Fronts for Periodic KPP Equations

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Content

- Fronts and Asymptotic Spreading
- Variational Formulation for the Speed
- Qualitative Consequences
- Generalizations

KPP in periodic media

Consider the KPP equation in a periodic medium:

$$\text{(KPP)} \qquad \begin{array}{c} \partial_t u(t,x) = \nabla(A(x)\nabla u(t,x)) + f(x,u(t,x)) \\ t > 0, \quad x \in R^n \qquad u(0,x) = u_0(x) \geq 0 \end{array}$$

A, b, f are C^1 and 1-periodic in each direction x_i
$A(x) \in R^{n \times n}$ positive definite, not necessarily symmetric
$f(x,0) = f(x,1) = 0$
$f(x,u)/u$ is decreasing for $u > 0$
$\partial_u f(x,0) =: \mu(x) > 0$ for $0 < u < 1$

e.g. $f(x,u) = \mu(x)u(1-u)$

Classical case $n = 1$: $u_t = Au_{xx} + \mu u(1-u)$

Travelling waves

A travelling wave solution in direction $e \in R^n, |e| = 1$ with speed c satisfies with $\xi = ct + e \cdot x \in R$:

$$u(t,x) = U(\xi, x), \quad U(\xi, x) \quad \text{is periodic w.r.t.} \quad x_i$$

$$\text{(TW)} \qquad \begin{gathered} c\partial_\xi U(\xi,x) = (\nabla + e\partial_\xi)(A(x)(\nabla + e\partial_\xi)U(\xi,x)) + f(x, U(\xi,x)) \\ U(-\infty, x) = 0, \quad U(\infty, x) = 1 \end{gathered}$$

We need the following linear operator.
With $\mu(x) = \partial_u f(x, 0)$ and $\lambda \geq 0$ let

$$(L_\lambda \phi)(x) = \nabla(A(x)\nabla\phi(x)) + \mu(x)\phi(x)$$

$$\phi(x)e^{-\lambda e \cdot x} \quad \text{is 1-periodic}$$

Due to the last condition L_λ is not selfadjoint !
Let $k(\lambda)$ be the principal eigenvalue with corresponding eigenfunction $\phi(x) > 0$.

Theorem (Berestycki, Hamel '02)

A travelling wave exist iff

$$\boxed{c \geq c(e) := \min_{\lambda>0} \frac{k(\lambda)}{\lambda}}$$

Explanation: Let $(U(\xi, x), c)$ be a travelling wave.

$$U(\xi, x) \sim e^{\lambda\xi} v(x), \quad \xi \to -\infty \quad \lambda > 0, \quad v \text{ periodic}$$

Then $\phi(x) := e^{\lambda e \cdot x} v(x) > 0$ satisfies

$$L\phi(x) = \lambda c \phi(x)$$

Hence $c = \frac{k(\lambda)}{\lambda}$. Construction of upper and lower solutions shows, that every c in the range of $\frac{k(\lambda)}{\lambda}$ occurs.
How does $c(e)$ depend on A, μ, e ?

Asymptotic Spreading

Consider an initial value $0 \le u_0(x) \le 1$ for KPP with compact, nonempty support and let $u(t,x)$ be the solution.

Theorem (Weinberger '02, Beresytcki, Hamel, Nadin '08))

For

$$\boxed{w(e) = \min_{f\cdot e>0} \frac{c(f)}{f\cdot e}}$$

the following holds

$$\lim_{t\to\infty} u(t, x + wte) = 0, \quad x \in R^n, w > w(e)$$
$$\lim_{t\to\infty} u(t, x + wte) = 1, \quad x \in R^n, 0 \le w < w(e)$$

$w(e)$ is called the spreading speed in direction e.

We will see that the formula for $w(e)$ can be inverted:

$$c(e) = \sup_f (e\cdot f)w(f)$$

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Variational Formulation of the Speed

There exists a variational formulation of $k(\lambda)$ using the maximum principle:

$$k(\lambda) = \inf_{0<\phi e^{-\lambda e \cdot x} \in C^2_{per}} \sup_x \frac{(L_\lambda \phi)(x)}{\phi(x)}$$

Difficult to use for qualitative analysis. Seek an integral variational principle.
The adjoint operator of L_λ is

$$(L^*_\lambda \psi) = \nabla(A(x)^T \nabla \psi(x)) + \mu(x)\psi(x)$$

$$\psi(x)e^{\lambda e \cdot x} \quad \text{is 1-periodic}$$

Observe that $k(\lambda)$ is a critical value of:

$$\int_C (-\nabla\psi A \nabla\phi + \mu\phi\psi)\, dx \to \text{critical} \quad \text{with constraint} \quad \int_C \phi\psi\, dx = 1$$

Goal: Transform s.t. convex and concave part are seperated:

$$\phi(x) = \alpha(x)e^{\lambda\rho(x)}, \quad \psi(x) = \alpha(x)e^{-\lambda\rho(x)}$$
$$\alpha(x),\ \rho(x) - e \cdot x \text{ are 1-periodic}$$

$$G(\alpha, \rho, \lambda) := \int_C \Big(-(\nabla\alpha - \lambda\alpha\nabla\rho)A(\nabla\alpha + \lambda\alpha\nabla\rho) + \mu\alpha^2 \Big)$$

$$= \int_C \Big(\lambda^2\alpha^2\nabla\rho A_s \nabla\rho + 2\lambda\alpha\nabla\rho A_a \nabla\alpha - \nabla\alpha A_s \nabla\alpha + \mu\alpha^2 \Big)$$

$$\to \text{critical point} \quad \text{with constriant } \int_C \alpha^2 = 1$$

where $A_s := \frac{1}{2}(A + A^T), \quad A_a := \frac{1}{2}(A - A^T)$

Now the following saddle point property for $k(\lambda)$ follows:

Theorem (Donsker-Varadhan '76, Holland '78)

$$k(\lambda) = \sup_{\alpha} \inf_{\rho} G(\alpha, \rho, \lambda) = \inf_{\rho} \sup_{\alpha} G(\alpha, \rho, \lambda)$$

$$\alpha, \rho - e \cdot x \in C^1_{per}(R^n), \alpha > 0, \int_C \alpha^2 \, dx = 1$$

The Euler Lagrange equations are:

$$\nabla(A_s \nabla \alpha) + \lambda^2 \alpha \nabla \rho A_s \nabla \rho + \lambda \alpha \nabla \cdot A_a \nabla \rho + \mu \alpha = k(\lambda) \alpha$$

$$\lambda \nabla(\alpha^2 A_s \nabla \rho) + \nabla \cdot (A_a \nabla \alpha^2) = 0$$

This is a selfadjoint problem for α coupled to a Poisson equation for ρ.

Saddle point principle for the speed

Goal: Eliminate λ in $c(e) = \inf\limits_{\lambda>0} \frac{k(\lambda)}{\lambda}$.

Idea: Consider

$$J(\alpha, \rho) := \inf_{\lambda>0} \frac{G(\alpha, \rho, \lambda)}{\lambda}$$

$$= 2\left(\int_C (\mu\alpha^2 - \nabla\alpha A_s \nabla\alpha) \int_C \alpha^2 \nabla\rho A_s \nabla\rho\right)^{1/2} + \int_C \nabla\rho A_a \nabla\alpha^2,$$

if $\int_C (\mu\alpha^2 - \nabla\alpha A_s \nabla\alpha) \geq 0$ and $-\infty$ otherwise.

Let $(\alpha_\lambda, \rho_\lambda)$ be the saddle point of $G(\alpha, \rho, \lambda)$. and let $\lambda^* > 0$ be the unique minimizer of $k(\lambda)/\lambda$.

Have to show, that $J(\alpha_{\lambda^*}, \rho_{\lambda^*}) > -\infty$ holds.

This follows from

$$0 = \frac{d}{d\lambda}\frac{k(\lambda)}{\lambda}\Big|_{\lambda^*} = \int_C \alpha_{\lambda^*}^2 \nabla\rho_{\lambda^*} A_s \nabla\rho_{\lambda^*} - \frac{1}{\lambda^{*2}} \int_C (\mu\alpha_{\lambda^*}^2 - \nabla\alpha_{\lambda^*} A_s \nabla\alpha_{\lambda^*})$$

Since J is convex, w.r.t. λ, ρ and concave w.r.t. α we obtain:

Theorem: (Saddle point principle)

$$c(e) = \sup_{\alpha} \inf_{\rho} J(\alpha, \rho) = \inf_{\rho} \sup_{\alpha} J(\alpha, \rho)$$

$$\alpha, \rho - e \cdot x \in C^1_{per}(R^n), \alpha > 0, \int_C \alpha^2 \, dx = 1$$

Dual variational principle

Goal: Dualize the infimum into a supremum:

For simplicity assume $A_a = 0$.
Let $v(x)$ be a nontrivial periodic divergence free C^1 vector field.

$$\left(\int_C e \cdot v\right)^2 = \left(\int_C v\nabla\rho\right)^2 \leq \int_C \frac{vA^{-1}v}{\alpha^2} \int_C \alpha^2 \nabla\rho A \nabla\rho$$

and equality holds iff $v = \gamma\alpha^2 A\nabla\rho$ for some $\gamma \in R$. This is the Euler Lagrange equation for ρ.

Theorem (Maximization principle)

$$c(e)^2/4 = \sup_{\alpha,v} \frac{\left(\int_C v \cdot e\right)^2 \int_C (\mu\alpha^2 - \nabla\alpha A \nabla\alpha)}{\int_C \frac{vA^{-1}v}{\alpha^2}}$$

$$\alpha, v \in C^1_{per}(R^n), \alpha > 0, \int_C \alpha^2\, dx = 1, \nabla \cdot v = 0, v \neq 0$$

For $n = 1$ this simplyfies to:

$$c(e)^2/4 = \sup_\alpha \frac{\int_0^1 (\mu\alpha^2 - A\alpha'^2)}{\int_0^1 \frac{1}{A\alpha^2}}$$

Qualitative Consequences

Smooth Dependence

The principle eigenvalue $k(\lambda)$ is simple. Hence it depends smoothly on parameters.
One can show that

$$\frac{d}{d\lambda}\frac{k(\lambda)}{\lambda}\Big|_{\lambda^*} = 0 \quad \text{implies} \quad \frac{d^2}{d^2\lambda}\frac{k(\lambda)}{\lambda}\Big|_{\lambda^*} > 0.$$

Hence the minimal speed also depends smoothly on parameters.
Suppose that the functional J depends on a parameter t and (α_t, ρ_t) is the saddle point. Then

$$\frac{d}{dt}c(t) = \partial_t J(\alpha_t, \rho_t, t)$$

Dependence on the direction

Extend the definition of $c(e)$ and $w(e)$ as a homogeneous function to all $e \in R^n$ of degree 1 and degree -1 respectively.

Theorem

$c(e)$ and $w(e)$ satisfy:

1. $c(e), w(e) > 0, \quad e \neq 0$
2. $c(se) = sc(e), \quad s \geq 0$
3. $c(e+f) \leq c(e) + c(f)$
4. *$c(e)$ is the support function of a convex body.*
5. *$1/w(e)$ is the support function of the polar convex body, i.e.* $1/w(e) = \sup_{f \neq 0} \frac{e \cdot f}{c(f)}$
6. $c(e) = \sup_{f \neq 0} e \cdot f \; w(f)$

Dependence on the period

Consider the symmetric case $A_a = 0$.
For $L > 0$ replace in (KPP) $A(x)$ and $f(x, u)$ by $A(x/L)$ and $f(x/L, u)$. A rescaling of x gives:

$$c(L)^2/4 = \sup_\alpha \inf_\rho \int_C (\mu\alpha^2 - \frac{1}{L^2}\nabla\alpha A^s \nabla\alpha) \int_C \alpha^2 \nabla\rho A^s \nabla\rho$$

Let (α_L, ρ_L) be the saddle point of J.

Theorem

$$\frac{d}{dL}c(L) = \frac{2}{L^3}\int_C \nabla\alpha_L A \nabla\alpha_L \geq 0$$

holds. Equality holds for some $L_0 > 0$ or equvalently for all $L >$ iff

$$\frac{\nabla\rho A \nabla\rho}{\int_C \nabla\rho A \nabla\rho} + \frac{\mu}{\int_C \mu} = 2$$

where ρ solves

$$\nabla(A\nabla\rho) = 0, \quad \rho(x) - e\cdot x \quad \text{is 1-periodic}$$

Remark: There exist nonconstant media, s.t. the speed is independent of the period and the direction.
(test case for numerics)

Homogenization

For simplicity consider the case $A_a = 0$ only.

Theorem

The following limit exists

$$\lim_{L \to 0} c(L) =: c_0 = 2\sqrt{\overline{\mu} e A^h e}$$

and equals the minimal speed of the homogenized equation.

Proof: Choosing $\alpha = 1$ in the variational principle gives:

$$c(L)^2/4 \geq \int_C \mu \inf_\rho \int_C \nabla \rho A \nabla \rho = 2\sqrt{\overline{\mu} e A^h e}$$

Let (α_L, ρ_L) be the critical point of $J(\alpha, \rho, L)$. We know:

$$\int_C (\mu \alpha_L^2 - \frac{1}{L^2} \nabla \alpha_L A \nabla \alpha_L) \geq 0$$

This implies $\alpha_L \to 1$ in H^1_{per}. We have for every ρ:

$$c(L)^2/4 = J(\alpha_L, \rho_L) \leq J(\alpha_L, \rho)$$

$$\leq \int_C \mu \alpha_L^2 \int_C \alpha_L^2 \nabla \rho A \nabla \rho \to \int_C \mu \int_C \nabla \rho A \nabla \rho$$

Minimizing over ρ completes the proof.

Large period limit

Theorem

Suppose $A_a = 0$. *Then*

$$\lim_{L\to\infty} c(L) = c_\infty = \sup_{\alpha, v} \frac{\left(\int_C v \cdot e\right)^2 \int_C \mu\alpha^2}{\int_C \frac{vA^{-1}v}{\alpha^2}}$$

In 1-D we have $$c_\infty = \sup_\alpha \int_0^1 \mu\alpha^2 \left(\int_0^1 \frac{1}{A\alpha^2}\right)^{-1}$$

The supremum can be evaluated, conjectured by Hamel, Fayard, Roques '10.

Theorem

Let $\mu^* = \sup \mu(x)$ *and assume*

$$\mu^* \int_0^1 (a(\mu^* - \mu))^{-1/2} \geq 2 \int_0^1 \left(\frac{\mu^* - \mu}{a}\right)^{1/2}$$

Then

$$c_\infty = \inf_{\eta > \mu_+} \frac{\eta}{\int_0^1 \sqrt{\frac{\eta - \mu}{a}}}$$

This holds e.g. if μ *is piecewise* C^2*. If* μ *is constant this gives*

$$c_\infty = 2\sqrt{\mu} \left(\int_0^1 \frac{1}{\sqrt{a}}\right)^{-1}$$

Proof: The lower is obtained by choosing $\alpha = (a(\eta - \mu))^{1/4}$ in the variational principle and maximizing over η.
The upper bound follows from

$$\left(\int_C \sqrt{\frac{\psi - \mu}{a}}\right)^2 \leq \int \alpha^2(\eta - \mu) \int_C \frac{1}{a\alpha^2}$$

and minimizing over η. Equality of these bounds holds iff the condition above holds.

Dependence on μ

Theorem

If A is constant, then $\mu \mapsto \gamma(\mu) := c(\mu)^2$ is increasing and convex.
In particular if $\mu_2(x) = \mu_1(x + a)$ holds, then

$$c(\frac{\mu_1 + \mu_2}{2}) \leq c(\mu_1)$$

follows, i.e. fragmentation decreases the minimal speed.

Proof: Let (α, v) be the maximizer of $J^*(\alpha, v, (\mu_1 + \mu_2)/2)$ where J^* is the functional of the dual variational principle.

$$c(\frac{\mu_1 + \mu_2}{2})^2 = J(\alpha, v, \frac{\mu_1 + \mu_2}{2})^2$$

$$= \frac{1}{2}J(\alpha, v, \mu_1)^2 + \frac{1}{2}J(\alpha, v, \mu_2)^2 \leq \frac{c(\mu_1)^2 + c(\mu_2)^2}{2}$$

Optimization over μ

Theorem

Suppose that A is constant and let $\overline{\mu} > 0$ be given. Then

$$\sup_{\int_0^1 \mu = \overline{\mu}} c(\mu) = c(\overline{\mu}\delta_{x_0})$$

where δ_{x_0} is the Dirac functional at x_0.

In 1-D this is finite.

Proof: The convexity implies for every periodic ν with integral one:

$$c(\mu * \nu) \leq c(\mu)$$

The variational formula can be extended to measures. Choosing $\mu = \overline{\mu}\delta_{x_0}$ completes the proof.

Related rearrangement results: Nadim (2010)

Generalizations

- Less regular data A, f, μ, e.g. μ could be a characterstic function.
- $f(x, u) < 0$ for $u > M$ instead of $f(x, 1) = 0$. Then $U^+(x)$ is a periodic function instead of $U^+ = 1$.
- Instead of $\mu(x) \geq 0$, assume $k(0) > 0$, e.g. $\int_C \mu > 0$ is enough.
- Perforated domains.
- Space and time periodic data.