Transport inequalities for Poisson point processes

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Goal

- Show you some transport-entropy inequalities Poisson point processes (and convince you they are true!).

Motivation

- Functional inequalities are a powerful tool to understand in an analytic way the geometric properties of a space.
- Concentration of measure is one instance of these geometric-functional inequalities (books by M. Ledoux, M. Gromov).
- But lacks one crucial feature: concentration of measure does not scale well with dimension.
- In the 1990s, K. Marton and M. Talagrand invent transport-entropy inequalities and study connection with dimension-free concentration of measure.
- Understanding the geometry of infinite-dimensional (non-product) spaces is an active field of research.
- Here we study the space of discrete measures equipped with the law of a Poisson point process.

Optimal transport

- (E, τ) Polish space.
- $\omega \colon E \times E \to [0, \infty]$ lower semi-continuous:

$$T_\omega(\nu_1, \nu_2) = \inf \left\{ \int \omega(x, y) q(\mathrm{d}x\mathrm{d}x),\ q \in Cpl(\nu_1, \nu_2) \right\}.$$

- $c \colon E \times \mathcal{P}(E) \to [0, \infty]$ lower semi-continuous:

$$T_c(\nu_2|\nu_1) = \inf \left\{ \int c(x, p_x) \nu_1(\mathrm{d}x),\ p_x(\mathrm{d}y)\nu_1(\mathrm{d}x) \in Cpl(\nu_1, \nu_2) \right\}.$$

- $\alpha \colon [0, \infty] \to [0, \infty]$, $\rho \colon E \times E \to [0, \infty]$ lsc:

$$\widetilde{T}_{\alpha,\rho}(\nu_2|\nu_1) = \inf \left\{ \int \alpha \left(\int \rho(x, y) p_x(\mathrm{d}y) \right) \nu_1(\mathrm{d}x) \right\}.$$

Transport inequalities

- (c, γ) satisfies a transport-entropy inequality if
$$T_c(\nu_2|\nu_1) \leq a_1 H(\nu_1|\gamma) + a_2 H(\nu_2|\gamma).$$

- Equivalent to dimension free concentration of measure, Nathael and GRST:
$$\mu^n(E^n \setminus A_t^n)^{a_2} \mu^n(A)^{a_1} \leq K\,\mathrm{e}^{-t},$$
$$A_t^n = \{c_A^n \leq t\},$$
$$c_A^n(x) = \inf\{\sum c(x_i, p_i) : p(A) = 1\}.$$

- Formally equivalent to Maurey
$$\gamma(\exp(Q_c\phi))\gamma(\exp(-\phi)) \leq 1,$$
$$Q_c\phi(x) = \inf_{p \in \mathcal{P}(E)} \{p(\phi) + c(x, p)\}.$$

Examples

- Talagrand: $\omega(x, y) = 4|x - y|^2$, $\gamma \sim \mathbf{N}(0, id)$ satisfies a transport-entropy inequality.
- Marton, Dembo: for all $t \in [0, 1]$, for some α_t explicit, $\forall \gamma \in \mathcal{P}(E)$:

$$\widetilde{T}_{\alpha_t, d_H}(\nu_2|\nu_1) \leq \frac{1}{t} H(\nu_1|\gamma) + \frac{1}{1-t} H(\nu_2|\gamma).$$

One can take $\alpha_{\frac{1}{2}}(u) = \frac{u^2}{2}$, in which case

$$\widetilde{T}_{\alpha_{\frac{1}{2}}, d_H}(\nu_2|\nu_1) = \frac{1}{2} \int \left[1 - \frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_2}\right]_+^2 \nu_2.$$

By Jensen, this yields an improvement of Pinsker's inequality.

Stability of transport-entropy inequalities

- If (ν, c) satisfies a transport-entropy inequality, so does (ν^n, c^n);

$$c^n((x_1, \ldots, x_n), p) = \sum c(x_i, p_i).$$

- $S\colon E \to E'$, if (ν, c) satisfies a transport-entropy inequality then so does $(S_\# \nu, S_\# c)$;

$$S_\# c(y, q) = \inf\{c(x, p) : S(x) = y,\ S_\# p = q\}.$$

- If c is lower semi-continuous, $\nu_n(A) \to \nu(A)$, and (ν_n, c) satisfies a transport-entropy inequality, so does (ν, c).

In all cases, the constants are preserved.

Configuration space

- (Z, d) separable complete metric space.
- $\mathcal{M}_{\mathbb{N}}(Z) = \{\sum_{i=1}^{I} \delta_{x_i}\}$.
- $\eta \in \mathcal{M}_{\bar{\mathbb{N}}}(Z)$ if and only if $\eta_{|B} \in \mathcal{M}_{\mathbb{N}}(B)$ for all balls B.
- A point process is any random variable with value in $\mathcal{M}_{\mathbb{N}}(Z)$.
- A point process η is a Poisson point process with σ-finite intensity ν if
$$\mathbb{E}\, e^{-\eta(u)} = \exp(\nu(e^{-u} - 1)).$$
- Write Π_ν for its law.

Universal transport-entropy inequality for Π_ν

Theorem

For all $t \in [0,1]$, $\xi \in \mathcal{M}_{\bar{\mathbb{N}}}(Z)$, $\Pi \in \mathcal{P}(\mathcal{M}_{\bar{\mathbb{N}}}(Z))$,

$$c_t(\xi, \Pi) = \int \alpha_t \left(\int \left[1 - \frac{\chi(x)}{\xi(x)} \right]_+ \Pi(\mathrm{d}\chi) \right) \xi(\mathrm{d}x).$$

For all $\nu \in \mathcal{P}(Z)$, *for all* $\Pi_1, \Pi_2 \in \mathcal{P}(\mathcal{M}_{\bar{\mathbb{N}}}(Z))$*:*

$$T_{c_t}(\Pi_2|\Pi_1) \leq \frac{1}{t} H(\Pi_1|\Pi_\nu) + \frac{1}{1-t} H(\Pi_2|\Pi_\nu).$$

Idea of the proof

- Lift Dembo's inequality to the Poisson point process.
- First proof it for $\nu(Z) = 1$ and for $B_{n,\nu} = (\sum_{i=1}^{n} \delta_{x_i})_{\#}\nu^n$. Ok since transport-entropy inequalities are closed under taking tensor product and push forward.
- Via a thinning procedure one can strongly approximate Π_ν by a binomial process. Transport-entropy inequalities are also stable under approximation.
- Case $\nu(Z) = \infty$ also by strong approximation.

Applications

- General concentration of measure Reitzner.
- Concentration of measure for U-statistics Reitzner & Bachmann.
- Modified logarithmic Sobolev inequality (next slide).

Modified logarithmic Sobolev inequality

- $F\colon \mathcal{M}_{\bar{\mathbb{N}}}(Z) \to \mathbb{R}$, $D_x^+ F(\xi) = F(\xi + \delta_x) - F(\xi)$.
- F is non-decreasing if $D_x^+ F \geq 0$ and convex if $D_x^+ D_y^+ F \geq 0$.

Theorem

Let $\phi(u) = s - \log(1+s)$ and F non-decreasing convex, $\eta \sim \Pi_\nu$.

$$\mathbf{Ent}\, e^{F(\eta)} \leq \mathbb{E}\, e^{F(\eta)} \int \phi\left(\frac{F(\eta) - F(\eta - \delta_x)}{2}\right) \eta(\mathrm{d}x).$$

Doe not improve upon another modified logarithmic Sobolev inequality by Wu (but of same order).

Geometric transport-entropy inequality for Π_ν

Theorem

If (ν, ω) satisfies Talagrand, then (Π_ν, T_ω) satisfies Talagrand. That is: if $T_\omega(\nu_1, \nu_2) \leq a_1 H(\nu_1|\nu) + a_2 H(\nu_2|\nu)$ then Π_ν satisfies

$$T_{T_\omega}(\Pi_1, \Pi_2) \leq a_1 H(\Pi_1|\Pi_\nu) + a_2 H(\Pi_2|\Pi_\nu),$$
$$T_{T_\omega}(\Pi_1, \Pi_2) = \inf\{T_\omega(\xi_1, \xi_2) : \xi_1 \sim \Pi_1,\ \xi_2 \sim \Pi_2\}.$$

- Not clear how to interpret / apply this inequality: even on $Z = \mathbb{R}^d$, $\omega(x, y) = |x - y|^2$, $\nu =$ Gaussian, $M_{\bar{\mathbb{N}}}(Z)$ equipped with $W_2 = T_\omega^{\frac{1}{2}}$ has bad properties. It is only an extended metric space . Reminiscent of a result by Erbar & Huesmann: they show that the RCD condition lifts from a manifold to the configuration space with Π_{vol} but not clear what are the consequences in this extended setting.

Open questions

- Deeper links with modified logarithmic Sobolev inequalities in the spirit Otto & Villani; Bobkov, Gentil & Ledoux; Gozlan, Roberto & Samson.
- Links with discrete displacement convexity GRST?
- Links with semigroup methods?
- What about interacting particles system?
- What about other random measures?