leq i\leq n}\|(\mathbb{1}_E(i,j))_j\|_2 + \max_{1\leq j\leq n}\|(\mathbb{1}_E(i,j))_i\|_2 + \max_{1\leq k\leq n}\min_{I\subset[n],|I|\leq k}\sup_{\|s\|_2,\|t\|_2\leq 1}\Big\|\sum_{i,j\notin I}\mathbb{1}_E(i,j)\varepsilon_{i,j}s_it_j\Big\|_{\mathrm{Log}\,k}. \qquad (4)$$

Inequality (3) for $\{0,1\}$-weights is a consequence of the more general Theorem 5 below, applied to the symmetric $2m\times 2m$ {0,1}-matrix $A=\begin{pmatrix} 0 & A\\ A^T & 0\end{pmatrix}$. Estimate (4) follows from (3) as in the proof of [8, Remark 4.5].

Remark 2. [8, Proposition 1.4] gives an equivalent (up to a constant) form of $R_A(p)$ for $\{0,1\}$-matrices:

$$\sup_{\|s\|_2,\|t\|_2\leq 1}\Big\|\sum_{i,j}\mathbb{1}_E(i,j)\varepsilon_{ij}s_it_j\Big\|_p \sim \max_{F\subset E,|F|\leq p}\|(\mathbb{1}_{\{(i,j)\in F\}})\|.$$

Hence the first part of Theorem 1 gives a positive answer to the question posed by Ramon van Handel (private communication):

$$\mathbb{E}\|(\mathbb{1}_E(i,j)\varepsilon_{i,j})_{i,j\leq n}\| \lesssim \max_{1\leq i\leq n}\|(\mathbb{1}_E(i,j))_j\|_2 + \max_{1\leq j\leq n}\|(\mathbb{1}_E(i,j))_i\|_2 + \sup_{F\subset E,|F|\leq\mathrm{Log}\,n}\|(\mathbb{1}_{\{(i,j)\in F\}})_{i,j}\|.$$

One may also state the two-sided estimate (4) in the equivalent way as

$$\mathbb{E}\|(\mathbb{1}_E(i,j)\varepsilon_{i,j})_{i,j\leq n}\| \sim \max_{1\leq i\leq n}\|(\mathbb{1}_E(i,j))_j\|_2 + \max_{1\leq j\leq n}\|(\mathbb{1}_E(i,j))_i\|_2 + \max_{1\leq k\leq n}\min_{I\subset[n],|I|\leq k}\max_{F\subset E,|F|\leq\mathrm{Log}\,k}\|(\mathbb{1}_{\{(i,j)\in F, i,j\notin I\}})_{i,j}\|.$$

Remark 3. Two-sided bound on moments of norms of Rademacher vectors [7] gives that for every $p\geq 1$,

$$\Big(\mathbb{E}\|(a_{i,j}\varepsilon_{i,j})_{i,j\leq n}\|^p\Big)^{1/p} \sim \mathbb{E}\|(a_{i,j}\varepsilon_{i,j})_{i,j\leq n}\| + R_A(p).$$

Thus, the first part of Theorem 1 might be equivalently stated as

$$(\mathbb{E}\|(\mathbb{1}_E(i,j)\varepsilon_{i,j})_{i,j\le n}\|^{2\lfloor \mathrm{Log}\ n\rfloor})^{1/2\lfloor \mathrm{Log} n\rfloor} \sim \max_{1\le i\le n}\|(\mathbb{1}_E(i,j))_j\|_2 + \max_{1\le j\le n}\|(\mathbb{1}_E(i,j))_i\|_2 + \max_{F\subset E, |F|\le \mathrm{Log}\, n}\|(\mathbb{1}_{\{(i,j)\in F\}})_{i,j}\|. \tag{5}$$

It is quite tempting to show (5) for symmetric sets E via a combinatorial method, since for $n\times n$ symmetric matrix A and $k=\lfloor \mathrm{Log}\, n\rfloor$, $\|A\|\sim(\mathrm{tr}(A^{2k}))^{1/2k}$. Such an approach worked for Gaussian matrices [4], but we were not able to apply it in the Rademacher case.

Remark 4. Signed adjacency matrices were studied in [6] in connection with 2-lifts of graphs. [6, Lemma 3.1] shows that to each signed adjacency matrix of a graph G one may associate the 2-lift of G with the set of eigenvalues being the union of the eigenvalues of G and of the signed matrix. Hence Theorem 1 provides an average uniform bound on new eigenvalues of random 2-lifts.

To state results for general matrices we need to introduce some additional notation. We associate to a symmetric matrix $(a_{i,j})_{i,j\le n}$ a graph $G_A=([n],E_A)$, where $(i,j)\in E_A$ iff $i\ne j$ and $a_{i,j}\ne 0$. By d_A we denote the maximal degree of vertices in G_A. Observe that in the case of $\{0,1\}$-matrices $\sqrt{d_A}\|(a_{i,j})\|_\infty=\sqrt{d_A}\le \max_i\|(a_{i,j})_j\|_2$.

Theorem 5. *For any symmetric matrix* $(a_{i,j})_{i,j\le n}$,

$$\mathbb{E}\|(a_{i,j}\varepsilon_{i,j})_{i,j\le n}\| \lesssim \max_i \|(a_{i,j})_j\|_2 + R_A(\mathrm{Log} n) + d_A^{19/40}\|(a_{i,j})\|_\infty. \tag{6}$$

Remark 6. Since $\|(a_{i,i}\varepsilon_{i,i})\|=\max_i|a_{i,i}|$ we may only consider matrices with zero diagonal. Moreover, for any unit vectors s,t we have

$$\begin{aligned}\Big|\sum_{i\ne j}a_{i,j}\varepsilon_{ij}s_it_j\Big| &\le \|(a_{i,j})\|_\infty\sum_{i,j}\mathbb{1}_{\{(i,j)\in E_A\}}\frac{1}{2}(s_i^2+t_j^2)\\ &= \frac{\|(a_{i,j})\|_\infty}{2}\Big(\sum_i s_i^2\sum_j\mathbb{1}_{\{(i,j)\in E_A\}}+\sum_j t_j^2\sum_i\mathbb{1}_{\{(i,j)\in E_A\}}\Big)\\ &\le d_A\|(a_{i,j})\|_\infty.\end{aligned}$$

Hence,

$$\mathbb{E}\|(a_{i,j}\mathbb{1}_{\{i\ne j\}}\varepsilon_{i,j})_{i,j}\|\le d_A\|(a_{i,j})\|_\infty \tag{7}$$

and it is enough to consider only the case $n\ge d_A\ge 3$.

The proof of Theorem 5 takes the most part of the paper. Here we briefly sketch the main ideas of this proof. Bernoulli conjecture, formulated by Talagrand and proven in [5], states that to estimate a supremum of the Bernoulli process one needs to decompose the index set into two parts and estimate supremum over the first part using the uniform bound and over the second part by the supremum of the Gaussian process. Unfortunately, there is no algorithmic method for making such a decomposition – a rule of thumb is that the uniform bound works well for large coefficients and the Gaussian bound for small ones. We try to follow this informal recipe, decompose vectors $s,t\in B_2^n$ into almost "flat" parts and use the uniform bound when infinity norms of these parts are far apart. When they are of the same order we make some further technical adjustments (using properties of the graph G_A) and apply the Gaussian bound. The crucial tool used to estimate the corresponding Gaussian process is an improvement of van Handel's bound [12], provided in Section 2.1.

We postpone the details of the proof till the end of the paper and discuss now some consequences of Theorem 5.

Theorem 7. *For any symmetric matrix $(a_{i,j})_{i,j\le n}$,*
$$\mathbb{E}\|(a_{i,j}\varepsilon_{i,j})_{i,j\le n}\| \le \mathrm{LogLog}(d_A)\Big(\max_i \|(a_{i,j})_j\|_2 + R_A(\mathrm{Log}\, n)\Big).$$

Proof. Let $M := \max_i \|(a_{i,j})_j\|_2$, $u_0 = 1$ and $u_k := \exp(-(20/19)^k)$ for $k = 1, 2, \ldots$. Let k_0 be the smallest integer such that $(\frac{20}{19})^{k_0} \ge \mathrm{Log}(d_A)$. Then $k_0 \sim \mathrm{LogLog}(d_A)$ and $u_{k_0} \le d_A^{-1}$. We have
$$\mathbb{E}\|(a_{i,j}\varepsilon_{i,j})\| \le \mathbb{E}\|(a_{i,j}\mathbb{1}_{\{|a_{i,j}|\le u_{k_0}M\}}\varepsilon_{i,j})\| + \sum_{k=1}^{k_0}\mathbb{E}\|(a_{i,j}\mathbb{1}_{\{u_kM<|a_{i,j}|\le u_{k-1}M\}}\varepsilon_{i,j})\|.$$
For any k,
$$d_k := \max_i |\{j\colon\ |a_{i,j}| > u_kM\}| \le u_k^{-2},$$
so by Theorem 5
$$\mathbb{E}\|(a_{i,j}\mathbb{1}_{\{u_kM<|a_{i,j}|\le u_{k-1}M\}}\varepsilon_{i,j})\| \lesssim M + R_A(\mathrm{Log}\, n) + d_k^{19/40}u_{k-1}M \lesssim M + R_A(\mathrm{Log}\, n).$$
Moreover, using again Theorem 5
$$\mathbb{E}\|(a_{i,j}\mathbb{1}_{\{|a_{i,j}|\le u_{k_0}M\}}\varepsilon_{i,j})\| \lesssim M + R_A(\mathrm{Log}\, n) + d_A^{19/40}u_{k_0}M \lesssim M + R_A(\mathrm{Log}\, n) \qquad \square$$

Remark 8. In Theorems 5 and 7 we do not assume the symmetry of $(\varepsilon_{i,j})_{i,j}$. However analogous bounds holds for $\mathbb{E}\|(a_{i,j}\tilde{\varepsilon}_{i,j})_{i,j}\|$, where $(\tilde{\varepsilon}_{i,j})_{i,j}$ is the symmetric Rademacher matrix (i.e., $\tilde{\varepsilon}_{i,j} = \tilde{\varepsilon}_{j,i} = \varepsilon_{i,j}$ for $i \ge j$), since
$$\begin{aligned}\mathbb{E}\|(a_{i,j}\tilde{\varepsilon}_{i,j})_{i,j}\| &\le \mathbb{E}\|(a_{i,j}\tilde{\varepsilon}_{i,j}\mathbb{1}_{\{i\le j\}})_{i,j}\| + \mathbb{E}\|(a_{i,j}\tilde{\varepsilon}_{i,j}\mathbb{1}_{\{i>j\}})_{i,j}\|\\ &= \mathbb{E}\|(a_{i,j}\varepsilon_{i,j}\mathbb{1}_{\{i\le j\}})_{i,j}\| + \mathbb{E}\|(a_{i,j}\varepsilon_{i,j}\mathbb{1}_{\{i>j\}})_{i,j}\| \le 2\mathbb{E}\|(a_{i,j}\varepsilon_{i,j})_{i,j}\|.\end{aligned}$$

Obviously, $d_A \le n$, so Theorem 7 (together with the standard symmetrization argument) implies that bounds (3) and (2) hold up double logarithms of n. However, decomposing matrix into two parts and using the Bandeira-van Handel bound one may derive conjectured upper bounds up to triple logarithms.

Theorem 9. *For any matrix $(a_{i,j})_{i,j\le n}$,*
$$\mathbb{E}\|(a_{i,j}\varepsilon_{i,j})_{i,j\le n}\| \lesssim \mathrm{LogLogLog}\, n\Big(\max_{1\le i\le n}\|(a_{i,j})_j\|_2 + \max_{1\le j\le n}\|(a_{i,j})_i\|_2 + R_A(\mathrm{Log}\, n)\Big)$$
and
$$\begin{aligned}\mathbb{E}\|(a_{i,j}\varepsilon_{i,j})_{i,j\le n}\| \lesssim \mathrm{LogLogLog}\, n\Bigg(&\max_{1\le i\le n}\|(a_{i,j})_j\|_2 + \max_{1\le j\le n}\|(a_{i,j})_i\|_2\\ &+ \max_{1\le k\le n}\ \min_{I\subset[n],|I|\le k}\ \sup_{\|s\|_2,\|t\|_2\le 1}\Bigg\|\sum_{i,j\notin I}a_{i,j}\varepsilon_{i,j}s_it_j\Bigg\|_{\mathrm{Log}\, k}\Bigg).\end{aligned}$$

Proof. Assume first that the matrix $(a_{i,j})$ is symmetric. Let $g_{i,j}$ be iid $\mathcal{N}(0,1)$ r.v's. The result of Bandeira and van Handel [4] implies
$$\mathbb{E}\|(a_{i,j}\varepsilon_{i,j})_{i,j\le n}\| \lesssim \mathbb{E}\|(a_{i,j}g_{i,j})_{i,j\le n}\| \lesssim \max_{1\le i\le m}\|(a_{i,j})_j\|_2 + \sqrt{\log n}\|(a_{i,j})\|_\infty. \tag{8}$$

Put $M := \max_{1\le i\le m}\|(a_{i,j})_j\|_2$. Estimate (8) yields
$$\mathbb{E}\|(a_{i,j}\mathbb{1}_{\{|a_{i,j}|\le M\mathrm{Log}^{-1/2}n\}}\varepsilon_{i,j})_{i,j\le n}\| \lesssim \max_{1\le i\le m}\|(a_{i,j})_j\|_2.$$

We have
$$\max_i |\{j\colon\ |a_{i,j}| > M\mathrm{Log}^{-1/2}n\}| \le \mathrm{Log}\, n,$$

hence Theorem 7, applied to a matrix $(a_{i,j}\mathbb{1}_{\{|a_{i,j}|>M\mathrm{Log}^{-1/2}n\}})_{i,j\le n}$ implies

$$\mathbb{E}\|(a_{i,j}\mathbb{1}_{\{|a_{i,j}|>M\mathrm{Log}^{-1/2}n\}}\varepsilon_{i,j})_{i,j\le n}\| \lesssim \mathrm{LogLogLog}\, n\Big(\max_{1\le i\le m}\|(a_{i,j})_j\|_2 + R_A(\mathrm{Log}\, n)\Big).$$

Therefore, for any symmetric matrix $(a_{i,j})$,

$$\mathbb{E}\|(a_{i,j}\varepsilon_{i,j})_{i,j\le n}\| \lesssim \mathrm{LogLogLog}\, n\Big(\max_{1\le i\le m}\|(a_{i,j})_j\|_2 + R_A(\mathrm{Log}\, n)\Big). \tag{9}$$

Now, supppose that matrix $(a_{i,j})$ is arbitrary. Applying (9) to the symmetric $2n\times 2n$ matrix $\begin{pmatrix} 0 & A \\ A^T & 0\end{pmatrix}$ we get the first part of the assertion.

The second part follows from the first one as in the proof of [8, Remark 4.5]. □

Organization of the paper. In Section 2 we discuss basic tools used in the sequel, including an improvement of the van Handel bound for norms of Gaussian matrices from [12]. In Section 3 we derive a weaker version of Theorem 7 with $\log(d_A)$ instead of $\log\log(d_A)$ factors. The last section is devoted to the proof of Theorem 5.

2. Tools

We will use the following estimate for suprema of Rademachers. It is a special case of [1, Lemma 5.10].

Proposition 10. *Let $T_1,\ldots,T_n$ be nonempty bounded subsets of $\mathbb{R}^N$. Then*

$$\mathbb{E}\max_{k\le n}\sup_{t\in T_k}\sum_{i=1}^N t_i\varepsilon_i \lesssim \max_{k\le n}\mathbb{E}\sup_{t\in T_k}\sum_{i=1}^N t_i\varepsilon_i + \max_{k\le n}\sup_{t\in T_k}\Big\|\sum_{i=1}^N t_i\varepsilon_i\Big\|_{\mathrm{Log}\, n}.$$

Another useful result is the estimate on the number of connected subsets of a graph.

Lemma 11. *Let $H=(V_H,E_H)$ be a graph with n_H vertices and maximal degree d_H.*
i) For a fixed $v\in V$ the number of connected subsets $I\subset V_H$ with cardinality k containing v is at most $(4d_H)^{k-1}$.
ii) The number of all connected subsets $I\subset V_H$ with cardinality k is not bigger than $n_H(4d_H)^{k-1}$.

Proof. i) The connected subset I may be chosen by first choosing its spanning tree rooted at v and then labelling the vertices of the tree. The number of unlabelled rooted trees is less than the number of oriented trees with k vertices, i.e., less than the $(k-1)$-th Catalan number $C_{k-1}\le 4^{k-1}$. The root of the tree is v and the rest of vertices may be labelled in at most d_H^{k-1} ways.
Part i) of the assertion immediately yields part ii). □

2.1. **Van Handel-type bound.** In this part we will establish the following improvement on van Handel's bound [12].

Proposition 12. *For any $n\times m$ matrix $(a_{i,j})_{i\le m,j\le n}$ and $b\in(0,1]$ we have*

$$\mathbb{E}\sup_{s\in B_2^m\cap bB_\infty^m}\sup_{t\in B_2^n\cap bB_\infty^n}\sum_{i\le m,j\le n}a_{i,j}\varepsilon_{i,j}s_it_j \lesssim \max_i\|(a_{i,j})_j\|_2+\max_j\|(a_{i,j})_i\|_2 + \mathrm{Log}((n+m)b^2)\|(a_{i,j})_{i,j}\|_\infty.$$

Let us first formulate and prove a symmetric variant of Propostion 12.

Proposition 13. *Let $(\tilde{\varepsilon}_{i,j})_{i,j}$ be a symmetric Rademacher matrix. Then for any symmetric matrix $(a_{i,j})_{i,j\le n}$ and any $b\in(0,1]$,*

$$\mathbb{E}\sup_{s,t\in B_2^n\cap bB_\infty^n}\sum_{i,j\le n}a_{i,j}\tilde{\varepsilon}_{i,j}s_it_j\lesssim\max_i\|(a_{i,j})_j\|_2+\mathrm{Log}(nb^2)\|(a_{i,j})_{i,j}\|_\infty.$$

The proof uses the following, quite standard, technical lemma.

Lemma 14. *Let $Y_1,\ldots,Y_n$ be r.v's and $m_i,\sigma_i\ge 0$ be such that*

$$\mathbb{P}(|Y_i|\ge m_i+u\sigma_i)\le e^{-u^2/2}\quad\textit{for every } u\ge 0 \textit{ and } i=1,\ldots,n.$$

Then

$$\mathbb{E}\sup_{s\in B_2^n\cap bB_\infty^n}\sum_{i=1}^n s_i^2Y_i\lesssim\max_i m_i+\sqrt{\mathrm{Log}(nb^2)}\max_i\sigma_i$$

and

$$\mathbb{E}\sup_{s\in B_2^n\cap bB_\infty^n}\sqrt{\sum_{i=1}^n s_i^2Y_i^2}\lesssim\max_i m_i+\sqrt{\mathrm{Log}(nb^2)}\max_i\sigma_i.$$

Proof. Let $(Y_1^*,\ldots,Y_n^*)$ be a nondecreasing rearrangement of $|Y_1|,\ldots,|Y_n|$. We set $k=n$ if $b^2\le 1/n$, otherwise we choose $1\le k\le n-1$ such that $\frac{1}{k+1}<b^2\le\frac{1}{k}$. Then $\mathrm{Log}(nb^2)\sim\mathrm{Log}(n/k)$ and

$$\sup_{s\in B_2^n\cap bB_\infty^n}\sum_{i=1}^n s_i^2Y_i\le\frac{1}{k}(Y_1^*+\ldots+Y_k^*).$$

A standard argument shows that $\mathbb{E}Y_l^*\le(\mathbb{E}|Y_l^*|^2)^{1/2}\lesssim\max_i m_i+\mathrm{Log}^{1/2}(n/l)\max_i\sigma_i$. Thus

$$\mathbb{E}\sup_{s\in B_2^n\cap bB_\infty^n}\sum_{i=1}^n s_i^2Y_i\lesssim\max_i m_i+\frac{1}{k}\sum_{l=1}^k\sqrt{\mathrm{Log}(\frac{n}{l})}\max_i\sigma_i\lesssim\max_i m_i+\sqrt{\mathrm{Log}(\frac{n}{k})}\max_i\sigma_i$$

and

$$\begin{aligned}\mathbb{E}\sup_{s\in B_2^n\cap bB_\infty^n}\sqrt{\sum_{i=1}^n s_i^2Y_i^2}&\le\mathbb{E}\sqrt{\frac{1}{k}(|Y_1^*|^2+\ldots+|Y_k^*|^2)}\le\sqrt{\frac{1}{k}\sum_{l=1}^k\mathbb{E}|Y_l^*|^2}\\&\lesssim\max_i m_i+\sqrt{\frac{1}{k}\sum_{l=1}^k\mathrm{Log}(\frac{n}{l})}\max_i\sigma_i\\&\lesssim\max_i m_i+\sqrt{\mathrm{Log}(\frac{n}{k})}\max_i\sigma_i.\end{aligned}$$

□

Proof of Proposition 13. Let $(g_{i,j})_{i,j\le n}$ be a symmetric Gaussian matrix (i.e., $g_{i,j}=g_{j,i}$ and $(g_{i,j})_{i\ge j}$ are iid $\mathcal{N}(0,1)$ r.v's), independent of $\tilde{\varepsilon}_{i,j}$. We have for any matrix norm $\|\cdot\|$,

$$\mathbb{E}\|(a_{i,j}g_{i,j})\|=\mathbb{E}\|(a_{i,j}\tilde{\varepsilon}_{i,j}|g_{i,j}|)\|\ge\mathbb{E}\|(a_{i,j}\tilde{\varepsilon}_{i,j}\mathbb{E}|g_{i,j}|)\|=\sqrt{\frac{2}{\pi}}\mathbb{E}\|(a_{i,j}\tilde{\varepsilon}_{i,j})\|.$$

For any symmetric matrix B we have $\langle Bs,t\rangle=\frac14\big(\langle B(s+t),s+t\rangle-\langle B(s-t),s-t\rangle\big)$, hence,

$$\sup_{s,t\in B_2^n\cap bB_\infty^n}\langle Bs,t\rangle\le 2\sup_{s\in B_2^n\cap bB_\infty^n}|\langle Bs,s\rangle|.$$

Therefore,

$$\begin{aligned}\mathbb{E}\sup_{s\in B_2^n\cap bB_\infty^n}\sum_{i,j\le n}a_{i,j}\tilde{\varepsilon}_{i,j}s_it_j &\lesssim \mathbb{E}\sup_{s\in B_2^n\cap bB_\infty^n}\Big|\sum_{i,j\le n}a_{i,j}g_{i,j}s_is_j\Big|\\ &\le \mathbb{E}\sup_{s\in B_2^n\cap bB_\infty^n}\sum_{i,j\le n}a_{i,j}g_{i,j}s_is_j+\mathbb{E}\sup_{s\in B_2^n\cap bB_\infty^n}\sum_{i,j\le n}(-a_{i,j}g_{i,j}s_is_j)\\ &= 2\mathbb{E}\sup_{s\in B_2^n\cap bB_\infty^n}\sum_{i,j\le n}a_{i,j}g_{i,j}s_is_j.\end{aligned}$$

Now we follow van Handel's approach from [12]. Let $g_1,g_2,\ldots,g_n$ be iid $\mathcal{N}(0,1)$ r.v's and $Y=(Y_1,\ldots,Y_n)\sim\mathcal{N}(0,B_-)$, where B_- is the negative part of $B=(a_{i,j}^2)$. Define the new Gaussian process Z_s by

$$Z_s=2\sum_{i=1}^n s_ig_i\sqrt{\sum_{j=1}^n a_{ij}^2s_j^2}+\sum_{i=1}^n s_i^2Y_i.$$

It is shown in [12] (see the proof of Theorem 4.1 therein) that for any $s,s'\in\mathbb{R}^n$

$$\mathbb{E}\Big|\sum_{i,j\le n}a_{i,j}g_{i,j}(s_is_j-s_i's_j')\Big|^2\le\mathbb{E}|Z_s-Z_{s'}|^2.$$

Hence the Slepian-Fernique inequality [10, Theorem 3.15]. yields

$$\mathbb{E}\sup_{s\in B_2^n\cap bB_\infty^n}\sum_{i,j\le n}a_{i,j}g_{i,j}s_is_j\le\mathbb{E}\sup_{s\in B_2^n\cap bB_\infty^n}Z_s.$$

Variables Y_i are centered Gaussian and (see the proof of Corollary 4.2 in [12]) $(\mathbb{E}Y_i^2)^{1/2}\le\|(a_{i,j})_j\|_4$. Hence Lemma 14 applied with $m_i=0$ and $\sigma_i=\|(a_{i,j})_j\|_4$ yields

$$\begin{aligned}\mathbb{E}\sup_{s\in B_2^n\cap bB_\infty^n}\sum_{i=1}^n s_i^2Y_i &\lesssim \sqrt{\mathrm{Log}(nb^2)}\max_i\|(a_{i,j})_j\|_4\\ &\le \sqrt{\mathrm{Log}(nb^2)}\max_{i,j}|a_{i,j}|^{1/2}\max_j\|(a_{i,j})_j\|_2^{1/2}\\ &\le \max_i\|(a_{i,j})_j\|_2+\mathrm{Log}(nb^2)\|(a_{i,j})_{i,j}\|_\infty.\end{aligned}$$

We have

$$\mathbb{E}\sup_{s\in B_2^n\cap bB_\infty^n}\sum_{i=1}^n s_ig_i\sqrt{\sum_j a_{ij}^2s_j^2}\le\mathbb{E}\sup_{s\in B_2^n\cap bB_\infty^n}\sqrt{\sum_{i,j}a_{ij}^2s_j^2g_i^2}=\mathbb{E}\sup_{s\in B_2^n\cap bB_\infty^n}\sqrt{\sum_j s_j^2V_j^2},$$

where $V_j=\sqrt{\sum_i a_{ij}^2g_i^2}$. The Gaussian concentration [10, Lemma 3.1] yields

$$\mathbb{P}(|V_j|\ge\|(a_{i,j})_i\|_2+t\|(a_{i,j})_i\|_\infty)\le e^{-t^2/2},$$

so Lemma 14 applied with $Y_j=V_j$, $m_j=\|(a_{i,j})_i\|_2$ and $\sigma_j=\|(a_{i,j})_i\|_\infty$ yields

$$\mathbb{E}\sup_{s\in B_2^n\cap bB_\infty^n}\sum_{i=1}^n s_i\sqrt{\sum_j a_{ij}^2s_j^2}g_i\lesssim\max_j\|(a_{i,j})_i\|_2+\sqrt{\mathrm{Log}(nb^2)}\|(a_{i,j})_{i,j}\|_\infty.\qquad\square$$

Proof of Proposition 12. We apply Proposition 13 to the symmetric $(n+m)\times(n+m)$ matrix $\tilde{A}=(\tilde{a}_{i,j})$ of the form $\tilde{A}=\begin{pmatrix}0 & A\\ A^T & 0\end{pmatrix}$. Observe that $\max_{i,j}|\tilde{a}_{i,j}|=\max_{i,j}|a_{i,j}|$,

$$\max_i\|(\tilde{a}_{i,j})_j\|_2=\max\Big\{\max_i\|(a_{i,j})_j\|_2,\max_j\|(a_{i,j})_i\|_2\Big\}$$

and

$$\mathbb{E} \sup_{s,t\in B_2^{n+m}\cap bB_\infty^{n+m}} \sum_{i,j\leq n+m} \tilde{a}_{i,j}\tilde{\varepsilon}_{i,j}s_it_j \geq \mathbb{E} \sup_{s\in B_2^m\cap bB_\infty^m} \sup_{t\in B_2^n\cap bB_\infty^n} \sum_{i\leq m,j\leq n} a_{i,j}\varepsilon_{i,j}s_it_j. \qquad \square$$

3. Bounds up to polylog factors

In this section we derive weaker estimates than in Theorem 7 (with powers of $\log d_A$ instead of $\log\log(d_A)$). They will be used in the proof of Theorem 5 to estimate the parts of Bernoulli process $(\sum_{i,j} a_{i,j}\varepsilon_{i,j}s_it_j)_{s,t\in B_2^n}$, where coefficients s_i and t_j are of the same order.

Let us first introduce the notation which will be used till the end of the paper. Recall that $G_A = ([n], E_A)$ is a graph associated to a given symmetric matrix $A = (a_{i,j})_{i,j\leq n}$. By $\rho = \rho_A$ we denote the distance on $[n]$ induced by E_A. For $r = 1, 2, \ldots$ we put $G_r = G_r(A) = ([n], E_{A,r})$, where $(i,j) \in E_{A,r}$ iff $\rho(i,j) \leq r$. In particular $G_1 = ([n], E_A)$ and the maximal degree of G_r is at most $d_A + d_A(d_A-1) + \ldots + d_A(d_A-1)^{r-1} \leq d_A^r$. We say that a subset of $[n]$ is *r-connected* if it is connected in G_r.

We denote by $\mathcal{I}(k) = \mathcal{I}(k,n)$ the family of all subsets of $[n]$ of cardinality k and by $\mathcal{I}_r(k) = \mathcal{I}_r(k,A)$ the family of all r-connected subsets of $[n]$ of cardinality k.

For a set $I \subset [n]$ and a vertex $j \in [n]$ we write $I \sim_A j$ if $(i,j) \in E_A$ for some $i \in I$. By $I' = I'(A)$ we denote the set of all neighbours of I in G_1 and by $I'' = I''(A)$ the set of all neighbours of I' in G_1, i.e.,

$$I' = \{j \in [n]\colon\ \exists_{i\in I}\ (i,j) \in E_A\}, \quad I'' = \{i \in [n]\colon\ \exists_{i_0\in I, j\in[n]}\ (i_0,j),(i,j) \in E_A\}. \tag{10}$$

Observe that I is a subset of I'', but does not have to be a subset of I'. Moreover $|I'| \leq d_A|I|$ and $|I''| \leq d_A^2|I|$.

By Remark 6 we may and will assume that $a_{i,i} = 0$ for all i.

For $1 \leq k,l \leq n$ define random variables

$$X_{k,l} = X_{k,l}(A) := \frac{1}{\sqrt{kl}} \max_{I\in\mathcal{I}(k),J\in\mathcal{I}(l)} \max_{\eta_i,\eta_j'=\pm1} \sum_{i\in I,j\in J} a_{i,j}\varepsilon_{i,j}\eta_i\eta_j'$$

and their 4-connected counteparts

$$\overline{X}_{k,l} = \overline{X}_{k,l}(A) := \frac{1}{\sqrt{kl}} \max_{I\in\mathcal{I}_4(k),J\in\mathcal{I}_4(l)} \max_{\eta_i,\eta_j'=\pm1} \sum_{i\in I,j\in J} a_{i,j}\varepsilon_{i,j}\eta_i\eta_j'.$$

Set

$$X = X(A) := \max_{1\leq k,l\leq n} X_{k,l} = \max_{\emptyset\neq I,J\subset V} \frac{1}{\sqrt{|I||J|}} \max_{\eta_i,\eta_j'=\pm1} \sum_{i\in I,j\in J} a_{i,j}\varepsilon_{i,j}\eta_i\eta_j. \tag{11}$$

Variables $\overline{X}_{k,l}$ are easier to estimate than $X_{k,l}$, since the number of 4-connected subsets is much smaller than the number of all subsets – calculations based on this idea are made in Lemma 16. Lemma 15 shows that in expectation these variables do not differ too much.

Lemma 15. *For any* $1 \leq k,l \leq n$,

$$\mathbb{E}X_{k,l} \lesssim \max_{1\leq k'\leq k,1\leq l'\leq l} \mathbb{E}\overline{X}_{k',l'} + R_A(\mathrm{Log}(kl)).$$

Proof. Let us first fix sets $I \in \mathcal{I}(k)$ and $J \in \mathcal{I}(l)$. Let $I_1, \ldots, I_r$ be connected components of $I \cap J'$ in G_2 and $J_u := J \cap I_u'$. Then sets $J_1, \ldots, J_r$ are disjoint and 4-connected subsets of J.

Hence, for every $\eta_i, \eta'_j = \pm 1$ we have

$$\sum_{i\in I, j\in J} a_{i,j}\varepsilon_{i,j}\eta_i\eta'_j = \sum_{i\in I\cap J', j\in J} a_{i,j}\varepsilon_{i,j}\eta_i\eta'_j = \sum_{u=1}^{r}\sum_{i\in I_u, j\in J_u} a_{i,j}\varepsilon_{i,j}\eta_i\eta'_j \le \sum_{u=1}^{r}\overline{X}_{|I_u|,|J_u|}\sqrt{|I_u||J_u|}$$
$$\le \max_{k'\le k, l'\le l}\overline{X}_{k',l'}\sum_{u=1}^{r}\sqrt{|I_u||J_u|} \le \max_{k'\le k, l'\le l}\overline{X}_{k',l'}\Big(\sum_{u=1}^{r}|I_u|\Big)^{1/2}\Big(\sum_{u=1}^{r}|J_u|\Big)^{1/2}$$
$$\le \max_{k'\le k, l'\le l}\overline{X}_{k',l'}\sqrt{|I||J|}.$$

Taking the supremum over all sets $I \in \mathcal{I}(k)$, $J \in \mathcal{I}(l)$ and $\eta_i, \eta'_j = \pm 1$ we get

$$X_{k,l} \le \max_{k'\le k, l'\le l}\overline{X}_{k',l'}. \tag{12}$$

Observe that

$$\max_{I\in\mathcal{I}_4(k'), J\in\mathcal{I}_4(l')}\max_{\eta_i,\eta'_j=\pm1}\frac{1}{\sqrt{k'l'}}\Big\|\sum_{i\in I, j\in J} a_{i,j}\varepsilon_{i,j}\eta_i\eta'_j\Big\|_{\mathrm{Log}(kl)} \le \sup_{\|s\|_2,\|t\|_2\le 1}\Big\|\sum_{i,j} a_{i,j}\varepsilon_{i,j}s_it_j\Big\|_{\mathrm{Log}(kl)}$$
$$= R_A(\mathrm{Log}(kl)).$$

Thus, by Proposition 10

$$\mathbb{E}\max_{k'\le k, l'\le l}\overline{X}_{k',l'} \lesssim \max_{k'\le k, l'\le l}\mathbb{E}\overline{X}_{k',l'} + R_A(\mathrm{Log}(kl)).$$

□

Lemma 16. *We have for any* $1 \le k, l \le n$,

$$\mathbb{E}\overline{X}_{k,l} \lesssim \sqrt{\mathrm{Log}\, d_A}\max_i \|(a_{i,j})_j\|_2 + R_A(\mathrm{Log}\, n).$$

Proof. Obviously $\overline{X}_{k',l'} \le \|(a_{i,j}\varepsilon_{i,j})\|$, so by (7) we may assume that $n \ge d_A \ge 3$. By the symmetry it is enough to consider only the case $l \ge k$. By Lemma 11, $2^k|\mathcal{I}_4(k)| \le n(8d_A^4)^k \le nd_A^{6k}$.

We have

$$\overline{X}_{k,l} \le \frac{1}{\sqrt{k}}\max_{I\in\mathcal{I}_4(k)}\max_{\eta_i=\pm1}\sup_{\|t\|_2\le1}\sum_{i\in I}\sum_j a_{i,j}\varepsilon_{i,j}\eta_it_j.$$

For any fixed $I \in \mathcal{I}_4(k)$ and $\eta_i = \pm 1$,

$$\mathbb{E}\frac{1}{\sqrt{k}}\sup_{\|t\|_2\le1}\sum_{i\in I}\sum_j a_{i,j}\varepsilon_{i,j}\eta_it_j = \frac{1}{\sqrt{k}}\mathbb{E}\Big(\sum_j\Big(\sum_{i\in I}a_{i,j}\varepsilon_{i,j}\eta_i\Big)^2\Big)^{1/2}$$
$$\le \frac{1}{\sqrt{k}}\Big(\sum_j\mathbb{E}\Big(\sum_{i\in I}a_{i,j}\varepsilon_{i,j}\eta_i\Big)^2\Big)^{1/2} = \frac{1}{\sqrt{k}}\Big(\sum_j\sum_{i\in I}a_{i,j}^2\Big)^{1/2}$$
$$= \frac{1}{\sqrt{k}}\Big(\sum_{i\in I}\sum_j a_{i,j}^2\Big)^{1/2} \le \max_i\|(a_{i,j})_j\|_2.$$

In the case $n \ge d_A^{6k}$, $\log(2^k|\mathcal{I}_4(k)|) \lesssim \log n$ and

$$\max_{I\in\mathcal{I}_4(k)}\max_{\eta_i=\pm1}\sup_{\|t\|_2\le1}\frac{1}{\sqrt{k}}\Big\|\sum_{i\in I}\sum_j a_{i,j}\varepsilon_{i,j}\eta_it_j\Big\|_{\mathrm{Log}(2^k|\mathcal{I}_4(k)|)} \lesssim R_A(\log n).$$

In the case $n \le d_A^{6k}$ we have $\log(2^k|\mathcal{I}_4(k)|) \lesssim k\log d_A$ and

$$\begin{aligned}
\max_{I\in\mathcal{I}_4(k)} \max_{\eta_i=\pm1} \sup_{\|t\|_2\le 1} \frac{1}{\sqrt{k}} \Big\| \sum_{i\in I}\sum_j a_{i,j}\varepsilon_{i,j}\eta_i t_j \Big\|_{\mathrm{Log}(2^k|\mathcal{I}_4(k)|)} \\
\lesssim \max_{I\in\mathcal{I}_4(k)} \max_{\eta_i=\pm1} \sup_{\|t\|_2\le 1} \sqrt{\log d_A}\Big(\sum_{i\in I}\sum_j (a_{i,j}\eta_i t_j)^2\Big)^{1/2} \\
= \max_{I\in\mathcal{I}_4(k)} \max_j \sqrt{\log d_A}\Big(\sum_{i\in I} a_{i,j}^2\Big)^{1/2} \le \sqrt{\log d_A}\max_j \|(a_{i,j})_i\|_2.
\end{aligned}$$

The assertion follows by Proposition 10. □

Corollary 17. *We have*

$$\mathbb{E}X = \mathbb{E}\max_{\emptyset\neq I,J\subset V} \frac{1}{\sqrt{|I||J|}} \max_{\eta_i,\eta_j'=\pm1} \sum_{i\in I,j\in J} a_{i,j}\varepsilon_{i,j}\eta_i\eta_j \lesssim \sqrt{\mathrm{Log}\, d_A}\max_i \|(a_{i,j})_j\|_2 + R_A(\mathrm{Log}\, n).$$

Proof. Lemmas 15 and 16 imply that for a fixed $1\le k,l\le n$

$$\mathbb{E}X_{k,l} \lesssim \max_{k'\le k, l'\le l} \mathbb{E}\overline{X}_{k',l'} + R_A(\mathrm{Log}(kl)) \lesssim \sqrt{\mathrm{Log}\, d_A}\max_i \|(a_{i,j})_j\|_2 + R_A(\mathrm{Log}\, n).$$

Moreover,

$$\begin{aligned}
\max_{1\le k,l\le n} \max_{I\in\mathcal{I}(k),J\in\mathcal{I}(l)} \frac{1}{\sqrt{kl}} \max_{\eta_i,\eta_j'=\pm1} \Big\| \sum_{i\in I,j\in J} a_{i,j}\varepsilon_{i,j}\eta_i\eta_j \Big\|_{\mathrm{Log}(n^2)} \\
\le \sup_{\|t\|_2\le1,\|s\|_2\le1} \Big\| \sum_{i,j\in V} a_{i,j}\varepsilon_{i,j}t_i s_j \Big\|_{2\mathrm{Log}\, n} \lesssim R_A(\mathrm{Log}\, n)
\end{aligned}$$

and the assertion follows by Proposition 10. □

Proposition 18. *For any symmeric matrix $(a_{i,j})_{i,j\le n}$ we have*

$$\mathbb{E}\big\|(a_{i,j}\varepsilon_{i,j})_{i,j\le n}\big\| \lesssim \mathrm{Log}^{3/2}(d_A)\max_i \|(a_{i,j})_j\|_2 + \mathrm{Log}(d_A)R_A(\mathrm{Log}\, n).$$

Proof. By Remark 6 we may assume that $a_{i,i}=0$ for all i and $n\ge d_A\ge 3$.

For vectors s,t and integers k,l we define sets

$$I_k(s) = \{i\in V\colon\ e^{-k-1} < |s_i| \le e^{-k}\},\quad J_l(t) = \{j\in V\colon\ e^{-l-1} < |t_j| \le e^{-l}\}.$$

Observe that for any s,t,k,l,

$$\sum_{i\in I_k(s),j\in J_l(t)} a_{i,j}\varepsilon_{i,j}s_i t_j \le e^{-k-l} \max_{\eta_i,\eta_j'=\pm1} \sum_{i\in I_k(s),j\in J_l(t)} a_{i,j}\varepsilon_{i,j}\eta_i\eta_j',$$

therefore

$$\big\|(a_{i,j}\varepsilon_{i,j})_{i,j\in V}\big\| \le \sup_{\|s\|_2,\|t\|_2\le1} \sum_{k,l} e^{-k-l} \sup_{\eta_i,\eta_j'=\pm1} \sum_{i\in I_k(s),j\in J_l(t)} a_{i,j}\varepsilon_{i,j}\eta_i\eta_j'.$$

We have

$$
\begin{aligned}
\sum_k \sum_{l\geq k+\log d_A} e^{-k-l} &\max_{\eta_i,\eta'_j=\pm 1} \sum_{i\in I_k(s), j\in J_l(t)} a_{i,j}\varepsilon_{i,j}\eta_i\eta'_j \\
&\leq \sum_k \sum_{l\geq k+\log d_A} e^{-k} \sum_{i\in I_k(s), j\in J_l(t)} |a_{i,j}| e^{-l} \\
&\leq \sum_k e^{-k} \sum_{i\in I_k(s)} \sum_j |a_{i,j}| \sum_{l\geq k+\log d_A} e^{-l} \mathbb{1}_{\{j\in J_l(t)\}} \\
&\leq \sum_k \sum_{i\in I_k(s)} \sum_j |a_{i,j}| e^{-2k-\log d_A} \leq \|(a_{i,j})\|_\infty \sum_k \sum_{i\in I_k(s)} e^{-2k} \\
&\leq \|(a_{i,j})\|_\infty \sum_k \sum_{i\in I_k(s)} e^2 s_i^2 = e^2\|s\|_2^2 \|(a_{i,j})\|_\infty.
\end{aligned}
$$

In the same way we show that

$$
\sum_l \sum_{k\geq l+\log d_A} e^{-k-l} \max_{\eta_i,\eta'_j=\pm 1} \sum_{i\in I_k(s), j\in J_l(t)} a_{i,j}\varepsilon_{i,j}\eta_i\eta'_j \leq e^2\|t\|_2^2\|(a_{i,j})\|_\infty.
$$

Moreover, for any s,t,

$$
\begin{aligned}
\sum_{k,l\colon |k-l|<\log d_A} e^{-k-l} &\sup_{\eta_i,\eta'_j=\pm 1} \sum_{i\in I_k(s), j\in J_l(t)} a_{i,j}\varepsilon_{i,j}\eta_i\eta'_j \\
&\leq X \sum_{k,l\colon |k-l|<\log d_A}^{\infty} e^{-k-l}\sqrt{|I_k(s)||J_l(t)|}.
\end{aligned}
$$

For any fixed integer r

$$
\begin{aligned}
\sum_k e^{-k-(k+r)}\sqrt{|I_k(s)||J_{k+r}(t)|} &\leq \Big(\sum_k e^{-2k}|I_k(s)|\Big)^{1/2}\Big(\sum_k e^{-2(k+r)}|J_{k+r}(t)|\Big)^{1/2} \\
&\leq e^2\|t\|_2\|s\|_2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\big\|(a_{i,j}\varepsilon_{i,j})_{i,j\leq n}\big\| &\leq \sup_{\|s\|_2,\|t\|_2\leq 1} e^2((\|s\|^2+\|t\|^2)\|(a_{i,j})\|_\infty + (2\log d_A+1)X\|t\|_2\|s\|_2) \\
&\leq e^2(2\|(a_{i,j})\|_\infty + (2\log d_A+1)X)
\end{aligned}
$$

and the assertion follows by Corollary 17. $\square$

4. Proof of Theorem 5

By Remark 6 we may assume that $a_{i,i}=0$ for all i and $n\geq d_A\geq 3$.
For $k=1,2,\dots$ and $t,s\in B_2^n$ we define

$$
I_k(s) := \{i\colon\ d_A^{-k/40} < |s_i| \leq d_A^{(1-k)/40}\},\quad J_l(t) := \{j\colon\ d_A^{-l/40} < |t_j| \leq d_A^{(1-l)/40}\}.
$$

Then

$$
\sum_{k\geq 1} d_A^{-k/20}|I_k(s)| \leq \|s\|_2^2,\quad \sum_{l\geq 1} d_A^{-l/20}|J_l(t)| \leq \|t\|_2^2 \tag{13}
$$

and

$$
\mathbb{E}\|(a_{i,j}\varepsilon_{i,j})_{i,j\leq n}\| = \mathbb{E}\sup_{\|s\|_2\leq 1}\sup_{\|t\|_2\leq 1}\sum_{k,l\geq 1}\sum_{i\in I_k(s)}\sum_{j\in J_l(t)} a_{i,j}\varepsilon_{i,j}s_it_j.
$$

Observe that for any $s,t\in B_2^n$,

$$\begin{aligned}\Big|\sum_{k\geq 1}\sum_{l\geq k+41}\sum_{i\in I_k(s)}\sum_{j\in J_l(t)}a_{i,j}\varepsilon_{i,j}s_it_j\Big| &\leq \sum_{k\geq 1}\sum_{i\in I_k(s)}|s_i|\sum_{l\geq k+41}\sum_{j\in J_l(t)}|a_{i,j}||t_j|\\ &\leq \sum_{k\geq 1}\sum_{i\in I_k(s)}|s_i|d_A^{-(k+40)/40}\sum_j|a_{i,j}|\\ &\leq \|(a_{i,j})\|_\infty\sum_{k\geq 1}\sum_{i\in I_k(s)}s_i^2\leq \|(a_{i,j})\|_\infty.\end{aligned}$$

Similarily,

$$\Big|\sum_{l\geq 1}\sum_{k\geq l+41}\sum_{i\in I_k(s)}\sum_{j\in J_l(t)}a_{i,j}\varepsilon_{i,j}s_it_j\Big|\leq \|(a_{i,j})\|_\infty.$$

Hence it is enough to estimate

$$\sum_{r=-40}^{40}\mathbb{E}\sup_{\|s\|_2\leq}\sup_{\|t\|_2\leq 1}\sum_{\substack{k,l\geq 1\\ l-k=r}}\sum_{i\in I_k(s)}\sum_{j\in J_l(t)}a_{i,j}\varepsilon_{i,j}s_it_j.$$

By symmetry it is enough to bound only the terms with $r\geq 0$. Let X be defined by (11).

Then for a fixed $r\geq 0$ and $\alpha>0$,

$$\begin{aligned}&\sup_{\|s\|_2\leq 1}\sup_{\|t\|_2\leq 1}\sum_{k\geq 1}\sum_{i\in I_k(s)}\sum_{j\in J_{k+r}(t)}\mathbb{1}_{\{|J_{k+r}(t)|\geq\alpha|I_k(s)|\}}a_{i,j}\varepsilon_{i,j}s_it_j\\ &\quad\leq \sup_{\|s\|_2\leq 1}\sup_{\|t\|_2\leq 1}\max_{\eta_i,\eta_j'=\pm1}\sum_{k\geq 1}\sum_{i\in I_k(s)}\sum_{j\in J_{k+r}(t)}d_A^{(2-2k-r)/40}\mathbb{1}_{\{|J_{k+r}(t)|\geq\alpha|I_k(s)|\}}a_{i,j}\varepsilon_{i,j}\eta_i\eta_j'\\ &\quad\leq X\sup_{\|s\|_2\leq 1}\sup_{\|t\|_2\leq 1}\sum_{k\geq 1}d_A^{(2-2k-r)/40}\sqrt{|I_k(s)||J_{k+r}(t)|}\mathbb{1}_{\{|J_{k+r}(t)|\geq\alpha|I_k(s)|\}}\\ &\quad\leq \alpha^{-1/2}X\sup_{\|s\|_2\leq 1}\sup_{\|t\|_2\leq 1}\sum_{k\geq 1}d_A^{(2-2k-r)/40}|J_{k+r}(t)|\leq \alpha^{-1/2}d_A^{(r+2)/40}X,\end{aligned}$$

where the last inequality follows by (13).

Hence Corollary 17 yields

$$\begin{aligned}\mathbb{E}\sup_{\|s\|_2\leq 1}\sup_{\|t\|_2\leq 1}\sum_{k\geq 1}\sum_{i\in I_k(s)}\sum_{j\in J_{k+r}(t)}&\mathbb{1}_{\{|J_{k+r}(t)|\geq d_A^{(2r+5)/40}|I_k(s)|\}}a_{i,j}\varepsilon_{i,j}s_it_j\\ &\lesssim \max_i\|(a_{i,j})_j\|_2+R_A(\log n).\end{aligned}$$

In a similar way we show that

$$\begin{aligned}\sup_{\|s\|_2\leq 1}\sup_{\|t\|_2\leq 1}\sum_{k\geq 1}\sum_{i\in I_k(s)}\sum_{j\in J_{k+r}(t)}&\mathbb{1}_{\{|J_{k+r}(t)|\leq\alpha|I_k(s)|\}}a_{i,j}\varepsilon_{i,j}s_it_j\\ &\leq \alpha^{1/2}X\sup_{\|s\|_2\leq 1}\sup_{\|t\|_2\leq 1}\sum_{k\geq 1}d_A^{(2-2k-r)/40}|I_k(s)|\leq \alpha^{1/2}d_A^{(2-r)/40}X\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\sup_{\|s\|_2\leq 1}\sup_{\|t\|_2\leq 1}\sum_{k\geq 1}\sum_{i\in I_k(s)}\sum_{j\in J_{k+r}(t)}&\mathbb{1}_{\{|J_{k+r}(t)|\geq d_A^{(2r-5)/40}|I_k(s)|\}}a_{i,j}\varepsilon_{i,j}s_it_j\\ &\lesssim \max_i\|(a_{i,j})_j\|_2+R_A(\log n).\end{aligned}$$

Hence it is enough to bound for $r = 0, 1, \ldots, 40$,

$$\mathbb{E} \sup_{\|s\|_2 \le 1} \sup_{\|t\|_2 \le 1} \sum_{k \ge 1} \sum_{i \in I_k(s)} \sum_{j \in J_{k+r}(t)} \mathbb{1}_{\left\{d_A^{(2r-5)/40} < |J_{k+r}(t)|/|I_k(s)| < d_A^{(2r+5)/40}\right\}} a_{i,j} \varepsilon_{i,j} s_i t_j.$$

Recall definition (10) of sets I' and I''. Let us fix $0 \le r \le 40$ and k, t, s such that $d_A^{(2r-5)/40} < |J_{k+r}(t)|/|I_k(s)| < d_A^{(2r+5)/40}$. Let $|I_k(s)| = m$. Let us consider the following greedy algorithm with output being a subset $\{i_1, \ldots, i_M\}$ of $I_k(s)$ of size $M \le m$

- In the first step we pick a vertex $i_1 \in I_k(s)$ with maximal number of neighbours in $J_{k+r}(t)$.
- Once we have $\{i_1, \ldots, i_N\}$ and $N < M$, we pick $i_{N+1} \in I_k(s) \setminus \{i_1, \ldots, i_N\}$ with maximal number of neighbours in $J_{k+r}(t) \setminus \{i_1, \ldots, i_N\}'$.

If l_N is the number of neighbours of i_N in $J_{k+r}(t) \setminus \{i_1, \ldots, i_{N-1}\}'$, then $l_1 \ge l_2 \ge \ldots \ge l_M$, so $Ml_M \le |J_{k+r}(t)|$. Hence, using this algorithm we may find a subset $I \subset I_k(s)$ with cardinality $|I| \le d_A^{-(r+18)/40}|J_{k+r}(t)| \le d_A^{(r-13)/40} m$ such that for every $i \in I_k(s) \setminus I$, $|\{j \in J_{k+r}(t) \setminus I' \colon i \sim_A j\}| \le \lceil d_A^{(r+18)/40} \rceil \le 2d_A^{(r+18)/40}$. Note that if $(i,j) \in E_A$ and $(i,j) \in (I_k(s) \times J_{k+r}(t)) \setminus (I'' \times I')$, then $j \in J_{k+r}(t) \setminus I'$. Therefore,

$$\begin{aligned} \sum_{(i,j) \in I_k(s) \times J_{k+r}(t)) \setminus (I'' \times I')} |a_{i,j}||s_i t_j| &\le 2\|(a_{i,j})\|_\infty \sum_{i \in I_k(s)} |s_i| d_A^{(r+18)/40} d_A^{(1-k-r)/40} \\ &\le 2\|(a_{i,j})\|_\infty d_A^{19/40} \sum_{i \in I_k(s)} s_i^2. \end{aligned} \tag{14}$$

Observe that if $d_A^{-(r+18)/40}|J_{k+r}(t)| > m$ we may take $I = I_k(s)$ and then $a_{i,j} = 0$ for $(i,j) \in I_k(s) \times J_{k+r}(t)) \setminus (I'' \times I')$, so estimate (14) is also valid in this case.

Let

$$s' = (s'_i)_{i \in I'' \cap I_k(s)}, \quad t' = (t'_j)_{j \in I' \cap J_{k+r}(s)},$$

where

$$s'_i := \frac{s_i}{\|(s_i)_{i \in I_k(s)}\|_2}, \quad t'_j := \frac{t_j}{\|(t_j)_{j \in J_{k+r}(t)}\|_2}.$$

Then

$$\begin{aligned} &\|s'\|_2 \le 1, \quad \|s'\|_\infty \le d_A^{1/40}|I_k(s)|^{-1/2} = d_A^{1/40} m^{-1/2}, \\ &\|t'\|_2 \le 1, \quad \|t'\|_\infty \le d_A^{1/40}|J_{k+r}(t)|^{-1/2} \le d_A^{(7-2r)/80} m^{-1/2}. \end{aligned}$$

Hence,

$$\sum_{(i,j) \in (I_k(s) \times J_{k+r}(t)) \cap (I'' \times I')} \varepsilon_{i,j} a_{i,j} s_i t_j \le Y_{m,r} \|(s_i)_{i \in I_k(s)}\|_2 \|(t_j)_{j \in J_{k+r}(t)}\|_2, \tag{15}$$

where

$$Y_{m,r} := \max_{|I| \le d_A^{(r-13)/40} m} \sup_{s \in B_2^n \cap d_A^{1/40} m^{-1/2} B_\infty^n} \sup_{t \in B_2^n \cap d_A^{(7-2r)/80} m^{-1/2} B_\infty^n} \sum_{i \in I'', j \in I'} a_{i,j} \varepsilon_{i,j} s_i t_j.$$

Define $Y_r := \max_{1\le m\le n} Y_{m,r}$. Estimates (14) and (15) yield

$$\begin{aligned}
&\mathbb{E}\sup_{\|s\|_2\le 1}\sup_{\|t\|_2\le 1}\sum_{k\ge 1}\sum_{i\in I_k(s)}\sum_{j\in J_{k+r}(t)}\mathbb{1}_{\left\{d_A^{(2r-5)/40}<|J_{k+r}(t)|/|I_k(s)|<d_A^{(2r+5)/40}\right\}}a_{i,j}\varepsilon_{i,j}s_it_j\\
&\le \sup_{\|s\|_2\le 1}\sup_{\|t\|_2\le 1}\left(\sum_{k\ge 1}2d_A^{19/40}\|(a_{i,j})\|_\infty\sum_{i\in I_k(s)}s_i^2+\mathbb{E}Y_r\sum_{k\ge 1}\|(s_i)_{i\in I_k(s)}\|_2\|(t_j)_{j\in J_{k+r}(t)}\|_2\right)\\
&\le 2d_A^{19/40}\|(a_{i,j})\|_\infty+\mathbb{E}Y_r\sup_{\|s\|_2\le 1}\Big(\sum_{k\ge 1}\|(s_i)_{i\in I_k(s)}\|_2^2\Big)^{1/2}\sup_{\|t\|_2\le 1}\Big(\sum_{k\ge 1}\|(t_j)_{j\in J_{k+r}(t)}\|_2^2\Big)^{1/2}\\
&\le 2d_A^{19/40}\|(a_{i,j})\|_\infty+\mathbb{E}Y_r.
\end{aligned}$$

Therefore, to establish Theorem 5 it is enough to prove the following lemma.

Lemma 19. *For every $0\le r\le 40$,*

$$\mathbb{E}Y_r=\mathbb{E}\max_{1\le m\le n}Y_{m,r}\lesssim \max_i\|(a_{i,j})_j\|_2+R_A(\log n)+\mathrm{Log}(d_A)\|(a_{i,j})\|_\infty.$$

First we show a connected counterpart to Lemma 19.

Lemma 20. *We have*

$$\begin{aligned}
&\mathbb{E}\max_{1\le k\le n}\max_{I\in\mathcal{I}_4(k)}\sup_{s\in B_2^n\cap d_A^{3/8}k^{-1/2}B_\infty^n}\ \sup_{t\in B_2^n\cap d_A^{-1/16}k^{-1/2}B_\infty^n}\sum_{i\in I'',j\in I'}a_{i,j}\varepsilon_{i,j}s_it_j\\
&\qquad\lesssim \max_i\|(a_{i,j})_j\|_2+R_A(\log n)+\mathrm{Log}(d_A)\|(a_{i,j})\|_\infty.
\end{aligned}$$

Proof. Let us first fix k and $I\in\mathcal{I}_4(k)$. Then $|I'|\le d_Ak$ and $|I''|\le d_A^2k$. Proposition 12, applied with $(a_{i,j})=(a_{i,j})_{i\in I'',j\in I'}$, $n=|I''|$, $m=|I'|$, and $b=d_A^{3/8}k^{-1/2}$ yields

$$\begin{aligned}
&\mathbb{E}\sup_{s\in B_2^n\cap d_A^{3/8}k^{-1/2}B_\infty^n}\ \sup_{t\in B_2^n\cap d_A^{-1/16}k^{-1/2}B_\infty^n}\sum_{i\in I'',j\in I'}a_{i,j}\varepsilon_{i,j}s_it_j\\
&\qquad\lesssim \max_i\|(a_{i,j})_j\|_2+\mathrm{Log}(d_A)\|(a_{i,j})\|_\infty.
\end{aligned}$$

By Lemma 11, $|\mathcal{I}_4(k)|\le n(4d_A^4)^k\le\max\{n^2,d_A^{12k}\}$ (recall that we assume that $d_A\ge 3$). We have

$$\begin{aligned}
&\sup_{s\in B_2^n\cap d_A^{3/8}k^{-1/2}B_\infty^n}\ \sup_{t\in B_2^n\cap d_A^{-1/16}k^{-1/2}B_\infty^n}\Big\|\sum_{i,j}a_{i,j}\varepsilon_{i,j}s_it_j\Big\|_{\mathrm{Log}(|\mathcal{I}_4(k)|)}\\
&\le \sup_{s,t\in B_2^n}\Big\|\sum_{i,j}a_{i,j}\varepsilon_{i,j}s_it_j\Big\|_{2\mathrm{Log}(n)}+\sup_{s\in B_2^n}\ \sup_{t\in B_2^n\cap d_A^{-1/16}k^{-1/2}B_\infty^n}\Big\|\sum_{i,j}a_{i,j}\varepsilon_{i,j}s_it_j\Big\|_{12k\mathrm{Log}(d_A)}\\
&\le R_A(2\mathrm{Log}n)+\sup_{s\in B_2^n}\ \sup_{t\in B_2^n\cap d_A^{-1/16}k^{-1/2}B_\infty^n}\sqrt{12k\mathrm{Log}(d_A)}\Big(\sum_{i,j}a_{i,j}^2s_i^2t_j^2\Big)^{1/2}\\
&\lesssim R_A(\mathrm{Log}\,n)+\sqrt{k\mathrm{Log}(d_A)}\max_i d_A^{-1/16}k^{-1/2}\Big(\sum_j a_{i,j}^2\Big)^{1/2}\\
&\le R_A(\mathrm{Log}\,n)+\sqrt{\mathrm{Log}(d_A)}d_A^{-1/16}\max_i\|(a_{i,j})_j\|_2\lesssim R_A(\log n)+\max_i\|(a_{i,j})_j\|_2.
\end{aligned}$$

Hence, by Proposition 10,

$$\begin{aligned}
&\mathbb{E}\max_{I\in\mathcal{I}_4(k)}\sup_{s\in B_2^n\cap d_A^{3/8}k^{-1/2}B_\infty^n}\ \sup_{t\in B_2^n\cap d_A^{-1/16}k^{-1/2}B_\infty^n}\sum_{i\in I'',j\in I'}a_{i,j}\varepsilon_{i,j}s_it_j\\
&\qquad\lesssim \max_i\|(a_{i,j})_j\|_2+R_A(\log n)+\mathrm{Log}(d_A)\|(a_{i,j})\|_\infty.
\end{aligned}$$

Applying again Proposition 10 and observing that

$$\max_{1\le k\le n}\sup_{s\in B_2^n\cap d_A^{3/8}k^{-1/2}B_\infty^n}\sup_{t\in B_2^n\cap d_A^{-1/16}k^{-1/2}B_\infty^n}\Big\|\sum_{i,j}a_{i,j}\varepsilon_{i,j}s_it_j\Big\|_{\mathrm{Log}(n)}\le R_A(\log n)$$

we get the assertion. □

Proof of Lemma 19. Let

$$Z_k:=\max_{I\in\mathcal{I}_4(k)}\sup_{s\in B_2^n\cap d_A^{3/8}k^{-1/2}B_\infty^n}\sup_{t\in B_2^n\cap d_A^{-1/16}k^{-1/2}B_\infty^n}\sum_{i\in I'',j\in I'}a_{i,j}\varepsilon_{i,j}s_it_j.$$

Let us fix $r\ge 0$, $I\subset V$ such that $|I|\le d_A^{(r-13)/40}m$, $s\in B_2^n\cap d_A^{1/40}m^{-1/2}B_\infty^n$ and $t\in B_2^n\cap d_A^{(7-2r)/80}m^{-1/2}B_\infty^n$. Let $I_1,\ldots,I_l$ be 4-connected components of I. Then $\{I_1',\ldots,I_l'\}$ is a partition of I', $\{I_1'',\ldots,I_l''\}$ is a partition of I'' and

$$\sum_{i\in I'',j\in I'}a_{i,j}\varepsilon_{i,j}s_it_j=\sum_{u=1}^{l}\sum_{i\in I_u'',j\in I_u'}a_{i,j}\varepsilon_{i,j}s_it_j. \tag{16}$$

Define

$$U_1:=\big\{1\le u\le l\colon\ \|(s_i)_{i\in I_u''}\|_2\ge d_A^{(12-r)/80}m^{-1/2}\sqrt{|I_u|}\big\},$$
$$U_2:=\big\{1\le u\le l\colon\ \|(t_j)_{j\in I_u'}\|_2\ge d_A^{(12-r)/80}m^{-1/2}\sqrt{|I_u|}\big\}.$$

For $u\in U_1\cap U_2$ define vectors

$$\tilde{s}(u):=\frac{(s_i)_{i\in I_u''}}{\|(s_i)_{i\in I_u''}\|_2},\quad \tilde{t}(u):=\frac{(t_j)_{j\in I_u'}}{\|(t_j)_{j\in I_u'}\|_2}.$$

Then $\|\tilde{s}(u)\|_2=\|\tilde{t}(u)\|_2=1$,

$$\|\tilde{s}(u)\|_\infty\le d_A^{(r-12)/80}m^{1/2}|I_u|^{-1/2}\|s\|_\infty\le d_A^{(r-10)/80}|I_u|^{-1/2}\le d_A^{3/8}|I_u|^{-1/2},$$
$$\|\tilde{t}(u)\|_\infty\le d_A^{(r-12)/80}m^{1/2}|I_u|^{-1/2}\|t\|_\infty\le d_A^{-(r+5)/80}|I_u|^{-1/2}\le d_A^{-1/16}|I_u|^{-1/2}.$$

Hence

$$\begin{aligned}\sum_{u\in U_1\cap U_2}\sum_{i\in I_u'',j\in I_u'}a_{i,j}\varepsilon_{i,j}s_it_j&\le\sum_{u\in U_1\cap U_2}Z_{|I_u|}\|(s_i)_{i\in I_u''}\|_2\|(t_j)_{j\in I_u'}\|_2\\&\le\max_k Z_k\Big(\sum_{u\le l}\|(s_i)_{i\in I_u''}\|_2^2\Big)^{1/2}\Big(\sum_{u\le l}\|(t_j)_{j\in I_u'}\|_2^2\Big)^{1/2}\\&\le\max_k Z_k.\end{aligned}\tag{17}$$

Observe that

$$\sum_{u\notin U_1}\|(s_i)_{i\in I_u''}\|_2^2\le\sum_u d_A^{(12-r)/40}m^{-1}|I_u|=d_A^{(12-r)/40}m^{-1}|I|\le d_A^{-1/40}$$

and by the same token

$$\sum_{u\notin U_2}\|(t_j)_{j\in I_u'}\|_2^2\le d_A^{-1/40}.$$

Hence

$$\begin{aligned}\sum_{u\notin U_1}\sum_{i\in I_u'',j\in I_u'} a_{i,j}\varepsilon_{i,j}s_it_j &\leq \|(a_{i,j}\varepsilon_{i,j})_{i,j}\|\sum_{u\notin U_1}\|(s_i)_{i\in I_u''}\|_2\|(t_j)_{j\in I_u'}\|_2\\ &\leq \|(a_{i,j}\varepsilon_{i,j})_{i,j}\|\Big(\sum_{u\notin U_1}\|(s_i)_{i\in I_u''}\|_2^2\Big)^{1/2}\Big(\sum_{u\leq l}\|(t_j)_{j\in I_u'}\|_2^2\Big)^{1/2}\\ &\leq d_A^{-1/80}\|(a_{i,j}\varepsilon_{i,j})_{i,j}\| \end{aligned}\tag{18}$$

and

$$\begin{aligned}\sum_{u\in U_1\setminus U_2}\sum_{i\in I_u'',j\in I_u'} a_{i,j}\varepsilon_{i,j}s_it_j &\leq \|(a_{i,j}\varepsilon_{i,j})_{i,j}\|\sum_{u\notin U_2}\|(s_i)_{i\in I_u''}\|_2\|(t_j)_{j\in I_u'}\|_2\\ &\leq \|(a_{i,j}\varepsilon_{i,j})_{i,j}\|\Big(\sum_{u\leq l}\|(s_i)_{i\in I_u''}\|_2^2\Big)^{1/2}\Big(\sum_{u\notin U_2}\|(t_j)_{j\in I_u'}\|_2^2\Big)^{1/2}\\ &\leq d_A^{-1/80}\|(a_{i,j}\varepsilon_{i,j})_{i,j}\|. \end{aligned}\tag{19}$$

Bounds (16)-(19) yield

$$\mathbb{E}\max_m Y_{m,r} \leq \mathbb{E}\max_k Z_k + 2d_A^{-1/80}\mathbb{E}\|(a_{i,j}\varepsilon_{i,j})_{i,j}\|$$

and the assertion follows by Lemma 20 and Proposition 18. □

Acknowledgements. Part of this work was carried out while the author was visiting the Hausdorff Research Institute for Mathematics, Univerity of Bonn. The hospitality of HIM and of the organizers of the program *Synergies between modern probability, geometric analysis and stochastic geometry* is gratefully acknowledged. The author would like to thank Marta Strzelecka for careful readings of various versions of the manuscript and numerous valuable comments and Ramon van Handel for pointing out the reference [6].

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