Single-use register automata and orbit-finite semigroups for total-order atoms

Mikołaj Bojańczyk Nathan Lhote Rafał Stefański

Highlights 2021

Work in progress

Single-use register automata and orbit-finite semigroups for total-order atoms

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Background

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    - 1-way automata = 2-way automata = orbit-finite monoids
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- Now we would like to extend those results to $\mathbb{A} = \langle \mathbb{Q}, \leq \rangle$

Register automata

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3 2 2 1 5 2

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The first letter is greater than other letters

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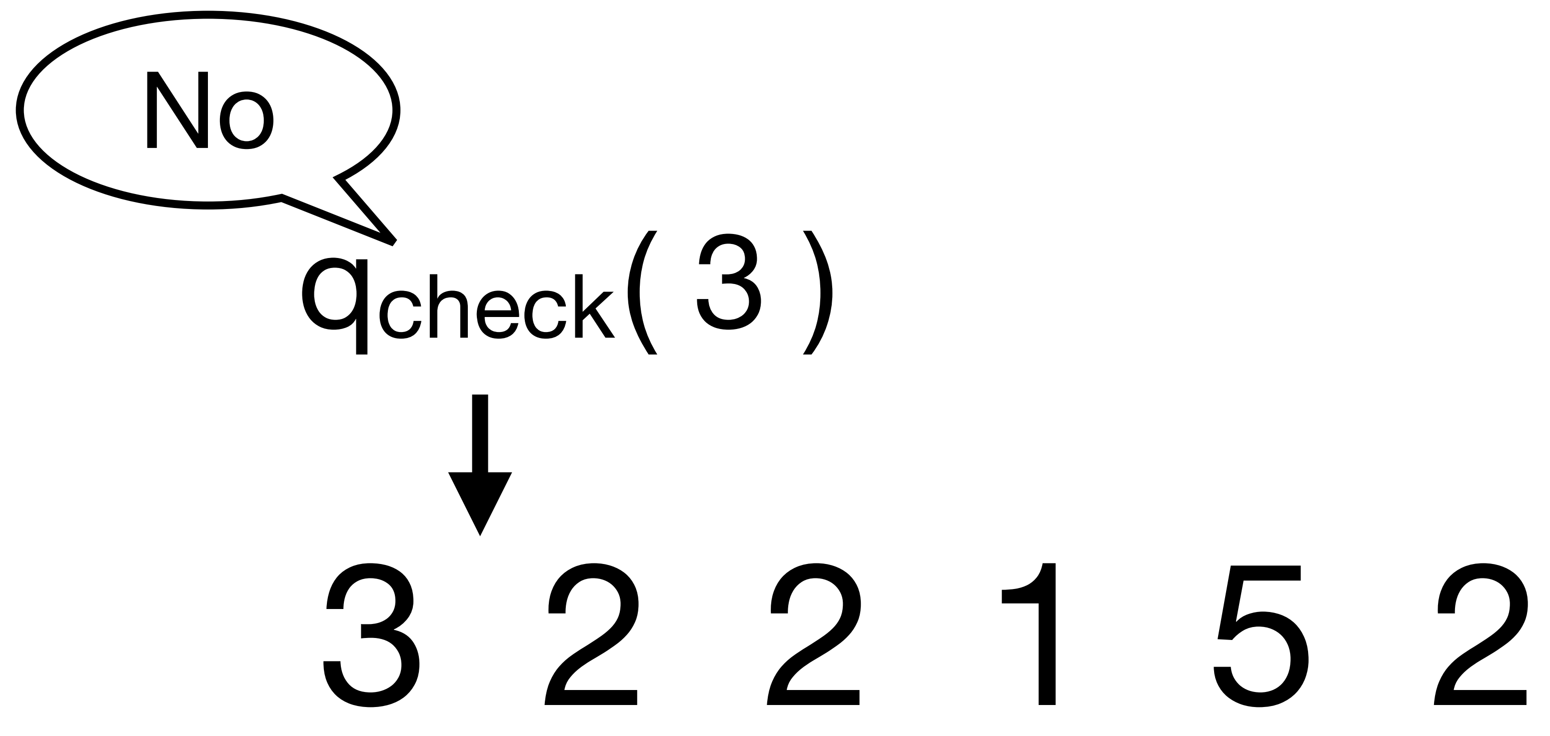

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$5 \stackrel{?}{\leq} 2$

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q_{fail} ( )

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Register automata

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Reject

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Single-use restriction

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Every read access to a register has the side effect of destroying its contents

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Recognised by an orbit-finite monoid of finite J-height
$\neq$
Syntactic monoid is orbit-finite and has finite J-height

Single-use automata $=$ Orbit-finite monoids of finite J-height