Two-pan balance and generalized counterfeit coin problem

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Hugo Steinhaus in „Mathematical Snapshots" ([7]) offers the following problems:

We have nine nickels, including one counterfeit coin, which can only be told apart by its weight being different from the others. How can one tell in no more than two weightings which one it is? The balance we are allowed to use only gives information whether two masses have the same weight or – if not – which one is heavier or lighter.

and

We have thirteen nickels, including exactly one counterfeit coin. We do not know whether it is heavier or lighter. We aim to find the counterfeit nickel in three weightings using a two-pan balance.

The second problem is often referred to as the counterfeit coin problem and it is a classical puzzle. Various modifications of this problem can be found in many publications (cf. References).

In this paper I consider a wide class of possible generalizations of the counterfeit coin problem. Due to a large number of variants and their mutual similarity I will introduce all of them at once. Particular variants will be identified by a five letter code, in the following way.

We are given two natural numbers n and k.

We know that among n coins

A) there is for sure *B) there may be*

a counterfeit coin which can be distinguished only by its weight.

C) We know *D) We do not know*

whether it is heavier or lighter than a genuine coin.

We also have

E) 0 *F) 1* *G) infinitely many*

additional genuine coins.

Our aim is to state whether it is possible

H) to find out if any of the coins is counterfeit and if it is so – to indicate this coin

I) to find out if any of the coins is counterfeit and if it is so – to indicate this coin and tell its weight

in k weightings using a two-pan balance without weights (we are allowed to use coins only).

In addition all the weightings

J) are required *K) are not required*

to be planned before the first weighting. Statement J means that the choice of the coins participating in subsequent weightings should not depend on the results of previous weightings.

The particular variant is determined by a choice of five letters: one of the letters A, B,

one of the letters C, D, one of the letters E, F or G,

one of the letters H, I and one of the letters J, K.

For example, the above Steihaus' problems correspond to ACEHK with n=9, k=2 and ADEHK with n=13, k=3.

There are 2·2·3·2·2=48 possible variations of the original puzzle ((A/B)(C/D)(E/F/G)(H/I)(J/K)), but some of them are very similar, hence I will often write about a couple of the variants together. In such cases some letters will be substituted with the asterisk (*) to tell that the letter can be chosen freely. For example the group AC*** consists of all problems in which we know that the counterfeit coin exists and we know whether it is heavier or lighter than other coins.

I give a full solution to all possible variants of the problem for every n and k. When the answer is positive, I present the required algorithm of weightings. When it is negative, I prove that no such algorithm exists.

Below I state the results, which will be proved in this paper. First I consider the problems in which we know whether counterfeit coin is heavier or lighter than the genuine nickels (*C***). The answer here is positive if and only if n and k satisfy the following inequalities:

	E (no additional genuine coins)		F/G (we are given additional genuine coins)
	J (weightings are planned in advance)	K (weightings are not planned in advance)	
A (we know that the counterfeit coin exists) **The problem makes sense only if $n\geq1$**	$n\leq3^k$	$n\leq3^k$	$n\leq3^k$
B (it is possible that the counterfeit coin does not exist)	$n\leq3^k-1$ $n\neq1$ $n\neq3^k-2$	$n\leq3^k-1$ $n\neq1$	$n\leq3^k-1$

The problems *D*** (we do not know whether counterfeit coin is heavier or lighter than genuine nickel) have the positive answer when n and k satisfy the inequalities:

		E (no additional genuine coins)	F/G (we are given additional genuine coins)
A (we know that the counterfeit coin exists) **The problem makes sense only if $n\geq1$**	H (we do not have to find out if the counterfeit coin is heavier or lighter than the genuine ones)	$n\leq(3^k-1)/2$ $n\neq2$	$n\leq(3^k+1)/2$
	I (we have to find out if the counterfeit coin is heavier or lighter than the genuine ones)	$n\leq(3^k-3)/2$ $n\neq1$ $n\neq2$	$n\leq(3^k-1)/2$
B (it is possible that counterfeit coin does not exist)		$n\leq(3^k-3)/2$ or $n=0$ $n\neq1$ $n\neq2$	$n\leq(3^k-1)/2$

Example: It is impossible to find a counterfeit coin among 40 nickels in 4 weightings without additional genuine coins when we do not know whether it exists and if it is heavier or lighter than genuine nickels ($40>39=(3^4-3)/2$). This answer remains correct for each of the variants BDE**.

It is worth to notice that the variants A**** of the problem make sense only if $n\geq 1$, because there must be at least one coin (the counterfeit coin). The authors of [5] write that they do not know the answer of the variant ADEHK.

The results presented in the tables show that if we have any additional genuine coin it does not matter whether we have one or infinite amount of them. We can always solve the problem using only one additional genuine coin. Another conclusion is that in general it is possible to make the choice of the coins participating in subsequent weightings without knowledge of the results of previous weightings. The only exceptions are variants BCE*K with $n=3^k-2$.

I shall prove that conditions presented in the tables are necessary and sufficient to perform required weightings.

Necessity of the conditions presented in the tables

The method of proving the necessity of the conditions is following: I shall compare the number of possible results of k weightings with the number of possible answers to a particular variant of the problem. If the number of possible results of weightings is less than the number of possible answers then we will not be able to give an answer.

*We know whether the counterfeit coin is heavier or lighter than genuine ones (variants *C***)*

In variants AC*** (when we know that the counterfeit coin exists and we know whether it is heavier or lighter than genuine ones) I shall prove that if $n>3^k$ then we cannot perform required weightings. Indeed, in one weighting we can get one of three possible results (the right hand side of balance is down or the left one is down or the pans are balanced). k weightings give one of 3^k possible results. The counterfeit coin is one of $n>3^k$ nickels, so the information received in k weightings is not sufficient to indicate the counterfeit coin.

In variants BC*** there are 3^k possible results of k weightings while the number of possible answers is equal to n+1 (either one of n coins is counterfeit or there is no counterfeit nickel). If $n>3^k-1$ then $n+1>3^k$, so we are not able to indicate the counterfeit coin.

Condition $n\neq 1$ in variants BCE** is a consequence of the fact that if there is only one coin then we have nothing to compare it with. In variants BCE*J we have to prove that k weightings is not enough to indicate the counterfeit coin among 3^k-2 nickels. We assume, tending to the contradiction, that we can plan required k weightings. Clearly, in each weighting the number of coins put onto the left and right pan should be equal. Let us assign to the coins numbers $1,2,...,3^k-2$. If we want to plan k weightings, it is enough to assign to each coin a k-tuple being an element of $\{R,L,0\}^k$. These k-tuples have the following meaning: Let $(w^i_1,w^i_2,\ldots,w^i_k)$ be a k-tuple assigned to the i-th coin ($i=1,2,\ldots, 3^k-2$) and let $j=1,2,\ldots,k$. Then $w^i_j=R$ means than during the j-th weighting the i-th coin should be put onto the right pan; $w^i_j=L$ means then during the j-th weighting the i-th coin should be put onto the left pan. Finally, $w^i_j=0$ means that the i-th coin does not participate in the j-th weighting. Clearly, k-tuples assigned to particular coins should be different. Moreover, the zero k-tuple, $(0,0,\ldots,0)$, should not be assigned to any coin. The number of non-zero k-tuples is equal to 3^k-1, while the number of coins is equal to 3^k-2. It follows that there exists exactly one non-zero k-tuple $(w_1,w_2,\ldots,w_k)$, which is not assigned to any coin. We have $w_j\neq 0$ for some $j=1,2,...,k$. We may assume that $w_j=L$ (the reasoning is similar when $w_j=R$). The set $\{i: w^i_j=L\}$ has $3^{k-1}-1$ elements, while the set $\{i: w^i_j=R\}$ has 3^{k-1} elements. It follows that during the j-th weighting there is one coin more put onto the right pan. This contradiction shows that k weightings are not enough to indicate the counterfeit coin among 3^k-2 nickels in variants BCE*J.

*We do not know if the counterfeit coin is heavier or lighter than genuine ones (variants *D***)*

First I shall consider the variants AD(F/G)I*. We know that one of the coins is counterfeit and we have to find out if it is heavier or lighter than genuine nickels. It follows that during at least one of the weightings the pans should not be balanced – otherwise we would not be able to find out if the counterfeit coin is heavier or lighter. Hence there are 3^k-1 possible results of weightings. The number of possible answers is equal to 2n (we have to indicate one of n coins and find out if it is heavier or lighter). If $n>(3^k-1)/2$ then $2n>3^k-1$, so we are not able to point the counterfeit coin in k weightings.

In variants BD(F/G)** we do not need to find out if the counterfeit coin is heavier or lighter than the genuine ones. However, we will receive this information during the weightings. The number of possible results of weightings is equal to 3^k, while the number of possible answers is equal to $2n+1$ (either there is no counterfeit coin or any among n coins is counterfeit and is heavier or lighter than the others). If $n>(3^k-1)/2$ then $2n+1>3^k$, so then k weightings are not enough

Variants AD(F/G)H*. We assume that k weightings are enough to solve the problem. There are 3^k possible results of weightings. If in all k weightings the pans are balanced, we will not be able to tell if the counterfeit coin is heavier or lighter. However, if in at least one weighting any pan of the balance rises, we will point the counterfeit coin and in addition we will be able to tell if it is heavier or lighter than the genuine nickels. Hence the inequality $2(n-1)+1\leq 3^k$, which is the same as $n\leq(3^k+1)/2$.

Condition $n\neq 2$ in variants *DE** is a consequence of the fact that if we have exactly 2 coins and we do not know whether the counterfeit coin is heavier or lighter, then we cannot identify the counterfeit coin. Condition $n\neq 1$ in variants BDE** is a consequence of the fact that if there is only one coin then we have nothing to compare it with. Similarly, in variants ADEI*: although we know that one coin is counterfeit, still we are not able to find out whether it is heavier or lighter than the genuine nickels (we do not possess any additional genuine coins).

Proof of inequality $n \le (3^k-3)/2$ for variants ADEI*: Assume that for specified n and k we can perform required weightings. Let m be the number of coins, which will be put in the first weighting on each pan. If in the first weighting the pans are balanced, the counterfeit coin is one of n-2m nickels not participating in this weighting. Remaining k-1 weightings must be enough to identify the counterfeit coin among n-2m nickels and to find out if it is heavier or lighter. It results in the following inequality:

(1) $$2(n-2m) \le 3^{k-1}.$$

However, if in the first weighting left or right pan moved up, the counterfeit coin is among 2m nickels, which participated in this weighting. Remaining k-1 weightings must be enough to point the counterfeit coin among these 2m nickels and to find out if it is heavier or lighter. We obtain the following inequality $2\cdot 2m \le 2\cdot 3^{k-1}$ ($2\cdot 3^{k-1}$ – because in the first weighting either left or right pan could rise), which is the same as:

(2) $$2m \le 3^{k-1}.$$

As 3^{k-1} is odd, we can write inequalities (1) and (2) as:

(3) $$2(n-2m) \le 3^{k-1}-1$$

(4) $$2m \le 3^{k-1}-1.$$

By adding inequalities (3) and doubled (4) we receive $2n \le 3\cdot 3^{k-1}-3$, which is the same as $n \le (3^k-3)/2$.

In case BDE** let us notice that a detection of a counterfeit coin means, that in one of the weightings the pans were not balanced. That means that we have not only identified the false coin, but we also learned whether it is heavier or lighter. Therefore the inequality $n \le (3^k-3)/2$, proved already for ADEI* is valid here.

The proof of $n \le (3^k-1)/2$ for ADEH* is similar to the proof of $n \le (3^k-3)/2$ for the variant ADEI*. The only difference is that in the case of balanced pans, with the remaining k-1 weightings we must either detect the counterfeit coin among n-2m coins or decide that all coins are genuine. Inequality (1) will take the form $2(n-2m-1)+1 \le 3^{k-1}$. (Notice that if the pans were not balanced all the time, we will not only indicate the false coin but also we will know whether it is heavier or lighter.) In this way we have proved that the conditions given in the tables are necessary for the existence of the weightings satisfying the conditions of the particular variants. Now we have to prove that these conditions are also sufficient.

Sufficiency of the conditions presented in the tables

Variants BCEIJ and BDEIJ

We start from variants BCEIJ and BDEIJ, as they include the most constraints. We have to plan the weightings in advance, we do not know whether a false coin exists, we do not know whether it is heavier of lighter. Moreover, we have no additional coins at our disposal. In the other variants we prove the sufficiency of our conditions by using a suitable construction for the variants BCEIJ and BDEIJ.

In both variants we have to plan k weightings beforehand. It means that we have to assign to each of the n coins a k-tuple $(w_1,w_2,\ldots,w_k)\subset\{R,L,0\}^k$. We have $w_j=R$, $w_j=L$ or $w_j=0$ which respectively means that the given coin will be on the right pan, on the left pan or it will not take part in the j-th weighting. Different k-tuples must correspond to different coins – otherwise it would not be possible to distinguish a pair of coins by this sequence of weightings. The assigned k-tuples form an n-element subset $A\subset\{R,L,0\}^k$. Moreover, we require that in each weighting the number of coins lying on both pans is the same. This means that the sets $\{(w_1,w_2,\ldots,w_k)\in A\colon w_j=R\}$, $\{(w_1,w_2,\ldots,w_k)\in A\colon w_j=L\}$ must have the same number of elements for $j=1, 2, \ldots,k$.

Let us turn to variant BCEIJ. Without loss of generality we may assume that the counterfeit coin (if it exists) is heavier than the others. Let us also assume that for each pair k, n satisfying the conditions from the table there exists an n-element subset $\Delta^n{}_k\subset\{R,L,0\}^k$ such that $(0,0,\ldots,0)\notin\Delta^n{}_k$ and for any $j\in\{1,2,\ldots,k\}$ the sets $\{(w_1,w_2,\ldots,w_k)\in\Delta^n{}_k\colon w_j=R\}$, $\{(w_1,w_2,\ldots,w_k)\in\Delta^n{}_k\colon w_j=L\}$ have equal number of elements. Let us assign to each coin a different element of the set $\Delta^n{}_k$. Notice that the sequence of weightings designed in this way allows detecting the counterfeit coin, if it exists. In fact, let us write the result of k weightings as a k-tuple $(w_1,w_2,\ldots,w_k)$, where $w_j=R$, $w_j=L$, $w_j=0$ means that in the j-th weighting the right pane was lower, higher or both pans were balanced, respectively. If all coins are fair, we get the zero k-tuple $(0,0,\ldots,0)$. Otherwise we get a k-tuple, which coincides with the k-tuple assigned to the counterfeit coin while planning the weightings. It remains to prove that:

For any $k\geq 0$ and n such that $2\leq n\leq 3^k-3$ or $n=0$ or $n=3^k-1$ there exists an n-element set $\Delta^n{}_k\subset\{R,L,0\}^k$, such that $(0,0,\ldots,0)\notin\Delta^n{}_k$ and for all $j\in\{1,2,\ldots,k\}$ the sets $\{(w_1,w_2,\ldots,w_k)\in\Delta^n{}_k\colon w_j=R\}$ and $\{(w_1,w_2,\ldots,w_k)\in\Delta^n{}_k\colon w_j=L\}$ have equal number of elements.

For every $k\geq 0$ we set $\Delta^0{}_k=\varnothing$.

For every $k\geq 1$ we set $\Delta^2{}_k=\{(L,0,\ldots,0), (R,0,\ldots,0)\}$.

For every $k\geq 2$ we set $\Delta^3{}_k=\{(L,R,0,\ldots,0), (R,0,0,\ldots,0), (0,L,0,\ldots,0)\}$,

$\Delta^4{}_k=\{(R,0,0,\ldots,0), (L,0,0,\ldots,0), (0,R,0,\ldots,0), (0,L,0,\ldots,0)\}$,

$\Delta^5{}_k=\{(L,R,0,\ldots,0), (R,0,0,\ldots,0), (0,L,0,\ldots,0), (R,R,0,\ldots,0), (L,L,0,\ldots,0)\}$.

We shall construct the other sets Δ^n_k using the induction on k. We have already defined all the requested sets Δ^n_k for k=0 and k=1. Let $k\geq 2$ and $2\leq n\leq 3^k-3$ or n=0 or $n=3^k-1$. We assume that for every m<k the sets Δ^n_m have already been constructed. Since Δ^0_k, Δ^2_k, Δ^3_k, Δ^4_k, Δ^5_k are already defined we only need to construct the sets Δ^n_k for $6\leq n\leq 3^k-1$, $n\neq 3^k-2$.

Number n can be represented as $n=n_1+2n_2$, $2\leq n_1\leq 3^{k-1}-1$, $n_1\neq 3^{k-1}-2$, $2\leq n_2\leq 3^{k-1}$. Numbers n_1, n_2 can be determined as follows:

$n_1=n/3$	and	$n_2=n/3$	if $n\equiv 0 \pmod 3$
$n_1=(n-1)/3 + 1$	and	$n_2=(n-1)/3$	if $n\equiv 1 \pmod 3$
$n_1=(n-2)/3$	and	$n_2=(n-2)/3 + 1$	if $n\equiv 2 \pmod 3$, then.

Such a choice of n_1 and n_2 does not guarantee that $n_1\neq 3^{k-1}-2$. To ensure that $n_1\neq 3^{k-1}-2$ we set:

$n_1=3^{k-1}-4$	and	$n_2=3^{k-1}$	if $n=3^k-4$,
$n_1=3^{k-1}-4$	and	$n_2=3^{k-1}-1$	if $n=3^k-6$,
$n_1=3^{k-1}-4$	and	$n_2=3^{k-1}-2$	if $n=3^k-8$.

Finally we set:

$$\Delta^n_k = (\Delta^{n_2}_{k-1}\times\{L,R\})\cup(\Delta^{n_1}_{k-1}\times\{0\}) \quad when \quad n_2\neq 3^{k-1} \; and \; n_2\neq 3^{k-1}-2$$

or $$\Delta^n_k = (\Delta^{n_2-1}_{k-1}\times\{L,R\})\cup(\Delta^{n_1}_{k-1}\times\{0\})\cup\{(0,\ldots,0,L),(0,\ldots,0,R)\} \quad when \quad n_2 = 3^{k-1} \; or \; n_2 = 3^{k-1}-2$$

■

Let us turn to variant BDEIJ. For $\mathbf{w}=(w_1,w_2,\ldots,w_k)\in\{R,L,0\}^k$ we denote by $-\mathbf{w}$ the k-tuple with all L replaced by R and vice versa, e.g – (L,0,R,R,L)=(R,0,L,L,R). We will call $-\mathbf{w}$ the inverse of $\mathbf{w}$. Moreover, for $A\subset\{R,L,0\}^k$ we denote by $-A$ the set $\{-\mathbf{w}: \mathbf{w}\in A\}$. Assume that for all k, n satisfying the condition from the table there exists an n-element set $\Gamma^n_k\subset\{R,L,0\}^k$ such that $(0,0,\ldots,0)\notin\Gamma^n_k$, $\Gamma^n_k\cap(-\Gamma^n_k)=\varnothing$ and for all $j\in\{1,2,\ldots,k\}$ the sets $\{(w_1,w_2,\ldots,w_k)\in\Gamma^n_k: w_j=R\}$ and $\{(w_1,w_2,\ldots,w_k)\in\Gamma^n_k: w_j=L\}$ have the same number of elements. We assign to each coin a different element from Γ^n_k. Notice that the sequence of k weightings planned in this way allows us to indicate the counterfeit coin (if exists) and to determine whether it is heavier or lighter. In fact, let us write the results of the k weightings in the form $(w_1,w_2,\ldots,w_k)$, with $w_j=R$, $w_j=L$, $w_j=0$, as before. If all coins are fair we get (0,0,…,0). If not and the counterfeit coin is heavier, we get a k-tuple assigned to it when planning the weightings. If the false coin is lighter, we get the inverse of the same tuple. As $\Gamma^n_k\cap(-\Gamma^n_k)=\varnothing$, we get a different result in each of those situations, which allows to indicate the counterfeit coin and determine whether it is heavier or lighter. It remains to prove that

For any $k\geq 0$ and n such that $3\leq n\leq(3^k-3)/2$ or n=0 there exists an n-element set $\Gamma^n_k\subset\{L,R,0\}^k$ such that $\Gamma^n_k\cap(-\Gamma^n_k)=\varnothing$ and for all $j\in\{1,2,\ldots,k\}$ the sets $\{(w_1,w_2,\ldots,w_k)\in\Gamma^n_k: w_j=R\}$ and $\{(w_1,w_2,\ldots,w_k)\in\Gamma^n_k: w_j=L\}$ have the same number of elements.

The sets Γ^n_k will be constructed using induction on k. First we consider some special cases. Let $\Gamma^0_k=\varnothing$ for every k.

For every k≥2 we set: $\Gamma^3_k=\{(L,R,0,\dots,0), (R,0,0,\dots,0), (0,L,0,\dots,0)\}$.

For every k≥3 we set:

$\Gamma^4_k=\{(R,R,L,0,\dots,0), (R,L,P,0,\dots,0), (L,R,R,0,\dots,0), (L,L,L,0,\dots,0)\}$

$\Gamma^5_k=\{(0,0,R,0,\dots,0), (P,0,0,0,\dots,0), (L,0,R,0,\dots,0), (0,L,L,0,\dots,0), (0,R,L,0,\dots,0)\}$

$\Gamma^6_k=\{(L,R,L,0,\dots,0), (R,0,L,0,\dots,0), (0,L,L,0,\dots,0), (L,R,R,0,\dots,0), (R,0,R,0,\dots,0), (0,L,R,0,\dots,0)\}$

$\Gamma^7_k=\{(L,R,0,0,\dots,0), (R,0,0,0,\dots,0), (0,L,0,0,\dots,0), (R,R,L,0,\dots,0), (R,L,R,0,\dots,0), (L,R,R,0,\dots,0),(L,L,L,0,\dots,0)\}$

$\Gamma^8_k=\{(L,L,L,0,\dots,0), (L,R,L,0,\dots,0), (R,L,L,0,\dots,0), (R,R,L,0,\dots,0), (0,R,R,0,\dots,0), (0,L,R,0,\dots,0), (R,0,R,0,\dots,0),$
$(L,0,R,0,\dots,0)\}$

The sets $\Gamma_k^{(3^k-3)/2}$ will be defined using induction on k≥1. The set $\Gamma^0_1=\varnothing$ is already constructed. Let k>1. We set

$$\Gamma_k^{(3^k-3)/2}=\Gamma_{k-1}^{(3^{k-1}-3)/2}\times\{L,R,0\}\cup\{(L,L,\dots,L,R),\ (R,R,\dots,R,0),(0,0,\dots,0,L)\}$$

The constructed above sets $\Gamma_k^{(3^k-3)/2}$ satisfy the following conditions:

1° $(R,\dots,R), (L,\dots,L)\notin \Gamma_k^{(3^k-3)/2}$ for k≥0,

2° $(L,R,0,0,\dots,0), (R,R,0,0,\dots,0), (0,L,0,0,\dots,0), (L,\dots,L,R,0), (0,\dots,0,0,L), (0,\dots,0,L,0)\in \Gamma_k^{(3^k-3)/2}$ for k≥3,

3° $(0,R,0,0,\dots,0), (0,0,\dots,0,0,R,0)\notin \Gamma_k^{(3^k-3)/2}$ for k≥3.

These conditions can be easily proved by induction. The conditions 1°, 2°, 3° imply that we may define the sets $\Gamma_k^{(3^k-5)/2}$ and $\Gamma_k^{(3^k-7)/2}$ by the following formulas:

$$\Gamma_k^{(3^k-5)/2}=\Gamma_k^{(3^k-3)}\setminus\{(L,\dots,L,R,0),(0,\dots,0,0,L),(0,\dots,0,L,0)\}\cup\{(L,\dots,L,L,L),(0,\dots,0,R,0)\}$$
$$\Gamma_k^{(3^k-7)/2}=\Gamma_k^{(3^k-3)}\setminus\{(L,R,0,\dots,0),(R,R,0,\dots,0),(0,L,0,\dots,0)\}\cup\{(0,R,0,\dots,0)\}.$$

Now we may pass to the main part of the construction. We use induction on k. The sets Γ^n_k, where k=0, k=1 and k=2, are already constructed. Let k≥3 and $0\le n\le(3^k-3)/2$, $n\neq1,2$.

If n<9 or $n=(3^k-3)/2$ or $n=(3^k-5)/2$ or $n=(3^k-7)/2$ then sets Γ^n_k have already been determined, so we can assume $9\le n\le(3^k-9)/2$.

Number n can be represented as $n=n_1+2n_2$, $3\le n_1,n_2\le(3^{k-1}-3)/2$, where n_1 and n_2 may be set as:

$n_1= n/3$	and	$n_2=n/3$	if $n\equiv0 \pmod 3$,
$n_1=(n-1)/3 + 1$	and	$n_2=(n-1)/3$	if $n\equiv1 \pmod 3$,
$n_1=(n-2)/3$	and	$n_2=(n-2)/3 +1$	if $n\equiv2 \pmod 3$.

Finally we define $\Gamma_k^n=\Gamma_{k-1}^{n_2}\times\{L,R\}\cup\Gamma_{k-1}^{n_1}\times\{0\}$ ■

The remaining variants

The remaining variants can be easily reduced to one of BCEIJ or BDEIJ. As an example we will consider variants BCF** and BCE*K. Variants BCF**: If n=1 then we compare the only one coin with additional genuine one, which we posses (letter F). If $n=3^k-2$ then we use additional coin and we perform the weightings similarly to variant BCEIJ with $n+1=3^k-1$ coins and k weightings. If $n\leq 3^k-1$, $n\neq 1$ and $n\neq 3^k-2$, we follow variant BCEIJ.

Variants BCE*K. If n and k satisfy conditions of BCEIJ, i.e., $n\leq 3^k-1$, $n\neq 1$ and $n\neq 3^k-2$, we follow variant BCEIJ. If $n=3^k-2$ and $k\geq 2$, we split n coins into 3 groups: group A consisting of $3^{k-1}-2$ coins and groups B and C of 3^{k-1} coins each. In the first weighting we put groups B and C onto the pans. If any pan rises then the counterfeit coin is among one of these groups (we know which one). The remaining k-1 weightings are enough to indicate the counterfeit coin among 3^{k-1} coins (the sequence of weightings is the same as in variants ACE*K, which can be reduced to BCEIJ). If the pans are balanced the counterfeit coin (if it exists) is among group A. We perform the remaining k-1 weightings following variants BCF** for group A (we know that coins from groups B and C are genuine so we may use them as additional genuine coins).

In the remaining cases we proceed in a similar way (cf. the second of the examples below for the variant BDFHJ.)

Examples

We are given 19 nickels including one counterfeit coin, which is lighter than the genuine ones. We have to find out which one it is in no more than three weightings using a two-pan balance.

This puzzle corresponds to the variant ACEHK of the problem with n=19 and k=3. Let us number all the coins with 1, 2,…, 19. In the first weighting we should put coins 1, 2, 3, 4, 5, 6 onto the left pan and coins 7, 8, 9, 10, 11, 12 onto the right pan.

1° If the left pan rises then we know that the counterfeit coin is among 1, 2, 3, 4, 5, 6. In the second weighting we compare coins 1, 2 with 3, 4. The third weighting allows us to indicate the counterfeit coin.

2° If the right pan rises then we know that the counterfeit coin is among 7, 8, 9, 10, 11, 12. In the second weighting we compare coins 7, 8 with 9, 10. The third weighting allows us to indicate the counterfeit coin.

3° If the pans are balanced then we know the counterfeit coin is one among 13, 14, 15, 16, 17, 18, 19. In the second weighting we compare coins 13, 14 with 15, 16. The third weighting allows indicating the counterfeit coin.

We are given 40 coins including at most one counterfeit nickel. We do not know if it is heavier or lighter. We have arbitrary many additional genuine coins. We want to know if any of the coins is counterfeit and if it is so – indicate this coin and tell whether it is heavier or lighter. We may use two-pan balance not more than four times.

We will solve the problem using only one additional genuine coin. We will plan all the weightings in advance. This is variant BDFHJ of generalized problem. We know we can solve the problem because the numbers of coins and of weightings satisfy the inequality presented in the table for variant BDFHJ ($40 \le (3^4-1)/2$). Let us number the coins: 1, 2, 3, ... ,40. We will find elements of $\Gamma^{39}{}_4$, what is helpful to plan the weightings.

$\Gamma^{39}{}_4=\Gamma^{12}{}_3\times\{L,R,0\}\cup\{(L,L,L,R), (R,R,R,0), (0,0,0,L)\}$

$\Gamma^{12}{}_3=\Gamma^{3}{}_2\times\{L,R,0\}\cup\{(L,L,R), (R,R,0), (0,0,L)\}$

$\Gamma^{3}{}_2=\{(L,R), (R,0), (0,L)\}$

Finally $\Gamma^{39}{}_4$={(L,R,L,L), (R,0,L,L), (0,L,L,L), (L,R,R,L), (R,0,R,L), (0,L,R,L), (L,R,0,L), (R,0,0,L), (0,L,0,L), (L,L,R,L), (R,R,0,L), (0,0,L,L), (L,R,L,R), (R,0,L,R), (0,L,L,R), (L,R,R,R), (R,0,R,R), (0,L,R,R), (L,R,0,R), (R,0,0,R), (0,L,0,R), (L,L,R,R), (R,R,0,R), (0,0,L,R), (L,R,L,0), (R,0,L,0), (0,L,L,0), (L,R,R,0), (R,0,R,0), (0,L,R,0), (L,R,0,0), (R,0,0,0), (0,L,0,0), (L,L,R,0), (R,R,0,0), (0,0,L,0), (L,L,L,R), (R,R,R,0), (0,0,0,L)}.

For each coin numbered with 1,2,…,39 we map exactly one element of $\Gamma^{39}{}_4$, e.g. we assign (L,R,L,L) to coin 1, (R,0,L,L) to coin 2, etc, (0,0,0,L) to coin 39. We assign (L,L,L,L) to coin 40 and, finally, (R,R,R,R) to the additional genuine coin. In this way all weightings are planned (0 corresponds to the additional coin) as follows:

	The left pan	The right pan
The 1st weighting	1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, 37, 40	0, 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35, 38
The 2nd weighting	3, 6, 9, 10, 15, 18, 21, 22, 27, 30, 33, 34, 37, 40	0, 1, 4, 7, 11, 13, 16, 19, 23, 25, 28, 31, 35, 38
The 3rd weighting	1, 2, 3, 12, 13, 14, 15, 24, 25, 26, 27, 36, 37, 40	0, 4, 5, 6, 10, 16, 17, 18, 22, 28, 29, 30, 34, 38
The 4th weighting	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 39, 40	0, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 37

The weighting results we code as quadruples (w_1,w_2,w_3,w_4), as before. If we get (0,0,0,0), all coins are genuine. Otherwise we get the quadruple assigned earlier to the counterfeit coin (if it is heavier) or its inverse (if it is lighter than genuine ones).

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