

Partial Differential Equations in Mathematical Biology

**On the Keller-Segel System with
External Application of Chemoattractant**

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Contents

First Problem: A System with external application of chemoattractant

$$\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u - \chi \nabla \cdot (u \nabla v), & x \in \mathbb{R}^2 & \quad t > 0 \\ -\Delta v &= u + 2\pi F_0 \delta(x) & x \in \mathbb{R}^2\end{aligned}$$

Second Problem: On a Chemotaxis system with logistic term

$$\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u - \chi \nabla \cdot (u \nabla v) + \lambda u(1 - u), & x \in \Omega & \quad t > 0 \\ -\Delta v + v &= u & x \in \Omega\end{aligned}$$

First Part

1-. Introduction

2-. Auxiliary Problem

$$W(s, t) := \frac{1}{\pi} \int_{B_{\sqrt{s}}(0)} \rho u(\rho, t) d\rho, \quad s > 0, \quad t > 0,$$

3-. Instantaneous Blow up

4-. Case

$$\int_{\mathbb{R}^2} u_0 > 8\pi - 4\pi F_0$$

5-. Case

$$\int_{\mathbb{R}^2} u_0 < 8\pi - 4\pi F_0$$

1 Introduction

Chemotaxis is the ability of microorganisms to respond to **chemical signals** by moving along the gradient of the chemical substance, either toward the higher concentration (**positive taxis**) or away from it (**negative taxis**).

- **Dictyostelium discoideum** Bacteria. See Keller-Segel [1970], [1971]
- **Tumour-induced angiogenesis**. See Anderson-Chaplain [1997], [1998] A. Kubo in the conference (Tuesday Afternoon)
- **Astrophysics and gravitational interaction of particles**. See Biler [1995], Biler-Hilhorst-Nadzieja [1994]
- **Morphogenesis** formation of the embryo. See Merkin-Needham-Sleeman [2005], Bollenbach-Kruse-Pantazis-Gonzalez Gaitan-Julicher [2007], C. Stinner in the Poster Session of the conference.

Mathematical models of chemotaxis

Keller and Segel [1970], [1971] (after Patlak [1953])

p = bacteria, w = chemoattractant

$$\begin{aligned}\frac{\partial u}{\partial t} &= Q(u, v) + \nabla \cdot D(u, v) \nabla u - u \chi(v) \nabla v, \\ \frac{\partial v}{\partial t} &= d \Delta v + h(u, v) \quad x \in \Omega \quad t > 0\end{aligned}$$

We consider $\Omega = \mathbb{R}^2$, fast diffusion, and the simplified system

$$\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u - \chi \nabla \cdot (u \nabla v), & x \in \mathbb{R}^2 \quad t > 0 \\ -\Delta w &= u + f(x) & x \in \mathbb{R}^2 \\ & \text{+initial datum for } u\end{aligned}$$

where f is an external application of chemoattractant.

Jäger and Luckhaus [1992], $\Omega \subset \mathbb{R}^2$

$$\frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \nabla v), \quad \Omega \times (0, T)$$

$$-\Delta v = u - 1 \quad \Omega \times (0, T)$$

zero flux on the boundary

- if $\int_{\Omega} u_0 < 8\pi$, then global solutions exist
- if $\int_{\Omega} u_0 > 8\pi$, solution blows up in finite time
- See also Herrero-Velazquez [1996]

Case $-\Delta v = u$ in \mathbb{R}^2 . See Biler [1995a], Blanchet-Dolbeault-Perthame [2006] Blanchet-Carrillo-Masmoundi [2008], Naito-Suzyki [2004], Velázquez [2002], [2004].

Under assumptions

$$\int_{\mathbb{R}^2} (1 + |x|^2) u_0 < \infty, \quad \int_{\mathbb{R}^2} u_0 \log u_0 < \infty$$

- The subcritical case $\int_{\mathbb{R}^2} u_0 < 8\pi \longrightarrow$ global existence. See Blanchet-Dolbeault-Perthame [2006].
- The supercritical case $\int_{\mathbb{R}^2} u_0 > 8\pi$ solutions blows up as a dirac functions. Velázquez [2002]
- The critical case $\int_{\mathbb{R}^2} u_0 = 8\pi$. The blow up at $t = \infty$ with similar profile than in the supercritical case. Blanchet-Carrillo-Masmoundi [2008].

The mathematical model

- u concentration of living organisms
- v concentration of chemoattractant substance
- we assume $\chi = 1$

$$\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u - \nabla \cdot (u \nabla v) & x \in \mathbb{R}^2, t > 0 \\ -\Delta v &= u + f(x), & x \in \mathbb{R}^2,\end{aligned}$$

where

$$f(x) = \lim_{\varepsilon \rightarrow 0} \frac{f_0}{|\omega_\varepsilon|} I_{\omega_\varepsilon} = f_0 \delta(x)$$

where $f_0 \geq 0$.

We introduce higher gradient of v . We don't increase the mass of u .

2 Auxiliary Problem

Radially symmetric solution $u = u(r, t)$

$$W(s, t) := \frac{1}{\pi} \int_{B_{\sqrt{s}}(0)} \rho u(\rho, t) d\rho, \quad s > 0, t > 0,$$

which satisfies

$$\begin{cases} W_t = 4sW_{ss} + WW_s + 2F_0W_s, & s > 0, t > 0, \\ W(0, t) = 0, \quad \lim_{s \rightarrow \infty} W(s, t) = \frac{\mu}{\pi}, & t > 0, \\ W(s, 0) = W_0(s), & s > 0, \end{cases}$$

with $F_0 := \frac{f_0}{2\pi}$ and

$$W_0(s) := 2 \int_0^{\sqrt{s}} \rho u_0(\rho) d\rho, \quad s > 0.$$

We may consider the cut-off function

$$\chi \in C^\infty([0, \infty)), \quad \chi \equiv 0 \quad \text{on} \quad \left[0, \frac{1}{2}\right],$$

$$\chi \equiv 1 \quad \text{on} \quad [1, \infty), \quad \chi' \geq 0 \quad \text{on} \quad [0, \infty)$$

$$\varepsilon \in (0, 1) \quad \chi^{(\varepsilon)}(s) := \chi\left(\frac{s}{\varepsilon}\right), \quad s \geq 0.$$

Then

$$\begin{cases} W_t^{(\varepsilon)} = 4sW_{ss}^{(\varepsilon)} + \chi^{(\varepsilon)}(s)W^{(\varepsilon)}W_s^{(\varepsilon)} + 2F_0\chi^{(\varepsilon)}(s)W_s^{(\varepsilon)}, & s > 0, \quad t > 0, \\ W^{(\varepsilon)}(0, t) = 0, \quad \lim_{s \rightarrow \infty} W^{(\varepsilon)}(s, t) = \frac{\mu}{\pi}, & t > 0, \\ W^{(\varepsilon)}(s, 0) = W_0(s), & s > 0, \end{cases}$$

- There exists a unique solution $W^{(\varepsilon)}$ to the ε -problem
- $W^{(\varepsilon)}$ is non-increasing with respect to ε
- $W^{(\varepsilon)} \rightarrow W$ pointwise and C^2 in Compact sets.
- $W \in L^\infty((0, \infty) \times (0, \infty))$ is a weak solution of the problem in the following sense

$$W(s, t) \rightarrow \frac{\mu}{\pi} \text{ as } s \rightarrow \infty \text{ for all } t > 0$$

$$-\int_0^\infty \int_0^\infty \zeta_t W - \int_0^\infty \zeta(\cdot, 0) W_0 = 4 \int_0^\infty \int_0^\infty (s\zeta)_{ss} W - \frac{1}{2} \int_0^\infty \int_0^\infty \zeta_s W^2 - \frac{f_0}{2\pi} \int_0^\infty \int_0^\infty \zeta_s W$$

for $\zeta \in C_0^\infty([0, \infty) \times [0, \infty))$,

where $W_0(s) := \frac{1}{\pi} \int_{B_{\sqrt{s}}(0)} u_0(x) dx$ for $s \geq 0$.

3 Instantaneous Blow up

Lemma 1

Let $F_0 > 0$, $\delta > \frac{2-F_0}{2}$ such that $\delta \in (0, 1)$. Then there exist positive constants a, b, ξ, k_0, K_0 such that for any $\gamma > 0$,

$$\varphi(s) := \begin{cases} \frac{a}{\gamma^\delta} s^{-\delta} - b & \text{if } 0 < s < \frac{\xi}{\gamma}, \\ e^{-\gamma s} & \text{if } s \geq \frac{\xi}{\gamma}, \end{cases}$$

$$\varphi(s) \in W_{loc}^{2,\infty}((0, \infty))$$

$$4s\varphi_{ss} + (8 - 2F_0)\varphi_s \geq k_0\gamma\varphi \quad \text{a.e. in } (0, \infty)$$

$$\int_0^\infty \varphi^2(s) |\varphi_s(s)|^{-1} ds \leq \frac{K_0}{\gamma^2}.$$

Theorem 1. Let $F_0 > 0$, and that W_0

- $W_0 \in W^{1,\infty}((0, \infty))$,
- $W_{0s} \geq 0$ in $(0, \infty)$
- $W_0(s) \rightarrow \frac{\mu}{\pi}$ as $s \rightarrow \infty$ for some $\mu > 0$.

Then for any positive $\alpha > \frac{2-F_0}{2}$ and any $t_0 \geq 0$,

$$\sup_{s>0, t \in (t_0, t_0+\tau)} \frac{W(s, t)}{s^\alpha} = \infty \quad \text{for all } \tau > 0.$$

In particular

$$\|W_s\|_{L^\infty((0, \infty) \times (t_0, t_0+\tau))} = \infty \quad \text{for all } \tau > 0.$$

which implies

$$\|u\|_{L^\infty(\mathbb{R}^2 \times (t_0, t_0+\tau))} = \infty \quad \text{for all } \tau > 0.$$

Idea of the proof.

We consider

$$y(t) := \int_0^\infty \varphi(s)W(s,t)ds, \quad t > 0,$$

and

$$y^{(\varepsilon)}(t) := \int_0^\infty \varphi(s)\chi_\varepsilon W(s,t)ds, \quad t > 0,$$

multiply the ε - problem by $\chi_\varepsilon\varphi$ and integrate by parts and take limits as $\varepsilon \rightarrow 0$ after technical estimates we arrive to

$$y(t) \geq y(t_1) + k_0\gamma \int_{t_1}^t \int_0^\infty \varphi W - \frac{1}{2} \int_{t_1}^t \int_0^\infty \varphi_s W^2 \quad \text{for all } t \in (t_1, t_0 + \tau).$$

Notice that

$$y^2(t) = (\int_0^\infty \varphi W)^2 \leq (\int_0^\infty \varphi^2 |\varphi_s|^{-1}) \cdot (\int_0^\infty |\varphi_s| W^2) \leq \frac{K_0}{\gamma^2} \cdot \int_0^\infty |\varphi_s| W^2 \quad \text{for } t > 0,$$

therefore we have

$$y(t) \geq y(t_1) + \int_{t_1}^t Ay(\bar{t}) + By^2(\bar{t})d\bar{t} \quad \text{for all } t \in (t_1, t_0 + \tau).$$

for

$$A := k_0\gamma \quad \text{and} \quad B := \frac{\gamma^2}{K_0}.$$

By comparison with the following equation

$$z' = Az + Bz^2 \quad z(t_1) = y(t_1)$$

we have finite time blow up for $T \leq \frac{C}{\gamma}$.

4 Formation of Dirac-type singularities for $\mu > 8\pi - 4\pi F_0$

Theorem 2. Let $F_0 \geq 0$

$$\mu := \int_{\mathbb{R}^2} u_0(x) dx < \infty,$$

such that $\mu > 0$. Then, for $\mu > 8\pi - 4\pi F_0$, u satisfies

$$u(x, t) \rightarrow \mu \delta(x) \quad \text{as } t \rightarrow \infty.$$

The Proof of the Theorem is given in 3 steps.

Lemma 3

Let $F_0 \geq 0$, and W_0 such that

$$\begin{cases} \text{(H1)} & W_0 \in W^{1,\infty}((0, \infty)), \\ \text{(H2)} & W_{0s} \geq 0 \text{ in } (0, \infty) \quad \text{as well as} \\ \text{(H3)} & W_0(s) \rightarrow \frac{\mu}{\pi} \quad \text{as } s \rightarrow \infty. \end{cases}$$

Then for all $\hat{\mu} \in (8\pi - 4\pi F_0, \mu)$ there exist $s_0 > 0$ and \underline{W}_0

$$\underline{W}_0(s) := \begin{cases} 0 & \text{if } s \in [0, s_0], \\ a - \frac{1}{b+cs\beta} & \text{if } s > s_0, \end{cases}$$

such that

$$\begin{aligned} & \underline{W}_0 \in W^{1,\infty}((0, \infty)) \cap C^2([0, \infty) \setminus \{s_0\}) \\ & \liminf_{s \searrow s_0} \underline{W}_{0s}(s) > 0, \quad \underline{W}_0(s) \rightarrow \frac{\hat{\mu}}{\pi} \text{ as } s \rightarrow \infty, \\ & 4s\underline{W}_{0ss} + \underline{W}_0\underline{W}_{0s} + 2F_0\underline{W}_{0s} = 0 \quad \text{in } (s_0, \infty) \\ & \underline{W}_0 \leq W_0 \quad \text{in } (0, \infty) \end{aligned}$$

For

$$a := \frac{\hat{\mu}}{\pi} \quad b := \frac{1}{2(a + 2F_0 - 4)} \quad \beta := \frac{a + 2F_0 - 4}{4} \quad s_0 := \left(\frac{\frac{1}{a} - b}{c} \right)^{\frac{1}{\beta}} \geq s_1.$$

for some large $s_1 > 0$ such that

$$W_0(s) \geq a \quad \text{for all } s \geq s_1,$$

and $c > 0$.

Lemma 4

Let $F_0 \geq 0$, and assume that $\psi \in C^2((0, \infty))$ is a nonnegative solution of

$$0 = 4s\psi_{ss} + \psi\psi_s + 2F_0\psi_s, \quad s > 0,$$

with the additional properties $\psi_s \geq 0$ on $(0, \infty)$ and

$$\psi(s) \nearrow \frac{\mu}{\pi} \quad \text{as } s \rightarrow \infty$$

with some $\mu \geq 0$. In that case, if

$$\mu > 8\pi - 4\pi F_0,$$

then

$$\psi \equiv \frac{\mu}{\pi} \quad \text{in } (0, \infty).$$

Lemma 5

Let $F_0 \geq 0$, and for some

$$\mu > 8\pi - 4\pi F_0$$

we have

$$\left\{ \begin{array}{l} \text{(H1)} \quad W_0 \in W^{1,\infty}((0, \infty)), \\ \text{(H2)} \quad W_{0s} \geq 0 \text{ in } (0, \infty) \quad \text{as well as} \\ \text{(H3)} \quad W_0(s) \rightarrow \frac{\mu}{\pi} \quad \text{as } s \rightarrow \infty. \end{array} \right.$$

Then, W satisfies

$$W(s, t) \rightarrow \frac{\mu}{\pi} \quad \text{as } t \rightarrow \infty,$$

the convergence being uniform on compact subsets of $(0, \infty)$.

Idea of the proof:

- Let \underline{W} be the solution for initial data \underline{W}_0
- Since $\underline{W}_0 < W_0 \implies \underline{W} < W$
- $\underline{W}_t \geq 0$ in $(0, \infty) \times (0, \infty)$.
- $\underline{W}(s, t) \nearrow \psi(s)$ as $t \rightarrow \infty$
- Lemma 4 ends the proof.

5 Emergence of mild singularities for $\mu < 8\pi - 4\pi F_0$

Theorem 3: Let $F_0 \geq 0$, and W_0 such that

$$\left\{ \begin{array}{l} \text{(H1)} \quad W_0 \in W^{1,\infty}((0, \infty)), \\ \text{(H2)} \quad W_{0s} \geq 0 \text{ in } (0, \infty) \quad \text{as well as} \\ \text{(H3)} \quad W_0(s) \rightarrow \frac{\mu}{\pi} \quad \text{as } s \rightarrow \infty. \end{array} \right.$$

for $\mu < 8\pi - 4\pi F_0$. Then, for all $\tau > 0$ there exists $C > 0$ such that

$$W_s(s, t) \leq C(1 + s^{-\frac{F_0}{2}}) \quad \text{for all } s > 0 \text{ and any } t > \tau.$$

therefore

$$u(x, t) \leq C|x|^{-F_0} \quad \text{for all } x \in \mathbb{R}^2 \text{ and } t \geq \tau.$$

Moreover, for $p \in [1, \frac{2}{F_0})$ and $\tau > 0$

$$\|u(\cdot, t)\|_{L^p(B_1(0))} \leq C \quad \text{for all } t \geq \tau.$$

Second Part

Contents

- 1.- Global bounded solutions
- 2.- Weak global solutions
- 3.- Steady states
- 4.- Asymptotic behaviour

The mathematical model

$$\frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \chi \nabla v) + \lambda u(1 - u), \quad x \in \Omega, t > 0$$

$$-\Delta v + v = u, \quad x \in \Omega,$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad x \in \partial\Omega,$$

$$u(x, 0) = u_0(x) \geq 0, \quad x \in \Omega,$$

Global bounded solutions

– Assumptions:

$$- 0 \leq u_0 \leq c < \infty;$$

$$- \lambda > 0 \text{ for } n = 1, 2;$$

$$- \lambda > \frac{n-2}{n} \chi \text{ for } n \geq 3.$$

Global Classical

Solution exists

\implies

$\lambda = 0$ blows up for $n \geq 3$ and for $n = 2$ if $\int_{\Omega} u_0 > c(\Omega)$ (Herrero-Velázquez 96)

The result is valid if we replace $u(1-u)$ by $h(u)$ satisfying $h(u) \leq s_0 - s_1 u^2$.

2.-Global weak solutions for arbitrary $\lambda > 0$

(u, v) is a *weak solution* to the problem in $(0, T)$ if

$$u \in L^1((0, T); W^{1,1}(\Omega)), \quad v \in L^1((0, T); W^{1,1}(\Omega));$$

such that

$$u \nabla v \in L^1((0, T); L^1(\Omega)), \quad \lambda u(1 - u) \in L^1((0, T); L^1(\Omega))$$

and

$$\begin{aligned}
 - \int_0^T \int_{\Omega} u \varphi_t + \int_0^T \int_{\Omega} \nabla u \cdot \nabla \varphi - \chi \int_0^T \int_{\Omega} u \nabla v \cdot \nabla \varphi &= \int_{\Omega} u_0 \varphi(0) + \int_0^T \int_{\Omega} \lambda u(1 - u) \varphi; \\
 \int_0^T \int_{\Omega} \nabla v \cdot \nabla \psi + \int_0^T \int_{\Omega} v \psi &= \int_0^T \int_{\Omega} u \psi
 \end{aligned} \tag{1}$$

for $\varphi, \psi \in C_0^\infty(\bar{\Omega} \times [0, T])$.

If $u_0 \in L^\gamma(\Omega)$ for $\gamma \in (1, \frac{\chi}{\chi-\lambda})_+$ then there exists a global weak solution satisfying

$$u \in L^\infty((0, \infty); L^\gamma(\Omega)) \cap L_{loc}^{\gamma+1}([0, \infty); L^{\gamma+1}(\Omega)) \cap L_{loc}^p([0, \infty); W^{1,p}(\Omega)),$$

$$\nabla u^{\frac{\chi}{2}} \in L_{loc}^2([0, \infty); L^2(\Omega))$$

$$v \in L^\infty((0, \infty); W^{2,\gamma}(\Omega)) \cap L_{loc}^{\gamma+1}([0, \infty); W^{2,\gamma+1}(\Omega))$$

for

$$p \in (1, \frac{2}{3}(1 + \min\{2, \frac{\chi}{\chi-\lambda}\})_+).$$

3.- Steady states

$$\left\{ \begin{array}{l} 0 = \Delta u - \chi \nabla \cdot (u \nabla v) + \lambda u(1 - u) \quad \text{in } \Omega, \\ 0 = \Delta v - v + u \quad \quad \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad \quad \quad \text{in } \partial\Omega \end{array} \right.$$

Definition. (u, v) is a steady state of the problem if $u, v \geq 0$;

$$u, v \in W^{1,1}(\Omega), \quad u \nabla v \in L^1(\Omega), \quad u(1 - u) \in L^1(\Omega),$$

and satisfy the identities

$$\int_{\Omega} \nabla u \cdot \nabla \varphi - \chi \int_{\Omega} u \nabla v \cdot \nabla \varphi = \int_{\Omega} \lambda u(1 - u) \varphi \quad \text{and} \\ \int_{\Omega} \nabla v \cdot \nabla \psi + \int_{\Omega} v \psi = \int_{\Omega} u \psi$$

for all $\varphi \in C^\infty(\bar{\Omega})$ and $\psi \in C^\infty(\bar{\Omega})$.

Regularity of steady states

Lemma

i) Under assumption

$$\lambda > 0 \text{ and } n \leq 4 \quad \text{or} \quad \lambda > \frac{n-4}{n-2}\chi \text{ and } n > 4;$$

the solution is bounded and $u, v \in C^{1,\alpha}(\overline{\Omega})$ for $\alpha \in (0, 1)$.

ii) For any nonconstant solution (u, v) , we have

$$\exp\{\chi(\min_{x \in \Omega} v(x) - \max_{x \in \Omega} v(x))\} \leq u(x) \leq \exp\{\chi(\max_{x \in \Omega} v(x) - \min_{x \in \Omega} v(x))\} \quad \text{in } \Omega.$$

In particular, if v is bounded $u, v \in C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in (0, 1)$.

4.- Asymptotic behavior

Theorem. Under assumptions

$$\lambda > 2\chi, \quad u_0 \in C_0(\Omega), \quad 0 < u < c$$

the unique solution (u, v) satisfies

$$\|u - 1\|_{L^\infty(\Omega)} + \|v - 1\|_{L^\infty(\Omega)} \longrightarrow 0 \text{ as } t \longrightarrow \infty.$$

Idea of the proof:

$$\begin{aligned} u_t - \Delta u &= -\chi \nabla u \cdot \nabla v + \chi u(u - v) + \lambda u(1 - u) \\ &\quad - \Delta v + v = u \end{aligned}$$

We consider the following system of equations

$$\bar{u}_t = \chi \bar{u}(\bar{u} - \underline{u} + \frac{\lambda}{\chi}(1 - \bar{u})),$$

$$\underline{u}_t = \chi \underline{u}(\underline{u} - \bar{u} + \frac{\lambda}{\chi}(1 - \underline{u})).$$

Step 1.- \underline{u} and \bar{u} exist for $t \in (0, \infty)$;

Step 2.- $\underline{u} \leq \bar{u}$;

Step 3.- $0 < \underline{u} \leq 1 \leq \bar{u}$;

Step 4.- $\lim_{t \rightarrow \infty} |\bar{u} - \underline{u}| = 0$.

$$\frac{\bar{u}_t}{\bar{u}} = \chi(\bar{u} - \underline{u} + \frac{\lambda}{\chi}(1 - \bar{u})),$$

$$\frac{\underline{u}_t}{\underline{u}} = \chi(\underline{u} - \bar{u} + \frac{\lambda}{\chi}(1 - \underline{u})).$$

Subtracting

$$\frac{d}{dt} \left(Ln \frac{\bar{u}}{\underline{u}} \right) = \chi \left(2(\bar{u} - \underline{u}) + \frac{\lambda}{\chi} (\underline{u} - \bar{u}) \right) = (2\chi - \lambda)(\bar{u} - \underline{u}).$$

Integrating

$$Ln \frac{\bar{u}}{\underline{u}} \leq e^{-\alpha\alpha_0 t} Ln \frac{\bar{u}_0}{\underline{u}_0}$$

and taking limits, we obtain

$$\lim_{t \rightarrow \infty} Ln \frac{\bar{u}}{\underline{u}} = 0.$$

Step 5.- (3) + (4) implies

$$|\bar{u} - 1| + |\underline{u} - 1| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

By comparison, if $\underline{u}_0 < u_0 < \bar{u}_0$, \underline{u} is a subsolution and \bar{u} is a supersolution.

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