Chapter 13

Spectral Analysis

13.1 Spectral analysis

The spectral density $f(\cdot)$ of a stationary time series $\{X_t : t \in \mathbb{Z}\}$ was introduced in Definition 2.3:

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda}\gamma(h), \quad -\pi \leq \lambda \leq \pi.$$

If $\sum_{h=-\infty}^{\infty} |\gamma(h)| < +\infty$, then it is clear that

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda)\, d\lambda. \tag{13.1}$$

The following theorem gives a complete characterisation of the class of functions which are spectral densities of stationary time series.

Theorem 13.1. *A real-valued function $f(\cdot)$ on $(-\pi, \pi]$ is the spectral density of a stationary time series if and only if*

1. $f(\lambda) = f(-\lambda)$,
2. $f(\lambda) \geq 0$,
3. $\int_{-\pi}^{\pi} f(\lambda)\, d\lambda < \infty$.

Proof Firstly, proving that if the conditions on f are satisfied then it is the spectral density of a stationary time series: Define $\gamma : \mathbb{Z} \to \mathbb{R}$ by Equation (13.1). For any n vector $(a_1, \ldots, a_n)$, set $A(\lambda) = \sum_{j=1}^{n} a_j e^{ij\lambda}$. Then

$$\sum_{j,k=1}^{n} a_j a_k \gamma(j-k) = \int_{-\pi}^{\pi} \left(\sum_{j,k} a_j a_k e^{ij\lambda} e^{-ik\lambda} \right) f(\lambda) d\lambda = \int_{-\pi}^{\pi} |A(\lambda)|^2 f(\lambda) d\lambda.$$

It follows that if f satisfies the conditions above, then this is clearly well defined and non-negative and hence γ is an ACVF by Theorem 1.12.

Now assume that f is the spectral density of a stationary time series and prove that it satisfies the conditions:

If f is a spectral density, then f satisfies equation (13.1) for some ACVF. It follows directly from equation (13.1) that $f(\lambda) = f(-\lambda)$ and $\int_{-\pi}^{\pi} f(\lambda)d\lambda = \gamma(0) < +\infty$. To show that $f(\lambda)$ is non negative: since γ is non negative definite, it follows that for any complex numbers a_j satisfying $\sum_j |a_j|^2 < +\infty$, $\sum_{j,k} a_j \overline{a}_k \gamma(j-k) \geq 0$. Let (c_k) be any real numbers satisfying $\sum_k c_k^2 < +\infty$. Then

$$0 \leq \sum_{j,k} c_j c_k e^{i\lambda(j-k)} \gamma(j-k) = \sum_k \sum_p c_k c_{p+k} e^{i\lambda p} \gamma(p) = \sum_h \left(\sum_k c_k c_{h+k} \right) e^{i\lambda h} \gamma(h).$$

A sequence of numbers $(c_{n,k})$ can be chosen so that for each h $\lim_{n\to+\infty} \sum_k c_{n,k} c_{n,h+k} = 1$. For example,

$$c_{n,0} = \frac{1}{(1 + 2\sum_{j=1}^{\infty} j^{-(1+(1/n))})^{1/2}} \qquad c_{n,k} = \frac{|k|^{-(1+(1/n))/2}}{(1 + 2\sum_{j=1}^{\infty} j^{-(1+(1/n))})^{1/2}} \qquad k \neq 0.$$

□

13.2 The spectral distribution

In some cases, the spectral density is not well defined. The following theorem gives an important generalisation:

Theorem 13.2 (Herglotz's theorem). *A complex-valued function $\gamma(\cdot)$ defined on $\mathbb{Z}$ is non-negative definite if and only if*

$$\gamma(h) = \int_{(-\pi,\pi]} e^{ih\nu}\, dF(\nu) \quad \textit{for all } h \in \mathbb{Z},$$

where $F(\cdot)$ is a right-continuous, non-decreasing, bounded function on $[-\pi, \pi]$ and $F(-\pi) = 0$.

Sketch of Proof Firstly, if

$$\gamma(h) = \int_{(-\pi,\pi]} e^{ih\nu}\, dF(\nu),$$

where F is right continuous, non decreasing, bounded on $[-\pi, \pi]$, then it is clear that γ is non-negative definite.

For the converse, assume that γ is non-negative definite and define $f_N(\nu)$ as:

$$f_N(\nu) := \frac{1}{2\pi N} \sum_{r,s=1}^{N} e^{-ir\nu} \gamma(r-s) e^{is\nu}.$$

Then

$$f_N(\nu) = \frac{1}{2\pi N} \sum_{|m|<N} (N - |m|)\gamma(m) e^{-im\nu} \geq 0 \quad \text{for all } \nu \in [-\pi, \pi].$$

Set

$$F_N(\lambda) = \int_{-\pi}^{\lambda} f_N(\nu)\, d\nu.$$

Then

$$\begin{aligned}\int_{(-\pi,\pi]} e^{ih\nu}\, dF_N(\nu) &= \frac{1}{2\pi} \sum_{|m|<N} \left(1 - \frac{|m|}{N}\right) \gamma(m) \int_{-\pi}^{\pi} e^{i(h-m)\nu}\, d\nu \\ &= \begin{cases} \left(1 - \frac{|h|}{N}\right)\gamma(h), & |h| < N, \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

There exists a distribution function F and a subsequence $\{N_k\}$ such that

$$\int_{(-\pi,\pi]} g(\nu)\, dF_{N_k}(\nu) \to \int_{(-\pi,\pi]} g(\nu)\, dF(\nu) \quad \text{as } k \to \infty,$$

for all continuous and bounded functions g. For each h, use the function

$$g(\nu) = e^{ih\nu}.$$

□

Let $\{X_t\}$ be a stationary time series with autocovariance function $\gamma_X(\cdot)$. It follows directly from Theorem 13.2 that $\gamma_X(\cdot)$ has a spectral representation

$$\gamma_X(h) = \int_{(-\pi,\pi]} e^{ih\nu}\, dF_X(\nu).$$

The function F is called *the spectral distribution function* of γ. If $F(\lambda) = \int_{-\pi}^{\lambda} f(\nu)\, d\nu$, then f is the spectral density of γ. If $\sum_{-\infty}^{\infty} |\gamma_X(h)| < \infty$ it follows that the spectral density ,f_X is well defined and $F_X(\lambda) = \int_{-\pi}^{\lambda} f_X(\nu)\, d\nu$ and

$$\gamma_X(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f_X(\lambda)\, d\lambda,$$

from which

$$f_X(\lambda) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{-in\lambda} \gamma_X(n).$$

For a real valued time series, the spectral distribution is symmetric; that is,

$$F_X(\lambda) = F_X(\pi^-) - F_X(-\lambda^-).$$

This may be seen by considering the limiting sequence of densities f_{N_k} which satisfy $f_{N_k}(\lambda) = f_{N_k}(-\lambda)$ so that

$$F_{N_k}(\lambda) = \int_{(-\pi,\lambda]} f_{N_k}(-x) dx = \int_{[-\lambda,\pi)} f_{N_k}(x) dx = F_{N_k}(\pi^-) - F_{N_k}(-\lambda^-)$$

from which the result follows. $\square$

When F_X has a density f_X it follows that $f_X(\lambda) = f_X(-\lambda)$ so that

$$\gamma_X(h) = \int_0^{\pi} \left(e^{ih\lambda} + e^{-ih\lambda}\right) f_X(\lambda)\, d\lambda = 2\int_0^{\pi} \cos(h\lambda) f_X(\lambda)\, d\lambda.$$

13.3 Spectral representation of a time series

The aim is to show that a time series $\{X_t\}$ has representation in terms of a process Z:

$$X_t = \int_{(-\pi,\pi]} e^{it\nu} dZ(\nu).$$

The process Z in the construction is a so-called *orthogonal increment process.* The following gives the definition of an orthogonal increment process and also the definition of integration with respect to such a process.

Definition 13.3 (Orthogonal-increment process)**.** *An orthogonal-increment process on $[-\pi, \pi]$ is a complex-valued process $\{Z(\lambda)\}$ such that the following three conditions hold:*

$$\begin{cases} \langle Z(\lambda), Z(\lambda)\rangle < \infty, & -\pi \le \lambda \le \pi, \\ \langle Z(\lambda), 1\rangle = 0, & -\pi \le \lambda \le \pi, \\ \langle Z(\lambda_4) - Z(\lambda_3), Z(\lambda_2) - Z(\lambda_1)\rangle = 0, & \textit{if } (\lambda_1, \lambda_2] \cap (\lambda_3, \lambda_4] = \emptyset \end{cases} \tag{13.2}$$

where $\langle X, Y\rangle = \mathbb{E}\left[X\overline{Y}\right]$.

The process $\{Z(\lambda)\}$ is assumed to be right continuous. That is,

$$\|Z(\lambda + \delta) - Z(\lambda)\| = \mathbb{E}\left[|Z(\lambda + \delta) - Z(\lambda)|^2\right] \to 0 \quad \text{as } \delta \downarrow 0.$$

Proposition 13.4. *Let $\{Z(\lambda) : -\pi \le \lambda \le \pi\}$ be an orthogonal-increment process. There exists a unique spectral distribution function F such that*

$$F(\lambda) = 0 \qquad \lambda \le -\pi,$$

$$F(\lambda) = F(\pi) \qquad \lambda \ge \pi$$

and

$$F(\mu) - F(\lambda) = \|Z(\mu) - Z(\lambda)\|^2, \quad -\pi \le \lambda \le \mu \le \pi. \tag{13.3}$$

Proof of Proposition 13.4 For F to satisfy the prescribed conditions, it is clear, setting $\lambda = -\pi$, that

$$F(\mu) = \|Z(\mu) - Z(-\pi)\|^2 \qquad -\pi \le \mu \le \pi.$$

To check that the function is non decreasing, use the orthgonality of $Z(\mu) - Z(\lambda)$ and $Z(\lambda) - Z(-\pi)$ for $-\pi \le \lambda \le \mu \le \pi$. This gives

$$\begin{aligned} F(\mu) &= \|Z(\mu) - Z(\lambda) + Z(\lambda) - Z(-\pi)\|^2 \\ &= \|Z(\mu) - Z(\lambda)\|^2 + \|Z(\lambda) - Z(-\pi)\|^2 \geq F(\lambda) \end{aligned}$$

so that the function is non decreasing. Furthermore, this also gives

$$F(\mu) = \|Z(\mu) - Z(\lambda)\|^2 + F(\lambda)$$

so that

$$F(\mu) - F(\lambda) = \|Z(\mu) - Z(\lambda)\|^2.$$

The same calculation gives

$$F(\mu + \delta) - F(\mu) = \|Z(\mu + \delta) - Z(\mu)\|^2 \to 0 \qquad \delta \downarrow 0.$$

□

13.3.1 Integration with respect to an Orthogonal Increment Process

The aim is to define an integral with respect to an orthogonal increment process:

$$I(f) = \int_{(-\pi,\pi]} f(\nu) dZ(\nu) \tag{13.4}$$

where $\{Z(\lambda) : -\pi \leq \lambda \leq \pi\}$ is an orthogonal increment process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and f is any function on $[-\pi, \pi]$ square integrable with respect to the distribution F associated with Z. Consider the two Hilbert spaces $L^2(\Omega,\, \mathcal{F},\, \mathbb{P})$ of all square-integrable random variables defined on $(\Omega,\, \mathcal{F},\, \mathbb{P})$ and $L^2([-\pi, \pi],\, \mathcal{B},\, F) = L^2(F)$ of all functions f such that $\int_{(-\pi,\pi]} |f(\nu)|^2 \, dF(\nu) < \infty$. The inner-product in $L^2(F)$ is defined by:

$$\langle f, g \rangle = \int_{(-\pi,\pi]} f(\nu)\overline{g}(\nu) \, dF(\nu).$$

Let $\mathcal{D} \subseteq L^2(F)$ be the set of all functions f of the form

$$f(\lambda) = \sum_{i=0}^{n} f_i \mathbf{1}_{(\lambda_i,\lambda_{i+1}]}(\lambda), \quad -\pi = \lambda_0 < \ldots < \lambda_{n+1} = \pi.$$

Let $I(f)$ denote the integration operation which, $f \in \mathcal{D}$, is defined by:

$$I(f) = \sum_{i=0}^{n} f_i \left(Z(\lambda_{i+1}) - Z(\lambda_i) \right).$$

The idea is to extend I to an isomorphism of $L^2([-\pi, \pi], \mathcal{B}, F) \equiv L^2(F)$ onto a subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

For $f, g \in \mathcal{D}$, there is a representation

$$f(\lambda) = \sum_{i=0}^{n} f_i \mathbf{1}_{(\lambda_i,\lambda_{i+1}]}(\lambda) \qquad g(\lambda) = \sum_{i=0}^{n} g_i \mathbf{1}_{(\lambda_i,\lambda_{i+1}]}(\lambda),$$

and with this representation,

$$\begin{aligned}\langle I(f), I(g)\rangle &= \langle\sum_{i=0}^{n} f_i(Z(\lambda_{i+1})-Z(\lambda_i)), \sum_{i=0}^{n} g_i(Z(\lambda_{i+1})-Z(\lambda_i))\rangle \\ &= \sum_{i=0}^{n} f_i\overline{g}_i(F(\lambda_{i+1})-F(\lambda_i)) = \int_{(-\pi,\pi]} f(\nu)\overline{g}(\nu)dF(\nu) = \langle f, g\rangle_{L^2(F)}.\end{aligned}$$

Now let $\overline{\mathcal{D}}$ denote the closure in $L^2(F)$ of $\mathcal{D}$ For any $f \in \overline{\mathcal{D}}$, there is a Cauchy sequence (f_n) in $\mathcal{D}$ such that $\|f_n - f\|_{L^2(F)} \to 0$. Therefore, define

$$I(f) = \lim_{n\to+\infty} I(f_n)$$

in the mean squared sense. To check,

$$\|I(f_n) - I(f_m)\| = \|I(f_n - f_m)\| = \|f_n - f_m\|_{L^2(F)}.$$

From this, the sequence $I(f_n)$ is Cauchy and hence convergent in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ hence $I(f)$ is well defined for $f \in \overline{\mathcal{D}}$. The mapping I is clearly an isomorphism and it can be extended to an isomorphism of $\overline{\text{spa}}\{\mathcal{D}\}$ onto a subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore $\mathcal{D}$ is dense in $L^2(F)$ and thus $\overline{\text{spa}}\{\mathcal{D}\} = L^2(F)$. It follows that I is an isomorphism of $L^2(F)$ onto the subspace $I(L^2(F))$ of $L^2(\Omega, \mathcal{F}, \mathbb{P})$. This is the formal definition of Equation (13.4).

Let $\{Z(\lambda) : -\pi \leq \lambda \leq \pi\}$ be an orthogonal increment process with associated distribution function F. Let

$$X_t = I(e^{it\cdot}) = \int_{(-\pi,\pi]} e^{it\nu} dZ(\nu) \qquad t \in \mathbb{Z}$$

then X is a stationary mean zero process with autocovariance function

$$\mathbb{E}[X_t\overline{X}_{t+h}] = \int_{(-\pi,\pi]} e^{i\nu h} dF(\nu).$$

Having established the basic definitions, the main result of this section may now be stated and proved.

Theorem 13.5 (The Spectral Representation Theorem)**.** *Let $\{X_t\}$ be a stationary time series with mean 0 and spectral distribution function F. Then there exists a right continuous orthogonal increment process such that:*

1.

$$\mathbb{E}[(Z(\lambda) - Z(-\pi))^2] = F(\lambda) \qquad -\pi \leq \lambda \leq \pi$$

2.

$$X_t = \int_{(-\pi,\pi]} e^{it\nu} dZ(\nu) \qquad \textit{with probability 1.} \tag{13.5}$$

Firstly, we need to define Hilbert space isomorphisms.

Definition 13.6 (Hilbert space isomorphisms). *An isomorphism of the Hilbert space $\mathcal{H}_1$ onto the Hilbert space $\mathcal{H}_2$ is a one to one mapping T of $\mathcal{H}_1$ onto $\mathcal{H}_2$ such that for all $f_1,\, f_2 \in \mathcal{H}_1$,*

$$T(af_1 + bf_2) = aTf_1 + bTf_2 \quad \textit{for all scalars } a \textit{ and } b$$

and

$$\langle Tf_1, Tf_2\rangle_{\mathcal{H}_2} = \langle f_1, f_2\rangle_{\mathcal{H}_1}.$$

The spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are said to be isomorphic if there is an isomorphism T of $\mathcal{H}_1$ onto $\mathcal{H}_2$. The inverse mapping T^{-1} is then an isomorphism of $\mathcal{H}_2$ onto $\mathcal{H}_1$.

The following properties are clear:

- $\|Tx\|_{\mathcal{H}_2} = \|x\|_{\mathcal{H}_1}$;
- $\|Tx_n - Tx\|_{\mathcal{H}_2} \to 0$ if and only if $\|x_n - x\|_{\mathcal{H}_1} \to 0$;
- $\{Tx_n\}$ is a Cauchy sequence in $\mathcal{H}_2$ if and only if $\{x_n\}$ is a Cauchy sequence in $\mathcal{H}_1$.
- For any set Λ, $TP_{\overline{\text{spa}}\{x_\lambda,\,\lambda\in\Lambda\}}x = P_{\overline{\text{spa}}\{Tx_\lambda,\,\lambda\in\Lambda\}}Tx$.

Let $\{X_t\}$ be a stationary time series defined on a probability space $(\Omega,\, \mathcal{F},\, \mathbb{P})$ with spectral distribution function F. Let $\mathcal{H}$ and $\mathcal{K}$ denote all finite linear combinations of $\{X_t\}$ and $\{e^{it\cdot}\}$ respectively. Their closures are:

$$\overline{\mathcal{H}} = \overline{\text{spa}}\{X_t,\, t\in\mathbb{Z}\} \subset L^2(\Omega,\, \mathcal{F},\, \mathbb{P}) \quad \text{and} \quad \overline{\mathcal{K}} = \overline{\text{spa}}\{e^{it\cdot},\, t\in\mathbb{Z}\} \subseteq L^2(F).$$

We lift the standard result from Fourier analysis that $\overline{\mathcal{K}} = L^2(F)$. The mapping

$$T\Bigg(\sum_{j=1}^{n} a_j X_{t_j}\Bigg) = \sum_{j=1}^{n} a_j e^{it_j\cdot}$$

is an isomorphism between $\mathcal{H}$ and $\mathcal{K}$ since

$$\begin{aligned}
&\left\langle T\Bigg(\sum_{j=1}^{n} a_j X_{t_j}\Bigg), T\Bigg(\sum_{k=1}^{m} b_k X_{s_k}\Bigg)\right\rangle \\
&\quad = \left\langle \sum_{j=1}^{n} a_j e^{it_j\cdot} \sum_{k=1}^{m} b_k e^{is_k\cdot}\right\rangle_{L^2(F)} = \sum_{j=1}^{n}\sum_{k=1}^{m} a_j \bar{b}_k \left\langle e^{it_j\cdot}, e^{is_k\cdot}\right\rangle_{L^2(F)} \\
&\quad = \sum_{j=1}^{n}\sum_{k=1}^{m} a_j \bar{b}_k \int_{(-\pi,\pi]} e^{i(t_j - s_k)\nu}\, dF(\nu) = \sum_{j=1}^{n}\sum_{k=1}^{m} a_j \bar{b}_k \left\langle X_{t_j}, X_{s_k}\right\rangle_{L^2(\Omega,\,\mathcal{F},\,\mathbb{P})} = \left\langle \sum_{j=1}^{n} a_j X_{t_j}, \sum_{k=1}^{m} b_k X_{s_k}\right\rangle.
\end{aligned}$$

T can be extended to an isomorphism of $\overline{\mathcal{H}}$ onto $L^2(F)$. The aim now is to find functions $g_\lambda(\nu) \in L^2(F)$

such that $T^{-1}g_\lambda = Z(\lambda)$ where $\{Z(\lambda)\}$ is an orthogonal-increment process with distribution function F. Such functions must satisfy:

$$\int_{(-\pi,\pi]} \Big(g_{\lambda_2}(\nu) - g_{\mu_2}(\nu)\Big)\Big(\,\overline{g_{\lambda_1}(\nu) - g_{\mu_1}(\nu)}\,\Big)\, dF(\nu) = \begin{cases} 0, & \text{if } \mu_1 < \lambda_1 < \mu_2 < \lambda_2, \\ F(\lambda_1) - F(\mu_2), & \text{if } \mu_1 < \mu_2 < \lambda_1 < \lambda_2. \end{cases}$$

This is obtained if, for $\mu < \lambda$,

$$g_\lambda(\nu) - g_\mu(\nu) = \mathbf{1}_{(\mu,\lambda]}(\nu) = \mathbf{1}_{(-\pi,\lambda]}(\nu) - \mathbf{1}_{(-\pi,\mu]}(\nu).$$

It is therefore it is natural to define

$$Z(\lambda) = T^{-1}\mathbf{1}_{(-\pi,\lambda]} \tag{13.6}$$

since clearly $\mathbf{1}_{(-\pi,\lambda]} \in L^2(F)$. For any $f \in \mathcal{D}$, i.e. for any f of the form

$$f(\lambda) = \sum_{i=0}^{n} f_i \mathbf{1}_{(\lambda_i,\lambda_{i+1}]}(\lambda), \quad -\pi = \lambda_0 < \ldots < \lambda_{n+1} = \pi,$$

it follows that:

$$I(f) = \sum_{i=0}^{n} f_i \left(Z(\lambda_{i+1}) - Z(\lambda_i)\right) = \sum_{i=0}^{n} f_i T^{-1}\mathbf{1}_{(\lambda_i,\lambda_{i+1}]} = T^{-1}f.$$

Since both I and T^{-1} can be extended to $L^2(F)$,

$$I = T^{-1} \quad \text{on } L^2(F).$$

Using $\{Z(\lambda)\}$ defined by Equation (13.6) gives:

$$\left\| X_t - \int_{(-\pi,\pi]} e^{it\nu}\, dZ(\nu) \right\|^2 = \left\| X_t - I(e^{it\cdot}) \right\|^2 = \left\| TX_t - TI(e^{it\cdot}) \right\|^2_{L^2(F)} = \left\| e^{it\cdot} - e^{it\cdot} \right\|^2_{L^2(F)} = 0$$

and hence the integral with respect to the orthogonal integral process of Equation (13.4) is established.

□

Remark Any $Y \in \overline{\text{spa}}\{X_t,\, t \in \mathbb{Z}\}$ has the representation

$$\int_{(-\pi,\pi]} f(\nu)\, dZ(\nu) \quad \text{for some } f \in L^2(F).$$

This follows from:

$$Y = IT(Y) = \int_{(-\pi,\pi]} TY(\nu)\, dZ(\nu) \quad \text{for } f = TY.$$

□

Remark Equation (13.5) has been derived only by using Hilbert spaces, i.e. by using 'geometric' or covariance properties. Distributional properties follow from the fact that

$$Z(\lambda) \in \overline{\text{spa}}\{X_t,\, t \in \mathbb{Z}\}.$$

If, for example, $\{X_t\}$ is Gaussian then, since linear combinations of (multivariate) normal random variables are normal, it follows that $\{Z(\lambda)\}$ is a Gaussian process. □

13.4 Prediction in the frequency domain

In Section 8.3, starting at page 133, it was mentioned that prediction based on infinitely many observations is best treated in the framework of spectral properties of the underlying time series. Let us first consider prediction based on finitely many observations. As usual, let $\{X_t\}$ be a zero-mean stationary time series and assume that we have observed $X_1, \ldots, X_n$ and want to predict X_{n+h}. Then

$$\widehat{X}_{n+h} = P_{\overline{\text{spa}}\{X_1,\ldots,X_n\}} X_{n+h} = \alpha_0 X_n + \ldots \alpha_{n-1} X_1$$

for some constants $\alpha_0, \ldots \alpha_{n-1}$. By Theorem 13.5 (the spectral representation theorem) Equation (13.5), it follows that:

$$\widehat{X}_{n+h} = \int_{(-\pi,\pi]} g(\nu)\, dZ(\nu) \tag{13.7}$$

where $g(\nu) = \sum_{k=0}^{n-1} \alpha_k e^{i(n-k)\nu}$. Assume now that we have infinitely many observations $(X_{n-j})_{j\geq 0}$. In Equation (8.28), we assumed that the predictor could be represented as an infinite sum. This is not always the case, but the predictor does always have a spectral representation of the form (13.7). The aim is now to determine the function $g(\cdot)$.

Let $\{X_t\}$ be a zero-mean stationary time series with spectral distribution function F and associated orthogonal-increment process $\{Z(\lambda)\}$. Recall from Section 13.3 that the mapping I defined by

$$I(g) = \int_{(-\pi,\pi]} g(\nu)\, dZ(\nu)$$

is an isomorphism of $L^2(F)$ onto the subspace $\overline{\mathcal{H}} = \overline{\text{spa}}\{X_t,\, t \in \mathbb{Z}\}$ such that

$$I(e^{it\cdot}) = X_t.$$

The idea is to compute projections, i.e. predictors, in $L^2(F)$ and then apply I. More precisely:

$$P_{\overline{\text{spa}}\{X_t,\ t\leq n\}} X_{n+h} = I\left(P_{\overline{\text{spa}}\{e^{it\cdot},\ t\leq n\}} e^{i(n+h)\cdot}\right).$$

Solution to the Prediction Problem The solution to the prediction problem is the function

$$g(.) := P_{\overline{\text{spa}}\{e^{it\cdot},\ t\leq n\}} e^{i(n+h)\cdot}.$$

There is *always* a solution of this form, even when the infinite sum is not well defined. Of course, when the infinite sum is not well defined, this solution may not be so useful, since it does not produce coefficients needed for a well defined linear combination of the original series. It does, however, give the coefficients up to some 'proportionality'. For $X_t = Z_t + Z_{t+1}$ (the MA(1) process), let $\widehat{X}_1 = P_{\overline{\text{spa}}(X_n: n\leq 0)}(X_1)$. Then the linear predictor may be regarded as the limit of $Y^{(n)} = \frac{1}{n}\sum_{j=-n}^{0} X_j$. This may be expressed in terms of a function $g(\nu)$, but not in terms of an infinite series.

13.4.1 Spectral Representation for Predicting a Causal Invertible ARMA(p,q)

The causal invertible ARMA(p,q) process has a representation of the predictor in terms of an infinite sum. The fact that the predictor has such a representation follows from Theorem 8.6; the following argument shows how to compute the coefficients from the spectral representation.

Consider a causal invertible ARMA(p,q) process $\{X_t\}$

$$\phi(B)X_t = \theta(B)Z_t, \qquad \{Z_t\} \sim \mathrm{WN}(0,\sigma^2).$$

From Equation (2.4) of Theorem 2.4,

$$f_X(\lambda) = \frac{\sigma^2}{2\pi}\left|\frac{\theta(e^{-i\lambda})}{\phi(e^{-i\lambda})}\right|^2 = a(\lambda)\overline{a(\lambda)}.$$

where the notation $\theta(x) = \sum \theta_j x^j$ and the quantity $a(\lambda)$ is defined as:

$$a(\lambda) = \frac{\sigma}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\psi_k e^{-ik\lambda} \qquad \text{where } \sum_{k=0}^{\infty}\psi_k z^k = \frac{\theta(z)}{\phi(z)}, \quad |z| \le 1.$$

From the projection theorem, it follows that $g(\cdot) := P_{\overline{\mathrm{spa}}\{e^{it\cdot},\ t\le n\}} e^{i(n+h)\cdot}$ satisfies:

$$\begin{aligned}\left\langle e^{i(n+h)\cdot} - g(\cdot), e^{im\cdot}\right\rangle_{L^2(F)} &= \int_{-\pi}^{\pi}\left(e^{i(n+h)\lambda} - g(\lambda)\right)\overline{e^{im\lambda}} f_X(\lambda)\,d\lambda \\ &= \int_{-\pi}^{\pi}\left(e^{i(n+h)\lambda} - g(\lambda)\right)e^{-im\lambda}a(\lambda)\overline{a(\lambda)}\,d\lambda = 0 \qquad \text{for } m \le n.\end{aligned}$$

It follows that:

$$\left(e^{i(n+h)\lambda} - g(\lambda)\right)a(\lambda)\overline{a(\lambda)} \in \mathcal{M}_+ := \overline{\mathrm{spa}}\{e^{im\cdot},\ m > n\} \subset L^2(d\lambda).$$

Since $\{X_t\}$ is invertible, it follows that $\frac{1}{a(\lambda)} = \frac{\sqrt{2\pi}}{\sigma}\sum_{k=0}^{\infty}\pi_k e^{-ik\lambda}$ and hence:

$$\frac{1}{a(\cdot)} \in \overline{\mathrm{spa}}\{e^{im\cdot},\ m \le 0\} \subset L^2(d\lambda),$$

from which:

$$\frac{1}{\overline{a(\cdot)}} \in \overline{\mathrm{spa}}\{e^{im\cdot},\ m \ge 0\} \subset L^2(d\lambda).$$

It follows that

$$\left(e^{i(n+h)\cdot} - g(\cdot)\right)a(\cdot) = \left(e^{i(n+h)\cdot} - g(\cdot)\right)a(\cdot)\overline{a(\cdot)}\cdot\frac{1}{\overline{a(\cdot)}} \in \mathcal{M}_+.$$

Set:

$$e^{i(n+h)\lambda}a(\lambda) = g(\lambda)a(\lambda) + \left(e^{i(n+h)\lambda} - g(\lambda)\right)a(\lambda).$$

Since $g(\cdot)a(\cdot) \in \overline{\text{spa}}\{e^{im\cdot},\ m \leq n\}$ and $\overline{\text{spa}}\{e^{im\cdot},\ m \leq n\} \perp \mathcal{M}_+$ in $L^2(d\lambda)$, and furthermore an element, here $e^{i(n+h)\cdot}a(\cdot)$, has a unique decomposition in two orthogonal Hilbert spaces, it follows that:

$$g(\lambda)a(\lambda) = \frac{\sigma}{\sqrt{2\pi}} e^{in\lambda} \sum_{k=0}^{\infty} \psi_{k+h} e^{-ik\lambda}.$$

It follows that:

$$g(\lambda) = e^{in\lambda} \frac{\sum_{k=0}^{\infty} \psi_{k+h} e^{-ik\lambda}}{\frac{\theta(e^{-i\lambda})}{\phi(e^{-i\lambda})}} := \sum_{j=0}^{\infty} \alpha_j e^{i(n-j)\lambda}.$$

An application of the map I gives:

$$P_{\overline{\text{spa}}\{X_t,\ t \leq n\}} X_{n+h} = \sum_{j=0}^{\infty} \alpha_j X_{n-j}.$$

Example 13.1 (AR(1) process)**.**

From Example 8.1, $\widehat{X}_{n+1} = \phi_1 X_n$ from which it follows directly that $\widehat{X}_{n+h} = \phi_1^h X_n$. This also follows from the derivation above. For the AR(1), $\theta(z) = 1$, $\phi(z) = 1 - \phi_1 z$ and $\psi_k = \phi_1^k$. It follows that:

$$g(\lambda) = e^{in\lambda} \frac{\sum_{k=0}^{\infty} \phi_1^{k+h} e^{-ik\lambda}}{\frac{1}{1-\phi_1 e^{-i\lambda}}} = \phi_1^h e^{in\lambda},$$

and the predictor follows. □

Spectral Analysis: Written Exercises

1. Let $\{X_t\}$ and $\{Y_t\}$ be stationary processes where

$$\begin{cases} X_t - \alpha X_{t-1} = W_t & \{W_t\} \sim \mathrm{WN}(0, \sigma^2) \\ Y_t - \alpha Y_{t-1} = X_t + Z_t & \{Z_t\} \sim \mathrm{WN}(0, \sigma^2) \end{cases}$$

where $|\alpha| < 1$ and $\{W_t\}$ and $\{Z_t\}$ are uncorrelated. Compute the spectral density of $\{Y_t\}$.

2. Let $\{X_t\}$ be the MA(1) process:

$$X_t = Z_t - 2Z_{t-1} \qquad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$$

Given $\epsilon > 0$, find a positive integer $k(\epsilon)$ and constants $a_0 = 1, a_1, \ldots, a_k$ such that the spectral density of the process

$$Y_t = \sum_{j=0}^{k} a_j X_{t-j}$$

satisfies

$$\sup_{-\pi \le \lambda \le \pi} \left| f_Y(\lambda) - \frac{1}{2\pi} \mathbf{V}(Y_t) \right| < \epsilon.$$

3. Let $\{X_t\}$ be a stationary time series with spectral density f satisfying $0 \le f(\lambda) \le K$ and $f(\pi) \neq 0$. Let f_n denote the spectral density of the series $Y_t = (1-B)^n X_t$.

   (a) Express $f_n(\lambda)$ in terms of $f_{n-1}(\lambda)$ and hence evaluate $f_n(\lambda)$.

   (b) Show that $\lim_{n\to+\infty} \frac{f_n(\lambda)}{f_n(\pi)} = 0$ for each $\lambda \in (-\pi, \pi)$.

4. Let $\{Z(\nu) : -\pi \le \nu \le \pi\}$ be an orthogonal increment process with associated distribution function F. Let $\psi \in L^2(F)$.

   (a) Prove that

   $$W(\nu) = \int_{(-\pi,\nu]} \psi(\lambda) dZ(\lambda) \qquad -\pi \le \nu \le \pi$$

   is an orthogonal increment process with associated distribution function

   $$G(\nu) = \int_{(-\pi,\nu]} |\psi(\lambda)|^2 dF(\lambda)$$

   (b) Prove that if $g \in L^2(G)$ then $g\psi \in L^2(F)$ and that

   $$\int_{(-\pi,\pi]} g(\lambda) dW(\lambda) = \int_{(-\pi,\pi]} g(\lambda)\psi(\lambda) dZ(\lambda).$$

   (c) Prove that if $|\psi| > 0$ (except possibly on a set of F-measure zero) then

   $$Z(\nu) - Z(-\pi) = \int_{(-\pi,\nu]} \frac{1}{\psi(\lambda)} dW(\lambda) \qquad -\pi \le \nu \le \pi$$

5. Let $\{X_t\}$ be a stationary process with spectral representation

$$X_t = \int_{(-\pi,\pi]} e^{it\nu} dZ_X(\nu) \qquad t = 0, \pm 1, \pm 2, \ldots$$

where

$$\lim_{h \downarrow 0} \frac{1}{F(\nu+h) - F(\nu-h)} \mathbb{E}\left[|Z_X(\nu+h) - Z_X(\nu-h)|^2 \right] = |\phi(\nu)|^2$$

F is a distribution function on $[-\pi, \pi]$, $\phi \neq 0$ F-almost everywhere, $\phi \in L^2(F)$. Prove that

$$Y_t = \int_{(-\pi,\pi]} e^{it\nu} \frac{1}{\phi(\nu)} dZ_X(\nu) \qquad t \in \mathbb{Z}$$

is a stationary process and compute its spectral density f.

6. Let $\{X_t\}$ be the moving average

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j} \qquad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$$

where

$$\psi_k = \begin{cases} \frac{1}{\pi}\left(\frac{\sin k}{k}\right) & k \neq 0 \\ \psi_0 & k = 0 \end{cases}$$

Find the spectral density of $\{X_t\}$.

7. Suppose that

$$X_t = A\cos\left(\frac{\pi t}{3}\right) + B\sin\left(\frac{\pi t}{3}\right) + Z_t + \frac{1}{2} Z_{t-1} \qquad t = 0, \pm 1, \pm 2, \ldots$$

where $\{Z_t\} \sim \mathrm{WN}(0, 1)$ and A and B are uncorrelated random variables with mean 0, variance 4 and satisfying $\mathbb{E}[AZ_t] = \mathbb{E}[BZ_t] = 0$ for each $t \in \mathbb{Z}$. Find the best linear predictor of X_{t+1} based on X_t and X_{t-1}. What is the mean squared error of the best linear predictor of X_{t+1} based on $\{X_j : -\infty < j \leq t\}$?

8. Recall the definition of *deterministic* and recall the Wold decomposition. Let $\{Y_t\}$ be the MA(1) process

$$Y_t = Z_t + 2.5 Z_{t-1} \qquad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2).$$

Define:

$$X_t = A\cos(\omega t) + B\sin(\omega t) + Y_t$$

where A and B are uncorrelated $(0, \sigma_1^2)$ variables and are uncorrelated with $\{Y_t\}$.

(a) Show that $\{X_t\}$ is non-deterministic.

(b) Determine the Wold decomposition of $\{X_t\}$.

(c) What are the components of the spectral distribution function of $\{X_t\}$ corresponding to the deterministic and purely non-deterministic components of the Wold decomposition?

Answers

1. Firstly, the spectral density for X may be computed by:

$$(X_{t+h} - \alpha X_{t+h-1})(X_t - \alpha X_t) = W_{t+h} W_t$$

giving

$$(1+\alpha^2)\gamma_X(h) - \alpha(\gamma_X(h+1) + \gamma_X(h-1)) = \sigma^2 \mathbf{1}_0(h)$$

so that

$$(1+\alpha^2) f_X(\lambda) - \alpha(e^{i\lambda} + e^{-i\lambda}) f_X(\lambda) = \frac{\sigma^2}{2\pi}$$

$$f_X(\lambda) = \frac{\sigma^2}{2\pi(1+\alpha^2 - 2\alpha\cos(\lambda))}.$$

For the spectral density of Y,

$$(1+\alpha^2)\gamma_Y(h) - \alpha(\gamma_Y(h+1) + \gamma_Y(h-1)) = \gamma_X(h) + \sigma^2 \mathbf{1}_0(h)$$

$$(1+\alpha^2 - 2\alpha\cos(\lambda)) f_Y(\lambda) = f_X(\lambda) + \frac{\sigma^2}{2\pi}$$

$$f_Y(\lambda) = \frac{\sigma^2}{2\pi(1+\alpha^2 - 2\alpha\cos(\lambda))^2} + \frac{\sigma^2}{2\pi(1+\alpha^2 - 2\alpha\cos(\lambda))}.$$

2. Here

$$\gamma_X(h) = \begin{cases} 5\sigma^2 & h = 0 \\ -2\sigma^2 & h = \pm 1 \\ 0 & |h| \geq 2 \end{cases}$$

so

$$f_Y(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{i\lambda h} \sum_{j_1=0}^{k} \sum_{j_2=0}^{k} a_{j_1} a_{j_2} \gamma_X(h - j_1 + j_2)$$

giving

$$f_Y(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{i\lambda h} \sum_{j=0}^{k} a_j \left(5a_{j+h} - 2a_{j+h-1} - 2a_{j+h+1}\right).$$

($a_0 = 1$, $a_j = 0$ for $j \geq k+1$. Solving $2 - 5x + 2x^2 = 0$, for $|x| < 1$ gives $x = 1.25 - \sqrt{1.25^2 - 1}$. Set

$$a_j = x^j \qquad j = 0, \ldots, k$$

for this value of x, then

$$\begin{aligned} f_Y(\lambda) &= \frac{\sigma^2}{2\pi}\Big((5-2x) + x^{2k-1}(5x-2)\Big) \\ &+ \frac{\sigma^2}{2\pi} \sum_{h=1}^{k} \Big(e^{i\lambda h} + e^{-i\lambda h})(5x^k - 4x^{k-1})\Big) - 2x^k \frac{\sigma^2}{2\pi}\Big(e^{i(\lambda(k+1))} + e^{-i(\lambda(k+1))}\Big) \\ &= \frac{1}{2\pi}\mathbf{V}(Y_t) + \frac{\sigma^2}{\pi}(5x^k - 4x^{k-1}) \sum_{h=1}^{k} \cos(\lambda h) - x^k \frac{2\sigma^2}{\pi}\cos(\lambda(k+1)). \end{aligned}$$

so that

$$|f_Y(\lambda) - \frac{1}{2\pi}\mathbf{V}(Y_t)| \le \frac{\sigma^2}{\pi}kx^{k-1}(5x-4) + \frac{2\sigma^2}{\pi}x^k$$

from which the result may easily be obtained. □

3. Let $Y = (I - B)X$, then

$$f_Y(\lambda) = 2(1 - \cos(\lambda))f_X(\lambda)$$

so

$$f_n(\lambda) = 2(1 - \cos(\lambda))f_{n-1}(\lambda), \qquad f_n(\lambda) = 2^n(1 - \cos(\lambda))^n f_X(\lambda).$$

$$\frac{f_n(\lambda)}{f_n(\pi)} = \frac{1}{2^n}(1 - \cos(\lambda))^n \frac{f_X(\lambda)}{f_X(\pi)}$$

from which the result follows.

4. (a) This follows from approximations:

$$W^{(n)}(\nu) = \sum_{\lambda_j \le \nu} \psi(\lambda_{n,j})\left(Z(\lambda_{n,j+1}) - Z(\lambda_{n,j})\right).$$

Clearly mean zero, orthogonal increments, and prescribed variance:

$$\mathbb{E}[W(\nu)\overline{W}(\nu)] = \int_{(-\pi,\nu]} |\psi(\lambda)|^2 dF(\lambda).$$

(b)

$$+\infty > \int |g|^2 dG = \int |g|^2|\psi|^2 dF \Rightarrow g\psi \in L^2(F)$$

(c) From previous part, require: $\frac{1}{\psi}\mathbf{1}_{(-\pi,\nu]} \in L^2(G)$ for each ν. Let $K = \text{ess min}|\psi(\lambda)|$ then

$$\int \frac{1}{|\psi(\lambda)|^2} dG(\lambda) \le \frac{1}{K^2}\int_{(-\pi,\pi]} |\psi(\lambda)|^2 dF(\lambda) < +\infty$$

because $\psi \in L^2(F)$. The result follows:

$$\int_{(-\pi,\nu]} \frac{1}{\psi(\lambda)} dW(\lambda) = \int_{(-\pi,\nu]} dZ(\lambda) = Z(\nu) - Z(-\pi).$$

5. Clearly $\mathbb{E}[Y_t] = 0$ and

$$\begin{aligned}\mathbb{E}[Y_t\overline{Y}_{t+s}] &= \int_{(-\pi,\pi]} e^{it\nu}e^{-i(t+s)\nu}\frac{1}{|\phi(\nu)|^2}dF(\nu) = \int_{(-\pi,\pi]} e^{-is\nu}\frac{1}{|\phi(\nu)|^2}dF(\nu) \\ &= \int_{-\pi}^{\pi} e^{-is\nu}d\nu = \begin{cases} 2\pi & s = 0 \\ 0 & s = \pm 1, \pm 2, \dots \end{cases}\end{aligned}$$

which does not depend on t, hence stationary, and

$$f_Y(\lambda) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{ih\lambda}\gamma_Y(h) = 1 \qquad \lambda \in (-\pi, \pi].$$

6.
$$f_X(\lambda) = \frac{\sigma^2}{2\pi} |\sum_j \psi_j e^{-ij\lambda}|^2$$

$$\begin{aligned}
\sum_j \psi_j e^{-ij\lambda} &= \psi_0 + \frac{1}{\pi} \sum_{j\neq 0} \frac{\sin(j)}{j} e^{-ij\lambda} = \psi_0 + \frac{1}{2\pi} \left(\sum_{j\neq 0} \frac{1}{2ij} \left(e^{ij(1-\lambda)} - e^{-ij(1+\lambda)} \right) \right) \\
&= \psi_0 + \frac{1}{2\pi} \left(\sum_{j=1}^{\infty} \frac{1}{2ij} \left(e^{ij(1-\lambda)} - e^{-ij(1-\lambda)} + e^{ij(1+\lambda)} - e^{-ij(1+\lambda)} \right) \right)
\end{aligned}$$

Now use:
$$\int_0^\infty e^{-(\alpha+i\beta)j} d\alpha = \frac{1}{j} e^{-i\beta j}.$$

so that:
$$\sum_{j=1}^{\infty} \frac{1}{j} e^{-i\beta j} = \sum_{j=1}^{\infty} \int_0^\infty e^{-(\alpha+i\beta)j} d\alpha = \int_0^\infty \frac{e^{-(\alpha+i\beta)}}{1-e^{-(\alpha+i\beta)}} d\alpha = \ln \frac{1}{1-e^{-i\beta}}$$

giving
$$\begin{aligned}
\sum_j \psi_j e^{-i\lambda j} &= \psi_0 + \frac{1}{4\pi i} \left(-\ln(1-e^{i(1-\lambda)}) + \ln(1-e^{-i(1-\lambda)}) - \ln(1-e^{i(1+\lambda)}) + \ln(1-e^{-i(1+\lambda)}) \right) \\
&= \psi_0 - \frac{1}{2\pi}
\end{aligned}$$

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \left(\psi_0 - \frac{1}{2\pi} \right)^2.$$

7.
$$\widehat{X}_{t+1} = \alpha X_t + \beta X_{t-1}.$$

From the definition,
$$\gamma(0) = 5.25 \qquad \gamma(1) = 4\cos\frac{\pi}{3} + \frac{1}{2} = 2.5$$
$$\mathbf{C}(\widehat{X}_{t+1}, X_t) = \mathbf{C}(X_{t+1}, X_t) = \gamma(1) = \alpha\gamma(0) + \beta\gamma(1) \Rightarrow 5.25\alpha + 2.5\beta = 2.5$$
$$\mathbf{C}(\widehat{X}_{t+1}, X_{t-1}) = 4\cos\frac{2\pi}{3} = -2 = \alpha\gamma(1) + \beta\gamma(0) \Rightarrow 2.5\alpha + 5.25\beta = -2.$$
$$\alpha \simeq 0.8504 \qquad \beta \simeq -0.7859$$

For the second part: $\mathbb{V}(Z_t) = 1$.

8. (a)
$$\mathcal{M}_{-\infty} = \{A, B\}$$

hence non-deterministic (since $\mathcal{M}_{-\infty}$ is non empty).

(b) For Wold decomposition, $V_t = A\cos(\omega t) + B\sin(\omega t)$, $\psi_0 = 1$, $\psi_1 = 2.5$, $\psi_k = 0$ for $k \neq 0, 1$.

$$X_t = V_t + \sum_j \psi_j Z_{t-j}$$

$$f_X(\lambda) = f_V(\lambda) + f_Y(\lambda)$$

$$\gamma_V(h) = \sigma_1^2 \cos(\omega h)$$

Spectral density: sum is not well defined; instead consider

$$G(\lambda) = \sigma_1^2 \sum_{h\neq 0} \frac{1}{ih} e^{-ih\lambda}\gamma_V(h) = \frac{1}{2}\sum_{h\neq 0}\frac{1}{ih}\left(e^{-i(\lambda-\omega)h} + e^{-i(\lambda+\omega)h}\right).$$

Now use (previous question)

$$\sum_{h=1}^{\infty} \frac{1}{ih} e^{i\beta h} = i\ln(1-e^{i\beta})$$

$$G(\lambda) = \frac{i}{2}\left(\ln(1-e^{-i(\lambda-\omega)}) - \ln(1-e^{i(\lambda-\omega)}) + \ln(1-e^{-i(\lambda+\omega)}) - \ln(1-e^{i(\lambda+\omega)})\right) = \lambda.$$

so that

$$f_V(\lambda) = \frac{d}{d\lambda}G(\lambda) = \frac{\sigma^2}{2\pi}$$

$$\gamma_Y(h) = \begin{cases} 7.25\sigma^2 & h = 0 \\ 2.5\sigma^2 & h = \pm 1 \\ 0 & |h| \geq 2. \end{cases}$$

$$f_Y(\lambda) = \frac{7.25\sigma^2}{2\pi} + 5\frac{\sigma^2}{2\pi}\cos(\lambda)$$
$$f_X = f_Y + f_V.$$