Spatio-Temporal Chaos in Chemotaxis Models

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with K.J. Painter, to appear in Physica D

A chemotaxis model with growth

$u(x,t)$: cell density
$v(x,t)$: chemical signal

$$\begin{aligned} u_t &= \nabla\left(D\nabla u - \chi u \nabla v\right) + ru(1-u)\,, \\ v_t &= \Delta v + u - v\,. \end{aligned} \tag{1}$$

- Consider on $[0, L]$ with homogeneous boundary conditions.

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- Initial conditions

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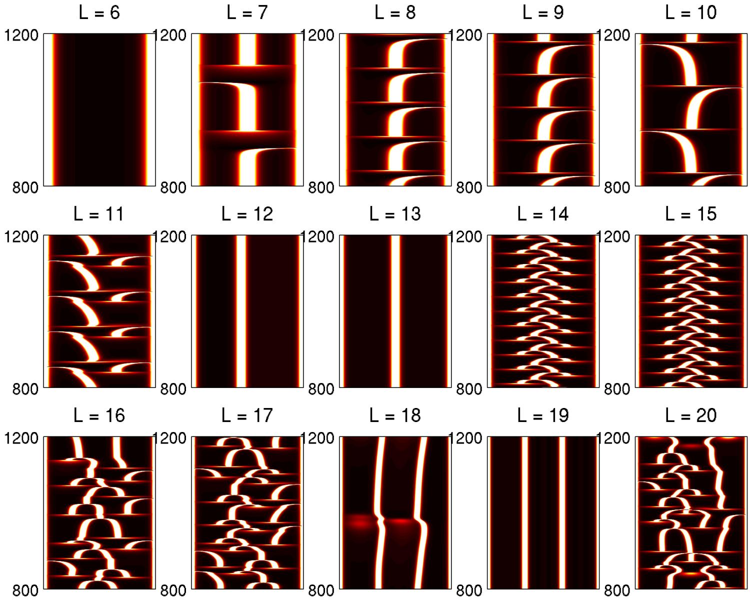
- Fix $\chi = 10$, $D = 1$ and vary L:

Varying Interval Length



Outline

Applications

- Murray et al. 1989-1999:
  - embryonic pattern formation
  - pigmentation of alligator and snake skin

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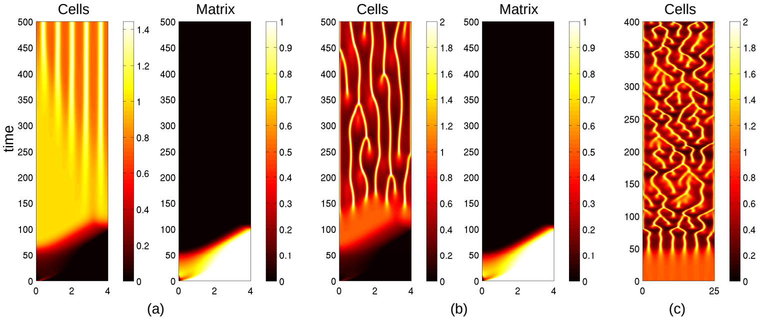
- Orme, Chaplan 1996:
  - Capillary sprouting in tumor angiogenesis

Chaplain, Lolas, Gerisch et al. (2005, 06, 10):
Tumor invasion into extracellular matrix



Mathematically

- Mimura et al. 1996:
  - Allee like nonlinearity $f(u) = u(1-u)(u-a)$.
  - domain separates into regions where the solution
  is close to 1 and regions where the solution is close to 0.
  - they develop an asymptotic theory for movement of the
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  - **Theorem:** $\dim A \geq$ number of unstable modes of $(1,1)$.
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  - **Theorem:** dim $A \geq$ number of unstable modes of $(1,1)$.
  - 2D simulations of periodic or irregular behavior.
- Tello, Winkler 2007, Winkler 2010:
  - existence of unique global solutions for (1) in any space dimension, for smooth enough initial data.

Including volume filling

Painter, H', 2002

$$
\begin{aligned}
u_t &= \nabla \left(D\nabla u - \chi u(2-u)\nabla v \right) + ru(1-u)\,, \\
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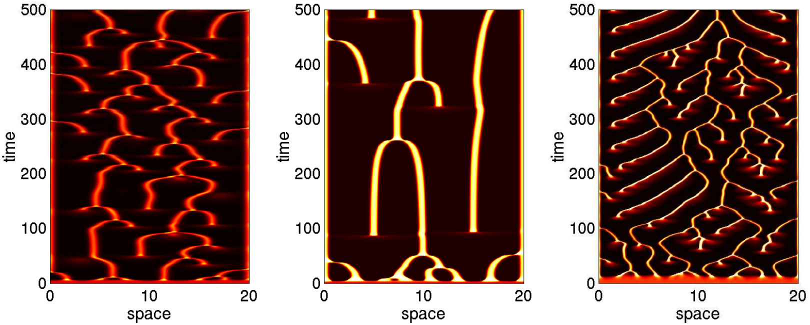
$$

First observation of these irregular patterns.



Including squeezing

Wang, H', 2007: Include a squeezing probability $q(u)$:

$$\begin{aligned} u_t &= \nabla\left(D(q(u) - q'(u)u)\nabla u - \chi u q(u)\nabla v\right) + ru(1-u) \\ v_t &= \Delta v + u - v\,. \end{aligned}$$

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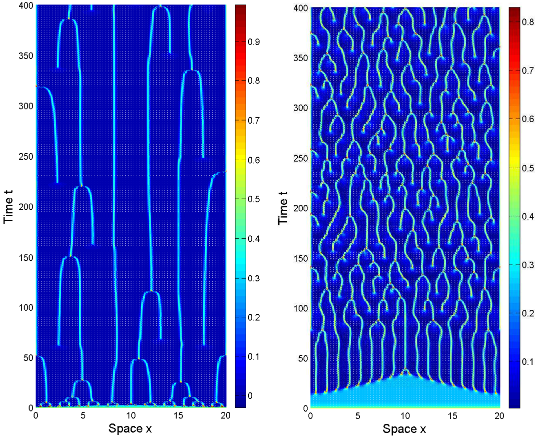
Case 2:

$$q(u) = (2 - u)_+^\alpha, \quad \alpha \leq 1$$

Case 2 leads to a fast diffusion problem, see Wrzosek et al.

Pattern with squeezing $q(u)$



Outline

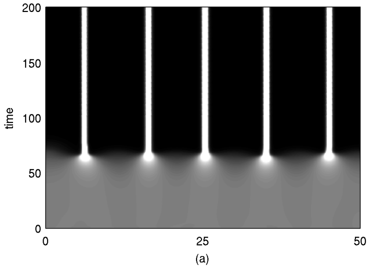
Merging or Coarsening

$r = 0$: the Minimal model:

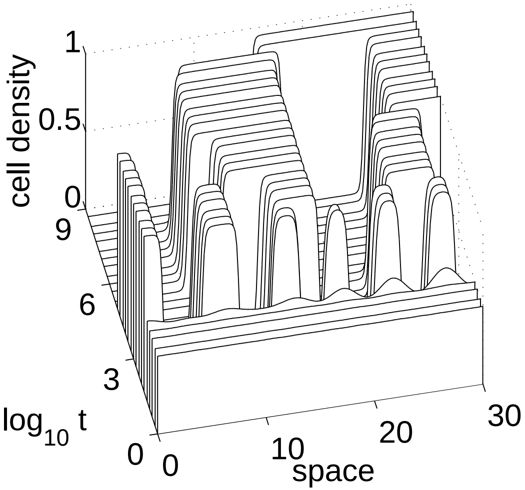
$$\begin{aligned} u_t &= \nabla\left(D\nabla u - \chi u \nabla v\right) , \\ v_t &= \Delta v + u - v \, . \end{aligned}$$

- $1 - D$: global existence and spikes.
- $n - D$, $n > 1$: blow-up possible.

Coarsening dynamics



Volume filling with $r = 0$



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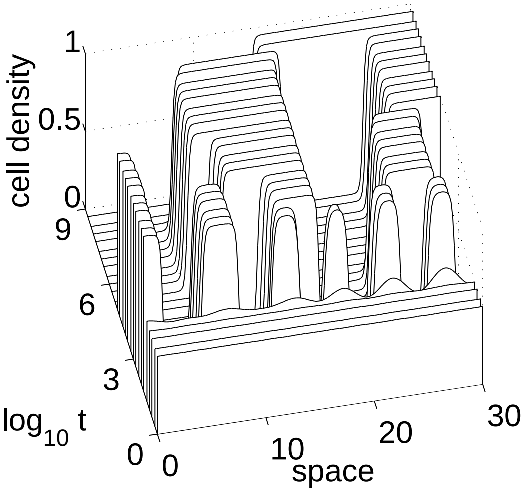
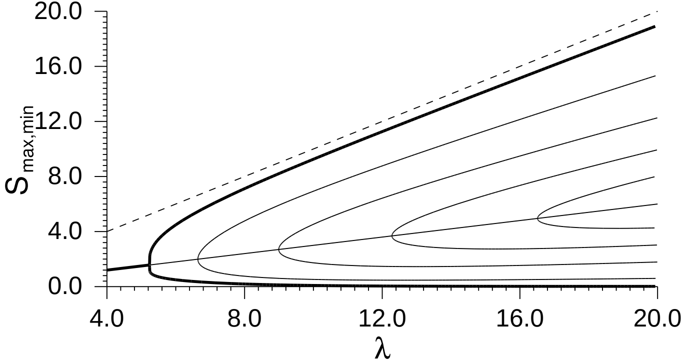


Biologically observed in bacteria aggregations in liquid (Budrene, Berg, 1991, 1995).

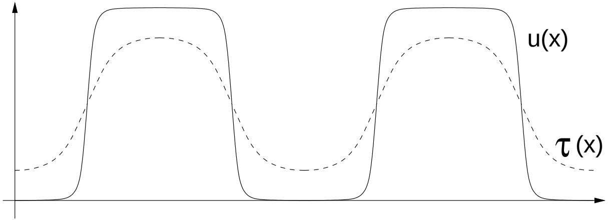
Bifurcation Analysis

- minimal model: Schaaf, 1985
- volume fillineg: Potapov, H', 2005



Movement of transition layers

Dolak, Schmeiser, 2005: Singular perturbation for the movement of transition layers for the volume filling model.



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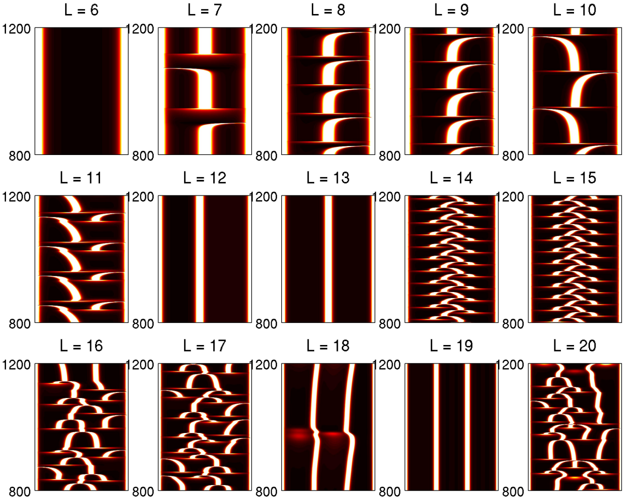
- Nontrivial solutions exist for

$$l > l_e := 2\pi\sqrt{\frac{D}{r}}.$$

Example

$$\chi = 10, \quad D = r = 1, \quad l_e = 2\pi \approx 6.28$$

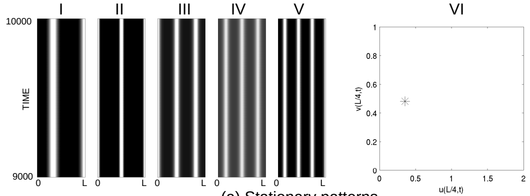


Outline

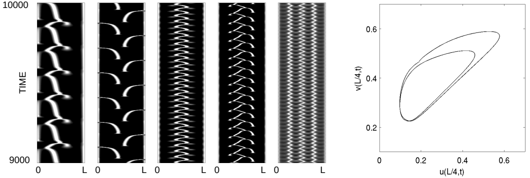
Classification of Solutions

- H-solutions: homogeneous steady states (no pattern)
- S-solutions: stationary patterns
- P-solutions: periodic patterns
- I-solutions: irregular patterns

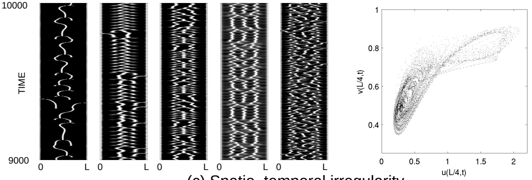
Solution Types



(a) Stationary patterns

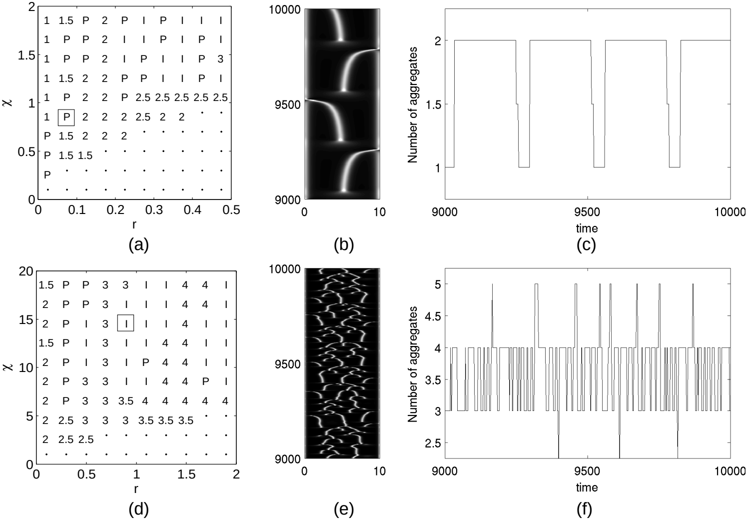


(b) Spatio−temporal periodicity



(c) Spatio−temporal irregularity

Solution Types



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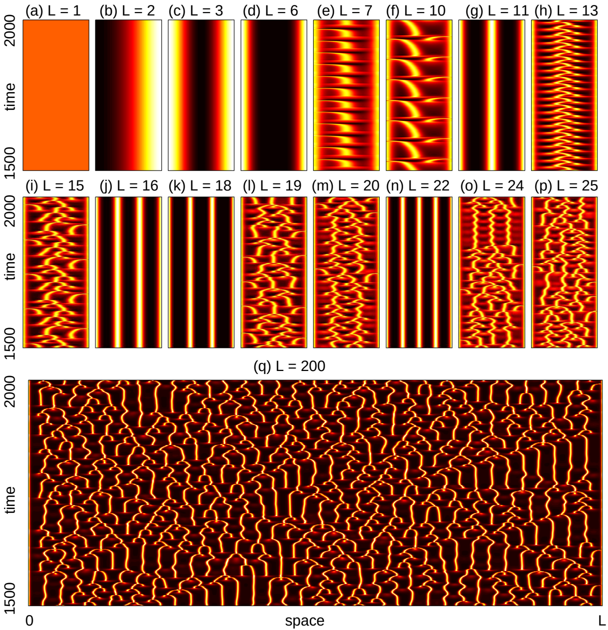

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Lyapunov Exponent $\mathcal{L}$

The Lyapunov exponent is a measure for the sensitive dependence on intitial conditions.

Take two solutions $u(x,t)$, and $u_{pert}(x,t)$, then

$$\mathcal{L}(t) = \log_{10}\left(\frac{1}{L}\int_0^L |u(x,t) - u_{pert}(x,t)|dx\right)$$

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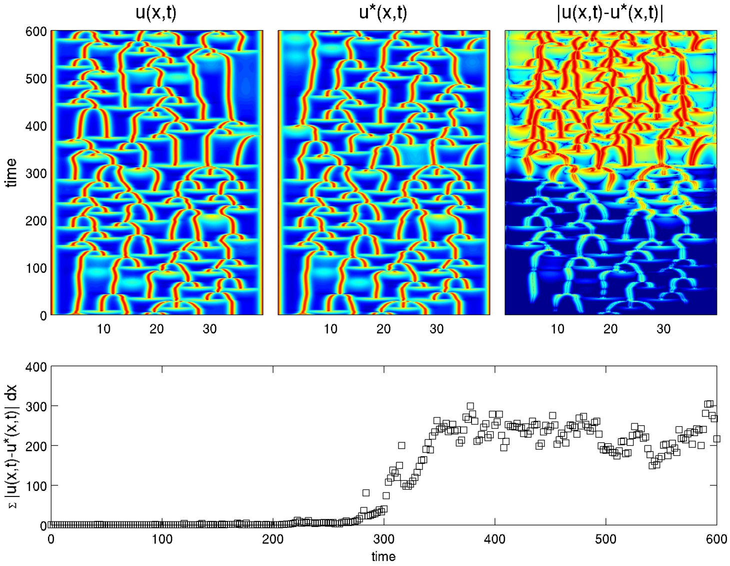
$$\mathcal{L}(t) = \log_{10}\left(\frac{1}{L}\int_0^L |u(x,t) - u_{pert}(x,t)|dx\right)$$

Fit $\mathcal{L}(t)$ by a linear curve with slope λ.

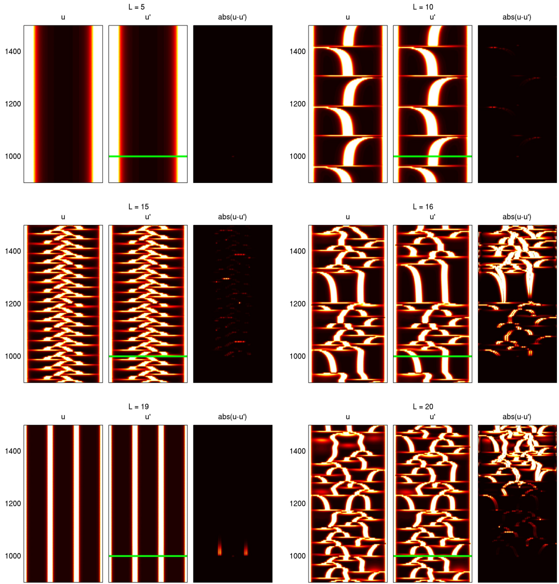
- If $\lambda < 0$, then solutions converge exponentially (stability).
- If $\lambda \approx 0$, then solutions keep their distance (e.g. periodic).
- If $\lambda > 0$, then solutions diverge exponentially.

Comparison



u(x,t)
u*(x,t)
|u(x,t)-u*(x,t)|
time
600
500
400
300
200
100
0
10
20
30
Σ |u(x,t)-u*(x,t)| dx
400
300
200
100
0
0
100
200
300
400
500
600
time

Comparison


L = 5
u
u'
abs(u-u')
1400
1200
1000
L = 10
u
u'
abs(u-u')
1400
1200
1000
L = 15
u
u'
abs(u-u')
1400
1200
1000
L = 16
u
u'
abs(u-u')
1400
1200
1000
L = 19
u
u'
abs(u-u')
1400
1200
1000
L = 20
u
u'
abs(u-u')
1400
1200
1000

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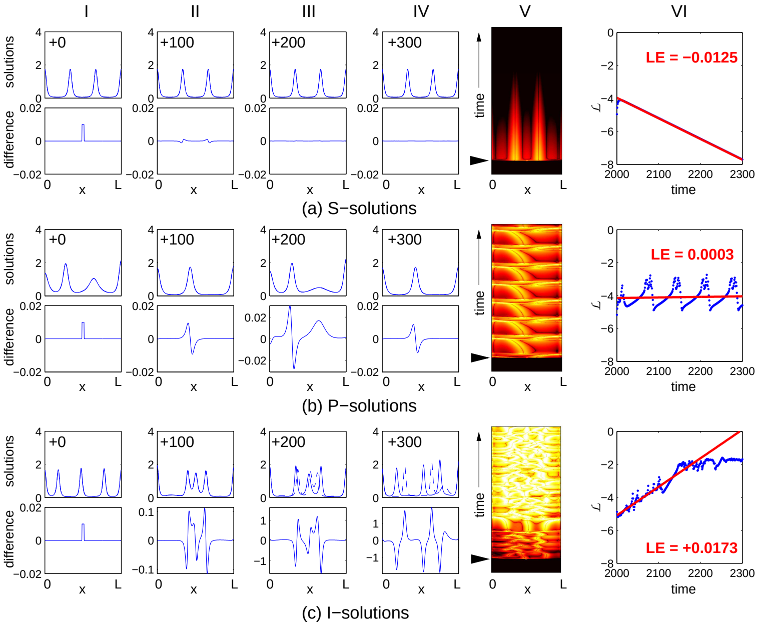
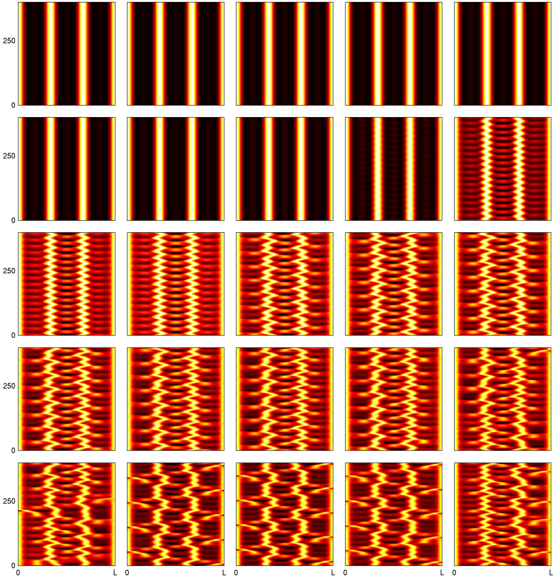
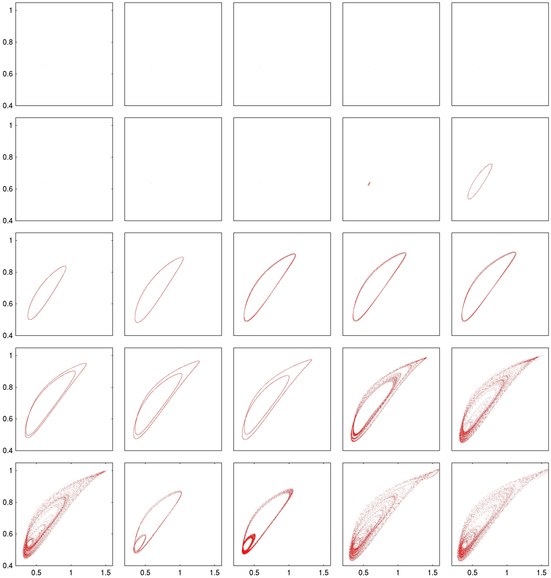


I
II
III
IV
V
VI
solutions
difference
+0
+100
+200
+300
time
x
L
LE = −0.0125
(a) S−solutions
LE = 0.0003
(b) P−solutions
LE = +0.0173
(c) I−solutions

Outline

Period Doubling, increase χ




1
0.8
0.6
0.4
0.5
1
1.5

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Conclusions

- The logistic Keller-Segel model in one spatial dimension shows many characteristics which are typically associated to chaotic dynamics.

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- We gave some evidence of chaotic like behavior (positive Lyapunov exponent, period doubling). We tested many more parameter sets and they show the same behavior. However, a final proof of chaos is still open.
- There are many routes for further investigation:
  - Is there a way to characterize a merging distance?
  - Better understand steady states and their stability
  - Geometric analysis of the attractor?
  - Higher dimensions

Thank You

- Thanks to you
- Thanks to the organizers
- Thanks to my coauthor K.J. Painter
- My work is supported by NSERC and MITACS