Steady-state patterns of the shadow system with nontrivial basic production terms

Kanako Suzuki[1] Izumi Takagi[2]

[1]Institute for International Advanced Interdisciplinary Research, Tohoku University

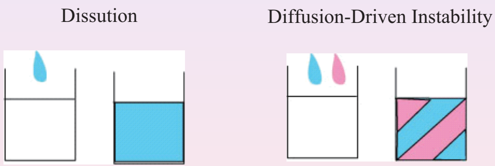
[2]Mathematical Institute, Tohoku University

September 13, 2010

Diffusion-driven instability

Turing's idea(1952): "Diffusion-Driven Instability"

- When two chemicals with different diffusion rates interact and diffuse, the spatially homogeneous state may become unstable, as a result spatially nontrivial structure can be formed autonomously.



Questions

- What kind of reaction should be considered?
- What kind of spatially pattern is caused by the reaction?

1D Gierer-Meinhardt system

Activator-inhibitor system with different sources

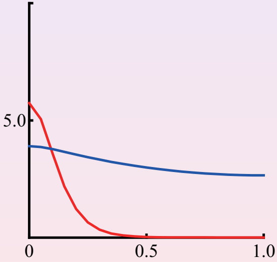
Gierer and Meinhardt in 1972 developed Turing 's idea and proposed a reaction-diffusion system in order to simulate the transplantation experiment on *hydra*.

$$\begin{cases} \dfrac{\partial a}{\partial t} = \varepsilon^2 \dfrac{\partial^2 a}{\partial x^2} - \mu a + c\rho \dfrac{a^2}{h} + \rho_0 \rho & \text{for } 0 < x < l,\ t > 0, \\ \dfrac{\partial h}{\partial t} = D \dfrac{\partial^2 h}{\partial x^2} - \nu h + c' \rho' a^2 & \text{for } 0 < x < l,\ t > 0. \end{cases}$$

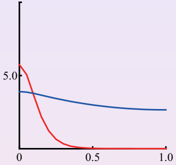
- ε, D, c, c', ρ_0 are positive constants;
- $\mu(x)$, $\nu(x)$, $\rho(x)$, $\rho'(x)$ are positive functions;
- $a = a(x,t)$ and $h = h(x,t)$: the concentrations at point x and time t of chemicals called an activator and an inhibitor, respectively;

1D Gierer-Meinhardt system

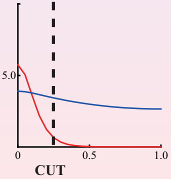
It is suspected that the head-activating substance is present in hydra as a gradient from the head to the foot, which is high in the head, and low toward the food.



1D Gierer-Meinhardt system



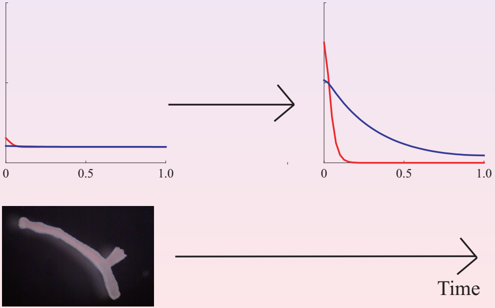
$\Downarrow$



1D Gierer-Meinhardt system

Starting from almost homogeneous state, we would like to obtain a strongly localized pattern of the activator concentration.

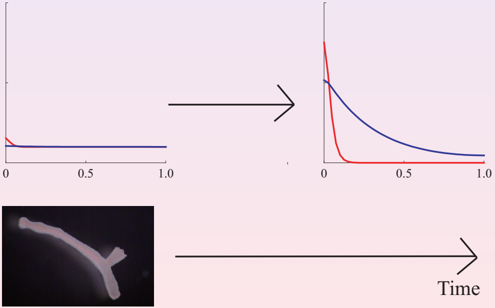
⟹ Head regenerates in the region where the activator concentration is high.



1D Gierer-Meinhardt system

Starting from almost homogeneous state, we would like to obtain a strongly localized pattern of the activator concentration.

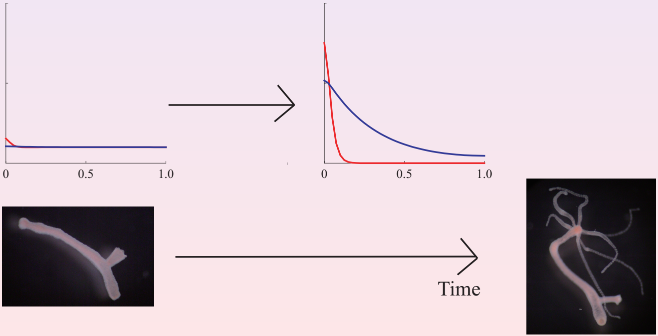
$\Longrightarrow$ Head regenerates in the region where the activator concentration is high.



1D Gierer-Meinhardt system

Starting from almost homogeneous state, we would like to obtain a strongly localized pattern of the activator concentration.

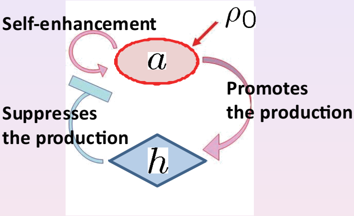
$\Longrightarrow$ Head regenerates in the region where the activator concentration is high.



1D Gierer-Meinhardt system

Mechanism

$$
\begin{cases}
\dfrac{\partial a}{\partial t} = \varepsilon^2 \dfrac{\partial^2 a}{\partial x^2} - \mu a + c\rho \dfrac{a^2}{h} + \rho_0 \rho, \\
\dfrac{\partial h}{\partial t} = D \dfrac{\partial^2 h}{\partial x^2} - \nu h + c' \rho' a^2.
\end{cases}
$$



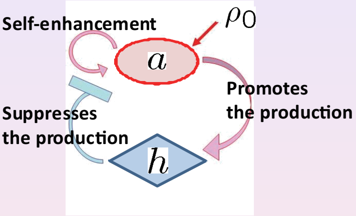
- The inhibitor diffuses much faster than the activator;
- The reaction terms have good relation;

We can obtain a pattern.

1D Gierer-Meinhardt system

Mechanism

$$\begin{cases} \dfrac{\partial a}{\partial t} = \varepsilon^2 \dfrac{\partial^2 a}{\partial x^2} - \mu a + c\rho \dfrac{a^2}{h} + \rho_0 \rho, \\ \dfrac{\partial h}{\partial t} = D \dfrac{\partial^2 h}{\partial x^2} - \nu h + c' \rho' a^2. \end{cases}$$

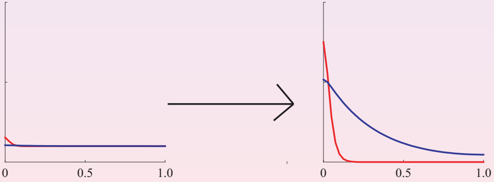


- The inhibitor diffuses much faster than the activator;
- The reaction terms have good relation;

We can obtain a pattern.

1D Gierer-Meinhardt system

$$\begin{cases} \dfrac{\partial a}{\partial t} = \varepsilon^2 \dfrac{\partial^2 a}{\partial x^2} - \mu a + c\rho \dfrac{a^2}{h} + \rho_0 \rho & \text{for } 0 < x < l,\ t > 0, \\ \dfrac{\partial h}{\partial t} = D \dfrac{\partial^2 h}{\partial x^2} - \nu h + c' \rho' a^2 & \text{for } 0 < x < l,\ t > 0. \end{cases}$$



It must depend on a choice of exponents and parameters to obtain a typical pattern.

Generalized Gierer-Meinhardt system

$$\begin{cases} \dfrac{\partial A}{\partial t} = \varepsilon^2 \dfrac{\partial^2 A}{\partial x^2} - A + \dfrac{A^p}{H^q} + \sigma_a(x) & \text{for } 0 < x < l,\ t > 0, \\ \tau \dfrac{\partial H}{\partial t} = D \dfrac{\partial^2 H}{\partial x^2} - H + \dfrac{A^r}{H^s} + \sigma_h(x) & \text{for } 0 < x < l,\ t > 0, \\ \dfrac{\partial A}{\partial x}(0,t) = \dfrac{\partial A}{\partial x}(l,t) = \dfrac{\partial H}{\partial x}(0,t) = \dfrac{\partial H}{\partial x}(l,t) = 0 & \text{for } t > 0. \end{cases} \tag{GM}$$

- ε, τ, D are positive constants;
- $\sigma_a(x) \geq 0$, $\sigma_h(x) \geq 0$: basic production terms;
- $p > 1$, $q > 0$, $r > 0$, $s \geq 0$ satisfy $0 < \dfrac{p-1}{r} < \dfrac{q}{s+1}$.

Purpose of this talk

We study the role of basic production terms $\sigma_a(x)$ and $\sigma_h(x)$.

- the amount of the activator and the inhibitor produced by cells in a unit time;
- they are independent of the interaction;

1. The effect of basic production terms on the dynamics of (GM). Collapse of patterns
   - $\sigma_a(x) \equiv 0 \implies$ collapse of patterns occurs.

   To avoid the collapse of patterns, is it sufficient to set $\sigma_a(x) \not\equiv 0$?

   No.

   When $\sigma_h(x) \not\equiv 0$, the collapse of patterns occurs in some sense.
   - The dynamics of (GM) becomes more complicated when $\sigma_h(x) \not\equiv 0$.
   - The simplest way to avoid the collapse of patterns is to take $\sigma_a(x) \not\equiv 0$ and $\sigma_h(x) \equiv 0$.

Purpose of this talk

We study the role of basic production terms $\sigma_a(x)$ and $\sigma_h(x)$.

- the amount of the activator and the inhibitor produced by cells in a unit time;
- they are independent of the interaction;

1. The effect of basic production terms on the dynamics of (GM).

   Collapse of patterns

   - $\sigma_a(x) \equiv 0 \implies$ collapse of patterns occurs.

   To avoid the collapse of patterns, is it sufficient to set $\sigma_a(x) \not\equiv 0$?

   No.

   When $\sigma_h(x) \not\equiv 0$, the collapse of patterns occurs in some sense.

   - The dynamics of (GM) becomes more complicated when $\sigma_h(x) \not\equiv 0$.
   - The simplest way to avoid the collapse of patterns is to take $\sigma_a(x) \not\equiv 0$ and $\sigma_h(x) \equiv 0$.

Purpose of this talk

We study the role of basic production terms $\sigma_a(x)$ and $\sigma_h(x)$.

- the amount of the activator and the inhibitor produced by cells in a unit time;
- they are independent of the interaction;

1. The effect of basic production terms on the dynamics of (GM).
   Collapse of patterns
   - $\sigma_a(x) \equiv 0 \quad \Longrightarrow \quad$ collapse of patterns occurs.

   To avoid the collapse of patterns, is it sufficient to set $\sigma_a(x) \not\equiv 0$?

   No.

   When $\sigma_h(x) \not\equiv 0$, the collapse of patterns occurs in some sense.
   - The dynamics of (GM) becomes more complicated when $\sigma_h(x) \not\equiv 0$.
   - The simplest way to avoid the collapse of patterns is to take $\sigma_a(x) \not\equiv 0$ and $\sigma_h(x) \equiv 0$.

Purpose of this talk

2. The effect of basic production terms $\sigma_a(x)$ and $\sigma_h(x)$ on the shape of stationary solutions.

We consider the stationary problem for 1D shadow system.

1D shadow system

$$\begin{cases} \dfrac{\partial A}{\partial t} = \varepsilon^2 \dfrac{\partial^2 A}{\partial x^2} - A + \dfrac{A^p}{H^q} + \sigma_a(x) & \text{for } 0 < x < l,\ t > 0, \\ \tau \dfrac{\partial H}{\partial t} = D \dfrac{\partial^2 H}{\partial x^2} - H + \dfrac{A^r}{H^s} + \sigma_h(x) & \text{for } 0 < x < l,\ t > 0, \\ \dfrac{\partial A}{\partial x}(0,t) = \dfrac{\partial A}{\partial x}(l,t) = \dfrac{\partial H}{\partial x}(0,t) = \dfrac{\partial H}{\partial x}(l,t) = 0 & \text{for } t > 0. \end{cases} \qquad \text{(GM)}$$

- ε, τ, D are positive constants;
- $\sigma_a(x) \geq 0$, $\sigma_h(x) \geq 0$: basic production terms;
- $p > 1$, $q > 0$, $r > 0$, $s \geq 0$ satisfy $0 < \dfrac{p-1}{r} < \dfrac{q}{s+1}$.

1D shadow system

Note $H(x,t) \to \xi(t)$ as $D \to +\infty$. The equation satisfied by the limit $(A(x,t), \xi(t))$ is called the *shadow system*:

$$\begin{cases} \dfrac{\partial A}{\partial t} = \varepsilon^2 \dfrac{\partial^2 A}{\partial x^2} - A + \dfrac{A^p}{\xi^q} + \sigma_a(x) & \text{for } 0 < x < l,\ t > 0, \\ \tau \dfrac{d\xi}{dt} = -\xi + \dfrac{1}{l}\displaystyle\int_0^l \dfrac{A^r}{\xi^s}\,dx + \dfrac{1}{l}\displaystyle\int_0^l \sigma_h(x)\,dx & \text{for } t > 0, \\ \dfrac{\partial A}{\partial x}(0,t) = \dfrac{\partial A}{\partial x}(l,t) = 0, & \text{for } t > 0, \end{cases} \qquad \text{(SS)}$$

1D shadow system

Known results

When $\sigma_h(x) \equiv 0$ and σ_a is a nonnegative constant, there are several results:

- Existence of stationary solutions with boundary spikes
  Takagi(1986), Lin-Ni-Takagi(1988), Ni-Takagi(1986, 1991, 1993, 1995), Wei(1997), Ni-Takagi-Yanagida(preprint),...
- Stability
  Nishiura(1994), Ni-Poláčik-Yanagida(2001), Miyamoto(2005), Ni-Takagi-Yanagida(preprint),...

In 1D case, if $\sigma_a(x) \equiv \sigma_h(x) \equiv 0$, then for ε sufficiently small there are no other stable stationary solutions except for constant solution and monotone solutions.

1D shadow system

Known results

When $\sigma_h(x) \equiv 0$ and σ_a is a nonnegative constant, there are several results:

- Existence of stationary solutions with boundary spikes
  Takagi(1986), Lin-Ni-Takagi(1988), Ni-Takagi(1986, 1991, 1993, 1995), Wei(1997), Ni-Takagi-Yanagida(preprint),...
- Stability
  Nishiura(1994), Ni-Poláčik-Yanagida(2001), Miyamoto(2005), Ni-Takagi-Yanagida(preprint),...

In 1D case, if $\sigma_a(x) \equiv \sigma_h(x) \equiv 0$, then for ε sufficiently small there are no other stable stationary solutions except for constant solution and monotone solutions.

1D shadow system

Known results

We consider the following:

- the case where $\sigma_a(x)$ and $\sigma_h(x)$ are functions of x;
- the existence of stationary solutions of (SS);
- the influence of $\sigma_a(x)$ and $\sigma_h(x)$ upon the shape of stationary solutions, especially solutions with boundary spike.

1D shadow system

Existence

Let $(A(x), \xi)$ be a stationary solution of (SS). Put $A(x) = \xi^{q/(p-1)} u(x)$. The stationary problem for (SS) is equivalent to the following problem:

$$\varepsilon^2 u'' - u + u^p + \xi^{-q/(p-1)} \sigma_a(x) = 0, \tag{1}$$

$$-1 + \frac{\xi^\alpha}{l} \int_0^l u^r \, dx + \frac{1}{\xi} \overline{\sigma}_h = 0, \tag{2}$$

$$u'(0) = u'(l) = 0, \tag{3}$$

where

$$\alpha = \frac{qr}{p-1} - (s+1) > 0, \quad \overline{\sigma}_h = \frac{1}{l} \int_0^l \sigma_h(x) \, dx.$$

Idea for existence of solutions to (1)–(3): the perturbation theory

1D shadow system

Existence

Let $(A(x), \xi)$ be a stationary solution of (SS). Put $A(x) = \xi^{q/(p-1)}u(x)$. The stationary problem for (SS) is equivalent to the following problem:

$$\varepsilon^2 u'' - u + u^p + \xi^{-q/(p-1)}\sigma_a(x) = 0, \quad (1)$$

$$-1 + \frac{\xi^\alpha}{l}\int_0^l u^r \, dx + \frac{1}{\xi}\overline{\sigma}_h = 0, \quad (2)$$

$$u'(0) = u'(l) = 0, \quad (3)$$

where

$$\alpha = \frac{qr}{p-1} - (s+1) > 0, \quad \overline{\sigma}_h = \frac{1}{l}\int_0^l \sigma_h(x)\, dx.$$

Idea for existence of solutions to (1)–(3): the perturbation theory

1D shadow system

Existence

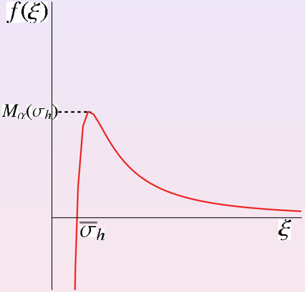
(2) $\Longrightarrow$

$$f(\xi) = \xi^{-\alpha-1}(\xi - \overline{\sigma}_h) = \frac{1}{l}\int_0^l u^r \, dx$$

$f(\xi)$ attains the maximum

$$M_\alpha(\sigma_h) = \left(\frac{\alpha}{\overline{\sigma}_h}\right)^\alpha \cdot \left(\frac{1}{\alpha+1}\right)^{\alpha+1}$$

at $\xi = \overline{\sigma}_h(\alpha+1)/\alpha$.



The equation $f(\xi) = \frac{1}{l}\int_0^l u^r \, dx$ has exactly two positive roots if

$$0 < \frac{1}{l}\int_0^l u^r \, dx < M_\alpha(\sigma_h).$$

1D shadow system

Existence

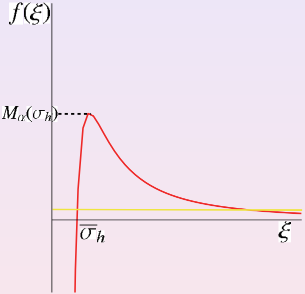
(2) $\Longrightarrow$

$$f(\xi) = \xi^{-\alpha-1}(\xi - \overline{\sigma}_h) = \frac{1}{l}\int_0^l u^r\,dx$$

$f(\xi)$ attains the maximum

$$M_\alpha(\sigma_h) = \left(\frac{\alpha}{\overline{\sigma}_h}\right)^\alpha \cdot \left(\frac{1}{\alpha+1}\right)^{\alpha+1}$$

at $\xi = \overline{\sigma}_h(\alpha+1)/\alpha$.



The equation $f(\xi) = \frac{1}{l}\int_0^l u^r\,dx$ has exactly two positive roots if

$$0 < \frac{1}{l}\int_0^l u^r\,dx < M_\alpha(\sigma_h).$$

1D shadow system

Existence of boundary spike

The case where ξ is large

$$\varepsilon^2 u'' - u + u^p + \xi^{-q/(p-1)}\sigma_a(x) = 0, \quad (1)$$

$$-1 + \frac{\xi^\alpha}{l}\int_0^l u^r\,dx + \frac{1}{\xi}\overline{\sigma}_h = 0, \quad (2)$$

$$u'(0) = u'(l) = 0. \quad (3)$$

Consider the perturbation theory for the boundary value problem

$$\begin{cases} \varepsilon^2 u_0'' - u_0 + u_0^p = 0 & \text{for } 0 < x < l, \\ u_0'(0) = u_0'(l) = 0, \end{cases} \quad (4)$$

- (4) has a unique strictly decreasing solution for ε sufficiently small;
- $u_0(\varepsilon y) \to w(y)$ uniformly on $[0, 1/\varepsilon]$ as $\varepsilon \downarrow 0$.

1D shadow system

Existence of boundary spike

The case where ξ is large

$$\varepsilon^2 u'' - u + u^p + \xi^{-q/(p-1)}\sigma_a(x) = 0, \qquad (1)$$

$$-1 + \frac{\xi^\alpha}{l}\int_0^l u^r\,dx + \frac{1}{\xi}\overline{\sigma}_h = 0, \qquad (2)$$

$$u'(0) = u'(l) = 0. \qquad (3)$$

Consider the perturbation theory for the boundary value problem

$$\begin{cases} \varepsilon^2 u_0'' - u_0 + u_0^p = 0 & \text{for } 0 < x < l, \\ u_0'(0) = u_0'(l) = 0, \end{cases} \qquad (4)$$

- (4) has a unique strictly decreasing solution for ε sufficiently small;
- $u_0(\varepsilon y) \to w(y)$ uniformly on $[0, 1/\varepsilon]$ as $\varepsilon \downarrow 0$.

1D shadow system
Existence of boundary spike

Here w is a solution of

$$\begin{cases} w'' - w + w^p = 0 \quad \text{and} \quad w > 0 \quad \text{for } 0 < y < +\infty, \\ w'(0) = 0, \quad \lim_{y \to +\infty} w(y) = 0. \end{cases}$$

- w is unique and decays exponentially as $y \uparrow +\infty$: $\sup_{0<y<+\infty} e^y w(y) < +\infty$.

Let $\phi(y)$ be a solution of

$$\begin{cases} \Phi'' - \Phi + pw^{p-1}\Phi + pw^{p-1} = 0 \quad \text{for } 0 < y < +\infty, \\ \Phi'(0) = 0, \quad \lim_{y \to +\infty} \Phi(y) = 0. \end{cases}$$

- the solution ϕ is unique.

1D shadow system
Existence of boundary spike

Here w is a solution of

$$\begin{cases} w'' - w + w^p = 0 \quad \text{and} \quad w > 0 \quad \text{for } 0 < y < +\infty, \\ w'(0) = 0, \quad \lim_{y\to+\infty} w(y) = 0. \end{cases}$$

- w is unique and decays exponentially as $y \uparrow +\infty$: $\sup_{0<y<+\infty} e^y w(y) < +\infty$.

Let $\phi(y)$ be a solution of

$$\begin{cases} \Phi'' - \Phi + pw^{p-1}\Phi + pw^{p-1} = 0 \quad \text{for } 0 < y < +\infty, \\ \Phi'(0) = 0, \quad \lim_{y\to+\infty} \Phi(y) = 0. \end{cases}$$

- the solution ϕ is unique.

1D shadow system

Theorem

Assume that (i) $\max_{0\le x\le l}\sigma_a(x) > 0$ *and (ii)* $\min_{0\le x\le l}\sigma_a(x) > 0$ *if* $0 < r < 1$. *There exists an* $\varepsilon_0 > 0$ *such that for each* $\varepsilon \in (0,\varepsilon_0)$ (SS) *has a pair of stationary solutions* $(A_{1,\varepsilon},\xi_{1,\varepsilon})$ *and* $(A_{2,\varepsilon},\xi_{2,\varepsilon})$ *satisfying*

$$A_{1,\varepsilon}(x) = \xi_{1,\varepsilon}^{q/(p-1)}\left\{w\left(\frac{x}{\varepsilon}\right) + o(1)\right\} + \sigma_a(x) + \sigma_a(0)\Phi\left(\frac{x}{\varepsilon}\right) + o(1),$$

$$\xi_{1,\varepsilon} = \left\{\varepsilon\left(\frac{1}{l}\int_0^\infty w(z)^r\,dz + o(1)\right)\right\}^{-(p-1)/[qr-(p-1)(s+1)]},$$

$$A_{2,\varepsilon}(x) = \xi_{2,\varepsilon}^{q/(p-1)}\left\{w\left(\frac{l-x}{\varepsilon}\right) + o(1)\right\} + \sigma_a(x) + \sigma_a(l)\Phi\left(\frac{l-x}{\varepsilon}\right) + o(1),$$

$$\xi_{2,\varepsilon} = \left\{\varepsilon\left(\frac{1}{l}\int_0^\infty w(z)^r\,dz + o(1)\right)\right\}^{-(p-1)/[qr-(p-1)(s+1)]},$$

as $\varepsilon \downarrow 0$. *Here, the terms* $o(1)$ *are uniform in* $x \in [0,l]$.

1D shadow system

Stability of solutions with a boundary spike

Theorem

Let $r = 2$ *and* $1 < p < 5$. *For each* $\alpha \in (0, \alpha_0)$ *and* $\varepsilon \in (0, \varepsilon_0)$ *there exist* $\tau_1 > 0$ *and* τ_2 *such that*

(i) *if* $0 < \tau < \tau_1$, *then* $(A_{1,\varepsilon}, \xi_{1,\varepsilon})$ *is asymptotically stable, and if* $0 < \tau < \tau_2$, *then* $(A_{2,\varepsilon}, \xi_{2,\varepsilon})$ *is asymptotically stable.*

(ii) $(A_{1,\varepsilon}, \xi_{1,\varepsilon})$ *is unstable if* $\tau > \tau_1$ *and* $(A_{2,\varepsilon}, \xi_{2,\varepsilon})$ *is unstable if* $\tau > \tau_2$.

For a proof, we investigate the spectrum of the linearized operator

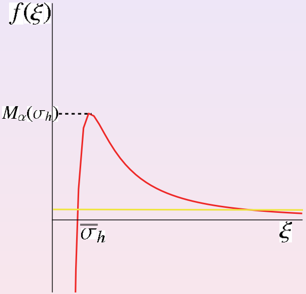
$$\mathcal{L}_{\varepsilon,\infty} = \begin{pmatrix} \varepsilon^2 \frac{d^2}{dx^2} - 1 + p\xi_\varepsilon^{-q} A_\varepsilon(x)^{p-1} & -q\xi_\varepsilon^{-(q+1)} A_\varepsilon(x)^p \\ \tau^{-1} r \xi_\varepsilon^{-s} \int_0^l A_\varepsilon(x)^{r-1} \cdot \, dx & \tau^{-1}\left(-1 - s\xi_\varepsilon^{-(s+1)} \int_0^l A_\varepsilon(x)^r \, dx\right) \end{pmatrix}.$$

1D shadow system

Existence of minimal positive solution

The case where $\xi \sim \overline{\sigma}_h$

$$\varepsilon^2 u'' - u + u^p + \xi^{-q/(p-1)}\sigma_a(x) = 0,$$
$$-1 + \frac{\xi^\alpha}{l}\int_0^l u^r\,dx + \frac{1}{\xi}\overline{\sigma}_h = 0,$$
$$u'(0) = u'(l) = 0.$$



From (2) $\Longrightarrow$ $\frac{1}{l}\int_0^l u^r\,dx \sim 0$

We can find a solution in a neighborhood of $u \equiv 0$.

1D shadow system

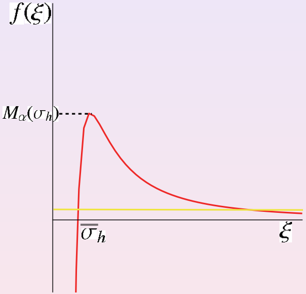
Existence of minimal positive solution

The case where $\xi \sim \overline{\sigma}_h$

$$\varepsilon^2 u'' - u + u^p + \xi^{-q/(p-1)} \sigma_a(x) = 0,$$

$$-1 + \frac{\xi^\alpha}{l} \int_0^l u^r \, dx + \frac{1}{\xi} \overline{\sigma}_h = 0,$$

$$u'(0) = u'(l) = 0.$$



From (2) $\Longrightarrow$ $\frac{1}{l} \int_0^l u^r \, dx \sim 0$

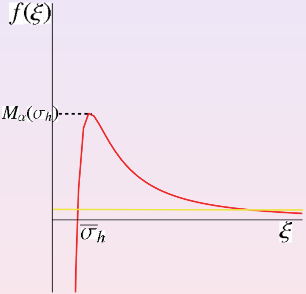
We can find a solution in a neighborhood of $u \equiv 0$.

1D shadow system

Existence of minimal positive solution

The case where $\xi \sim \overline{\sigma}_h$

$$\varepsilon^2 u'' - u + u^p + \xi^{-q/(p-1)} \sigma_a(x) = 0,$$
$$-1 + \frac{\xi^\alpha}{l} \int_0^l u^r \, dx + \frac{1}{\xi} \overline{\sigma}_h = 0,$$
$$u'(0) = u'(l) = 0.$$



From (2) $\Longrightarrow$ $\frac{1}{l} \int_0^l u^r \, dx \sim 0$

We can find a solution in a neighborhood of $u \equiv 0$.

1D shadow system

Existence of minimal positive solution

GENERAL THEORY: There exists a $\kappa_* > 0$ such that for $0 < \kappa < \kappa_*$ a boundary value problem

$$\begin{cases} \varepsilon^2 u'' - u + u^p + \kappa\sigma_a(x) = 0, \\ u'(0) = u'(l) = 0. \end{cases}$$

has a minimal positive solution $u(x;\kappa)$.

If κ is sufficiently small, we see the minimal solution satisfies

$$\sup_{0<x<l} |u(x;\kappa) - \kappa\Sigma_{a,\varepsilon}(x)| \leq C\kappa^p,$$

where $C > 0$ is a constant and $\Sigma_{a,\varepsilon}$ is the unique solution of

$$\begin{cases} \varepsilon^2 \Sigma_{a,\varepsilon}'' - \Sigma_{a,\varepsilon} + \kappa\sigma_a(x) = 0, \\ \Sigma_{a,\varepsilon}'(0) = \Sigma_{a,\varepsilon}'(l) = 0. \end{cases}$$

1D shadow system

Existence of minimal positive solution

GENERAL THEORY: There exists a $\kappa_* > 0$ such that for $0 < \kappa < \kappa_*$ a boundary value problem

$$\begin{cases} \varepsilon^2 u'' - u + u^p + \kappa\sigma_a(x) = 0, \\ u'(0) = u'(l) = 0. \end{cases}$$

has a minimal positive solution $u(x;\kappa)$.

If κ is sufficiently small, we see the minimal solution satisfies

$$\sup_{0<x<l} |u(x;\kappa) - \kappa\Sigma_{a,\varepsilon}(x)| \le C\kappa^p,$$

where $C > 0$ is a constant and $\Sigma_{a,\varepsilon}$ is the unique solution of

$$\begin{cases} \varepsilon^2 \Sigma''_{a,\varepsilon} - \Sigma_{a,\varepsilon} + \kappa\sigma_a(x) = 0, \\ \Sigma'_{a,\varepsilon}(0) = \Sigma'_{a,\varepsilon}(l) = 0. \end{cases}$$

1-D Shadow system

Existence of minimal positive solution

Finding a solution of (1)–(3) is equivalent to finding a $\kappa > 0$ satisfying

$$\kappa = \xi^{-q/(p-1)}, \tag{4}$$

$$f(\xi) = \frac{1}{l}\int_0^l u(x;\kappa)^r\,dx, \tag{5}$$

where $f(\xi) = \xi^{-\alpha-1}(\xi - \overline{\sigma}_h)$.

$$(5) \implies \frac{1}{l}\int_0^l u(x;\kappa)^r\,dx \le \frac{1}{l}\kappa^r \int_0^l \left\{\Sigma_{a,\varepsilon}(x) + O(\kappa^{p-1})\right\}^r\,dx \;\le C_0\kappa^r.$$

We see

$$f(\kappa^{-(p-1)/q}) - \frac{1}{l}\int_0^l u(x;\kappa)^r\,dx > 0 \quad \text{if } \kappa > 0 \text{ is sufficiently small.}$$

1-D Shadow system

Existence of minimal positive solution

Finding a solution of (1)–(3) is equivalent to finding a $\kappa > 0$ satisfying

$$\kappa = \xi^{-q/(p-1)}, \tag{4}$$

$$f(\xi) = \frac{1}{l}\int_0^l u(x;\kappa)^r\,dx, \tag{5}$$

where $f(\xi) = \xi^{-\alpha-1}(\xi - \overline{\sigma}_h)$.

$$(5) \implies \frac{1}{l}\int_0^l u(x;\kappa)^r\,dx \le \frac{1}{l}\kappa^r \int_0^l \left\{\Sigma_{a,\varepsilon}(x) + O(\kappa^{p-1})\right\}^r\,dx \ \le C_0\kappa^r.$$

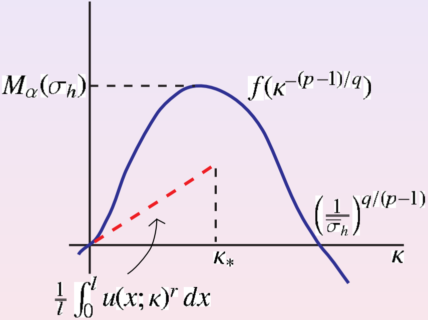
We see

$$f(\kappa^{-(p-1)/q}) - \frac{1}{l}\int_0^l u(x;\kappa)^r\,dx > 0 \quad \text{if } \kappa > 0 \text{ is sufficiently small.}$$

1-D Shadow system

Existence of minimal positive solution

$$f(\kappa^{-(p-1)/q}) - \frac{1}{l}\int_0^l u(x;\kappa)^r\,dx > 0$$



If $\overline{\sigma}_h$ is large, then there exists a root κ_1 of the equation

$$f(\kappa^{-(p-1)/q}) = \frac{1}{l}\int_0^l u(x;\kappa)^r\,dx.$$

1-D Shadow system

The case $\sigma_a(x) \geq 0$ and $\sigma_h(x) \geq 0$:

There exist stationary solutions with boundary spike.

- The contribution of $\sigma_a(x)$ is relatively small.

The minimal stationary solution can exist.

- When $\sigma_h(x) \equiv 0$, it cannot exist.

Under certain condition, both stationary solutions can be stable.

1-D Shadow system

The case $\sigma_a(x) \geq 0$ and $\sigma_h(x) \geq 0$:

There exist stationary solutions with boundary spike.

- The contribution of $\sigma_a(x)$ is relatively small.

The minimal stationary solution can exist.

- When $\sigma_h(x) \equiv 0$, it cannot exist.

Under certain condition, both stationary solutions can be stable.

1-D Shadow system

The case $\sigma_a(x) \geq 0$ and $\sigma_h(x) \geq 0$:

There exist stationary solutions with boundary spike.

- The contribution of $\sigma_a(x)$ is relatively small.

The minimal stationary solution can exist.

- When $\sigma_h(x) \equiv 0$, it cannot exist.

Under certain condition, both stationary solutions can be stable.

1D shadow system
Least-energy solutions

Assume $\sigma_a(x) \equiv \sigma_h(x) \equiv 0$:

$$\varepsilon^2 u'' - u + \rho_a(x) u^p = 0, \tag{6}$$

$$-1 + \frac{\xi^\alpha}{l} \int_0^l \rho_h(x) u^r \, dx = 0, \tag{7}$$

$$u'(0) = u'(l) = 0. \tag{8}$$

Here the interaction coefficients $\rho_a(x)$ and $\rho_h(x)$ are continuous and positive for $0 \le x \le l$.

Least-energy solution: $u(x)$ is a critical point corresponding to the minimal positive critical value of the functional

$$J_\varepsilon(v) = \int_0^l \left(\frac{1}{2} \left\{ \varepsilon^2 (v')^2 + v^2 \right\} - \frac{\rho_a(x)}{p+1} v_+^{p+1} \right) dx,$$

where $v_+(x) = \max\{v(x), 0\}$.

1D shadow system

Least-energy solutions

Assume $\sigma_a(x) \equiv \sigma_h(x) \equiv 0$:

$$\varepsilon^2 u'' - u + \rho_a(x) u^p = 0, \tag{6}$$

$$-1 + \frac{\xi^\alpha}{l} \int_0^l \rho_h(x) u^r \, dx = 0, \tag{7}$$

$$u'(0) = u'(l) = 0. \tag{8}$$

Here the interaction coefficients $\rho_a(x)$ and $\rho_h(x)$ are continuous and positive for $0 \le x \le l$.

Least-energy solution: $u(x)$ is a critical point corresponding to the minimal positive critical value of the functional

$$J_\varepsilon(v) = \int_0^l \left(\frac{1}{2} \left\{ \varepsilon^2 (v')^2 + v^2 \right\} - \frac{\rho_a(x)}{p+1} v_+^{p+1} \right) dx,$$

where $v_+(x) = \max\{v(x), 0\}$.

1D shadow system

Least-energy solutions

Known Results: Assume $\rho_a(x) \equiv 1$. the least-energy solution has a boundary spike as $\varepsilon \downarrow 0$.

Theorem

The least-energy solution $(A_\varepsilon(x), \xi_\varepsilon)$ concentrates at a single point in $0 \leq x \leq l$:

(i) *if* $\max_{0 \leq x \leq l} \rho_a(x) > 2^{(p-1)/2} \max\{\rho_a(0), \rho_a(l)\}$, *then* $A_\varepsilon(x)$ *concentrates around the interior maximum point of* $\rho_a(x)$*;*

(ii) *if* $\max_{0 \leq x \leq l} \rho_a(x) < 2^{(p-1)/2} \max\{\rho_a(0), \rho_a(l)\}$, *then* $A_\varepsilon(x)$ *concentrates around the boundary point that attains* $\max\{\rho_a(0), \rho_a(l)\}$.

See X. Ren(1993).

1D shadow system

Least-energy solutions

Known Results: Assume $\rho_a(x) \equiv 1$. the least-energy solution has a boundary spike as $\varepsilon \downarrow 0$.

Theorem

The least-energy solution $(A_\varepsilon(x), \xi_\varepsilon)$ concentrates at a single point in $0 \le x \le l$:

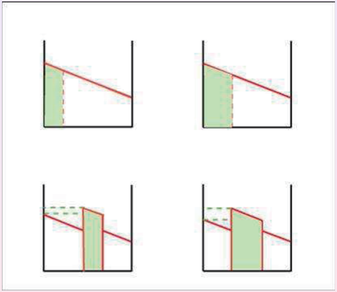
(i) *if $\max_{0\le x\le l} \rho_a(x) > 2^{(p-1)/2} \max\{\rho_a(0), \rho_a(l)\}$, then $A_\varepsilon(x)$ concentrates around the interior maximum point of $\rho_a(x)$;*

(ii) *if $\max_{0\le x\le l} \rho_a(x) < 2^{(p-1)/2} \max\{\rho_a(0), \rho_a(l)\}$, then $A_\varepsilon(x)$ concentrates around the boundary point that attains $\max\{\rho_a(0), \rho_a(l)\}$.*

See X. Ren(1993).

1D shadow system

Least-energy solutions



The interior maximum need to be significantly larger than the boundary value of the coefficient $\rho_a(x)$ in order to generate a head at an interior point.