Single-use automata and orbit-finite monoids for total order atoms

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Observation: There are such monoids M and N, that M has finite $\mathcal{J}$-height, N has infinite $\mathcal{J}$-height, and $N \subseteq M$

Letters are strictly increasing

This language is recognised by the following morphism:

$$1\ 3\ 7\ 9 \;\mapsto\; (1, 9)$$

first last

Its monoid is defined as follows:

$$N = \mathbb{Q}^2 + \bot + 1$$

$$(a_1, b_1)\cdot(a_2, b_2) = \begin{cases} (a_1, b_2) & \text{if } a_2 < b_1 \\ \bot & \text{otherwise} \end{cases}$$

It has an infinite infix chain: $(-1, 1), (-2, 2), (-3, 3), \ldots$

Letters between #'s are strictly increasing

The morphism is almost the same:

$$3\ 4\ 7\ 9\ \#\ 3\ 5\ 7\ \#\ 1\ 2 \;\mapsto\; (3, 2)$$

first last

The monoids are also very similar:

$$M = (\mathbb{Q} + \#)^2 + \bot + 1$$

$$(a_1, b_1)\cdot(a_2, b_2) = \begin{cases} (a_1, b_2) & \text{if } a_2 < b_1 \text{ or } a_2 = \# \text{ or } b_1 = \# \\ \bot & \text{otherwise} \end{cases}$$

But now every two elements from $(\mathbb{Q} + \#)^2$ are each other's infixes:

$$(3, \#)\cdot(x, y)\cdot(\#, 2) = (3, 2)$$