Highly nonlinear large-competition limits of elliptic systems

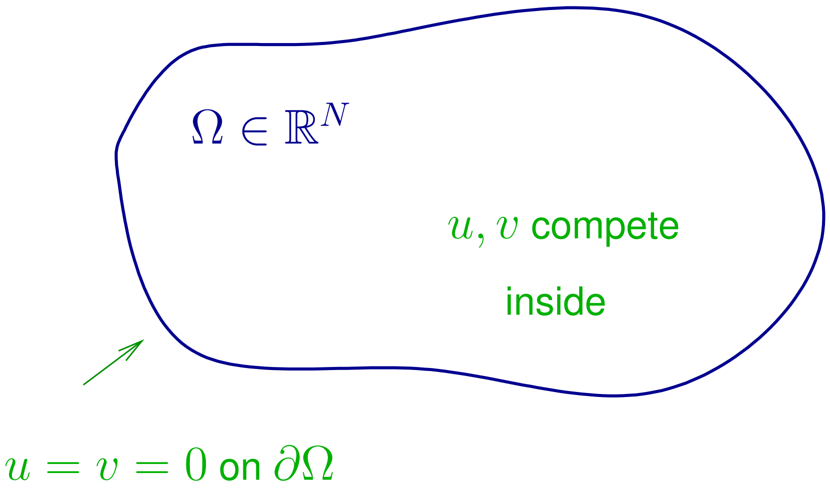
Elaine Crooks
Swansea

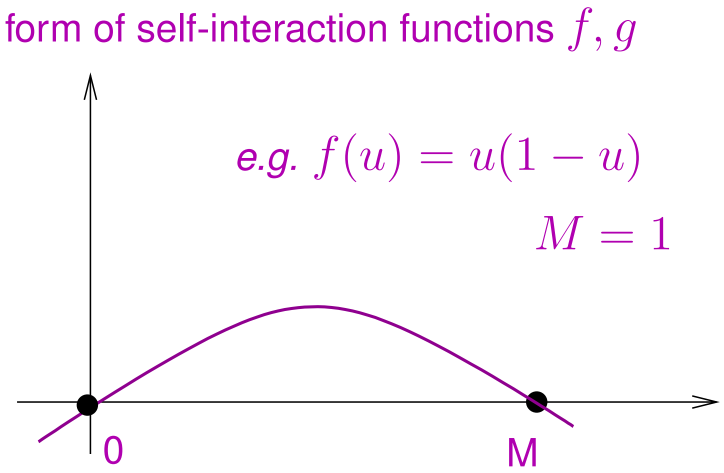
Joint work with Norman Dancer, Sydney.

- Parabolic systems of form

$$\begin{aligned} u_t &= d_1\,\Delta u + f(u) - kuv, \quad x \in \Omega, \quad t \geq 0, \\ v_t &= d_2\,\Delta v + g(v) - kuv, \quad x \in \Omega, \quad t \geq 0, \\ u(x) &= v(x) = 0, \qquad\qquad\qquad x \in \partial\Omega \end{aligned}$$

model populations of densities u, v that compete in $\Omega \in \mathbb{R}^N$





- Elliptic systems of form

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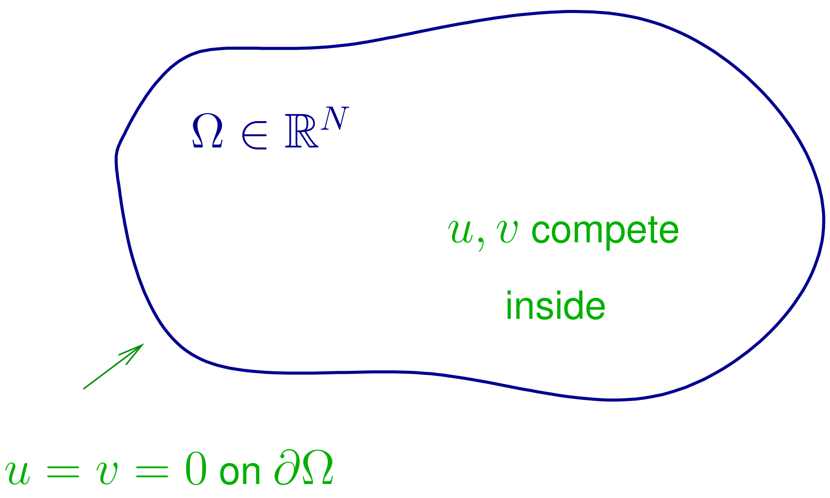

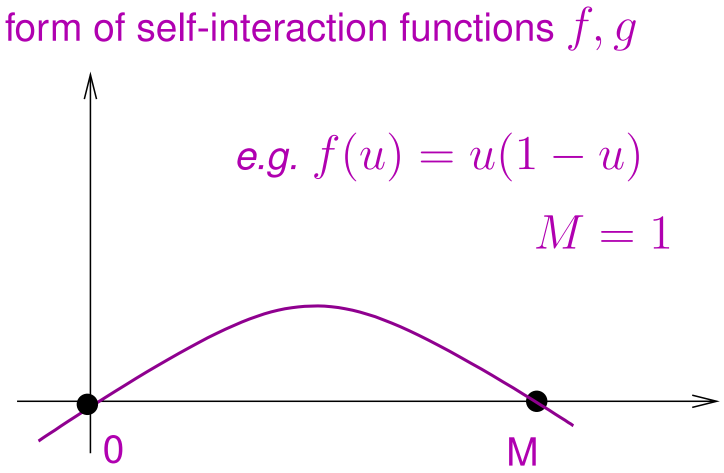

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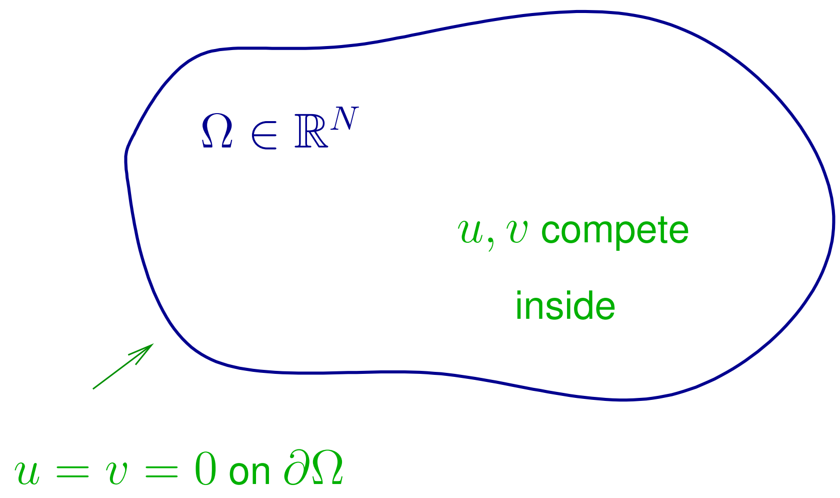

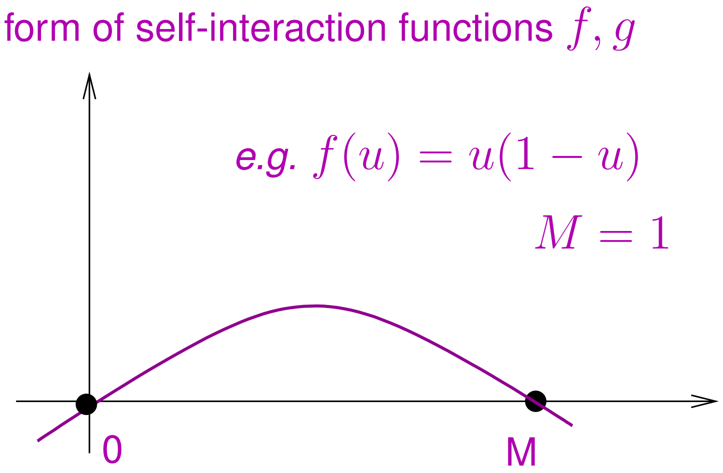

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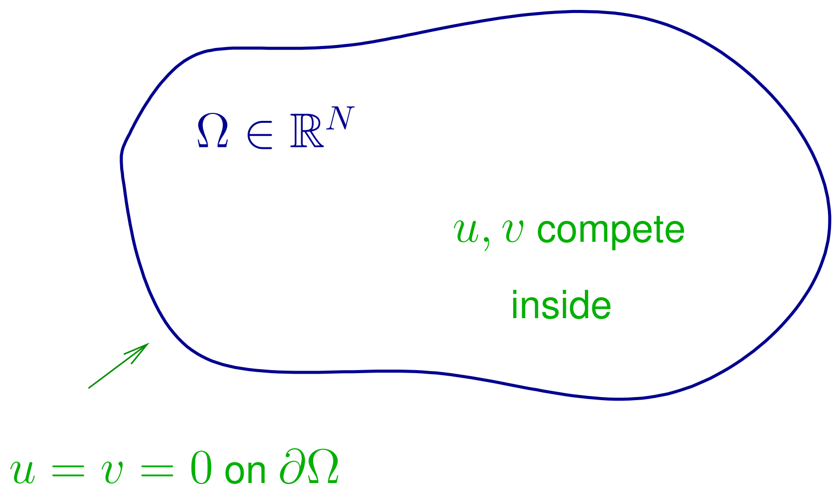

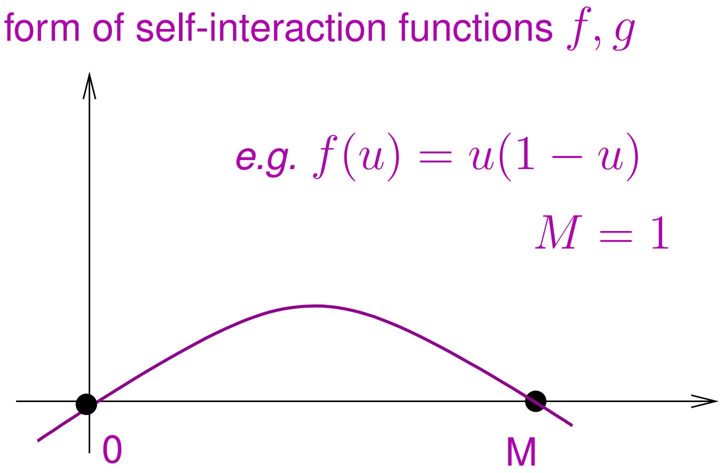

• densities non-negative $\Rightarrow u \geq 0, v \geq 0$

• competition parameter $k > 0$

- Interest in the large-competition ($k \longrightarrow \infty$) limit comes from

(i) the k-dependent system is difficult to analyse; for example, it is not in general the Euler-Lagrange equations of an energy functional variational, whereas the limit problem is a scalar equation

(ii) the $k \longrightarrow \infty$ limit is linked to

- spatial segregation in population dynamics
- phase separation in, for example, Bose-Einstein condensates

both of which are of importance in applications

Large-competition limit $k \to \infty$ of solutions (u^k, v^k)

Seminal ref: Dancer and Du, Journal Diff. Eqs. 114 (1994) 434-475

- (u^k, v^k) converge to the positive and negative parts resp. of a limit function w satisfying the scalar equation

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- Key ingredients:

(i) the linear combination $w^k := u^k - v^k$ satisfies

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which does not depend explicitly on k $\Rightarrow$ good bounds for w^k independent of k

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(ii) u^k, v^k converge in some sense as $k \to \infty$

(iii) u^k and v^k segregate , since $k\ u^k v^k$ bounded $\Rightarrow$ $u^k v^k \to 0$ as $k \to \infty$

and

$$\left.\begin{array}{l} uv = 0 \quad a.e. \\ u, v \geq 0 \\ w = u - v \end{array}\right\} \Rightarrow \begin{array}{l} u = w^+ \quad a.e. \\ v = -w^- \end{array}$$

- Note: there are two aspects to large-interaction limit problem

(i) to show that (u^k, v^k) converges as $k \to \infty$ to a solution of the limit problem

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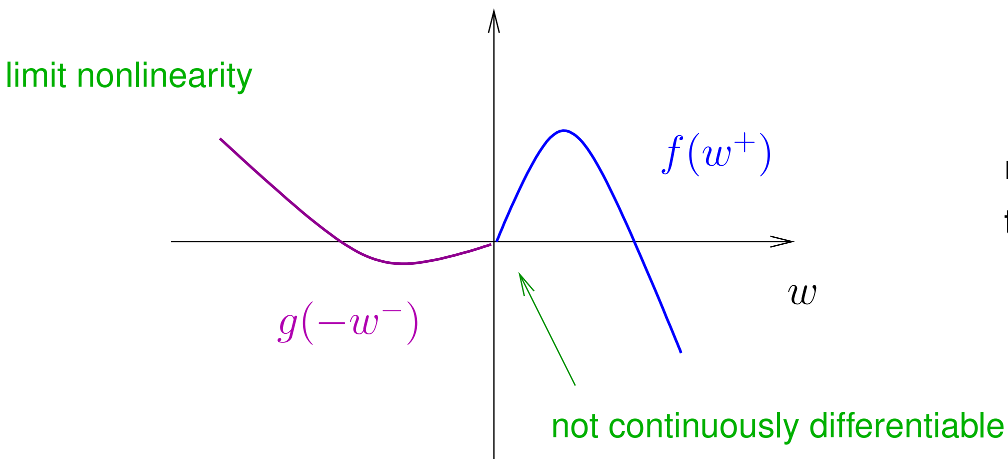

methods: *e.g.* degree theory in cones

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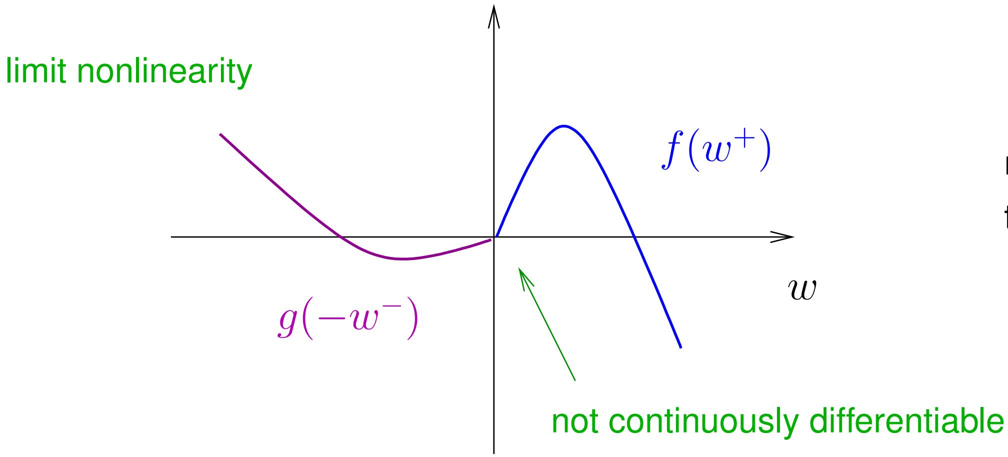

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Focus on (i) here

- Key property that allows cancellation of competition terms "kuv" is that the same term occurs in both equations

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- Question: to what types of system with different competition terms in the two equations can this "cancellation" approach be extended?

Our two prototype classes of system

1. Non-autonomous system

$$\begin{aligned} \Delta u + f(u) - \alpha_1(x)kuv &= 0, \quad x \in \Omega, \\ \Delta v + g(v) - \alpha_2(x)kuv &= 0, \quad x \in \Omega, \\ u(x) = v(x) &= 0, \quad x \in \partial\Omega \end{aligned}$$

where $\alpha_1, \alpha_2 \in C^2(\Omega, [\alpha_0, \infty))$ for some constant $\alpha_0 > 0$

2. “Nonlinear” competition system

$$\begin{aligned} \Delta u + f(u) - kuv &= 0, \quad x \in \Omega, \\ \Delta v + g(v) - k(1+u^2)uv &= 0, \quad x \in \Omega, \\ u(x) = v(x) &= 0, \quad x \in \partial\Omega \end{aligned}$$

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Key feature: each competition term is of form "kuv" multiplied by a positive function that is bounded below by a strictly positive constant

- systems (1.) and (2.) are special cases of the general system

$$\begin{aligned} \Delta u + f(u) - k\alpha_1(x)\gamma_1(v)uv &= 0, \quad x \in \Omega, \\ \Delta v + g(v) - k\alpha_2(x)\gamma_2(u)uv &= 0, \quad x \in \Omega \end{aligned}$$

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• the system

$$\begin{aligned}\Delta u + f(u) - kuv^2 &= 0, \quad x \in \Omega,\\ \Delta v + g(v) - ku^2v &= 0, \quad x \in \Omega\end{aligned}$$

is unfortunately excluded from our framework; it

- arises in modelling phase separation in Bose-Einstein condensates
- is variational, being the Euler-Lagrange equations of a functional of form

$$J(u,v) = \int_\Omega \frac{1}{2}(|\nabla u|^2 + |\nabla v|^2) - F(u) - G(v) + \frac{1}{2}ku^2v^2\, dx$$

(references: Conti, Terracini, Verzini, Squassina, ...)

Preliminary "cancellation" calculations

System 1.

Given a solution (u^k, v^k) of system 1,

$$\begin{aligned} \Delta u + f(u) - \alpha_1(x)kuv &= 0, \quad x \in \Omega, \\ \Delta v + g(v) - \alpha_2(x)kuv &= 0, \quad x \in \Omega, \\ u(x) = v(x) &= 0, \quad x \in \partial\Omega, \end{aligned}$$

define

$$w^k = \alpha_2 u^k - \alpha_1 v^k.$$

Then w^k satisfies the equation

$$\begin{aligned} \Delta w^k &= 2\nabla\alpha_2 \cdot \nabla u^k - 2\nabla\alpha_1 \cdot \nabla v^k \\ &\quad + u^k\Delta\alpha_2 - v^k\Delta\alpha_1 - \alpha_2 f(u^k) + \alpha_1 g(v^k) \quad \text{in } \Omega, \\ w^k &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

because

$$\Delta w^k = \alpha_2\Delta u^k - \alpha_1\Delta v^k + 2\nabla\alpha_2 \cdot \nabla u^k - 2\nabla\alpha_1 \cdot \nabla v^k + u^k\Delta\alpha_2 - v^k\Delta\alpha_1$$

System 2.

Given a solution (u^k, v^k) of system 2,

$$\begin{aligned} \Delta u + f(u) - kuv &= 0, \quad x \in \Omega, \\ \Delta v + g(v) - k(1+u^2)uv &= 0, \quad x \in \Omega, \\ u(x) = v(x) &= 0, \quad x \in \partial\Omega, \end{aligned}$$

define

$$y^k = u^k + \frac{(u^k)^3}{3} - v^k.$$

Then y^k satisfies the equation

$$\begin{aligned} \Delta y^k &= 2u^k |\nabla u^k|^2 - (1 + (u^k)^2) f(u^k) + g(v^k) \quad \text{in } \Omega, \\ y^k &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

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Note:

(i) no terms involving second derivatives of u^k or v^k in eqn for y^k

(ii) $u \mapsto u + \frac{u^3}{3}$ is invertible, since $\frac{d}{du}\left(u + \frac{u^3}{3}\right) = 1 + u^2 \geq 1$ for all u

Why? - form of system is

$$\begin{aligned} \Delta u + f(u) - kuv &= 0, \quad x \in \Omega, \\ \Delta v + g(v) - k\,\gamma(u)\,uv &= 0, \quad x \in \Omega \end{aligned}$$

where

$$\gamma(u) = 1 + u^2 \geq \gamma_0 > 0$$

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and

$$\nabla y^k = \gamma(u^k)\,\nabla u^k - \nabla v^k$$

$$\begin{aligned}\Rightarrow \qquad \Delta y^k &= \gamma'(u^k)|\nabla u^k|^2 + \gamma(u^k)\Delta u^k - \Delta v^k \\ &= \gamma'(u^k)|\nabla u^k|^2 - \gamma(u^k)f(u^k) + g(v^k)\end{aligned}$$

- Also,

$$\Gamma(u) = \int_0^u \gamma(s)\, ds > 0 \text{ if } u > 0$$

and

$$\Gamma(0) = 0$$

- thus

$$\left. \begin{array}{c} uv = 0 \ \ a.e. \\ u, v \geq 0 \\ y = \Gamma(u) - v \end{array} \right\} \Rightarrow \begin{array}{c} y^+ = \Gamma(u) \\ y^- = -v \end{array} \Rightarrow \begin{array}{c} \Gamma^{-1}(y^+) = u \\ y^- = v \end{array}$$

i.e. segregation of u and v implies that u and v can be written in terms of the positive and negative parts of y

Theorem Given a sequence of non-negative solutions (u^k, v^k) of either system 1 or 2, there exist subsequences $\{u^{k_n}\}$, $\{v^{k_n}\}$ and non-negative functions $u, v \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$ such that

- $u^{k_n} \to u, \quad v^{k_n} \to v$ in $W_0^{1,2}(\Omega)$ as $k_n \to \infty$;
- $uv = 0 \ \ a.e.$ in Ω.

• In the case of system 1, the function $w := \alpha_2 u - \alpha_1 v$ is such that $w^+ = \alpha_2 u$, $w^- = -\alpha_1 v$ and w is a weak solution of the equation

$$\Delta w = 2\nabla\alpha_2 \cdot \nabla(\alpha_2^{-1} w^+) - 2\nabla\alpha_1 \cdot \nabla(-\alpha_1^{-1} w^-)$$
$$+\alpha_2^{-1} w^+ \Delta\alpha_2 - \alpha_2 f(\alpha_2^{-1} w^+) + \alpha_1^{-1} w^- \Delta\alpha_1 + \alpha_1 g(-\alpha_1 w^-) \quad \text{in } \Omega,$$
$$w = 0 \quad \text{on } \partial\Omega$$

• In the case of system 2, the function $y := \Gamma(u) - v$, where $\Gamma(u) := u + \frac{u^3}{3}$, is such that $y^+ = \Gamma(u)$, $y^- = -v$ and y is a weak solution of the equation

$$\Delta y = \frac{2\Gamma^{-1}(y^+)}{(1 + \Gamma^{-1}(y^+)^2)^2} |\nabla y^+|^2 + (1 + \Gamma^{-1}(y^+)^2) f(\Gamma^{-1}(y^+)) + g(-y^-) \quad \text{in } \Omega,$$
$$y = 0 \quad \text{on } \partial\Omega.$$

Basic estimates on solutions (u^k, v^k) of system 1 or 2

(i) L^∞-bound

$$0 \le u^k, v^k \le M \quad \text{for all } x \in \Omega, k > 0$$

by maximum principle, since $f(u), g(v) < 0$ when $u, v > M$ and so if, say, u^k attains a maximum value $u^k(x_0) > M$, then

$$-\Delta u^k(x_0) \le f(u^k(x_0)) < 0,$$

which is impossible

(ii) L^2-gradient bound there exists $K_1 > 0$ such that

$$\int_\Omega |\nabla u^k(x)|^2\, dx,\ \int_\Omega |\nabla v^k(x)|^2\, dx \le K_1 \quad \text{for all} \quad k > 0$$

since, *e.g.*, multiplication of u^k equation by u^k and integration over Ω gives

$$-\int_\Omega |\nabla u^k|^2\, dx + \int_\Omega u^k f(u^k)\, dx \ge 0$$

(iii) normal derivative bound there exists $K_2 > 0$ such that

$$\left|\frac{\partial u^k}{\partial \nu}\right|(x),\ \left|\frac{\partial v^k}{\partial \nu}\right|(x) \leq K_2 \ \text{ for all } x \in \partial\Omega, k > 0$$

since

$$-\Delta u^k \leq f(u^k), \quad x \in \Omega, \quad u^k = 0 \ \text{ on } \ \partial\Omega,$$

and so $0 \leq u^k \leq \overline{u}$, where $\overline{u}$ is the maximal solution in $[0, M]$ of

$$-\Delta u = f(u), \ \ x \in \Omega, \ \ u = 0 \ \text{ on } \ \partial\Omega$$

which, as $u^k = \overline{u} = 0$ on $\partial\Omega$, then implies

$$\left|\frac{\partial u^k}{\partial \nu}\right|(x) \leq \left|\frac{\partial \overline{u}}{\partial \nu}\right|(x) \text{ for all } x \in \partial\Omega$$

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(i), (ii) and (iii) use the sign of the competition term

(iv) basic segregation bound there exists $K_3 > 0$ such that

$$\int_\Omega k u^k v^k \, dx \le K_3$$

since

$$\begin{aligned} 0 \le \min\{1, \alpha_0\} \int_\Omega k u^k v^k \, dx &\le \int_\Omega \Delta u^k + f(u^k) \, dx \\ &= \int_{\partial\Omega} \frac{\partial u^k}{\partial \nu} \, dx + \int_\Omega f(u^k) \, dx \\ &\le C \end{aligned}$$

Note: (iv) uses key feature that α, etc are bounded below by a positive const

Key lemma $\nabla u^{k_n} \to \nabla u, \quad \nabla v^{k_n} \to \nabla v$ in $L^2(\Omega)$ as $k_n \to \infty.$

Idea of proof for system 1

- have to prove that

$$\limsup_{k_n \to \infty} \int_\Omega |\nabla u^{k_n}|^2 \, dx \le \int_\Omega |\nabla u|^2 \, dx$$

- multiplication of v^{k_n} equation by limit u and integration over Ω yields

$$-\int_\Omega \nabla u \cdot \nabla v^{k_n} \, dx + \int_\Omega u g(v^{k_n}) \, dx - k_n \int_\Omega u u^{k_n} v^{k_n} \alpha_2 \, dx = 0$$

- then as $k_n \to \infty$,

$$\int_\Omega \nabla u \cdot \nabla v^{k_n} \, dx \to \int_\Omega \nabla u \cdot \nabla v \, dx = 0, \quad \int_\Omega u g(v^{k_n}) \, dx \to \int_\Omega u g(v) \, dx = 0$$

so that

$$k_n \int_\Omega u u^{k_n} v^{k_n} \alpha_2 \, dx \to 0 \text{ as } k_n \to \infty$$

$\Rightarrow$

$$k_n \int_\Omega u u^{k_n} v^{k_n} \alpha_1 \, dx \to 0 \text{ as } k_n \to \infty$$

Idea of proof contd....

- now by multiplication of u^{k_n} equation by the limit u and integration over Ω,

$$-\int_\Omega \nabla u^{k_n} \cdot \nabla u \, dx + \int_\Omega u f(u^{k_n}) \, dx - k_n \int_\Omega u u^{k_n} v^{k_n} \alpha_1 \, dx = 0,$$

  and then letting $k_n \to \infty$ gives

$$\int_\Omega |\nabla u|^2 \, dx = \int_\Omega u f(u) \, dx$$

- multiplication of u^{k_n} equation by u^{k_n} and integration over Ω gives

$$-\int_\Omega |\nabla u^{k_n}|^2 \, dx + \int_\Omega u^{k_n} f(u^{k_n}) \, dx - k_n \int_\Omega (u^{k_n})^2 v^{k_n} \alpha_1 \, dx = 0,$$

  which, since α_1 and v^{k_n} are non-negative, implies that

$$\begin{aligned} \int_\Omega |\nabla u^{k_n}|^2 \, dx &\leq \int_\Omega u^{k_n} f(u^{k_n}) \, dx \\ &\to \int_\Omega u f(u) \, dx \\ &= \int_\Omega |\nabla u|^2 \, dx \qquad \square \end{aligned}$$

Remark : improved segregation

Lemma Let $\varepsilon > 0$. Then there exists $k_0 \in \mathbb{N}$ such that if $k \geq k_0$ and (u^k, v^k) is a non-negative solution of

$$\begin{aligned} \Delta u + f(u) - k\alpha_1(x)\gamma_1(v)uv &= 0, \quad x \in \Omega, \\ \Delta v + g(v) - k\alpha_2(x)\gamma_2(u)uv &= 0, \quad x \in \Omega, \\ u = v &= 0, \quad x \in \partial\Omega, \end{aligned}$$

then given $x \in \Omega$,

$$u^k(x) \leq \varepsilon_0 \quad \text{or} \quad v^k(x) \leq \varepsilon_0$$

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Idea of proof : Suppose not. Then there exist $\varepsilon_0 > 0$ and sequences $k_j \to \infty$ and $x_j \in \Omega$ such that

$$u^{k_j}(x_j) \geq \varepsilon_0 \;\text{ and }\; v^{k_j}(x_j) \geq \varepsilon_0.$$

Rescale

$$(U^{k_j}, V^{k_j})(\sqrt{k_j}(x - x_j)) = (u^{k_j}, v^{k_j})(x), \quad x \in \Omega$$

satisfies

$$\Delta U^{k_j} + k_j^{-1} f(U^{k_j}) - \alpha_1(x_j + \frac{x'}{\sqrt{k_j}})\gamma_1(V^{k_j})U^{k_j}V^{k_j} = 0 \text{ in } \Omega_j,$$

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$$U^{k_j} = V^{k_j} = 0 \qquad \text{on } \partial\Omega_j$$

and

$$0 \le U^{k_j}, V^{k_j} \le M, \quad 0 \in \Omega_j, \quad U^{k_j}(0) \ge \varepsilon_0 \text{ and } V^{k_j}(0) \ge \varepsilon_0$$

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$k_j \to \infty$ limit system

$$\left.\begin{aligned}
\Delta U &= \alpha_1(\overline{x})\gamma_1(V)UV,\\
\Delta V &= \alpha_2(\overline{x})\gamma_2(U)UV,
\end{aligned}\right\} \quad \text{in } \mathbb{R}^N$$

and

$$0 \le U, V \le M, \quad U(0) \ge \varepsilon_0 \ \text{ and } \ V(0) \ge \varepsilon_0$$

Then on the one hand....

$$\Delta U \geq 0 \quad \text{and} \quad U \quad \text{is bounded on} \quad \mathbb{R}^N;$$
$$\Delta V \geq 0 \quad \text{and} \quad V \quad \text{is bounded on} \quad \mathbb{R}^N,$$

$\Rightarrow$ $\exists$ direction $\{\lambda\xi : \xi \in S^{n-1}, \quad \lambda \geq 0\}$ along which

$$U(x) \to \sup U \quad \text{and} \quad V(x) \to \sup V \quad \text{as} \quad |x| \to \infty,$$

by properties of subharmonic functions

$\therefore$ limit $(\tilde{U}, \tilde{V})$ of translates $U(\cdot + x_n)$, $V(\cdot + x_n)$ along this direction satisfies

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But on the other hand....

$$\Delta\tilde{U}(0) = \alpha_1(\overline{x})\gamma_1(\tilde{V}(0))\, \tilde{U}(0)\tilde{V}(0) \;\; > 0$$

$\therefore$ contradiction

here also use feature that α, etc are bounded below by a positive const

Remark : regularity for limit equation of System 2

- $y \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ is a weak solution of limit equation

$$\begin{aligned} \Delta y &= \frac{2\Gamma^{-1}(y^+)}{(1+\Gamma^{-1}(y^+)^2)^2}|\nabla y^+|^2 + (1+\Gamma^{-1}(y^+)^2) f(\Gamma^{-1}(y^+)) + g(-y^-) \ \text{ in } \ \Omega, \\ y &= 0 \quad \text{on} \ \ \partial\Omega \end{aligned}$$

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• change of variables : let $r : \mathbb{R} \longrightarrow \mathbb{R}$ be such that

$$\begin{aligned} r'(t) &= e^{H(r(t))}, \quad t \in \mathbb{R}, \\ r(0) &= 0, \end{aligned}$$

where $H' = h$, and note that

$$r''(t) - h(r(t))r'(t)^2 = 0, \quad t \in \mathbb{R}.$$

- define $s : \Omega \to \mathbb{R}$ by

$$s(x) = r^{-1}(y(x)), \quad x \in \Omega,$$

where $y \in W_0^{1,2}(\Omega)$ is a solution of the limit equation

- then s satisfies

$$\begin{aligned} \Delta s &= \frac{d(s(x))}{r'(s(x))}, \quad x \in \Omega \\ s &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

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- hence

$$\begin{aligned} \Delta s \in L^\infty(\Omega),\ s = 0 \text{ on } \partial\Omega \ &\Rightarrow\ s \in W^{2,p}(\Omega) \ \text{ for all } \ p \in [1, \infty) \\ &\Rightarrow\ s \in C^{1,\mu}(\Omega) \ \text{ for all } \ \mu \in (0,1) \\ &\Rightarrow\ y \in C^{1,\mu}(\Omega) \ \text{ for all } \ \mu \in (0,1) \end{aligned}$$

Main open question...

- to better understand solutions of the scalar limit problems, especially sign-changing solutions of the limit problems

- in particular, to understand which sign-changing solutions arises as the limit as $k \to \infty$ of co-existence states of the k-dependent system

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Thank you for your attention ...