Mathematical Statistics
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Time Series

Examination

Deadline: Saturday 6th July 2024, 13.00

1. Let $\{X_t\}_{t=-\infty}^{\infty}$ be a stationary AR(1) process defined by:

$$X_t + 0.25X_{t-1} = Z_t \qquad \{Z_t\} \sim \text{WN}(0,1).$$

Suppose that X_1 and X_3 are observed, but the value X_2 is missed. The value X_2 is then estimated by a linear combination:

$$\widehat{X}_2 = a_1 X_1 + a_2 X_3.$$

(a) Find a_1 and a_2 such that the mean squared error $\mathbb{E}\left[\left(X_2 - \widehat{X}_2\right)^2\right]$ is minimised. (5pts)

(b) Find the value of the optimal mean squared error. (5pts)

2. Let $\{X_t : t \in \mathbb{Z}\}$ be an MA(1) process

$$X_t = Z_t + \theta Z_{t-1} \qquad \{Z_t\} \sim \text{WN}(0, \sigma^2).$$

(a) Find the value θ^* for θ such that the correlation between X_t and X_{t-1} is maximised. (7pts)

(b) Find the spectral density of $\{X_t : t \in \mathbb{Z}\}$ when $\theta = \theta^*$. Are high or low frequencies dominating the spectrum? How does this change when θ moves away from θ^*? (3pts)

3. (a) Which of the following processes $\{Y_t : t \in \mathbb{Z}\}$ are causal ARMA(1,1) process? Justify your answer.

i.

$$Y_t - \frac{5}{6}Y_{t-1} = Z_t - \frac{9}{20}Z_{t-1} \qquad \{Z_t\} \sim \text{WN}(0,1)$$

(1pt)

ii.

$$Y_t - Y_{t-1} + \frac{1}{4}Y_{t-2} = Z_t - \frac{5}{4}Z_{t-1} + \frac{3}{8}Z_{t-2} \qquad \{Z_t\} \sim \text{WN}(0,1)$$

(2pts)

iii.

$$Y_t + \frac{1}{2}Y_{t-1} - \frac{1}{2}Y_{t-2} = Z_t - \frac{5}{4}Z_{t-1} + \frac{3}{8}Z_{t-2} \qquad \{Z_t\} \sim \text{WN}(0,1)$$

(2pts)

(b) Find a *state space representation* of the process $\{Y_t : t \in \mathbb{Z}\}$ defined by

$$Y_t - \frac{5}{6}Y_{t-1} = Z_t - \frac{9}{20}Z_{t-1} \qquad \{Z_t\} \sim \text{WN}(0,1).$$

Justify your answer. (5pts)

4. Let $\{X_t : t \in \mathbb{Z}\}$ be a time series that satisfies:

$$\begin{cases} X_t = b_t - (\theta_1 + \theta_2)Z_t - \theta_2 Z_{t-1} \\ b_t = b_{t-1} + (1 + \theta_1 + \theta_2)Z_t \qquad \{Z_t\} \sim \text{WN}(0,1). \end{cases}$$

Show that $\{X_t : t \in \mathbb{Z}\}$ is an ARIMA(0,1,2) process. (5pts)

5. For this exercise, use R for the matrix computations.
Let $\{X_t : t = 0, 1, 2, \ldots\}$ be a stationary mean zero time series. Let

$$\widehat{X}_t = \begin{cases} 0 & t = 1 \\ P_{t-1}X_t & t = 2, 3, \ldots \end{cases}$$

where $P_{t-1}X_t$ is the optimal projection of X_t onto the linear span of $(X_1, \ldots, X_{t-1})$. In other words,

$$\widehat{X}_t := P_{t-1}X_t = \phi_{t-1,1}X_{t-1} + \phi_{t-2,2}X_{t-2} + \ldots + \phi_{t-1,t-1}X_1.$$

Let

$$\epsilon_t := X_t - \widehat{X}_t$$

These are the one step prediction errors. Let

$$v_{t-1} = \mathbb{E}\left[\left(X_t - \widehat{X}_t\right)^2\right].$$

Set

$$\underline{\epsilon}_n = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_{n-1} \\ \epsilon_n \end{pmatrix}, \qquad \underline{X}_n = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{pmatrix}, \qquad \widehat{X}_n = \begin{pmatrix} \widehat{X}_1 \\ \widehat{X}_2 \\ \vdots \\ \widehat{X}_{n-1} \\ \widehat{X}_n \end{pmatrix},$$

$$\Phi_n = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ -\phi_{11} & 1 & 0 & \ldots & 0 \\ -\phi_{22} & -\phi_{21} & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -\phi_{n-1,n-1} & -\phi_{n-1,n-2} & -\phi_{n-1,n-3} & \ldots & 1 \end{pmatrix}$$

(a) Let $C_n = \Phi_n^{-1}$. The matrix C_n is lower triangular with $C_{n;jj} = 1$ for each j. Show that the covariance matrix of $\underline{X}_n$ is given by

$$\Gamma_n = \mathbb{E}[\underline{X}_n \underline{X}_n^t] = C_n D_n C_n^t$$

where

$$D_n = \begin{pmatrix} v_0 & 0 & 0 & \dots & 0 \\ 0 & v_1 & 0 & \dots & 0 \\ 0 & 0 & v_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & v_{n-1} \end{pmatrix}.$$

Justify your solution. (2pts)

(b) Express $\widehat{\underline{X}}_n$ in terms of $\underline{\epsilon}_n$ via a matrix operation. (2pts)

(c) The MA(1) process $\{X_t : t \in \mathbb{Z}\}$ has zero mean and satisfies

$$X_t = Z_t - 1.2 Z_{t-1} \qquad \{Z_t\} \sim \mathrm{WN}(0, 1).$$

(2pts)

i. Express $\widehat{\underline{X}}_4$ in terms of $\underline{\epsilon}_4$.

ii. Find the values of v_1 and v_2. (4pts)

6. Let $\{X_t\}$ be an invertible MA(q) process:

$$X_t = Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q}, \qquad \{Z_t\} \sim WN(0, \sigma^2)$$

Let $\widehat{X}_t$ denote the one step predictor based on $(X_1, \ldots, X_{t-1})$ and let $\theta_{t,j}$ denote the coefficients in the expansion

$$\widehat{X}_{t+1} = \sum_{j=1}^{t} \theta_{t,j} (X_{t+1-j} - \widehat{X}_{t+1-j}).$$

Let

$$v_t := \mathbb{E}\left[\left(X_{t+1} - \widehat{X}_{t+1}\right)^2\right].$$

Show that, as $t \to +\infty$,

(a)

$$\mathbb{E}\left[\left(X_t - \widehat{X}_t - Z_t\right)^2\right] \to 0$$

(7pts)

(b)

$$v_t \to \sigma^2$$

(3pts)