Tutorial 11

Arch-Garch Processes

AG1. Derive multistep-ahead forecasts of the volatility for a GARCH(1,2) model at the forecast origin h. Do this by noting that $\sigma_h^2(j) = \mathbb{E}\left[\sigma_{h+j}^2 | \mathcal{F}_h\right]$, using $\mathbb{E}\left[\mathbb{E}\left[X|Y\right]\right] = \mathbb{E}\left[X\right]$ and that $\mathbb{E}[X_t^2|\mathcal{F}_{t-1}] = \sigma_t^2$. Obtain a recursive formula by computing $\sigma_h^2(1)$, $\sigma_h^2(2)$ and $\sigma_h^2(j)$ for $j > 2$.

Compute $\sigma_h^2(\infty)$, assuming it exists and state conditions such that the answer is well defined.

AG2. Derive multistep-ahead forecasts of the volatility for a GARCH(2,1) model at the origin h.

Comute $\sigma_h^2(\infty)$ assuming it exists and state conditions such that the answer is well defined.

AG3. Suppose that $r_1, \ldots, r_n$ are observations of a return series that follows the AR(1)-GARCH(1,1) model:

$$\begin{cases} X_t = \sigma_t Z_t & \{Z_t\} \sim \text{IID}N(0,1) \\ \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2 & \\ R_t = \mu + \phi_1 R_{t-1} + X_t & \end{cases}$$

Derive the conditional log-likelihood function of the data. Base this on $r_3, \ldots, r_n$ and use as an initialisation $r_{-1} = r_0 = \sigma_0^2 = 0$.

AG4. Consider the AR(1)-GARCH(1,1) model of the previous exercise, with the difference that $\{Z_t\}$ is IID t_ν. Derive the conditional log-likelihood function of the data.

Kalman Filtering

K1. Let $\{Y_t\}$ be the MA(1) process

$$Y_t = Z_t + \theta Z_{t-1} \qquad \{Z_t\} \sim \text{WN}(0, \sigma^2)$$

Show that $\{Y_t\}$ has the state space representation

$$Y_t = (1 \quad 0)\underline{X}_t$$

where $\{\underline{X}_t\}$ is the unique stationary solution of

$$\underline{X}_{t+1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \underline{X}_t + \begin{pmatrix} 1 \\ \theta \end{pmatrix} Z_{t+1}.$$

In particular, show that the state vector $\underline{X}$ may be written as:

$$\underline{X}_t = \begin{pmatrix} 1 & \theta \\ \theta & 0 \end{pmatrix} \begin{pmatrix} Z_t \\ Z_{t-1} \end{pmatrix}$$

K2. Let $\{Y_t\}$ be an ARIMA(p,d,q) process. By using a suitable state space model, show that $\{Y_t\}$ has representation

$$\begin{cases} Y_t = G\underline{X}_t \\ \underline{X}_{t+1} = F\underline{X}_t + HZ_{t+1} \end{cases}$$

for $t \in \mathbb{N}$ and suitably chosen matrices F, G and H.

K3. Consider the state space representation for a causal AR(p) process:

$$\begin{cases} Y_t = (0 \quad 0 \quad 0 \quad \dots 1)\underline{X}_t & t \in \mathbb{Z} \\ \underline{X}_{t+1} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \phi_p & \phi_{p-1} & \phi_{p-2} & \dots & \phi_1 \end{pmatrix} \underline{X}_t + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} Z_{t+1} & t \in \mathbb{Z} \end{cases}$$

of the form $\underline{X}_{t+1} = F\underline{X}_t + H\underline{Z}_{t+1}$. Show the stability of the equation for $\underline{X}$ by showing that the eigenvalues of F are equal to the reciprocals of the autoregressive polynomial $\phi(z)$. In particular, show that

$$\det(zI - F) = z^p \phi(z^{-1}).$$

Answers

ARCH-GARCH Processes

AG1. For the GARCH(1,2),

$$\begin{cases} X_t = \sigma_t Z_t & \{Z_t\} \sim \text{IID N}(0,1) \\ \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-2}^2 \end{cases}$$

It follows that

$$\mathbb{E}[X_t^2 | \mathcal{F}_{t-1}] = \sigma_t^2$$

so that

$$\sigma_h^2(1) = \alpha_0 + \alpha_1 X_h^2 + \beta_1 \sigma_h^2 + \beta_2 \sigma_{h-1}^2$$

$$\begin{aligned} \sigma_h^2(2) &= \alpha_0 + \alpha_1 \sigma_h^2(1) + \beta_1 \sigma_h^2(1) + \beta_2 \sigma_h^2 \\ &= \alpha_0 + (\alpha_1 + \beta_1)\sigma_h^2(1) + \beta_2 \sigma_h^2 \\ &= \alpha_0 + (\alpha_1 + \beta_1)(\alpha_0 + \alpha_1 X_h^2 + \beta_2 \sigma_{h-1}^2) + (\beta_1(\alpha_1 + \beta_1) + \beta_2)\sigma_h^2 \end{aligned}$$

For $j \geq 3$,

$$\begin{aligned} \sigma_h^2(j) &= \alpha_0 + \alpha_1 \mathbb{E}[X_{h+j-1}^2 | \mathcal{F}_h] + \beta_1 \sigma_h^2(j-1) + \beta_2 \sigma_h^2(j-2) \\ &= \alpha_0 + (\alpha_1 + \beta_1)\sigma_h^2(j-1) + \beta_2 \sigma^2(j-2). \end{aligned}$$

$$\sigma_h^2(\infty)\,(1 - \alpha_1 - \beta_1 - \beta_2) = \alpha_0 \Rightarrow \sigma_h^2(\infty) = \frac{\alpha_0}{1 - \alpha_1 - \beta_1 - \beta_2}.$$

AG2.

$$\begin{cases} \sigma_h^2(1) = \alpha_0 + \alpha_1 X_h^2 + \alpha_2 X_{h-1}^2 + \beta_1 \sigma_h^2 \\ \sigma_h^2(2) = \alpha_0 + \alpha_1 \sigma_h^2(1) + \alpha_2 X_h^2 + \beta_1 \sigma_h^2(1) \\ \alpha_h(j) = \alpha_0 + (\alpha_1 + \beta_1)\sigma_h^2(j-1) + \alpha_2 \sigma_h^2(j-2) \end{cases}$$

$$\sigma_h^2(\infty) = \frac{\alpha_0}{1 - \alpha_1 - \alpha_2 - \beta_1}$$

(if and only if $\alpha_1 + \alpha_2 + \beta_1 < 1$)

AG3. Let $\underline{\theta} = (\alpha_0, \alpha_1, \beta_1, \mu, \phi_1)$ denote the vector of parameters and let $f(r_1, \ldots, r_n)$ be the joint density, then

$$f(r_1, \ldots, r_n | \underline{\theta}) = f(r_1, r_2 | \underline{\theta}) \prod_{j=3}^{n} f(r_j | \underline{\theta}, r_1, \ldots, r_{j-1})$$

Take $\mathcal{L}(\underline{\theta}) = f(r_3, \ldots, r_n | \underline{\theta}, r_1, r_2)$, then

$$\mathcal{L}(\underline{\theta}) = -\frac{n-2}{2}\ln(2\pi) - \sum_{j=3}^{n} \ln \sigma_j^2 - \frac{1}{2}\sum_{j=3}^{n} \frac{(r_j - \mu - \phi_1 r_{j-1})^2}{\sigma_j^2}$$

$$\sigma_t^2 = \beta_1 \sigma_{t-1}^2 + \alpha_0 + \alpha_1 (r_{t-1} - \mu - \phi_1 r_{t-2})^2.$$

Use an initialisation of $\sigma_0^2 = 0$, $r_0 = r_{-1} = 0$.

AG4. The student t on ν degrees of freedom has density proportional to:

$$f(x) \propto \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2}$$

and the model is:

$$\frac{X_t}{\sigma_t} := \frac{(R_t - \nu - \phi_1 R_{t-1})}{\sigma_t} \sim t_\nu$$

so that

$$\mathcal{L}(\underline{\theta}) = \text{const} - \frac{\nu+1}{2} \sum_{j=3}^{n} \ln\left(1 + \frac{(r_j - \nu - \phi_1 r_{j-1})^2}{\nu \sigma_t^2}\right)$$

$$\sigma_t^2 = \beta_1 \sigma_{t-1}^2 + \alpha_0 + \alpha_1 (r_{t-1} - \mu - \phi_1 r_{t-2})^2.$$

Use an initialisation of $\sigma_0^2 = 0$, $r_0 = r_{-1} = 0$.

K1 Let $\underline{X}_t = \binom{X_{1t}}{X_{2t}}$ then the unique stationary solution of the equation written satisfies:

$$X_{1,t+1} = X_{2,t} + Z_{t+1} \qquad X_{2,t+1} = \theta Z_{t+1} \Rightarrow X_{2,t} = \theta Z_t \Rightarrow X_{1,t+1} = Z_{t+1} + \theta Z_t = Y_t$$

so that $Y_t = (1,0)\underline{X}_t = X_{1,t}$ as required. Rewriting gives:

$$\begin{cases} X_{1,t} = Z_t + \theta Z_{t-1} \\ X_{2,t} = \theta Z_t \end{cases} \Rightarrow \underline{X}_t = \begin{pmatrix} 1 & \theta \\ \theta & 0 \end{pmatrix} \begin{pmatrix} Z_t \\ Z_{t-1} \end{pmatrix}$$

as required.

K2 If $\{Y_t\}$ is an ARIMA(p,d,q) process, then it has representation:

$$(1-B)^d \phi(B) Y_t = \theta(B) Z_k.$$

Now suppose that U_t is a process that satisfies $(1-B)^d \phi(B) U_t = Z_t$, then $Y_t = \theta(B) U_t$:

$$(1-B)^d \phi(B) Y_t = (1-B)^d \phi(B) \theta(B) U_t = \theta(B)(1-B)^d \phi(B) U_t = \theta(B) Z_t.$$

Let

$$\psi(z) = 1 + \sum_{j=1}^{p+d} \psi_j z^j = (1-z)^d \phi(z) = 1 + \sum_{l=1}^{p+d} \left(\sum_{k=(l-d)\vee 0}^{l \wedge p} (-1)^{l-k} \begin{pmatrix} d \\ l-k \end{pmatrix} \phi_k \right) z^l$$

so

$$\psi_l = \sum_{k=(l-d)\vee 0}^{l \wedge p} (-1)^{l-k} \begin{pmatrix} d \\ l-k \end{pmatrix} \phi_k \qquad l = 1, \ldots, p+d$$

Derive state space rep. from:

$$\begin{cases} Y_t = \theta(B)U_t \\ \psi(B)U_{t+1} = Z_{t+1} \Rightarrow U_{t+1} = \sum_{j=1}^{p+d} \psi_j U_{t+1-j} + Z_{t+1} \end{cases}$$

The state space representation is now the same as that for the ARMA(p,q) given in lectures, with $r = \max(p+d, q+1)$.

K3 Let $D_p = |zI - F|$ denote the determinant of the $p \times p$ matrix thus defined; note that

$$zI - F = \begin{pmatrix} z & -1 & 0 & \dots & 0 & 0 \\ 0 & z & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & z & -1 \\ -\phi_p & -\phi_{p-1} & -\phi_{p-2} & \dots & -\phi_2 & z - \phi_1 \end{pmatrix}$$

then it is clear (consider p even and odd separately - in both cases) that

$$D_p = zD_{p-1} + \phi_p \Rightarrow \frac{D_p}{z^p} = \frac{D_{p-1}}{z^{p-1}} + \frac{\phi_p}{z^p} \Rightarrow D_p = z^p \sum_{j=1}^{p} \frac{\phi_j}{z^j}.$$

It follows that each eigenvalue λ_j satisfies either $\lambda_j = 0$ or

$$0 = |\lambda_j I - F| = \sum_{k=1}^{p} \frac{\phi_k}{\lambda_j^k}.$$

Since the process is causal, it follows that $\phi(z) := \sum_{k=1}^{p} \phi_k z^k$ does not have roots satisfying $|z| \leq 1$, hence that $\frac{1}{|\lambda_j|} > 1$, so that $|\lambda_j| < 1$. Hence stability.