Critical mass in a parabolic-elliptic Patlak-Keller-Segel equation with nonlinear diffusion

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The generalized Patlak-Keller-Segel (PKS) equation I

$$\begin{aligned} \partial_t u &= \operatorname{div}\left(\nabla A(u) - u\,\nabla\varphi\right), \quad (t,x) \in (0,\infty)\times\mathbb{R}^d, \\ \varphi &= E_d * u, \quad (t,x)\in(0,\infty)\times\mathbb{R}^d, \end{aligned}$$

with

- $d \geq 2$,
- $A(u) := u^m$ with $m \geq 1$,
- E_d is the Poisson kernel

$$E_2(x) := -\frac{1}{2\pi}\ \ln|x| \quad \text{or} \quad E_d := c_d\,|x|^{-(d-2)} \quad \text{if} \quad d \geq 3\,,$$

(so that $-\Delta\varphi = u$).

The generalized PKS equation II

- Self-gravitating particles (in astrophysics).
- Parabolic-elliptic Keller-Segel model for chemotaxis.

$\rightarrow$ $d = 2$, $m = 1$: classical parabolic-elliptic PKS equation,

$\rightarrow$ Derivation from a Vlasov-Poisson-Fokker-Planck kinetic equation (with non-gaussian equilibria if $m > 1$) as a diffusion limit,

$\rightarrow$ Prevention of crowding if $m > 1$ (diffusion coefficient $u^{m-1} \rightarrow \infty$ as $u \rightarrow \infty$).

Properties of u

$$\partial_t u = \text{ div } \left(\nabla u^m - u\,\nabla(E_d * u)\right)\,, \quad (t,x) \in (0,\infty) \times \mathbb{R}^d\,.$$

- Non-negativity: $u \geq 0$ (if $u_0 \geq 0$).

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- Conservation of mass: $\|u(t)\|_1 = M_0 := \|u_0\|_1$.

Properties of u

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- Non-negativity: $u \geq 0$ (if $u_0 \geq 0$).
- Conservation of mass: $\|u(t)\|_1 = M_0 := \|u_0\|_1$.
- Competition between the diffusive term Δu^m (spreading) and the drift term (concentrating) $\operatorname{div}\left(u\,\nabla(E_d * u)\right)$: global existence or finite time blowup.

$$\partial_t u = \Delta u^m - \nabla u \cdot \nabla(E_d * u) + u^2 , \quad (t,x) \in (0,\infty) \times \mathbb{R}^d .$$

A Liapunov functional I

The functional

$$\mathcal{F}[u(t)] := \int_{\mathbb{R}^d} \mathcal{A}(u(t,x))\, dx - \frac{1}{2} \int_{\mathbb{R}^d} (E_d * u)(t,x)\, u(t,x)\, dx\,,$$

with

- $\mathcal{A}(r) = r \ln r - r \geq -1$ if $m = 1$,
- $\mathcal{A}(r) = r^m/(m-1) \geq 0$ if $m > 1$.

Two competing terms in $\mathcal{F}$

A Liapunov functional II

Liapunov functional:

$$\mathcal{F}[u(t)] := \int_{\mathbb{R}^d} \mathcal{A}(u(t,x))\, dx - \frac{1}{2} \int_{\mathbb{R}^d} (E_d * u)(t,x)\, u(t,x)\, dx\,.$$

At first glance, the "negative" term is quadratic in u and the positive term might dominate it if $\mathcal{A}$ increases faster than quadratically ($m > 2$).

In fact,

$$m > m_d := \frac{2(d-1)}{d} \quad (m_2 = 1)$$

guarantees global existence while there are solutions blowing up in finite time if

$$1 < m \le m_d := \frac{2(d-1)}{d}\,.$$

[Sugiyama & Kunii (2006), Sugiyama (2007), Cieślak & Winkler (2008)]

Virial identity

$$M_2(t) := \int_{\mathbb{R}^d} |x|^2 \, u(t,x) \, dx \,.$$

- $d = 2$ and $m = m_2 = 1$ ($M_0 = \|u_0\|_1$):

$$\frac{dM_2}{dt}(t) = -\frac{M_0}{4\pi} \, (M_0 - 8\pi) \,,$$

$\longrightarrow$ non-existence of global solutions for $M_0 > 8\pi$.

- $d \geq 3$ and $m = m_d$:

$$\frac{dM_2}{dt}(t) = 2(d-2) \, \mathcal{F}[u(t)] \leq 2(d-2) \, \mathcal{F}[u_0] \,,$$

$\longrightarrow$ non-existence of global solutions for u_0 such that $\mathcal{F}[u_0] < 0$.

The generalized PKS equation with $m = 1$ in $\mathbb{R}^2$

$$\partial_t u = \operatorname{div}\left(\nabla u - u\,\nabla(E_2 * u)\right)\,, \quad (t,x) \in (0,\infty) \times \mathbb{R}^2\,.$$

The Liapunov functional:

$$\begin{aligned}\mathcal{F}[u(t)] \quad := \quad & \int_{\mathbb{R}^2} \left(u(t,x) \ln u(t,x) - u(t,x)\right)\, dx \\ & - \frac{1}{2} \int_{\mathbb{R}^2} (E_2 * u)(t,x)\, u(t,x)\, dx\,.\end{aligned}$$

Global existence: $m = 1$ and $d = 2$

- Finite time blow-up if $\|u_0\|_1 > 8\pi$ (virial identity).
- Global existence if $\|u_0\|_1 < M_* < 8\pi$ by Gagliardo-Nirenberg inequalities. [Jäger & Luckhaus (1992)]
- Global existence if $\|u_0\|_1 < 8\pi$ by symmetrization techniques. [Diaz, Nagai & Rakotoson (1998)]
- Global existence if $\|u_0\|_1 < 8\pi$ by the logarithmic Hardy-Littlewood-Sobolev inequality. [Dolbeault & Perthame (2004)]

The logarithmic Hardy-Littlewood-Sobolev inequality

$$\int_{\mathbb{R}^2} h \, \ln h \, dx \quad - \quad \frac{4\pi}{\|h\|_1} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} E_2(x-y) \, h(x) \, h(y) \, dydx$$
$$\geq \quad -\|h\|_1 \, \left(1 + \ln \pi - \ln \|h\|_1\right)$$

[Carlen & Loss (1992), Beckner (2003)]

Then

$$\mathcal{F}[u] \quad \geq \quad \left(1 - \frac{\|u_0\|_1}{8\pi}\right) \int_{\mathbb{R}^2} u \, \ln u \, dx$$
$$- \frac{\|u_0\|_1^2}{8\pi} \, \left(1 + \ln \pi - \ln \|u_0\|_1\right) ,$$

hence a control of $u \, \ln u$ in L^1 $\longrightarrow$ global existence.

The generalized PKS equation with $m = m_d$ in $\mathbb{R}^d$, $d \geq 3$

$$m = m_d = \frac{2(d-1)}{d}$$

$$\partial_t u = \text{ div } \left(\nabla u^m - u \, \nabla (E_d * u) \right) , \quad (t,x) \in (0,\infty) \times \mathbb{R}^d .$$

The Liapunov functional:

$$\mathcal{F}[u(t)] := \int_{\mathbb{R}^d} \frac{u^m(t,x)}{m-1}) \, dx - \frac{1}{2} \int_{\mathbb{R}^d} (E_d * u)(t,x) \, u(t,x) \, dx .$$

Global existence and blowup I

- Global existence if $\|u_0\|_1 < M_1$ by Gagliardo-Nirenberg inequalities.
- Finite time blowup if $\|u_0\|_1 > M_2 > M_1$.

[Sugiyama (2007)]

A modified Hardy-Littlewood-Sobolev inequality

Approach: find a functional inequality characterizing the critical mass.

Combining Hölder and Hardy-Littlewood-Sobolev inequalities, there exists $C_* > 0$ such that

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{h(x)\, h(y)}{|x-y|^{d-2}}\, dxdy \right| \leq C_* \, \|h\|_m^m \, \|h\|_1^{2/d}$$

for all $h \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)$.

[Blanchet, Carrillo & L. (2009)]

A bound from below for $\mathcal{F}$

Introducing the critical mass

$$M_c := \left[\frac{2}{(m-1)\, C_*\, c_d}\right]^{d/2},$$

we have

$$\frac{C_*\, c_d}{2} \left(M_c^{2/d} - \|h\|_1^{2/d}\right) \|h\|_m^m \leq \mathcal{F}[h]$$

for all $h \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)$.

- If $\|u_0\|_1 \leq M_c$ then $\mathcal{F}[u_0] \geq 0$,

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for all $h \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)$.

- If $\|u_0\|_1 \leq M_c$ then $\mathcal{F}[u_0] \geq 0$,
- If $\|u_0\|_1 < M_c$ then control on the L^m-norm $\longrightarrow$ global existence.

Global existence and blowup II

- Global existence if $\|u_0\|_1 < M_c$ by the modified Hardy-Littlewood-Sobolev inequality.
- If $M > M_c$, then an argument from Weinstein (1986) gives

$$\mu_M := \inf \left\{ \mathcal{F}[h] \ : \ h \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d) \ , \ \|h\|_1 = M \right\} = -\infty \, ,$$

and thus finite time blowup if $\mathcal{F}[u_0] < 0$ by the virial identity.

[Blanchet, Carrillo & L. (2009), Suzuki & Takahashi (2009)]

What happens if $\|u_0\|_1 = M_c$?

Stationary solutions

There is a two-parameter family $\{V_{z,R}\}$ of non-negative and compactly supported stationary solutions such that

$$\|V_{z,R}\|_1 = M_c\,, \quad z \in \mathbb{R}^d\,,\ R > 0\,.$$

Minimisers of $\mathcal{F}$ in $\left\{h \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)\ :\ \|h\|_1 = M_c\right\}$.
[Chavanis & Sire (2008), Blanchet, Carrillo & L. (2009)]

Recall that, if $d = 2$ and $m = 1$, existence of a two-parameter family of stationary solutions:

$$\frac{8b}{\left(b + |x - z|^2\right)^2} \in L^1(\mathbb{R}^2) \setminus L^1\left(\mathbb{R}^2; |x|^2\,dx\right)\,, \quad z \in \mathbb{R}^d\,,\ b > 0\,.$$

Global existence: $\|u_0\|_1 = M_c$

If $\|u_0\|_1 = M_c$, there is a global solution u.
[Blanchet, Carrillo & L. (2009)]

Open question: If $\|u_0\|_1 = M_c$, what is the large time behaviour of the global solution:

- Convergence to a steady state?
- Blowup in infinite time and concentration to a Dirac mass?

Recall that, if $d = 2$, $m = 1$, and $\|u_0\|_1 = 8\pi$, there is a global solution u and $u(t) \rightharpoonup 8\pi\, \delta_{x_m}$ as $t \to \infty$, x_m being the center of mass of u_0.
[Biler, Karch, L. & Nadzieja (2006), Blanchet, Carrillo & Masmoudi (2008)] .

Self-similar blowing-up solutions I

Look for (radially symmetric) solutions of the form

$$u(t,x) = \frac{1}{s(t)^d}\, U\left(\frac{|x|}{s(t)}\right) \quad \text{and} \quad \varphi(t,x) = \frac{1}{s(t)^{d-2}}\, \Phi\left(\frac{|x|}{s(t)}\right)$$

for $(t,x) \in [0,T) \times \mathbb{R}^d$ with $s(t) := [d(T-t)]^{1/d}$, $T > 0$. Then

$$U(r)\, \frac{dJ}{dr}(r) = 0\,, \quad J(r) := \frac{2(d-1)}{d-2}\, U^{(d-2)/d}(r) - \Phi(r) - \frac{r^2}{2}\,.$$

$\longrightarrow$ J is constant on connected components of

$$\mathcal{P} := \{r > 0 \ :\ U(r) > 0\}\,.$$

Self-similar blowing-up solutions II

Assume further that U is non-increasing so that

$$\mathcal{P} = \{r > 0 \ : \ U(r) > 0\} = [0, R_s) , \quad R_s \in (0, \infty] .$$

Then

- J is constant in $[0, R_s)$,

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- $J'' = 0$ in $[0, R_s)$,

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Then

- J is constant in $[0, R_s)$,
- $J'' = 0$ in $[0, R_s)$,
- $\xi : r \mapsto U^{(d-2)/d}(r/\mu_d)/\lambda_d$ solves

$$\begin{cases} \xi''(r) + \dfrac{d-1}{r}\, \xi'(r) + \xi(r)^{d/(d-2)} - 1 = 0\,, & r \in [0, \mu_d R_s)\,, \\ \xi'(0) = \xi(\mu_d R_s) = 0\,, & \end{cases}$$

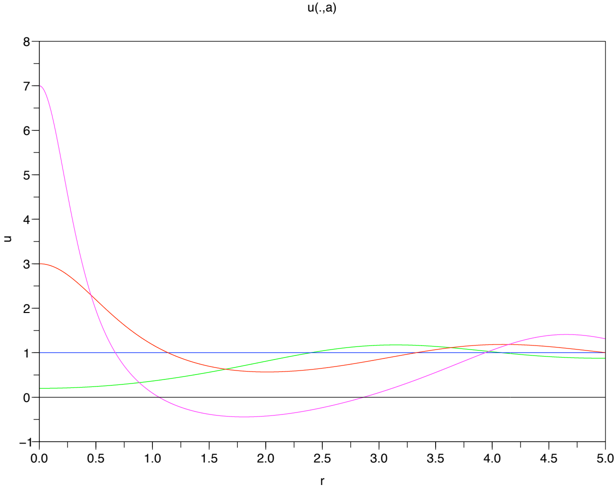
An auxiliary ODE

For $a \in \mathbb{R}$, let $\eta(.,a) \in \mathcal{C}^1([0, r_{\max}(a)))$ denote the maximal solution to the Cauchy problem

$$\begin{cases} \eta''(r,a) + \dfrac{d-1}{r}\,\eta'(r,a) + |\eta(r,a)|^{2/(d-2)}\eta(r,a) - 1 = 0\,, \\ \eta(0,a) = a\,, \quad \eta'(0,a) = 0\,. \end{cases}$$

- $\eta(.,1) \equiv 1$,
- $r_{\max}(a) = \infty$ and $\eta(.,a)$ oscillates around the value 1 with decreasing amplitude.

Oscillating behaviour of $\eta(., a)$ for $a \in \{0.2, 1, 3, 7\}$



Vanishing solutions I

There is a constant $a_c > 1$ such that

- if $a \in (0, a_c)$, then $\eta(r, a) > 0$ for all $r \geq 0$ (and $\eta(r, a) \to 1$ as $r \to \infty$),
- if $a = a_c$, then there is $R(a_c) > 0$ such that $\eta(r, a_c) > 0$ for all $r \in [0, R(a_c))$, and

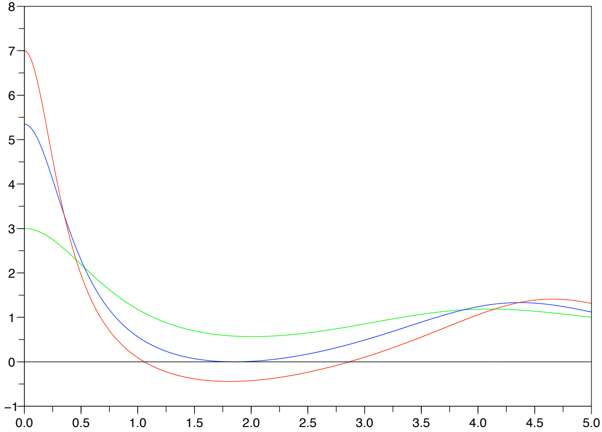
$$\eta(R(a_c), a_c) = \eta'(R(a_c), a_c) = 0\,,$$

- if $a \in (a_c, \infty)$, then there is $R(a) > 0$ such that $\eta(r, a) > 0$ for all $r \in [0, R(a))$, and

$$\eta(R(a), a) = 0\,, \quad \eta'(R(a), a) < 0\,.$$

[Blanchet & L.]

Vanishing solutions II



Self-similar blowing-up solutions III

Given $a \in [a_c, \infty)$ and $T > 0$,

$$U_a(r) := \begin{cases} \lambda_d^{d/(d-2)} \, \eta(\mu_d r, a)^{d/(d-2)} & \text{for} \quad r \in [0, R(a)/\mu_d] \\ 0 \quad \text{for} \quad r \geq R(a)/\mu_d \,, & \end{cases}$$

gives a self-similar blowing-up solution (u_a, φ_a) which satisfies for $t \in [0, T)$:

$$\lim_{a \to \infty} \|u_a(t)\|_1 = \lim_{a \to \infty} \sigma_d \int_0^\infty U_a(r) \, r^{d-1} \, dr = M_c \,,$$

$$\sup_{a \in [a_c, \infty)} \|u_a(t)\|_1 = \sup_{a \in [a_c, \infty)} \sigma_d \int_0^\infty U_a(r) \, r^{d-1} \, dr \leq M_2 < \infty \,.$$