Lower bounds on expectations of generalized order statistics from restricted families of distributions

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Generalized order statistics (gOSs) (Kamps, 1995)

gOSs unify various models of ordered random variables i.a.

- order statistics,
- sequential order statistics,
- type II progressively censored order statistics,
- records, k-th records,
- Pfeifer's records.

Generalized order statistics (gOSs) (Kamps, 1995)

Let $n \in \mathbf{N}$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$, $\gamma_1, \ldots, \gamma_n > 0$.

$X_\gamma^{(1)}, \ldots, X_\gamma^{(n)}$ are called **generalized order statistics (gOSs)** with parameter γ based on distribution function F if

$$X_\gamma^{(r)} \stackrel{d}{=} F^{-1}\left(1 - \prod_{i=1}^{r} U_i^{1/\gamma_i}\right), \quad r = 1, \ldots, n,$$

where $U_1, \ldots, U_n \stackrel{iid}{\sim} U(0,1)$ (Cramer & Kamps (2003)).

Another representation of gOSs:

$$X_\gamma^{(r)} = F^{-1}\left(U_\gamma^{(r)}\right),$$

where $U_\gamma^{(1)}, \dots, U_\gamma^{(n)}$ - uniform gOSs based on γ.

Notation:

$$f_{\gamma,r} \text{ - pdf } U_\gamma^{(r)},$$
$$F_{\gamma,r} \text{ - cdf } U_\gamma^{(r)}.$$

Examples of gOSs

- order statistics $X_{1:n}, \ldots, X_{n:n}$ based on $X_1, \ldots, X_n \overset{iid}{\sim} F$

$$\gamma_j = n - j + 1, \quad j = 1, \ldots, n$$

- n first k-th (upper) record values $X^{(k)}_{L(1)}, \ldots, X^{(k)}_{L(n)}$ based on $(X_i)_{i \in \mathbb{N}} \overset{iid}{\sim} F$

$$\gamma_j = k, \quad j = 1, \ldots, n$$

Generalized Pareto distributions

For a fixed $\alpha > -\frac{1}{2}$, GPD is defined as follows

$$W_\alpha(x) = \begin{cases} 1-(1-\alpha x)^{1/\alpha}, & \text{for } x \geqslant 0 \text{ if } \alpha < 0, \\ & \text{for } 0 \leqslant x \leqslant \frac{1}{\alpha} \text{ if } \alpha > 0, \\ 1-\mathrm{e}^{-x}, & \text{for } x \geqslant 0 \text{ if } \alpha = 0. \end{cases}$$

Let $F \succ_c W_\alpha \iff W_\alpha^{-1} F$ - concave on the support of F and if F is absolutely continuous with pdf f,

$$(W_\alpha^{-1} F)'(y) = (1 - F(y))^{\alpha - 1} f(y)$$

is decreasing.

Distributions with the decreasing generalized failure rate (DGFR)

Bieniek (2008) introduced the family of DGFR distributions

$$\mathrm{DGFR}(\alpha) = \{F : F \succ_c W_\alpha\},$$

with the generalized failure rate of an absolutely continuous F, defined as

$$\gamma_\alpha(y) = (1 - F(y))^{\alpha-1} f(y),$$

- $\alpha = 1 \quad \Longrightarrow \quad W_1 = U \quad \Longrightarrow \quad$ DGFR(0)=DFR
- $\alpha = 0 \quad \Longrightarrow \quad W_0 = E \quad \Longrightarrow \quad$ DGFR(1)=DD

PROBLEM

Assumptions

$X_1, \ldots, X_n$ are i.i.d. $\sim F$ with finite moments

$$\begin{aligned}\mu &= \mathbb{E}X_1 = \int_0^1 F^{-1}(x)dx,\\ \sigma^2 &= VarX_1 = \mathbb{E}|X_1 - \mu|^2.\end{aligned}$$

Find the lower non-positive bounds on

$$\mathbb{E}\frac{X_\gamma^{(r)} - \mu}{\sigma}, \quad 1 \leqslant p \leqslant \infty,$$

where $F \in DGFR$.

Procedure

Fix W - cdf on $[0, d)$, $d \leqslant \infty$, with pdf w and define

$$\mathcal{C}_W = \{g : [0, d) \longrightarrow \mathbb{R} : \int_0^d g^2(u) w(u) du < \infty, g \text{ is nondecreasing and convex}\},$$

and P_W - the projection onto $\mathcal{C}_W$. Let

$$\begin{aligned} \hat{f}_{\gamma,r} &= f_{\gamma,r} \circ W, \\ \hat{h}_{\gamma,r} &= 1 - \hat{f}_{\gamma,r}. \end{aligned}$$

Procedure, cont.

Since $\int\limits_0^d \hat{h}_{\gamma,r}(u)w(u)du = 0$, we have

$$-(\mathbb{E}X_\gamma^{(r)} - \mu) = \int\limits_0^d (F^{-1}W(u) - \mu)\hat{h}_{\gamma,r}w(u)du \leqslant \int\limits_0^d (F^{-1}W(u) - \mu)P_W\hat{h}_{\gamma,r}w(u)du.$$

Therefore

$$\frac{\mathbb{E}X_\gamma^{(r)} - \mu}{\sigma} \geqslant -||P_W\hat{h}_{\gamma,r}||_W,$$

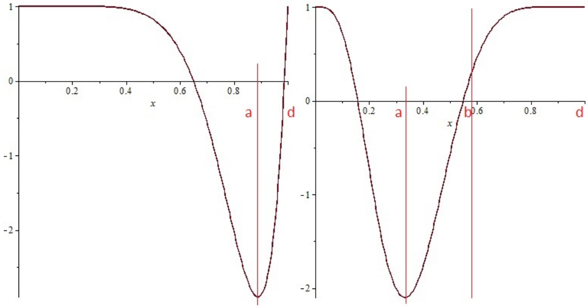
where

$$||P_W\hat{h}_{\gamma,r}||_W = \left(\int\limits_0^d |P_W\hat{h}_{\gamma,r}(u)|^2 w(u)du\right)^{1/2}.$$

"$=$" holds for F satisfying $\frac{F^{-1}W(u)-\mu}{\sigma} = \frac{P_W\hat{h}_{\gamma,r}}{||P_W\hat{h}_{\gamma,r}||_W}$.

Assumptions

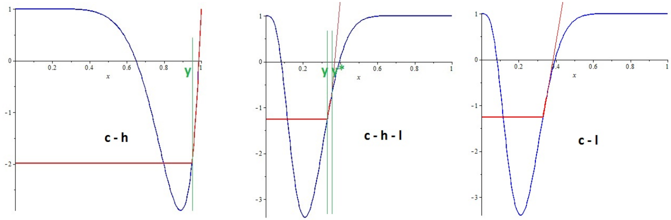
(A) Let h be a bounded, twice differentiable function on $[0, d)$, such that $h(0) = \lim_{x \nearrow d} h(d) \geqslant 0$ and $\int_0^d h(x)w(x)dx = 0$, where w is a positive weight function satisfying $\int_0^d w(x)dx = 1$. Moreover, h is decreasing on $(0, a)$, convex increasing on (a, b), and concave increasing on (b, d), for some $0 < a < b \leqslant d$.



Shape of functions $\hat{h}_{\gamma,r}$

Functions $\hat{h}_{\gamma,r}$ satisfies assumptions (A):

- $r \geqslant 2$, $1 < \gamma_r \leqslant 1 + \alpha$: $\hat{h}_{\gamma,r}$ is decreasing on $(0, a)$, then convex increasing on (a, d)
- $r \geqslant 2$, $\gamma_r > 1 + \alpha$: $\hat{h}_{\gamma,r}$ is decreasing on $(0, a)$, convex increasing on (a, b) and concave increasing on (b, d)



Auxiliary functions, $\gamma_r > 1 + \alpha$

Consider

$$\lambda(\beta) = \frac{\int_\beta^d (x-\beta)(\hat{h}_{\gamma,r}(x) - \hat{h}_{\gamma,r}(\beta))w(x)dx}{\int_\beta^d (x-\beta)^2 w(x)dx},$$

$$K(\beta) = \lambda(\beta) - \hat{h}'_{\gamma,r}(\beta),$$

$$L(\beta) = \int_\beta^d [\hat{h}_{\gamma,r}(x) - \lambda(\beta)(x-\beta) - \hat{h}_{\gamma,r}(\beta)]w(x)dx.$$

Proposition, $\gamma_r > 1 + \alpha$

Let $y > a$ satisfy condition

$$F_{\gamma,r}W(y) = W(y)f_{\gamma,r}W(y)$$

and β_* be the only solution of $K(\beta) = 0$, $\beta \in (a, b)$.
If y satisfies

$$K(y) > 0 \text{ and } L(y) < 0 < L(\beta_*),$$

then for $y_* \in (y, \beta_*)$ satisfying $L(y_*) = 0$ we have

$$P_W \hat{h}_{\gamma,r}(x) = \begin{cases} \hat{h}_{\gamma,r}(y), & 0 \leqslant x \leqslant y, \\ \hat{h}_{\gamma,r}(x), & y < x \leqslant y^*, \qquad (c-h-l) \\ \hat{h}_{\gamma,r}(y^*) + \lambda(y^*)(x - y^*), & y^* < x < d. \end{cases}$$

Proposition, $\gamma_r > 1+\alpha$, cont.

Otherwise we have

$$P_W \hat{h}_{\gamma,r}(x) = \frac{F_{\gamma,r}(W(\beta)) - W(\beta)}{W(\beta)} \left[\frac{(x-\beta)\mathbf{1}_{[\beta,d)}(x)}{\frac{1}{1+\alpha}(1-\alpha\beta)^{1+1/\alpha}} - 1 \right], \quad (c-l)$$

for the greatest $0 < \beta \leqslant y$, which satisfies the following condition

$$\sum_{j=1}^{r} \frac{\sigma_{j,r}(\alpha)}{\gamma_j} \hat{f}_{\gamma,j}(\beta) = \frac{1}{(1-W(\beta))^{1+\alpha}} \left[\frac{(1-\alpha\beta)^{1+\frac{1}{\alpha}}}{1+\alpha} + \frac{W(\beta) - F_{\gamma,r}(W(\beta))}{W(\beta)} \cdot (1-\alpha\beta)\left(\frac{2}{1+2\alpha} - \frac{(1-\alpha\beta)^{1/\alpha}}{1+\alpha} \right) \right], \tag{1}$$

where

$$\sigma_{j,r}(\alpha) = \begin{cases} \frac{1}{\alpha}(1 - \prod_{i=j}^{r} \frac{\gamma_i}{\gamma_i+\alpha}), & \alpha \neq 0, \\ \sum_{i=j}^{r} \frac{1}{\gamma_i}, & \alpha = 0. \end{cases}$$

Results: $0 < \gamma_r \leqslant 1 + \alpha$

Proposition

Let $r \geqslant 2$, $F \in \mathrm{DGFR}(\alpha)$, where $W = W_\alpha$.
If $0 < \gamma_r \leqslant 1$, then $\mathbb{E}X_\gamma^{(r)} \geqslant \mu$.
If $1 < \gamma_r \leqslant 1 + \alpha$, then

$$\mathbb{E}\frac{X_\gamma^{(r)} - \mu}{\sigma} \geqslant -B_1,$$

$$B_1^2 = (1 - \hat{f}_{\gamma,r}(y))^2 W(y) + 1 - W(y) - 2(1 - F_{\gamma,r}(W(y))) + \int_y^d \hat{f}_{\gamma,r}^2(x) w(x) dx.$$

"=" holds for F satisfying

$$F^{-1}(W(x)) = \begin{cases} \mu + \frac{\sigma}{B_1}(1 - \hat{f}_{\gamma,r}(y)), & 0 < x < y, \\ \mu + \frac{\sigma}{B_1}(1 - \hat{f}_{\gamma,r}(x)), & y \leqslant x < d. \end{cases}$$

Results: $\gamma_r > 1 + \alpha$

Proposition, cont.

Let $y > a$, satisfy condition $F_{\gamma,r}W(y) = W(y)f_{\gamma,r}W(y)$ and β_* be the only solution of $K(\beta) = 0$, $\beta \in (a, b)$. If y satisfies

$$K(y) > 0 \text{ and } L(y) < 0 < L(\beta_*),$$

then for $y_* \in (y, \beta_*)$ such that $L(y_*) = 0$ we have the following bound

$$\mathbb{E}\frac{X_\gamma^{(r)} - \mu}{\sigma} \geqslant -B_2,$$

$$\begin{aligned} B_2^2 &= (1 - \hat{f}_{\gamma,r}(y))^2 W(y) + (1 - \hat{f}_{\gamma,r}(y^*))^2 (1 - W(y^*)) + W(y^*) - W(y) - 2F_{\gamma,r}(W(y^*)) \\ &\quad -2F_{\gamma,r}(W(y)) + 2\lambda(y^*)\frac{(1-\alpha y^*)^{1+1/\alpha}}{1+\alpha}\left[1 - \hat{f}_{\gamma,r}(y^*) + \lambda(y^*)\frac{1-\alpha y^*}{1+2\alpha}\right] \\ &\quad + \int_y^{y^*} \hat{f}_{\gamma,r}(x)w(x)dx. \end{aligned}$$

Results: $\gamma_r > 1 + \alpha$

Proposition, cont.

"=" holds for F satistfying

$$F^{-1}(W(x)) = \begin{cases} \mu + \frac{\sigma}{B_2}(1 - \hat{f}_{\gamma,r}(y)), & 0 \leqslant x < y, \\ \mu + \frac{\sigma}{B_2}(1 - \hat{f}_{\gamma,r}(x)), & y \leqslant x < y^*, \\ \mu + \frac{\sigma}{B_2}[1 - \hat{f}_{\gamma,r}(y^*) + \lambda(y^*)(x - y^*)], & y^* \leqslant x < d. \end{cases}$$

Results: $\gamma_r > 1 + \alpha$

Proposition, cont.

Otherwise we have

$$\mathbb{E}\frac{X_\gamma^{(r)} - \mu}{\sigma} \geqslant -B_3,$$

$$B_3^2 = \frac{(W(\beta) - F_{\gamma,r}(W(\beta)))^2}{W(\beta)^2}\left[2\frac{1+\alpha}{1+2\alpha}(1-\alpha\beta)^{-1/\alpha} - 1\right].$$

for the greatest $0 < \beta \leqslant y$, satisfying (1).

Results: $\gamma_r > 1 + \alpha$

Proposition, cont.

"=" holds for F satistfying

$$F^{-1}(W(x)) = \begin{cases} \mu + \frac{\sigma}{B_3}(1 - \hat{f}_{\gamma,r}(y)), & 0 \leqslant x < y, \\ \mu + \frac{\sigma}{B_3}(1 - \hat{f}_{\gamma,r}(x)), & y \leqslant x < y^*, \\ \mu + \frac{\sigma}{B_3}[1 - \hat{f}_{\gamma,r}(y^*) + \lambda(y^*)(x - y^*)], & y^* \leqslant x < d. \end{cases}$$

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