Size-structured population models with coagulation and fragmentation

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The reproduction operator (R. Rudnicki)

We consider an evolving population of individuals characterized by some attribute $x \in \Omega \subset \mathbb{R}_+$. The population may be described by the density $u(x,t)$. Alternatively, we can consider the measure $\mu_{x,t}$ describing the evolution of the attribute x if at $t = 0$ we had only one individual with attribute x and look at the evolution of

$$\phi(x,t) = \int\limits_{\Omega} f(y)\mu_{x,t}(dy).$$

If the evolution only occurs due to growth with intrinsic rate r, then

$$\partial_t u = -\partial_x(ru), \qquad u(x,0) = \mathring{u}(x)$$

(forward Kolmogorov) and

$$\partial_t \phi = r\partial_x \phi, \qquad \phi(x,0) = \mathring{f}(x)$$

(backward Kolmogorov), which are adjoint to each other (at least formally).

We introduce the set function

$$\mathcal{P}(x, A)$$

which is the rate at which an individual with attribute x which produces descendants which at birth have attribute from the set $A \subset \Omega$. Then

$$\partial_t \phi = r \partial_x \phi - \mu \phi + \int_\Omega \phi(y,t) \mathcal{P}(\cdot, dy), \qquad \phi(x,0) = \overset{\circ}{f}(x),$$

where μ is the total death rate.

Example 1. Cell division into two equal parts at the rate $b(x)$:

$$\mathcal{P}(x,A) = 2b(x) \text{ if } x/2 \in A \qquad \mathcal{P}(x,A) = 0 \text{ if } x/2 \notin A.$$

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Example 2. Continuously distributed attributes.

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Example 3. McKendrick model (all descendants have attribute 0).

$$\mathcal{P}(x,A) = b(x)\chi_A(0)$$

where χ_A is the characteristic function of A.

To write the equation for the density, we should find the pre-dual to

$$B^*\phi = \int_{\Omega} \phi(y,t)\mathcal{P}(\cdot, dy).$$

If $P(x, A) = 0$ almost everywhere for any set A of Lebesgue measure zero, then using Radon-Nikodym theorem, we can define the pre-dual operator B by

$$\int_{A} [Bf](x)dx = \int_{\Omega} \mathcal{P}(x, A)f(x)dx.$$

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If not, we split $\mathcal{P}(x, A)$ into singular and regular part.

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$$[Bf](x) = \int_{\Omega} k(x,y)f(y)dy$$

in the continuous attribute distribution case.

In the McKendrick case, we can calculate that if

$Au = -\mu u - (ru)_x$ on the domain

$$D(A) = \{u \in W^{1,1}(\Omega);\ u(0) = \int\limits_{\Omega} b(x)u(x)dx\}$$

then

$$A^*\phi = -\mu\phi + r\phi_x + b(x)\phi(0).$$

Thus, at least formally,

$$[B^*\phi](x) = b(x)\phi(0) = \int\limits_{\Omega} \phi(y)\mathcal{P}(x, dy)$$

The model.

In our model, we consider $\Omega = [x_0, \infty)$ with $x_0 \geq 0$ and the general reproduction operator with $\mathcal{P}(x, A)$ having both continuous and singular part with the latter concentrated at x_0. We also consider a nonlinear perturbation describing coagulation of particles. The resulting coagulation-fragmentation equation with decay or growth is typically written in the form

$$
\begin{aligned}
\partial_t u(x,t) &= -\partial_x[r(x)u(x,t)] - \mu(x)u(x,t) \\
&\quad -a(x)u(x,t) + \int\limits_{x_0+x}^{\infty} a(y)b(x|y)u(y,t)dy \\
&\quad -u(x,t)\int\limits_{x_0}^{\infty} k(x,y)u(y,t)dy \qquad (1) \\
&\quad +\frac{1}{2}\int\limits_{x_0}^{x-x_0} k(x-y,y)u(x-y,t)u(y,t)dy,
\end{aligned}
$$

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$$\int\limits_{x_0}^{y} xb(x|y)dx = \int\limits_{x_0}^{y-x_0} xb(x|y)dx = y \tag{2}$$

The expected number of particles in a fragmentation event is

$$n_0(y) = \int\limits_{x_0}^{y} b(x|y)dx < +\infty, \quad \text{any } y > x_0. \tag{3}$$

The ‘stickiness function’ $k(x, y)$ represents the likelihood of an aggregate of size x sticking to an aggregate of size y. We assume

$$k \in L_\infty([x_0, x_1) \times [x_0, x_1)). \tag{4}$$

and $k(x, y) \neq 0$ if $x, y > x_0$.

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r is the rate of continuous mass growth defined so that $r(m(t)) = dm/dt$ for a particle of time-dependent mass $m(t)$.

The function μ represents the death term; it is assumed that

$$0 \leq \mu \in L_{\infty,loc}([x_0, \infty)).$$

Abstract formulation

The problem is formulated as an abstract differential equation

$$\partial_t u = T_{0,b}u + Fu + Nu = T_{0,b}u + Au + Bu + Nu = T_b u + Bu + Nu \tag{5}$$

in an appropriate Banach space X. Here $T_{0,b}$ is a realization of the original growth term subject to appropriate boundary conditions, A is the loss and B the gain term of the fragmentation operator F and N is the coagulation operator.

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- Use standard Lipschitz perturbation techniques to prove local (in time) solvability of the full equation;
- Use characterization of the generator of the linear semigroup to derive moment estimates;
- Show global existence or blow up, if possible.

Interlude: quasi substochastic semigroup theory

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A C_0-semigroup $(G(t))_{t\geq 0}$ is (quasi) *substochastic* if it is positive and (quasi) contractive, that is,

$$\|G(t)\| \leq e^{\omega t}$$

for some ω.

Let $(T, D(T))$ and $(B, D(B))$ be linear operators on X such that $(T, D(T))$ generates a quasi-substochastic semigroup $(G_T(t))_{t\geq 0}$, $(B, D(B))$ is positive on $D(B)_+$ with $D(B) \supset D(T)$ and

$$\int_{\Omega} (T+B)u\, d\mu = c(u) := -c_-(u) + \omega' \|u\|_X, \qquad 0 \leq u \in D(T), \tag{6}$$

where c_- is a positive functional defined on $D(T)$ (zero in the formally conservative case) and $\omega' \in \mathbb{R}$.

Theorem

Under the above assumptions, there exists a smallest quasi substochastic semigroup $(G_K(t))_{t\geq 0}$ generated by an extension K of the operator $T + B$.

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Under the above assumptions, there exists a smallest quasi substochastic semigroup $(G_K(t))_{t\geq 0}$ generated by an extension K of the operator $T + B$. Furthermore, the functional c (and thus c_-) extends to a continuous functional on $D(K)$ by monotone limits of elements of $D(T)_+$ as well as by continuity in the graph norm of $D(K)$.

Denote $u(t) = G_K(t)f$ with $f \in D(K)_+$. Then always

$$\frac{d}{dt}\int\limits_{\Omega} u(t)\, d\mu = \frac{d}{dt}\|u(t)\| \le c\,(u(t))\,. \qquad (7)$$

If there is $' =' $ in (7), then we say that $(G_K(t))_{t\ge 0}$ is *honest*.

A characterization of honesty is given in the following theorem.

Theorem

The following are equivalent:

(a) The semigroup $(G_K(t))_{t\geq 0}$ *is honest.*

(b) $K = \overline{T + B}$.

(c) $\int\limits_{\Omega} Ku \, d\mu \geq c(u) = -c_{-}(u) + \|u\|_X, \quad u \in D(K)_{+}.$

Extension techniques

Using the fact that X is embedded in the lattice of measurable functions and the either the operators involved or their resolvents are positive integral operators, we can define various extensions by monotonicity. This technique has proved to be very useful in characterizing the domain of K. In particular, Theorem 2 c) is most often used through the following corollary.

Theorem

Let $\mathcal{K}$ be an extension of the generator K acting in X. If

$$\int_{\Omega} \mathcal{K}u \, d\mu \geq c(u) \quad u \in D(\mathcal{K})_+$$

then the semigroup is honest.

Typical state spaces.

In

$$X_1 := L_1([x_0, \infty), xdx)$$

the norm of a nonnegative element u, which represents a state of the system; that is, a distribution of mass among particles, given by

$$\|u\|_1 = \int_{x_0}^{\infty} u(x)xdx$$

represents the total mass of the system.

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Note that if $x_0 > 0$, then $X_1 \subset X_0$.

Choice of the state space.

Example 1. Fragmentation equation with unbounded fragmentation rate a is well posed in $X_1 := L_1(\mathbb{R}_+, xdx)$. However, it is not well posed in $X_0 := L_1(\mathbb{R}_+, dx)$. Indeed, the function

$$u(t,x) = e^{-xt}\left(f(x) + \int_x^\infty f(y)[2t + t^2(y-x)]dy\right)$$

is the solution to

$$\partial_t u(x,t) = -xu(x,t) + 2\int_x^\infty u(y,t)dy,$$

($a(x) = x$ and $b(x|y) = 2/y$) and initial condition f.

However, for the norm X_0 we have

$$\|u(1,\cdot)\|_0 = \int\limits_0^\infty f(y)(y-1+e^{-y})dy$$

which is finite only if $f \in X_0$.

Example 2. Consider

$$\partial_t u(x,t) = -\partial_x u(x,t), \quad t > 0, x > 0$$

with initial and boundary conditions $u(x,0) = f(x)$, $u(0,t) = 0$.

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with initial and boundary conditions $u(x,0) = f(x)$, $u(0,t) = 0$.Then $u(x,t) = f(x-t)$ for $x > t$, $u(x,t) = 0$ for $x < t$ and

$$\|u\|_1 = \int\limits_t^{\infty} f(x-t)xdx = \int\limits_0^{\infty} f(y)ydy + t\int\limits_0^{\infty} f(y)dy$$

and we see that $\|u\|_1$ may be infinite if $f \notin X_0$. Thus, X_1 is not a good state space if $r(0) \neq 0$.

Example 3.

$$u_t(t,x) = -(x^2u)_x, \quad u(0,x) = f(x), \quad x > 0. \tag{8}$$

Since $1/x^2$ is not integrable at 0, there is no need to impose any boundary conditions. The solution is given by

$$u(t,x) = \frac{1}{(1+xt)^2} f\left(\frac{x}{1+xt}\right). \tag{9}$$

The solution is defined for all $t, x > 0$, the process is dissipative in X_0:

$$\int_0^\infty \frac{1}{(1+xt)^2} f\left(\frac{x}{1+xt}\right) dx = \int_0^{1/t} f(\xi)d\xi \leq \int_0^\infty u_0(\xi)d\xi.$$

On the other hand the norm of the solution in X_1 is given by

$$\int_0^\infty \frac{x}{(1+xt)^2} u_0\left(\frac{x}{1+xt}\right) dx = \int_0^{1/t} \frac{\xi u_0(\xi)}{1-\xi t} d\xi$$

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and the right hand side has a non integrable singularity at $\xi = 1/t$ which shows that (9) cannot define a semigroup in X_1.We note that if $0 \leq r(x) \leq r_0(1+x)$, then the characteristics of the transport equation are defined for all $t > 0$. This property is crucial for the generation of a strongly continuous semigroup X_1. However, it can be proved that (9) (extended by 0) defines a strongly continuous semigroup in X_0.

The coagulation operator does not behave well in X_1 but it can be controlled in X_0. Hence analysis is usually carried out in the Banach space $X_{0,1} := L_1([x_0, \infty), (1 + x)dx)$ in which both the total mass and the number of particles are controlled.

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The analysis in $X_{0,1}$ is well-developed but requires that rather severe constraints on a and β, such as sublinearity, be imposed. It is unlikely that one can build an $X_{0,1}$ theory with a and β with faster growth at infinity.

Introduce the mth moment of the solution u

$$M_m(t) = \int_{x_0}^{\infty} x^m u(x,t)dx.$$

Formally (with, say, $x_0 = 0$, $k = const$) we have

$$\begin{aligned} \frac{d}{dt}M_0(t) &= \int_0^{\infty} \beta(x)u(x,t)dx + \int_0^{\infty}(n_0(x)-1)a(x)u(x,t)dx \\ &\quad -\frac{k}{2}M_0^2, \end{aligned} \tag{10}$$

and we see that if β and an_0 are not sublinear, then the zeroth moment is controlled by higher order moments.

The answer: build a theory in higher moments spaces

$$
\begin{aligned}
X_{0,m} &= X_0 \cap X_m = L_1([x_0,\infty), dx) \cap L_1([x_0,\infty), x^m dx) \\
&= L_1([x_0,\infty), (1+x^m)dx). \qquad (11)
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(i) Does the transport part generates a quasi-contractive semigroup on $X_{0,m}$?

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Problems:

(i) Does the transport part generates a quasi-contractive semigroup on $X_{0,m}$?

(ii) Does the fragmentation operator behave well in $X_{0,m}$?

(iii) Can we combine these two?

The transport part.

$$\partial_t u(x,t) = \pm\partial_x[r(x)u(x,t)] - q(x)u(x,t), \qquad x \geq x_0,$$

$q = \mu + a$.

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- If we have mass loss, then there is (almost) no need for boundary conditions.
- If we have mass growth and

$$\int_{x_0^+} \frac{dx}{r(x)} = \infty,$$

then there is no need for boundary condition.

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We consider general boundary condition

$$\lim_{x\to x_0} r(x)u(x,t) = \int_{x_0}^{\infty} \beta(x)u(x,t)dx \qquad (12)$$

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which incorporates both homogeneous Dirichlet ($\beta = 0$) and renewal boundary conditions ($\beta \geq 0$). If the smallest particles only are created by fragmentation, then

$$\beta(x) = a(x)b(x_0|x).$$

We focus on the growth case and consider

$$[\mathcal{T}^g u](x,t) = -\partial_x[r(x)u(x,t)] - q(x)u(x,t), \qquad (13)$$

for $t > 0, x > x_0$, where $0 \leq q \in L_{\infty,loc}([x_0,\infty))$, We focus on the growth case and consider

$$[\mathcal{T}^g u](x,t) = -\partial_x[r(x)u(x,t)] - q(x)u(x,t), \tag{13}$$

for $t > 0, x > x_0$, where $0 \le q \in L_{\infty,loc}([x_0,\infty))$, $r(x) > 0$ on (x_0,∞), $r \in AC((x_0,\infty))$, $\sup_{x \ge x_0}(1+x)^{-1}r(x) < \infty$. In the case when r is not integrable at x_0 we assume

$$\|r\|^*_{1,\infty} = \sup_{x \ge x_0}(x-x_0)^{-1}r(x) < \infty. \tag{14}$$

When $1/r$ is integrable, we consider (13) with

$$\lim_{x\to x_0} r(x)u(x,t) = \int\limits_{x_0}^{\infty} \beta(x)u(x,t)dx. \tag{15}$$

This gives the operator $T^g_{m,\beta}$ as the restriction of $\mathcal{T}^g$ to

$$\begin{aligned} D(T^g_{m,\beta}) &= \{u \in X_{0,m};\ ru \in AC((x_0,\infty)),\ (ru)_x, qu \in X_{0,m}, \\ & \quad \text{(15) holds}\}. \end{aligned}$$

Theorem

The operator $(T^g_{m,\beta}, D(T^g_{m,\beta}))$ *is the generator of a strongly continuous positive semi-contractive semigroup, say* $(G_{T^g_{m,\beta}}(t))_{t\geq 0}$, *on* $X_{0,m}$ *which satisfies*

$$\|G_{T^g_{m,\beta}}(t)\|_{0,m} \leq e^{Ct},$$

where $C = C(r, \beta, x_0, m)$ *is a constant.*

The fragmentation part.

We impose assumptions:

$$
\begin{aligned}
0 &\le a(x) \le a_0(1 + x^k), \\
n_0(x) &= \int_0^x b(y|x)dy \le b_0(1 + x^l),
\end{aligned} \tag{16}
$$

for any $x \in [x_0, \infty)$, some $k, l \in \mathbb{N}_0$ and $a_0 > 0$ and $b_0 \ge 1$.

Note. These assumptions are only necessary if $x_0 = 0$ since, as we shall see later, they enter solely into the estimates of the zeroth moment, see (10), which for $x_0 > 0$ is controlled by the higher moments and there is no need to use it.

We note that (3) implies

$$n_m(y) := \int_{x_0}^{y} b(x|y)x^m dx \le y^m \int_{x_0}^{y} b(x|y)dx = y^m n_0(y) < +\infty$$

for any $m \in \mathbb{N}_0$ and $y \in [x_0, \infty)$. We denote

$$N_m(y) = n_m(y) - y^m.$$

Observe that, by (2), for $m \ge 1$

$$N_m(y) = \int_{0}^{y} b(x|y)x^m dx - y^m \le y^{m-1} \int_{0}^{y} b(x|y)x dx - y^m \le 0 \tag{17}$$

while

$$N_0 = n_0(y) - 1 \ge 0. \tag{18}$$

For any m, let $A_m u = au$ on

$$D(A_m) = \{u \in X_{0,m};\ au \in X_{0,m}\}$$

and consider the expression

$$[\mathcal{B}u](x) = \int_x^\infty a(y)b(x|y)u(y)dy. \tag{19}$$

Then

Lemma

If $0 \le f \in D(A_m)$ *with* $m \ge k + l$*, then*

$$\|\mathcal{B}f\|_{0,m} = \int_0^\infty a(x)(n_m(x) + n_0(x))f(x)dx < +\infty. \tag{20}$$

Fragmentation equation with growth/decay.

Again we shall consider $T^{g}_{m,\beta}$ with $q = \mu + a$. Let k, l be specified in (16) and let $m \geq k + l$. To shorten notation, we drop the indices g and β. Then for $u \in D(T_m)$ we have

$$\int_{x_0}^{\infty} (T_m + B_m)u(x)(1 + x^m)dx = -c_m(u) := \tag{21}$$

$$- \int_{x_0}^{\infty} \mu(x)u(x)(1 + x^m)dx + (1 + x_0^m) \int_{x_0}^{\infty} \beta(x)u(x)dx$$

$$+m \int_{x_0}^{\infty} r(x)u(x)x^{m-1}dx + \int_{x_0}^{\infty} (N_m(x) + N_0(x))a(x)u(x)dx$$

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$$

$$
+m\int_{x_0}^{\infty}r(x)u(x)x^{m-1}dx + \int_{x_0}^{\infty}(N_m(x) + N_0(x))a(x)u(x)dx
$$

$$
\leq -c_m^-(u) + C_m\|u\|_{0,m}.
$$

Thus there is an extension K_m of $T_m + B_m$ which generates a positive quasi-contractive semigroup $(G_{T_m}(t))_{t\geq 0}$ satisfying

$$\|G_{T_m}(t)\|_{0,m} \leq e^{C_m t}. \tag{22}$$

Moreover, the functional $-c_m^-$ extends to $D(K_m)$ by density in the graph norm of $D(K_m)$, and also to $D(K_m)_+$ by monotonic limits, from $D(T_m)$, respectively $D(T_m)_+$. Thus, in particular, for $u \in D(K_m)_+$ we have

$$\int_{x_0}^{\infty} \mu(x)u(x)(1 + x^m)dx < +\infty$$

and

$$-\int_{x_0}^{\infty} N_m(x)a(x)u(x)dx < +\infty. \qquad (23)$$

By (17) we have

$$|N_m(x)| \leq 2x^m.$$

If N_m is precisely of order x^m, then (23) gives $D(K_m) \subset D(A_m)$ and, in fact, using theory of extensions, one can prove

$$K_m = T_m + B_m.$$

This is true in an important in applications homogeneous fragmentation, where $b(x|y) = y^{-1}h(x/y)$.

Characterization of the generator

In all cases it can be proved that

$$K_m = \overline{T_m + B_m}. \qquad (24)$$

This result shows that all moments

$$\int\limits_{x_0}^{\infty} [K_m u](x) x^k dx, \qquad u \in D(K_m),$$

for $k \leq m$ can be calculated by direct evaluation for $u \in D(T_m)$ and extended to $D(K_m)$.

Trotter formula

Moreover, the following holds

$$G_{K_m}(t)u = \lim_{n\to\infty}\left(G_{T_{0,m}}(t/n)G_{\bar{F}}(t/n)\right)^n u, \quad u \in X_{0,m}, \qquad (25)$$

uniformly in t on bounded intervals.

Local solvability of the complete equation.

We introduce the coagulation operator defined by

$$\begin{aligned}[N_m f](x) \quad &:= \quad \frac{1}{2}\int_{x_0}^{x-x_0} k(x-y,y)f(x-y)f(y)\,dy \\ &\quad -f(x)\int_{x_0}^{\infty} k(x,y)f(y)\,dy\,, \quad x > x_0\end{aligned}$$

on $X_{0,m} \times X_{0,m}$.

Moments of $N_m f$ satisfy

$$\int_{x_0}^{\infty} x^p[N_m(f)](x)\,dx$$
$$= \frac{1}{2}\int_{x_0}^{\infty}\int_{x_0}^{\infty}((x+y)^p - x^p - y^p)k(x,y)f(x)f(y)\,dx\,dy.$$

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$$(x+y)^\beta \leq 2^\beta(x^\beta + y^\beta), \qquad \beta \geq 0, \tag{26}$$

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Basic inequalities used are

$$(x+y)^\beta \le 2^\beta(x^\beta + y^\beta), \qquad \beta \ge 0, \tag{26}$$

and, for $p \ge 2$,

$$0 \le (x+y)^p - x^p - y^p \le (2^p - 1)(xy^{p-1} + yx^{p-1}).$$

Theorem

N_m is continuously Fréchet differentiable at any point $f \in X_{0,m}$ and therefore locally Lipschitz on $X_{0,m}$. Consequently, for any $0 \leq u_0 \in D(K_m)$ there exists a unique non-negative strict solution $u(t)$ to

$$u_t(t) = K_m u(t) + N_m u(t), \tag{27}$$

defined on its maximal interval of existence $[0, t(u_0))$, where $t(u_0) > 0$.

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Define

$$M_m(t) = \int_{x_0}^{\infty} x^m u(x,t)dx.$$

Moments' estimates.

a) Decay case:

$$M_0' \le \int_{x_0}^{\infty} (n_0(x) - 1)a(x)u(x,t)dx$$

$$- \int_{x_0}^{\infty}\int_{x_0}^{\infty} k(x,y)u(x,t)u(y,t)dxdy \le C_0(M_0 + M_m)$$

$$M_1' \le 0$$

$$M_m' \le \frac{1}{2}\int_{x_0}^{\infty}\int_{x_0}^{\infty} ((x+y)^m - x^m - y^m)k(x,y)u(x,t)u(y,t)dxdy$$

$$\le C_m(M_1^2 + 2M_1M_m)$$

b) Growth with no boundary conditions or $\beta = 0$:

$$\begin{aligned} M_0' &\leq C_0(M_0 + M_m), \\ M_1' &\leq \int_{x_0}^{\infty} r(x)u(x,t)dx, \\ M_m' &\leq C_m(M_1^2 + 2M_1M_m) + C_{m,r}(M_0 + M_1). \end{aligned}$$

b) Growth with boundary conditions $\beta \neq 0$:

$$\begin{aligned} M_0' &\leq C_0'(M_0 + M_m), \\ M_1' &\leq x_0 \int_{x_0}^{\infty} \beta(x) u(x,t) dx + \int_{x_0}^{\infty} r(x) u(x,t) dx, \\ M_m' &\leq C_m(M_1^2 + 2M_1 M_m) + C_{m,r,\beta}(M_0 + M_1). \end{aligned}$$

Global solvability.

a) In the decay case, $M_1 \leq M_1(0)$ is bounded, thus M_m and, consequently, M_0 are bounded on each time interval, giving thus global solvability of the DFG equation.

Global solvability.

a) In the decay case, $M_1 \leq M_1(0)$ is bounded, thus M_m and, consequently, M_0 are bounded on each time interval, giving thus global solvability of the DFG equation.

b) Growth with $1/r$ non-integrable at x_0. Then

$$\int_0^\infty r(x)u(x,t)dx \leq \|r\|^*_{1,\infty} M_1$$

and thus M_1, growing exponentially, yields M_m to be globally defined and therefore M_0 is globally defined and the solution is global.

c) The case of growth with $1/r$ integrable at x_0. Assume $x_0 > 0$, $r(x), \beta(x)$ sublinear. Then, similarly to the above,

$$\int_{x_0}^{\infty} r(x)u(x,t)dx \leq \|r\|_{1,\infty} \int_{x_0}^{\infty} (1+x)u(x,t)dx$$
$$\leq \|r\|_{1,\infty}(M_1 + M_0) \leq \|r\|_{1,\infty}(1+x_0)x_0^{-1}M_1$$

and

$$\int_{x_0}^{\infty} \beta(x)u(x,t)dx \leq \|\beta\|_{1,\infty}(1+x_0)x_0^{-1}M_1$$

so

$$M_1' \leq CM_1$$

for some constant C and the argument as above yields global solvability.

A case of a finite time blow-up.

We consider the following example. Let $x_0 = 1$, $k = 1$, $a(x) = 1$. Consider a uniform binary fragmentation on $[1, \infty)$. Then we have

$$b(x|y) = \frac{2\chi_{[1,y-1]}(x)}{y-2}$$

for $y > 2$ expressing the fact that no particles of size $y < 2$ can fragment into two particles of size at least 1 and no particle of size greater than $y - 1$ can emerge after fragmentation. The clearly we have

$$n_0(x) = \frac{2}{y-2}\int_1^{y-1} dx = 2, \quad n_1(x) = \frac{2}{y-2}\int_1^{y-1} x dx = y$$

Furthermore,

$$n_2(y) = \frac{2}{y-2}\int_1^{y-1} x^2 dx = \frac{2}{3}(y^2 - y + 1)$$

so that $-N_2(y) \sim \frac{1}{3}y^2$ uniformly for large y. Thus, we know that $D(K_m) \subset D(A_m)$ and, using the earlier results, we can integrate termwise which leads to the system

$$\begin{aligned} M_1' &= M_2 + M_1, \\ M_2' &= -\frac{2}{3}M_1 + \frac{2}{3}M_0 + \frac{8}{3}M_2 + M_1^2 \end{aligned}$$

on the maximal interval of existence of the solution.

Then

$$M_1'' \geq M_1^2.$$

which, for any initial value u_0 such that

$M_2(0) + M_1(0) - M_1^3(0) \geq 0$ gives

$$M_1' \geq \sqrt{\frac{2}{3} M_1^3}$$

and yields the estimate of the blow-up time

$$t_b \leq \sqrt{\frac{3}{2}} \int\limits_{M_1(0)}^{\infty} \frac{ds}{s^{3/2}} = \sqrt{\frac{3}{8}} \frac{1}{\sqrt{M_1(0)}}.$$