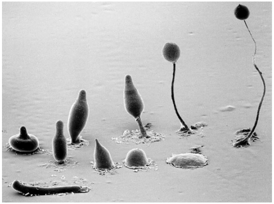
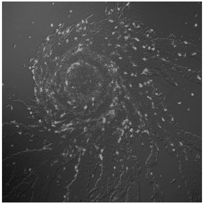
# A simple 2D-model of cell sorting induced by propagation of chemical signals along spiral waves

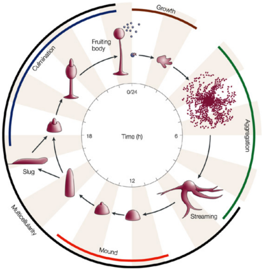
K. Kang - Department of Mathematics and Institute of Basic Science, "Sungkyunkwan University" (Suwon, South Korea)
I. Primi - Institute for Applied Mathematics + Bioquant Center, "Ruprecht Karls Universität" (Heidelberg, Germany)
J.J.L. Velázquez - ICMAT (CSIC-UAM-UC3M-UCM), "Universidad Complutense" (Madrid, Spain)

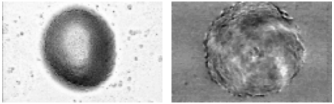




From: http://dictybase.org/ ©M.J. Grimson & R.L. Blanton (left), R. M. Bickford & D. A. Knecht (right)

# Sorting in the mound stage of *Dictyostelium discoideum*



From: R.L. Chisholm, R.A. Firtel *Insights into morphogenesis from a simple developmental system*, Nat. Rev. Mol. Cell Biol. 5, 531-541 (2004)

From: F. Siegert, C.J. Weijer *Spiral and concentric waves organize multicellular Dictyostelium mounds* Curr. Biol. 5, 937-943 (1995)

# A simple model for the sorting at the mound stage

- Two-dimensional model
- Propagation of cAMP along a train of spiral waves with constant angular speed $-\omega_0$
- Two type of cells, both with positive chemotactic sensitivity but different reaction strengths
- Continuos cell distribution in the whole plane with densities $m$ and $n$, $m + n = N_0$
- $\Omega_0$: constant angular speed of cell rotation
- No influence of cell motion / differentiation on propagation of the chemical signal
- Very thin action range of the chemical

Chemical waves with moderately increasing wavelength $\Longrightarrow$ Trapped cell states with most of the more sensitive cells in a bounded region

# A simple model for the sorting at the mound stage

- Two-dimensional model
- Propagation of cAMP along a train of spiral waves with constant angular speed $-\omega_0$
- Two type of cells, both with positive chemotactic sensitivity but different reaction strengths
- Continuos cell distribution in the whole plane with densities $m$ and $n$, $m + n = N_0$
- $\Omega_0$: constant angular speed of cell rotation
- No influence of cell motion / differentiation on propagation of the chemical signal
- Very thin action range of the chemical

Chemical waves with moderately increasing wavelength $\Longrightarrow$ Trapped cell states with most of the more sensitive cells in a bounded region

# A simple model for the sorting at the mound stage

- Two-dimensional model
- Propagation of cAMP along a train of spiral waves with constant angular speed $-\omega_0$
- Two type of cells, both with positive chemotactic sensitivity but different reaction strengths
- Continuos cell distribution in the whole plane with densities $m$ and $n$, $m + n = N_0$
- $\Omega_0$: constant angular speed of cell rotation
- No influence of cell motion / differentiation on propagation of the chemical signal
- Very thin action range of the chemical

Chemical waves with moderately increasing wavelength $\Longrightarrow$ Trapped cell states with most of the more sensitive cells in a bounded region

# A simple model for the sorting at the mound stage

- Two-dimensional model
- Propagation of cAMP along a train of spiral waves with constant angular speed $-\omega_0$
- Two type of cells, both with positive chemotactic sensitivity but different reaction strengths
- Continuos cell distribution in the whole plane with densities $m$ and $n$, $m + n = N_0$
- $\Omega_0$: constant angular speed of cell rotation
- No influence of cell motion / differentiation on propagation of the chemical signal
- Very thin action range of the chemical

Chemical waves with moderately increasing wavelength $\Longrightarrow$ Trapped cell states with most of the more sensitive cells in a bounded region

# A simple model for the sorting at the mound stage

- Two-dimensional model
- Propagation of cAMP along a train of spiral waves with constant angular speed $-\omega_0$
- Two type of cells, both with positive chemotactic sensitivity but different reaction strengths
- Continuos cell distribution in the whole plane with densities $m$ and $n$, $m + n = N_0$
- $\Omega_0$: constant angular speed of cell rotation
- No influence of cell motion / differentiation on propagation of the chemical signal
- Very thin action range of the chemical

Chemical waves with moderately increasing wavelength $\Longrightarrow$ Trapped cell states with most of the more sensitive cells in a bounded region

# A simple model for the sorting at the mound stage

- Two-dimensional model
- Propagation of cAMP along a train of spiral waves with constant angular speed $-\omega_0$
- Two type of cells, both with positive chemotactic sensitivity but different reaction strengths
- Continuos cell distribution in the whole plane with densities $m$ and $n$, $m + n = N_0$
- $\Omega_0$: constant angular speed of cell rotation
- No influence of cell motion / differentiation on propagation of the chemical signal
- Very thin action range of the chemical

Chemical waves with moderately increasing wavelength $\Longrightarrow$ Trapped cell states with most of the more sensitive cells in a bounded region

# A simple model for the sorting at the mound stage

- Two-dimensional model
- Propagation of cAMP along a train of spiral waves with constant angular speed $-\omega_0$
- Two type of cells, both with positive chemotactic sensitivity but different reaction strengths
- Continuos cell distribution in the whole plane with densities $m$ and $n$, $m + n = N_0$
- $\Omega_0$: constant angular speed of cell rotation
- No influence of cell motion / differentiation on propagation of the chemical signal
- Very thin action range of the chemical

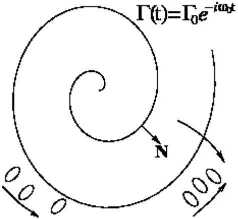
Chemical waves with moderately increasing wavelength $\Longrightarrow$ Trapped cell states with most of the more sensitive cells in a bounded region

# A simple model for the sorting at the mound stage

- Two-dimensional model
- Propagation of cAMP along a train of spiral waves with constant angular speed $-\omega_0$
- Two type of cells, both with positive chemotactic sensitivity but different reaction strengths
- Continuos cell distribution in the whole plane with densities $m$ and $n$, $m + n = N_0$
- $\Omega_0$: constant angular speed of cell rotation
- No influence of cell motion / differentiation on propagation of the chemical signal
- Very thin action range of the chemical

Chemical waves with moderately increasing wavelength $\Longrightarrow$ Trapped cell states with most of the more sensitive cells in a bounded region

# Model derivation from the assumptions




$$\frac{\partial m}{\partial t} + \mathrm{div}(j_m) = 0, \quad \frac{\partial n}{\partial t} + \mathrm{div}(j_n) = 0 \quad \text{with} \quad \mathrm{div}(j_m + j_n) = 0$$

Fluxes: $j_m = -D\nabla m + m\,\mathbf{b} + J_{chem}$ $j_n = -D\nabla n + n\,\mathbf{b} - J_{chem}$

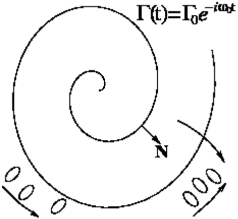
$$\mathbf{b}(x_1, x_2) = (-\Omega_0 x_2, \Omega_0 x_1)$$

Choice of a coordinate system rotating with the spiral wave

$\Rightarrow$ wave at rest & cell rotation at speed $\Omega := \Omega_0 + \omega_0$

$\rightarrow \mathbf{b}(x_1, x_2) := (-\Omega x_2, \Omega x_1)$

# Model derivation from the assumptions




$\frac{\partial m}{\partial t} + \mathrm{div}(j_m) = 0, \quad \frac{\partial n}{\partial t} + \mathrm{div}(j_n) = 0$ with $\mathrm{div}(j_m + j_n) = 0$

Fluxes: $j_m = -D\nabla m + m\,\mathbf{b} + J_{chem} \quad j_n = -D\nabla n + n\,\mathbf{b} - J_{chem}$

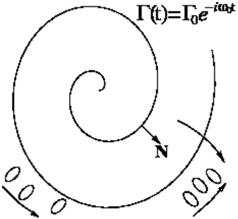
$\mathbf{b}(x_1, x_2) = (-\Omega_0 x_2, \Omega_0 x_1)$

Choice of a coordinate system rotating with the spiral wave

$\Rightarrow$ wave at rest & cell rotation at speed $\Omega := \Omega_0 + \omega_0$

$\rightarrow \mathbf{b}(x_1, x_2) := (-\Omega x_2, \Omega x_1)$

# Model derivation from the assumptions




$$\frac{\partial m}{\partial t} + \mathrm{div}(j_m) = 0, \quad \frac{\partial n}{\partial t} + \mathrm{div}(j_n) = 0 \qquad \text{with} \quad \mathrm{div}(j_m + j_n) = 0$$

Fluxes: $j_m = -D\nabla m + m\,\mathbf{b} + J_{chem} \quad j_n = -D\nabla n + n\,\mathbf{b} - J_{chem}$

$$\mathbf{b}(x_1, x_2) = (-\Omega_0 x_2, \Omega_0 x_1)$$

Choice of a coordinate system rotating with the spiral wave

$\Rightarrow$ wave at rest & cell rotation at speed $\Omega := \Omega_0 + \omega_0$

$\rightarrow \mathbf{b}(x_1, x_2) := (-\Omega x_2, \Omega x_1)$

# Form of $J_{chem}$

$$J_{chem} = \chi(m^+, n^+)\, \delta_\Gamma \mathbf{N} = \chi(m^+, N_0 - m^+)\, \delta_\Gamma \mathbf{N} \quad \text{with}$$

- $\mathbf{N} = \mathbf{N}(x)$: outer normal unit vector to the spiral at point $x \in \Gamma$;
- $\delta_\Gamma$: measure concentrated on the spiral $\Gamma$;
- $\chi(0, n) = \chi(m, 0) = 0$;
- $(\cdot)^+$, $(\cdot)^-$: limit values from outer, inner side respectively.

Ansatz: $\chi(m, n) = Amn$, $A : \Gamma \longrightarrow \mathbb{R}$,

$$\Longrightarrow \frac{\partial m}{\partial t} - D\Delta m + \mathrm{div}(m\,\mathbf{b}) + \mathrm{div}(Am^+(N - m^+)\delta_\Gamma \mathbf{N}) = 0$$

with (simplification) $A = Const. < 0$, $|A| \ll 1$.

In our first investigation $J_{chem} = Am^+\delta_\Gamma \mathbf{N}$

# Form of $J_{chem}$

$$J_{chem} = \chi(m^+, n^+)\,\delta_\Gamma \mathbf{N} = \chi(m^+, N_0 - m^+)\,\delta_\Gamma \mathbf{N} \quad \text{with}$$

- $\mathbf{N} = \mathbf{N}(x)$: outer normal unit vector to the spiral at point $x \in \Gamma$;
- $\delta_\Gamma$: measure concentrated on the spiral $\Gamma$;
- $\chi(0, n) = \chi(m, 0) = 0$;
- $(\cdot)^+$, $(\cdot)^-$: limit values from outer, inner side respectively.

Ansatz: $\chi(m, n) = Amn$, $A : \Gamma \longrightarrow \mathbb{R}$,

$$\Longrightarrow \; \frac{\partial m}{\partial t} - D\Delta m + \mathrm{div}(m\,\mathbf{b}) + \mathrm{div}(Am^+(N - m^+)\delta_\Gamma \mathbf{N}) = 0$$

with (simplification) $A = Const. < 0$, $|A| \ll 1$.

In our first investigation $J_{chem} = Am^+\delta_\Gamma \mathbf{N}$

# Form of $J_{chem}$

$$J_{chem} = \chi(m^+, n^+)\, \delta_\Gamma \mathbf{N} = \chi(m^+, N_0 - m^+)\, \delta_\Gamma \mathbf{N} \quad \text{with}$$

- $\mathbf{N} = \mathbf{N}(x)$: outer normal unit vector to the spiral at point $x \in \Gamma$;
- $\delta_\Gamma$: measure concentrated on the spiral $\Gamma$;
- $\chi(0, n) = \chi(m, 0) = 0$;
- $(\cdot)^+$, $(\cdot)^-$: limit values from outer, inner side respectively.

Ansatz: $\chi(m, n) = Amn$, $A : \Gamma \longrightarrow \mathbb{R}$,

$$\Longrightarrow \frac{\partial m}{\partial t} - D\Delta m + \mathrm{div}(m\,\mathbf{b}) + \mathrm{div}(Am^+(N - m^+)\delta_\Gamma \mathbf{N}) = 0$$

with (simplification) $A = Const. < 0$, $|A| \ll 1$.

In our first investigation $J_{chem} = Am^+\delta_\Gamma \mathbf{N}$

# Form of $J_{chem}$

$$J_{chem} = \chi(m^+, n^+)\,\delta_\Gamma \mathbf{N} = \chi(m^+, N_0 - m^+)\,\delta_\Gamma \mathbf{N} \quad \text{with}$$

- $\mathbf{N} = \mathbf{N}(x)$: outer normal unit vector to the spiral at point $x \in \Gamma$;
- $\delta_\Gamma$: measure concentrated on the spiral $\Gamma$;
- $\chi(0, n) = \chi(m, 0) = 0$;
- $(\cdot)^+$, $(\cdot)^-$: limit values from outer, inner side respectively.

Ansatz: $\chi(m, n) = Amn$, $A : \Gamma \longrightarrow \mathbb{R}$,

$$\Longrightarrow \frac{\partial m}{\partial t} - D\Delta m + \mathrm{div}(m\,\mathbf{b}) + \mathrm{div}(Am^+(N - m^+)\delta_\Gamma \mathbf{N}) = 0$$

with (simplification) $A = Const. < 0$, $|A| \ll 1$.

In our first investigation $J_{chem} = Am^+\delta_\Gamma \mathbf{N}$

# Form of $J_{chem}$

$$J_{chem} = \chi(m^+, n^+)\,\delta_\Gamma \mathbf{N} = \chi(m^+, N_0 - m^+)\,\delta_\Gamma \mathbf{N} \quad \text{with}$$

- $\mathbf{N} = \mathbf{N}(x)$: outer normal unit vector to the spiral at point $x \in \Gamma$;
- $\delta_\Gamma$: measure concentrated on the spiral $\Gamma$;
- $\chi(0, n) = \chi(m, 0) = 0$;
- $(\cdot)^+$, $(\cdot)^-$: limit values from outer, inner side respectively.

Ansatz: $\chi(m, n) = Amn$, $A : \Gamma \longrightarrow \mathbb{R}$,

$$\Longrightarrow \frac{\partial m}{\partial t} - D\Delta m + \mathrm{div}(m\,\mathbf{b}) + \mathrm{div}(Am^+(N - m^+)\delta_\Gamma \mathbf{N}) = 0$$

with (simplification) $A = Const. < 0$, $|A| \ll 1$.

In our first investigation $J_{chem} = Am^+\delta_\Gamma \mathbf{N}$

# Form of $J_{chem}$

$$J_{chem} = \chi(m^+, n^+)\,\delta_\Gamma \mathbf{N} = \chi(m^+, N_0 - m^+)\,\delta_\Gamma \mathbf{N} \quad \text{with}$$

- $\mathbf{N} = \mathbf{N}(x)$: outer normal unit vector to the spiral at point $x \in \Gamma$;
- $\delta_\Gamma$: measure concentrated on the spiral $\Gamma$;
- $\chi(0, n) = \chi(m, 0) = 0$;
- $(\cdot)^+$, $(\cdot)^-$: limit values from outer, inner side respectively.

Ansatz: $\chi(m, n) = Amn$, $A : \Gamma \longrightarrow \mathbb{R}$,

$$\Longrightarrow \frac{\partial m}{\partial t} - D\Delta m + \mathrm{div}(m\,\mathbf{b}) + \mathrm{div}(Am^+(N - m^+)\delta_\Gamma \mathbf{N}) = 0$$

with (simplification) $A = Const. < 0$, $|A| \ll 1$.

In our first investigation $J_{chem} = Am^+\delta_\Gamma \mathbf{N}$

# Form of $J_{chem}$

$$J_{chem} = \chi(m^+, n^+)\,\delta_\Gamma \mathbf{N} = \chi(m^+, N_0 - m^+)\,\delta_\Gamma \mathbf{N} \quad \text{with}$$

- $\mathbf{N} = \mathbf{N}(x)$: outer normal unit vector to the spiral at point $x \in \Gamma$;
- $\delta_\Gamma$: measure concentrated on the spiral $\Gamma$;
- $\chi(0, n) = \chi(m, 0) = 0$;
- $(\cdot)^+$, $(\cdot)^-$: limit values from outer, inner side respectively.

Ansatz: $\chi(m, n) = Amn$, $A : \Gamma \longrightarrow \mathbb{R}$,

$$\Longrightarrow \frac{\partial m}{\partial t} - D\Delta m + \mathrm{div}(m\,\mathbf{b}) + \mathrm{div}(Am^+(N - m^+)\delta_\Gamma \mathbf{N}) = 0$$

with (simplification) $A = Const. < 0$, $|A| \ll 1$.

In our first investigation $J_{chem} = Am^+\delta_\Gamma \mathbf{N}$

# Form of $J_{chem}$

$$J_{chem} = \chi(m^+, n^+)\,\delta_\Gamma \mathbf{N} = \chi(m^+, N_0 - m^+)\,\delta_\Gamma \mathbf{N} \quad \text{with}$$

- $\mathbf{N} = \mathbf{N}(x)$: outer normal unit vector to the spiral at point $x \in \Gamma$;
- $\delta_\Gamma$: measure concentrated on the spiral $\Gamma$;
- $\chi(0, n) = \chi(m, 0) = 0$;
- $(\cdot)^+$, $(\cdot)^-$: limit values from outer, inner side respectively.

Ansatz: $\chi(m, n) = Amn$, $A : \Gamma \longrightarrow \mathbb{R}$,

$$\Longrightarrow \frac{\partial m}{\partial t} - D\Delta m + \mathrm{div}(m\,\mathbf{b}) + \mathrm{div}(Am^+(N - m^+)\delta_\Gamma \mathbf{N}) = 0$$

with (simplification) $A = Const. < 0$, $|A| \ll 1$.

In our first investigation $J_{chem} = Am^+\delta_\Gamma \mathbf{N}$

# The evolutive problem

$\Gamma: \quad \rho = \rho(\theta)$

$$\begin{cases} \frac{\partial m}{\partial t} = D\Delta m - \mathrm{div}(\mathbf{b}(x)m) & t > 0,\ x = (x_1, x_2) \in \mathbb{R}^2 \setminus \Gamma \\ m^+ - m^- = Am^+ & t > 0,\ x \in \Gamma \\ \frac{\partial m}{\partial N}^+ - \frac{\partial m}{\partial N}^- = Bm^+ & t > 0,\ x \in \Gamma \end{cases}$$

$$\mathbf{b}(x) = (-\Omega x_2, \Omega x_1), \quad \Omega: \text{ angular speed}$$

$$\frac{d}{dt}\left(\int_{\mathbb{R}^2\setminus\Gamma} m(x,t)dx\right) = 0 \quad \Rightarrow \quad B = \frac{A}{D}\mathbf{b}(x)\cdot\mathbf{N}$$

Up to time-rescale $\quad D = 1 \quad \Rightarrow \quad B = A\,\mathbf{b}(x)\cdot\mathbf{N}$

# The evolutive problem

$\Gamma: \quad \rho = \rho(\theta)$

$$\begin{cases} \frac{\partial m}{\partial t} = D\Delta m - \operatorname{div}(\mathbf{b}(x)m) & t > 0,\ x = (x_1, x_2) \in \mathbb{R}^2 \setminus \Gamma \\ m^+ - m^- = Am^+ & t > 0,\ x \in \Gamma \\ \frac{\partial m}{\partial N}^+ - \frac{\partial m}{\partial N}^- = Bm^+ & t > 0,\ x \in \Gamma \end{cases}$$

$\mathbf{b}(x) = (-\Omega x_2, \Omega x_1), \quad \Omega:$ angular speed

$$\frac{d}{dt}\left(\int_{\mathbb{R}^2\setminus\Gamma} m(x,t)dx\right) = 0 \quad \Rightarrow \quad B = \frac{A}{D}\mathbf{b}(x)\cdot\mathbf{N}$$

Up to time-rescale $\quad D = 1 \quad \Rightarrow \quad B = A\,\mathbf{b}(x)\cdot\mathbf{N}$

# The evolutive problem

$$\Gamma : \quad \rho = \rho(\theta)$$

$$\begin{cases} \frac{\partial m}{\partial t} = D\Delta m - \mathrm{div}(\mathbf{b}(x)m) & t > 0,\ x = (x_1, x_2) \in \mathbb{R}^2 \setminus \Gamma \\ m^+ - m^- = Am^+ & t > 0,\ x \in \Gamma \\ \frac{\partial m}{\partial N}^+ - \frac{\partial m}{\partial N}^- = Bm^+ & t > 0,\ x \in \Gamma \end{cases}$$

$$\mathbf{b}(x) = (-\Omega x_2, \Omega x_1), \quad \Omega : \text{ angular speed}$$

$$\frac{d}{dt}\left(\int_{\mathbb{R}^2\setminus\Gamma} m(x,t)dx\right) = 0 \quad \Rightarrow \quad B = \frac{A}{D}\mathbf{b}(x)\cdot\mathbf{N}$$

Up to time-rescale $D = 1 \quad \Rightarrow \quad B = A\,\mathbf{b}(x)\cdot\mathbf{N}$

# The evolutive problem

$$\Gamma : \quad \rho = \rho(\theta)$$

$$\begin{cases} \frac{\partial m}{\partial t} = D\Delta m - \mathrm{div}(\mathbf{b}(x)m) & t > 0,\ x = (x_1, x_2) \in \mathbb{R}^2 \setminus \Gamma \\ m^+ - m^- = Am^+ & t > 0,\ x \in \Gamma \\ \frac{\partial m}{\partial N}^+ - \frac{\partial m}{\partial N}^- = Bm^+ & t > 0,\ x \in \Gamma \end{cases}$$

$$\mathbf{b}(x) = (-\Omega x_2, \Omega x_1), \quad \Omega : \text{ angular speed}$$

$$\frac{d}{dt}\left(\int_{\mathbb{R}^2\setminus\Gamma} m(x,t)dx\right) = 0 \quad \Rightarrow \quad B = \frac{A}{D}\mathbf{b}(x)\cdot\mathbf{N}$$

Up to time-rescale $\quad D = 1 \quad \Rightarrow \quad B = A\,\mathbf{b}(x)\cdot\mathbf{N}$

# Stationary problem and its approximation

$$\begin{cases} -\Delta m + \mathrm{div}(\mathbf{b}(x)m) = 0 & x = (x_1, x_2) \in \mathbb{R}^2 \setminus \Gamma \\ m^+ - m^- = Am^+ & x \in \Gamma \\ \frac{\partial m}{\partial N}^+ - \frac{\partial m}{\partial N}^- = Bm^+ & x \in \Gamma \end{cases}$$

$$m > 0, \quad \int_{\mathbb{R}^2} m(x)\, dx < \infty$$

$$\Omega \frac{\partial p}{\partial \theta} = \frac{\partial^2 p}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial p}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 p}{\partial \theta^2} \overset{\rho \to \infty}{\approx} \Omega \frac{\partial p}{\partial \theta} = \frac{\partial^2 p}{\partial \rho^2} \quad (p(\rho, \theta) := m(\rho e^{i\theta}))$$

$$\begin{cases} \Omega p_\theta = p_{\rho\rho} & (\rho, \theta) \in S \\ p(\rho(\theta), \theta + 2\pi) - p(\rho(\theta), \theta) = Ap(\rho(\theta), \theta + 2\pi) & \theta \geq 0 \\ \text{Second boundary condition in polar coordinates} \end{cases}$$

$$S = \{(\theta, \rho) \mid \theta > 0,\ \max(\rho(\theta - 2\pi), 0) < \rho < \rho(\theta)\} \text{ (polar domain)}$$

$$p > 0, \ \int_S p(\rho, \theta)\, \rho\, d\rho\, d\theta < \infty$$

# Stationary problem and its approximation

$$\begin{cases} -\Delta m + \operatorname{div}(\mathbf{b}(x)m) = 0 & x = (x_1, x_2) \in \mathbb{R}^2 \setminus \Gamma \\ m^+ - m^- = Am^+ & x \in \Gamma \\ \frac{\partial m}{\partial N}^+ - \frac{\partial m}{\partial N}^- = Bm^+ & x \in \Gamma \end{cases}$$

$$m > 0, \quad \int_{\mathbb{R}^2} m(x)\, dx < \infty$$

$$\Omega \frac{\partial p}{\partial \theta} = \frac{\partial^2 p}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial p}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 p}{\partial \theta^2} \overset{\rho \to \infty}{\approx} \Omega \frac{\partial p}{\partial \theta} = \frac{\partial^2 p}{\partial \rho^2} \quad (p(\rho, \theta) := m(\rho e^{i\theta}))$$

$$\begin{cases} \Omega p_\theta = p_{\rho\rho} & (\rho, \theta) \in S \\ p(\rho(\theta), \theta + 2\pi) - p(\rho(\theta), \theta) = Ap(\rho(\theta), \theta + 2\pi) & \theta \geq 0 \\ \text{Second boundary condition in polar coordinates} & \end{cases}$$

$$S = \{(\theta, \rho) \mid \theta > 0, \max(\rho(\theta - 2\pi), 0) < \rho < \rho(\theta)\} \text{ (polar domain)}$$

$$p > 0, \ \int_S p(\rho, \theta)\, \rho\, d\rho\, d\theta < \infty$$

# Stationary problem and its approximation

$$\begin{cases} -\Delta m + \operatorname{div}(\mathbf{b}(x)m) = 0 & x = (x_1, x_2) \in \mathbb{R}^2 \setminus \Gamma \\ m^+ - m^- = Am^+ & x \in \Gamma \\ \frac{\partial m}{\partial N}^+ - \frac{\partial m}{\partial N}^- = Bm^+ & x \in \Gamma \end{cases}$$

$$m > 0, \quad \int_{\mathbb{R}^2} m(x)\, dx < \infty$$

$$\Omega \frac{\partial p}{\partial \theta} = \frac{\partial^2 p}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial p}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 p}{\partial \theta^2} \overset{\rho \to \infty}{\approx} \Omega \frac{\partial p}{\partial \theta} = \frac{\partial^2 p}{\partial \rho^2} \quad (p(\rho, \theta) := m(\rho e^{i\theta}))$$

$$\begin{cases} \Omega p_\theta = p_{\rho\rho} & (\rho, \theta) \in S \\ p(\rho(\theta), \theta + 2\pi) - p(\rho(\theta), \theta) = Ap(\rho(\theta), \theta + 2\pi) & \theta \geq 0 \\ \text{Second boundary condition in polar coordinates} \end{cases}$$

$$S = \{(\theta, \rho) \mid \theta > 0, \ \max(\rho(\theta - 2\pi), 0) < \rho < \rho(\theta)\} \text{ (polar domain)}$$

$$p > 0, \ \int_S p(\rho, \theta)\, \rho\, d\rho\, d\theta < \infty$$

# Distributional formulation

$$A < 0, |A| \ll 1 \;\&\; \rho(\theta) = \theta \;\Rightarrow \exists \text{ positive solution } p(\rho,\theta) \overset{\theta\to\infty}{\sim} e^{\frac{A\theta}{2\pi}}$$

Since now on $\rho(\theta) = \theta$

Distributional formulation of s.s. problem

$$-\Delta m + \mathrm{div}(\mathbf{b}(x)m) + \mathrm{div}(Am^+(x)\delta_\Gamma(x)\mathbf{N}(x)) = 0$$

Superposition principle $\Rightarrow$

$$m(x) = C + A\int_\Gamma m^+(y)\,\nabla_y G(x,y)\cdot\mathbf{N}(y)\,d\sigma(y)\,, \qquad C \in \mathbb{R}^+, x \notin \Gamma$$

with

$$-\Delta_x G + \mathrm{div}_x(\mathbf{b}(x)G) = \delta_y(x) \qquad x \in \mathbb{R}^2 \setminus \{y\}$$

# Distributional formulation

$$A < 0, |A| \ll 1 \;\; \& \;\; \rho(\theta) = \theta \;\; \Rightarrow \exists \;\; \text{positive solution } p(\rho, \theta) \overset{\theta \to \infty}{\sim} e^{\frac{A\theta}{2\pi}}$$

Since now on $\rho(\theta) = \theta$

Distributional formulation of s.s. problem

$$-\Delta m + \mathrm{div}(\mathbf{b}(x)m) + \mathrm{div}(Am^+(x)\delta_\Gamma(x)\mathbf{N}(x)) = 0$$

Superposition principle $\Rightarrow$

$$m(x) = C + A \int_\Gamma m^+(y) \nabla_y G(x,y) \cdot \mathbf{N}(y)\, d\sigma(y), \qquad C \in \mathbb{R}^+, x \notin \Gamma$$

with

$$-\Delta_x G + \mathrm{div}_x(\mathbf{b}(x)G) = \delta_y(x) \qquad x \in \mathbb{R}^2 \setminus \{y\}$$

# Distributional formulation

$$A < 0, |A| \ll 1 \ \& \ \rho(\theta) = \theta \ \Rightarrow \exists \ \text{positive solution } p(\rho,\theta) \overset{\theta\to\infty}{\sim} e^{\frac{A\theta}{2\pi}}$$

Since now on $\rho(\theta) = \theta$

Distributional formulation of s.s. problem

$$-\Delta m + \mathrm{div}(\mathbf{b}(x)m) + \mathrm{div}(Am^{+}(x)\delta_{\Gamma}(x)\mathbf{N}(x)) = 0$$

Superposition principle $\Rightarrow$

$$m(x) = C + A\int_{\Gamma} m^{+}(y)\,\nabla_y G(x,y)\cdot\mathbf{N}(y)\,d\sigma(y)\,, \qquad C \in \mathbb{R}^{+}, x \notin \Gamma$$

with

$$-\Delta_x G + \mathrm{div}_x(\mathbf{b}(x)G) = \delta_y(x) \qquad x \in \mathbb{R}^2 \setminus \{y\}$$

# Distributional formulation

$A < 0, |A| \ll 1$ & $\rho(\theta) = \theta \Rightarrow \exists$ positive solution $p(\rho,\theta) \overset{\theta\to\infty}{\sim} e^{\frac{A\theta}{2\pi}}$

Since now on $\rho(\theta) = \theta$

Distributional formulation of s.s. problem

$$-\Delta m + \mathrm{div}(\mathbf{b}(x)m) + \mathrm{div}(Am^+(x)\delta_\Gamma(x)\mathbf{N}(x)) = 0$$

Superposition principle $\Rightarrow$

$$m(x) = C + A\int_\Gamma m^+(y)\,\nabla_y G(x,y)\cdot\mathbf{N}(y)\,d\sigma(y)\,, \qquad C \in \mathbb{R}^+, x \notin \Gamma$$

with

$$-\Delta_x G + \mathrm{div}_x(\mathbf{b}(x)G) = \delta_y(x) \qquad x \in \mathbb{R}^2 \setminus \{y\}$$

# Reduction to an integral equation

$$G(x,y) = H(x,y) - \frac{1}{2\pi}\log\|x-y\| + \frac{1}{2\pi}\log\|y\|$$

with ( $q \in [1,2)$ )

$$x \mapsto H(x,y) \in W^{2,q}_{loc}(\mathbb{R}^2), \quad \text{and } y \mapsto \|H(\cdot,y)\|_{W^{2,q}_{loc}} \in C^0(\mathbb{R}^2)$$
$$y \mapsto H(x,y) \in W^{2,q}_{loc}(\mathbb{R}^2), \quad \text{and } x \mapsto \|H(x,\cdot)\|_{W^{2,q}_{loc}} \in C^0(\mathbb{R}^2)$$

$$m^+(x) = C + \frac{A}{2}m^+(x) + A\int_\Gamma m^+(y)\,\nabla_y G(x,y)\cdot \mathbf{N}(y)\,d\sigma(y)$$

$\longrightarrow$ Existence problem for this integral equation with

$$m^+(\theta\cos\theta, \theta\sin\theta) \sim e^{-\mu\theta} \qquad \mu \in \left(0, \frac{|A|}{2\pi}\right)$$

# Reduction to an integral equation

$$G(x,y) = H(x,y) - \frac{1}{2\pi}\log\|x-y\| + \frac{1}{2\pi}\log\|y\|$$

with ( $q \in [1,2)$ )

$$x \mapsto H(x,y) \in W^{2,q}_{loc}(\mathbb{R}^2), \quad \text{and } y \mapsto \|H(\cdot,y)\|_{W^{2,q}_{loc}} \in C^0(\mathbb{R}^2)$$
$$y \mapsto H(x,y) \in W^{2,q}_{loc}(\mathbb{R}^2), \quad \text{and } x \mapsto \|H(x,\cdot)\|_{W^{2,q}_{loc}} \in C^0(\mathbb{R}^2)$$

$$m^+(x) = C + \frac{A}{2}m^+(x) + A\int_\Gamma m^+(y)\,\nabla_y G(x,y)\cdot\mathbf{N}(y)\;d\sigma(y)$$

$\longrightarrow$ Existence problem for this integral equation with

$$m^+(\theta\cos\theta, \theta\sin\theta) \sim e^{-\mu\theta} \qquad \mu \in \left(0, \frac{|A|}{2\pi}\right)$$

# Reduction to an integral equation

$$G(x,y) = H(x,y) - \frac{1}{2\pi}\log\|x-y\| + \frac{1}{2\pi}\log\|y\|$$

with ( $q \in [1,2)$ )

$$x \mapsto H(x,y) \in W^{2,q}_{loc}(\mathbb{R}^2), \quad \text{and } y \mapsto \|H(\cdot,y)\|_{W^{2,q}_{loc}} \in C^0(\mathbb{R}^2)$$
$$y \mapsto H(x,y) \in W^{2,q}_{loc}(\mathbb{R}^2), \quad \text{and } x \mapsto \|H(x,\cdot)\|_{W^{2,q}_{loc}} \in C^0(\mathbb{R}^2)$$

$$m^+(x) = C + \frac{A}{2}m^+(x) + A\int_\Gamma m^+(y)\,\nabla_y G(x,y)\cdot \mathbf{N}(y)\;d\sigma(y)$$

$\longrightarrow$ Existence problem for this integral equation with

$$m^+(\theta\cos\theta, \theta\sin\theta) \sim e^{-\mu\theta} \qquad \mu \in \left(0, \frac{|A|}{2\pi}\right)$$

# Sometimes polar coordinates are better

$$G(\rho,\theta;R,\xi) := G(\rho e^{i\theta}, Re^{i\xi}) = -\frac{\chi_{[0,\infty)}(\rho - R)}{2\pi}\log\left(\frac{\rho}{R}\right) + W(\rho,\theta;R,\xi)\,,$$

$$W(\rho,\theta;R,\xi) = \begin{cases} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{K_n(\sqrt{i\Omega n}R)e^{in(\theta-\xi)}}{2\pi} I_n(\sqrt{i\Omega n}\rho) & \rho \le R \\ \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{I_n(\sqrt{i\Omega n}R)e^{in(\theta-\xi)}}{2\pi} K_n(\sqrt{i\Omega n}\rho) & \rho > R\,, \end{cases}$$

$I_n, K_n$: modified Bessel functions of integer order $n$

$$m^+(\theta) = C + \frac{A}{2}m^+(\theta) + \frac{A}{2\pi}\int_0^\theta m^+(\xi)d\xi + A\int_0^\infty f_\Gamma(\theta,\xi)m^+(\xi)d\xi$$

$$m^+(\theta) := m^+(\theta e^{i\theta}), \quad f_\Gamma(\theta,\xi) = \xi\frac{\partial W}{\partial R}\bigg|_{(\theta,\theta;\xi,\xi)} - \frac{1}{\xi}\frac{\partial W}{\partial \xi}\bigg|_{(\theta,\theta;\xi,\xi)}$$

Formal computation:

$$m^+(\theta) \overset{\theta\to\infty}{\sim} e^{-\mu\theta} \Rightarrow \int_0^\infty f_\Gamma(\theta,\xi)m^+(\xi)d\xi \overset{\theta\to\infty}{\sim} e^{-\mu\theta} \quad (\mu > 0)$$

# Sometimes polar coordinates are better

$$G(\rho,\theta;R,\xi) := G(\rho e^{i\theta}, Re^{i\xi}) = -\frac{\chi_{[0,\infty)}(\rho - R)}{2\pi}\log\left(\frac{\rho}{R}\right) + W(\rho,\theta;R,\xi)\,.$$

$$W(\rho,\theta;R,\xi) = \begin{cases} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{K_n(\sqrt{i\Omega n}R)e^{in(\theta-\xi)}}{2\pi} I_n(\sqrt{i\Omega n}\rho) & \rho \le R \\ \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{I_n(\sqrt{i\Omega n}R)e^{in(\theta-\xi)}}{2\pi} K_n(\sqrt{i\Omega n}\rho) & \rho > R\,, \end{cases}$$

$I_n, K_n$: modified Bessel functions of integer order $n$

$$m^+(\theta) = C + \frac{A}{2}m^+(\theta) + \frac{A}{2\pi}\int_0^\theta m^+(\xi)d\xi + A\int_0^\infty f_\Gamma(\theta,\xi)m^+(\xi)d\xi$$

$$m^+(\theta) := m^+(\theta e^{i\theta}), \quad f_\Gamma(\theta,\xi) = \xi\frac{\partial W}{\partial R}\Big|_{(\theta,\theta;\xi,\xi)} - \frac{1}{\xi}\frac{\partial W}{\partial \xi}\Big|_{(\theta,\theta;\xi,\xi)}$$

Formal computation:

$$m^+(\theta) \overset{\theta\to\infty}{\sim} e^{-\mu\theta} \Rightarrow \int_0^\infty f_\Gamma(\theta,\xi)m^+(\xi)d\xi \overset{\theta\to\infty}{\sim} e^{-\mu\theta} \quad (\mu > 0)$$

# Sometimes polar coordinates are better

$$G(\rho,\theta;R,\xi) := G(\rho e^{i\theta}, Re^{i\xi}) = -\frac{\chi_{[0,\infty)}(\rho - R)}{2\pi}\log\left(\frac{\rho}{R}\right) + W(\rho,\theta;R,\xi)$$

$$W(\rho,\theta;R,\xi) = \begin{cases} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{K_n(\sqrt{i\Omega n}R)e^{in(\theta-\xi)}}{2\pi} I_n(\sqrt{i\Omega n}\rho) & \rho \le R \\ \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{I_n(\sqrt{i\Omega n}R)e^{in(\theta-\xi)}}{2\pi} K_n(\sqrt{i\Omega n}\rho) & \rho > R, \end{cases}$$

$I_n, K_n$: modified Bessel functions of integer order $n$

$$m^+(\theta) = C + \frac{A}{2}m^+(\theta) + \frac{A}{2\pi}\int_0^\theta m^+(\xi)d\xi + A\int_0^\infty f_\Gamma(\theta,\xi)m^+(\xi)d\xi$$

$$m^+(\theta) := m^+(\theta e^{i\theta}), \quad f_\Gamma(\theta,\xi) = \xi\frac{\partial W}{\partial R}\bigg|_{(\theta,\theta;\xi,\xi)} - \frac{1}{\xi}\frac{\partial W}{\partial \xi}\bigg|_{(\theta,\theta;\xi,\xi)}$$

Formal computation:

$$m^+(\theta) \overset{\theta\to\infty}{\sim} e^{-\mu\theta} \Rightarrow \int_0^\infty f_\Gamma(\theta,\xi)m^+(\xi)d\xi \overset{\theta\to\infty}{\sim} e^{-\mu\theta} \quad (\mu > 0)$$

# Fundamental properties of the kernel $f_\Gamma$

$$X_\mu := \left\{ \lambda \in C^0([0,\infty)) \;\middle|\; \|\lambda\|_\mu := \sup_{\theta \geq 0} |\lambda(\theta) e^{\mu\theta}| < \infty \right\} \quad (\mu > 0)$$

N. C. for existence of a solution in $X_\mu$: $C + \frac{A}{2\pi} \int_0^\infty m^+(\xi) d\xi = 0$

Lemma:

- $f_\Gamma$ is regular outside $\{\theta = \xi\}$, and $f_\Gamma(0,\xi) \equiv 0$,
- $\exists M_\Omega > 0$ s.t. $\forall \mu \in (0, M_\Omega]$

$$\begin{array}{rccc} T: & X_\mu & \longrightarrow & X_\mu \\ & \lambda & \mapsto & T(\lambda)(\theta) := \int_0^\infty f_\Gamma(\theta,\xi)\lambda(\xi) d\xi \end{array}$$

well defined, and $T \in \mathcal{L}(X_\mu)$ with $\|T\| \leq C_\Omega$

Consequences:

$m^+ > 0 \Rightarrow (1 - A/2)m^+(0) = C \Rightarrow C > 0$

$\frac{A}{2\pi} \int_0^\infty m^+(\xi) d\xi = -C \Rightarrow A < 0$

# Fundamental properties of the kernel $f_\Gamma$

$$X_\mu := \left\{ \lambda \in C^0([0,\infty)) \;\Big|\; \|\lambda\|_\mu := \sup_{\theta \geq 0} |\lambda(\theta) e^{\mu\theta}| < \infty \right\} \quad (\mu > 0)$$

N. C. for existence of a solution in $X_\mu$: $C + \frac{A}{2\pi} \int_0^\infty m^+(\xi) d\xi = 0$

Lemma:

- $f_\Gamma$ is regular outside $\{\theta = \xi\}$, and $f_\Gamma(0,\xi) \equiv 0$,
- $\exists M_\Omega > 0$ s.t. $\forall \mu \in (0, M_\Omega]$

$$\begin{array}{rccl} T: & X_\mu & \longrightarrow & X_\mu \\ & \lambda & \mapsto & T(\lambda)(\theta) := \int_0^\infty f_\Gamma(\theta,\xi)\lambda(\xi) d\xi \end{array}$$

well defined, and $T \in \mathcal{L}(X_\mu)$ with $\|T\| \leq C_\Omega$

Consequences:

$m^+ > 0 \Rightarrow (1 - A/2) m^+(0) = C \Rightarrow C > 0$

$\frac{A}{2\pi} \int_0^\infty m^+(\xi) d\xi = -C \Rightarrow A < 0$

# Fundamental properties of the kernel $f_\Gamma$

$$X_\mu := \left\{ \lambda \in C^0([0,\infty)) \;\middle|\; \|\lambda\|_\mu := \sup_{\theta \geq 0} |\lambda(\theta) e^{\mu\theta}| < \infty \right\} \quad (\mu > 0)$$

N. C. for existence of a solution in $X_\mu$: $C + \frac{A}{2\pi} \int_0^\infty m^+(\xi) d\xi = 0$

Lemma:

- $f_\Gamma$ is regular outside $\{\theta = \xi\}$, and $f_\Gamma(0,\xi) \equiv 0$,
- $\exists M_\Omega > 0$ s.t. $\forall \mu \in (0, M_\Omega]$

$$\begin{array}{cccc} T: & X_\mu & \longrightarrow & X_\mu \\ & \lambda & \mapsto & T(\lambda)(\theta) := \int_0^\infty f_\Gamma(\theta,\xi)\lambda(\xi) d\xi \end{array}$$

well defined, and $T \in \mathcal{L}(X_\mu)$ with $\|T\| \leq C_\Omega$

Consequences:

$m^+ > 0 \;\Rightarrow\; (1 - A/2)m^+(0) = C \;\Rightarrow\; C > 0$

$\frac{A}{2\pi} \int_0^\infty m^+(\xi) d\xi = -C \;\Rightarrow\; A < 0$

# Fundamental properties of the kernel $f_\Gamma$

$$X_\mu := \left\{ \lambda \in C^0([0,\infty)) \;\middle|\; \|\lambda\|_\mu := \sup_{\theta\geq 0} |\lambda(\theta)e^{\mu\theta}| < \infty \right\} \quad (\mu > 0)$$

N. C. for existence of a solution in $X_\mu$: $C + \frac{A}{2\pi}\int_0^\infty m^+(\xi)d\xi = 0$

Lemma:

- $f_\Gamma$ is regular outside $\{\theta = \xi\}$, and $f_\Gamma(0,\xi) \equiv 0$,
- $\exists M_\Omega > 0$ s.t. $\forall \mu \in (0, M_\Omega]$

$$\begin{array}{rccl} T: & X_\mu & \longrightarrow & X_\mu \\ & \lambda & \mapsto & T(\lambda)(\theta) := \int_0^\infty f_\Gamma(\theta,\xi)\lambda(\xi)d\xi \end{array}$$

well defined, and $T \in \mathcal{L}(X_\mu)$ with $\|T\| \leq C_\Omega$

Consequences:

$m^+ > 0 \;\Rightarrow\; (1 - A/2)m^+(0) = C \;\Rightarrow\; C > 0$

$\frac{A}{2\pi}\int_0^\infty m^+(\xi)d\xi = -C \;\Rightarrow\; A < 0$

# Fundamental properties of the kernel $f_\Gamma$

$$X_\mu := \left\{ \lambda \in C^0([0,\infty)) \;\middle|\; \|\lambda\|_\mu := \sup_{\theta \geq 0} |\lambda(\theta) e^{\mu\theta}| < \infty \right\} \quad (\mu > 0)$$

N. C. for existence of a solution in $X_\mu$: $C + \frac{A}{2\pi} \int_0^\infty m^+(\xi) d\xi = 0$

Lemma:

- $f_\Gamma$ is regular outside $\{\theta = \xi\}$, and $f_\Gamma(0,\xi) \equiv 0$,
- $\exists M_\Omega > 0$ s.t. $\forall \mu \in (0, M_\Omega]$

$$\begin{array}{rccl} T: & X_\mu & \longrightarrow & X_\mu \\ & \lambda & \mapsto & T(\lambda)(\theta) := \int_0^\infty f_\Gamma(\theta,\xi)\lambda(\xi) d\xi \end{array}$$

well defined, and $T \in \mathcal{L}(X_\mu)$ with $\|T\| \leq C_\Omega$

Consequences:

$m^+ > 0 \;\Rightarrow\; (1 - A/2) m^+(0) = C \;\Rightarrow\; C > 0$

$\frac{A}{2\pi} \int_0^\infty m^+(\xi) d\xi = -C \;\Rightarrow\; A < 0$

# Statement of the main theorem

Let $C = 1$

Theorem: There exists $A_\Omega > 0$ s.t. for every $A \in (-A_\Omega, 0)$ the integral equation has a positive solution $m^+ \in X_{\mu(A)}$ with $\mu(A) \in \left(0, \frac{|A|}{2\pi}\right)$

Sketch of the Proof:

If $m^+$ in $X_\mu$ with $\mu < \frac{|A|}{\pi(2-A)} \leq M_\Omega \Rightarrow \frac{|A|}{2\pi} \int_0^\infty m^+(\xi) d\xi = 1$, and polar integral equation equivalent to

$$\begin{cases} m^+(\theta) = \frac{A}{2} m^+(\theta) - \frac{A}{2\pi} \int_\theta^\infty m^+(\xi) d\xi + A\,T(m^+)(\theta) & \theta \geq 0 \\ \frac{|A|}{2\pi} \int_0^\infty m^+(\xi) d\xi = 1 \end{cases}$$

Construction of sequence $\{\lambda_n\}_{n \in \mathbb{N}_0}$ of approximate solutions:

$$\lambda_0(\theta) = \frac{2}{2-A} \exp\left(-\frac{|A|\theta}{\pi(2-A)}\right)$$

# Statement of the main theorem

Let $C = 1$

Theorem: There exists $A_\Omega > 0$ s.t. for every $A \in (-A_\Omega, 0)$ the integral equation has a positive solution $m^+ \in X_{\mu(A)}$ with $\mu(A) \in \left(0, \frac{|A|}{2\pi}\right)$

Sketch of the Proof:

If $m^+$ in $X_\mu$ with $\mu < \frac{|A|}{\pi(2-A)} \leq M_\Omega \Rightarrow \frac{|A|}{2\pi}\int_0^\infty m^+(\xi)d\xi = 1$, and polar integral equation equivalent to

$$\begin{cases} m^+(\theta) = \frac{A}{2}m^+(\theta) - \frac{A}{2\pi}\int_\theta^\infty m^+(\xi)d\xi + A\,T(m^+)(\theta) & \theta \geq 0 \\ \frac{|A|}{2\pi}\int_0^\infty m^+(\xi)d\xi = 1 \end{cases}$$

Construction of sequence $\{\lambda_n\}_{n\in\mathbb{N}_0}$ of approximate solutions:

$$\lambda_0(\theta) = \frac{2}{2-A}\exp\left(-\frac{|A|\theta}{\pi(2-A)}\right)$$

# Statement of the main theorem

Let $C = 1$

Theorem: There exists $A_\Omega > 0$ s.t. for every $A \in (-A_\Omega, 0)$ the integral equation has a positive solution $m^+ \in X_{\mu(A)}$ with $\mu(A) \in \left(0, \frac{|A|}{2\pi}\right)$

Sketch of the Proof:

If $m^+$ in $X_\mu$ with $\mu < \frac{|A|}{\pi(2-A)} \leq M_\Omega \Rightarrow \frac{|A|}{2\pi}\int_0^\infty m^+(\xi)d\xi = 1$, and polar integral equation equivalent to

$$\begin{cases} m^+(\theta) = \frac{A}{2}m^+(\theta) - \frac{A}{2\pi}\int_\theta^\infty m^+(\xi)d\xi + A\,T(m^+)(\theta) & \theta \geq 0 \\ \frac{|A|}{2\pi}\int_0^\infty m^+(\xi)d\xi = 1 \end{cases}$$

Construction of sequence $\{\lambda_n\}_{n\in\mathbb{N}_0}$ of approximate solutions:

$$\lambda_0(\theta) = \frac{2}{2-A}\exp\left(-\frac{|A|\theta}{\pi(2-A)}\right)$$

# Sketch of the proof

$$\begin{cases} \lambda_{n+1}(\theta) = \frac{A}{2}\lambda_{n+1}(\theta) - \frac{A}{2\pi}\int_\theta^\infty \lambda_{n+1}(\xi)d\xi + A\,T(\lambda_n)(\theta) \quad \theta \geq 0 \\ \frac{|A|}{2\pi}\int_0^\infty \lambda_{n+1}(\xi)d\xi = 1 \end{cases}$$

Variation of constants formula: $\lambda_{n+1}(\theta) =$

$$\frac{2}{2-A}\left( e^{-\frac{|A|}{\pi}\frac{\theta}{2-A}} - |A|\,T(\lambda_n)(\theta) + \frac{|A|^2}{\pi(2-A)}\int_0^\theta T(\lambda_n)(\sigma)e^{\frac{|A|}{\pi}\frac{\sigma-\theta}{2-A}}d\sigma \right)$$

$\forall n \in \mathbb{N}_0 :\ \lambda_n \in X_\mu \ \Rightarrow\ \lambda_{n+1} \in X_\mu$ and

$$\|\lambda_{n+1} - \lambda_n\| \leq \frac{2|A|C_\Omega}{2-A}\left(1 + \frac{|A|}{\pi(2-A)}\frac{1}{\frac{|A|}{\pi(2-A)} - \mu}\right)\|\lambda_n - \lambda_{n-1}\|$$

$\mu(\text{fixed}) < \frac{|A|}{\pi(2-A)} \Rightarrow \lambda_0 \in X_\mu$ and, up to choose $A_\Omega$ smaller,

$$\forall n \in \mathbb{N}_0 :\ \lambda_n \in X_\mu \text{ and } \|\lambda_{n+1} - \lambda_n\| \leq \frac{1}{2}\|\lambda_n - \lambda_{n-1}\| \quad (\lambda_{-1} := 0)$$

$$m^+(\theta) := \lim_{n\to\infty} \lambda_n$$

# Sketch of the proof

$$\begin{cases} \lambda_{n+1}(\theta) = \frac{A}{2}\lambda_{n+1}(\theta) - \frac{A}{2\pi}\int_\theta^\infty \lambda_{n+1}(\xi)d\xi + A\,T(\lambda_n)(\theta) & \theta \geq 0 \\ \frac{|A|}{2\pi}\int_0^\infty \lambda_{n+1}(\xi)d\xi = 1 \end{cases}$$

Variation of constants formula: $\lambda_{n+1}(\theta) =$

$$\frac{2}{2-A}\left(e^{-\frac{|A|}{\pi}\frac{\theta}{2-A}} - |A|\,T(\lambda_n)(\theta) + \frac{|A|^2}{\pi(2-A)}\int_0^\theta T(\lambda_n)(\sigma)e^{\frac{|A|}{\pi}\frac{\sigma-\theta}{2-A}}d\sigma\right)$$

$$\forall n \in \mathbb{N}_0 : \ \lambda_n \in X_\mu \ \Rightarrow \ \lambda_{n+1} \in X_\mu \text{ and}$$

$$\|\lambda_{n+1} - \lambda_n\| \leq \frac{2|A|C_\Omega}{2-A}\left(1 + \frac{|A|}{\pi(2-A)}\frac{1}{\frac{|A|}{\pi(2-A)} - \mu}\right)\|\lambda_n - \lambda_{n-1}\|$$

$\mu(\text{fixed}) < \frac{|A|}{\pi(2-A)} \Rightarrow \lambda_0 \in X_\mu$ and, up to choose $A_\Omega$ smaller,

$$\forall n \in \mathbb{N}_0 : \ \lambda_n \in X_\mu \text{ and } \|\lambda_{n+1} - \lambda_n\| \leq \frac{1}{2}\|\lambda_n - \lambda_{n-1}\| \quad (\lambda_{-1} := 0)$$

$$m^+(\theta) := \lim_{n\to\infty} \lambda_n$$

# Sketch of the proof

$$\begin{cases} \lambda_{n+1}(\theta) = \frac{A}{2}\lambda_{n+1}(\theta) - \frac{A}{2\pi}\int_\theta^\infty \lambda_{n+1}(\xi)d\xi + A\,T(\lambda_n)(\theta) & \theta \geq 0 \\ \frac{|A|}{2\pi}\int_0^\infty \lambda_{n+1}(\xi)d\xi = 1 \end{cases}$$

Variation of constants formula: $\lambda_{n+1}(\theta) =$

$$\frac{2}{2-A}\left(e^{-\frac{|A|}{\pi}\frac{\theta}{2-A}} - |A|T(\lambda_n)(\theta) + \frac{|A|^2}{\pi(2-A)}\int_0^\theta T(\lambda_n)(\sigma)e^{\frac{|A|}{\pi}\frac{\sigma-\theta}{2-A}}d\sigma\right)$$

$$\forall n \in \mathbb{N}_0 :\ \lambda_n \in X_\mu \ \Rightarrow\ \lambda_{n+1} \in X_\mu \text{ and}$$

$$\|\lambda_{n+1} - \lambda_n\| \leq \frac{2|A|C_\Omega}{2-A}\left(1 + \frac{|A|}{\pi(2-A)}\frac{1}{\frac{|A|}{\pi(2-A)} - \mu}\right)\|\lambda_n - \lambda_{n-1}\|$$

$\mu(\text{fixed}) < \frac{|A|}{\pi(2-A)} \Rightarrow \lambda_0 \in X_\mu$ and, up to choose $A_\Omega$ smaller,

$$\forall n \in \mathbb{N}_0 :\ \lambda_n \in X_\mu \text{ and } \|\lambda_{n+1} - \lambda_n\| \leq \frac{1}{2}\|\lambda_n - \lambda_{n-1}\| \quad (\lambda_{-1} := 0)$$

$$m^+(\theta) := \lim_{n\to\infty} \lambda_n$$

# Bibliografy

- M. Abramovich & I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover, New York, 1974.
- D. Gilbarg & N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1977
- F. Siegert & C.J. Weijer, *Spiral and concentric waves organize multicellular Dictyostelium mounds*, Curr. Biol. **5** (1995), no. 8, 937-943
- D. Traynor, R. H. Kessin & J. G. Williams, *Chemotactic sorting to cAMP in the multicellular stages of Dictyostelium development*, Proc. Natl. Acad. Sci. USA, **89** (1992), 8303-8307, Developmental Biology
- A. T. Winfree *Spiral waves of chemical activity*, Science, **175** (1972), 634-636